Beginning Course
Lecture 3

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1 Introduction

Last time we defined the classical invariants of oriented Legendrian knots, the Thurston-Bennequin and rotation numbers. Our definitions were formulae in terms of features of the front projection, and we proved that these formulae deliver the same values when we take front projections corresponding to isotopic knots. However, at this point it's not clear why these particular numbers should give us good information: if we wrote down lots of formulas that could be computed from the front projection, would we expect to get lots of invariants? Another way to present this is to ask if the classical invariants actually mean something.

Question 1. What does the Thurston-Bennequin number measure?

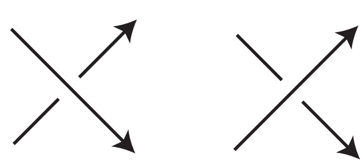
The goal for today is proving the following two statements:

- The Thurston-Bennequin number is a linking number.
- A linking number counts intersections.

2 Linking numbers

Recall the following definition from yesterday's problem set:

Definition 1. The *linking number* of two knots $\mathrm{lk}(K_1, K_2)$ is one-half the sum over the number of positive crossings minus the number of negative crossings.



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You were asked to show that the linking number of a pair of knots is an invariant of the pair, but there's a nice intuitive characterization of what such a number describes. Given two knots in space, we'd like to characterize how entangled they are. If they can be separated by a wall, we say that the link is *split*, whereas the "links" in chain are clearly stuck together.

2.1 The Thurston-Bennequin number as a linking number

Theorem 1. Given a Legendrian knot K, let K' be the parallel copy of K formed by pushing K off itself in a direction transverse to ξ_{std}. Then $\text{tb}(K) = \text{lk}(K, K')$.

(Note that K' inherits an orientation from the orientation of K.)

Heuristically, we want to know how many times K' winds around K.

Proof. At each point on K, we want to pick a vector that's transverse to the contact plane. For convenience, let's choose our vector field to point in the positive z direction. (Clearly, this is transverse to the plane spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}$).

In order to draw a front projection of K', then, we just translate K vertically. The resulting diagram is a front projection for K', and since both components of the new link are Legendrian, the slope still determines the y-coordinate, we can identify the sign of each crossing from the projection.

Each crossing of K in the original diagram results in a pair of crossings between K and K'. Similarly, each cusp of $f(K)$ leads to a negative crossing between K and K'.

We see that

$$\text{lk}(K, K') = \frac{1}{2}[2\#P_{K,K} - 2\#N_{K,K} - \#C_K] = w(K) - \frac{1}{2}\#\text{cusps} = \text{tb}(K).$$

Here the subscript K, K indicates a crossing between K and itself in the original diagram. □

2.2 Orientations and intersection numbers

The preceding theorem gives us a way to interpret the integer we compute for the Thurston-Bennequin number, but the statement that the linking number describes the entanglement of two knots is still rather vague. With the rest of today's lecture, we'll develop an extremely concrete geometric interpretation for $\text{lk}(K_1, K_2)$.

Definition 2. A *Seifert surface* for a knot K is an orientable surface whose boundary is K. ◇

Theorem 2. Every knot has (infinitely-many) Seifert surfaces.

An oriented curve induces a transverse orientation on any orientable surface it bounds:

- counter-clockwise orientation is positive;
- clockwise orientation is negative.

The intersection between an oriented curve and an oriented surface is assigned a sign. If the curve C pierces the negative side of the surface S and comes out on the positive side, the intersection is positive. If C pierces the negative side of S and comes out on the negatives side, the intersection is negative.

Example 1. Consider the unknot U in $\mathbb{R}^3$ parameterized by $(\cos t, \sin t, 0$. The z-axis intersects the disc bounded by U positively.

2.3 Linking numbers and intersection

Theorem 3. $\mathrm{lk}(K_1, K_2)$ is the signed intersection number between K_2 and any Seifert surface for K_1.

Example 2. A split unlink has linking number 0, whereas the Hopf link has linking number ± 1, depending on how the components are oriented.

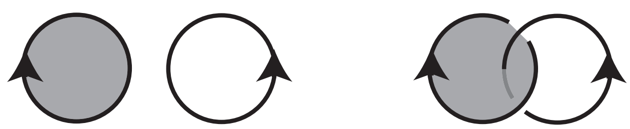


Figure 2.0. In the right-hand figure, $lk(U_1, U_2) = 1$.

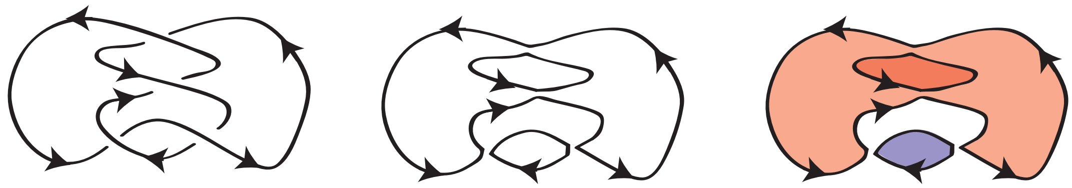
So, if we take any Seifert surface for K, the Thurston-Bennequin number describes how many (signed) times K' intersects that surface. This is extremely concrete!

One way to think about this is the following: the Seifert surface is something associated to the topological type of K; it's not sensitive to the Legendrian condition. However, the Legendrian push-off is defined using this more rigid structure. So in some sense, $tb(K)$ compares the Legendrian type of K to the underlying topological knot. Thinking about this for unknots may be illuminating.

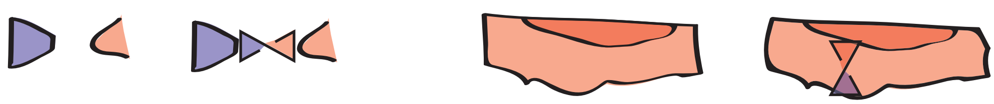
Proof. In order to prove the theorem, we need a way to build an oriented surface bounded by any given knot. We'll start with a projection of the knot K_1 and use this projection to construct our surface $\Sigma(K_2)$. This is the hard part: once we understand our surface in terms of the projection, it won't be hard to see that the signed crossings counted by the formula for linking number correspond to signed intersections between K_2 and $\Sigma(K_1)$.

There are three steps to Seifert's Algorithm:

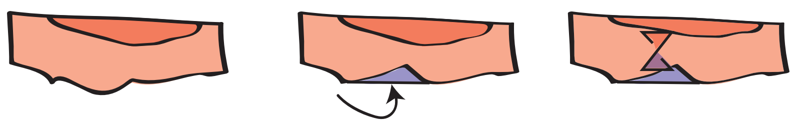
1. Use the orientation of K_1 to resolve each crossing. This replaces the knot projection with a collection of disjoint circles, called *cycles*.



2. Each cycle bounds a disc whose transverse orientation comes from the orientation of the boundary.
3. Reattach the discs together using a twisted band at each point where a crossing was resolved. If your resolution leads to two concentric cycles with the same orientation,



   Step 2 will result in two discs stacked on top of each other, as shown on the right. It's a bit harder to visualize the twisted band in this case, so it might be helpful to think about folding up an edge of the bottom disc first.



To complete the proof, show that crossings between K_1 and K_2 correspond to intersections between K_2 and the Seifert surface you construct for K_1.

□

3 Exercises

1. Draw a few knot projections (or use some of the fronts from yesterday's exercises) and use Seifert's Algorithm to construct a Seifert surface for each one.
2. You've shown that tb(K) is not affected by the choice of orientation of K. Reconcile that observation with the two new perspectives on the Thurston-Bennequin number:
   (a) tb(K) as a linking number between K and its push-off;
   (b) tb(K) as a signed count of intersections between K' and any Seifert surface for K.
3. For any two knots K_1 and K_2, prove that the signed intersection number between K_1 and any Siefert surface for K_2 is equal to the signed intersection number between K_2 and any Siefert surface for K_1.

4. Why does Seifert's Algorithm always yield an orientable surface?

5. For any n, construct a two-component link with the property that $lk(K_1, K_2) = 0$, but every Seifert surface for K_1 will be intersected $2n$ times by K_2. Can you make both components of your example Legendrian?

6. This question provides another perspective on the Thurston-Bennequin number.

   Definition 3. A *framing* of a knot K is a choice of a parallel copy of K. More precisely, it's a section of the unit normal bundle of K. $\diamond$

   We define framings up to isotopy in the complement of K. That is, you can deform a framing F as long as this deformation doesn't pass F through K.

   (a) Given any two framings of an oriented knot, we can compare them using a single integer. Why? What does this number mean?

   Hint: Try drawing a couple of different framings for the unknot.

   (b) If K has a Seifert surface S, we get a new push-off K formed by sliding K into the interior of S. This is called the *Seifert framing*. What is the linking number between K and its Seifert framing?

   (c) When K is a Legendrian knot, the push-off K' that we defined above is called the *contact framing*.

   Show that the Thurston-Bennequin number of a Legendrian knot K is the difference between the Seifert and contact framings.

7. Complete the proof that $\text{lk}(K_1, K_2)$ is the signed intersection number between K_2 and a Seifert surface for K_1.

8. In order to compute $\text{tb}(K)$, we defined K' as a push-off of K along a vector field transverse to the contact planes. Now consider K'', the push-off of K along a vector field that lies in the contact planes. Show that $\text{lk}(K, K') = \text{lk}(K, K'')$.

   Remark 3.1. K'' is called a *Legendrian push-off* of K.