1 Lecture 2

1.1 Knot Invariants

Let's start by recalling the definition of a Legendrian knot.

Definition 1. A smooth embedding $K : S^1 \to \mathbb{R}^3$ is *Legendrian* with respect to the standard contact structure if the tangent to K at the point (x, y, z) is parallel to the plane spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}$.

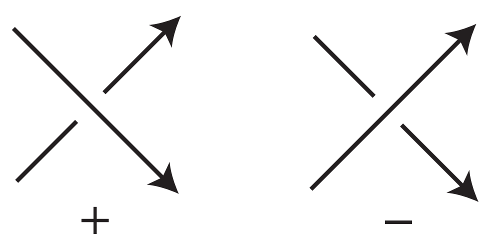
Yesterday we stated that two Legendrian knots are Legendrian isotopic if and only if their front projections (i.e., projections to the xz-plane) can be related by a sequence of Legendrian Reidemeister moves.

If you're given two front projections and you can find a connecting sequence, then you know the corresponding knots are Legendrian isotopic. But what if you can't find such a sequence?

The solution to this dilemma lies in the notion of a *knot invariant*, an object associated (somehow!) to a knot. The value of a knot invariant is that it allows you to distinguish knots which are not Legendrian isotopic.

1.2 The Thurston-Bennequin Number

Orient K in order to identify each crossing of the front projection as either *positive* or *negative*.



Definition 2. The *writhe* of a knot diagram D, $w(D)$, is number of positive crossings of the diagram minus the number of negative crossings of the diagram.

Definition 3. The Thurston-Bennequin number of a front diagram D is the integer

$$\mathrm{tb}(D) = w(D) - \frac{1}{2}(\text{number of cusps}).$$

Proposition 1.1. Every front diagram for the oriented Legendrian knot K has the same Thurston-Bennequin number.

The proposition implies that the Thurston-Bennequin number is an invariant of the Legendrian knot.

Proof. Recall yesterday's theorem, which stated that all front projetions of K are related by Legendrian Reideister moves. It suffices to show that no Reidemeister move changes the Thurston-Bennequin number of the diagram.

- Reidemeister I creates a new positive crossing, but it also adds two new cusps.
- Reidemeister II creates a positive and a negative crossing simultaneously and preserves the number of cusps.
- Reidemeister III preserves the number of crossings of each type and preserves the number of cusps.

□

So, this allows us to distinguish some Legendrian knots that we have no hope of connecting via a sequence of Reidemeister moves.

Example 1. Yesterday, you were asked whether various front projections could correspond to Legendrian isotopic knots. Four of the diagrams you were presented with corresponded to topological unknots:

However, we can easily compute that for two diagrams, $tb = -2$, whereas for the other two diagrams, $tb = -4$. It follows that there's no Legendrian isotopy between the knots corresponding to, say, the first and second diagrams.

Once you've defined an invariant, you can ask whether it's a good invariant: can it tell apart knots that you couldn't previously tell apart? (Also, you might want it to be easy to compute.)

Since the Thurston-Bennequin number is our first invariant, it's not hard for it to give us new information. We'll come back to this invariant tomorrow to get a more geometric perspective on what it's measuring.

1.3 The rotation number

Let's finish up by looking at an invariant of oriented Legendrian knots, the rotation number. As with the Thurston-Bennequin number, there's both a formula for computing rot(K) from a front projection and a geometric interpretation of what the number means.

Definition 4. The *rotation number* of an oriented front projection is defined as $\frac{1}{2}(D - U)$, where D and U represent the number of cusps oriented down and up, respectively.

Proposition 1.2. Rotation number is an invariant of oriented Legendrian knots.

Proof. This is even easier than the Thurston-Bennequin number:

- Reidemeister I creates a pair of cusps, one up and one down.
- Reidemeister II and III don't change the number of cusps.

□

1.4 Classification

The Thurston-Bennequin and rotation numbers have been recognized as invariants for a long time, and toegether with the topological type of the Legendrian knot, they are sometimes called the *classical invariants*.

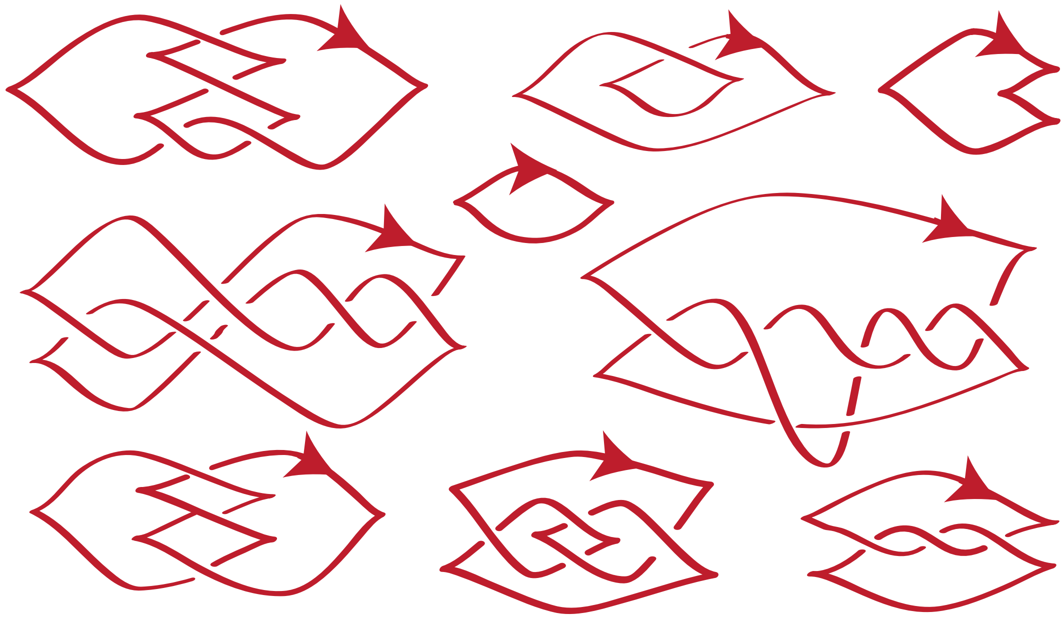
Theorem 1. (Eliashberg-Fraser) Two unknots in $(\mathbb{R}^3, \xi_{std})$ are Legendrian isotopic if and only if they have the same Thurston-Bennequin and rotation numbers.

However, it's not true that the classical invariants will always suffice to classify Legendrian knots!

This was suspected for some time, but it wasn't until 2002 that Y. Chekanov first proved that there are two Legendrian knots with the same classical invariants which are not Legendrian isotopic. There are now a variety of so-called "non-classical" invariants of Legendrian knots, and they're all more algebraically complex (polynomials, vector spaces, homology theories) than the classical invariants.

1.5 Exercises

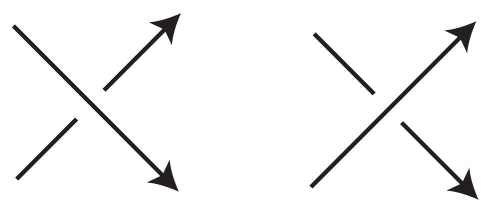
1. Sort the front projections shown below by their classical invariants.



2. How does changing the orientation of K affect $\text{tb}(K)$ and $\text{rot}(K)$?

3. * For what pairs (r, t) can you find a Legendrian unknot U in $(\mathbb{R}^3, \xi_{\text{std}})$ with $\text{rot}(U) = r$ and $\text{tb}(U) = t$? According to Eliashberg's theorem, your answer gives a complete list of all possible Legendrian unknots.

4. * Given a projection of a pair of oriented knots K_1 and K_2, we define their *linking number* $\text{lk}(K_1, K_2)$ as one-half the sum over the number of positive crossings between K_1 and K_2 minus the number of negative crossings between K_1 and K_2. (Crossings between K_i and itself don't contribute anything.)



$+$ $\qquad\qquad$ $-$

Show that $\text{lk}(K_1, K_2)$ is an invariant of the two-component link.

Remark 1.1. Linking number makes sense for a pair of topological (as opposed to necessarily Legendrian) knots. For this problem, you can consider either the case when both knots are Legendrian, or you could use the topological Reidemeister moves from Exercise 6 to study the topological case.

5. Rulings

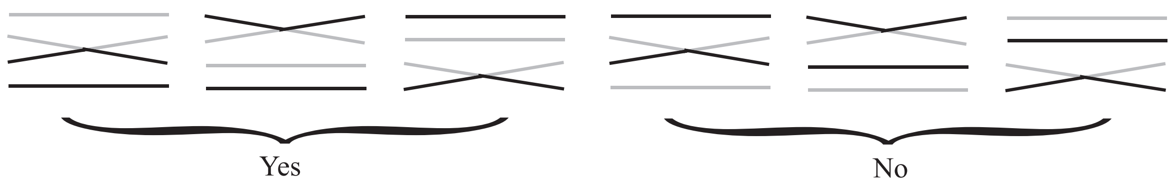
   A *ruling* of a front projection is a 1-to-1 correspondence between left and right cusps together with a pair of paths in the front joining each pair of cusps such that

   - the paths in a pair meets only at their shared endpoints, and
   - any two paths meet only at crossings or cusps.

   (a) Find all possible rulings for a few of the front projections given in Exercise 1.

   (b) Some diagrams don't admit any rulings. What features of a front projection will guarantee that it has no rulings?

   (c) A *switch* in a ruling is a crossing in the original diagram where two paths meet but don't cross. We say that a ruling is *normal* if the switches match the models on the left, rather than on the right, in the figure below. (You may have to isotope your diagram so that the x-coordinates of the crossings are distinct.)



   Which of the rulings you constructed in the first part are normal?

   (d) If $\mathrm{rot}(K) = 0$, we can assign an integer to each switch of a ruling. Starting at one of the two rays leaving the switch with positive slope, trace out the knot until you return to the original crossing for the first time. Glue the ends of your path together and treat the curve you get as a front projection of some new knot. The rotation number of this knot is the *degree* of the switch.

   A normal ruling is *graded* if all its switch crossings have degree 0. Which of the normal rulings you've found are graded?

   (e) To each graded normal ruling R, assign the value

   $$f(R) = \#\text{right cusps} - \#\text{switches}.$$

   Let $g(k) = \{\text{rulings } R : f(R) = k\}$. Then the *graded ruling polynomial* of a front D is

   $$R_D(z) = \sum_k |g(1-k)| z^k.$$

   Compute the ruling polynomial for each of the examples you've considered.

   (f) If $\mathrm{rot}(K) \neq 0$, what could go wrong with the definition in 5d? Can you define a (possibly weaker) versions of the degree of a switch?

Remark 1.2. The ruling polynomial is an example of a non-classical Legendrian knot invariant which successfully distinguishes some pairs of non-isotopic Legendrian knots which have the classical invariants. You can find such a pair in the set of knots given above.

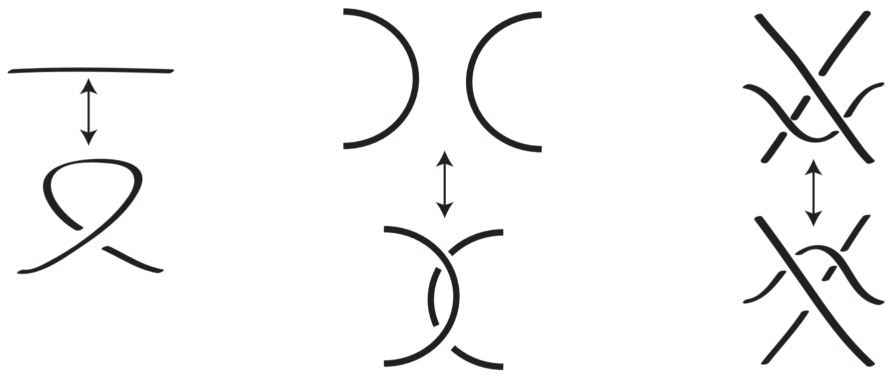
6. Colorability

   Definition 5. A diagram is 3-*colorable* if each arc of the diagram can be labeled either R, B, or G so that all three colors are used at least once, and at each crossing, the three strands either have a single label or all three labels.

   (a) Prove that 3-colorability is a Legendrian knot invariant.

   (b) Does this invariant allow you to distinguish any of the Legendrian knot types you grouped by classical invariants earlier?

   (c) Two knot projections represent topologically isotopic knots if and only if they can be related by a sequence of reflections or rotations of the topological Reidemeister moves:

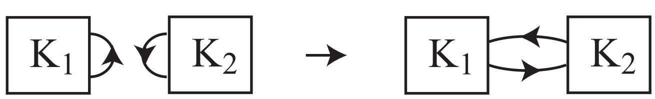
   

   Will colorability distinguish any Legendrian knots with the same underlying topological type?

   (d) Can you generalize this definition to define n-colorability for other values of n? Does your generalization distinguish any knots that you can't distinguish with three-colorability?

7. The *connected sum* of two oriented knots K_1 and K_2 is the knot formed by removing an interval from each knot and splicing the ends together so that the orientations match.

   (a) Prove that when K_1 and K_2 are topological knots, the topological isotopy class of $K_1 \# K_2$ is independent of the intervals chosen.

   (b) The connected sum of a pair of Legendrian knots is also well-defined, although it's harder to prove this than in the topological case. What could the connect sum operated look like in terms of front projections?

   (c) What can you say about the classical invariants of a connected sum?

* Starred question(s) will show up in later lectures.