Let $G$ be a finitely generated group generated by a finite symmetric generating set $S$. For an element $g \in G$, we denote by $\ell_S(g)$ the word length of $g$ with respect to $S$. We let $B_S(n) = \{ g \in G : \ell_S(g) \leq n \}$ be the ball of radius $n$ around the identity in the Cayley graph $\text{Cay}(G, S)$.

Recall that the growth rate of $G$ with respect to $S$ is $\omega_S(G) = \lim_{n \to \infty} \sqrt[n]{|B_S(n)|}$.

The group $G$ has exponential growth if for some (equiv. any) finite symmetric generating set $S$, the growth rate $\omega_S(G) > 1$. Otherwise, $G$ has subexponential growth.

**Exercise 0.1.** Prove that a finitely generated group of subexponential growth is amenable.

The next exercise shows that the converse to Exercise 0.1 does not hold. Let $G$ and $H$ be groups. Recall that the (restricted) wreath product $G \wr H$ is the semidirect product $\bigoplus_{h \in H} G \rtimes H$. The action of $H$ on $\bigoplus_{h \in H} G$ is defined as follows. Let $f \in \bigoplus_{h \in H} G$. Then $f$ can be viewed as a finitely supported function $f : H \to G$. Then, for $k \in H$, $(k * f)(h) = f(k^{-1}h)$. The wreath product $G \wr H$ consists of elements of the form $(f, k)$ for $f \in \bigoplus_{h \in H} G$ and $k \in H$. The product is defined as follows.

$$(f_1, k_1)(f_2, k_2) = (f_1(k_1 * f_2), k_1 + k_2).$$

**Exercise 0.2** (The lamplighter group - an amenable group of exponential growth). The lamplighter group $L_2$ is defined to be the (restricted) wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. Show that

(a) $L_2$ is 2-generated.

(b) $L_2$ has exponential growth.

(c) $L_2$ is amenable.

**Exercise 0.3** (The first Grigorchuk group). The first Grigorchuk group $\Gamma$ is a self-similar group generated by four automorphisms of the binary tree $T$. The automorphism $a$ is the switch at the root $a = (1, 1)\sigma$ where $\sigma$ is the non-trivial element of the symmetric group $S_2$. The automorphisms $b, c$ and $d$ are defined recursively by the formulas

$$b = (a, b); \quad c = (a, d); \quad d = (1, b).$$

Thus, for example, $b$ acts trivially at the root and acts as $a$ on the subtree rooted at the vertex 0 of the first level and as $c$ on the subtree rooted at 1.
(a) Show that the subgroup of $\Gamma$ generated by $\{b, c, d\}$ consists of the elements $\{1, b, c, d\}$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Use this to represent elements of $\Gamma$ by reduced words in the alphabet $a, b, c, d$.

(b) For $k \geq 1$, consider the stabilizer $\text{Stab}_\Gamma(k)$ of the vertices of the tree of level $\leq k$ and denote by $H$ the stabilizer of the first level $\text{Stab}_\Gamma(1)$, a subgroup of index 2 consisting of all words with even number of $a$'s. Show that $H$ is generated by $\{b, c, d, aba,aca, ada\}$.

(c) Consider the homomorphism

$$\varphi = (\varphi_0, \varphi_1): H \to \text{Aut}(T) \times \text{Aut}(T)$$

which sends each element in the stabilizer of the first level to the couple of its restrictions to the subtrees rooted at the vertices of the first level. Show that $\varphi(H) \leq \Gamma \times \Gamma$ and that the projection of $\varphi(H)$ onto each of the two components is the whole $\Gamma$ (i.e., that $\varphi_0(H) = \Gamma$ and $\varphi_1(H) = \Gamma$). Deduce that $\Gamma$ is infinite.

(d) Use $\varphi$ to show that $\Gamma$ is a 2-group, that is that for any $g \in \Gamma$ there exists an integer $N \geq 0$ such that $g^{2^N} = 1$. Hint: Use induction on the length of $g$ in terms of $a, b, c, d$. There are several cases to consider but all of them essentially boil down to the fact that the homomorphism $\varphi$ is length decreasing. More precisely,

$$|\varphi_j(g)| \leq \frac{|g| + 1}{2},$$

where $|\cdot|$ stands for the word length with respect to $\{a, b, c, d\}$.

Remark 0.4. The last exercise shows that $\Gamma$ is an infinite finitely generated torsion group. It is known (Chuo) that such groups cannot be elementary amenable. However $\Gamma$ is amenable being of subexponential growth. Alternatively, to see that $\Gamma$ is not elementary amenable one can use the fact that it has intermediate growth (i.e., subexponential but superpolynomial) and Chuo’s result that finitely generated elementary amenable groups have either polynomial or exponential growth.