A linear map \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a map of the form
\[
L \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} \ldots a_{1n} \\ \vdots \\ a_{m1} \ldots a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
\]
\[a_{ij} \in \mathbb{R}\]

Example:
\[
L : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}.
\]

An affine map \( A : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a map of the form \( A(\overrightarrow{x}) = L(\overrightarrow{x}) + \overrightarrow{y}_0 \), \( \overrightarrow{y}_0 \in \mathbb{R}^n \).

Example:
\[
A : \mathbb{R}^3 \rightarrow \mathbb{R}^2, A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} x + 2y + 3z + 7 \\ 4x + 5y + 6z + 8 \end{pmatrix}.
\]

Maximal Rank. We will need to know what affine maps look like in the “nice case”, i.e. the case where \( L \) has maximal rank. The rank of \( L \) is the number of linearly independent rows (or columns) of the matrix \((a_{ij})\). The maximum possible rank is \( \min(m,n) \).

a Let \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be linear. Then \( L \) has max rank (i.e. \( n \))
\[ \iff L \text{ is injective} \]
\[ \iff L \text{ is surjective} \]
\[ \iff L \text{ has a continuous inverse} \]
\[ \iff \det(a_{ij}) \neq 0. \]

b Let \( L : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k} \) be linear. Then \( L \) has max rank (i.e. \( n \))
\[ \iff L \text{ is injective} \]
\[ \iff \text{The image of } L \text{ is an } n \text{-dimensional plane in } \mathbb{R}^{n+k}. \]

c Let \( L : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n \) be linear. Then \( L \) has max rank (i.e. \( n \))
\[ \iff L \text{ is surjective} \]
\[ \iff \text{the preimage } f^{-1}(\overrightarrow{y}_0) \text{ is a } k \text{-dimensional plane in } \mathbb{R}^{n+k} \text{ if } \overrightarrow{y}_0 \neq 0. \]
HINTS for finding the rank of a matrix:
A square matrix has max rank $\Leftrightarrow$ determinant $\neq 0$.
A matrix with 1 row (or column) has max rank (1) unless every entry is 0.
A matrix with 2 rows (or columns) has max rank (2) unless one row (column) is a multiple of the other.

$^*$ Inverse of a $2 \times 2$ matrix:
$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1} = \begin{pmatrix}
d & -b \\
\frac{-a}{ad-bc} & \frac{1}{ad-bc}
\end{pmatrix}.
$$

Facts about determinants
$Det : \mathbb{R}^n \rightarrow \mathbb{R}$

1. $Det$ is a group homomorphism from the group of invertible $n \times n$ matrices (with matrix multiplication) to the group of non zero real numbers (with multiplication). Thus
$$
Det(A \cdot B) = Det(A) \cdot Det(B)
$$
(Note: the multiplication on the left is matrix multiplication. The multiplication on the right is multiplication of real numbers.) And
$$
Det(I) = 1
$$
where
$$
I = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

2. $Det$ is conjugation invariant, i.e. $Det(A^{-1}BA) = DetB$ (this follows from 1 above; do you see why?)

3. $Det(A) = 0 \Leftrightarrow$ the rows of $A$ are linearly dependent
   $\Leftrightarrow$ the columns of $A$ are linearly dependent
   $\Leftrightarrow$ the associated linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is not injective
   $\Leftrightarrow$ the associated linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is not surjective
   $\Leftrightarrow 0$ is an eigenvalue of $A$

EIGENVALUES
$\lambda$ is an eigenvalue of $A \Leftrightarrow A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \in \mathbb{R}^n, \vec{v} \neq 0$
   $\Leftrightarrow (A - \lambda I)\vec{v} = 0$
   $\Leftrightarrow det(A - \lambda I) = 0$
(What does it mean geometrically if $A\overrightarrow{v} = \lambda\overrightarrow{v}$? What is the significance of the sign?)

Let $A$ be an $n \times n$ matrix.

The characteristic polynomial $P(t) = \det(A - tI)$ is a polynomial of degree $n$. Ex: If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then

$$P(t) = \det \begin{pmatrix} 1-t & 2 \\ 3 & 4-t \end{pmatrix} = t^2 - 5t - 2.$$

FACT: $A$ always satisfies its characteristic polynomial, i.e. $P(A) = 0$. In above example you can check

$$A^2 - 5A - 2 = 0$$

If $P(t)$ has $n$ distinct (real) roots, then $\mathbb{R}^n$ has a basis of (real) eigenvectors for $A$. That is, we can find a basis $\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n$ for $\mathbb{R}^n$ with

$$A\overrightarrow{v}_i = \lambda_i \overrightarrow{v}_i$$

for each $i$, where the eigenvalues $\lambda_i$ are the roots of the characteristic polynomial. In this case $A$ is conjugate to a diagonal matrix, i.e. there is a matrix $B$ (invertible) with $B^{-1}AB = D$ with $D$ diagonal; in fact $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$.

The columns of $B$ are the eigenvectors $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n$.

Special matrices.

$O(n) =$ Orthogonal Group

$= \{ A | A^tA = I \}$

$A$ is orthogonal $\iff A^tA = I \iff$ the rows of $A$ form an orthonormal basis for $\mathbb{R}^n$

$\iff$ the columns of $A$ form an orthonormal basis for $\mathbb{R}^n$.

$\iff$ The linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ preserves the inner product $\langle , \rangle$ on $\mathbb{R}^n$, i.e. $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \langle A\overrightarrow{x}, A\overrightarrow{y} \rangle \forall \overrightarrow{x}, \overrightarrow{y}$

$\iff$ The image of the standard orthonormal basis $\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$ of $\mathbb{R}^n$ is an orthonormal basis for $\mathbb{R}^n$.

Exercise: Show that $O(n)$ is a group, with the operation of matrix multiplication.

$SO(n) =$ Special Orthogonal Group

$= \{ A | A^tA = I \text{ and } \text{Det}A = 1 \}$

$A$ is symmetric if $A^t = A$. 

3
FACT: If $A$ is symmetric, then all eigenvalues are real and $A$ is conjugate by an orthogonal matrix $\Theta$ to a diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$.

That is, $\Theta^{-1} A \Theta = D$.

This means that there is an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{R}^n$ consisting of eigenvectors for $A$. The vectors $v_1$ are the columns of $\Theta$.

This means that the linear map $A$ does just what you would expect a diagonal map $D$ to do.