Extreme eigenvalue fluctutations for GUE

C. Donati-Martin

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1 Introduction

Random matrices were introduced in multivariate statistics, in the thirties by Wishart [Wis] and in theoretical physics by Wigner [Wig] in the fifties. Since then, the theory developed in a wide range of mathematics fields and physical mathematics. These lectures give a brief introduction to a well studied model : the Gaussian Unitary ensemble (GUE). The GUE is both a Wigner matrix (independent entries) and a model invariant by unitary conjugation.
The Gaussian structure enable to compute explicitely some quantities leading to a complete description of the global and local behavior of the spectrum.
In particular, in the asymptotics $N \to \infty$ where N is the size of the matrix, we shall study the fluctuation of the largest eigenvalue around its deterministic limit and prove a central limit theorem towards the so called Tracy-Widom distribution.

2 The Gaussian unitary ensemble - Definition

Let $\mathcal{H}_N$ be the space of Hermitian matrices of size N i.e. matrices M such that $M = M^*$. $\mathcal{H}_N$ is a real vector space of dimension N^2.

Definition 2.1 $\mathbf{X}_N$ *is a gaussian unitary matrix of size* N*, variance* σ^2 *(denoted by GUE(*$N; \sigma^2$*)) if :*
- $\mathbf{X}_N \in \mathcal{H}_N$*,*
- The entries of $\mathbf{X}_N$ *satisfy :*

$\mathbf{X}_N(i,i), i \leq N$, $\sqrt{2}\Re\mathbf{X}_N(j,k); j < k$, $\sqrt{2}\Im\mathbf{X}_N(j,k); j < k$ *are independent and distributed as* $N(0,\sigma^2)$.

One can give an equivalent definition (exercice):

Definition 2.2 *The distribution* P_{N,σ^2} *of GUE(*N*;* σ^2*) is given by*

$$dP_{N,\sigma^2}(M) = \frac{1}{Z_{N,\sigma^2}}\exp(-\frac{1}{2\sigma^2}\operatorname{Tr}(M^2))dM \tag{2.1}$$

where dM *denotes the Lebesgue measure on* $\mathcal{H}_N$ *given by*

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1\leq i<j\leq N} d\Re M_{ij} d\Im M_{ij}$$

and Z_{N,σ^2} *is a normalizing constant.*

We are interesting in the behavior of GUE(N; σ^2) in the asymptotics $N \longrightarrow \infty$. The good normalization to "see something" at the limit is to take $\sigma^2 = \frac{1}{N}$. In the following, we take $\sigma^2 = \frac{1}{N}$ and denote $P_{N,\frac{1}{N}}$ by P_N.

Remark 2.1 *1) GUE(*N*;* $\frac{1}{N}$*) is a Wigner matrix (independent entries above the diagonal).*

Proposition 2.1 *The distribution* P_N *is invariant under unitary conjugation : if* $\mathbf{X}_N$ *is* P_N *distributed, then* $U\mathbf{X}_N U^*$ *is also* P_N *distributed, for all unitary matrix* U.

Proof: We make the change of variable in (2.1) :

$$T_U : M \mapsto U^* M U, \quad \text{for } U \text{ unitary.}$$

T_U is an isometry on $\mathcal{H}_N$ since :

$$\|T_U(M)\|^2 = \operatorname{Tr}((T_U(M))^2) = \operatorname{Tr}(U^*MUU^*MU) = \operatorname{Tr}(M^2) = \|M\|^2.$$

Thus $|\det(T_U)| = 1$. On the other hand,

$$\exp(-\frac{1}{2\sigma^2}\operatorname{Tr}((U^*MU)^2) = \exp(-\frac{1}{2\sigma^2}\operatorname{Tr}(M^2)).$$

We conclude, by the change of variable formula, that :

$$\mathbb{E}(f(U\mathbf{X}_N U^*)) = \mathbb{E}(f(\mathbf{X}_N)). \square$$

2.1 Distribution of the eigenvalues of GUE(N, $\frac{1}{N}$)

Let $\mathbf{X}_N$ a random matrix distributed as GUE(N, $\frac{1}{N}$)) and we denote by $\lambda_1(\mathbf{X}_N) \leq \cdots \leq \lambda_N(\mathbf{X}_N)$ the ranked eigenvalues of $\mathbf{X}_N$.

Proposition 2.2 *The joint distribution of the eigenvalues $\lambda_1(\mathbf{X}_N) \leq \cdots \leq \lambda_N(\mathbf{X}_N)$ has a density with respect to Lebesgue measure equal to*

$$p_N(x) = \frac{1}{\bar{Z}_N} \prod_{1 \leq i < j \leq N} (x_j - x_i)^2 \exp(-\frac{N}{2} \sum_{i=1}^{N} x_i^2) 1_{x_1 \leq \cdots \leq x_N} \tag{2.2}$$

$\Delta(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$ is called the Vandermonde determinant and equals $\det(x_i^{j-1})_{1 \leq i,j \leq N}$.
We refer to Mehta [Me, Chap. 3], Deift [D, Chap. 5], Anderson-Guionnet-Zeitouni [AGZ, Chap. 2] for the proof of this proposition. It relies on the expression of the N^2 components of M in (2.1) in terms of the N eigenvalues (x_i) and $N(N-1)$ independent parameters (p_i) which parametrize the unitary matrix U in the decomposition $M = U\mathrm{diag}(x)U^*$. Heuristically, the term $\exp(-\frac{N}{2}\sum_{i=1}^N x_i^2)$ comes from the $\exp(-\frac{1}{2}N\,\mathrm{Tr}(M^2))$ in P_N and the square of the Vandermonde determinant comes from the Jacobian of the map $M \mapsto ((x_i), U)$ after integration on U on the unitary group.

Corollary 2.1 *If f is a bounded function of $\mathcal{H}_N$, invariant by the unitary transformations, that is $f(M) = f(UMU^*)$ for all unitary matrix U then $f(M) = f(\lambda_1(M), \cdots, \lambda_N(M))$ is a symmetric function of the eigenvalues and*

$$\begin{aligned}
\mathbb{E}[f(\mathbf{X}_N)] &= \frac{1}{Z_N} \int_{\mathcal{H}_N} f(M) \exp(-\frac{1}{2} N\,\mathrm{Tr}(M^2)) dM \\
&= \frac{1}{\bar{Z}_N} \int_{x_1 \leq \cdots \leq x_N} f(x_1, \cdots, x_N) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \exp(-\frac{N}{2} \sum_{i=1}^{N} x_i^2) d^N x \\
&= \frac{1}{N! \bar{Z}_N} \int_{\mathbb{R}^N} f(x_1, \cdots, x_N) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \exp(-\frac{N}{2} \sum_{i=1}^{N} x_i^2) d^N x
\end{aligned}$$

2.2 k-point correlation functions of the GUE

Let ρ_N a symmetric density distribution on $\mathbb{R}^N$, considered as the distribution of N particles X_i.

Definition 2.3 *Let $k \leq N$. The k- point correlation functions of (X_i) are defined by*

$$R_{N,k}(x_1, \cdots x_k) = \frac{N!}{(N-k)!} \int_{\mathbb{R}^{N-k}} \rho_N(x_1, \cdots x_N) dx_{k+1} \cdots dx_N. \tag{2.3}$$

The correlation functions are, up to a constant, the marginal distributions of ρ_N. Heuristically, $R_{N,k}$ is the probability of finding a particle at x_1, ... a particle at x_k. The factor $\frac{N!}{(N-k)!}$ comes from the choice of the k particles and the symmetry of ρ_N (see the computation below). We have, using the symmetry of ρ_N,

$$\begin{aligned}
\mathbb{E}[\prod_{i=1}^N (1 + f(X_i)] &= \mathbb{E}[\sum_{k=0}^N \sum_{i_1 < \cdots < i_k} f(X_{i_1}) \cdots f(X_{i_k})] \\
&= \sum_{k=0}^N \mathbb{E}[\sum_{i_1 < \cdots < i_k} f(X_{i_1}) \cdots f(X_{i_k})] \\
&= \sum_{k=0}^N \left(\begin{array}{c} N \\ k \end{array} \right) \mathbb{E}[f(X_1) \cdots f(X_k)] \\
&= \sum_{k=0}^N \frac{1}{k!} \frac{N!}{(N-k)!} \mathbb{E}[f(X_1) \cdots f(X_k)]
\end{aligned}$$

and thus,

$$\mathbb{E}\left[\prod_{i=1}^N (1 + f(X_i))\right] = \sum_{k=0}^N \frac{1}{k!} \int_{\mathbb{R}^k} f(x_1) \cdots f(x_k) R_{N,k}(x_1, \cdots x_N) dx_1 \cdots dx_k. \tag{2.4}$$

The correlation functions enables to express probabilistic quantities as:

1) *The hole probability:*

Take $f(x) = 1_{\mathbb{R} \setminus I} - 1$ where I is a Borel set of $\mathbb{R}$. Then, the left-hand side of (2.4) is the probability of having no particles in I. Therefore,

$$\mathbb{P}(\forall i, X_i \notin I) = \sum_{k=0}^N \frac{(-1)^k}{k!} \int_{I^k} R_{N,k}(x_1, \cdots x_k) dx_1 \cdots dx_k.$$

In particular, for $I =]a, +\infty[$,

$$\mathbb{P}(\max X_i \leq a) = \sum_{k=0}^N \frac{(-1)^k}{k!} \int_{[a,\infty[^k} R_{N,k}(x_1, \cdots x_k) dx_1 \cdots dx_k. \tag{2.5}$$

2) *the density of state:*

$$\mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(X_i)] = \frac{1}{N}\int_{\mathbb{R}} f(x)R_{N,1}(x)dx$$

that is $\frac{1}{N}R_{N,1}(x)dx$ represents the expectation of the empirical distribution $\mathbb{E}[\frac{1}{N}\sum_{i=1}^{N}\delta_{X_i}]$.

We now compute the correlation functions associated to the symmetric density of the (unordered) eigenvalues of the GUE

$$\rho_N(x) = \frac{1}{N!\bar{Z}_N}\prod_{1\le i<j\le N}(x_i - x_j)^2 \exp(-\frac{N}{2}\sum_{i=1}^{N} x_i^2).$$

Proposition 2.3 *The correlation functions of the eigenvalues of GUE(N, $\frac{1}{N}$) are given by*

$$R_{N,k}(x_1,\cdots x_k) = \det(K_N(x_i,x_j))_{1\le i,j\le k} \tag{2.6}$$

where the kernel K_N is given by

$$K_N(x,y) = \exp(-\frac{N}{4}(x^2+y^2))\sum_{l=0}^{N-1} q_l(x)q_l(y) \tag{2.7}$$

where

$$q_l(x) = \left(\frac{N}{2\pi}\right)^{1/4}\frac{1}{\sqrt{2^l l!}}h_l(\sqrt{N/2}\ x) \tag{2.8}$$

where h_l are the Hermite polynomials.

The process of the eigenvalues of GUE is said to be a **determinantal process**.

Proof: Since the value of a determinant does not change if we replace a column by the column + a linear combination of the others, we have that the Vandermonde determinant $\Delta(x) = \det(P_{j-1}(x_i))$ if P_j denotes a polynomial of degree j with higher coefficient equal to 1.

Let $w(x) = \exp(-\frac{N}{2}x^2)$ and define the orthonormal polynomials q_l with respect to w such that:

- q_l is of degree l, $q_l(x) = a_l x^l + \dots$ with $a_l > 0$.

- $\int_{\mathbb{R}} q_l(x)q_p(x)w(x)dx = \delta_{pl}$.
$(q_l)_l$ also depends on N and is up to a scaling factor the family of Hermite polynomials (to be discussed later).
Thus, $\Delta(x) = C_N \det(q_{j-1}(x_i))$ and, using $(\det(A))^2 = \det(A)\det(A^T)$, we have:

$$\begin{aligned}\rho_N(x) &= \frac{1}{\tilde{Z}_N}\prod_{1\le i\le N} w(x_i)\left(\det(q_{j-1}(x_i))\right)^2 \\ &= \frac{1}{\tilde{Z}_N}\prod_{1\le i\le N} w(x_i)\det\left(\sum_{l=1}^{N} q_{l-1}(x_i)q_{l-1}(x_j)\right)_{i,j\le N} \\ &= \frac{1}{\tilde{Z}_N}\det\left(K_N(x_i,x_j)\right)_{i,j\le N}\end{aligned}$$

where

$$K_N(x,y) = \sqrt{w(x)}\sqrt{w(y)}\sum_{l=1}^{N} q_{l-1}(x)q_{l-1}(y) = \sum_{l=0}^{N-1}\phi_l(x)\phi_l(y)$$

where $\phi_l(x) = \sqrt{w(x)}q_l(x)$. The sequence $(\phi_l)_l$ is orthornormal for the Lebesgue measure dx. From the orthonormality of (ϕ_l), it is easy to show that the kernel K_N satisfies the properties:

$$\int_{\mathbb{R}} K_N(x,x)dx = N$$

$$\int_{\mathbb{R}} K_N(x,y)K_N(y,z)dy = K_N(x,z).$$

This proves (2.6) for $k = N$ (up to a constant). The general case follows from the Lemma:

Lemma 2.1 *Let $J_N = (J_{ij})$ a matrix of size N of the form $J_{ij} = f(x_i,x_j)$ with f satisfying:*

1. $\int_{\mathbb{R}} f(x,x)dx = C$
2. $\int_{\mathbb{R}} f(x,y)f(y,z)dy = f(x,z)$

Then,

$$\int_{\mathbb{R}} \det(J_N)dx_N = (C-N+1)\det(J_{N-1})$$

where J_{N-1} is a matrix of size $N-1$ obtained from J_N by removing the last row and column containing x_N.

Proof of Lemma 2.1: Exercice
Hint: use the formula giving the determinant :

$$\det(J_N) = \sum_{\sigma \in \Sigma_N} \epsilon(\sigma) \prod_{i=1}^{N} f(x_i, x_{\sigma(i)})$$

where Σ_N is the set of permutations on $\{1, \dots, N\}$ and ϵ stands for the signature of a permutation.
Next, consider two cases for σ : $\sigma(N) = N$ and $\sigma(N) \neq N$. $\square$

In the case of GUE, $J = (K_N(x_i, x_j))$ satisfies the hypothesis of the lemma with $C = N$.

$$\int_{\mathbb{R}} \det(K_N(x_i, x_j))_{i,j \leq N} \; dx_N = (N - N - 1) \det(K_N(x_i, x_j))_{i,j \leq N-1}$$

$$\int_{\mathbb{R}} \det(K_N(x_i, x_j))_{i,j \leq N-1} \; dx_{N-1} = (N - N - 2) \det(K_N(x_i, x_j))_{i,j \leq N-2}.$$

Integrating over all the variables gives:

$$\int_{\mathbb{R}} \det(K_N(x_i, x_j))_{i,j \leq N} \; dx_1 \dots dx_N = N!$$

and therefore, $\tilde{Z}_N = N!$. Integrating over the $N - k$ variables $dx_{k+1}, \dots dx_N$ gives:

$$\int_{\mathbb{R}} \det(K_N(x_i, x_j))_{i,j \leq N} dx_1 \dots dx_N = (N - k)! \det(K_N(x_i, x_j))_{i,j \leq k}$$

and

$$\begin{aligned} R_{k,N}(x_1, \dots x_k) &= \frac{N!}{(N-k)!} \int \rho_N(x_1, \dots x_N) dx_{k+1} \dots dx_N \\ &= \frac{1}{(N-k)!} \int \det(K_N(x_i, x_j))_{i,j \leq N} dx_{k+1} \dots dx_N \\ &= \det(K_N(x_i, x_j))_{i,j \leq k} \end{aligned}$$

This proves (2.6) and (2.7). It remains to determine the polynomials q_l. Let h_l the Hermite polynomial of degree l defined by:

$$h_l(x) = (-1)^l e^{x^2} \left(\frac{d}{dx} \right)^l (e^{-x^2}).$$

These polynomials (see [Sz]) are orthogonal with respect to $e^{-x^2}dx$, $\int_{\mathbb{R}} h_l^2(x)e^{-x^2}dx = 2^l l!\sqrt{\pi}$ and the coefficient of x^l in h_l is 2^l. Then, it is easy to see that q_l given by (2.8) are orthonormal with respect to $\exp(-\frac{N}{2}x^2)dx$. □

Corollary 2.2 *Let $\bar{\mu}_{H_N}(dx) = \mathbb{E}[\mu_{H_N}(dx)]$ where μ_{H_N} is the spectral distribution of GUE(N, $\frac{1}{N}$), then $\bar{\mu}_{H_N}(dx)$ is absolutely continuous with respect to Lebesgue measure with density f_N given by:*

$$f_N(x) = \frac{1}{N}R_{N,1}(x,x) = \frac{1}{N}K_N(x,x),\ x \in \mathbb{R}.$$

f_N is called the density of state.

2.3 The local regime

Let us denote, for I a Borel set of $\mathbb{R}$, $\nu_N(I) = \#\{i \leq N; \lambda_i \in I\} = N\mu_{H_N}(I)$ where λ_i are the eigenvalues of GUE(N, $\frac{1}{N}$). From Wigner's theorem, as $N \longrightarrow \infty$, $\nu_N(I) \sim N \int_I f_{sc}(x)dx)$ a.s. where f_{sc} is the density of the semi-circular distribution μ_{sc}. The spacing between eigenvalues is of order $1/N$. In the local regime, we consider an interval I_N whose size tends to 0 as $N \longrightarrow \infty$. Two cases have to be considered.

a) Inside the bulk: Take $I_N = [u - \varepsilon_N, u + \varepsilon_N]$ with u such that $f_{sc}(u) > 0$ that is $u \in]-2,2[$. Then, $\nu_N(I_N)$ has the order of a constant for $\varepsilon_N \sim \frac{1}{N}$. This suggest to introduce new random variables (renormalisation) l_i by

$$\lambda_i = u + \frac{l_i}{Nf_{sc}(u)},\ i = 1, \ldots N.$$

The mean spacing between the rescaled eigenvalues l_i is 1. Straightforward computations give:

Lemma 2.2 *The correlation functions R^{bulk} of the distribution of $(l_1, \ldots, l_N)$ are given in terms of the correlation functions of the (λ_i) by*

$$R^{bulk}_{N,k}(y_1, \ldots, y_k) = \frac{1}{(Nf_{sc}(u))^k} R_{N,k}(u + \frac{y_1}{Nf_{sc}(u)}, \ldots, u + \frac{y_k}{Nf_{sc}(u)}). \quad (2.9)$$

We shall see in the next subsection the asymptotic of the correlation functions R^{bulk} (or the kernel K_N).

b) At the edge of the spectrum: $u = 2$ (or -2). $f_{sc}(u) = 0$.

$$\nu_N([2-\varepsilon_N, 2]) = \frac{N}{2\pi}\int_{2-\varepsilon}^{2}\sqrt{4-x^2}dx = \frac{N}{2\pi}\int_0^{\varepsilon}\sqrt{4y-y^2}dy \sim CN\varepsilon^{3/2}.$$

So the normalisation at the edge is $\varepsilon = \frac{1}{N^{2/3}}$ and we define the rescaled correlation functions by:

$$R_{N,k}^{edge}(y_1,\dots,y_k) = \frac{1}{(N^{2/3})^k}R_{N,k}(2+\frac{y_1}{N^{2/3}},\dots,2+\frac{y_k}{N^{2/3}}). \tag{2.10}$$

From (2.5) and (2.10),

$$\mathbb{P}[N^{2/3}(\lambda_{max}-2) \le a] = \sum_{k=0}^{N}\frac{(-1)^k}{k!}\int_{[a,\infty[^k} R_{N,k}^{edge}(x_1,\cdots x_k)dx_1\cdots dx_k. \tag{2.11}$$

where λ_{max} is the maximal eigenvalue of the GUE.
The asymptotic of R^{edge} will be given in the next section.

2.4 Limit kernel

The asymptotic of the correlation functions relies on asymptotic formulas for the orthonormal polynomials q_l for $l \sim N$. We have the following:

Proposition 2.4 *(Plancherel - Rotach formulas, [Sz])*
Let $(h_n)_n$ denote the Hermite polynomials.
1) If $x = \sqrt{2n+1}\cos(\Phi)$ with $\varepsilon \le \Phi \le \pi - \varepsilon$,

$$\exp(-x^2/2)h_n(x) = b_n(\sin(\Phi))^{-1/2}\{\sin[(\frac{n}{2}+\frac{1}{4})(\sin(2\Phi)-2\Phi)+3\pi/4]+O(\frac{1}{n})\} \tag{2.12}$$

where $b_n = 2^{n/2+1/4}(n!)^{1/2}(\pi n)^{-1/4}$.
2) If $x = \sqrt{2n+1} + 2^{-1/2}n^{-1/6}t$, t bounded in $\mathbb{C}$,

$$\exp(-x^2/2)h_n(x) = \pi^{1/4}2^{n/2+1/4}(n!)^{1/2}(n)^{-1/12}\{Ai(t)+O(\frac{1}{n})\}$$

where Ai is Airy's function, that is the solution of the differential equation

$$y'' = xy \text{ with } y(x) \sim_{x\to+\infty} \frac{1}{2\sqrt{\pi}}x^{-1/4}\exp(-\frac{2}{3}x^{3/2}).$$

From these formulas, one can show:

Theorem 2.1

$$\lim_{N\to\infty} R_{n,k}^{bulk}(y_1,\dots,y_k) = \det(K^{bulk}(y_i,y_j))_{i,j\le k} \tag{2.13}$$

where

$$K^{bulk}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} \tag{2.14}$$

$$\lim_{N\to\infty} R_{n,k}^{edge}(y_1,\dots,y_k) = \det(K^{edge}(y_i,y_j))_{i,j\le k} \tag{2.15}$$

where

$$K^{edge}(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{(x-y)} \tag{2.16}$$

Sketch of Proof of (2.13)**:** From (2.9), (2.6), we may find the limit of

$$\frac{1}{Nf_{sc}(u)} K_N(u + \frac{s}{Nf_{sc}(u)}, u + \frac{t}{Nf_{sc}(u)}).$$

We express the kernel K_N given by (2.7) thanks to Cristoffel-Darboux formula (see Appendix)

$$\begin{aligned} K_N(X,Y) &= \frac{k_{N-1}}{k_N}\frac{q_N(X)q_{N-1}(Y) - q_N(Y)q_{N-1}(X)}{X-Y}\exp(-\frac{N}{4}(X^2+Y^2)) \\ K_N(X,Y) &= \frac{1}{2^N(N-1)!\sqrt{\pi}}\frac{h_N(\sqrt{N/2}\;X)h_{N-1}(\sqrt{N/2}\;Y) - h_N(\sqrt{N/2}\;Y)h_{N-1}(\sqrt{N/2}\;X)}{X-Y} \\ &\quad \times\exp(-\frac{N}{4}(X^2+Y^2)) \end{aligned}$$

with k_N the highest coefficient in q_N. Then, set $X = u + \frac{s}{Nf_{sc}(u)}$, $Y = u + \frac{t}{Nf_{sc}(u)}$, $u = 2\cos(\Phi)$. Then, $f_{sc}(u) = \frac{\sin(\Phi)}{\pi}$ and

$$x = \sqrt{N/2}X = \sqrt{2N}(\cos(\Phi) + \frac{\pi s}{2N\sin(\Phi)}).$$

In order to use Plancherel-Rotach formulas, we express x as

$$x = \sqrt{2N+1}\cos(\Phi_N).$$

A development gives

$$\Phi_N = \Phi + \frac{a}{2N} + O(\frac{1}{N^2})$$

with $a = \frac{1}{2\tan(\Phi)} - \frac{\pi s}{\sin^2(\Phi)}$. Then,

$$\sin(2\Phi_N) - 2\Phi_N = (\sin(2\Phi) - 2\Phi) + \frac{a}{N}(\cos(2\Phi) - 1) + O(\frac{1}{N^2})$$

and

$$(\sin(\Phi_N))^{-1/2} = (\sin(\Phi))^{-1/2}(1 + O(1/N)).$$

Formula (2.12) gives:

$$\exp(-x^2/2)h_N(x) = b_N(\sin(\Phi))^{-1/2}\{\sin[(\frac{N}{2}+\frac{1}{4})(\sin(2\Phi)-2\Phi)+\frac{a}{2}(\cos(2\Phi)-1)+\frac{3\pi}{4}]+O(1/N)\}$$

We make the same transformations for $e^{-x^2/2}h_{N-1}(x)$, $e^{-y^2/2}h_N(y)$, $e^{-y^2/2}h_{N-1}(y)$ giving ϕ'_N, Ψ_N and Ψ'_N associated respectively to:

$$a' = -\frac{1}{2\tan(\Phi)} - \frac{\pi s}{\sin^2(\Phi)},\ b = \frac{1}{2\tan(\Phi)} - \frac{\pi t}{\sin^2(\Phi)},\ b' = -\frac{1}{2\tan(\Phi)} - \frac{\pi t}{\sin^2(\Phi)}.$$

Then, we replace in the product $h_N(x)h_{N-1}(y)$ the product of two sinus by a trigonometric formula and then in the difference, we obtain a linear combination of cosinus, The difference of two of them cancels using that $a' + b = a + b'$. Then, we use again a trigonometric formula. After some computations, the kernel K^{bulk} appears. The Airy kernel appears, using the second formula of Plancherel-Rotach.

Corollary 2.3 *(Fluctuations of λ_{max})*
The fluctuations of the largest eigenvalue of the GUE around 2 are given by:

$$\mathbb{P}(N^{2/3}(\lambda_{max} - 2) \leq x) = F_2(x)$$

where F_2 is called the Tracy-Widom distribution and is given by

$$F_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{]x,\infty[^k} \det(K^{edge}(y_i, y_j))_{1\leq i,j\leq k} d^k y.$$

F_2 can be written $F_2(x) = \det(I - \mathbf{K})_{L^2(x,\infty)}$ where $\mathbf{K}$ is the integral operator on L^2 with kernel $K^{edge}(x, y)$ and the det *is the Fredholm determinant.*

2.5 Comments

1. The computation of the correlation functions which have a determinantal form is specific to the unitary case and do not hold for the GOE case.

2. We refer to [D], [Me] for others computations involving correlation functions such as the spacing distribution.

3. The Tracy-Widom distribution can also be expressed as

$$F_2(x) = \exp\left(-\int_x^\infty (y-x)q^2(y)dy)\right)$$

   where $q''(x) = xq(x) + q^3(x)$ with $q(x) = Ai(x)(1 + o(1))$ as $x \longrightarrow \infty$. The function q is called the solution of Painlevé II equation (see [TW]).

4. One of the important ideas of the theory is that of **universality**. This idea is that the asymptotic distribution of some statistic of the eigenvalues in the local scale does not depend very much on the ensemble (like in the TCL), that is the sine kernel (2.14) or the Airy kernel is "universal" and appears in other models of Hermitian random matrices.
   This has been shown for

   - Hermitian Wigner matrices: Soshnikov [So] (for the edge), Johansson [J] for a particular class of matrices. The universality in the bulk was proved in great generality independently by two teams : Erdos, Schlein, Yau [ESY] and their collaborators on one hand and Tao, Vu [TV] on the other hand.

   - unitary invariant ensemble of the form

$$P_N(dM) = C_N \exp(-N \operatorname{Tr}(V(M)))dM$$

   for a weight V satisfying some assumptions. See [DKMVZ], [PS]. Note that the GUE corresponds to the quadratic weight $V(x) = \frac{1}{2}x^2$. For example, for the Wishart ensemble (associated to the Laguerre polynomials), we have the same asymptotic kernel as in the GUE, while the density of state is not universal (semicircular for GUE and Marchenko-Pastur distribution for Wishart). The main difficulty for general V is to derive the asymptotics of orthogonal polynomials. This can be done using Riemann-Hilbert techniques (see [D]).

3 Appendix

3.1 Change of variable formula

Let U and V two open sets in $\mathbb{R}^d$, g a C^1 diffeomorphism from U to V. If ϕ is a measurable function on V, positive or Lebesgue integrable , then

$$\int_U \phi(g(x)) \left| \frac{Dg}{Dx}(x) \right| dx = \int_V f(y) dy$$

where $\frac{Dg}{Dx}(x)$ is the Jacobian of g, that is the determinant of the Jacobian matrix

$$\left(\frac{\partial g_i}{\partial x_j} \right)_{1 \leq i,j \leq d}.$$

3.2 Van der Monde determinant

Recall the van der Monde determinant:

$$\det(x_i^{j-1})_{1 \leq i,j \leq n} = \prod_{i<j} (x_j - x_i).$$

3.3 Orthogonal polynomials (see [D], [Sz])

Let $w(x)$ a positive function on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^m w(x) dx < \infty$ for all $m \geq 0$. On the space of real polynomials $P[X]$, we consider the scalar product

$$(P|Q) = \int_{\mathbb{R}} P(x)Q(x)w(x)dx.$$

Then the orthogonalisation procedure of Schmidt enables to construct of sequence of orthogonal polynomials (p_l): p_l is of degree l and

$$\int_{\mathbb{R}} p_m(x)p_n(x)w(x)dx = 0 \text{ if } m \neq n.$$

We denote by a_l the coefficient of x^l in $p_l(x)$ and $d_l = \int_{\mathbb{R}} p_l(x)^2 w(x) dx$.
Example: If $w(x) = \exp(-x^2)$, the Hermite polynomials h_l are orthogonal with $a_l = 2^l$ and $d_l = 2^l l! \sqrt{\pi}$.

Christoffel-Darboux formula: We consider a family of orthonormal polynomials (p_l) $(d_l = 1)$ for the weight w. We denote by K_n the kernel defined by:

$$K_n(x,y) = \sum_{l=0}^{n-1} p_l(x)p_l(y).$$

K_n is the kernel associated to the orthogonal projection in the space of polynomials of degree less than $n-1$. This kernel has a simple expression based upon a three terms recurrence relation between the (p_l):

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \alpha_{n-1} p_{n-1}(x)$$

for some coefficients $\alpha_n = \frac{a_n}{a_{n+1}}$ and β_n (depending on a_n and the coefficient b_n of x^{n-1} in p_n).
From this relation, one obtains:

$$K_n(x,y) = \alpha_{n-1} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y} \tag{3.1}$$

For the orthonormal polynomials q_l defined in (2.8),

$$a_l (= a_{N,l}) = \left(\frac{N}{2\pi}\right)^{1/4} \left(\sqrt{\frac{N}{l!}}\right)^l$$

and $\alpha_{N-1} = 1$.

3.4 Fredholm determinant

Let $K(x,y)$ a bounded measurable kernel on a space (X,μ) where μ is a finite measure on X. The Fredholm determinant of K is defined by

$$D(\lambda) = \det(I - \lambda K) := 1 + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \int_{X^k} \det(K(x_i,x_j))_{1\le i,j\le k}\, \mu(dx_1)\dots\mu(dx_k).$$

The serie converges for all λ.

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