THE FLEXIBILITY AND RIGIDITY OF LAGRANGIAN AND LEGENDRIAN SUBMANIFOLDS - ADDITIONAL TOPICS FOR THE PROBLEM SESSION

1. Relation between generating functions and the action functional

In order to discuss the relation between generating functions and the action functional, we need a definition of generationg functions which is slightly more general than the one given in the lectures.

1.1. **Definition of generating functions.** Let $p : E \to B$ be a fiber bundle, and consider a smooth function $S : E \to \mathbb{R}$ on the total space. Let $\Sigma_S \subset E$ be the set of *fiber critical points* of S, i.e. the set of points e of E that are critical points of the restriction of S to the fiber through e:

$$\Sigma_S := \{\, e \in E \,|\, e \text{ critical point of } S|_{p^{-1}\big(p(e)\big)} \,\}.$$

- Note that, in general, Σ_S is not necessarily a smooth submanifold of E. Show that if $dS : E \to T^*E$ is transverse to the *fiber normal bundle*

$$N_E := \{\, (e, \mu) \in T^*E \,|\, \mu = 0 \text{ on } \ker dp(e) \,\}.$$

  then Σ_S is a smooth submanifold of E, of dimension equal to the dimension of B. In the following, we will always assume this condition.

Given a point e of Σ_S, we can associate to it an element $v^*(e)$ of $T^*_{p(e)}B$ by defining

$$v^*(e) := dS(\hat{X})$$

for $X \in T_{p(e)}B$, where $\hat{X}$ is any vector in T_eE with $p_*(\hat{X}) = X$. The element $v^*(e)$ of $T^*_{p(e)}B$ is sometimes called the *Lagrange multiplier* of e.

- Show that $v^*(e)$ is well-defined, i.e. it does not depend on the choice of the lift $\hat{X}$ of X.
- Show that the map $i_S : \Sigma_S \to T^*B$, $e \mapsto \big(p(e), v^*(e)\big)$ is a Lagrangian immersion. [*Hint*: Show that ${i_S}^*\lambda_{\text{can}} = d(S|\Sigma_S)$].
- Show that the map $j_S : \Sigma_S \to J^1B$, $e \mapsto \big(p(e), v^*(e), S(e)\big)$ is a Legendrian immersion.

S is called a generating function for the Lagrangian submanifold $L := i_S(\Sigma_S)$ of T^*B and for the Legendrian submanifold $\Lambda := j_S(\Sigma_S)$ of J^1B.

- Show that if $E = B$ and $p : E \to B$ is the identity then L can be identified to the graph of the differential of S, and Λ to the graph of the 1-jet of S.
- Assume that $p : E \to B$ is a trivial vector bundle, i.e. $E = B \times \mathbb{R}^N$. Show that in this case we recover the definition given in the lectures, i.e. we have that

$$L = \{\, (q, p) \in T^*B \,|\, \exists\, \xi \in \mathbb{R}^N \text{ such that } \frac{\partial S}{\partial \xi}(q, \xi) = 0 \text{ and } \frac{\partial S}{\partial q}(q, \xi) = p \,\}.$$

  and

$$\Lambda = \{\, (q, p, z) \in J^1B \,|\, \exists\, \xi \in \mathbb{R}^N \text{ such that } \frac{\partial S}{\partial \xi}(q, \xi) = 0\,,\ \frac{\partial S}{\partial q}(q, \xi) = p \text{ and } S(q, \xi) = z \,\}.$$

1.2. **The Action Functional.** Consider an exact symplectic manifold $(M, \omega = -d\lambda)$ and let $H_t : M \to \mathbb{R}$ be a time-dependent Hamiltonian. Then H_t determines a functional $\mathcal{A}_H$ on the space of paths $\gamma : [t_0, t_1] \to M$. It is called the *action functional* and is defined by

$$\mathcal{A}_H(\gamma) := \int_{t_0}^{t_1} \Big(\lambda\big(\dot\gamma(t)\big) + H_t\big(\gamma(t)\big) \Big)\, dt\,.$$

- Show that γ is a critical point of $\mathcal{A}_H$ (with respect to variations with fixed endpoints) if and only if it is a trajectory of the Hamiltonian flow of H_t.

Let now B be a smooth manifold, and consider the space E of all paths $\gamma : [0,1] \to T^*B$ that begin at the 0-section. E can be seen as a fiber bundle over B, with projection $p : E \to B$ given by $\gamma \mapsto \pi\big(\gamma(1)\big)$ where π is the projection of T^*B into B. Given a time-dependent Hamiltonian $H_t : T^*B \to \mathbb{R}$ we consider the function $S : E \to \mathbb{R}$ given by $S(\gamma) := \mathcal{A}_H(\gamma)$.

- Show that the set $\Sigma_S \subset E$ of fiber critical points of $S : E \to \mathbb{R}$ is given by the set of trajectories of the Hamiltonian flow of H_t.
- Show that, given a fiber critical point γ, the Lagrange multiplier $v^*(\gamma)$ is the vertical component of $\gamma(1)$.
- Conclude that S is a "generating function"for the image of the 0-section by the time-1 map of the Hamiltonian flow of H_t.

Note that S is not a generating function in the sense of the definition given in 1.1, because E is not a finite-dimensional manifold. In the next section we will show how to construct a finite-dimensional appoximation of E, and thus obtain a true generating function for any Lagrangian submanifold of T^*B which is Hamiltonian isotopic to the 0-section.

1.3. **Finite-dimensional reduction.** The following construction was given by Laudenbach and Sikorav [3]. It is inspired by the method of "broken geodesics "of Morse theory [4, 1].

Let H_t be a time-dependent Hamiltonian on T^*B. For every integer N we will define the space E_N of *broken Hamiltonian trajectories* of H_t with $N-1$ singularities and N smooth pieces. Element of E_N will be of the form

$$e = (q_0, X, P)$$

where q_0 is a point of B, $X = (X_1, \cdots, X_{N-1})$ is an $(N-1)$-tuple of vectors $X_i \in T_{q_0}B$ and $P = (P_1, \cdots, P_{N-1})$ is an $(N-1)$-tuple of linear maps $P_i \in T_{q_0}{}^*B$. The broken Hamiltonian trajectory of H_t associated to e is defined as follows. The first smooth piece, for $t \in [0, \frac{1}{N}]$, is obtained by following the Hamiltonian flow of H_t in T^*B starting at the point $(q_0, 0)$ of the 0-section. The endpoint of this first smooth piece will be some other point of T^*B, that we denote by ${z_1}^-$. The second smooth piece of the broken Hamiltonian trajectory will not necessarily start from ${z_1}^-$ but from a point ${z_1}^+$ which is uniquely determined by ${z_1}^-$, X_1 and P_1 in a way that we will describe later. The second smooth piece of the broken Hamiltonian trajectory is obtained by following the flow of H_t for $t \in [\frac{1}{N}, \frac{2}{N}]$, starting from ${z_1}^+$. The endpoint will be some point ${z_2}^-$ of T^*B. The third smooth piece of the broken Hamiltonian trajectory is obtained by following the Hamiltonian flow of H_t for $t \in [\frac{2}{N}, \frac{3}{N}]$, starting at the point ${z_2}^+$ that is uniquely determined by ${z_2}^-$, X_2 and P_2 by the procedure we are going to describe later. We continue in this way to describe the whole broken trajectory for $t \in [0,1]$. It has $N-1$ jumps for $t = \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}$ and N smooth pieces from ${z_i}^+$ to ${z_i}^-$ for $t \in [\frac{i}{N}, \frac{i+1}{N}]$, $i = 0, \cdots N-1$. In order to describe the jumps we need to fix a Riemannian metric on M. Then TM and T^*M have the associated Levi-Civita connection. We describe now the first jump, from ${z_1}^-$ to ${z_1}^+$. The point ${z_1}^+ = ({q_1}^+, {p_1}^+)$ is determined by ${z_1}^- = ({q_1}^-, {p_1}^-)$, $X_1 \in T_{q_0}B$ and $P_1 \in T_{q_0}{}^*B$ as follows. Denote by $\gamma(t) = \big(q(t), p(t)\big)$ for $t \in [0, \frac{1}{N}]$ the first smooth piece of the broken Hamiltonian trajectory, from $(q_0, 0)$ to ${z_1}^- = ({q_1}^-, {p_1}^-)$. In particular, $q(t)$ for $t \in [0, \frac{1}{N}]$ is a smooth path in B. We take the vector $\overline{X}_1 \in T_{{q_1}^-}B$ and the 1-form $\overline{P}_1 \in T_{{q_1}^-}{}^*B$ that are obtained by parallel transport, with respect to the Levi-Civita connection, of $X_1 \in T_{q_0}B$ and $P_1 \in {T_{q_0}}^*B$ along $q(t)$, $t \in [0, \frac{1}{N}]$. The point ${z_1}^+$ is then defined to be

$$ {z_1}^+ = ({q_1}^+, {p_1}^+) $$

where ${q_1}^+ := \exp_{{q_1}^-}(\overline{X}_1)$ and ${p_1}^+ := {}^t\big(d\exp_{{q_1}^-}(\overline{X}_1)\big)^{-1}(\overline{p}_1)$. The other jumps are defined similarly.

Consider the projection $p : E_N \to B$ that sends e to the projection to B of the endpoint of the broken Hamiltonian trajectory associated to e.

We define a function $S : E_N \to \mathbb{R}$ by

$$ S(e) := \sum_{i=1}^{N-1} < p_i, X_i > + \mathcal{A}_H(\gamma_i) $$

where γ_i denotes the i-th smooth piece of the broken Hamiltonian trajectoy of H associated to e.

- Show that the fiber critical points of $S : E_N \to \mathbb{R}$ are the unbroken Hamiltonian trajectories of H_t.
- Show that S is a generating function for the image of the 0-section by the time-1 map of the Hamiltonian flow of H_t.
- The above construction works only if N is sufficiently big. Try to understand where this is needed.
- Is there an analogue construction also in the contact case, for any Legendrian submanifold of J^1B which is contact isotopic to the 0-section?

2. Symplectic rigidity in $\mathbb{R}^{2n}$

2.1. **Generating functions for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$.** Recall from the lectures that if B is a compact smooth manifold then any Lagrangian submanifold L of the cotangent bundle T^*B has a (unique, up to equivalence) generating function quadratic at infinity. We will now see how this result can be applied to obtain a generating function quadratic at infinity for any compactly supported Hamiltonian symplectomorphism ϕ of $\mathbb{R}^{2n}$. We will do this by associating to ϕ a Lagrangian submanifold Γ_ϕ of T^*S^{2n}, and then by taking the generating function of Γ_ϕ.

Let ϕ be a Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ (not necessarily compactly supported, for the moment). We will now see how to associate to ϕ a Lagrangian submanifold Γ_ϕ of $T^*\mathbb{R}^{2n}$.

Recall first that the graph of ϕ is a Lagrangian submanifold of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$. It is defined by

$$ \mathrm{gr}(\phi) = \{ \big(q, \phi(q)\big) \mid q \in M \}. $$

- Show that the map $\tau : \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$, $\tau(x, y, X, Y) = (x, Y, Y - y, x - X)$ is a symplectomorphism. Show that it sends the diagonal to the 0-section.

We define $\Gamma_\phi := \tau\big(\mathrm{gr}(\phi)\big)$. It is a Lagrangian submanifold of $T^*\mathbb{R}^{2n}$.

- Let Ψ_ϕ be the symplectomorphism of $T^*\mathbb{R}^{2n}$ defined by the diagram

$$ \begin{array}{ccc} \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} & \xrightarrow{\mathrm{id}\times\phi} & \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \\ \downarrow \tau & & \downarrow \tau \\ T^*\mathbb{R}^{2n} & \xrightarrow[\Psi_\phi]{} & T^*\mathbb{R}^{2n}. \end{array} $$

  Show that $\Gamma_\phi = \Psi_\phi\,(\text{0-section})$. Use this to conclude that Γ_ϕ is Hamiltonian isotopic to the 0-section.

Assume now that ϕ is compactly supported.

- Show that Γ_ϕ coincides with the 0-section outside a compact set. Using this, explain how we can identify Γ_ϕ to a Lagrangian submanifold of T^*S^{2n} (that we will still denote by Γ_ϕ).

We have thus seen that we can associate to any compactly supported Hamiltonian symplectomorphism ϕ of $\mathbb{R}^{2n}$ a Lagrangian submanifold Γ_ϕ of T^*S^{2n}. Since S^{2n} is compact, by the existence and uniqueness theorems of generating functions we have that Γ_ϕ has a (unique, up to equivalence) generating function $S : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$ quadratic at infinity, i.e. $S = S_0 + Q_\infty$ with S_0 compactly supported and Q_∞ a non-degenerate quadratic form on $\mathbb{R}^N$.

We will now see that critical points of S correspond to fixed points of ϕ, and that the critical values coincide with the symplectic action of the corresponding fixed points.

Definition 2.1. *Let ϕ be a Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. The **symplectic action** of a fixed point q of ϕ is defined by*

$$\mathcal{A}_\phi(q) := \mathcal{A}_H\left(\phi_t(q)\right) = \int_0^1 \left(\lambda_0(X_t) + H_t\right)\left(\phi_t(q)\right) dt$$

*where ϕ_t is a Hamiltonian isotopy joining ϕ to the identity, X_t the vector field generating it and H_t the corresponding Hamiltonian. The **action spectrum** of ϕ is the set $\Lambda(\phi)$ of all values of $\mathcal{A}_\phi$ at fixed points of ϕ.*

(Recall the definition of the *action functional* $\mathcal{A}_H$ given in Part 1)

Let ϕ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$, with generating function S. Recall that this means that Γ_ϕ is the image of $i_S : \Sigma_S \to T^*\mathbb{R}^{2n}$.

- Show that fixed points of ϕ correspond to critical points of S. More precisely, show that a point q of $\mathbb{R}^{2n}$ is a fixed point of ϕ if and only if ${i_S}^{-1}(q, 0)$ is a critical point of S.

Suppose now that we have a fixed point q of ϕ, and take a point p in $\mathbb{R}^{2n}$ outside the support of ϕ.

- Show that
$$\mathcal{A}_\phi(q) = -\int_{\gamma \sqcup \phi(\gamma)^{-1}} \lambda_0$$
where γ is any path in $\mathbb{R}^{2n}$ joining p to q. [*Hint*: consider the map $u : [0,1] \times [0,1] \to \mathbb{R}^{2n}$, $u(s,t) = \phi_t\left(\gamma(s)\right)$ and apply Stokes' theorem to $u^*\omega_0 = -d(u^*\lambda_0)$.]
- Show that $S\left({i_S}^{-1}(q, 0)\right) = -\int_{\gamma \sqcup \phi(\gamma)^{-1}} \lambda_0$. [*Hint*: Show first that if a Lagrangian submanifold L of T^*B is generated by $S : E \to \mathbb{R}$, i.e. L is the image of $i_S : \Sigma_S \to T^*B$ then we have that $\int_\gamma \lambda_{\mathrm{can}} = S\left({i_S}^{-1}(y)\right) - S\left({i_S}^{-1}(x)\right)$ for any path γ in L joining two points x and y.]

This concludes the proof that $S\left({i_S}^{-1}(q, 0)\right) = \mathcal{A}_\phi(q)$, i.e. that the critical value of a critical point of the generating function coincides with the symplectic action of the corresponding fixed point.

2.2. **Symplectic Homology.** We will associate homology groups first to compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$, by considering relative homology of sublevel sets of the generating function, and then, by a limit process, to domains of $\mathbb{R}^{2n}$.

Let ϕ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. Given real numbers a, b not belonging to the action spectrum of ϕ and such that $-\infty < a < b \leq \infty$, we define the **$k$-th symplectic homology group** of ϕ with respect to the values a, b by

$${G_k}^{(a,b]}(\phi) := H_{k+\iota}\left(E^b, E^a\right)$$

where E^c, for $c \in \mathbb{R}$, denotes the sublevel set $\{\, x \in E \,|\, S(x) \leq c \,\}$ of a generating function $S : E \to \mathbb{R}$ for ϕ and ι is the index of the quadratic at infinity part of S.

- Show that the groups ${G_k}^{(a,b]}(\phi)$ are well-defined, i.e. do not depend on the choice of the generating function.

We will now see that they are invariant by conjugation with a Hamiltonian symplectomorphism. We need the following two lemmas.

Lemma 2.2. *Let ψ be a symplectomorphism of $\mathbb{R}^{2n}$. Then $\Lambda(\psi\phi\psi^{-1}) = \Lambda(\phi)$.*

(A proof of this fundamental fact can be found for example in [2, 5.2].)

Lemma 2.3. *Let f_t, $t \in [0,1]$, be a continuous 1-parameter family of functions defined on a compact manifold M. Suppose that $a \in \mathbb{R}$ is a regular value of all f_t. Then there exists an isotopy θ_t of M such that $\theta_t({M^a}_0) = {M^a}_t$, where ${M^a}_t := \{\, x \in M \mid f_t(x) \leq a \,\}$.*

- Using Lemmas 2.2 and 2.3, prove that for any ϕ and ψ in $\mathrm{Ham}^c(\mathbb{R}^{2n})$ we have an induced isomorphism
$$\psi^* : {G_*}^{(a,b]}(\psi\phi\psi^{-1}) \longrightarrow {G_*}^{(a,b]}(\phi).$$

Consider now a domain $\mathcal{U}$ of $\mathbb{R}^{2n}$. Given $a, b \in \mathbb{R}$ we denote by ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ the set of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ that are the time-1 map of a Hamiltonian function which is supported in $\mathcal{U}$ and whose action spectrum does not contain a and b. We will consider the partial order $\leq$ on ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ defined as follows. We say that $\phi_1 \leq \phi_2$ if $\phi_2\, {\phi_1}^{-1}$ can be written as the time-1 flow of a non-negative Hamiltonian function. It can be proved that if $\phi_0 \leq \phi_1$ then there are generating functions S_0, $S_1 : E \longrightarrow \mathbb{R}$ for Γ_{ϕ_0}, Γ_{ϕ_1} respectively such that $S_0 \leq S_1$.

- Show that if $\phi_1 \leq \phi_2$ than we have an induced homomorphism
$${\lambda_1}^2 : {G_k}^{(a,b]}(\phi_2) \longrightarrow {G_k}^{(a,b]}(\phi_1).$$
- Show that, given ϕ_1, ϕ_2, ϕ_3 in ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ with $\phi_1 \leq \phi_2 \leq \phi_3$, it holds ${\lambda_3}^2 \circ {\lambda_2}^1 = {\lambda_3}^1$ and ${\lambda_i}^i = \mathrm{id}$.
- Conclude that $\{{G_k}^{(a,b]}(\phi_i)\}_{\phi_i \in {\mathrm{Ham}_{a,b}}^c(\mathcal{U})}$ is an inversely directed system of groups[1].

We define the **k-th symplectic homology group** ${G_k}^{(a,b]}(\mathcal{U})$ of $\mathcal{U}$ with respect to the values a, b to be the inverse limit of the inversely directed system $\{{G_k}^{(a,b]}(\phi_i)\}_{\phi_i \in {\mathrm{Ham}_{a,b}}^c(\mathcal{U})}$. Note that ${G_k}^{(a,b]}(\mathcal{U})$ can be calculated by any sequence $\phi_1 \leq \phi_2 \leq \phi_3 \leq \cdots$ such that the associated Hamiltonians get arbitrarily large.

- Show that for any domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ and any Hamiltonian symplectomorphism ψ we have an induced isomorphism
$$\psi^* : {G_*}^{(a,b]}\big(\psi(\mathcal{U})\big) \longrightarrow {G_*}^{(a,b]}(\mathcal{U}).$$

[1] Recall the definition of an inversely directed system of groups. Let $(I, \leq)$ be a *directed partially ordered set*, i.e. a set I with a partial order $\leq$ such that for any two elements i and j of I there exists a third element k such that $i \leq k$ and $j \leq k$. A family of groups $\{A_i\}_{i\in I}$ is called an *inversely directed system of groups* if for every $i \leq j$ there exists a homomorphism $f_{ij} : A_j \to A_i$ such that the following properties are satisfied: $f_{ii} = \mathrm{id}$ and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$. The *inverse limit* of the inversely directed system $\{A_i\}_{i\in I}$ is then defined by
$$\varprojlim A_i := \{\underline{a} \in \prod_{i\in I} A_i \,|\, a_i = f_{ij}(a_j) \text{ for all } \ i \leq j \,\}.$$

2.3. **Spectral invariants.**

2.3.1. *Invariants for Lagrangian submanifolds.* Let B be a closed manifold and L a Lagrangian submanifolds of T^*B Hamiltonian isotopic to the 0-section. We know that L has a generating function quadratic at infinity $S : E \to \mathbb{R}$. We are going to see now how to define invariants for L by selecting critical values of its generating function S. Recall that S is only defined up to fiber-preserving diffeomorphism, stabilization and addition of a constant. While the first two operations do not affect the critical values of the function, addition of a constant does, so in order to get well-defined invariants we first need to normalize generating functions. This can be done by fixing a point P in B and only considering the set $\mathcal{L}_P$ of Lagrangian submanifolds L of T^*B which are Hamiltonian isotopic to the 0-section and intersect it at P. We can then normalize generating functions by requiring the critical value of the critical point corresponding to P to be 0.

Let L be an element of $\mathcal{L}_P$ with generating function $S : E \longrightarrow \mathbb{R}$. We will now explain how to use a cohomology class u of B to select a critical value of S, in order to get an invariant $c(u, L)$.

Recall that we can assume that $E = B \times \mathbb{R}^N$ and S is of the form $S = S_0 + Q_\infty$ where S_0 is compactly supported and Q_∞ is a non-degenerate quadratic form on $\mathbb{R}^N$. We denote by E^a, for $a \in \mathbb{R}$, the sublevel set of S at a i.e. $E^a = \{ x \in E \,|\, S(x) \leq a \}$ and by $E^{-\infty}$ the set E^{-a} for a big enough (i.e. such that $-a$ is smaller that all critical values of S_0). Note that up to homotopy equivalence $E^{-\infty}$ is the same for all elements of $\mathcal{L}_P$. We will study the inclusion $i_a : (E^a, E^{-\infty}) \hookrightarrow (E, E^{-\infty})$, and the induced map on cohomology

$$i_a{}^* : H^*(B) \equiv H^*(E, E^{-\infty}) \longrightarrow H^*(E^a, E^{-\infty}).$$

Here $H^*(B)$ is identified with $H^*(E, E^{-\infty})$ via the Thom isomorphism

$$T : H^*(B) \xrightarrow{\cong} H^*\big(D(E^-), S(E^-)\big)$$

where E^- denotes the subbundle of E where Q_∞ is negative definite. Note that this isomorphism shifts the grading by the index of Q_∞. Note also that by excision $H^*\big(D(E^-), S(E^-)\big)$ is isomorphic to $H^*(E, E^{-\infty})$. For $|a|$ big enough we have $H^*(E^a, E^{-\infty}) \equiv 0$ if $a < 0$, and $i_a{}^* = \mathrm{id}$ if $a > 0$. So we can define

$$c(u, L) := \inf \{ a \in \mathbb{R} \mid i_a{}^*(u) \neq 0 \}$$

for $u \neq 0$ in $H^*(B)$.

- Show that $c(u, L)$ is well-defined, i.e. it does not depend on the choice of the generating function used to calculate it.
- Show that $c(u, L)$ is a critical value of S.

2.3.2. *Spectral invariants for Hamiltonian symplectomorphisms of* $\mathbb{R}^{2n}$. We now apply the construction of the previous section to the special case of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$.

Consider a compactly supported Hamiltonian symplectomorphism ϕ of $\mathbb{R}^{2n}$. Define

$$c^+(\phi) := c(\mu, \Gamma_\phi) \text{ and } c^-(\phi) := c(1, \Gamma_\phi)$$

where Γ_ϕ is the Lagrangian submanifold of T^*S^{2n} constructed in Section 2.1 and μ and 1 are respectively the orientation and the unit classes of S^{2n}. Note that Γ_ϕ intersects the 0-section at the point at infinity of S^{2n}. This point plays the role of the point P of the previous section.

- Show that $c(\phi) = \mathcal{A}_\phi(q)$ for some fixed point q of ϕ. In particular, deduce that $c^+(\mathrm{id}) = c^-(\mathrm{id}) = 0$.

Moreover we have the following properties, that can be proved by methods of algebraic topology..

Lemma 2.4. *For all ϕ, ψ in $Ham^c\,(\mathbb{R}^{2n})$ it holds:*

(i) $c^+(\phi) \geq 0$ *and* $c^-(\phi) \leq 0$.
(ii) $c^+(\phi) = c^-(\phi) = 0$ *if and only if* ϕ *is the identity.*
(iii) $c^-(\phi) = -c^+(\phi^{-1})$.
(iv) $c^+(\phi\psi) \leq c^+(\phi) + c^+(\psi)$ *and* $c^-(\phi\psi) \geq c^-(\phi) + c^-(\psi)$.

We will now prove that c^+ and c^- are invariant by conjugation.

- Show that for all ϕ, ψ in $\mathrm{Ham}^c(\mathbb{R}^{2n})$ it holds $c(\phi) = c(\psi\phi\psi^{-1})$. [*Hint*: Let ψ be the time-1 flow of a Hamiltonian isotopy ψ_t, and consider the continuous map $t \mapsto c(\psi_t\phi{\psi_t}^{-1})$. Use Lemma 2.2.]

Recall that the relation $\leq$ on the group $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is defined by setting $\phi_1 \leq \phi_2$ if $\phi_2\,{\phi_1}^{-1}$ can be written as the time-1 flow of a non-negative Hamiltonian function.

- Show that c^+ and c^- are monotone with respect to $\leq$, i.e. that if $\phi_1 \leq \phi_2$, then $c(\phi_1) \leq c(\phi_2)$.

2.3.3. *Bi-invariant metric on Hamiltonian group.* Given two Hamiltonian symplectomorphisms ϕ and ψ of $\mathbb{R}^{2n}$, define

$$d(\phi, \psi) := c^+(\phi\psi^{-1}) - c^-(\phi\psi^{-1}).$$

- Show that d is a bi-invariant metric on $\mathrm{Ham}^c(\mathbb{R}^{2n})$.

This metric was discovered by Viterbo in [5].

2.4. **Symplectic capacity.** Given an open bounded domain $\mathcal{U}$ of $\mathbb{R}^{2n}$, its **Viterbo capacity** $c(\mathcal{U})$ is defined by

$$c(\mathcal{U}) := \sup\{\, c^+(\phi) \mid \phi \in \mathrm{Ham}(\mathcal{U}) \,\}$$

where $\mathrm{Ham}(\mathcal{U})$ denotes the set of time-1 maps of Hamiltonian functions supported in $\mathcal{U}$. It can be shown that $c(\mathcal{U})$ is a finite real number.

- Prove that for any Hamiltonian symplectomorphism ψ of $\mathbb{R}^{2n}$ we have
$$c(\psi(\mathcal{U})) = c(\mathcal{U}).$$
- Prove that if $\mathcal{U}_1 \subset \mathcal{U}_2$, then $c(\mathcal{U}_1) \leq c(\mathcal{U}_2)$.
- Prove that $c(\alpha\mathcal{U}) = \alpha^2 c(\mathcal{U})$ for any positive constant α. [*Hint:* You can use the fact that if ψ is a *conformal* symplectomorphism of $\mathbb{R}^{2n}$, i.e. $\psi^*\omega = \alpha\omega$ for some constant α, then $\Lambda(\psi\phi\psi^{-1}) = \alpha\Lambda(\phi)$ (see [2, 5.2]).]

Moreover it can be proved that $c(B^{2n}(1)) > 0$ and $c(C^{2n}(1)) < \infty$ where

$$B^{2n}(R) = \{\, \pi \sum_{i=1}^{n} x_i^2 + y_i^2 < R \,\}$$

and

$$C^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}.$$

All these property are summarized by saying that c is a *symplectic capacity* in $\mathbb{R}^{2n}$.

2.4.1. *The Symplectic Non-Squeezing Theorem.* It can be proved that the capacity of an ellipsoid

$$E(\alpha_1, \cdots, \alpha_n) := \{\, \sum_{i=1}^{n} \frac{\pi}{\alpha_i}({x_i}^2 + {y_i}^2) \; < 1 \,\}$$

is given by its smallest coefficient. In particular $c\big(B(R)\big) = R$. Since every bounded domain contained in the cylinder $C(R)$ is also contained in some ellipsoid with R as smallest coefficient, it follows from monotonicity that $c\big(C(R)\big) = R$.

Due to this result, the Viterbo capacity can be used to prove Gromov's Non-Squeezing Theorem, which says that if $R_2 < R_1$ then there is no symplectic embedding of $B(R_1)$ into $C(R_2)$.

- Prove the Non-Squeezing Theorem. [*Hint*: Use the *extension after restriction principle*: if there is an embedding $\Psi : B(R_1) \hookrightarrow C(R_2)$ then for any $\delta \in (0,1)$ we can find a compactly supported Hamiltonian symplectomorphism Ψ_δ of $\mathbb{R}^{2n}$ with $\Psi_\delta \equiv \Psi$ on $\delta\, B(R_1)$.]

Moreover, the Viterbo capacity has been used in [5] to prove the **Camel Theorem**, which says that there is no Hamiltonian isotopy ψ_t supported in $\left(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-1}\right) \cup B^{2n}(\epsilon)$ such that $\psi_0 = \mathrm{id}$ and ψ_1 sends a ball of radius $R > \epsilon$ contained in one component of $\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-1}$ into the other component.

References

[1] R. Bott, Lectures on Morse theory, old and new. *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations*, (Beijing, 1980), 169–218, 1982.

[2] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, 1994.

[3] F. Laudenbach, and J.-C. Sikorav, Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibré cotangent, *Invent. Math.* **82** (1985), 349–357.

[4] J. Milnor, *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 *Princeton University Press, Princeton, N.J.* 1963.

[5] C. Viterbo, Symplectic topology as the geometry of generating functions, *Math. Ann.* **292** (1992), 685–710.