Symplectic Techniques:

The moduli spaces of Holomorphic Maps

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Introduction

Consider (X^{2N}, ω) a (closed) symplectic manifold.

Q: How to gain some information about the symplectic structure on X?
(up to symplectic deformation)?

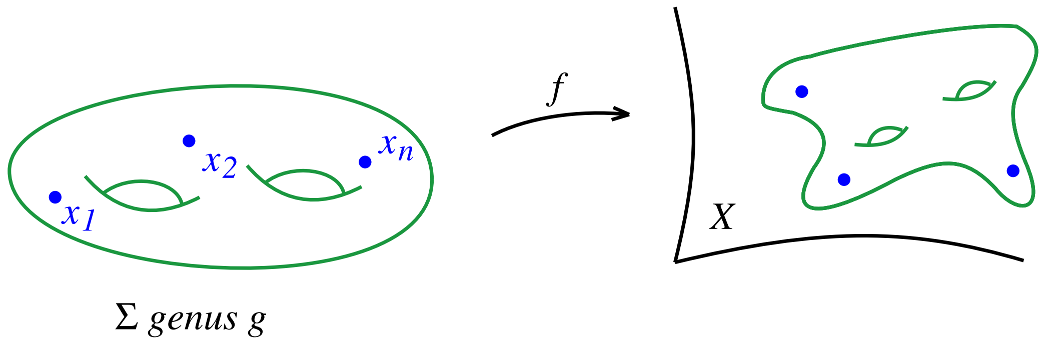
Motivating examples:

- Gromov's nonsqueezing theorem
- Gromov/Taubes Theorem that exotic symplectic $\mathbb{CP}^2$ do not exist.
- Arnold conjectures: existence of periodic orbits, Lagrangian intersections etc

Choose J a (compatible) almost complex structure;

The moduli space of holomorphic maps

$\mathcal{M}_{g,n,A}(X) = \{f : C \to X \mid \overline{\partial}_{jJ} f = 0,\ f_*[C] = A\}/Reparam$



- $C = (\Sigma, j, x_1, \ldots, x_n)$;
- j any complex structure;
- $x_1, \ldots, x_n$ distinct points;

Consider only equivalence classes up to reparametrization;

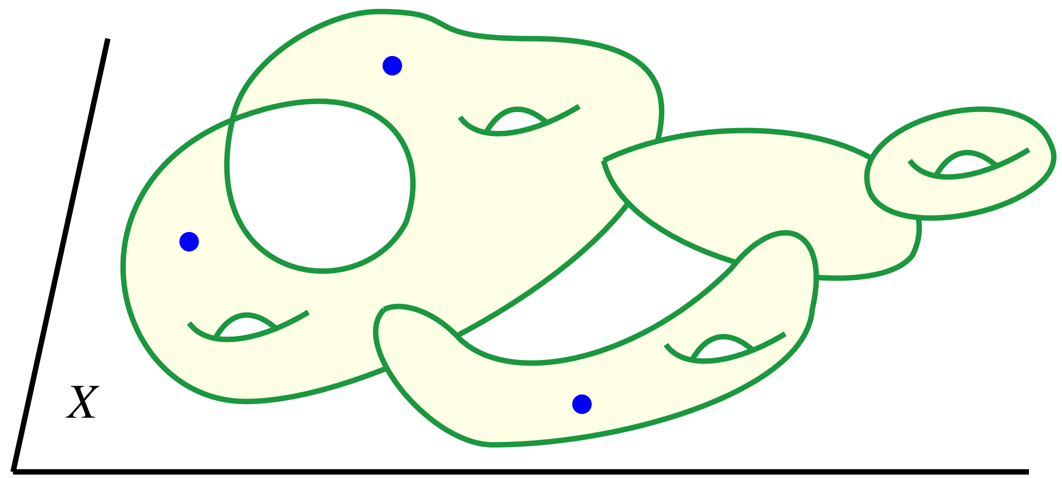
(f stable i.e. Aut$f$ finite)

Note: This is the space of solutions to a nonlinear elliptic PDE.

Generically* $\mathcal{M}(X)$ is a smooth*, oriented, finite dim manifold.

(orbifold if Aut$f \neq 1$; need to perturb the equation, etc)

Compactify: $\overline{\mathcal{M}}_{g,n,A}(X)$ = moduli space of *stable maps* (allow nodal domains)



Generically** $\overline{\mathcal{M}}(X)$ is a smooth*, compact, oriented, finite dim manifold (carries a "virtual fundamental cycle").

One can then extract invariants of the symplectic structure: the Gromov-Witten invariants

Generically, the image of the natural map

$$st \times ev : \overline{\mathcal{M}}_{g,n}(X, A) \to \overline{\mathcal{M}}_{g,n} \times X^n$$

represents a cycle $GW_{X,A,g,n} \in H_\iota(\overline{\mathcal{M}}_{g,n} \times X^n)$ in dimension

$$\iota = 2c_1(X) \cdot A + (\dim_{\mathbb{R}} X - 6)\,(1 - g) + 2n.$$

The GW-cycle is an invariant of the symplectic structure, i.e.

- independent of parameters J and perturbation ν;
- unchanged by a symplectic deformation of ω or symplectomorphism;

can be used to obtain symplecto-topological consequences, e.g:

- distinguish symplectic structures on X, etc.
- $GW \neq 0$ implies (global) existence of a J-holomorphic curve for any J!

Gromov's Nonsqueezing Thm: Assume $\varphi : (B^{2n}(1), \omega_0) \underset{\text{sympl}}{\hookrightarrow} (D^2(R) \times \mathbb{C}^{n-1}, \omega_0)$;

Then $R > 1$.

Key idea: Certain $GW \neq 0$ implies existence of a (closed) holomorphic curve C passing though $\varphi(0)$ and with area πR^2; but then $\varphi^{-1}(C)$ is a proper minimal surface in the ball passing though its center so its area is π;

Gromov/Taubes Thm Assume (X, ω) is a symplectic manifold and X is homeo to $\mathbb{CP}^2$. Then X is the standard $\mathbb{CP}^2$.

Key idea (Gromov): provided the moduli space is nonempty, can use it to show that X is swept by J-holomorphic "lines" just like $\mathbb{CP}^2$ is (and thus diffeo to $\mathbb{CP}^2$); Moser's argument implies X symplectomorphic to $(\mathbb{CP}^2, k\omega_{FS})$;

Taubes: used $SW(X)$ invariants to produce a J-holo curve;

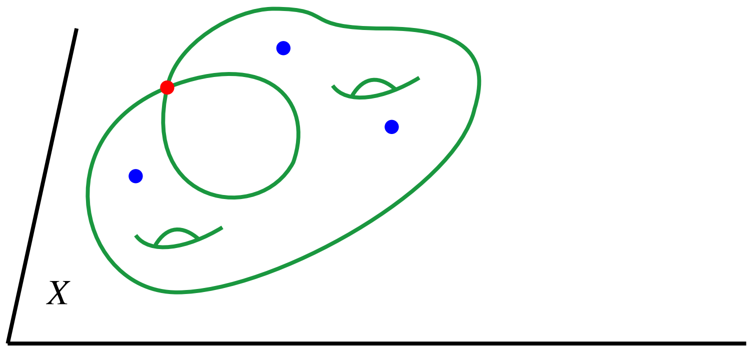
Q: How to compute GW? not easy!

(a) axioms/properties e.g. splitting axiom

(b) calculate for special J's, often degenerate cases (need obstruction bundle);

(c) equivariant localization (works so far only in the alg geom setting!)

Pull back relations from $\mathsf{H}^*\left(\overline{\mathcal{M}}_{g,n}\right)$ (eg splitting axiom):

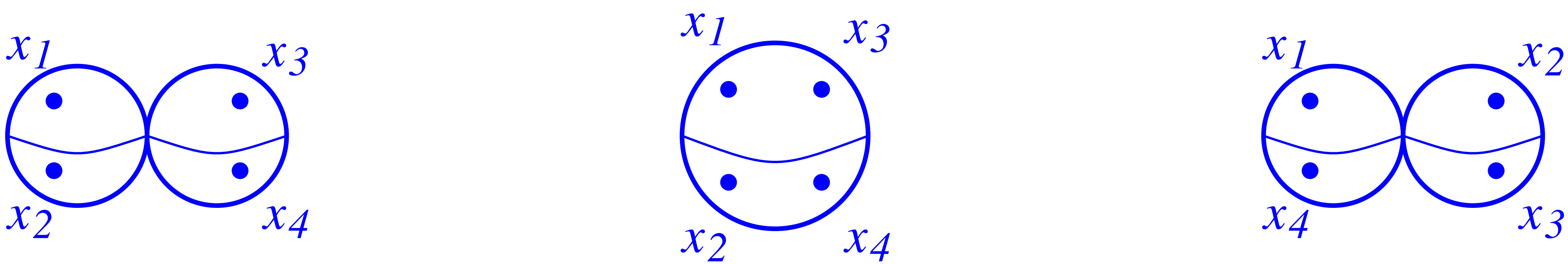


$$[\overline{\mathcal{M}}_{g,n,A}(X)] \cap \mathsf{st}^*\delta = [\mathsf{ev}^{-1}(\Delta)]$$

δ is the boundary divisor; Δ is the diagonal (describing the node);

Pull back relations from $\mathrm{H}^*\left(\overline{\mathcal{M}}_{g,n}\right)$ (eg divisor axiom);

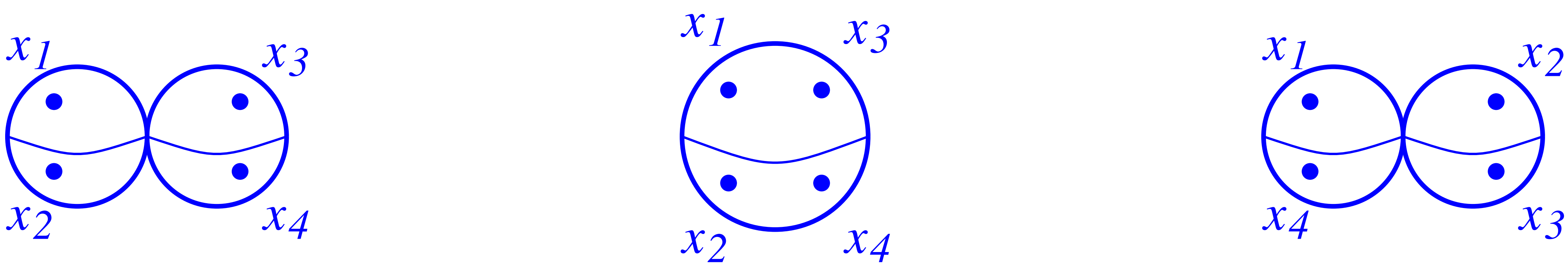
point relation in $\overline{\mathcal{M}}_{0,4}$:



(a) Konsevich's formula for the $g = 0$ GW of $\mathbb{CP}^2$:

$$N_d = \sum_{d_1+d_2=d} \left[\binom{3d-1}{3d_1-1} d_1^2 d_2^2 - \binom{3d-1}{3d_1-2} d_1^3 d_2 \right] N_{d_1} N_{d_2}$$

where $N_d = \#$ genus 0, degree d curves in $\mathbb{CP}^2$ passing through $3d-1$ points.



(b) the (small) quantum product $\alpha \cup_Q \beta$ on $H^{(}X)$:

$$\langle \alpha \cup_Q \beta,\ \gamma \rangle = \sum_{A \in H_2(X)} GW_{A,0}(\alpha, \beta, \gamma) q^A,$$

is associative.