FLAG VARIETIES

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There are far too many problems to do here in an hour, or maybe even a day. You should choose the problems that look interesting to you that you might have an idea about how to start.

(1) Write out two distinct points of $\mathcal{F}\ell\mathbb{C}^3$ that have the same projection onto $\mathrm{Gr}(2,3)$.
(2) Write out two distinct points of $\mathcal{F}\ell\mathbb{C}^4$ that have the same projections onto both $\mathrm{Gr}(1,4)$ and $\mathrm{Gr}(2,4)$.
(3) (a) Write down a particular element of $C_{[2413]}$.
    (b) What does the general element of $C_{[2413]}$ look like?
    (c) What are the conditions on the ranks of the sub matrices with upper left corner at the upper left corner of the whole matrix that elements of $C_{[2413]}$ meet?
(4) Let $F_\bullet = \langle e_2 + e_3, 2e_1 + e_3, e_1\rangle$ and $F'_\bullet = \langle 3e_1 + e_2 + 2e_3, 2e_1 + e_2 + e_3, -2e_1 - e_3\rangle$. What element of $GL_3\mathbb{C}$ acts to send $F_\bullet$ to $F'_\bullet$?
(5) Give a Schubert cell in $\mathcal{F}\ell\mathbb{C}^3$ that contains only one point of $\mathcal{F}\ell\mathbb{C}^3$. Is there such a cell in general in $\mathcal{F}\ell\mathbb{C}^n$?
(6) Let π be the product of the $n-1$ projections $\pi_k : \mathcal{F}\ell\mathbb{C}^n \twoheadrightarrow \mathrm{Gr}(k,n)$, where π_k remembers only the k-plane of a full flag. Is this map $\pi : \mathcal{F}\ell\mathbb{C}^n \to \prod_{k=1}^{n-1}\mathrm{Gr}(k,n)$ injective? Surjective? Bijective?
(7) Show that $GL_n\mathbb{C} = \coprod_{w\in S_n} C_w$, that is show that every matrix in $GL_n\mathbb{C}$ is a coset representative of some coset $B\backslash Bw$ times some b^T for $b \in B$.
(8) Define the diagram of a permutation w to be $D(w) = \{(i, w_j) : i < j \text{ and } w_i > w_j\}$.
    (a) Show that $|D(w)| = l(w)$.
    (b) Show that $\dim C_w = l(w)$.
(9) (This problem also makes sense to put off until Thursday.) Define the rank matrix of a permutation $w \in S_n$ to be the $n \times n$ matrix $r(w)$ with entries$r_{ij}(w)$ equal to the rank of the sub matrix of the permutation matrix of with corners at $(1,1)$ and (i,j).
    (a) Show that C_w is the set of all matrices with ranks in the sub matrix with corners $(1,1)$ and (i,j) given by the rank matrix.
    (b) Show that $\dim C_w = l(w)$.
    (c) Give equations for the (Zariski) closures of C_w in $\mathcal{F}\ell\mathbb{C}^n$.
(10) Prove that $B\backslash G$ is a smooth projective variety. (This problem is *hard*.)
(11) Let $\mathcal{F}\ell(d_1, d_2) = \{V_1 \subset V_2 \subset \mathbb{C}^n : \dim V_i = d_i\}$. Give a description of $\mathcal{F}\ell(d_1, d_2)$ as a quotient of $GL_n\mathbb{C}$.