Van Kampen Diagrams and Dehn Function

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Van Kampen diagram

Let $G = \langle S|R \rangle$ be a f.p. group. Suppose that relations are cyclically reduced. Let R^C be a cyclic closure of R.
Let Δ be a finite, connected, oriented, based, labelled, planar graph where each oriented edge is labelled by an element of S. The base point lies on the boundary of the unbounded region of $\mathbb{R}^2 - \Delta$. For each bounded region (face) the boundary dF (of the closure of F) is labelled by a word in R^C. Each label of the edge traversed is given a ± 1 exponent according to whether the direction of traversal coinsides with, or is opposite to, the orientation of the edge. The **boundary word** of the diagram Δ is the word w read on the boundary of the unbounded region of $\mathbb{R}^2 - \Delta$, starting from the base vertex. Then we say that Δ is a **van Kampen diagram** for w over the presentation $\langle S|R \rangle$.

van Kampen's Lemma. For a group Γ with finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$, and a word w, the following are equivalent.

(i) $w = 1$ in Γ,

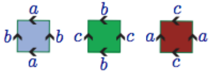
(ii) $w = \prod_{i=1}^{N} {u_i}^{-1} {r_i}^{\varepsilon_i} u_i$ in $F(\mathcal{A})$ for some $r_i \in \mathcal{R}$, $\varepsilon_i = \pm 1$, words u_i,

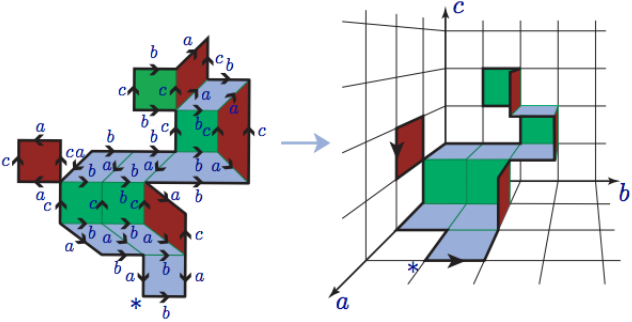
(iii) w admits a van Kampen diagram.

Technicalities hidden!

Moreover, $\mathrm{Area}(w)$ is the minimal N as occurring in (ii).

The diagram can be also viewed as a 2-complex, with a 2-cell attached to the graph for each bounded region. This constructs a combinatorial 2-complex. (Figures are from T. Riley's lectures)

$\mathbb{Z}^3 = \langle\, a, b, c \mid$  $\rangle$

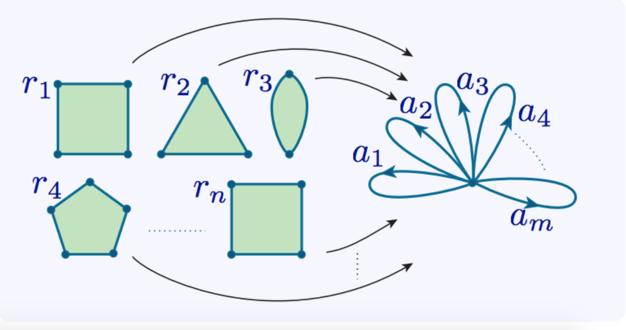


A *van Kampen diagram* for $ba^{-1}ca^{-1}bcb^{-1}ca^{-1}b^{-1}c^{-1}bc^{-1}b^{-2}acac^{-1}a^{-1}c^{-1}aba$.

A *van Kampen diagram* for a word w is a finite planar contractible 2-complex Δ with edges directed and labelled so that around each 2-cell one reads a defining relation and around $\partial\Delta$ one reads w.

Presentation Complex

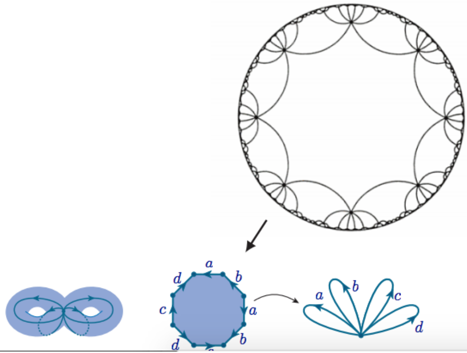
$\mathcal{P} = \langle a_1, \ldots, a_m | r_1, \ldots, r_n \rangle$ a finite presentation of G. The presentation 2-complex of $\mathcal{P}$:

K= 

$\pi_1(K) = G$. The universal cover $\tilde{K}$ is the Cayley 2-complex of $\mathcal{P}$. Its 1-skeleton $\tilde{K}^{(1)} = Cay(G, A)$.

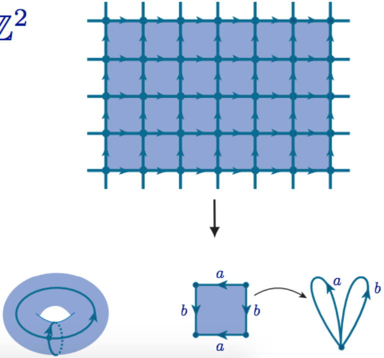
Presentation Complex

$$\langle a, b, c, d \mid a^{-1}b^{-1}ab\,c^{-1}d^{-1}cd \rangle$$



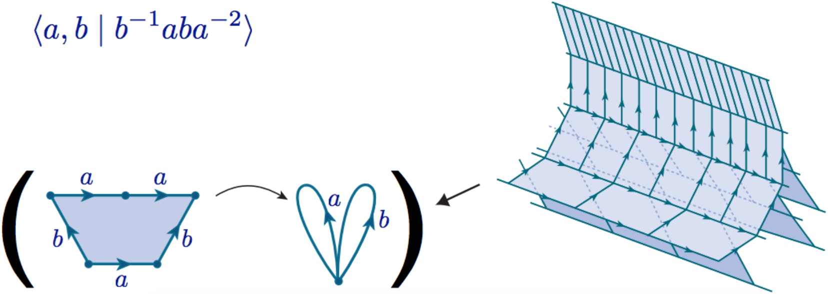
$$\langle a, b \mid a^{-1}b^{-1}ab \rangle$$

$\mathbb{Z}^2$



Presentation Complex

$$\langle a, b \mid b^{-1}aba^{-2} \rangle$$



Dehn Function

For an edge-loop ρ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\mathrm{Area}(\rho)$ is the minimum of $\mathrm{Area}(\Delta)$ over all van Kampen diagrams spanning ρ.

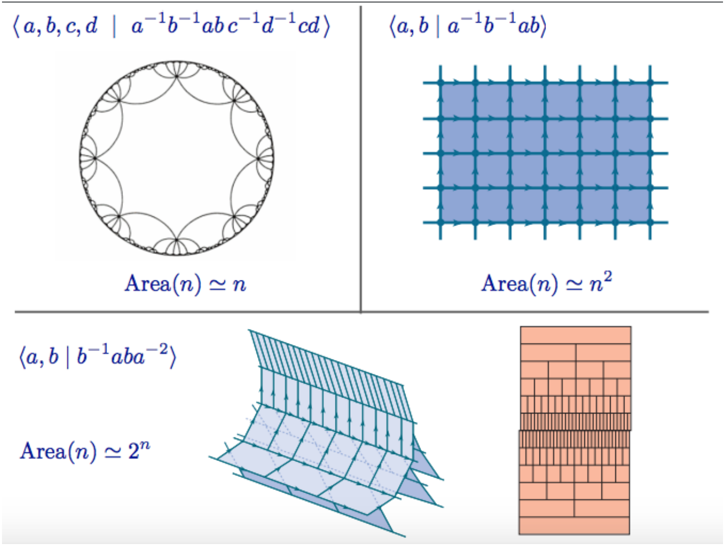
The *Dehn function* $\mathrm{Area}_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ of a finite presentation $\mathcal{P}$ with Cayley 2-complex $\widetilde{K}$ is

$$\mathrm{Area}_{\mathcal{P}}(n) \;=\; \max\{\mathrm{Area}(\rho) \mid \text{edge-loops } \rho \text{ in } \widetilde{K} \text{ with } \ell(\rho) \leq n\}.$$

The Filling Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold M then

$$\mathrm{Area}_{\mathcal{P}} \;\simeq\; \mathrm{Area}_{\widetilde{M}}.$$

Dehn Function



Dehn Function

Examples of Dehn functions

- finite groups: $\leq Cn$
- free groups: $\leq Cn$
- hyperbolic groups := groups with Dehn function $\leq Cn$
- finitely generated abelian groups: $\leq Cn^2$
- 3-dimensional integral Heisenberg group: $\simeq n^3$
- class c free nilpotent group on 2 letters: $\simeq n^{c+1}$

Dehn Functions and the Word Problem

For a finitely presented group, t.f.a.e.:

- the word problem is solvable,
- the Dehn function is recursive,
- the Dehn function is bounded from above by a recursive function.

But groups with large Dehn function may have efficient solutions to their word problem.