THE FLEXIBILITY AND RIGIDITY OF LAGRANGIAN AND LEGENDRIAN SUBMANIFOLDS

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Abstract. These are informal notes to accompany the first week of graduate lectures at the IAS Women and Mathematics Program. There are a number of topics covered in these notes that will not be covered in the lectures. These notes also contain some exercises which will hopefully start some good discussions at our afternoon problem sessions. We encourage you to read over all the notes and exercises. Sheila (Margherita) Sandon and I look forward to this symplectic and contact journey with you.

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1. Basic Objects and Important Questions

In this first lecture, we will define the basic objects in symplectic and contact topology and introduce some major motivating questions for the field. We will also briefly discuss the plans for the remainder of the week.

1.1. **Basic Objects.** Let us begin by defining symplectic and contact manifolds.

Definition 1.1.

(1) A **symplectic** manifold is a pair (X, ω) where X is a smooth $2n$-dimensional manifold and ω is a differential 2-form that is closed (meaning $dw = 0$) and non-degenerate in the sense that $\forall x \in X$, $\forall v \in T_xX$, if $v \neq 0$ then $\exists w \in T_xX$ so that $\omega(v, w) \neq 0$.

(2) A **contact** manifold is a pair (Y, ξ) where Y is a smooth $(2n + 1)$-dimensional manifold and ξ is a field of tangent hyperplanes (planes of dimension $2n$) that are maximally non-integrable: if ξ is locally given as $\ker \alpha$, then $\alpha \wedge (d\alpha)^n \neq 0$.

Remark 1.2.

(1) There are many equivalent ways to formulate non-degneracy of a symplectic form. Another common way is to say that $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form.

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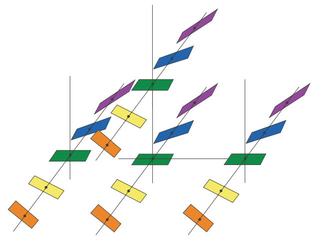

FIGURE 1. The standard contact structure on $\mathbb{R}^3$.

(2) The non-degeneracy of the symplectic form gives rise to an isomorphism $\theta : TX \to T^*X$ given by $\theta(v) = \omega(v, \cdot)$. This will be useful later when we discuss Hamiltonian diffeomorphisms.
(3) The Frobenius integrability theorem asserts that a hyperplane field ξ is integrable if and only if the sections of ξ are closed under the Lie bracket. When $\xi = \ker \alpha$, this means $\alpha([v, w]) = 0$ whenever $\alpha(v) = \alpha(w) = 0$, which is equivalent to $\alpha \wedge d\alpha = 0$. The contact condition is as far away from this as possible. For more information, see, for example, [26].
(4) Given a contact manifold, the maximally non-integrability condition says that it is not possible to find any $2n$-dimensional surface whose tangent planes coincide with the planes given by ξ. The Legendrian submanifolds we will study below are integral submanifolds of the largest possible dimension.

Example 1.3.

(1) **(Euclidean Space)**
   (a) $\left(\mathbb{R}^{2n} = \{(x_1, y_1, \dots, x_n, y_n)\}\right)$ has a canonical symplectic structure given by $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$.
   (b) $\left(\mathbb{R}^{2n+1} = \{(x_1, y_1, \dots, x_n, y_n, z)\}\right)$ has a canonical contact structure given by $\xi_0 = \ker(dz - \sum y_i dx_i)$.
(2) **(Cotangent bundles and 1-Jet Bundles)** For any smooth manifold M,
   (a) T^*M has a canonical symplectic structure given by $\omega = -d(\sum y_i dx_i)$, where $(x_1, \dots, x_n)$ are local coordinates on M and $y_1, \dots, y_n$ are local coordinates for T_q^*M.
   (b) $J^1M = T^*M \times \mathbb{R}$ has a canonical contact structure given by $\xi = \ker(dz - \sum y_i dx_i)$, where again $(x_1, \dots, x_n)$ are local coordinates on M and $y_1, \dots, y_n$ are local coordinates for T_q^*M.

Exercises 1.4.

(1) *Verify that ω_0 on $\mathbb{R}^{2n}$ is closed and non-degenerate.*
(2) *Verify that ξ_0 is a contact structure on $\mathbb{R}^{2n+1}$.*
(3) *Show that any orientable surface is a symplectic manifold.*

(4) *For a symplectic manifold (X, ω), $[\omega] \in H^2(X)$. Show that by Stokes' theorem, the non-degeneracy condition implies that when X is closed (compact and no boundary), $[\omega] \neq 0$. Can S^4 or S^6 have a symplectic structure?*

(5) *The following is a* **coordinate free definition of the canonical symplectic form** *on T^*M. Let M be a smooth manifold, and consider the cotangent bundle*

$$\pi : T^*M \to M.$$

Recall that, for every $q \in M$, $T^*_q M$ is the vector space dual to $T_q M$, i.e. the vector space of all linear maps $T_q M \to \mathbb{R}$. In other words, sections of $\pi : T^*M \to M$ are the 1-forms on M. Recall also that the smooth structure on the total space T^*M is given as follows. Let $x : \mathcal{U} \to \mathbb{R}^n$ be a local coordinate chart for M. Thus $x = (x_1, \cdots, x_n)$ where the x_i are functions $x_i : \mathcal{U} \to \mathbb{R}$. For any $q \in \mathcal{U}$, the differentials $dx_i(q) : T_q M \to \mathbb{R}$ form a basis for $T^*_q M$. Any element α_q of $T^*_q M$ can be written as $\alpha_q = \Sigma_{i=1}^n y_i \, dx_i$. Thus we get local coordinates $T^*\mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^n$, $(q, \alpha_q) \mapsto \big(x(q), y(q, \alpha_q)\big)$, which defines the smooth structure on T^*M.

The **Liouville 1-form** *(or* **tautological 1-form**, *or* **canonical 1-form***) λ_{can} on T^*M is defined by*

$$\lambda_{can}(X) := \alpha_q\big(\pi_*(X)\big)$$

for $X \in T_{(q,\alpha_q)}(T^*M)$.

- *Show the tautological property of λ_{can}, i.e. that for any 1-form σ on M, seen as a section $\sigma : M \to T^*M$, we have $\sigma^*\lambda_{can} = \sigma$.*
- *Show that in local coordinate $\lambda_{can} = \Sigma_{i=1}^n y_i \, dx_i$.*

The 2-form $\omega_{can} := -d\lambda_{can}$ is the canonical symplectic form on T^*M. In local coordinates it is given by $\omega_{can} = \Sigma_{i=1}^n dx_i \wedge dy_i$.

We will be interested in transformations of the space that preserve the symplectic/contact structure.

Definition 1.5.

(1) A **symplectic diffeomorphism** of (X, ω) (or **symplectomorphism**) is a diffeomorphism $\psi : X \to X$ so that $\psi^*\omega = \omega$. A **symplectic isotopy** consists of a 1-parameter family of symplectic diffeomorphisms ψ_t, $t \in [0, 1]$, where ψ_0 is the identity.

(2) A **contact diffeomorphism** (or **contactomorphism**) of (Y, ξ) is a diffeomorphism $\kappa : Y \to Y$ so that $\kappa_*(\xi_p) = \xi_{\kappa(p)}$. A **contact isotopy** consists of a 1-parameter family of contact diffeomorphisms κ_t, $t \in [0, 1]$, where κ_0 is the identity.

Exercises 1.6.

(1) *Show that a symplectic diffeomorphism preserves volume.*
(2) *Find some explicit symplectic diffeomorphisms of* $(\mathbb{R}^4, \omega_0)$*.*
(3) *Show that if* ξ *is globally given as the kernel a* 1*-form* α*, then* κ *is a contact diffeomorphism if and only if* $\kappa^*\alpha = f\alpha$*, for some non-vanishing function* f*.*
(4) *Find some explicit contact diffeomorphisms of* $(\mathbb{R}^3, \xi_0)$*.*

In fact, functions on the manifold give rise to symplectic and contact isotopies known as Hamiltonian isotopies and Reeb flows.

Example 1.7.

(1) Let (X, ω) be a symplectic manifold. Given a time-dependent function $H_t : X \to \mathbb{R}$, by the non-degeneracy condition, there exists a time-dependent vector field, v_t, called the **Hamiltonian vector field**, defined by
$$\omega(v_t, \cdot) = dH_t.$$
This time-dependent Hamiltonian vector field v_t defines a symplectic isotopy ϕ_t, called the **Hamiltonian isotopy** of X: $\frac{d}{dt}\phi_t = v_t \circ \phi_t$. A symplectomorphism ϕ is called **Hamiltonian** if there exists a Hamiltonain isotopy ϕ_t with $\phi_1 = \phi$. In fact, if $H^1(X) = 0$ (which is guaranteed when X is simply connected), then every symplectic isotopy is Hamiltonian.
(2) All contact forms for a contact structure ξ on Y differ by a non-vanishing function. There is a canonically defined flow associated to any contact form. Namely, given a contact form α, there exists a unique vector field $v = v_\alpha : Y \to TY$ such that
$$d\alpha(v_\alpha, \cdot) = 0, \quad \alpha(v_\alpha) = 1.$$
This vector field is called the **Reeb vector field** determined by α. One can think of v_α as a choice of a "perpendicular" to the contact planes. The flow of v_α preserves the contact form α and thus defines a contact isotopy. Notice that if one chooses a different contact form $f\alpha$, the corresponding Reeb vector field $v_{f\alpha}$ may be quite different from v_α, and so its flow may have very different dynamical properties.

Exercises 1.8.

(1) *Consider* $H : \mathbb{R}^4 \to \mathbb{R}$ *given by* $H(x_1, y_1, x_2, y_2) = x_1$*. What is the corresponding Hamiltonian vector field* v_t *and the corresponding Hamiltonian isotopy* ϕ_t*? Repeat for when* $H(x_1, y_1, x_2, y_2) = y_1$*.*
(2) *What is the Reeb vector field for the standard contact form* $\alpha_0 = dz - \sum y_i dx_i$ *on* $\mathbb{R}^{2n+1}$*?*

Projective space $\mathbb{C}P^n$, and more generally, Kähler manifolds with their Kähler form are other important sources of symplectic manifolds. However, during this first week, we will focus on the standard symplectic and contact manifolds, T^*M and J^1M. In fact, when $M = \mathbb{R}^n$, these standards are the local model for all:

Theorem 1.9 (Darboux). (1) *Every symplectic manifold is locally symplectomorphic to* $(\mathbb{R}^{2n}, \omega_0)$*: if* (X^{2n}, ω) *is a symplectic manifold and* $p \in X$*, then there exists local coordinates* $(x_1, y_1, \dots, x_n, y_n)$ *on a neighborhood* $U \subset X$ *of* p *where*

$$\omega|_U = \sum_i dx_i \wedge dy_i.$$

(2) *Every contact manifold is locally contactomorphic to* $(\mathbb{R}^{2n+1}, \xi_0)$*: if* α *is a contact form on* Y^{2n+1} *and* $p \in Y$*, then there are coordinates* $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ *on a neighborhood* $U \subset Y$ *of* p *so that*

$$\alpha|_U = dz - \sum_{i=1}^{n} y_i dx_i.$$

For proofs, see, for example, [26], [22].

Because of Darboux's theorem, there are no local invariants in symplectic or contact geometry. (This is in contrast to Riemannian geometry where there are a number of local curvature invariants.) *Symplectic/Contact Topology is the study of global phenomena in symplectic/contact manifolds.*

Special submanifolds of a symplectic/contact submanifold are examples of global phenomena to study.

Definition 1.10.

(1) An n-dimensional submanifold L of the symplectic manifold (X^{2n}, ω) is **Lagrangian** if $\omega(v, w) = 0$, for all $v, w \in T_pL$, for all $p \in L$;
(2) An n-dimensional submanifold Λ of the contact manifold (Y^{2n+1}, ξ) is **Legendrian** if $T_p\Lambda \subset \xi_p$, for all $p \in \Lambda$.

Example 1.11.

(1) **(Lagrangians in Cotangent Bundles)** For any smooth function $f : M \to \mathbb{R}$, Γ_{df} is a Lagrangian submanifold of T^*M.
(2) **(Legendrians in 1-Jet Bundles)** For any smooth function $f : M \to \mathbb{R}$, $j^1 f$ is a Legendrian submanifold of J^1M.

In fact, Legendrians in $J^1M = T^*M \times \mathbb{R}$ will project to immersed Lagrangians in T^*M. The **0-section**, L_0, is an important Lagrangian submanifold of T^*M.

Example 1.12. In fact, *symplectomorphisms can be viewed as Lagrangian submanifolds.* If (X, ω) is a symplectic manifold, $\overline{X} \times X$ is the symplectic manifold with symplectic structure $-\pi_1^*\omega + \pi_2^*\omega$, where $\pi_i : X \times X \to X$ is the projection to the i^{th} factor. Given $\psi : X \to X$, then the graph of ψ, $\Gamma_\psi = \{(p, \psi(p)\}$ is a Lagrangian submanifold of $\overline{X} \times X$.

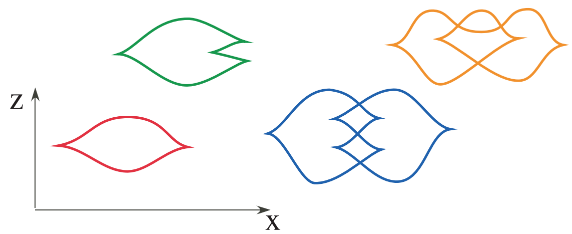

FIGURE 2. These curves in the xz-plane have lifts to Legendrian knots in $\mathbb{R}^3$.

Exercises 1.13.

(1) *Is the x_1y_1-plane a Lagrangian submanifold of $(\mathbb{R}^4, \omega_0)$? Is the x_1x_2-plane a Lagrangian submanifold of $(\mathbb{R}^4, \omega_0)$?*
(2) *Show that any embedded curve in an orientable surface is Lagrangian.*
(3) *Verify that $\Gamma_{df} \subset T^*M$ is a Lagrangian submanifold and $j^1 f$ is a Legendrian submanifold.*
(4) *Verify that the graph of a symplectomorphism $\Gamma_\psi \subset \overline{X} \times X$ is a Lagrangian submanifold.*
(5) *Show that for any $p \in M$, the fiber T_p^*M is a Lagrangian submanifold.*
(6) *Show that curves in Figure 2 can be lifted to Legendrians in $\mathbb{R}^3$ by setting the missing y coordinate equal to $\frac{dz}{dx}$.*
(7) *The Legendrians $\Lambda \subset \mathbb{R}^3 = J^1\mathbb{R} = T^*\mathbb{R} \times \mathbb{R}$ you constructed in the previous exercise will project to immersed Lagrangians $L_\Lambda \subset \mathbb{R}^2 = T^*\mathbb{R}$. Find these immersed Lagrangians.*

Just as there are no local invariants of symplectic/contact manifolds, there are no local invariants for Lagrangian/Legendrian submanifolds.

Theorem 1.14 (Weinstein)**.**

(1) *Let (X, ω) be a symplectic manifold and $L \subset X$ a compact Lagrangian submanifold. Then there exists a tubular neighborhood $\mathcal{U}$ of L in X that is contactomorphic to a tubular neighborhood $\mathcal{V}$ of the 0-section of T^*L.*
(2) *Let (Y, ξ) be a contact manifold and $\Lambda \subset Y$ a compact Legendrian submanifold. Then there exists a tubular neighborhood $\mathcal{U}$ of Λ in Y that is contactomorphic to a tubular neighborhood $\mathcal{V}$ of $j^1(0)$ in $J^1\Lambda$.*

Proofs of these can be found, for example, in [26].

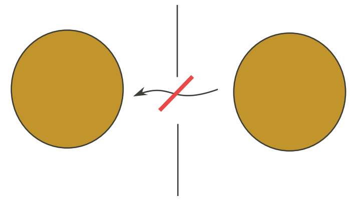

FIGURE 3. The symplectic camel.

1.2. Important Questions.

Question 1.15. *How flexible or rigid are symplectic/contact diffeomorphisms? How flexible or rigid are Lagrangian/Legendrian submanifolds?*

We know that symplectic diffeomorphisms must preserve volume. The "symplectic camel" illustrates that symplectic diffeomorphisms have properties that distinguish them from volume preserving diffeomorphisms.

Theorem 1.16. (Symplectic Camel) *(*[16]*,* [38] [27]*) Consider the* 4*-dimensional space E consisting of two half space and an open ball in the plane separating these two half spaces:*

$$E = \{x_1 < 0\} \cup \{x_1 > 0\} \cup \{(0, y_1, x_2, y_2) : y_1^2 + x_2^2 + y_2^2 < 1\}.$$

For $R > 1$, let $B(R)$ denote a 4*-dimensional ball of radius R in the half space $\{x_1 > 0\}$. Then there does not exist a symplectic isotopy ψ_t so that $\psi_1(B(R))$ is completely contained in the other half space, $\{x_1 < 0\}$.*

See Figure 3. In fact, it is possible to get an arbitrarily large percentage of the volume of the camel through that hole/eye of the needle! ([34]). In some sense, there are "ribs" that prevent the camel from completely passing through the hole.

Another important phenomena that illustrates the difference between symplectic and volume preserving diffeomorphisms is the "Non-Squeezing" Theorem; see Figure 4:

Theorem 1.17. (Symplectic Non-Squeezing) *Let $B^{2n}(r)$ denote the closed ball in $\mathbb{R}^{2n}$ of radius r, and let $Z^{2n}(r)$ denote the symplectic cylinder:*

$$Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{(x_1, y_1, \ldots, x_n, y_n) : x_1^2 + y_1^2 \leq 1\}.$$

Then, when $n \geq 2$ and $R > r$, there does not exist a symplectic embedding $\psi : B^{2n}(R) \to Z^{2n}(r)$.

The Non-Squeezing Theorem is a foundational theorem in symplectic topology. It can be proved by numerous techniques: variational methods, generating families, J-holomorphic curves. In the second week of the graduate course, you will see a proof using holomorphic curves. More recently, a contact version of non-squeezing has been discovered by both holomorphic and generating family techniques, [15], [32].

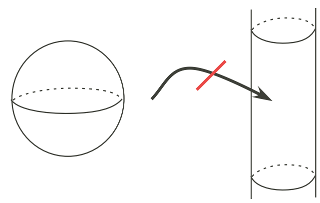

FIGURE 4. The symplectic non-squeezing theorem.

A great deal of research in symplectic and contact topology has been motivated by the Arnold conjectures, stated in [3]:

Conjecture 1.18 (Arnold Conjectures).

(1) *Let X be a compact symplectic manifold, and let* $\mathrm{Crit}(X)$ *denote the minimum number of critical points of a smooth, real-valued function on X. Then any Hamiltonian diffeomorphism ϕ of X has at least* $\mathrm{Crit}(X)$ *distinct fixed points.*

(2) *If L is a Lagrangian submanifold of (X, ω) and ϕ is a Hamiltonian diffeomorphism of X, then the number of intersection points in $L \cap \phi(L)$ is at least* $\mathrm{Crit}(L)$.

Much progress has been made on this conjecture. There are many different Arnold conjectures in the spirit of this original. Often one replaces $\mathrm{Crit}(X)$ with other measurements of X. For example, one can use $\mathrm{Crit}_{\mathrm{Morse}}(X)$, which is defined as the minimum number of critical points of a smooth, real-valued, Morse functions on X. From Morse theory, we know that $\mathrm{Crit}_{\mathrm{Morse}}(X)$ is bounded below by the sum of the Betti numbers of X. Arnold conjectures have been proved that use the sum of the Betti numbers as a lower bound for the number of fixed points. These results generally imply the existence of more fixed points than are guaranteed by the Lefschetz fixed point theorem, which says that the number of fixed points of a diffeomorphism is bounded below by the *alternating* sum of the Betti numbers.

In the second lecture, we will prove an Arnold conjecture where X is a cotangent bundle and $\mathrm{Crit}(L)$ is replaced by $\mathrm{Crit}_{\mathrm{stab}}(L)$, the minimum number of critical points of "stabilized" functions.

There are also Arnold conjectures regarding Reeb chords of a Legendrian:

Conjecture 1.19 (Arnold Chord Conjecture). *Every closed Legendrian curve in S^3 with the standard contact structure has for any choice of* 1-*form a Reeb chord (namely a trajectory of the Reeb flow that intersects the Legendrian at two different points).*

A weaker version of this conjecture in a more general setting has been proved by Mohnke, [28]. Recently, there have been investigations into the number of Reeb chords that must exist. In the third lecture, we will prove an Arnold conjecture that says that the number of Reeb chords for a compact

Legendrian in $\mathbb{R}^{2n+1}$ (using the standard contact form) is bounded below by half the sum of the Betti numbers of the Legendrian.

1.3. **Techniques to Address Questions.** As the Arnold conjectures highlight, there is a "rigidity" to the number of fixed points of a Hamiltonian diffeomorphism and the number of Reeb chords of a Legendrian. There are a number of major techniques in symplectic and contact topology to try to measure this rigidity. In the first week of this lecture course, we will focus on the technique of generating families; in the second week, the focus will be on the technique of J-holomorphic curves. In both of these techniques, one typically sets up a function where the critical points are the "geometrically significant objects" (fixed points of the Hamiltonian diffeomorphism or Reeb chords). In the generating family technique, this function is defined on a finite dimensional space, while in the holomorphic curve setting, this function is defined on an infinite dimensional space (often a space of loops). The generating family technique has the advantage that it is analytically simpler; the holomorphic technique has the advantage that it can apply to a wider class of symplectic and contact manifolds.

The development and achievements of generating family and pseudo-holomorphic curve techniques have often occurred in parallel. For example, the non-squeezing result can be proved using the technique of either pseudo-holomorphic curves or generating families. The Symplectic Camel was proved at approximately the same time using both holomorphic and generating family techniques. Another parallel is that symplectic homology for open subsets of $\mathbb{R}^{2n}$ can be defined via both holomorphic and generating family theories. On the contact side, almost simultaneously, polynomial invariants for Legendrian links were developed from both the generating family and holomorphic curve theories. Non-squeezing phenomena in contact topology can be detected from both holomorphic and generating family techniques.

In the remaining lectures this week, we will discuss things that can be done with generating families. In particular, in Lecture 2, we will discuss how the mere existence of a generating family implies an Arnold conjecture. In Lecture 3, we will discuss how generating families can be used to construct "generating family" homology groups for a Legendrian submanifold; these groups satisfy a duality that implies an Arnold conjecture. In Lecture 4, we will combine the topics of Legendrian and Lagrangian by studying obstructions to and constructions of Lagrangian cobordisms between Legendrian submanifolds.

2.
Lagrangian Submanifolds

In this second lecture, we will focus on proving versions of the symplectic Arnold conjectures. We will begin by proving an Arnold conjecture for C^1-small Hamiltonian diffeomorphisms. We will then show that an Arnold conjecture can be proved for other Hamiltonian diffeomorphisms through the technique of generating families.

2.1. Arnold Conjecture for C^1-small Hamiltonian Diffeomorphisms.

We begin by proving a simple case of the Arnold conjecture. Other C^1-small versions of Arnold's conjecture can be found in Weinstein's book, [39]. Recall that the 0-section, L_0, of T^*M is diffeomorhic to M.

Theorem 2.1 (C^1-small Arnold conjecture). *Let M be compact, let $L_0 \subset T^*M$ denote the 0-section, and let ϕ denote a Hamiltonian diffeomorphism of T^*M that is C^1-close to the identity. Then*

$$\#(\phi(L_0) \cap L_0) \geq \mathrm{Crit}(M).$$

Proof. If ϕ is C^1-close to the identity, then $\phi(L_0)$ is a section of the bundle $T^*M \to M$. Thus $\phi(L_0)$ corresponds to a 1-form. Since $\phi(L_0)$ is Lagrangian, this 1-form is closed. Moreover, since ϕ is Hamiltonian, this 1-form is exact. Thus there exists $f : M \to \mathbb{R}$ so that $\phi(L_0) = \Gamma_{df}$. It follows that $(\phi(L_0) \cap L_0)$ corresponds to the critical points of f, and the stated lower bound follows. □

The idea of this proof can be extended to more general Hamiltonian diffeomorphisms through the technique of generating families.

2.2. Definition of Generating Families.

We've seen that given a function $f : M \to \mathbb{R}$, the graph of df in T^*M is a Lagrangian submanifold and the 1-jet of f in J^1M is a Legendrian submanifold. Generating families extend this construction to "non-graphical" Lagrangians and Legendrians by expanding the domain to, for example, a trivial vector bundles $M \times \mathbb{R}^N$, for some potentially large N.

To get an idea of generating families, we will first show how one can explicitly construct a generating family for a Legendrian curve in $\mathbb{R}^3$ which is not the 1-jet of a function. Consider Λ which is obtained as the Legendrian lift of the graph of the "multi-valued" function shown in Figure 5. Λ is **not** the 1-jet of $f : \mathbb{R} \to \mathbb{R}$, but Λ can be viewed as the "reduced 1-jet" of

$$F : \mathbb{R} \times \mathbb{R} = \{(x, e)\} \to \mathbb{R}.$$

The idea is to construct a 1-parameter family of functions $F_x : \mathbb{R} = \{e\} \to \mathbb{R}$ so that plotting the critical values of F_x, as x varies, gives you back the picture in Figure 5. Figure 6 shows a schematic construction of a desired F.

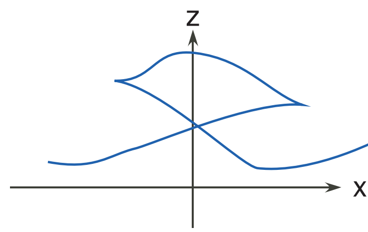

FIGURE 5. A Legendrian Λ with non-graphical front projection.

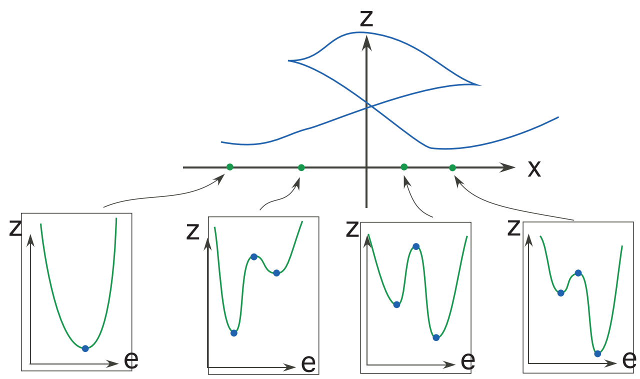

FIGURE 6. The construction of a 1-parameter family of functions (here with domain $\mathbb{R}$) whose critical values trace out the front projection of the Legendrian Λ of Figure 5.

So for this Λ, there exists $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that Λ is the "1-jet of F along the fiber critical submanifold":

$$\Lambda = \left\{ \left(x, \frac{\partial F}{\partial x}(x,e), F(x,e) \right) : \frac{\partial F}{\partial e}(x,e) = 0 \right\}.$$

Recall that under the projection map $J^1\mathbb{R} \to T^*\mathbb{R}$, the Legendrian Λ will be sent to a Lagrangian L. For the non-graphical Legendrian Λ in Figure 5, the associated Lagrangian will be non-graphical; see Figure 7. Notice that the F that generates Λ will also generate L:

$$L = \left\{ \left(x, \frac{\partial F}{\partial x}(x,e) \right) : \frac{\partial F}{\partial e}(x,e) = 0 \right\}.$$

Exercises 2.2. *This aims to get you familiar with the flexibility in the construction of generating families.*

(1) *In Figure 6, we constructed a generating family $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that generated the Legendrian curve in Figure 5 and the Lagrangian curve in Figure 7.*

(a) *What Legendrian/Lagrangian is generated by*

$$\widetilde{F}(x,\eta) = F(x,\eta-1)?$$

What Legendrian is generated by $\widehat{F}(x,\eta) = F(x,2\eta)$*? More generally, what Legendrian/Lagrangian will you generate if you precompose* F *with a "fiber-preserving diffeomorphism", a diffeomorphism* Φ *of* $\mathbb{R}\times\mathbb{R}$ *of the form* $\Phi(x,\eta) = (x,\phi_x(\eta))$*?*

(b) *What Legendrian/Lagrangian is generated by* $\widetilde{F} : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ *given by* $\widetilde{F}(x,\eta,\tilde{\eta}) = F(x,\eta) + (\widetilde{\eta})^2$*?*

(c) *What Legendrian/Lagrangian is generated by* $\widetilde{F} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *given by* $\widetilde{F}(x,\eta) = F(x,\eta) + 3$*?*

(2) *Convince yourself that the Legendrian "eye" and "flying saucer" in Figure 17 have generating families.*

More precisely, here is a formal definition of a generating family. (A more general definition of a generating family is given in an Appendix.)

Definition 2.3. Suppose F is a smooth function $F : M^n \times \mathbb{R}^N \to \mathbb{R}$ so that 0 is a regular value of $\frac{\partial F}{\partial \eta} : M\times\mathbb{R}^N \to \mathbb{R}^N$. Then the **fiber critical set** C_F of F is the n-dimensional submanifold

$$C_F = \left\{ (x,\eta) : \frac{\partial F}{\partial \eta}(x,\eta) = 0. \right\}$$

Define immersions $t_F : C_F \to T^*M$ and $j_F : C_F \to J^1M$ in local coordinates by:

$$\begin{aligned} t_F(x,\eta) &= (x, \partial_x F(x,\eta)), \\ j_F(x,\eta) &= (x, \partial_x F(x,\eta), F(x,\eta)). \end{aligned}$$

The image L of t_F is an immersed Lagrangian submanifold, while the image Λ of j_F is an immersed Legendrian submanifold. We say that F **generates** L and Λ, or that F is a **generating family (of functions)** for L and Λ.

Often we will start with an embedded Lagrangian or Legendrian and show that it has a generating family. Usually we will not do this explicitly but rather appeal to some general theorems.

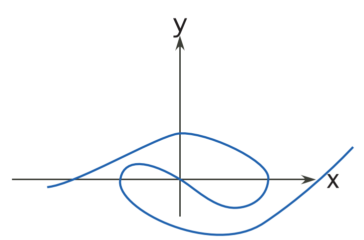

FIGURE 7. The Lagangian projection of the non-graphical Λ from Figure 5.

Critical points of a generating family detect interesting geometrical information about the Lagrangian generated:

Proposition 2.4. *Let L_0 denote the 0-section of T^*M. If $F : M \times \mathbb{R}^N \to \mathbb{R}$ generates $L \subset T^*M$, then critical points of F are in one-to-one correspondence with $L \cap L_0$.*

Proof. Notice

$$L = \left\{ \left(x, \frac{\partial F}{\partial x}(x,\eta) \right) : \frac{\partial F}{\partial \eta}(x,\eta) = 0 \right\}.$$

Thus

$$\begin{aligned} (x,\eta) \text{ is a critical point of } F &\iff (x,\eta) \in C_F \text{ and } \partial_x F(x,\eta) = 0 \\ &\iff t_F(x,\eta) \in L \cap L_0. \end{aligned}$$

□

Exercises 2.5. *Let $\Lambda_0^{\mathbb{R}}$ denote the 0-**wall** of J^1M:*

$$\Lambda_0^{\mathbb{R}} = L_0 \times \mathbb{R} \subset T^*M \times \mathbb{R} = J^1M.$$

Prove that if $F : M \times \mathbb{R}^N \to \mathbb{R}$ generates $\Lambda \subset J^1M$, then the critical points of F are in one-to-one correspondence with $\Lambda \cap \Lambda_0^{\mathbb{R}}$

Since the domain of F is not compact, we usually want to put some "taming condition" on the behavior of the function outside a compact set. This will guarantee that a number of standard Morse theoretic arguments will apply in this non-compact setting. In particular, to guarantee that the gradient vector field of F has a well-defined flow for all time, we want the generating family to be at worst "quadratic" outside a compact set. The condition at infinity will be dictated by the type of Legendrians/Lagrangians under consideration. When one wants to study Lagrangians in T^*M that are Hamiltonian isotopic to the 0-section or Legendrians in $J^1(M)$ that are Legendrian isotopic to the 1-jet of a function, we use quadratic-at-infinity generating families. Notice that the Lagrangian/Legendrian in T^*M or J^1M generated by a quadratic-at-infinity generating families will necessarily have a surjective project to M. Thus, for example, when one wants to study compact Legendrians in $J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}$, one wants to use linear-at-infinity generating families.

The conditions of linear-at-infinity or quadratic-at-infinity are defined precisely as follows.

Definition 2.6.

(1) A function $F : M \times \mathbb{R}^N \to \mathbb{R}$ is **linear-at-infinity** if F can be written as

$$F(x,\eta) = F^c(x,\eta) + A(\eta),$$

where F^c has compact support and A is a non-zero linear function.

(2) A function $F : M \times \mathbb{R}^N \to \mathbb{R}$ is **quadratic-at-infinity** if F can be written as

$$F(x, \eta) = F^c(x, \eta) + Q(\eta),$$

where F^c has compact support and Q is a non-degenerate quadratic form, $Q(\eta) = Q(\eta_1, \dots, \eta_n) = \sum_{i=1}^n \pm\eta_i^2$.

In the following, we will use the term **tame** to mean linear-at-infinity or quadratic-at-infinity.

2.3. **Existence of Generating Families.** As mentioned earlier, usually we do not explicitly construct generating families. Rather, we appeal to the following existence theorem.

Theorem 2.7 (**Existence of Generating Families**). *Let M be a compact manifold.*

(1) *For $t \in [0, 1]$, let $L_t \subset T^*M$ be the image of the zero section, L_0, under a compactly supported Hamiltonian isotopy ϕ_t of T^*M. Then there exists a quadratic-at-infinity generating family for L_t: if F is any quadratic-at-infinity generating family for L_0 then there exists a path of generating families $F_t : M \times \mathbb{R}^{2N} \to \mathbb{R}$ for $L_t = \phi_t(L_0)$ that agree with a quadratic of index N at infinity so that F_0 is a stabilization of F and $F_t = F_0$ outside a compact set.*

(2) *Let $\Lambda_0 \subset J^1M$ denote the 1-jet of the 0-function. If κ_t, $t \in [0, 1]$, is a compactly supported contact isotopy of $J^1(M)$, then there exists a smooth 1-parameter family of quadratic-at-infinity generating families $F_t : M \times \mathbb{R}^N \to \mathbb{R}$ for $\kappa_t(\Lambda_0)$.*

A **stabilization of a generating family** $F : M \times \mathbb{R}^N \to \mathbb{R}$ is a generating family of the form $\widetilde{F} : M \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$, where $\widetilde{F}(x, \eta, \widetilde{\eta}) = F(x, \eta) + Q(\widetilde{\eta})$, for a quadratic form Q.

Statement (1) for Lagrangians was shown by Laudenbach and Sikorav, [25]. There is also an alternative proof by Chekanov, [8]; his construction is sometimes referred to as Chekanov's formula. There also exists a Chekanov formula to prove (2); alternately Chaperon has a different construction, [7], [6]. An outline of Laudenbach and Sikorav's construction can be found in an Appendix and can be discussed at the problem sessions.

As a corollary of Theorem 2.7 (1), we can prove an Arnold conjecture. Recall, that $\mathrm{Crit}(M)$ denotes the minimal number of critical points of a smooth function on M. Define $\mathrm{Crit}_{\mathrm{stab}}(M)$ to be a stabilized version: $\mathrm{Crit}_{\mathrm{stab}}(M)$ is the minimal number of critical points of a function defined on $M \times B^{2N}$, where the function agrees with a quadratic form of index N near the boundary of the ball. Notice that

$$\mathrm{Crit}(M) \geq \mathrm{Crit}_{\mathrm{stab}}(M).$$

As a corollary to the existence of a generating family and Proposition 2.4, we have:

Corollary 2.8 (Arnold conjecture, [25])**.** *Let $L_0 \subset T^*M$ denote the 0-section and let ϕ denote a Hamiltonian diffeomorphism of T^*M. Then*

$$\#(\phi(L_0) \cap L_0) \geq \mathrm{Crit}_{\mathrm{stab}}(M).$$

Exercises 2.9. *Formulate and prove an Arnold conjecture for $\Lambda \cap \Lambda_0^{\mathbb{R}}$.*

2.4. **Uniqueness of Generating Families.** There are some natural equivalences to apply to generating families.

Definition 2.10.

(1) Given a generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$, let $Q : \mathbb{R}^K \to \mathbb{R}$ be a non-degenerate quadratic form: $Q(e_1, \ldots, e_K) = \sum_{j=1}^{K} \pm e_j^2$. Define $F \oplus Q : M \times \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ by $F \oplus Q(x, \eta, \eta') = F(x, \eta) + Q(\eta')$. Then $F \oplus Q$ is a **stabilization** of F.

(2) Given a generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$ and a constant $C \in \mathbb{R}$, $F' = F + C$ is said to be obtained from F by **addition of a constant**.

(3) Given a generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$, suppose $\Phi : M \times \mathbb{R}^N \to M \times \mathbb{R}^N$ is a fiber-preserving diffeomorphism, i.e., $\Phi(x, \eta) = (x, \phi_x(\eta))$ for diffeomorphisms ϕ_x . Then $F' = F \circ \Phi$ is said to be obtained from F by a **fiber-preserving diffeomorphism**.

Notice that two generating families that differ by fiber-preserving diffeomorphism and stabilization will generate the same Legendrian $\Lambda \subset J^1M$. Two generating families that differ by fiber-preserving diffeomorphism, stabilization, or the addition of a constant will generate the same Lagrangian $L \subset T^*M$.

Definition 2.11. Two generating families $F_i : M \times \mathbb{R}^{N_i} \to \mathbb{R}$, $i = 1, 2$ for a Lagrangian submanifold of T^*M are **equivalent** if they can be made equal after the operations of addition of a constant, fiber-preserving diffeomorphism, and stabilization. Two generating families $F_i : M \times \mathbb{R}^{N_i} \to \mathbb{R}$, $i = 1, 2$ for a Legendrian submanifold of J^1M are **equivalent** if they can be made equal after the operations fiber-preserving diffeomorphism, and stabilization.

There are some situations where the generating family is "unique" up to this notion of equivalence.

Theorem 2.12 (Uniqueness of Generating Families, [38], [33])**.** *Suppose M is compact.*

(1) *For $t \in [0, 1]$, let $L_t \subset T^*M$ be the image of the zero section, L_0, under a compactly supported Hamiltonian isotopy ϕ_t of T^*M. Then any two quadratic-at-infinity generating families for L_t are equivalent.*

(2) *For* $t \in [0,1]$*, let* $\Lambda_t \subset J^1M$ *be Legendrian isotopic to the* 1*-jet of a function of the* 0*-function. Then any two quadratic-at-infinity generating families for* Λ_t *are equivalent.*

Given this uniqueness, one can do a number of things. For example:

- For a Lagrangian $L \subset T^*M$ that is the image of L_0 under a Hamiltonian diffeomorphism, it is possible to define "capacities". These capacities are critical values of the generating family. One gets a capacity associated to each cohomology class $u \in H^k(M)$; [38].
- Since every compactly supported symplectic diffeomorphism (necessarily Hamiltonian) ψ of $\mathbb{R}^{2n}$ can be realized as a Lagrangian $L \subset T^*(S^{2n})$ which is the image of L_0 under a Hamiltonian diffeomorphism, one can define two "canonical capacities" $c_\pm(\psi)$. These capacities can be used, in particular, to define a bi-invariant metric on the space of symplectic diffeomorphisms of $\mathbb{R}^{2n}$; [38]. (Recently, Sandon has proved that there is an interger valued bi-invariant metric on the identity component of the group of compactly supported contact diffeomorphisms of $\mathbb{R}^{2n} \times S^1$, [31].)
- For an open set $U \subset \mathbb{R}^{2n}$, one can define its capacity $c(U)$ to be the supremum of $c_+(\psi)$, where ψ is a symplectic diffeomorphism supported in U; [38]. (In the contact world, Sandon has shown that it is possible to define capacities for $U \times S^1 \subset \mathbb{R}^{2n} \times S^1$, [32].)
- For a symplectic diffeomorphism ψ, in addition to assigned real numbers $c_\pm(\psi)$, one can assign homology groups, $GH_*^{(a,b]}(\psi)$, for every interval $(a,b]$ of real numbers. Namely, $GH^{(a,b]}(\psi)$ is defined to be the relative homology groups $H_*(F^b, F^a)$, where F is the unique generating family of the Lagrangian associated to ψ. Then for an open set $U \subset \mathbb{R}^{2n}$, by a limit over symplectomorphisms supported on U, one can then define symplectic homology groups $GH^{(a,b]}(U)$; [35]. (As a parallel result in the contact world, Sandon has shown that it is possible to assign symplectic homology groups, $GH^{(a,b]}(U \times S^1)$, to the set $U \times S^1 \subset \mathbb{R}^{2n} \times S^1$, [32])

Appendices to these notes outline some of these constructions.

3.
Legendrian Submanifolds

In this third lecture, we will briefly discuss some classical Legendrian invariants. Then we show how one can use generating families to construct non-classical invariants in the form of (co)homology groups and associated polynomials. We then show that a fundamental duality in these groups leads to a proof of an Arnold conjecture for Legendrians.

3.1. **Legendrian Equivalence and Classical Invariants.** Given two Legendrian submanifolds, Λ_0, Λ_1, it is natural to ask if they are "equivalent". A natural notion of equivalence is to say Λ_0 and Λ_1 are equivalent if there exists a smooth 1-parameter family of Legendrians Λ_t between Λ_0 and Λ_1. Another natural notion of equivalence is to say Λ_0 and Λ_1 are equivalent if there exists a contact isotopy κ_t so that $\kappa_1(\Lambda_0) = \Lambda_1$. In fact, these two notions of equivalence are the same. See, for example, [22].

For 1-dimensional Legendrians $\Lambda_0, \Lambda_1 \subset \mathbb{R}^3$, Λ_0 and Λ_1 are equivalent if and only if their front projections are equivalent by planar isotopies that do not introduce vertical tangents and the **Legendrian Reidemeister moves** as shown in Figure 8. Every Legendrian knot and link has a Legendrian representative. In fact, every Legendrian knot and link has an infinite number of different Legendrian representatives. For example, Figure 2 includes two different Legendrians that are topologically the unknot. In fact all Legendrian unknots can be obtained from the Legendrian unknot in the shape of an "eye" in Figure 17 by adding up or down **zig-zags** (also known as $\mp$ **stabilizations**). These unknots are classified by classical Legendrian invariants, the Thurston-Bennequin, tb, and rotation, r, numbers. These invariants can easily be computed from a front projection of the Legendrian

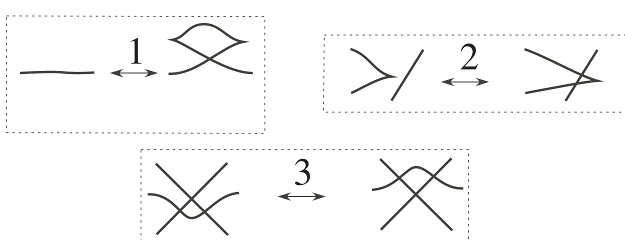

FIGURE 8. The three Legendrian Reidemeister moves: there is another type 1 move obtained by flipping the planar figure about a horizontal line, and there are three additional type 2 moves obtained by flipping the planar figure about a vertical, a horizontal, and both a vertical and horizontal line.

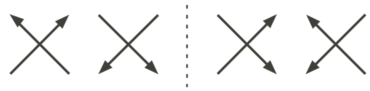

FIGURE 9. (a) Negative crossings; (b) Positive crossings.

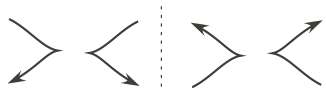

FIGURE 10. (a) Right and left down cusps; (b) Right and left up cusps.

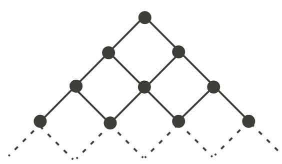

FIGURE 11. The tree of all Legendrian unknots.

link:

$$tb(\Lambda) = P - N - R, \qquad r(\Lambda) = \frac{1}{2}(D - U), \tag{3.1}$$

where P is the number of positive crossings, N is the number of negative crossings, R is the number of right cusps, D is the number of down cusps, and U is the number of up cusps in a generic front projection of Λ. (See Figures 9, 10.) Given that two front projections of equivalent Legendrian links differ by the Legendrian Reidemeister moves described in Figure 8, it is easy to verify that $tb(\Lambda)$ and $r(\Lambda)$ are Legendrian link invariants. (For more general definitions of these classical invariants, see, for example, [17].)

In general, it is an important question to understand the "geography" of other knot types. By work of Etnyre and Honda, [18] and Etnyre, Ng, and Vértesi, [19], we understand the trees/mountain ranges for all torus and twist knots. The Legendrian knot atlas of Chongchitmate and Ng, [9], gives the known and conjectured mountain ranges for all Legendrian knots with arc index at most 9; this includes all knot types with crossing number at most 7 and all non-alternating knots with crossing number at most 9.

Exercises 3.1.

(1) *Calculate the Thurston-Bennequin and rotation numbers for all the Legendrian knots in Figure 2.*
(2) *Check out the Legendrian knot atlas in* [9]. *Are there trees/mountain ranges that have different ancestors with the same tb and r values?*
(3) *Look up the general definition (non-combinatorial) of the Thurston-Bennequin and rotation numbers in* [17].
(4) *Look up the definition of the Thurston-Bennequin and rotation number invariants for higher dimensional Legendrians in* [12].

3.2. **Difference Function for Legendrian Submanifolds.** As mentioned earlier, the Reeb chords are an important rigid phenomena and appear in Arnold conjectures. We want to measure these Reeb chords with a generating family. As we saw in Exercise 2.5, the critical points of a generating family of a Legendrian knot Λ do *not* correspond to the Reeb chords of the knot; they do correspond to Reeb chords between Λ and Λ_0, the 1-jet of the 0 function. More generally, to measure the Reeb chords between disjoint Legendrian submanifolds or Reeb chords of a single Legendrian, we use a "difference" of generating families.

Definition 3.2. ([37], [24])

(1) Suppose $\Lambda_1, \Lambda_2 \subset J^1M$ are disjoint Legendrian submanifolds. If $f_i : M \times \mathbb{R}^{N_i} \to \mathbb{R}$ is a tame generating family for Λ_i, then the **difference function** of f_1, f_2 is $\delta_{f_1,f_2} : M \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$, defined by
$$\delta_{f_1,f_2}(x, \eta_1, \eta_2) = f_2(x, \eta_2) - f_1(x, \eta_1).$$
(2) Suppose $\Lambda \subset J^1M$ is a Legendrian submanifold. If $f : M \times \mathbb{R}^N \to \mathbb{R}$ is a tame generating family for Λ, then the **difference function** of f is $\delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, defined by
$$\delta_f(x, \eta_1, \eta_2) = f(x, \eta_2) - f(x, \eta_1).$$

Let $\ell(\gamma) > 0$ be the **length of the Reeb chord** γ (either between Λ_1 and Λ_2 or between Λ and itself), and let $\overline{\ell}$ (resp. $\underline{\ell}$) denote the maximum (resp. minimum) length of all Reeb chords of Λ.

Proposition 3.3 ([37], [20], [30], [24])**.**

(1) *For tame generating families f_1, f_2 of disjoint Legendrian submanifolds $\Lambda_1, \Lambda_2 \subset J^1M$,*
   (a) *the critical points of the difference function $\delta_{f_1,f_2} : M \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ are in one-to-one correspondence with Reeb chords γ between Λ_1 and Λ_2; the associated critical value is either $\ell(\gamma)$ or $-\ell(\gamma)$;*
   (b) *positive critical values of δ_{f_1,f_2} correspond to Reeb chords that begin on Λ_1 and end on Λ_2, while negative critical values correspond to Reeb chords that begin on Λ_2 and end on Λ_1;*

(c) 0 *is never a critical value of* δ_{f_1,f_2}.

(2) *For a tame generating family* f *that generates a Legendrian submanifold* $\Lambda \subset J^1M$, *the critical points of the difference function* $\delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *are of two types:*

(a) *For each Reeb chord* γ *of* Λ, *there are two critical points* (x, η_1, η_2) *and* (x, η_2, η_1) *of* δ_f *with nonzero critical values* $\pm\ell(\gamma)$.

(b) *The set*

$$\{(x, \eta, \eta) \,:\, (x, \eta) \in C_f\}$$

is a critical submanifold of δ_f *with critical value* 0.

For generic f_1, f_2, f, *these critical points and submanifolds are non-degenerate, and the critical submanifold has index* N.

Given this proposition, we want a way to detect the existence of critical points/values of δ. Morse theory is a natural tool to use to do this. Classical Morse theory is defined for real valued functions on compact manifolds. Given $f : M \to \mathbb{R}$, where M is a compact manifold, one examines the sublevel sets of f:

$$f^a := \{x \in M : f(x) \leq a\}.$$

Then the presence of a critical value of f in $[a, b]$ (and thus of a critical point of f) is detected by a change in the homology groups of f^b and f^a.

In a similar way, we will detect the presence of Reeb chords by measuring the relative homology (or cohomology groups) of sublevel sets of δ. The construction will vary slightly between looking at a disjoint union of Legendrians and a single Legendrian.

3.3. **GF Homology Groups for a disjoint union of Legendrian Submanifolds.** Assume that $\Lambda_1, \Lambda_2 \subset J^1M$ are disjoint Legendrian submanifolds with tame generating families f_1, f_2. Let $\delta_{f_1,f_2} : M \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ be the associated difference function. For $a \in \mathbb{R}$, let

$$\delta^a_{f_1,f_2} := \{(x, \eta_1, \eta_2) : \delta_{f_1,f_2}(x, \eta_1, \eta_2) \leq a\}.$$

Since the generating families are tame, there exists some constant $\omega > 0$ so that all critical values of δ_{f_1,f_2} lie in the interval $(-\omega, \omega)$. Since 0 can never be a critical value of δ_{f_1,f_2}; it is natural to examine the following relative homology groups. The **total generating family homology groups of** δ_{f_1,f_2} are defined as:

$$GH_k(f_1, f_2) = H_k\left(\delta^{\omega}_{f_1,f_2}, \delta^{-\omega}_{f_1,f_2}\right);$$

the **positive homology groups of** δ_{f_1,f_2} are defined as:

$$GH^+_k(f_1, f_2) = H_k\left(\delta^{\omega}_{f_1,f_2}, \delta^{0}_{f_1,f_2}\right);$$

and the **negative homology groups of** δ are defined as:

$$GH^-_k(f_1, f_2) = H_k\left(\delta^{0}_{f_1,f_2}, \delta^{-\omega}_{f_1,f_2}\right).$$

A classic algebraic topology argument ("critical non-crossings") shows that if one has a smooth 1-parameter family of generating families $f_1(t), f_2(t)$ that generate a Legendrian isotopy $\Lambda_1(t), \Lambda_2(t)$ with $\Lambda_1(t) \cap \Lambda_2(t) = \emptyset$, these groups $GH_k(f_1(t), f_2(t)), GH_k^+(f_2(t), f_2(t)), GH_k^-(f_1(t), f_2(t))$ will not change.

In fact, the total generating family homology groups are "not interesting", in the sense that they will not depend on the Legendrians. For example, when M is compact and f_1, f_2 are quadratic at infinity, $GH_k(f_1, f_2)$ records the homology groups of M; when $M = \mathbb{R}^{2n+1}$ and f_1, f_2 are linear-at-infinity, $GH_k(f_1, f_2)$ vanish for all k. This predicability can be useful: a long exact sequence associated to the triple $(\delta^{-\omega}, \delta^0, \delta^\omega)$ then shows that $GH_k^+(f_1, f_2)$ can be recovered from $GH_k^-(f_1, f_2)$.

A convenient way to record the negative (or positive) homology groups is through a Poincaré polynomial:

$$\Gamma_\delta^-(t) = \sum_{k=0}^{\infty} a_k t^k,$$

where a_k is the dimension of $GH_k^-(f_1, f_2)$. Then one can get a set of polynomials as an invariant for a disjoint union of Legendrian submanifolds.

$$\mathcal{P}^-(\Lambda_1, \Lambda_2) = \{\Gamma_\delta^-(t) : \quad \delta \text{ is a difference function of tame generating families } f_1, f_2 \text{ for } \Lambda_1, \Lambda_2\}.$$

When dealing with disjoint Legendrians, stabilizing one of the generating families will cause a shift of the index of all the homology groups. Thus for computational purposes, it is useful to "normalize" these polynomials (by multiplying by t^m, for some $m \in \mathbb{Z}$) so that, for example, there are no positive degree terms.

Exercises 3.4. *Consider the link of two components in J^1S^1 shown in Figure 12. Assume that each component has a unique generating family (up to fiber-preserving diffeomorphism and stabilization).*

1. *Suppose that f_2 generates the upper component and f_1 generates the lower component. What is $\mathcal{P}^-(\Lambda_1, \Lambda_2)$?*
2. *Now suppose that f_1 generates the upper component and f_2 generates the lower component. What is $\mathcal{P}^-(\Lambda_1, \Lambda_2)$?*

What do these calculations tell you? (Capacities for this link are defined and examined in [36]*. Generalizations of these calculations appear in* [37]*.)*

In the next subsection, we show how to modify this procedure when one only has a single Legendrian.

3.4. **Invariants for a Legendrian Submanifold.**

Assume that $\Lambda \subset J^1M$ is a Legendrian submanifold with a linear-at-infinity generating family f.

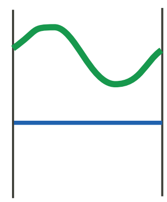

FIGURE 12. A link of 2 components in J^1S^1.

Let $\delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the associated difference function. For $a \in \mathbb{R}$, let

$$\delta_f^a := \{(x, \eta_1, \eta_2) : \delta_f(x, \eta_1, \eta_2) \leq a\}.$$

In contrast to the case of disjoint Legendrians, now 0 is always a critical value of δ_f. So now it is natural to choose ϵ and ω so that

$$0 < \epsilon < \underline{\ell} \leq \overline{\ell} < \omega, \tag{3.2}$$

and define the **total generating family cohomology** to be

$$\widetilde{GH}^k(f) = H^{k+N+1}(\delta_f^\omega, \delta_f^{-\epsilon}),$$

and the **relative generating family cohomology** to be

$$GH^k(f) = H^{k+N+1}(\delta_f^\omega, \delta_f^\epsilon).$$

There are also analogous definitions of the **total generating family homology**, $\widetilde{GH}_k(f)$, and **relative generating family homology**, $GH_k(f)$, using the same degree shift as above.

A classic algebraic topology argument ("critical non-crossings") shows that if one has a smooth 1-parameter family of generating families $f(t)$ that generate a Legendrian isotopy $\Lambda(t)$, these groups $\widetilde{GH}_k(f(t)), GH_k(f(t))$ will not change. Stabilizing the generating family will not shift the index of the cohomology groups.

A convenient way to record the homology groups is through a Poincaré polynomial. For example,

$$\Gamma_f(t) = \sum_{k=-\infty}^{\infty} a_k t^k,$$

where a_k is the dimension of $GH_k(f)$.

The full and relative generating family cohomology groups are related. By examing the long exact sequence of the triple $(\delta^\omega, \delta^\epsilon, \delta^{-\epsilon})$, we obtain:

Proposition 3.5 ([30]). *Let Λ^n be an orientable, Legendrian submanifold of J^1M with linear-at-infinity generating family f. There is a long exact sequence:*

$$\cdots \to H^k(\Lambda) \to GH^k(f) \to \widetilde{GH}^k(f) \to H^{k+1}(\Lambda) \to \dots.$$

If the groups are calculated with $\mathbb{Z}_2$ coefficients, the result holds without the orientability condition on the Legendrian.

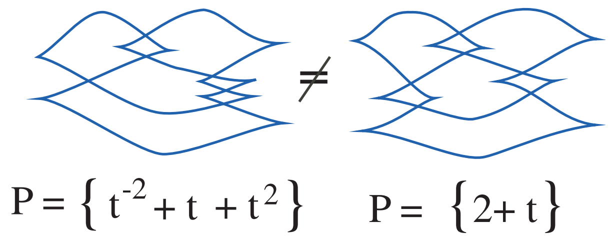

FIGURE 13. The Chekanov $m(5_2)$ knots can be distinguised by their generating family polynomials.

For a Legendrian, there may be non-equivalent linear-at-infinity generating families, so the generating family (co)homology of a linear-at-infinity generating family f, is not itself an invariant of the generated Legendrian Λ. However, we do have:

Proposition 3.6. *For a compact Legendrian submanifold $\Lambda \subset J^1M$, the sets*

$$\mathcal{GH}^k(\Lambda) = \{GH^k(f) \,:\, f \text{ is a linear-at-infinity generating family for } \Lambda\},$$
$$\mathcal{P}(\Lambda) = \{\Gamma_f(t) \,:\, f \text{ is a linear-at-infinity generating family for } \Lambda\},$$

are invariant under Legendrian isotopy.

Example 3.7. The Legendrian knots shown in Figure 13 have an underlying knot type of $m(5_2)$. These can be distinguished by the set of generating family polynomials which are shown in this figure. In fact, by a result of Fuchs and Rutherford, [20], whenever the set of linearized contact homology polynomials (defined through holomorphic curve theory) has a single element, this single element set must agree with the set of generating family polynomials. The set of linearized contact homology polynomials can be found, for example, in the Legendrian knot atlas, [9]. For example, there is a unique Legendrian knot that is topologically $m(3_1)$: it has a unique linearized contact homology polynomial of $2+t$, so it has a unique generating family polynomial of $2+t$. Table 3.7 lists some polynomials that can be realized as generating family polynomials (or linearized contact homology polynomials) of some Legendrian knots.

Exercises 3.8. *In the above table, all the Legendrians listed have a unique polynomial. Can you find an example of a Legendrian in the Legendrian knot atlas,* [9]*, where there are at least two different (linearized homology) polynomials?*

All of the polynomials that can arise as generating family polynomials in Table 3.7 for a Legendrian knot have a certain symmetry: they indicate that when $k \neq 1$, $GH_k(f) \simeq GH_{-k}(f)$. This will hold for any generating family

Table 1. Generating Family Polynomials

Legendrian	Knot Type	tb	$\|r\|$	GF Polynomial
Λ_1	$m(3_1)$	1	0	$2+t = t^{-0}+t+t^0$
Λ_2	4_1	-3	0	$t^{-1}+2t = t^{-1}+t+t^1$
Λ_3	$m(5_1)$	3	0	$4+t = 2t^{-0}+t+2t^0$
Λ_4	$m(5_2)$	1	0	$t^{-2}+t+t^2$
Λ_5	$m(5_2)$	1	0	$2+t = t^{-0}+t+t^0$
Λ_6	6_1	-5	0	$2t^{-1}+3t = 2t^{-1}+t+2t$
Λ_7	$m(6_1)$	-3	0	$t^{-3}+t+t^3$
Λ_8	$m(6_1)$	-3	0	$t^{-1}+2t = t^{-1}+t+t^1$

polynomial of a Legendrian knot and follows from a form of Alexander duality for generating family homology. For Legendrians of higher dimension, duality takes the form of a long exact sequence.

Theorem 3.9 (Duality,[30])**.** *If Λ is an Legendrian submanifold of J^1M with linear-at-infinity generating family f, then there is a long exact sequence:*

$$\cdots \to GH^{k-1}(f) \xrightarrow{\phi} GH_{n-k}(f) \to H^k(\Lambda) \to \cdots$$

This result parallels the theory of duality for the holomorphic curved based linearized contact homology of a horizontally-displaceable Legendrian submanifold in dimensions three [29] and higher [11].

From this duality, we can show that the topology of the Legendrian submanifold forces the existence of Reeb chords:

Corollary 3.10 (Arnold Conjecture, [30])**.** *Let Λ be a generic, n-dimensional Legendrian submanifold of J^1M with linear-at-infinity generating family f. Then*

$$\#\{\text{Reeb chords of } \Lambda\} \geq \frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda, \mathbb{F}),$$

where $b_i(\Lambda;\mathbb{F})$ is the i^{th} Betti number of Λ over a field $\mathbb{F}$

Proof of Corollary 3.10. We label the maps in the long exact sequence of Theorem 3.9 as follows:

$$\cdots \to GH_{n-k}(f) \xrightarrow{\rho_k} H^k(\Lambda) \xrightarrow{\sigma_k} GH^k(f) \to \cdots .$$

Let m_k denote the number of critical points of index $k+N+1$ of the difference function δ_f with positive critical value. We work over a field and denote the k^{th} Betti number by b_k: $b_k = \dim H^k(\Lambda)$. Finally, we compute:

$$\begin{aligned} b_k &= \dim \ker \sigma_k + \dim \operatorname{image} \sigma_k \\ &= \dim \operatorname{image} \rho_k + \dim \operatorname{image} \sigma_k \\ &\leq \dim GH_{n-k}(f) + \dim GH^k(f) \\ &\leq m_{n-k} + m_k. \end{aligned}$$

Since there is a one-to-one correspondence between Reeb chords and critical points of δ_f with positive critical values, we then get

$$\begin{aligned} \#\{\text{ Reeb chords of } \Lambda\} = \sum_{j=-\infty}^{\infty} m_j &\geq \sum_{j=0}^{n} m_j \\ &= \frac{1}{2} \sum_{k=0}^{n} m_{n-k} + m_k \\ &\geq \frac{1}{2} \sum_{k=0}^{n} b_k, \end{aligned}$$

as desired.

□

An analogous result to Corollary 3.10 has been proved using holomorphic techniques; see [13], [11].

4. Lagrangian Cobordisms between Legendrian Submanifolds

We now shift our attention from individual Lagrangian and Legendrian submanifolds to Lagrangian cobordisms between Legendrian submanifolds. We will show that Legendrian invariants, both the classical ones and the generating family polynomials, give obstructions to the existence of a Lagrangian cobordism. We then show some explicit constructions of Lagrangian cobordisms.

4.1. **The Cobordism Question.** Given the contact manifold J^1M with its standard contact structure $\xi = \ker\alpha$, where $\alpha = dz - \sum y_i dx_i$, the **symplectization** of J^1M is the symplectic manifold $(\mathbb{R} \times J^1M, d(e^s\alpha))$. Because the symplectic structure "grows" as s increases, it is common to draw this cylinder in a cone shape.

If Λ is a Legendrian submanifold of J^1M, then the **cylinder over** Λ, $\mathbb{R} \times \Lambda \subset \mathbb{R} \times J^1M$, is a Lagrangian submanifold. For us, a Lagrangian cobordism will be a Lagrangian that is cylindrical over Legendrians at its ends.

Question 4.1. *Given two Legendrian submanifolds $\Lambda_+, \Lambda_- \subset J^1M$, does there exist a Lagrangian cobordism between them? Namely, does there exist a Lagrangian submanifold $\overline{L} \subset \mathbb{R} \times J^1M$ so that at $+\infty$, $\overline{L}$ agrees with the cylinder over Λ_+ and at $-\infty$, $\overline{L}$ agrees with the cylinder over Λ_-? See Figure 14.*

Given a Lagrangian cobordism $\overline{L}$ between Λ_- and Λ_+, we will use the notation L to refer to a compact portion of $\overline{L}$: $\partial L = \Lambda_+ \cup \Lambda_-$. We will use

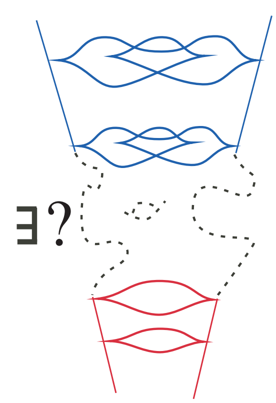

FIGURE 14. A Lagrangian cobordism $\overline{L} \subset \mathbb{R} \times J^1M$ is cylindrical over Legendrians at $\pm\infty$.

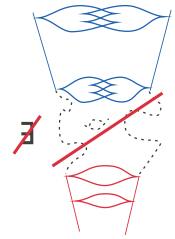

Figure 15. There is no Lagrangian cobordism between the Legendrian 3_1 with max tb and the unknot with maximal tb.

the notation $\Lambda_- \prec_L \Lambda_+$ to indicate that there exists a Lagrangian cobordism $\overline{L}$ between Λ_- and Λ_+. When $\Lambda_- = \emptyset$, we call $\overline{L}$ a **Lagrangian filling** of Λ_+.

We will also look at a generating family version of this question: we require that the Legendrians $\Lambda_\pm$ have generating families $f_\pm$ and that the Lagrangian cobordism $\overline{L}$ has a "compatible" generating family F. This notion of $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$ will be explained more below.

4.2. **Obstructions to the Existence of a Cobordism.** The first obstruction to the existence of a Lagrangian cobordism was found by Chantraine: for 1-dimensional Legendrian knots, he showed that classical invariants of Legendrian knots provide obstructions to the existence of cobordisms.

Theorem 4.2 ([5]). *Suppose $\Lambda_-, \Lambda_+ \subset \mathbb{R}^3$ are Legendrian knots in $\mathbb{R}^3$ and $\Lambda_- \prec_L \Lambda_+$. Then:*

(1) $r(\Lambda_-) = r(\Lambda_+)$, *and*
(2) $tb(\Lambda_+) - tb(\Lambda_-) = 2g(L)$.

In particular, there does not exist a Lagrangian cobordism between the Legendrian 3_1 and the unknot shown in Figure 15; as mentioned in [5], since the rotation numbers of these two Legendrians are different, it is not even possible to construct an immersed Lagrangian cobordism between them. Theorem 4.2 does not rule out the possibility of a Lagrangian cobordism between $m(3_1)$ and the unknot with maximal tb; notice that IF such a cobordism were to exist, the genus of this Lagrangian cobordism MUST be 2. This is in contrast to the smooth situation, where one can increase the genus by merely adding handles to a cobordism. This indicates a rigidity/efficiency of Lagrangian cobordisms.

Exercises 4.3. *For the Legendrians $\Lambda_1, \ldots, \Lambda_8$ in Table 3.7,*

(1) *Which are forbidden from being Lagrangian cobordant?*
(2) *For those that are potentially Lagrangian cobordant, what must be the genus of any potential cobordism?*

As described more precisely below, we will show that when the Legendrians $\Lambda_\pm$ have generating families, the generating family cohomology groups of a Legendrian (in any dimension) provide obstructions to the existence of a Lagrangian cobordism with a compatible generating family. For some of the statements in the results below, we assume that the Lagrangian is orientable; the hypothesis of orientability may be dropped if $\mathbb{Z}_2$ coefficients are used.

To apply the techniques of generating families, we need to be working in a cotangent bundle. We identify $\mathbb{R} \times J^1M$ with $T^*(\mathbb{R}_+ \times M)$ by the symplectomorphism

$$\begin{aligned} \theta : \mathbb{R} \times J^1M &\to T^*(\mathbb{R}_+ \times M) \\ (s, x, y, z) &\mapsto (e^s, x, z, e^s y). \end{aligned}$$

Given a Lagrangian cobordism of $\mathbb{R} \times J^1M$, by an abuse of notation, we will use $\overline{L}$ to refer to its image in $T^*(\mathbb{R}_+ \times M)$.

For a Lagrangian cobordism $\overline{L} \subset T^*(\mathbb{R}_+ \times M)$, we will be interested in the situation where $\overline{L} \subset T^*(\mathbb{R}_+ \times M)$ and $\Lambda_\pm \subset J^1M$ have "compatible" generating families.

Definition 4.4. Suppose $f_\pm : M \times \mathbb{R}^N \to \mathbb{R}$ are generating families for $\Lambda_\pm$. If $F : (\mathbb{R}_+ \times M) \times \mathbb{R}^N \to \mathbb{R}$ generates $\overline{L} \subset T^*(\mathbb{R}_+ \times M)$ and there exist $t_- < t_+$ so that

$$F(t, x, \eta) = \begin{cases} t f_-(x, \eta), & t \le t_- \\ t f_+(x, \eta), & t \ge t_+, \end{cases}$$

we say that F is **compatible** with $f_\pm$. Such a **compatible cobordism** will be denoted

$$(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+).$$

In addition, there are some "tameness" conditions on the generating families; for details, see [30].

We have the following obstruction to the existence of a Lagrangian filling. Recall, L denotes the compact portion of the cobordism.

Theorem 4.5 ([30])**.** *If $(\emptyset, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$, then:*

$$GH^k(f_+) \simeq H^{k+1}(L, \Lambda_+) \quad \textit{and} \quad \widetilde{GH}^k(f_+) \simeq H^{k+1}(L).$$

When we are considering 1-dimensional Legendrians, this means:

Corollary 4.6 ([30])**.** *If $(\emptyset, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$, then $\Gamma_{f_+}(t) = 2g + t$, where $g \ge 0$ is the genus of L.*

Exercises 4.7. *Which of the Legendrian knots from Table 3.7 cannot be filled with a Lagrangian (with a compatible generating family).*

Obstructions to more general Lagrangian cobordisms come in the form of a long exact sequence:

Theorem 4.8 ([30]). *Given $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$, there exists a homomorphism $\Psi_F : GH^k(f_-) \to GH^k(f_+)$ that fits into the following long exact sequence:*

$$\cdots \longrightarrow GH^k(f_-) \xrightarrow{\Psi_F} GH^k(f_+) \longrightarrow H^{k+1}(L, \Lambda_+) \longrightarrow \cdots .$$

The cobordism map Ψ_F satisfies some of the typical properties of a TQFT such as non-triviality, naturality, and functoriality. There are parallel result coming from the technique of holomorphic curves: [10, 14, 23]

Taking Euler characteristics of the long exact sequence (4.8) yields a generalization of Chantraine's 3-dimensional result about the relationships between the Thurston-Bennequin invariants of the Legendrian knots at the ends of a Lagrangian cobordism.

Corollary 4.9 ([30]). *Given $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$,*

$$tb(\Lambda_+) - tb(\Lambda_-) = (-1)^{\frac{1}{2}(n^2-3n)} \chi(L, \Lambda_+).$$

The asymmetry of the Lagrangian cobordism relation is evident from this corollary. If we only consider cobordisms that are actually concordances (i.e., genus 0), then we get:

Corollary 4.10 ([30]). *If $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$ and $\overline{L}$ is diffeomorphic to $\mathbb{R} \times \Lambda$, for some Λ, then Ψ_F is an isomorphism.*

Example 4.11. If K_1 and K_2 are the Legendrian $m(5_2)$ knots pictured in Figure 13, then there is no generating family compatible Lagrangian cobordism between them in either order. Since their Thurston-Bennequin invariants agree, Theorem 4.2 implies that a gf-compatible cobordism between them would necessarily have genus 0, but since their gf-polynomials are different, Corollary 4.10 forbids this.

Exercises 4.12. *Consider the Legendrians in Table 3.7. Could there be a generating family compatible Lagrangian cobordism between:*

(1) Λ_1 *and* Λ_4*?*
(2) Λ_1 *and* Λ_5*?*
(3) Λ_7 *and* Λ_8*?*

4.3. **Wrapped Generating Family Cohomology.** Theorems 4.5 and 4.8 arise from a notion of generating family cohomology for a Lagrangian cobordism, which we call Wrapped Generating Family Cohomology. (Wrapped Floer Homology, which is defined through holomorhic curves was introduced by by Abbondondolo and Schwarz [1] and developed further by Fukaya, Seidel and Smith [21] and by Abouzaid and Seidel [2].) For a Lagrangian cobordism $\overline{L}$, we would like to use generating families to define (co)homology groups that measures the intersections of the compact portion L and itself

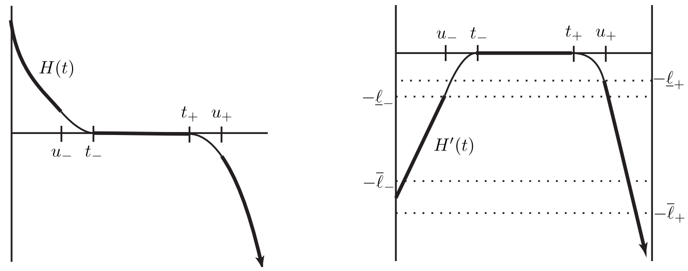

FIGURE 16. A schematic picture of H and $H'(t)$ for $H \in \mathcal{H}(\overline{L})$.

under a Hamiltonian diffeomorphism AND the Reeb chords of the Legendrians at the ends. This can be accomplished by looking at intersections of $\overline{L}$ under an appropriately defined Hamiltonian diffeomorphism.

We begin by specifying the Hamiltonian functions that will be used to convert Reeb chords of Legendrian submanifolds at the boundary to intersections of Lagrangians. Suppose a Lagrangian cobordism $\overline{L} \subset T^*(\mathbb{R}_+ \times M)$ is cylindrical over Λ_- when $t \leq t_-$ and cylindrical over Λ_+ when $t \geq t_+$. Then a **Hamiltonian shearing function** H consists of a decreasing smooth function $H : \mathbb{R}_+ \to \mathbb{R}$ that is 0 when $t \in [t_-, t_+]$ and is quadratic on $(0, u_-)$ and (u_+, ∞) for $u_- < t_- < t_+ < u_+$. A schematic figure for such an H is shown in Figure 16. The quadratics are chosen "sharp enough" so that all the Reeb chords will be detected as intersection points. A precise definition of the Hamiltonian shearing functions can be found in [30].

For $H \in \mathcal{H}\left(\overline{L}\right)$, the quadratic growth condition guarantees that the associated Hamiltonian vector field X_H will be integrable. If ϕ^1_H denotes the time-1 flow of this vector field and F generates $\overline{L}$, then it is easy to verify that $F(t, x, \eta) + H(t)$ generates $\phi^1_H(\overline{L})$.

In parallel to the definitions of the difference function δ, a shearing function H may be used to define the **sheared difference function** $\Delta_{F,H} : \mathbb{R}_+ \times M \times \mathbb{R}^N \times \mathbb{R}^{N'} \to \mathbb{R}$ as:

$$\Delta_{F,H}(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) + H(t) - F(t, x, \eta).$$

In parallel to Propositions 2.4 and 3.3, the critical points of $\Delta_{F,H}$ detect information about the intersection points of $\overline{L}$ and $\phi^1_H(\overline{L})$:

Proposition 4.13. *Suppose $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$, and H is a Hamiltonian shearing function. Then there is a one-to-one correspondence between intersection points in $\overline{L} \cap \phi^1_H\left(\overline{L}\right)$ and critical points of $\Delta_{F,H}$. Moreover, there is a one-to-one correspondence between Reeb chords $\gamma_\pm$ of $\Lambda_\pm$ and points in $\overline{L} \cap \phi^1_H\left(\overline{L}\right) \cap \{t \in (0, u_-) \cup (u_+, \infty)\}$. All other critical points lie in the critical submanifold*

$$C = \{(t, x, \eta, \eta) \ : \ (t, x, \eta) \in C_F \text{ with } t \in [t_-, t_+]\}.$$

The critical submanifold is diffeomorphic to $L = \overline{L} \cap \{t \in [t_-, t_+]\}$ *and has value* 0*; for generic* F*, the submanifold* C *is non-degenerate of index* N*.*

We are now ready to define generating family cohomology groups for Lagrangian cobordisms. Suppose $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$, and H is a Hamiltonian shearing function. For appropriate $0 < \mu \ll \Omega$, we define the **total wrapped generating family cohomology of** F to be:

$$\widetilde{WGH}^k(F) = H^{k+N}\left(\Delta^{\Omega}_{F,H}, \Delta^{-\mu}_{F,H}\right)$$

and the relative wrapped generating family cohomology of F to be:

$$WGH^k(F) = H^{k+N}\left(\Delta^{\Omega}_{F,H}, \Delta^{\mu}_{F,H}\right).$$

The constants μ, Ω are chose so that all critical values of $\Delta_{F,H}$ lie in $[-\mu, \Omega]$, and all critical values of $\Delta_{F,H}$ arising from the Reeb chords of the ends lie in $[\mu, \Omega]$; there are other technical restrictions on Ω and μ that can be found in [30].

In fact, the total wrapped generating family is "not interesting" (compare subsection 3.3); while the relative wrapped generating family detects the topology of $\overline{L}$:

Proposition 4.14. *If* $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$*, then:*

$$\widetilde{WGH}^k(F) = 0, \quad \forall k, \text{ and } \quad WGH^{k+1}(F) \simeq H^k(L, \partial L_+).$$

The key idea behind the proof of Theorem 4.8 (or 4.5) or is to view the relative pair $(\Delta^{\Omega}, \Delta^{\mu})$ which is used in the definition of relative wrapped generating family homology as a "relative" mapping cone. Recall that the cone of a space X, $C(X)$, is defined to be $X \times I / X \times \{1\}$, where $I = [0, 1]$. Given a map $f : X \to Y$, the mapping cone $C(f)$ is defined to be $C(X) \cup_f Y$, where $\cup_f$ indicates an identification of $(x, 0)$ with $f(x)$.

Definition 4.15. Given a pair (X, A), define the **relative cone** $C(X, A)$ to be the pair $(X \times I, A \times I \cup X \times \{1\})$. For a map $g : (X, A) \to (Y, B)$, let the **relative mapping cone** $C(g)$ be the pair $C(X, A) \cup_g (Y, B)$.

It is well-known that the classical mapping cone on $f : X \to Y$ induces a long exact sequence:

$$\cdots \to H^k(Y) \xrightarrow{f^*} H^k(X) \to H^{k+1}(C(f)) \to H^{k+1}(Y) \xrightarrow{f^*} \ldots.$$

A similar sequence exists for a relative mapping cone:

Lemma 4.16. *Given a map* $g : (X, A) \to (Y, B)$*, there is a long exact sequence in cohomology:*

$$\cdots \to H^k(Y, B) \xrightarrow{g^*} H^k(X, A) \to H^{k+1}(C(g)) \to H^{k+1}(Y, B) \xrightarrow{g^*} \cdots.$$

The long exact sequence in Theorem 4.8 is constructed by realizing the pair $\left(\Delta^{\Omega}, \Delta^{\mu}\right)$ as a relative mapping cone (where the associated (X, A) can be identified with $(\delta^{\omega}_{f_+}, \delta^{\epsilon}_{f_+})$ and (Y, B) can be identified with $(\delta^{\omega}_{f_-}, \delta^{\epsilon}_{f_-})$) by

FIGURE 17. (a) $\mathcal{S}^1$ is the Legendrian knot with $tb = -1$; (b)$\mathcal{S}^2$ is the "flying saucer".

examining the regions of $\mathbb{R}_+ \times M \times \mathbb{R}^{2N}$ over $(0, t_+]$ and $[t_+, \infty)$. This results in the following long exact sequence:

Proposition 4.17. *There exists a long exact sequence:*

$$\cdots \longrightarrow GH^k(f_-) \xrightarrow{\Psi_F^*} GH^k(f_+) \longrightarrow WGH^{k+2}(F) \longrightarrow \cdots .$$

As mentioned earlier, $WGH^{k+2}(F) \simeq H^{k+1}(L, \Lambda_+)$, and Theorem 4.8 follows.

4.4. **Constructing Lagrangian Cobordisms.** Above, the focus was on the "rigidity" of Lagrangian cobordisms. Now we show some ways of constructing Lagrangian cobordisms. First, isotopic Legendrian submanifolds are Lagrangian cobordant:

Proposition 4.18 ([4])**.** *Suppose that Λ_- is a Legendrian submanifold of J^1M with a generating family f_- and that Λ_+ is Legendrian isotopic to Λ_-. Then there exist generating families f_+, F so that $(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+)$.*

In addition, some basic Legendrians can be filled with Lagrangian disks/balls. Define $\mathcal{S}^1 \subset J^1(\mathbb{R}) = \mathbb{R}^3$ to be the Legendrian unknot with maximal Thurston-Bennequin invariant; then for $n \geq 2$, $\mathcal{S}^n \subset J^1(\mathbb{R}^n)$ is the n-dimensional Legendrian obtained by "spinning" $\mathcal{S}^{n-1}$. Figure 17 shows $\mathcal{S}^1$ and $\mathcal{S}^2$. Each of these Lagrangian spheres can be filled by a Lagrangian ball:

Proposition 4.19 ([4])**.** *For all $n \geq 1$, there exists f_-, f_+, F so that*

$$(\emptyset, f_-) \prec_{(L,F)} (\mathcal{S}^n, f_+).$$

The above proposition can be viewed as constructing a Lagrangian cobordism by attaching a 0-handle (to $\emptyset$). More generally, if one has a Lagrangian cobordism in $\{s \in \mathbb{R} : s \leq 0\} \times J^1M$ with Legendrian boundary $\Lambda_- \subset \{0\} \times J^1M$, we can extend this cobordism over $[0, 1] \times J^1M$ by attaching a q-handle; the resulting Legendrian $\Lambda_+ \subset \{1\} \times J^1M$ is obtained from Λ_- by a $(q-1)$-surgery. For this construction,the $(q-1)$-surgery to the Legendrian $\Lambda_- \subset J^1M$ is guided by a q-disk attached to the cusps of the front diagram of Λ_-.

We begin by specifying properties that the disk along which we perform surgery must satisfy. Let $\pi_{xz} : J^1M \to J^0M$ be the usual front projection

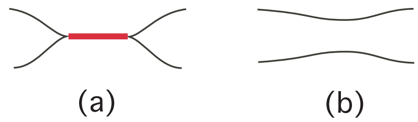

FIGURE 18. (a) A core disk for a 0-surgery of a 1-dimensional Legendrian in $\mathbb{R}^3 = J^1\mathbb{R}$, and (b)the result of embedded surgery along the core disk.

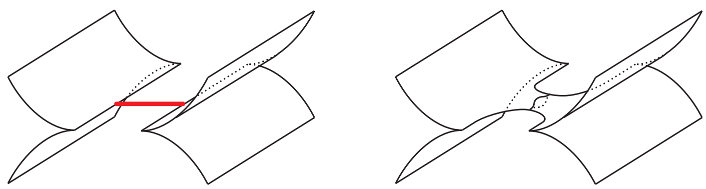

FIGURE 19. (a) A core disk for a 0-surgery of a 2-dimensional Legendrian in $\mathbb{R}^5 = J^1\mathbb{R}^2$, and (b) the result of embedded surgery along the core disk.

map, and let $\pi_x : J^0M \to M$ denote the projection to the base. The cusps of $\pi_{xz}(\Lambda)$ are denoted $\Lambda^{\succ}$, and if Λ has a generating family, let the cusps that represent births/deaths between critical points of indices j and $j+1$ be denoted by $\Lambda^{\succ j}$. In a neighborhood of a cusp point, the front of an n-dimensional Legendrian Λ has the form $C \times \mathbb{R}^{n-1}$, where C is the standard semi-cubical cusp in the plane. Thus, with respect to these coordinates, there is an open half-space H^+ that does not intersect the front of Λ.

We are now ready to define the disks along which we may perform surgery.

Definition 4.20 ([4])**.** An embedded q-disk $D \subset J^0M$ is a **gf-core disk** for a $(q-1)$-surgery on Λ if:

(1) D is disjoint from $\pi_{xz}(\Lambda)$ except that $\partial D \subset \Lambda^{\succ j}$, for some fixed $j \geq 0$,
(2) $D \subset H^+$ near its boundary, and
(3) T_pD never contains the vertical direction.

We refer to ∂D as the **attaching sphere**.

See Figures 18, 19, and 20 for examples of a core disks. We are now ready to formally state the surgery construction.

Theorem 4.21 ([4])**.** *Let Λ_- be a Legendrian submanifold of J^1M with tame generating family f_-. If there exists a q-dimensional gf-core disk D for Λ_-, then there exists Λ_+ with generating family f_+ and an F that generates a Lagrangian cobordism $\overline{L}$ so that $(\Lambda, f_-) - \prec_{(L,F)} (\Lambda_+, f_+)$, where*

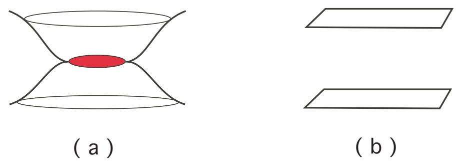

FIGURE 20. (a) A core disk for a 1-surgery of a 2-dimensional Legendrian in $\mathbb{R}^5 = J^1\mathbb{R}^2$, and (b)the result of embedded surgery along the core disk,

(1) *The cobordism* $\overline{L}$ *has the homotopy type of a cylinder over* Λ_- *with a q-cell attached, and*
(2) *The Legendrian* Λ_+ *is obtained from* Λ_- *by an embedded* $(q-1)$*-surgery along the attaching sphere of the gf-core disk.*

Example 4.22. From these constructions, we see that there does exist a Lagrangian cobordism between the Legendrian unknot and $m(3_1)$ illustrated in Figure 14. Namely, Figure 21 illustrated how it it possible to move between the Legendrian unknot with maximal tb and the Legendrian $m(3_1)$ with maximal tb using Legendrian isotopy and two 1 surgeries. The resulting cobordism has genus 1.

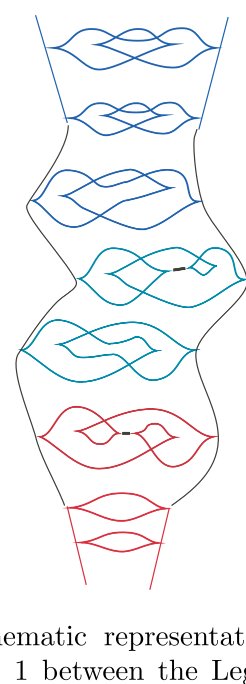

FIGURE 21. A schematic representation of a Lagrangian cobordism of genus 1 between the Legendrian unknot and Legendrian mirror trefoil.

Exercises 4.23. *In exercise 4.12, you found some Legendrians that were potentially Lagrangian cobordant. Can you find a Lagrangian cobordism using isotopies and surgeries?*

These construction techniques can also be applied to address some fundamental geography and botany questions for Legendrian submanifolds, see [4].

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