Beginning Course: Lecture 1

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1 Knots

Definition 1. A *knot* is a smooth embedding $K : S^1 \to \mathbb{R}^3$.

In general, it's more interesting to treat knots as "floppy" objects: we want a definition which models a physical loop of string. We can make this more formal:

Definition 2. Two knots K_0 and K_1 are *isotopic* if there is a homotopy $\kappa : S^1 \times [0,1] \to \mathbb{R}^3$ such that $\kappa \times \{0\} = K_0$, and $\kappa \times \{1\} = K_1$, and $\kappa \times \{t\}$ is an embedding for all t.

Less formally, two knots are isotopic if one can be smoothly deformed into the other.

2 The standard contact structure

Definition 3. The *standard contact structure* ξ_{std} (or just ξ) is the 2-plane field where the plane at (x, y, z)

- has normal vector equal to $\begin{bmatrix} y \\ 0 \\ -1 \end{bmatrix}$,
- or equivalently, is spanned by the pair of vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}$.

What does this look like?

- At the origin, we get a horizontal plane.
- As we change the x and z coordinates, nothing happens, so it's enough to understand what happens if we translate along the y axis.
- As y increases, the plane twists around the y axis. As y decreases, the plane twists in the opposite direction.
- Notice that the plane approaches verticality (i.e., parallel to the yz-plane) as $y \to \pm\infty$, but it won't actually reach it.

3 Legendrian knots

There's a rich field of mathematics devoted to the study of topological knots, but this week we're going to focus on a special class, called *Legendrian knots*. I'll give the precise definition momentarily, but the idea is that we only allow knots which satisfy a special geometric condition: the position of a point on the knot in $\mathbb{R}^3$ determines the tangent to the knot.

Definition 4. The knot K is *Legendrian* with respect to ξ_{std} if at every point on K, the tangent vector to K lies in ξ_{std}.

Two knots are *Legendrian isotopic* if there's an isotopy between them that preserves the property of being Legendrian at every stage.

Here are some links[1] you might find helpful in visualizing Legendrian knots:
http://www.math.duke.edu /∼ ng/knotgallery.html
http://www.youtube.com/watch?v=dNQwHcaJeCI

3.1 Projections

Studying knots in three-dimensional space is hard, so we generally study knots via *projections* to some plane.

If you project a topological knot to a plane, then you lose information about the normal direction. Usually you try to recover this by indicating over- and undercrossings.

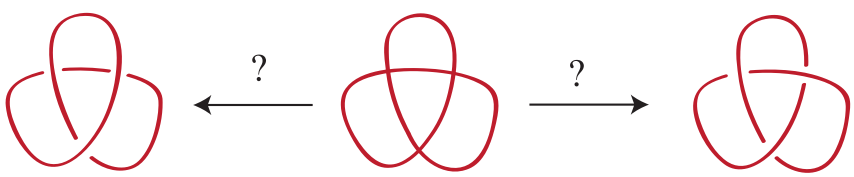


Figure 3.0. Without showing over- and under-crossings, it's not clear if the middle diagram is a projection of an unknot (left) or a trefoil (right).

However, Legendrian knots are more rigid than topological knots, so we don't actually need this extra information at the crossings. We call the projection of K to the xz plane the *front projection* of K. It has several interesting properties:

- If the tangent to the front projection of K at $p = (x_0, 0z_0)$ has $\frac{dz}{dz}$ slope m, then p is the projection of the point (x_0, m, z_0). This is a direct consequence of the Legendrian condition: the projection of the tangent line to (x_0, y_0, z_0) must lie in the projection of the contact plane at (x_0, y_0, z_0).

- The front projection of a Legendrian knot has no vertical tangents.

[1] If you're viewing these online, you might have to retype the address; I had trouble typesetting the URL.

- The front projection of a Legendrian knot is smooth away from finitely-many cusps.

Example 1. Our point of view is from the negative end of the y-axis, looking towards the xz-plane:

Notice that the segment with more negative slope will always pass over the segment with more positive slope.

Theorem 1. Any diagram without vertical tangents which is smooth away from finitely-many left and right cusps is planar isotopic[2] to the front projection of a Legendrian knot. Furthermore, two front projections which are planar isotopic are the projections of Legendrian isotopic knots.

In fact, we can say more!

Proposition 3.1. K_1 and K_2 are Legendrian isotopic if and only if their front projections are related by a sequence of *Legendrian Reidemeister moves*:

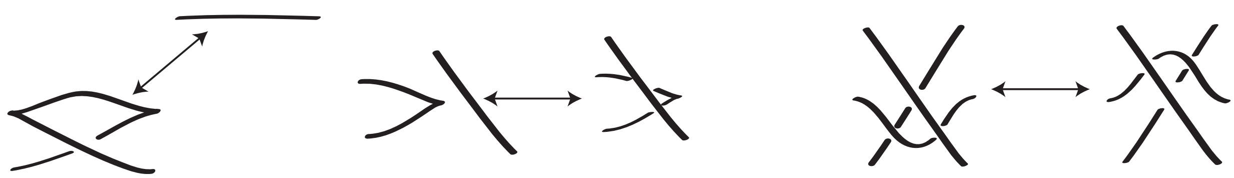
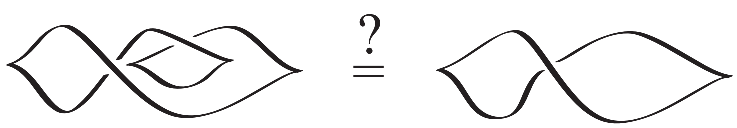


Figure 3.0. These 3 moves, as well as the rotation of each by π around each coordinate axis.

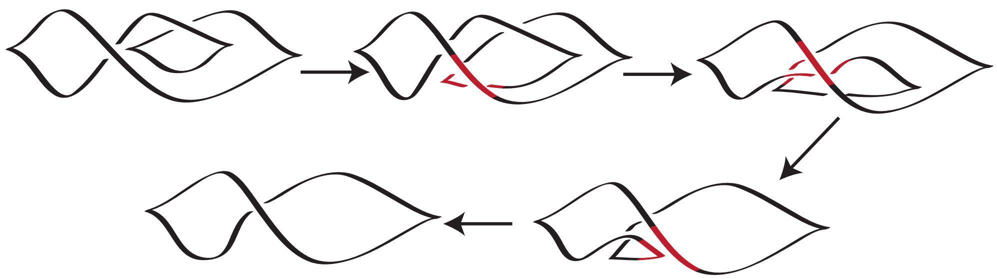
Example 2. Suppose we want to know if the two front projections below come from Legendrian isotopic knots.



We can find a sequence of Legendrian Reidemeister moves which connect the two diagrams:

In fact, there was a simpler sequence that could have been used...

[2]A planar isotopy is a continuous deformation of the planar diagram which preserves cusps, crossings, and the property of having no vertical tangents



4 Exercises

1. Draw the complete set of Legendrian Reidemeister moves.

2. The front projections shown below are arranged in three columns.

   (a) Use Legendrian Reidemeister moves to show that the two front projections in each column correspond to Legendrian isotopic knots.

   (This can be tricky: it's good practice to play around with this kind of exercise to get used to manipulating front projections, but don't spend all your time on it if you get stuck!)

   (b) Now compare the diagrams in different columns. Do you think any of them represent Legendrian isotopic knots?

3. Parameterized Legendrian knots

   Suppose that K is parameterized by some smooth function $\gamma : [0, 2\pi] \to \mathbb{R}^3$, so that $\gamma(t) = \big(x(t), y(t), z(t)\big)$.

   (a) Show that $z'(t) - y(t)x'(t) = 0$ for all values of $t \in (0, 2\pi)$.

   (b) Instead of projecting a Legendrian knot K to the xz-plane, we could project K to the xy-plane. This is called the *Lagrangian projection* of K. Show that for any t_0,

$$z(t_0) = z(0) + \int_0^{t_0} y(t)x'(t)dt.$$

   It follows that K can be recovered, up to an overall translation in the z-direction, from its Lagrangian projection.

(c) What can you say about the expression $\int_0^{2\pi} y(t)x'(t)dt$?

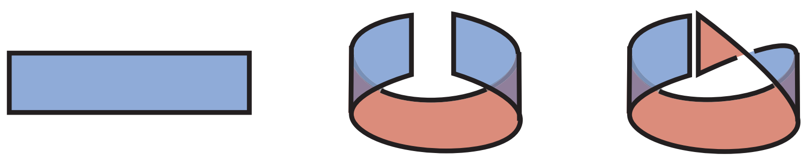
(d) Show that the loop below can't be part of the Lagrangian projection of any Legendrian knot.

Remark 4.1. Front diagrams are easier to work with than Lagrangian diagrams because the latter only satisfy a "weak Reidemeiser theorem":

> There exists a set of Lagrangian Reidemeister moves such that if K_1 and K_2 are Legendrian isotopic, then their Lagrangian projections are related by a sequence of Lagrangian Reidemeister moves.

4. * Orientable and non-orientable surfaces

(a) A *Möbius band* is an example of a non-orientable surface. If you've never seen a Möbius band before, make one out of paper before going on. Take a strip of paper and bring the ends together as if you're about to make a cylinder. Then twist one of the ends by π before attaching it to the other end. (See the diagram below.)



Although the original strip of paper had two sides, the Möbius strip has only one side: any two points can be connected by a continuous curve drawn on the paper.

(b) Instead of rotating one end of the strip by π, consider what happens if you rotate the end by $n\pi$ before gluing. Call the resulting object the n-twisted Möbius strip, M_n. For what values of n is M_n one-sided?

(c) The boundary of M_3 is a circle in $\mathbb{R}^3$ which we can view as a knot. This knot is called the *trefoil*. Can you find an orientable surface whose boundary is the trefoil?

(d) For any odd integer n, the boundary of M_n is a knot. Can you find an orientable surface whose boundary is the same knot?

5. Topological knots

If you don't require a knot to be Legendrian, it still makes sense to consider its projection to the xz-plane. However, now it's necessary to show over- and under-crossings, and the diagram can have vertical tangents. (In fact, the entire projection can be assumed to be smooth.)

(a) Replace each Legendrian Reidemeister move by a topological Reidemeister move. That is, draw a diagram move that comes from a topological isotopy and generalizes the move induced by a Legendrian isotopy.

(b) Look at the pairs of knot diagrams in Exercise 2. Use the topological Reidemeister moves you described in the first part to show that the two diagrams in each pair are projections of topologically isotopic knots. What about diagrams in different columns?

* Starred question(s) will show up in later lectures.