Amenability lecture 4

Percolation & amenability

\[ T = \] \[ \begin{array}{c|c|c|c|c} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 0 & & & & \\ \hline \end{array} \]

Map spreading: EX's disease spreading among an orchard, fires, coffee brewing (water & beans)

1957 Broadbent, Hammersley
Bernoulli bond percolation

\[ \forall \text{edge } e \in \text{Edges}(T) \]
\[ \text{toss a } (p, 1-p) \text{-coin} \]
\[ \text{Bernoulli RV} \]

Think of water flowing on the edges and declare the edge open or closed according to the result.

We obtain a Configuration Space:
\[ (0, 1)^{\text{Edges}} \]

Equipped with the prob. measure \[ P_p = \prod_{e \in \text{Edges}} P_p(e) \]

\[ P_p(e) = \begin{cases} p & \text{if } \gamma_p(e) = 1 \\ 1-p & \text{if } \gamma_p(e) = 0 \end{cases} \]

Note: \((0, 1)^{\text{Edges}}, P_p\) can be viewed as a "random subgraph" of \(T\)

\(P_p\) is \(G\)-invariant if \(T = \text{Cay}(G, S)\)

Remark: Such a prob space can be similarly defined \((\gamma_p \in [0, 1])\) and \(\gamma \in G, \gamma \in \text{gp}\)
and \(S\) a fin. symm. gen. set.
Moreover, it can similarly be defined on any transitive infinite-complement graph. Let a property that makes a graph similar to $\text{Cay}(G,S)$.

3. Given $p$, are we likely to have an infinite component in our subgraph (an infinite cluster)?

- $p=0 \implies \text{prob } 0 \text{ to have an inf. cl.}$
- $p=1 \implies \text{prob } 1 \text{ to have an inf. cl.}$

Define Percolation Function

$$\Theta(p) = P_p \{ \Omega \text{ an infinite cluster} \}$$

$$P_c = \sup \{ p : \Theta(p) = 0 \}$$

\[ P_c(\mathbb{Z}^2) = \frac{1}{2} \]

$$P_c(\mathbb{T}_d) = \frac{1}{d-1}$$

Critical value of percolation

- Its from left is open for $\mathbb{Z}^2$, $\mathbb{Z}^3$ but known for other $\mathbb{Z}^d$. 

$7/28/17$
Note \( p_c = \inf \{ p : \text{a.s. } \exists \text{ an inf. cluster} \} \) 
= \( \sup \{ p : \text{a.s. no inf. clusters} \} \)

Benjamini, Schramm 1996
"Percolation beyond \( \mathbb{Z}^d \): many questions & few answers"

Note \( p_c(\mathbb{Z}) = 1 \)

Conj \( p_c(\mathbb{G}, s) = 1 \) iff \( \mathbb{G} \) is virtually \( \mathbb{Z} \)

Proved for poly growth gps, exp growth gps, Grigorchuk's gps

If \( p > p_c \), how many inf. clusters do you have?

Thm \# of inf. clusters is almost surely \( 0, 1, \infty \)

Recall Stallings Thm \# of ends in a f.g. gp is \( 0, 1, 2 \) or \( \infty \)

Thm In \( \mathbb{Z}^2 \), \( \forall p > p_c \), \( \Pr_p (\exists \text{ inf. cluster}) = 1 \)

For amenable

Prop In a tree 
\( \Pr_p (\text{the inf. cluster is unique}) < 1 \) 
\( \forall p < 1 \)
Grimmet–Neuman \( \mathbb{Z} \times T \)

<table>
<thead>
<tr>
<th>0 inf. d.</th>
<th>inf. many</th>
<th>1 inf.</th>
<th>cl.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( P_c )</td>
<td>( P_u )</td>
<td>1</td>
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Unicity parameter

Conjecture (Benjamini–Schramm) (BS-amenable conj.)

\[ P_c = P_u \text{ iff } T \text{ is amenable} \]

N. Pack (’00) Proved a weak version of the conjecture:

If \( G \) is a non-amenable f.g. gp, then \( \exists S \subset G \) a finite gen set such that

\[ P_c(G, S) : P_c(T) < P_u(T) \]

Idea of proof combines Spectral Graph Theory & Geometric Group Theory

**Corollary.** A f.g. gp \( G \) is amenable iff

\[ \forall S \text{ f.g. gen set } \quad P_c(\text{Cay}(G, S)) = P_u(\text{Cay}(G, S)) \]

**Application:** Recall von Neumann’s Problem

Does every non-amenable \( G \) contain a free subgp \( F_2 \)? No

Thm (1999, K. Whyte) \( G \) non-amenable \( \iff \) its Cayley graph(s) can be partitioned into pieces which are uniformly bi-Lipschitz equivalent to \( T \).
Thm (Gaboriau-R. Lyons) 2013

G non-amenable $\Rightarrow$ G admits an action on a
prob. space such that almost
all orbits of this action
can be partitioned into
the orbits of an essentially
free action of $F_2$ on this space.

Idea of the proof: G non-amenable

weak version of B.S.

Let $S$ s.t. in Cay(G,S) = $T^d$, $P \leq P_n$

Take $p \in (p_c,p_n)$, $\tilde{\gamma} \mapsto (\gamma, \tilde{\gamma}_0, \cdots, \tilde{\gamma}_d)$, $P_p$

a.s. $\infty$ many infinite clusters

2. Use Stallings' Thm: Every conn'd graph with

$\forall x$ degrees $\leq 2d$ can be realized as
the Stallings graph of a subgp

in $F_d$. 