Beginning Course
Lecture 4

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1 Introduction

On Thursday we talked about the geometric definition of the Thurston-Bennequin number, and today I want to discuss some of the consequences of this. First of all, it's now clear that the Thurston-Bennequin number measures some geometric property of the knot, so this potentially makes it more interesting than if we could attach no meaning to this number. Second –and possibly more important– the theorem allows us to generalize the Thurston-Bennequin invariant to knots in different contact manifolds.

So far, we've considered only the standard contact $\mathbb{R}^3$, but if you have any three-manifold, there's a notion of a particular kind of two-plane field that we call a contact structure. The definition of a Legendrian knot makes sense in any such pair (M^3, ξ), but if we're working in contact manifolds besides $(\mathbb{R}^3, dz - ydx)$, there's generally not a nice notion of a front projection that lets us compute the Thurston-Bennequin number by counting cusps. Nevertheless, as long as the Legendrian knot K admits a Seifert surface, the geometric definition of $\text{tb}(K)$ makes sense.

2 Contact structures

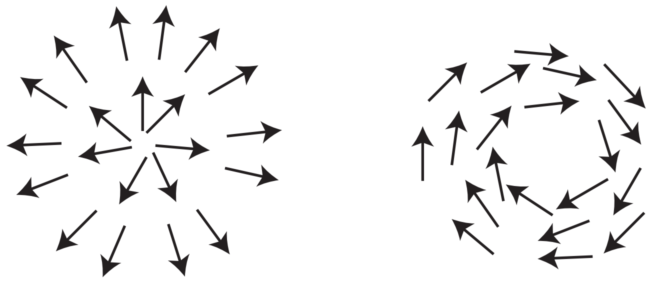
Today we're going to focus on a couple of different contact structures on $\mathbb{R}^3$. Alternatively, we could look at contact structures on different three-manifolds, but these are harder to picture if you've never encountered other three-manifolds before.

Definition 1. A *contact structure* ξ on M^3 is a nowhere integrable 2-plane field.

By "nowhere integrable," I mean that there's no surface in $\mathbb{R}^3$ whose tangent plane field agrees with ξ.

Example 1. You might find it helpful to compare plane fields to vector fields. For example, here are two vector fields in the punctured plane. In both cases, it's possible to find curves whose tangents are described by the given vector field.

These vector fields are integrable.



Recall that we defined the standard contact structure on $\mathbb{R}^3$ to be the plane field spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}$. These planes twist with translation in the y-direction, and stay constant with translation in the x and z directions.

It should seem at least plausible that this plane field is non-integrable, although this is difficult to prove without building up more machinery from differential topology.

Remark 2.1. In the language of differential forms, the plane field ξ_{std} is described as the kernel of the one-form $\alpha = dz - ydx$. Checking that a plane field defined by $\ker \alpha$ is non-integrable is equivalent to checking that $\alpha \wedge d\alpha \neq 0$.

2.1 Other contact structures on $\mathbb{R}^3$

Proving that a plane field is non-integrable is difficult to do without other tools, but we can nevertheless consider some other examples which twist enough to seem plausibly non-integral.

Example 2. Here's an example of a different contact structure on $\mathbb{R}^3$ which we express in (r, θ, z) polar coordinates. The two-plane field ξ_{rad} is spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -r^2 \end{bmatrix}$.

Geometrically, this plane field is

- horizontal at the origin;
- translation-invariant in the z-direction;
- invariant with respect to rotation around the z-axis; and
- along any horizontal ray from the origin, the planes rotate by $\frac{\pi}{2}$ as $r \to \infty$.

Remark 2.2. ξ_{rad} is the kernel of $\alpha_{\text{rad}} = dz + r^2 d\theta$.

Example 3. Let's look at yet another contact structure defined in terms of polar coordinates. This time, let's start with a geometric description:

- xy-plane at origin
- radially symmetric
- the planes twist in the same direction along rays, but more quickly: the plane is vertical at $r = \frac{\pi}{2}$ and then makes a complete rotation in each 2π interval of r.

In coordinates, this plane field is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \cos r \\ -r \sin r \end{bmatrix}$.

Although we're not going to prove that this satisfies the non-integrability condition, it turns out that any plane field that twists "enough" will be a contact structure.

Remark 2.3. The contact structure described above can be realized as the kernel of $\alpha_{\text{OT}} = \cos r dz + r \sin r d\theta$.

2.2 Tight and overtwisted contact structures

We've now seen three different contact structures on $\mathbb{R}^3$, but I claim that two of them are actually closely related. There's a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ that takes ξ_{rad} to ξ_{std}; such a map is called a *contactomorphism*, and in many situations, it's useful to treat contactomorphic contact structures as interchangeable.

Definition 2. If (M, ξ) is a contact manifold, we say that a knot $K \subset M$ is *Legendrian* if it's everywhere tangent to ξ.

A contactomorphism takes substructures of one contact manifold to substructures of the other. More precisely, suppose $\phi : (\mathbb{R}^3, \xi_1) \to (\mathbb{R}^3, \xi_2)$. Then if K is a Legendrian knot in $(\mathbb{R}^3, \xi_1)$, its image will be a Legendrian knot in $(\mathbb{R}^3, \xi_2)$.

The real payoff for the geometric definition of the Thurston-Bennequin number is that it makes sense for any Legendrian knot in any contact structure on $\mathbb{R}^3$: given a Seifert Σ surface for K, the Thurston-Bennequin number of K is the signed intersection number between Σ and a Legendrian push-off K'.

Since a contactomorphism is a homeomorphism, the intersection number between K' and a Seifert surface for K will be preserved. In other words, a contactomorphism sends a Legendrian knot in the original manifold to a Legendrian knot with the same Thurston-Bennequin number in the target contact manifold.

We can use this observation to prove that ξ_{OT} is *not* contactormorphic to ξ_{std}.

Definition 3. A contact structure is *overtwisted* if it admits a Legendrian unknot U with $\text{tb}(U) = 0$. A contact structure is called *tight* if it's not overtwisted. ◇

You were asked in the exercises to explore what integers can be realized as the Thurston-Bennequin number of a Legendrian unknot.

Theorem 1. Given any unknot U in $(\mathbb{R}^3, \xi_{\text{std}}$, tb$(U) \leq -1$.

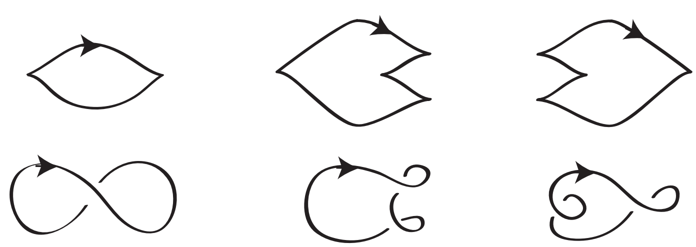
Since ξ_{std} and ξ_{rad} are contactomorphic, the same holds true for the latter. Therefore, these are examples of tight contact structures.

In contrast, I claim we can explicitly find an unknot in $(\mathbb{R}^3, \xi_{\text{OT}}$ with tb$(U) = 0$: take U to be the circle in the xy-plane with radius π. Since the contact planes are all horizontal along this circle, U is Legendrian. Note that U has an obvious Seifert surface: the disc in the $z = 0$ plane bounded by U. Furthermore, we can construct a Legendrian push-off of U by simply increasing the r coordinate of each point of U. The push-off is disjoint from the horizontal disc bounded by U, so the Thurston-Bennequin number is 0. This proves that ξ_{OT} is overtwisted. (Alternatively, we can find push off in the z direction which is transverse to ξ_{OT} and again has intersection number 0 with the horizontal disc bounded by U.)

As it turns out, it's usually much more interesting to study tight contact structures than overtwisted ones. This is a bit hard to motivate, but one way to think of a contact structure is as a plane field with "enough" twisting to be non-integrable. An overtwisted contact structure can be viewed as having more than the minimal amount of twisting, and the interesting phenomena arise on the edge between just enough and not enough.

3 Exercises

1. Just as the Thurston-Bennequin number has a geometric interpretation beyond the formula defining it from a front projection, the rotation number also has a geometric meaning. The figure below shows pairs of front and Lagrangian (i.e., xy) projections for some unknots in $(\mathbb{R}^3, \xi_{\text{std}})$..



   Compute the rotation number using the front projection and then use the corresponding Lagrangian projection to conjecture a geometric definition for the rotation number.

   Hint: consider the behavior of the tangent vector as you trace out the Lagrangian projection.

   Remark 3.1. The rotation number also has a definition that doesn't depend on a projection. See the reference suggested below.

2. The contact structure ξ_{std} has the special property that taking a front projection of a Legendrian knot doesn't lose any information. Can you find a surface in $(\mathbb{R}^3, \xi_{\text{rad}})$

that might play a similar role? That is, can you find a surface S with the property that when you project a Legendrian knot to S, you can fully recover the original knot? What about for ξ_{OT}?

3. If you're familiar with differential forms, write down an explicit diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ with the property that $\phi^*(\alpha_{\mathrm{std}}) = \alpha_{\mathrm{rad}}$.

To learn more about Legendrian knots and contact geometry, you'll probably want to use differential forms. If this is language you're comfortable with, I recommend John Etnyre's introductory reference on Legendrian knots, which you can find online at http://arxiv.org/abs/math.SG/0306256.