From Linear Algebra to the Non-squeezing Theorem of Symplectic Geometry

Margaret Symington *

May 25, 2012

Symplectic geometry is a geometry of even dimensional spaces in which area measurements, rather than length measurements, are the fundamental quantities. This course will introduce symplectic manifolds, starting with symplectic vector spaces and examples in $\mathbb{R}^{2n}$. It will culminate with the foundational non-squeezing theorem that reveals the ever-present tension between the rigid geometric and the soft topological features of symplectic manifolds.

1 Lecture 1: Linear symplectic transformations

The title of this series of lectures is "From Linear Algebra to the Non-Squeezing Theorem of Symplectic Geometry", but the subtitle should be "Ode to Ellipsoids". You'll see why, today and on Friday..

My goal in these five lectures is to give you a feel for some of the basic features of symplectic geometry, so I'm going to start with the historical origins of symplectic geometry.

1.1 Classical mechanics

As in mechanics, let's denote the time derivative of a quantity by a dot, so

$$\begin{aligned} x &= \text{position} \\ \dot{x} &= \text{velocity} \\ \ddot{x} &= \text{acceleration} \end{aligned}$$

Then Newton's 2nd Law is $F = m\ddot{x}$. Hamilton replaces Newton's 2nd order differential equation with a pair of 1st order differential equations that have some extra structure – a conserved quantity H, which is often the total energy of the system:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{aligned}$$

*Mercer University, symington_mf@mercer.edu

where $y = m\dot{x}$ is the momentum. The variables x and y are the coordinates of *phase space*. Since an initial value for position and velocity determine the evolution of the system, phase space is a natural space in which to study the trajectories of solutions to an initial value problem.

Let's see how this works in an example.

Example 1.1. Consider an undamped spring with mass m and spring constant k. Let x represent the displacement of the mass from its rest position, with positive values of x representing extension of the spring, as usual. Then we have

$$\begin{aligned} \text{Hooke's Law}: \;\; & F = -kx \\ \text{Kinetic energy}: \;\; & \frac{1}{2}m\dot{x}^2 \\ \text{Potential energy}: \;\; & \frac{1}{2}kx^2 \end{aligned}$$

For the Hamiltonian approach we need the momentum $y = m\dot{x}$.

Following Newton, the equation of motion for this system is $m\ddot{x} = -kx$. Let's check that Hamilton's equations agree if we let H be the total energy, i.e. $H = \frac{1}{2}\left(\frac{y^2}{m} + kx^2\right)$. Then

$$\dot{x} = \frac{\partial H}{\partial y} = \frac{y}{m},$$

which gives the relation between velocity and momentum, and

$$\dot{y} = -\frac{\partial H}{\partial x} = -kx,$$

which encodes Hooke's Law because $\dot{y} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$.

A strength of the Hamiltonian formalism is the flexibility of changes of coordinates, even changes that intertwiwne the position and momentum variables. We will see an example of this in Lecture 4. The intertwining is key. In fact, the adjective "symplectic" comes from the Greek word *symplektikos* which means "of intertwining".[1] But let's first try out a change of variables that is simpler. Trajectories in phase space (i.e. solution curves) lie on level sets of H; since the motion is one-dimensional – so our phase space is two-dimensional – the level sets of H **are** the solution curves. Notice that they are ellipses. Let's see if we can make them as nice as possible; let's try to make them circles.

We want new variables $\tilde{x}$, $\tilde{y}$ such that

$$H = \frac{c}{2}\left(\tilde{y}^2 + \tilde{x}^2\right)$$

for some $c > 0$. So let

$$\tilde{x} = \sqrt{\frac{k}{c}}x, \qquad \tilde{y} = \sqrt{\frac{1}{cm}}y.$$

Hamilton's 1st equation requires

$$\dot{\tilde{x}} = \frac{\partial H}{\partial \tilde{y}} = c\tilde{y}.$$

[1] This is from the Miriam Webster Unabridged Dictionary, accessed online, 2012.

Rewriting this in terms of x and y we find that

$$\sqrt{\frac{k}{c}}\dot{x} = \sqrt{\frac{c}{m}}y \quad \text{and hence} \quad \sqrt{\frac{k}{c}}\dot{x} = \sqrt{cm}\dot{x}.$$

Therefore we are not free to choose the radius of the circle! Solving for c we find $c = \sqrt{\frac{k}{m}} = \omega$, the *circular frequency* that appears in the general solution to Newton's 2nd order ODE: $x = A\cos\omega t + B\sin\omega t$. Thus, if we want to change variables to get a circular orbit, we must choose

$$\tilde{x} = (mk)^{\frac{1}{4}}, \qquad \tilde{y} = (mk)^{-\frac{1}{4}}y.$$

Notice that $\tilde{x}\tilde{y} = xy$ but $y = m\dot{x}$ while $\tilde{y} \neq m\dot{\tilde{x}}$. Why is the new $\tilde{y}$ no longer related to $\dot{\tilde{x}}$ in the way we would expect??!!

Exercise: Explain why this is good.

We also need to check that this coordinate change respects Hamilton's 2nd equation:

$$\dot{\tilde{y}} = \frac{\partial H}{\partial \tilde{x}} = -\omega\tilde{x}.$$

If we write this in terms of x and y we get

$$(mk)^{-\frac{1}{4}}\dot{y} = -\left(\frac{k}{m}\right)^{\frac{1}{2}}(mk)^{\frac{1}{4}}x$$

which reduces to $\dot{y} = -kx$ which is just Newton's equation since $\dot{y} = m\ddot{x}$.

In it original context, symplectic geometry is the geometry of phase space. Symplectic transformations are the transformations that preserve Hamilton's equations, and hence preserve the Hamiltonian (energy in this case), orbits (solution curves), periods of orbits, and the rates at which orbits converge or diverge from each other. In physics, symplectic transformations are typically known as *canonical transformations.*

In a more general setting there will be more degrees of freedom. Letting

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Hamilton's equations are the $2n$ equations encoded in

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -J_0 \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} = -J_0 \nabla H$$

where $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and ∇H is the gradient of H.

We will see this matrix J_0 a lot. Geometrically, it rotates vectors in each x_i-y_i plane 90 degrees counterclockwise (mapping the positive x_i-axis to the positive y_i-axis). Notice that phase space is always even dimensional and that there is a natural pairing among the coordinates, each position with the corresponding momentum. This pairing will last all week, so we'll call each x_i-y_i-plane a *symplectic coordinate plane.*

Canonical (symplectic) transformations are useful in physics because they

- sometimes allow one to solve a problem, using a coordinate change to transform it into a simpler or more recognizable problem and
- allow one to use local coordinates to understand complicated systems – since the canonical (symplectic) transformations encode the transitions between those coordinate charts.

The most basic symplectic transformations are the symplectic transformations of the standard $\mathbb{R}^{2n}$.

Let

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

be a linear coordinate change. Note that A is a $2n \times 2n$-matrix.

We need

$$\begin{pmatrix} \dot{\tilde{x}} \\ \\ \dot{\tilde{y}} \end{pmatrix} = -J_0 \begin{pmatrix} \frac{\partial H}{\partial \tilde{x}} \\ \\ \frac{\partial H}{\partial \tilde{y}} \end{pmatrix}$$

where the rightmost vector is the gradient of H with respect to $\tilde{x}$ and $\tilde{y}$.

Because of our change of variables,

$$\begin{pmatrix} \dot{\tilde{x}} \\ \\ \dot{\tilde{y}} \end{pmatrix} = A \begin{pmatrix} \dot{x} \\ \\ \dot{y} \end{pmatrix} = -AJ_0 \begin{pmatrix} \frac{\partial H}{\partial x} \\ \\ \frac{\partial H}{\partial y} \end{pmatrix} = -AJ_0A^T \begin{pmatrix} \frac{\partial H}{\partial \tilde{x}} \\ \\ \frac{\partial H}{\partial \tilde{y}} \end{pmatrix}$$

where the last equality requires working out how the gradient transforms under a change of variables.

Exercise: Verify the last equality.

This leads us to

Definition 1.2. A $2n \times 2n$-matrix A is symplectic if $AJ_0A^T = J_0$.

1.2 Symplectic matrices

Our goal now is to deduce some of the consequences of the defining equation for symplectic matrices, $AJ_0A^T = J_0$, and then understand its geometric meaning.

Proposition 1.3. *If A is symplectic, then so is A^{-1}. Also the product of any two symplectic matrices is symplectic.*
Note that this is enough to show that, for any given n, the set of $2n \times 2n$ symplectic matrices forms a group – a matrix group denoted by $\mathrm{Sp}(2n)$.

Proposition 1.4. *If A is symplectic then so is A^T.*

Proposition 1.5. *If λ is an eigenvalue of a symplectic matrix, then so is $1/\lambda$.*

A very important property of symplectic matrices takes a bit more work to prove:

Proposition 1.6. *For any symplectic matrix A, $\det A = 1$. Furthermore, if A is a $2x2$ matrix, then A is symplectic if and only if $\det A = 1$.*

Exercise: Prove these propositions. Note that it is easy to prove that $(\det A)^2 = 1$.

Now what about the geometric meaning of a matrix being symplectic? To clarify the question, compare $\mathrm{Sp}(2n)$ with two other matrix groups:

- The equation $A^T A = I$ characterizes orthogonal matrices which are the matrices that preserve lengths of vectors and hence angles between vectors. This group is denoted $O(n)$.
- The equation $AJ_0A^{-1} = J_0$ characterizes matrices that preserve the complex structure of $\mathbb{R}^{2n}$ viewed as $\mathbb{C}^n$ under the identification $z_j = x+j+iy_j$. This group is denoted $GL(n, \mathbb{C})$, while $GL(n, \mathbb{R})$ denotes the group of invertible matrices.[2]

1.3 Bilinear forms

The geometry of symplectic matrices is revealed by studying the geometry of the non-degenerate skew-symmetric bilinear form

$$\omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \quad \text{defined by} \quad \omega_0(u, v) := v^T J_0 u.$$

Exercise: Check that a matrix is symplectic if and only if it preserves ω_0, i.e. if and only if $\omega(Au, Av) = \omega(u, v)$ for all pairs of vectors $u, v \in \mathbb{R}^{2n}$.

Note the analogy with inner products which are non-degenerate symmetric bilinear forms that are also positive definite:

$$\langle u, v \rangle_B := v^T B u$$

where B is a symmetric matrix and the standard inner product comes by taking $B = I$.

Let's make sure we understand the vocabulary...

Definition 1.7. A *bilinear form* Ω on a vector space V is a map $\Omega : V \times V \to \mathbb{R}$ that is linear in each component, i.e.

$$\begin{aligned} \Omega(u_1 + u_2, v) &= \Omega(u_1, v) + \Omega(u_2, v) \\ \Omega(u, v_1 + v_2) &= \Omega(u, v_1) + \Omega(u, v_2) \\ \Omega(cu, v) &= c\Omega(u, v) = \Omega(u, cv) \quad \text{for any scalar } c \end{aligned}$$

[2] The "GL" stands for general linear. They are the matrices that represent all linear transformations of a vector space, complex and real respectively.

Definition 1.8. A bilinear form Ω on V is

- *symmetric* if $\Omega(v, u) = \Omega(u, v)$
- *skew-symmetric* if $\Omega(v, u) = -\Omega(u, v)$.

Definition 1.9. A bilinear form Ω on V is *non-degenerate* if for every $u \neq 0$ there is a v such that $\Omega(u, v) \neq 0$.

Definition 1.10. A symmetric bilinear form is *positive definite* if $A(v, v) > 0$ for all $v \neq 0$.

Exercise: Check that ω_0 is a non-degenerate skew-symmetric form.

Now we can define a general symplectic vector space.

Definition 1.11. A *symplectic vector space* is a vector space V equipped with a non-degenerate skew-symmetric bilinear form. In that case, the form itself is called the *symplectic form*.

In the next lecture we will begin our study of $(\mathbb{R}^{2n}, \omega_0)$ and later see that this symplectic vector space – the "standard" one – is completely general because symplectic vector spaces are classified by their dimension.

1.4 Exercises

There are a lot of exercises from this lecture. Don't worry. Some are easy and most are not pressing. I think the most important ones for future lectures are #2, #5b and #9. You should definitely do #3, #6. Problem #7 should be review; its purpose is for the analogy between inner products and symplectic forms. Problem #8 is just for discussion.

1. (This is a question to chew on and discuss.) Consider the change of variables $\tilde{x} = \lambda x$. This implies $\dot{\tilde{x}} = \lambda \dot{x}$, and hence $\tilde{y} = \lambda y$. But this disagrees with what Hamilton's equations requires. What went wrong?? Nothing really. Bear in mind the two purposes of symplectic changes of variables: 1) solving differential equations and 2) being able to transition between different local coordinate charts in a topologically non-trivial phase space. The second purpose dictates why we want to exclude this scaling, and excluding does not undermine the first purpose. Hint: Pay attention to the energy of the system.

2. Consider any invertible linear transformation $B : \mathbb{R}^n \to \mathbb{R}^n$ given by $\tilde{x} = Bx$ where x is an n-vector and any function $H(x)$. Show that

$$\frac{\partial H}{\partial x} = B^T \frac{\partial H}{\partial \tilde{x}},$$

in other words $\nabla H(x) = B^T \nabla \left(H \circ B^{-1}(\tilde{x})\right)$. This establishes

$$\begin{pmatrix} \frac{\partial H}{\partial x} \\ \\ \frac{\partial H}{\partial y} \end{pmatrix} = A^T \begin{pmatrix} \frac{\partial H}{\partial \tilde{x}} \\ \\ \frac{\partial H}{\partial \tilde{y}} \end{pmatrix}.$$

3. Show that if A is symplectic, then so are A^{-1} and A^T. Also the product of any two symplectic matrices is symplectic.

4. Show that if λ is an eigenvalue of a symplectic matrix A, then so is $1/\lambda$.

5. Prove that following facts about the determinant of a symplectic matrix:

   (a) If A is symplectic then $(\det A)^2 = 1$. (Easy)

   (b) If A is a 2×2-matrix, then $\det A = 1$ if and only if A is symplectic.

   (c) If A is symplectic, then $\det A = 1$. Here's a strategy: Since A is invertible, $A = PQ$ where P is positive definite and symmetric, and Q is orthogonal. (This is known as the *polar decomposition* of a non-degenerate matrix.) Prove that P is symplectic, and hence so is Q. Check that $\det P > 0$. Now show that Q – being both symplectic and orthogonal, and hence unitary – can be written in block form as $\left(\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right)$, which is in turn conjugate to the matrix $\left(\begin{smallmatrix} X+iY & 0 \\ 0 & X-iY \end{smallmatrix}\right)$, which has positive determinant since X and Y are both real matrices. Therefore, since the determinants of P and Q are positive, so is the determinant of $A = PQ$, and since we already know $\mathrm{detA} = \pm 1$ we have established that $\mathrm{detA} = 1$.

6. Write the $2n \times 2n$-matrix A in terms of $n \times n$ blocks, so $A = \left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$. Under what conditions on X, Y, Z and W is A symplectic?

7. Show that a matrix preserves the standard inner product if and only if it preserves lengths of vectors, i.e. A is orthogonal if and only if $|v| = |Av|$ for all v. Also show that if a matrix preserves the standard inner product, then it preserves angles between vectors.

8. Explain how the condition $AJ_0A^{-1} = J_0$ encodes the property that A preserves the complex structure of $\mathbb{R}^{2n} = \mathbb{C}^n$.

9. Check that a matrix is symplectic if and only if it preserves ω_0, i.e. if and only if $\omega_0(Au, Av) = \omega_0(u, v)$ for all pairs of vectors $u, v \in \mathbb{R}^{2n}$. (This is trivial, but is important so as to verify that our terminology for symplectic forms and symplectic matrices is consistent.)

10. Check that the standard symplectic form ω_0 is a non-degenerate skew-symmetric bilinear form.

11. Under what conditions on A does $(u, v) \to v^T Au$ define an inner product? A symplectic form? (The second question will be much easier later in the week.)

2 Lecture 2: Of Parallelograms and Planes

Goals for today:

- Understand what properties of $(\mathbb{R}^{2n}, \omega_0)$ are preserved under linear symplectic transformations (in analogy with the fact that lengths of vectors in $(\mathbb{R}^n, \langle\ ,\ \rangle)$ are preserved under orthogonal transformations).
- Understand the types of subspaces that exist in $(\mathbb{R}^{2n}, \omega_0)$.

2.1 Symplectic area

First let's restrict to $\mathbb{R}^2$.

Consider vectors $u = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$.

Then

$$\begin{aligned} v^T J_0\, u &= (c,d)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\ &= (c,d)\begin{pmatrix} -b \\ a \end{pmatrix} = -bc + ad \\ &= \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \det(uv) \end{aligned}$$

So, in dimension 2, $\omega_0(u,v)$ is the signed area of the parallelogram spanned by u and v, and hence linear symplectic transformations of $(\mathbb{R}^2, \omega_0)$ are precisely those transformations that preserve area and orientation.

Here are three ways to understand this:

1. If we think of u and v as being in $\mathbb{R}^2 \subset \mathbb{R}^3$, in the xy-coordinate plane in xyz-space then letting
$$\overline{u} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \qquad \overline{v} = \begin{pmatrix} c \\ d \\ 0 \end{pmatrix}$$
we have
$$\overline{u} \times \overline{v} = \begin{pmatrix} 0 \\ 0 \\ ad - bc \end{pmatrix}$$
and $|\overline{u} \times \overline{v}| = |\overline{u}|\,|\overline{v}| \sin\theta$ which is the signed area of P, the parallelogram with edges u and v. Note that the area is *signed* because $\overline{u} \times \overline{v} = -\overline{v} \times \overline{u}$.

2. Let $B = (uv) = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. Then $u = B\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $v = B\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $|\det B|$ measures the distortion in the area and, presuming B is non-degenerate, $\det B$ is positive or negative according to whether B preserves or reverses orientation. So since P is the image of a square of area 1, its area equals the distortion.

3. And to be really concrete... Check that you can cut up the parallelogram P and rearrange the pieces to form a rectangle whose area is $ad - bc$.

Exercise: Try that out. For simplicity you may want to choose u and v to both the in the first quadrant.

Now we are ready to consider: what does ω_0 measure in $\mathbb{R}^{2n}$?

First some notation. Let e_i and f_i be basis vectors

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad f_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1's are in the i^{th} and $n + i^{\text{th}}$ positions, respectively. We'll write a $2n$-vector u in several ways

$$u = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \sum_{i=1}^{n} a_i e_i + b_i f_i$$

Also, let's call the plane spanned by e_i and f_i the i^{th} *symplectic coordinate plane* and think of the projection of a vector u to the i^{th} symplectic coordinate plane as a 2-vector $a_i e_i + b_i f_i$, so $u_i = \left(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right)$.

Now in general we have

$$\begin{aligned}
\omega_0(u, v) &= (c^T, d^T) \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= (c^T, d^T) \begin{pmatrix} -b \\ a \end{pmatrix} = -c^T b + d^T a \\
&= -\langle b, c \rangle + \langle a, d \rangle \\
&= \sum_{i=1}^{n} a_i d_i - b_i c_i.
\end{aligned}$$

So what is this quantity? If P is the parallelogram in $\mathbb{R}^{2n}$ with edges u and v, then its projection to the i^{th} symplectic coordinate plane is the parallelogram spanned by $\left(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c_i \\ d_i \end{smallmatrix}\right)$. Now notice: $\omega_0(u, v)$ is the sum of the signed areas of these projections. We will call this the *symplectic area* of P.

So...

> Symplectic linear transformations preserve symplectic area, i.e. the sum of the signed areas of projections to the symplectic coordinate planes.

Recall that while orthogonal transformations are precisely those that preserve lengths of vectors, they also preserve angles between vectors. Analogously...

Theorem 2.1. *Linear symplectic transformations preserve volume.*

This is known as Liouville's theorem in mechanics. It means that if you consider a 2n-dimensional ball or box in phase space (think of a set of initial values) and then follow that ball (or box) as it evolves in time; it's volume will remain constant. This is a consequence of two things:

- $|\det A| = 1$ if A is symplectic and
- the determinant of a matrix $(v_1 v_2 \ldots v_k)$ whose columns are the vectors v_i of length k measures the volume of the parallelepiped (higher dimensional analog of a parallelogram) that has the v_i as edges.

2.2 Subspaces of symplectic vector spaces

Now here is how symplectic vector spaces are more interesting than vector spaces equipped with an inner product:

> Not all subspaces of a given dimension in (V, ω) are created equal!

Definition 2.2. A subspace $S \subset (V, \omega)$ is a *symplectic subspace* if $\omega|_S$, the restriction of ω to S, is non-degenerate. In other words, for all $u \in S$ such that $u \neq 0$ there is a $v \in S$ such that $\omega(u, v) \neq 0$.

Note that this agrees with our definition of a symplectic vector space as a vector space equipped with a non-degenerate skew-symmetric bilinear form – in this case $\omega|_S$.

Example 2.3. Consider two vectors u and v in $(\mathbb{R}^4, \omega_0)$ that lie in the plane L spanned by e_1 and e_2. Then, keeping with our convention on the names for the components of these vectors,

$$\begin{aligned} \omega_0(u, v) &= (c_1, c_2, 0, 0) \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix} \\ &= (c_1, c_2, 0, 0) \begin{pmatrix} 0 \\ 0 \\ a_1 \\ a_2 \end{pmatrix} = 0 \end{aligned}$$

This is because the projection of L to each of the symplectic coordinate planes is just a line, so there are no area contributions. Essentially, L is invisible to ω_0!

Definition 2.4. A subspace $L \subset (V, \omega)$ is *Lagrangian* if $\omega|_L = 0$ and $\dim L = \frac{1}{2} \dim V$.

In order to identify other interesting classes of subspaces we need to know how to take their complements – in a way that depends only on the symplectic form. We can't use orthogonal complements because we do not have an inner product. At least we can't use an inner product directly on its own, but in conjunction with Jz we can...

Definition 2.5. Given a subspace $W \subset (V, \omega)$, its *symplectic complement* W^ω is the subspace

$$W^\omega := \{v \in V \,|\, \omega(v, w) = 0 \text{ for all } w \in W\}.$$

If you would like to think of (V, ω) as $(\mathbb{R}^{2n}, \omega_0)$ you won't be missing anything because, as we will see, symplectic vector spaces are classified by their dimension.

Example 2.6. If W is the span of $e_1, \ldots e_n$ in $(\mathbb{R}^{2n}, \omega_0)$, then W is Lagrangian and $W^\omega = W$.

Hmm... so in what sense are W and W^ω complementary? Well, they are complementary in that their dimensions are as we would hope:

Proposition 2.7. *For any subspace $W \subset V$, $\dim W + \dim W^\omega = \dim V$.*

Proof: Since $\omega(v, w) = v^T A w$ for some non-degenerate matrix A (satisfying some additional properties you figured out if you did Exercise 11 from Lecture 1),

$$W^\omega = \{v \in V \,|\, v^T A w = 0 \text{ for all } w \in W\}.$$

but $v^T A w = \langle v, Aw \rangle$ so W^ω is the orthogonal complement of the subspace AW. Therefore,

$$\begin{aligned} \dim W^\omega &= \dim (AW)^\perp = \dim V - \dim AW \\ &= 2n - \dim W. \end{aligned}$$

Notice:

- If V is symplectic then $V^\omega \cap V = \{0\}$.
- If V is Lagrangian then $V^\omega = V$.
- The symplectic complement of a hyperplane is a one-dimensional subspace in the hyperplane.

Now we can define other distinguished subspaces:

Definition 2.8. A subspace W is *isotropic* if $W \subset W^\omega$, i.e. if $\omega|_W = 0$.

Notice that a Lagrangian subspace that is isotropic and has the largest dimension possible for an isotropic subspace.

Definition 2.9. A subspace W is *coisotropic* if $W^\omega \subset W$.

Next time we will prove:

Theorem 2.10. *Given any symplectic form ω on $\mathbb{R}^{2n}$, there is a basis with respect to which ω is standard, i.e. $\tilde{e}_1, \ldots \tilde{e}_n, \tilde{f}_1, \ldots \tilde{f}_n$ such that $\omega(\tilde{e}_i, \tilde{f}_i) = -\omega(\tilde{f}_i, \tilde{e}_i) = 1$ and ω evaluates to zero on all other pairs.*

This theorem implies that there is a linear transformation from $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\omega(u, v) = \omega(\Phi^{-1}u, \Phi^{-1}v)$, and hence any two symplectic forms on $\mathbb{R}^{2n}$ are equivalent up to a change of basis.

Corollary 2.11. *Symplectic vector spaces are classified by their dimension.*

2.3 Exercises

Doing exercises #1, and #3 will be helpful for upcoming lectures. Exercise #2 is good for building intuition. Exercises #6 and #7 are just to help you get used to the different types of subspaces.

1. Find vectors u and v in $\mathbb{R}^4$ that are the edges of a parallelogram whose symplectic area $\omega_0(u,v)$ is k times its Euclidean area. What are all possible values of k?

2. What are all the Lagrangian planes in $(\mathbb{R}^4, \omega_0)$ that contain the basis vector e_1? Describe the space of such planes. (In other words, suppose you needed to parameterize the set of planes. How many parameters would you need and what space would you be parameterizing?) Similarly, what is the space of symplectic planes containing e_1?

3. Consider a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then the graph of A in $\mathbb{R}^{2n}$ is the set

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid y = Ax \right\}$$

   where x and y are n-vectors. Under what conditions on A is the graph a Lagrangian subspace? Do this for $n = 2$ and then generalize.

4. Describe the Lagrangian planes in $(\mathbb{R}^4, \omega_0)$ that are not graphs, i.e. that intersect nontrivially with the Lagrangian plane spanned by $f_1, f_2, \ldots, f_n$.

5. Show that the space of Lagrangian planes is $(\mathrm{Sp}(2n) \cap \mathrm{O}(2n))/\mathrm{O}(n)$ where $\mathrm{Sp}(2n)$ is the group of symplectic $2n \times 2n$-matrices and $\mathrm{O}(k)$ is the group of orthogonal $k \times k$-matrices. (Note: If you know about unitary matrices, $\mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{Gl}(n, \mathbb{C})$ where we identify $z_j = x_j + iy_j$ with $x_j\, e_j + y_j f_j$.)

6. Verify that

   (a) the one-dimensional symplectic complement of a hyperplane lies in the hyperplane.

   (b) the dimension of a Lagrangian subspace is the largest possible dimension for an isotropic subspace.

7. Find subspaces of each of the different types in $(\mathbb{R}^{2n}, \omega_0)$. You may want to get specific, letting $n = 2$ or $n = 3$. Why is $n = 1$ less interesting?

3 Lecture 3: Submanifolds – Symplectic or Not

Today, after establishing that all symplectic vector spaces of a given dimension are equivalent up to a change of basis, we will move to submanifolds of $(\mathbb{R}^{2n}, \omega_0)$, the non-linear analog of subspaces. We will focus on symplectic and Lagrangian submanifolds.

3.1 General Symplectic Vector Spaces

See the handwritten notes of Jo Nelson.

3.2 Surfaces in $(\mathbb{R}^4, \omega_0)$

See the handwritten notes of Jo Nelson.

3.3 Exercises

In these exercises (and in the lectures), assume that there is a well defined tangent plane at each point in the graph of a function, i.e. that the function is smooth.

Note that you can now use our theorem that all symplectic forms on $\mathbb{R}^{2n}$ are equivalent up to a change of basis to do Exercise #11 from Lecture 1.

1. (Quick, filling in a detail in the notes.) Verify that the graph $(x, F(x))$ of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is Lagrangian if and only if DF, the Jacobian of F, is symmetric.

2. Consider the graph of a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x_2, y_2) = (f(x_1, y_1), g(x_1, y_1))$ in $\mathbb{R}^4$. Careful! The domain and target are not as they have been ... or will be ...

   (a) Show that if F is a holomorphic map, i.e. if the components of F satisfy the Cauchy Riemann equations
   $$\frac{\partial f}{\partial x_1} = \frac{\partial g}{\partial y_1}, \qquad \frac{\partial f}{\partial y_1} = -\frac{\partial g}{\partial x_1},$$
   then the graph of F is a symplectic submanifold of $(\mathbb{R}^4, \omega_0)$.

   (b) Equip the domain and target copies of $\mathbb{R}^2$ with the standard symplectic structure ω_0. Then we would say that the map F is a *symplectomorphism* from its domain to its image if, given any two vectors u, v at a point (x_1, y_1) in the domain, $\omega_0(u, v) = \omega_0(DFu, DFv)$ where DF is the Jacobian of F. Now put a symplectic structure on $\mathbb{R}^4$ given by
   $$\omega(u, v) = v^T \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix} u$$
   where we have reordered the coordinates as (x_1, y_1, x_2, y_2). Check that this is indeed a symplectic structure (usually denoted as $\omega = \omega_0 \oplus -\omega_0$), and then check that F is a symplectomorphism if and only if the graph of F is a Lagrangian submanifold of $(\mathbb{R}^4, \omega)$.

3. For each of the following surfaces in $(\mathbb{R}^4, \omega_0)$, determine if it is Lagrangian or symplectic, or if neither, then at what points its tangent planes are Lagrangian. If some tangent planes are Lagrangian and some not, represent the torus as a square with opposite edges identified and coordinates (α, β) and draw in the square the points at which the tangent planes are Lagrangian.

   (a) The torus that is the product of the unit circles in each of the symplectic coordinate planes, say parameterized as $(\cos\alpha, \cos\beta, \sin\alpha, \sin\beta)$.

   (b) The torus that is the product of the unit circles in each of the Lagrangian coordinate planes, say parameterized as $(\cos\alpha, \sin\alpha, \cos\beta, \sin\beta)$.

4. (Important for Lecture 4!) Calculate the line integral $\int_C x dy$ where C is the boundary of the triangle in the xy-plane that has two edges given by non-zero, non-parallel vectors $u = \binom{a}{b}$ and $v = \binom{c}{d}$, endowed with the usual counterclockwise orientation. (Without loss of generality, assume $\det(uv) > 0$.) The value of the integral should be the area of the triangle!

5. (A contact geometry problem you may even have done last week, but this phrasing is "differential form free" and emphasizes the symplectic connections.) Consider the unit 3-sphere in $\mathbb{R}^4$ as a level set of the function $H = \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2)$.

   (a) At each point of S^3, consider the tangent space $T_{(x,y)}$, which cannot be a symplectic subspace of $\mathbb{R}^4$ since it is odd dimensional. However, show that $J\nabla H$ is in $T_{(x,y)}$ and that its orthogonal complement $S_{(x,y)}$ inside that tangent space $T_{(x,y)}$ is a symplectic plane. (Yes, I mean orthogonal complement defined using the standard inner product on $\mathbb{R}^4$.)

   (b) Let α be the linear map from $T_{(x,y)}$ to $\mathbb{R}$ defined by $\alpha(v) = \omega_0(v, (x_1, x_2, y_1, y_2)^T)$ for each $v \in T_{(x,y)}$. Check that $\alpha(v) = 0$ for any $v \in S_{(x,y)}$ and $\alpha(\lambda J \nabla H) = 1$ for some $\lambda \neq 0$. Note that α is the 1-form $x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$ and that you have shown that the contact planes are the $S_{(x,y)}$ at each $(x, y) \in S^3$ and the Reeb vector field is $\lambda J_0 \nabla H$ for some $\lambda \neq 0$.

6. (For fun if you want another example.) Consider the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ inside $\mathbb{R}^3 \subset \mathbb{C}^3 = \mathbb{R}^6$. Then using complex coordinates, the set

$$\{(e^{\pi i t}x_1, e^{\pi i t}x_2, e^{\pi i t}x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1, t \in [0, 1]\}$$

   defines a 3-dimensional manifold in $\mathbb{C}^3$. Check that it is Lagrangian with respect to the standard symplectic form on $\mathbb{R}^6 = \mathbb{C}^3$, i.e. check that the tangent plane at each point is a Lagrangian subspace of $(\mathbb{R}^6, \omega_0)$. This manifold is the three-dimensional analog of a Klein bottle; can you see why? This example is due to Lalonde (J. Differential Geom. 36 (1992), no. 3, 747–764); it generalizes to any $\mathbb{R}^{2n}$ with n odd. Nemirovski proved that no such Lagrangian Klein bottle can be found in an $(\mathbb{R}^{2n}, \omega_0)$ if n is even. (Nemirovski, Stefan, *Lagrangian Klein bottles in R^{2n}*. Geom. Funct. Anal. 19 (2009), no. 3, 902909. http://arxiv.org/pdf/0712.1760v3.pdf)

7. (If you know about forms.) The cotangent bundle of $\mathbb{R}^n$ is $\mathbb{R}^{2n}$ and a 1-form $\alpha = \sum_{i=1}^n f_i(x) dx_i$ where $x = (x_1, x_2, ..., x_n)$ is closed, i.e. $d\alpha = 0$, if and only if the graph of $F = (f_1, f_2, \ldots f_n)$ is Lagrangian.

4 Lecture 4: Constructing Symplectic Manifolds

See the handwritten notes of Jo.

4.1 Exercises

1. (A corrected version of what had been #2 from Lecture 2. Useful for Lecture 5.) How does $|\omega_0(u,v)|$ compare in general with the product $|u||v|$? Note that it suffices to work in the standard symplectic plane $(\mathbb{R}^2, \omega_0)$. Hint: What is the value of $\omega_0(J_0\, u, u)$?

2. (Important for Lecture 5.) Show that for any invertible matrix A, $\sup_{|u|=1}\langle Au, v\rangle = |A^T v|$.

3. (Area preserving map from a sphere to a cylinder.) Consider the unit sphere given by

$$\{(r\cos\theta, r\sin\theta, z)\,|\, r^2 + z^2 = 1\}$$

and the open vertical cylinder that circumscribes it,

$$(\cos\theta, \sin\theta, z)\,|\, -1 < z < 1\}.$$

Show that the radial projection of points on the sphere outward from the z axis defines, off of the north pole $(0,0,1)$ and south pole $(0,0,-1)$, an area preserving map to the cylinder. (Archamedes proved that the surface area of a sphere is $2/3$ the surface area of a cylinder that circumscribes it. He was including the "top" and "bottom" disks of the cylinder, which we are excluding. He also proved that the volume of the sphere is $2/3$ the volume of a circumscribing cylinder.)

4. (Stereographic projection.) If you have never worked with stereographic projection, try writing down in coordinates the map ϕ_N for stereographic projection from the north pole of the unit sphere to an equatorial plane.

5 Lecture 5: Non-squeezing and Ellipsoids

See notes of Jo Nelson.

6 References

The *Introduction to symplectic topology* by McDuff and Salamon is THE introduction to the subject. There are three sets of notes on symplectic geometry that I find useful, each with its individual flavor: those by Cannas da Silva, Meinrenken and Gromov. The ones by Cannas da Silva are the easiest to manage with less mathematical background. While I have not used them, I expect that the notes by Jonathan Evans would also provide a good introduction, bridging the levels of the basic and advanced courses given this week. Note that Gromov discusses canonical forms for the matrices representing general bilinear forms with no constraint that they be non-degenerate, symmetric or skew symmetric.

There are several other books and sets of online notes; their absence from this list only reflects my intent to keep this list short.

Concerning proofs that the determinant of a symplectic matrix equals 1:
The most standard way to prove this fact is by considering the symplectic form on $\mathbb{R}^{2n}$ as a differential form and showing that its top wedge power is a volume form. Cannas da Silva outlines this proof for the bilinear form ω_0 using the exterior product on the vector space $\mathbb{R}^{2n}$. The thesis by Feitas contains a geometrically revealing proof (as Proof 2 on p.14). Meinrenken shows that the determinant of a symplectic matrix is 1 by showing that the group of symplectic matrices is connected. Meanwhile, several proofs are gathered in the paper by Mackey and Mackey.

Cannas da Silva, Ana. *Lectures on Symplectic Geometry.* Lecture Notes in Mathematics, 1764. Springer-Verlag, Berlin, 2001. http://www.math.ist.utl.pt/ acannas/Books/lsg.pdf

Evans, Jonathan. A First Course in Symplectic Topology.
http://www.math.ethz.ch/ evansj/sympcourse.htm

Freitas, Pedro Jorge. *On the Action of the Symplectic Group on the Siegel Upper Half Plane.* Thesis, University of Chicago, 1996. http://ptmat.fc.ul.pt/ pedro/thesis.pdf

Mackey, Nilfour; Mackey, D. Steven. "On the Determinant of Symplectic Matrices." Numerical Analysis Report No. 422, Manchester Centre for Computational Mathematics, Manchester, England. February 2003. http://math.nju.edu.cn/ guoxj/notes/detsymp.pdf

Meinrenken, Eckhard. *Symplectic Geometry, Lecture Notes, University of Toront.* Unpublished. http://www.math.toronto.edu/mein/teaching/sympl.pdf

Gromov, Mikhael. *Lectures on Symplectic Geometry.* Compiled by Richard Brown; edited by Robert Miner. Unpublished.

McDuff, Dusa; Salamon, Dietmar. *Introduction to symplectic topology.* Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.