SCHUBERT VARIETIES

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There are far too many problems to do here in an hour, or maybe even a day. You should choose the problems that look interesting to you that you might have an idea about how to start.

(1) (a) What equations cut X_{13} out of $\mathbb{P}^5$?
   (b) At what points in $\mathbb{P}^5$ is X_{23} singular?
   (c) What planes in $\mathrm{Gr}(2,4)$ do those points in $\mathbb{P}^5$ correspond to?

(2) How many Schubert varieties are there in $\mathrm{Gr}(2,4)$?

(3) List all of the Schubert varieties in $\mathrm{Gr}(2,4)$, giving each one a name (by partition) and giving a set of equations that cut them out.

(4) Give equations for the Schubert varieties corresponding to the partitions $(2,0)$ and $(1,1)$. What are the equations for the intersection of these two varieties?

(5) Prove that if $J + (j_1, \ldots, j_k)$ then $C_J \cong \mathbb{C}^d$, where $d = j_1 + \cdots + j_k - \frac{k(k+1)}{2}$.

(6) Prove that $X_J - C_J$ is a union of other Schubert cells C_I (and say which C_I!).

(7) What is the relationship between the number of boxes in a Young diagram λ and the entires in the corresponding sequence J?

(8) Show that the closure of the Schubert cell corresponding to λ in $\mathrm{Gr}(k,n)$ is the union of all Schubert cells with Young diagram μ such that $\mu \subseteq \lambda$.

(9) Let E_i represent the i-plane in the standard flag. Recall that the rank matrix of a permutation $w \in S_n$ to be the $n \times n$ matrix $r(w)$ with entries $r_{ij}(w)$ equal to the rank of the sub matrix of the permutation matrix of with corners at (n,n) and (i,j). One definition of C_w is $C_w = \{F \in \mathcal{F}\ell\mathbb{C}^n : \dim(E_i \cap F_j) = r_{ij}(w)\}$.
   (a) Show that $X_w = \{F \in \mathcal{F}\ell\mathbb{C}^n : \dim(E_i \cap F_j) \geq r_{ij}(w)\}$.
   (b) Give equations (you should have n^2 "sets" of equations) that cut out X_w.
   (c) Are all of these equations necessary? If not, which ones are? (This question is pretty doable in an example but is in general a tough theorem. If you want me to tell you the theorem, ask, it's one of my favorites. If you want to look it up, it is a result of Fulton's, published in 1992 in Duke Math. J.)