1. Relation between generating functions and the action functional

In order to discuss the relation between generating functions and the action functional, we need a definition of generating functions which is slightly more general than the one given in the lectures.

1.1. Definition of generating functions. Let $p : E \to B$ be a fiber bundle, and consider a smooth function $S : E \to \mathbb{R}$ on the total space. Let $\Sigma_S \subset E$ be the set of fiber critical points of $S$, i.e. the set of points $e$ of $E$ that are critical points of the restriction of $S$ to the fiber through $e$:

$$\Sigma_S := \{ e \in E | e \text{ critical point of } S \big|_{p^{-1}(p(e))} \}.$$  

- Note that, in general, $\Sigma_S$ is not necessarily a smooth submanifold of $E$. Show that if $dS : E \to T^*E$ is transverse to the fiber normal bundle $N_E := \{ (e, \mu) \in T^*E | \mu = 0 \text{ on } \ker dp(e) \}$, then $\Sigma_S$ is a smooth submanifold of $E$, of dimension equal to the dimension of $B$. In the following, we will always assume this condition.

Given a point $e$ of $\Sigma_S$, we can associate to it an element $v^*(e)$ of $T^{*}_{p(e)}B$ by defining

$$v^*(e) := dS(\hat{X})$$

for $X \in T_{p(e)}B$, where $\hat{X}$ is any vector in $T_eE$ with $p_*(\hat{X}) = X$. The element $v^*(e)$ of $T^{*}_{p(e)}B$ is sometimes called the Lagrange multiplier of $e$.

- Show that $v^*(e)$ is well-defined, i.e. it does not depend on the choice of the lift $\hat{X}$ of $X$.
- Show that the map $i_S : \Sigma_S \to T^*B, e \mapsto (p(e), v^*(e))$ is a Lagrangian immersion. [Hint: Show that $i_S^*\lambda_{\text{can}} = d(S|_{\Sigma_S})$].
- Show that the map $j_S : \Sigma_S \to J^1B, e \mapsto (p(e), v^*(e), S(e))$ is a Legendrian immersion.

$S$ is called a generating function for the Lagrangian submanifold $L := i_S(\Sigma_S)$ of $T^*B$ and for the Legendrian submanifold $\Lambda := j_S(\Sigma_S)$ of $J^1B$.

- Show that if $E = B$ and $p : E \to B$ is the identity then $L$ can be identified to the graph of the differential of $S$, and $\Lambda$ to the graph of the 1-jet of $S$.
- Assume that $p : E \to B$ is a trivial vector bundle, i.e. $E = B \times \mathbb{R}^N$. Show that in this case we recover the definition given in the lectures, i.e. we have that

$$L = \{ (q, p) \in T^*B | \exists \xi \in \mathbb{R}^N \text{ such that } \frac{\partial S}{\partial \xi}(q, \xi) = 0 \text{ and } \frac{\partial S}{\partial q}(q, \xi) = p \}.$$  

and

$$\Lambda = \{ (q, p, z) \in J^1B | \exists \xi \in \mathbb{R}^N \text{ such that } \frac{\partial S}{\partial \xi}(q, \xi) = 0, \frac{\partial S}{\partial q}(q, \xi) = p \text{ and } S(q, \xi) = z \}.$$
1.2. The Action Functional. Consider an exact symplectic manifold \((M, \omega = -d\lambda)\) and let \(H_t : M \to \mathbb{R}\) be a time-dependent Hamiltonian. Then \(H_t\) determines a functional \(\mathcal{A}_H\) on the space of paths \(\gamma : [t_0, t_1] \to M\). It is called the action functional and is defined by

\[
\mathcal{A}_H(\gamma) := \int_{t_0}^{t_1} \left(\lambda(\dot{\gamma}(t)) + H_t(\gamma(t))\right) dt.
\]

- Show that \(\gamma\) is a critical point of \(\mathcal{A}_H\) (with respect to variations with fixed endpoints) if and only if it is a trajectory of the Hamiltonian flow of \(H_t\).

Let now \(B\) be a smooth manifold, and consider the space \(E\) of all paths \(\gamma : [0, 1] \to T^*B\) that begin at the 0-section. \(E\) can be seen as a fiber bundle over \(B\), with projection \(p : E \to B\) given by \(\gamma \mapsto \pi(\gamma(1))\) where \(\pi\) is the projection of \(T^*B\) into \(B\). Given a time-dependent Hamiltonian \(H_t : T^*B \to \mathbb{R}\) we consider the function \(S : E \to \mathbb{R}\) given by \(S(\gamma) := \mathcal{A}_H(\gamma)\).

- Show that the set \(\Sigma_N \subset E\) of fiber critical points of \(S : E \to \mathbb{R}\) is given by the set of trajectories of the Hamiltonian flow of \(H_t\).
- Show that, given a fiber critical point \(\gamma\), the Lagrange multiplier \(\nu^*(\gamma)\) is the vertical component of \(\gamma(1)\).
- Conclude that \(S\) is a “generating function” for the image of the 0-section by the time-1 map of the Hamiltonian flow of \(H_t\).

Note that \(S\) is not a generating function in the sense of the definition given in 1.1, because \(E\) is not a finite-dimensional manifold. In the next section we will show how to construct a finite-dimensional approximation of \(E\), and thus obtain a true generating function for any Lagrangian submanifold of \(T^*B\) which is Hamiltonian isotopic to the 0-section.

1.3. Finite-dimensional reduction. The following construction was given by Laudenbach and Sikorav [3]. It is inspired by the method of “broken geodesics” of Morse theory [4, 1].

Let \(H_t\) be a time-dependent Hamiltonian on \(T^*B\). For every integer \(N\) we will define the space \(E_N\) of broken Hamiltonian trajectories of \(H_t\) with \(N - 1\) singularities and \(N\) smooth pieces. Element of \(E_N\) will be of the form

\[
e = (q_0, X, P)
\]

where \(q_0\) is a point of \(B\), \(X = (X_1, \ldots, X_{N-1})\) is an \((N - 1)\)-tuple of vectors \(X_i \in T_{q_0}B\) and \(P = (P_1, \ldots, P_{N-1})\) is an \((N - 1)\)-tuple of linear maps \(P_i \in T_{q_0}^*B\). The broken Hamiltonian trajectory of \(H_t\) associated to \(e\) is defined as follows. The first smooth piece, for \(t \in [0, \frac{1}{N}]\), is obtained by following the Hamiltonian flow of \(H_t\) in \(T^*B\) starting at the point \((q_0, 0)\) of the 0-section. The endpoint of this first smooth piece will be some other point of \(T^*B\), that we denote by \(z_1^-\). The second smooth piece of the broken Hamiltonian trajectory will not necessarily start from \(z_1^-\) but from a point \(z_1^+\) which is uniquely determined by \(z_1^-\), \(X_1\) and \(P_1\) in a way that we will describe later. The second smooth piece of the broken Hamiltonian trajectory is obtained by following the flow of \(H_t\) for \(t \in [\frac{1}{N}, \frac{2}{N}]\), starting from \(z_1^+\). The endpoint will be some point \(z_2^-\) of \(T^*B\). The third smooth piece of the broken Hamiltonian trajectory is obtained by following the Hamiltonian flow of \(H_t\) for \(t \in [\frac{2}{N}, \frac{3}{N}]\), starting at the point \(z_2^+\) that is uniquely determined by \(z_2^-\), \(X_2\) and \(P_2\) by the procedure we are going to describe later. We continue in this way to describe the whole broken trajectory for \(t \in [0, 1]\). It has \(N - 1\) jumps for \(t = \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\) and \(N\) smooth pieces from \(z_i^+\) to \(z_i^-\) for \(t \in [\frac{i}{N}, \frac{i+1}{N}]\), \(i = 0, \ldots, N - 1\). In order to describe the jumps we need to fix a Riemannian metric on \(M\). Then \(TM\) and \(T^*M\) have the associated Levi-Civita connection. We describe now the first jump, from \(z_1^-\) to \(z_1^+\). The point \(z_1^+ = (q_1^+, p_1^+)\) is determined by \(z_1^- = (q_1^-, p_1^-)\), \(X_1 \in T_{q_0}B\) and \(P_1 \in T_{q_0}^*B\) as follows. Denote by \(\gamma(t) = (q(t), p(t))\) for \(t \in [0, \frac{1}{N}]\) the first smooth piece of the broken Hamiltonian trajectory, from \((q_0, 0)\) to \(z_1^- = (q_1^-, p_1^-)\). In particular, \(q(t)\) for \(t \in [0, \frac{1}{N}]\) is a smooth path in \(B\). We take the vector \(\dot{X}_1 \in T_{q_1^-}B\) and the 1-form \(\dot{P}_1 \in T_{q_1^-}^*B\) that are obtained by parallel transport, with respect to the Levi-Civita connection,
of $X_1 \in T_{q_0}B$ and $P_1 \in T_{p_0}^*B$ along $q(t)$, $t \in [0, \frac{1}{N}]$. The point $z_1^+$ is then defined to be 
\[ z_1^+ = (q_1^+, p_1^+) \]
where $q_1^+ := \exp_{q_1}(-\bar{X}_1)$ and $p_1^+ := t(d\exp_{p_1}(-\bar{X}_1))^{-1}(\bar{p}_1)$. The other jumps are defined similarly.

Consider the projection $p : E_N \to B$ that sends $e$ to the projection to $B$ of the endpoint of the broken Hamiltonian trajectory associated to $e$.

We define a function $S : E_N \to \mathbb{R}$ by 
\[ S(e) := \sum_{i=1}^{N-1} < p_i, X_i > + A_H(\gamma_i) \]
where $\gamma_i$ denotes the $i$-th smooth piece of the broken Hamiltonian trajectory of $H$ associated to $e$.

- Show that the fiber critical points of $S : E_N \to \mathbb{R}$ are the unbroken Hamiltonian trajectories of $H$.
- Show that $S$ is a generating function for the image of the 0-section by the time-1 map of the Hamiltonian flow of $H$.
- The above construction works only if $N$ is sufficiently big. Try to understand where this is needed.
- Is there an analogue construction also in the contact case, for any Legendrian submanifold of $J^1B$ which is contact isotopic to the 0-section?

2. Symplectic rigidity in $\mathbb{R}^{2n}$

2.1. Generating functions for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. Recall from the lectures that if $B$ is a compact smooth manifold then any Lagrangian submanifold $L$ of the cotangent bundle $T^*B$ has a (unique, up to equivalence) generating function quadratic at infinity. We will now see how this result can be applied to obtain a generating function quadratic at infinity for any compactly supported Hamiltonian symplectomorphism $\phi$ of $\mathbb{R}^{2n}$. We will do this by associating to $\phi$ a Lagrangian submanifold $\Gamma_\phi$ of $T^*S^{2n}$, and then by taking the generating function of $\Gamma_\phi$.

Let $\phi$ be a Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ (not necessarily compactly supported, for the moment). We will now see how to associate to $\phi$ a Lagrangian submanifold $\Gamma_\phi$ of $T^*\mathbb{R}^{2n}$.

Recall first that the graph of $\phi$ is a Lagrangian submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. It is defined by 
\[ \text{gr}(\phi) = \{ (q, \phi(q)) \mid q \in M \} \]

- Show that the map $\tau : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$, $\tau(x, y, X, Y) = (x, Y, x - y, X - X)$ is a symplectomorphism. Show that it sends the diagonal to the 0-section.

We define $\Gamma_\phi := \tau(\text{gr}(\phi))$. It is a Lagrangian submanifold of $T^*\mathbb{R}^{2n}$.

- Let $\Psi_\phi$ be the symplectomorphism of $T^*\mathbb{R}^{2n}$ defined by the diagram

\[
\begin{array}{ccc}
\mathbb{R}^{2n} \times \mathbb{R}^{2n} & \xrightarrow{\text{id} \times \phi} & \mathbb{R}^{2n} \times \mathbb{R}^{2n} \\
\tau \downarrow & & \tau \\
T^*\mathbb{R}^{2n} & \xrightarrow{\Psi_\phi} & T^*\mathbb{R}^{2n}.
\end{array}
\]

Show that $\Gamma_\phi = \Psi_\phi(0\text{-section})$. Use this to conclude that $\Gamma_\phi$ is Hamiltonian isotopic to the 0-section.
Assume now that \( \phi \) is compactly supported.

- Show that \( \Gamma_\phi \) coincides with the 0-section outside a compact set. Using this, explain how we can identify \( \Gamma_\phi \) to a Lagrangian submanifold of \( T^*S^{2n} \) (that we will still denote by \( \Gamma_\phi \)).

We have thus seen that we can associate to any compactly supported Hamiltonian symplectomorphism \( \phi \) of \( \mathbb{R}^{2n} \) a Lagrangian submanifold \( \Gamma_\phi \) of \( T^*S^{2n} \). Since \( S^{2n} \) is compact, by the existence and uniqueness theorems of generating functions we have that \( \Gamma_\phi \) has a (unique, up to equivalence) generating function \( S: \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R} \) quadratic at infinity, i.e. \( S = S_0 + Q_\infty \) with \( S_0 \) compactly supported and \( Q_\infty \) a non-degenerate quadratic form on \( \mathbb{R}^N \).

We will now see that critical points of \( S \) correspond to fixed points of \( \phi \), and that the critical values coincide with the symplectic action of the corresponding fixed points.

**Definition 2.1.** Let \( \phi \) be a Hamiltonian symplectomorphism of \( \mathbb{R}^{2n} \). The *symplectic action* of a fixed point \( q \) of \( \phi \) is defined by

\[
A_\phi(q) := A_H(\phi_t(q)) = \int_0^1 \left( \lambda_0(X_t) + H_t \right) (\phi_t(q)) \, dt
\]

where \( \phi_t \) is a Hamiltonian isotopy joining \( \phi \) to the identity, \( X_t \) the vector field generating it and \( H_t \) the corresponding Hamiltonian. The *action spectrum* of \( \phi \) is the set \( \Lambda(\phi) \) of all values of \( A_\phi \) at fixed points of \( \phi \).

(Recall the definition of the *action functional* \( A_H \) given in Part 1)

Let \( \phi \) be a compactly supported Hamiltonian symplectomorphism of \( \mathbb{R}^{2n} \), with generating function \( S \). Recall that this means that \( \Gamma_\phi \) is the image of \( i_S: \Sigma_S \to T^*\mathbb{R}^{2n} \).

- Show that fixed points of \( \phi \) correspond to critical points of \( S \). More precisely, show that a point \( q \) of \( \mathbb{R}^{2n} \) is a fixed point of \( \phi \) if and only if \( i_S^{-1}(q,0) \) is a critical point of \( S \).

Suppose now that we have a fixed point \( q \) of \( \phi \), and take a point \( p \) in \( \mathbb{R}^{2n} \) outside the support of \( \phi \).

- Show that

\[
A_\phi(q) = -\int_{\gamma \in \phi(\gamma)^{-1}} \lambda_0
\]

where \( \gamma \) is any path in \( \mathbb{R}^{2n} \) joining \( p \) to \( q \). [Hint: consider the map \( u: [0, 1] \times [0, 1] \to \mathbb{R}^{2n} \), \( u(s,t) = \phi_t(\gamma(s)) \) and apply Stokes’ theorem to \( u^*\omega_0 = -d(u^*\lambda_0) \).]

- Show that \( S(i_S^{-1}(q,0)) = -\int_{\gamma \in \phi(\gamma)^{-1}} \lambda_0 \). [Hint: Show first that if a Lagrangian submanifold \( L \) of \( T^*B \) is generated by \( S: E \to \mathbb{R} \), i.e. \( L \) is the image of \( i_S: \Sigma_S \to T^*B \) then we have that \( \int_L \lambda_{can} = S(i_S^{-1}(y)) - S(i_S^{-1}(x)) \) for any path \( \gamma \) in \( L \) joining two points \( x \) and \( y \).]

This concludes the proof that \( S(i_S^{-1}(q,0)) = A_\phi(q) \), i.e. that the critical value of a critical point of the generating function coincides with the symplectic action of the corresponding fixed point.

### 2.2. Symplectic Homology

We will associate homology groups first to compactly supported Hamiltonian symplectomorphisms of \( \mathbb{R}^{2n} \), by considering relative homology of sublevel sets of the generating function, and then, by a limit process, to domains of \( \mathbb{R}^{2n} \).

Let \( \phi \) be a compactly supported Hamiltonian symplectomorphism of \( \mathbb{R}^{2n} \). Given real numbers \( a, b \) not belonging to the action spectrum of \( \phi \) and such that \( -\infty < a < b \leq \infty \), we define the \( k \)-th symplectic homology group of \( \phi \) with respect to the values \( a, b \) by

\[
G_k^{[a,b]}(\phi) := H_{k+i}(E^b, E^a)
\]
Lemma 2.2. We will now see that they are invariant by conjugation with a Hamiltonian symplectomorphism. Hamiltonians get arbitrarily large.

- Show that the groups \( G_k^{(a,b)}(\phi) \) are well-defined, i.e. do not depend on the choice of the generating function.

We will now see that they are invariant by conjugation with a Hamiltonian symplectomorphism. We need the following two lemmas.

**Lemma 2.2.** Let \( \psi \) be a symplectomorphism of \( \mathbb{R}^{2n} \). Then \( \Lambda(\psi \phi^{-1}) = \Lambda(\phi) \).

(A proof of this fundamental fact can be found for example in [2, 5.2].)

**Lemma 2.3.** Let \( f_t, t \in [0,1], \) be a continuous 1-parameter family of functions defined on a compact manifold \( M \). Suppose that \( a \in \mathbb{R} \) is a regular value of all \( f_t \). Then there exists an isotopy \( \theta_t \) of \( M \) such that \( \theta_t(M^n_a) = M^n_a \), where \( M^n_a := \{ x \in M \mid f_t(x) \leq a \} \).

- Using Lemmas 2.2 and 2.3, prove that for any \( \phi, \psi \) in \( \text{Ham}^c(\mathbb{R}^{2n}) \) we have an induced isomorphism

\[
\psi^* : G_*^{(a,b)}(\psi \phi^{-1}) \longrightarrow G_*^{(a,b)}(\phi).
\]

Consider now a domain \( U \) of \( \mathbb{R}^{2n} \). Given \( a, b \in \mathbb{R} \) we denote by \( \text{Ham}^c_{a,b}(U) \) the set of compactly supported Hamiltonian symplectomorphisms of \( \mathbb{R}^{2n} \) that are the time-1 map of a Hamiltonian function which is supported in \( U \) and whose action spectrum does not contain \( a \) and \( b \). We will consider the partial order \( \leq \) on \( \text{Ham}^c_{a,b}(U) \) defined as follows. We say that \( \phi_1 \leq \phi_2 \) if \( \phi_2 \circ \phi_1^{-1} \) can be written as the time-1 flow of a non-negative Hamiltonian function. It can be proved that if \( \phi_0 \leq \phi_1 \) then there are generating functions \( S_0, S_1 : E \longrightarrow \mathbb{R} \) for \( \Gamma_{\phi_0}, \Gamma_{\phi_1} \) respectively such that \( S_0 \leq S_1 \).

- Show that if \( \phi_1 \leq \phi_2 \) then we have an induced homomorphism

\[
\lambda_2^* : G_k^{(a,b)}(\phi_2) \longrightarrow G_k^{(a,b)}(\phi_1).
\]

- Show that, given \( \phi_1, \phi_2, \phi_3 \) in \( \text{Ham}^c_{a,b}(U) \) with \( \phi_1 \leq \phi_2 \leq \phi_3 \), it holds \( \lambda_3^2 \circ \lambda_2^1 = \lambda_3^1 \) and \( \lambda_i^j = \text{id} \).

- Conclude that \( \{ G_k^{(a,b)}(\phi_i) \}_{\phi_i \in \text{Ham}^c_{a,b}(U)} \) is an inversely directed system of groups\(^{1}\).

We define the \( k \)-th symplectic homology group \( G_k^{(a,b)}(U) \) of \( U \) with respect to the values \( a, b \) to be the inverse limit of the inversely directed system \( \{ G_k^{(a,b)}(\phi_i) \}_{\phi_i \in \text{Ham}^c_{a,b}(U)} \). Note that \( G_k^{(a,b)}(U) \) can be calculated by any sequence \( \phi_1 \leq \phi_2 \leq \phi_3 \leq \cdots \) such that the associated Hamiltonians get arbitrarily large.

- Show that for any domain \( U \) in \( \mathbb{R}^{2n} \) and any Hamiltonian symplectomorphism \( \psi \) we have an induced isomorphism

\[
\psi^* : G_*^{(a,b)}(\psi(U)) \longrightarrow G_*^{(a,b)}(U).
\]

\(^{1}\)Recall the definition of an inversely directed system of groups. Let \( (I, \leq) \) be a directed partially ordered set, i.e. a set \( I \) with a partial order \( \leq \) such that for any two elements \( i \) and \( j \) of \( I \) there exists a third element \( k \) such that \( i \leq k \) and \( j \leq k \). A family of groups \( \{ A_i \}_{i \in I} \) is called an inversely directed system of groups if for every \( i \leq j \) there exists a homomorphism \( f_{ij} : A_j \rightarrow A_i \) such that the following properties are satisfied: \( f_{ii} = \text{id} \) and \( f_{jk} \circ f_{ij} = f_{kj} \) for all \( i \leq j \leq k \). The inverse limit of the inversely directed system \( \{ A_i \}_{i \in I} \) is then defined by

\[
\lim_{i \in I} A_i := \{ a \in \prod_{i \in I} A_i \mid a_i = f_{ij}(a_j) \text{ for all } i \leq j \}.
\]
2.3. Spectral invariants.

2.3.1. Invariants for Lagrangian submanifolds. Let $B$ be a closed manifold and $L$ a Lagrangian submanifolds of $T^*B$ Hamiltonian isotopic to the 0-section. We know that $L$ has a generating function quadratic at infinity $S : E \to \mathbb{R}$. We are going to see now how to define invariants for $L$ by selecting critical values of its generating function $S$. Recall that $S$ is only defined up to fiber-preserving diffeomorphism, stabilization and addition of a constant. While the first two operations do not affect the critical values of the function, addition of a constant does, so in order to get well-defined invariants we first need to normalize generating functions. This can be done by fixing a point $P$ in $B$ and only considering the set $\mathcal{L}_P$ of Lagrangian submanifolds $L$ of $T^*B$ which are Hamiltonian isotopic to the 0-section and intersect it at $P$. We can then normalize generating functions by requiring the critical value of the critical point corresponding to $P$ to be 0.

Let $L$ be an element of $\mathcal{L}_P$ with generating function $S : E \to \mathbb{R}$. We will now explain how to use a cohomology class $u$ of $B$ to select a critical value of $S$, in order to get an invariant $c(u, L)$.

Recall that we can assume that $E = B \times \mathbb{R}^n$ and $S$ is of the form $S = S_0 + Q_\infty$ where $S_0$ is compactly supported and $Q_\infty$ is a non-degenerate quadratic form on $\mathbb{R}^n$. We denote by $E^a$, for $a \in \mathbb{R}$, the sublevel set of $S$ at $a$ i.e. $E^a = \{ x \in E | S(x) \leq a \}$ and by $E^{-\infty}$ the set $E^{-a}$ for $a$ big enough (i.e. such that $-a$ is smaller that all critical values of $S_0$). Note that up to homotopy equivalence $E^{-\infty}$ is the same for all elements of $\mathcal{L}_P$. We will study the inclusion $i_a : (E^a, E^{-\infty}) \hookrightarrow (E, E^{-\infty})$, and the induced map on cohomology

$$i_a^* : H^*(B) \equiv H^*(E, E^{-\infty}) \longrightarrow H^*(E^a, E^{-\infty}).$$

Here $H^*(B)$ is identified with $H^*(E, E^{-\infty})$ via the Thom isomorphism

$$T : H^*(B) \xrightarrow{\sim} H^*(D(E^-), S(E^-))$$

where $E^-$ denotes the subbundle of $E$ where $Q_\infty$ is negative definite. Note that this isomorphism shifts the grading by the index of $Q_\infty$. Note also that by excision $H^*(D(E^-), S(E^-))$ is isomorphic to $H^*(E, E^{-\infty})$. For $|a|$ big enough we have $H^*(E^a, E^{-\infty}) \equiv 0$ if $a < 0$, and $i_a^* = id$ if $a > 0$. So we can define

$$c(u, L) := \inf \{ a \in \mathbb{R} | i_a^*(u) \neq 0 \}$$

for $u \neq 0$ in $H^*(B)$.

- Show that $c(u, L)$ is well-defined, i.e. it does not depend on the choice of the generating function used to calculate it.
- Show that $c(u, L)$ is a critical value of $S$.

2.3.2. Spectral invariants for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. We now apply the construction of the previous section to the special case of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$.

Consider a compactly supported Hamiltonian symplectomorphism $\phi$ of $\mathbb{R}^{2n}$. Define

$$c^+(\phi) := c(\mu, \Gamma_\phi) \text{ and } c^-(-\phi) := c(1, \Gamma_\phi)$$

where $\Gamma_\phi$ is the Lagrangian submanifold of $T^*S^{2n}$ constructed in Section 2.1 and $\mu$ and 1 are respectively the orientation and the unit classes of $S^{2n}$. Note that $\Gamma_\phi$ intersects the 0-section at the point at infinity of $S^{2n}$. This point plays the role of the point $P$ of the previous section.

- Show that $c(\phi) = A_\phi(q)$ for some fixed point $q$ of $\phi$. In particular, deduce that $c^+(id) = c^-(id) = 0$.

Moreover we have the following properties, that can be proved by methods of algebraic topology..

Lemma 2.4. For all $\phi, \psi$ in $\text{Ham}^c(\mathbb{R}^{2n})$ it holds:
2.4.1. Moreover it can be proved that
\[ c^+(\phi) \geq 0 \text{ and } c^- (\phi) \leq 0. \]
(i) \[ c^+(\phi) = c^- (\phi) = 0 \text{ if and only if } \phi \text{ is the identity.} \]
(ii) \[ c^+ (\phi) = -c^- (\phi^{-1}). \]
(iii) \[ c^+(\phi\psi) \leq c^+(\phi) + c^+(\psi) \text{ and } c^-(\phi\psi) \geq c^- (\phi) + c^- (\psi). \]
(iv) c^+(\phi) \leq c^- (\phi^{-1}).

We will now prove that \( c^+ \) and \( c^- \) are invariant by conjugation.

- Show that for all \( \phi, \psi \in \text{Ham}^c (\mathbb{R}^{2n}) \) it holds \( c(\phi) = c(\psi\phi\psi^{-1}) \). [Hint: Let \( \psi \) be the time-1 flow of a Hamiltonian isotopy \( \psi_t \), and consider the continuous map \( t \mapsto c(\psi_t\phi\psi_t^{-1}) \). Use Lemma 2.2.]

Recall that the relation \( \leq \) on the group \( \text{Ham}^c (\mathbb{R}^{2n}) \) is defined by setting \( \phi_1 \leq \phi_2 \) if \( \phi_2 \phi_1^{-1} \) can be written as the time-1 flow of a non-negative Hamiltonian function.

- Show that \( c^+ \) and \( c^- \) are monotone with respect to \( \leq \), i.e. that if \( \phi_1 \leq \phi_2 \), then \( c(\phi_1) \leq c(\phi_2) \).

2.3.3. Bi-invariant metric on Hamiltonian group. Given two Hamiltonian symplectomorphisms \( \phi \) and \( \psi \) of \( \mathbb{R}^{2n} \), define
\[ d(\phi, \psi) := c^+(\phi\psi^{-1}) - c^- (\phi\psi^{-1}). \]
- Show that \( d \) is a bi-invariant metric on \( \text{Ham}^c (\mathbb{R}^{2n}) \).

This metric was discovered by Viterbo in [5].

2.4. Symplectic capacity. Given an open bounded domain \( \mathcal{U} \) of \( \mathbb{R}^{2n} \), its Viterbo capacity \( c(\mathcal{U}) \) is defined by
\[ c(\mathcal{U}) := \sup \{ c^+(\phi) \mid \phi \in \text{Ham} (\mathcal{U}) \} \]
where \( \text{Ham} (\mathcal{U}) \) denotes the set of time-1 maps of Hamiltonian functions supported in \( \mathcal{U} \). It can be shown that \( c(\mathcal{U}) \) is a finite real number.

- Prove that for any Hamiltonian symplectomorphism \( \psi \) of \( \mathbb{R}^{2n} \) we have
\[ c(\psi(\mathcal{U})) = c(\mathcal{U}). \]
- Prove that if \( \mathcal{U}_1 \subset \mathcal{U}_2 \), then \( c(\mathcal{U}_1) \leq c(\mathcal{U}_2) \).
- Prove that \( c(\alpha \mathcal{U}) = \alpha^2 c(\mathcal{U}) \) for any positive constant \( \alpha \). [Hint: You can use the fact that if \( \psi \) is a conformal symplectomorphism of \( \mathbb{R}^{2n} \), i.e. \( \psi^*\omega = \alpha \omega \) for some constant \( \alpha \), then \( \Lambda (\psi\phi\psi^{-1}) = \alpha \Lambda (\phi) \) (see [2, 5.2]).]

Moreover it can be proved that \( c(B^{2n}(1)) > 0 \) and \( c(C^{2n}(1)) < \infty \) where
\[ B^{2n}(R) = \{ \pi \sum_{i=1}^{n} x_i^2 + y_i^2 < R \} \]
and
\[ C^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}. \]
All these property are summarized by saying that \( c \) is a symplectic capacity in \( \mathbb{R}^{2n} \).

2.4.1. The Symplectic Non-Squeezing Theorem. It can be proved that the capacity of an ellipsoid
\[ E(\alpha_1, \cdots, \alpha_n) := \{ \sum_{i=1}^{n} \frac{\pi}{\alpha_i} (x_i^2 + y_i^2) < 1 \} \]
is given by its smallest coefficient. In particular \( c(B(R)) = R \). Since every bounded domain contained in the cylinder \( C(R) \) is also contained in some ellipsoid with \( R \) as smallest coefficient, it follows from monotonicity that \( c(C(R)) = R \).
Due to this result, the Viterbo capacity can be used to prove Gromov’s Non-Squeezing Theorem, which says that if $R_2 < R_1$ then there is no symplectic embedding of $B(R_1)$ into $C(R_2)$.

- Prove the Non-Squeezing Theorem. [*Hint: Use the extension after restriction principle: if there is an embedding $\Psi : B(R_1) \hookrightarrow C(R_2)$ then for any $\delta \in (0, 1)$ we can find a compactly supported Hamiltonian symplectomorphism $\Psi_\delta$ of $\mathbb{R}^{2n}$ with $\Psi_\delta \equiv \Psi$ on $\delta B(R_1).$]*

Moreover, the Viterbo capacity has been used in [5] to prove the Camel Theorem, which says that there is no Hamiltonian isotopy $\psi_t$ supported in $(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-1}) \cup B^n(\epsilon)$ such that $\psi_0 = \text{id}$ and $\psi_1$ sends a ball of radius $R > \epsilon$ contained in one component of $\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-1}$ into the other component.

REFERENCES