Symplectic Techniques: The moduli spaces of Holomorphic Maps

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Abstract

This is a DRAFT of some informal notes that are meant to accompany the second week of the graduate lectures at the 2012 IAS Women and Mathematics Program. Several topics in these notes will not be covered and some may be discussed only briefly in the lectures. The notes contain some exercises that could serve as the basis of the discussion in the afternoon problem sessions.

Lecture 1: Introduction

Question: Assume (X^{2n}, ω) is symplectic. How can we gain some more information about the symplectic structure on X?

We are interested in symplectic deformation equivalence: allow to deform ω through symplectic forms ω_t (and symplectomorphisms). Simplest deformation: scaling ω: $\omega_t = t\omega$ where $t \in (0, \infty)$. More generally, on product manifolds, scale each factor separately, etc.

Example 1.1 In 2 dim situation is clear: symplectic form= volume form (need orientable); any almost complex J is integrable ($N = 0$) etc.

Easy symplectic deformation (topological) invariants:

- the Chern classes $c_i(TX)$ where J is a compatible almost complex structure.
- obstruction to existence of ω (for closed manifolds):
  - existence of a class $a = [\omega] \in H^2(X, \mathbb{R})$ such that $a^n \neq 0$;
  - in 4 dim: existence of a class $h = [c_1(TX)] \in H^2(X, \mathbb{Z})$ with $h = w_2(M) \mod 2$ and $h^2 = 3\chi(M) + 2\sigma(M)$ (Euler characteristic and signature)

Further invariants: Use various flavors of moduli spaces of (perturbed) holomorphic curves; extract appropriate algebraic invariants; develop axioms/ways to calculate these invariants from the moduli spaces; use these invariants to get various symplecto-topological consequences.
Motivating examples:

- Gromov's nonsquezing theorem
- Gromov/Taubes Theorem that exotic symplectic $\mathbb{CP}^2$ do not exist.

- Arnold conjectures on the existence of periodic orbits, Lagrangian intersections etc

Extract algebraic invariants out of this moduli spaces e.g.: GW invariants, QH, Floer theories (of various flavors), SFT and contact homology, Fukaya categories and A_∞-structures, etc and use these to get further topological consequences e.g.:

- Taubes example: $\#^3\mathbb{CP}^2$ has an almost complex structure, but admits no symplectic structure (has $SW \equiv 0$ while $SW(\pm c_1(TM)) = \pm 1$ for symplectic manifolds).
- exotic symplectic structures: (first examples constructed by McDuff in 8 dim, a zoo of examples can be constructed in 6 dim from fibered knots);
- nonvanishing of the invariant implies existence of holomorphic curve (Gromov's proofs) or of periodic orbit (Weinstein conjecture) etc.

Note: Existence of solutions is the hardest. So far, the only known way: show the invariant is nonzero by calculating it using: (a) the properties or (b) show equal to other invar e.g. SW or Morse theory etc or (c) from a degenerate situation;

Main questions discussed in this lectures:

- set-up/definition of the moduli space of holomorphic curves
- its properties/structure
- extract some symplecto-topological invariants out of it;
- properties/structure of these invariants, including how to calculate them
- brief outline of some applications

Note: The moduli spaces of (perturbed) holomorphic maps have a beautiful and rich structure, not yet fully understood. But at the end of the day, they are only one of the techniques that can can be used to obtain information about the symplectic and contact structures.

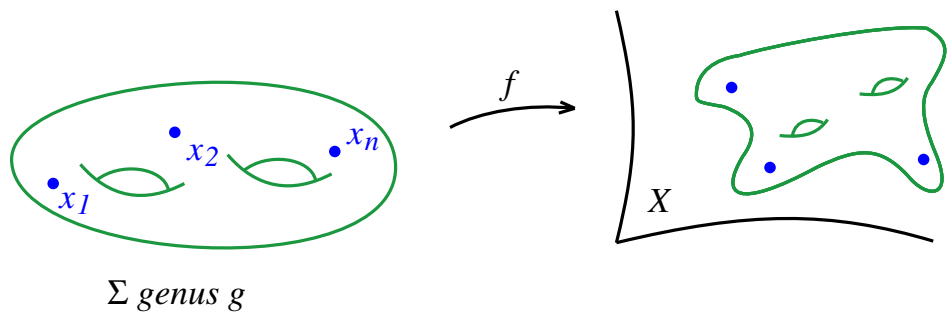
1.1 Moduli space of closed holomorphic curves

Assume (X^{2N}, ω) is a closed symplectic manifold. Choose a compatible (or tamed) almost complex structure J. Consider the moduli space of (smooth) holomorphic maps $f : C \to X$:

$$\mathcal{M}_{g,n,A}(X) = \{f : C \to X \mid \overline{\partial}_{jJ} f = 0,\ f_*[C] = A\}/Reparam \tag{1.1}$$

where

- the domain $C = (\Sigma, j, x_1, \ldots, x_n)$ is any smooth genus g Riemann surface with n distinct marked points and $A \in H_2(X)$;
- $f : (\Sigma, j) \to (X, J)$ is holomorphic, or more generally perturbed holomorphic $\bar{\partial}_{jJ} f = \nu$.



Note: We are considering *equivalence classes* of such maps up to reparametrizations of the domains. (f should be stable, i.e. $\mathrm{Aut}(f)$ is finite).
Note: This is (a family) of spaces of solutions to a nonlinear elliptic PDE;

Exercise 1.2 Show that the space of almost complex structures on X compatible with ω is contractible; discuss how they vary as we vary ω; what happens if J is only tamed? Show that the Chern classes of TX are symplectic deformation invariants.

The moduli spaces fit as the fiber for a fixed parameter $(J, \nu) \in \mathcal{J}(X)$ of the universal moduli space of solutions of the equation:

$$\pi : \mathcal{UM}_{g,n,A}(X) \to \mathcal{J}(X)$$

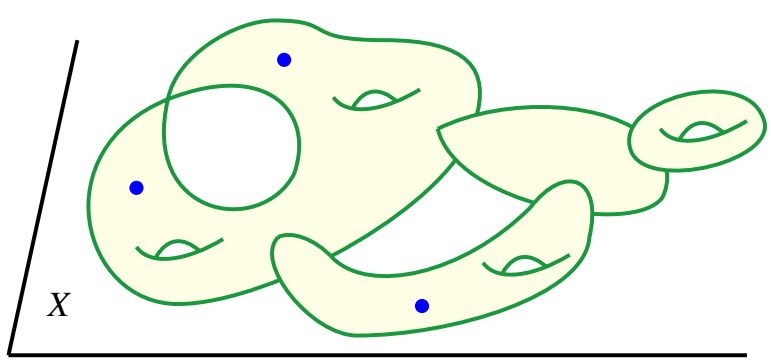
Claim: Generically* $\mathcal{M}_{g,n,A}(X)$ is a smooth*, oriented, finite dim manifold. Why?

- the universal moduli space is cut transversely (because of the variation in the parameters);
- Sard-Smale then implies that for generic parameter $\mathcal{UM}_{g,n,A}(X)$ is a smooth manifold modeled at f on $\mathrm{Ker} L_f$ (and $\mathrm{Coker} L_f = 0$);

Remark 1.3 We will get orbifold singularities at points with $\mathrm{Aut} f \neq 1$; need special care in setting up transversality.

The moduli space is usually not compact, but it has a natural compactification:

$$\overline{\mathcal{M}}_{g,n,A}(X) = \text{moduli space of } \textit{stable maps} \text{ (allow nodal domains)}$$



The moduli space $\overline{\mathcal{M}}_{A,g,n}(X)$ is compact and comes with two natural, continuous maps: the stabilization st that records the (stable model of the) domain and the evaluation ev at the n marked points:

$$\overline{\mathcal{M}}_{g,n} \xleftarrow{\text{st}} \overline{\mathcal{M}}_{A,g,n}(X) \xrightarrow{\text{ev}} X^n \tag{1.2}$$

Hope: Generically** $\overline{\mathcal{M}}_{A,g,n}(X)$ is a smooth*, compact, oriented, finite dim manifold (carries a "virtual fundamental cycle" $[\overline{\mathcal{M}}_{A,g,n}(X)]^{virt}$).

One can then extract invariants of the symplectic structure, the Gromov-Witten invariants. For example:

Theorem 1.4 *For generic perturbation $(J,\nu) \in J(X)$ the image of $\overline{\mathcal{M}}_{A,g,n}(X)$ under* st $\times$ ev *defines a homology class*

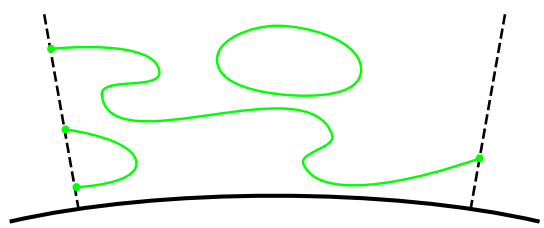
$$GW_{A,g,n}(X) \in H_{\dim}(\overline{\mathcal{M}}_{g,n} \times X^n)$$

in dimension

$$\dim = \dim_{\mathbb{R}} \overline{\mathcal{M}}_{A,g,n}(X) = 2c_1(TX)A + (\dim X - 6)(1-g) + 2n \tag{1.3}$$

which is invariant under the S_n action reordering the n marked points.

The class $GW_{A,g,n}(X)$ is independent of the perturbation ν and is invariant under smooth deformations of the pair (ω, J) through compatible (or tamed) structures; it is called the GW cycle of X.



Pictorial proof of independence of parameters

One can get numbers (the GW invariants) by evaluating this cycle. Pick any cohomology classes $\alpha_i \in H^*(X)$ and $\kappa \in H^*(\overline{\mathcal{M}}_{g,n})$ (of the appropriate total degree) and let

$$\begin{aligned} GW_{A,g,n}(\kappa; \alpha_1, \ldots, \alpha_n) &= \langle \kappa \times \alpha_1 \times \ldots \times \alpha_n,\ GW_{A,g,n}(X) \rangle \\ &= \int_{[\overline{\mathcal{M}}_{A,g,n}(X)]^{vir}} \text{st}^*\kappa \cup \text{ev}^*_{x_1}\alpha_1 \cup \cdots \cup \text{ev}^*_{x_n}\alpha_n \end{aligned}$$

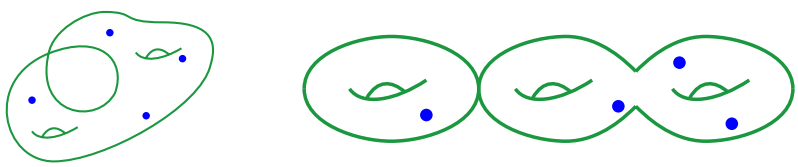
Note: This is in general only a rational number (because of automorphisms).

Geometric interpretation: fix geometric representatives for the Poincare duals (over $\mathbb{Q}$) for each cohomology class: $A_i \subset X$ for α_i and $K \subset \overline{\mathcal{M}}_{g,n}$ for κ. Then $GW_{A,g,n}(\kappa; \alpha_1, \ldots, \alpha_n)$ counts (with appropriate sign!) the number of stable maps $f : C \to X$ such that

$$C \in K \text{ and } f(x_i) \in A_i \text{ for all } i = 1, \ldots, n.$$

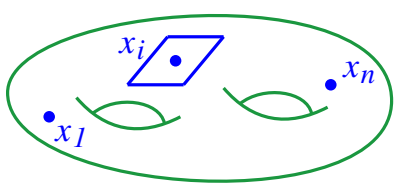
for a generic parameter (J, ν).

Example 1.5 If K is a point in $\overline{\mathcal{M}}_{g,n}$ this corresponds to asking that domain is fixed (up to isomorphism); if K is the pullback of a point in $\overline{\mathcal{M}}_g$ it corresponds to fixing the complex structure on the domain (but not the marked points); we could also take $[\delta] \in H_2(\overline{M})$ the class of one of the boundary divisors:



The irreducible boundary stratum δ_0 and a reducible one in $\overline{\mathcal{M}}_{3,4}$

Example 1.6 Normally one considers only descendent or ancestor classes from $\overline{\mathcal{M}}_{g,n}$: these are defined using $\varphi_i = c_1(\mathcal{L}_i)$ where $\mathcal{L}_i$ is the relative cotangent bundle at the i'th marked point (either over $\overline{\mathcal{M}}_{g,n,A}(X)$ or over $\overline{\mathcal{M}}_{g,n}$; the difference between the two is well understood).



Exercise 1.7 Describe the moduli space of stable holomorphic maps to X in the case X is 2 dim (real); show that these are mostly degree d branch covers of X except that there are some constant components (not cut transversely; why?). Restrict the discussion to branch covers. Show that the dimension of the moduli space is precisely the number of branch points, counted with multiplicity (Riemann-Hurwitz formula). Show that the generic branch cover has simple ramification index; the covers which have a point at which the local model of the cover is $z \to z^k$ (ramification index k) have codimension $k - 1$. For the experts: the branch covers are multiple covers, but are cut transversely in this case (why?). What happens if we fix the complex structure on the domain?

Example 1.8 The situation in X dim is also very nice. For example, the simple J-holomorphic curves are in fact generically immersed and have simple nodes (the number of nodes is topological, given by the adjunction formula). Taubes showed that the embedded count agrees with the SW invariant; this has been extended by Hutchings who defined ECH (embedded contact homology)

for 3 dimensional contact manifolds. Taubes and collaborators proved this agrees with the Seiberg Witten Floer theory and also used it to prove both the Weinstein conjecture (existence of a closed orbit, earlier proved by Hofer in many cases) and also (joint with Hutchings) the Arnold chord conjecture in 3-dim (existence of a Reeb chord).

We next briefly describe two of the applications: Gromov's proof of the Nonsqueezing theorem and Gromov/Taubes theorem about the nonexistence of an exotic $\mathbb{CP}^2$.

Gromov's Nonsqueezing Theorem Assume φ is a symplectic embedding of the closed unit ball B in $(\mathbb{C}^n, \omega_0)$ into the radius R open cylinder $D_R \times \mathbb{C}^{n-1} \subset (\mathbb{C}^n, \omega_0)$. Then $R > 1$.

Key ingredients/steps in the (holo curves) proof:

- the image $\varphi(B)$ lies in a compact region which then can be regarded as a subset of the compact manifold $X = S^2 \times T^{2n-2}$ with the product symplectic structure ω_X, where the symplectic area of the first factor is πR^2 (while the second factor could have arbitrarily large area K). Extend the image of the standard J_0 to an almost complex structure J on the entire X.
- there is at least one J holomorphic map $f : S^2 \to X$ representing $A = [S^2 \times pt]$ passing thorough each point $p \in X$ (since the corresponding $GW = 1$, calculated for the standard product complex structure on X; a priori the domain of f could be nodal, but that is not possible in this case for topological reasons);
- for $p = \varphi(0)$, the inverse image C of such holomorphic curve is a proper J_0 holomorphic and thus minimal surface in the ball; but the smallest area of proper surface in B passing though the center is π; on the other hand, the area of C is at least $\omega_X(A) = \pi R^2$.

Exercise 1.9 Fill in the details of this proof.

Theorem [Gromov/Taubes] Assume (X, ω) is a symplectic manifold such that X is homotopy equivalent to $\mathbb{CP}^2$. Then X is symplectomorphic to $\mathbb{CP}^2$ with a multiple of the standard Fubini-Study form.

Remark 1.10 For topological reasons, X must be homeomorphic to $\mathbb{CP}^2$. The hard part is to prove X must be diffeomorphic to $\mathbb{CP}^2$ (i.e. no exotic symplectic $\mathbb{CP}^2$'s).

Key ingredients/steps in the proof:

- Consider the moduli space of stable J-holomorphic maps $f : S^2 \to X$ representing the positive generator l of $H^2(X)$;
- Taubes theorem (using SW invariants) implies that this moduli space is nonempty and in fact the GW invariant is 1 (cut down by 2 distinct points)
- for topological reasons (including positivity of intersection):
  - for any two distinct points $p, q \in X$ there can be at most one f (and thus precisely one) passing through both points;

- the domain of all the f's are smooth and f is embedded in X (uses compactness of the moduli space);
- any two f intersect in precisely one point;

- the structure on X coming from this 2 complex dimensional family of J-holomorphic curves can be used to construct an explicit diffeomorphism from X to $\mathbb{CP}^2$; Moser's argument then implies the result.

Exercise 1.11 Fill in the details of this proof.

1.2 Properties of Gromov-Witten invariants

(1) **Divisor axiom:** Assume $A \neq 0$ or else $2g-2+n>0$ (stable range). For any $H \in H^2(X)$

$$\pi_*(\mathrm{ev}_{x_1}^* H \cap \pi^*[\overline{\mathcal{M}}_{g,n,A}(X)]) = (H \cap A) \cdot [\overline{\mathcal{M}}_{g,n,A}(X)] \tag{1.4}$$

where $\pi : \overline{\mathcal{M}}_{g,n+1,A}(X) \to \overline{\mathcal{M}}_{g,n,A}(X)$ is the map that forgets the first marked point. Therefore

$$GW_{A,g,n+1}(\pi^*\kappa; H, \alpha) = (H \cap A) \cdot GW_{A,g,n}(\kappa; \alpha)$$

This also extends to $H \in H^i(X)$ with $i \leq 1$ which case the RHS is zero by dimensional reasons.

Remark 1.12 This simply encodes the fact that generically a holomorphic curve representing A intersects H in $A \cdot H$ points counted with sign.

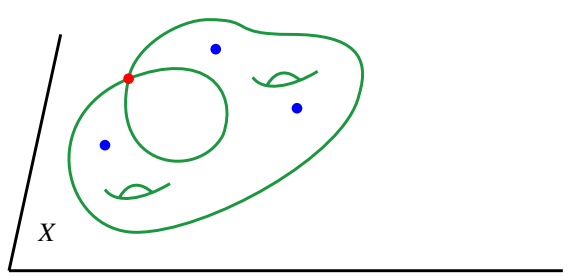
(2) **Splitting axiom:** Consider the moduli space of stable maps whose domain is in the boundary stratum δ. It also has a virtual fundamental cycle which is equal to:

$$[\overline{\mathcal{M}}_{g,n,A}(X)] \cap \mathrm{st}^*\delta = [\mathrm{ev}^{-1}(\Delta)] \tag{1.5}$$

where Δ is the diagonal corresponding to the two extra marked points forming the node. In particular, for the irreducible boundary stratum δ_0:

$$GW_{g,n,A}(\delta_0 \cap \kappa; \alpha) = \sum_i GW_{g-1,n+2,A}(\kappa; \alpha, H^i, H_i)$$

where H^i is a basis of $H^*(X)$, and $H_i \in H^*$ is the dual basis. ("split the diagonal"):



Exercise 1.13 State the corresponding splitting axiom for each reducible boundary stratum: Show that any GW invariant involving the class st^*pt where $pt \in H^{top}(\overline{\mathcal{M}}_g)$ when $g \geq 2$ or $H^{top}(\overline{\mathcal{M}}_{1,1})$ can be expressed entirely in terms of the $g = 0$ GW invariants. Show that any genus zero $n \geq 4$ invariant involving st^*pt where now $pt \in H^{top}(\overline{\mathcal{M}}_{0,n})$ can be expressed in terms of $g = 0$, $n = 3$ GW invariants.

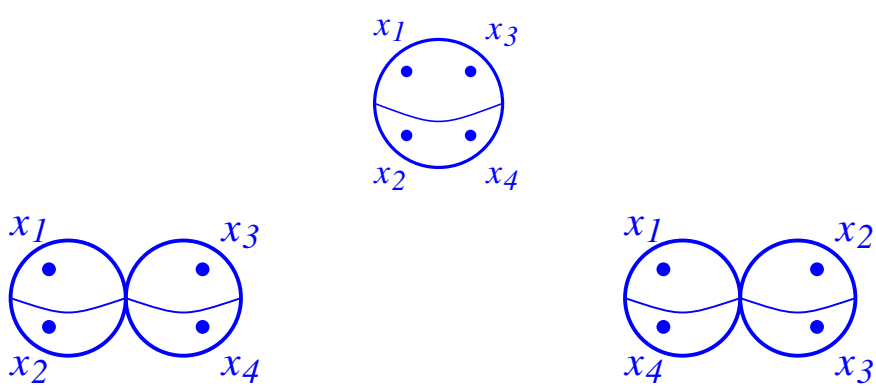
More generally, relations in $H^*(\overline{\mathcal{M}}_{g,n})$ induce relations among GW of all X via:

$$\overline{\mathcal{M}}_{g,n}(X, A) \xrightarrow{\text{st}} \overline{\mathcal{M}}_{g,n}$$

Exercise 1.14 Prove Konsevich's formula for the $g = 0$ GW of $\mathbb{CP}^2$:

$$N_d = \sum_{d_1+d_2=d} \left[\binom{3d-1}{3d_1-1} d_1^2 d_2^2 - \binom{3d-1}{3d_1-2} d_1^3 d_2 \right] N_{d_1} N_{d_2}$$

where N_d is the number of genus 0, degree d curves in $\mathbb{CP}^2$ passing through $3d-1$ points in general position. Hint: Use the point relation in $\overline{\mathcal{M}}_{0,4}$:



Quantum cohomology ring: for any α, $\beta \in H^*(X)$ the (small) quantum product $\alpha \cup_Q \beta$ is:

$$\langle \alpha \cup_Q \beta,\ \gamma \rangle = \sum_{A \in H_2(X)} GW_{A,0}(\alpha, \beta, \gamma) q^A,$$

where q is a formal variable recording the homology class A, with $q^A q^B = q^{A+B}$ for all $A, B \in H_2(X)$; for example could use $q^A = e^{t\omega(A)}$ where t is another formal variable.

Remark 1.15 This is a product on $H^*(X; \Lambda)$, with coefficients in Λ, the Novikov completion of the group ring of $H_2(X)$ i.e. Λ is the ring of formal power series $\sum_A n_A q^A$ such that for each constant C, the sum has only finitely many nonzero terms n_A with $\omega(A) < C$; we can then multiply two such formal power series. Gromov compactness implies we are in this case anyway.

Key properties:

- deformation of the cup product:

$$\alpha \cup_Q \beta = \alpha \cup \beta + \sum_{A \neq 0} c_A q^A$$

where $c_A = \mathrm{ev}_{0*}\mathrm{ev}^*(\alpha \times \beta)$ where $X \times X \xleftarrow{\mathrm{ev}} \overline{\mathcal{M}}_{0,3,A}(X) \xrightarrow{\mathrm{ev}_0} X$ ("pull-push" map)

- associative
- invariant of the symplectic structure

Exercise 1.16 Show that the quantum product is associative. Hint: use the point relation in $\overline{\mathcal{M}}_{0,4}$.

Exercise 1.17 Show that the (small) $QH(\mathbb{CP}^n) = \mathbb{Z}[h,q]/(h^{n+1} = q)$ where h is the generator of $H^2(\mathbb{CP}^n)$.

One can also define the large quantum product (that now involves the full genus zero GW invariants of X) with same key properties by:

$$\langle \alpha \cup_Q \beta,\ \gamma \rangle = \sum_{A \in H_2(X)} \sum_{n=0}^{\infty} \frac{1}{n!} GW_{A,0}(\alpha, \beta, \gamma, \underbrace{h, \ldots, h}_{n \text{ times}})q^A$$

where h is a formal variable on $H^*(X)$. For example, $h = \sum_i H^i t_i$ where H^i is a basis of $H^*(X)$, and t_i are formal variables. Careful about the odd dimensional cohomology – super commuting variables!; divisor axiom shows the variable q is now redundant.

We could also assemble all the Gromov-Witten invariants in (formal) generating functions:

$$F = \sum_{A \in H_2(X)} \sum_{g \geq 0} GW_{A,g}(e^t) q^A \lambda^{2g-2}$$

while the generating function of the disconnected ones is

$$Z = \exp(F) = \sum_{A \in H_2(X)} \sum_{\chi} GW^{\circ}_{A,\chi}(e^t) q^A \lambda^{-\chi}$$

where $GW^{\circ}_{A,\chi}$ is the GW-type invariant counting possibly disconnected curves of Euler characteristic χ and representing $A \in H_2(X)$.

Remark 1.18 These generating functions make sense if we use descendent/ancestors classes from $\overline{\mathcal{M}}_{g,n}$ so we can sum them over n (need a tensor algebra to talk about e^t);

Remark 1.19 Open question: finding a "home" for the virtual GW-cycle. In algebraic geometry it is a class in the Chow group $H_*(\overline{\mathcal{M}}(X))$ where $\overline{\mathcal{M}}(X)$ is the actual moduli space of stable maps for a fixed (typically non-generic) J.

Lecture 2: Brief outline of the analytical set-up

Key ingredients:

- the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ (describes how the domains vary);
- Gromov compactness and the moduli of stable maps $\overline{\mathcal{M}}_{A,g,n}(X)$;
- Fredhom theory (done on each stratum): transversality, index and orientations;
- gluing - describes how the strata fit together (the normal bundle to each stratum);

Upshot: the moduli space of stable maps is stratified by the topological type of the domain, and each nodal stratum can be described in terms of lower dimensional moduli spaces of stable maps.

Standard reference for most of these topics are McDuff and Salamon [MS1], [MS2] books on J-holo curves (except they consider only genus zero there);Arbarello-Cornalba [AC] have a nice survey on the construction of Teichmuller space in higher genus (which include classical references);

2.1 The Deligne-Mumford moduli space of stable curves

A nodal curve is a possibly singular complex algebraic curve C (of algebraic genus g), with finitely many singular points which are all simple double points (nodes) and a collection of n distinct marked points which are all smooth points of C. Such curve is stable if Aut$C$ is finite.

The Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves C (an element of it is an isomorphism class of stable, nodal curves); it is defined in the stable range $2g-2+n \geq 0$. Key properties of $\overline{\mathcal{M}}_{g,n}$:

- $\overline{\mathcal{M}}_{g,n}$ is a smooth, complex projective (global quotient) orbifold of complex dimension $3g-3+n$;
- it has a universal curve $\pi : \overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ whose fiber at $C \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to $C/\mathrm{Aut}C$;
- in fact, the universal curve $\overline{\mathcal{U}}_{g,n}$ can be taken to be $\overline{\mathcal{M}}_{g,n+1}$ where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the extra marked point; the map π is a (singular) fibration, with normal crossing singularities modeled on the equation $zw = \mu$ (or more precisely $(zw)^k = \mu$ at orbifold points);
- $\overline{\mathcal{M}}_{g,n}$ comes with several natural holomorphic bundles:
  - the relative cotangent bundle $\mathcal{L}_{x_i} \to \overline{\mathcal{M}}_{g,n}$ whose fiber at $C = (\Sigma, j, x_1, \ldots x_n)$ is $T^*_{x_i}\Sigma$
  - the dual of the Hodge bundle $E^* \to \overline{\mathcal{M}}_{g,n}$ whose fiber C is $H^1(C, \mathbb{C})$ (the rank g bundle of holomorphic differentials)
- the tangent bundle to $\overline{\mathcal{M}}_{g,n}$ is naturally isomorphic (via the Kodaira-Spencer map) to the rank $3g-3+n$ bundle whose fiber at C is $H^1(C, TC)$.

- $\overline{\mathcal{M}}_{g,n}$ comes with a natural stratification $\mathcal{M}_\Sigma$ by the topological type Σ of the curve C. The top stratum is $\mathcal{M}_{g,n}$ the moduli space of smooth stable curves (and the classifying space of the mapping class group); the boundary strata correspond to curves with $l \geq 1$ nodes and are complex codimension l.

- in fact, the boundary stratum is a normal crossing divisor: its strata are modeled on lower dimensional Deligne-Mumford moduli spaces and with normal bundle coming from the relative cotangent bundles (at the pair of points corresponding to a node).

For example, if we denote by $x_\pm$ the two points corresponding to the node, then the codimension 1 strata of $\overline{\mathcal{M}}_{g,n}$ are modeled by the images of the attaching maps

$$\begin{aligned} \xi &: \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n} \quad \text{and} \\ \xi &: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n} \end{aligned}$$

where $g_1 + g_2 = g$ and $n_1 + n_2 = n$ (and we consider all possible distributions of the original n marked points on the two components). The normal bundle to either one of this strata pulls back to $\mathcal{L}_{x_-} \otimes \mathcal{L}_{x_+}$, and the local model of the fibration $\pi : \overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ along its singular locus pulls back to the natural map

$$\mathcal{L}_{x_-} \oplus \mathcal{L}_{x_+} \to \mathcal{L}_{x_-} \otimes \mathcal{L}_{x_+}$$

Finally, in genus zero, $\mathrm{Aut}C = 1$ for any stable curve, and thus $\overline{\mathcal{M}}_{0,n}$ is smooth and the fiber of the universal curve at $C \in \overline{\mathcal{M}}_{0,n}$ is C. There are several ways to kill the automorphisms groups in higher genus. They all involve decorating C with a finite amount of topological information s such that $\mathrm{Aut}(C, s) = 1$. Furthermore:

- there is a compact, finite dimensional Deligne-Mumford type moduli space $\overline{\mathcal{M}}$ parametrizing all the (decorated) domains (C, s)

- $\overline{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ is a finite branched cover (even a Galois cover);

- there is a smooth, projective universal curve $\pi : \overline{\mathcal{U}} \to \overline{\mathcal{M}}$ whose fiber at $(C, s) \in \overline{\mathcal{M}}$ is isomorphic to C.

So we can always assume that all the stable curves C has trivial automorphisms, at the expense of going to a finite (branch) cover of $\overline{\mathcal{M}}_{g,n}$.

See appendix for more details and exercises for this section.

2.2 Basic properties of holomorphic curves

Fix next an almost complex structure J on X compatible with ω. Fix also a smooth complex (Kahler) curve C (possibly disconnected). This determines a metric both on the domain and also on the target so each map $f : C \to X$ has an energy

$$E(f) = \frac{1}{2} \int_C |df|^2 dvol_C$$

which is conformally invariant in the metric on the domain C, and is topological for jJ-holomorphic maps. In fact:

$$E(f) = \int_C f_*\omega + \frac{1}{2}\int_C |\overline{\partial}_{jJ} f|^2 dvol_C \geq \omega(f_*[C])$$

with equality iff f is jJ holomorphic. This has several consequences:

(a) all holomorphic maps in the same homology class A have the same energy $E(f) = \omega(A)$; furthermore $E(f) > 0$ unless $A = 0$ and f is constant.

(b) holomorphic maps minimize energy and also area.

The holomorphic map equation $\overline{\partial}_{jJ} f = 0$ is a nonlinear equation; the symbol of its linearization D_f is the same as that of a Cauchy-Riemann equation, so its solutions satisfy:

(c) elliptic regularity: as long as the coefficients are smooth, any solution in $W^{k,p}$ is smooth.

(d) unique continuation principle: as long as C is smooth and connected, any two solutions which agree on an open ball $B \subset C$ (or agree to infinite order at a point of C) must agree everywhere on C.

Exercise 2.1 Calculate explicitly the linearization D_f of the jJ-holomorphic map equation (in f) and show that has the form:

$$\begin{aligned} D_f &: \Gamma(f^*TX) \to \Lambda^{0,1}(f^*TX) \\ D_f\xi &= \overline{\partial}\xi + \frac{1}{2}\nabla_\xi J \circ df \circ j = \overline{\partial}\xi + a(\xi) \end{aligned} \tag{2.6}$$

where a is a 0'th order term (careful, the connection ∇ is typically not torsion free!). In fact, the complex linear part of the linearization induces a canonical holomorphic $\overline{\partial}$ operator on f^*TX, while the anticomplex linear part is 0'th order.

Remark 2.2 Properties (c) and (d) apply not only to solutions of the nonlinear equation $\overline{\partial}_J = 0$, but also to elements in the kernel and cokernel of D_f and more generally for the eigenspaces of the self adjoint (semi)-positive operators $D_f D_f^*$ and $D_f^* D_f$. As long as C is smooth, here are some useful consequences of (d):

(1) any two J-holomorphic maps which agree on an open ball $B \subset C$ must agree on its entire connected component of C;

(2) any J holomomorphic map without any constant components has (i) finite automorphism group (ii) $f^{-1}(p)$ is finite for each $p \in X$; (iii) f has finitely many critical points;

(3) positivity of intersections: e.g. if f, g are two nonconstant J-holomorphic maps defined of a disk into a 4-dimensional manifold, their images have positive local intersection multiplicity at each intersection point (unless their images agree to infinite order);

(5) if $D_f^*\eta = 0$ and η vanishes on an open ball $B \subset C$ then $\eta \equiv 0$ the entire component of C containing it.

We prefer to work in Banach spaces: C^k, Holder spaces $C^{k,\alpha}$ or Sobolev spaces $W^{k,p}$. The minimal regularity required is $f \in C^0$ (to define f^*TX) and $J \in C^1$; we will use Sobolev norms $W^{k,p}$ on $\mathcal{M}aps(\Sigma, X)$ (k derivatives in L^p) with $k \geq 1$, $p \geq 1$ and $kp \geq 2$ with $W^{1,2}$ the borderline case. Why?

- $W^{k,p}(\Sigma, X)$ is well defined, independent of charts on the domain;
- $W^{k,p} \hookrightarrow C^0$ (compact) embedding;
- to control the nonlinear terms (in particular D_f is indeed the linearization);

Such solutions f of the jJ-holomorphic map equation are then C^1 by elliptic regularity and the topology on the space of solutions is independent of the particular choice of such Sobolev norms.

With these choices the linearization $D_f : W^{k,p} \to W^{k-1,p}$ is Fredholm and its index $\mathrm{ind}D_f = \mathrm{dimKer}D_f - \mathrm{dimCoker}D_f$ is the same as that of $\overline{\partial}_{f^*TX}$, calculated by either Riemann-Roch or by Atiyah-Singer index theorem:

$$\mathrm{ind}_{\mathbb{R}} D_f = 2c_1(f^*TX) + (1-g)\mathrm{dim}_{\mathbb{R}} X \tag{2.7}$$

A point $x \in C$ is called a simple point of $f : C \to X$ if $df(x) \neq 0$ and $f^{-1}(f(x)) = \{x\}$); a map f is called simple if each component of its domain has a simple point (and thus an entire disk of such points).

Exercise 2.3 Show that a J-holomorphic map $f : C \to X$ (with smooth, but possibly disconnected domain) is either simple, or else it has multiply covered components. On these f is either constant, or else it factors as $g \circ \varphi$ where $\varphi : C \to \Sigma$ is a degree > 1 map and $g : \Sigma \to X$ is simple.

The crucial transversality result in this context is:

Proposition 2.4 *The universal moduli space is cut transversaly at simple maps. More precisely, the linearization of the jJ-holomorphic map equation in both (f, J) is*

$$\mathbf{D}_f(\xi, X) = D_f\xi + \frac{1}{2}X(f) \circ df \circ j$$

and is onto *for any J and any simple map $f : C \to X$.*

Outline of proof: pick an $\eta \in \mathrm{Coker}\mathbf{D}_f \subset \mathrm{Coker}D_f$. Then $\int_C \langle X \circ df \circ j, \eta\rangle = 0$ and so $\eta(x) = 0$ at any simple point x of f; unique continuation implies $\eta = 0$.

Exercise 2.5 Fill in the details of this proof.

Exercise 2.6 Show that when $A = 0$ and $(g, n) = (0, 3)$, the moduli space with is cut transversely and calculate the corresponding $GW(\alpha, \beta, \gamma)$. What happens in genus $g \geq 1$?

Exercise 2.7 Assume $X = \mathbb{CP}^2$ and $A = l$ the class of the line. Calculate by hand GW_A in genus zero; how about when $A = 2l$ or even $3l$?

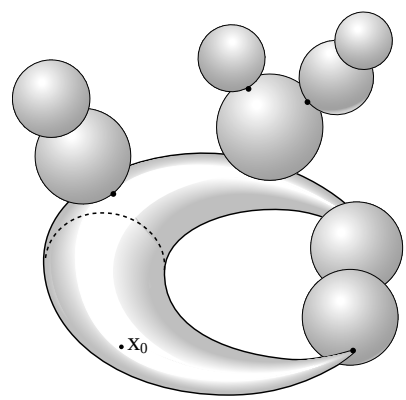
Exercise 2.8 Assume $X = \mathbb{CP}^1 \times T^{2n-2}$ and $A = [S^2 \times pt]$. Calculate $GW_A(pt)$ in genus zero.

2.3 Gromov Compactness

Gromov compactness: even if we were to fix a smooth domain C, the moduli space of jJ-holomorphic maps is typically not compact, as bubbling can occur. The technical reason for that is that bounded energy implies only $f_n \in W^{1,2}$ which is borderline for the Sobolev embedding. However, if a sequence of maps is *uniformly* bounded in $W^{1,p}$ for $p > 2$, then it has a convergent subsequence in C^0 and then we can bootstrap from there.

In general, we will get a priori convergence away from finitely many points, where bubbling could occur (energy concentration); there we need to inductively rescale (reparametrize) the domains to prevent this from happening and get a limit without loss of energy or topological data (like the homology class A).

Definition 2.9 *A* stable J-holomorphic map *is an equivalence class (under reparametrization) of pairs (f, C) where C is a nodal curve and $f : C \to X$ is a J-holomorphic map such that $Aut(f, C)$ is finite. Let $\overline{\mathcal{M}}_{g,n,A}(X)$ denote the* moduli space of stable J-holomorphic maps *whose images represent $A \in H_2(X)$.*



Theorem 2.10 (Gromov Compactness) *Assume $J_n \to J$ in C^∞ and $f_n : C_n \to X$ is a sequence of J_n holomorphic maps with uniformly bounded energy and topology (more precisely g, n and $\omega(f_{n*}[C_n])$ are all bounded). Then after passing to a subsequence and reparametrizing the domains C_n, f_n has a "Gromov-convergent sequence" to a limit $f : C \to X$ which is a stable J-holomorphic map. This means in particular that*

(a) $C_n \to \mathrm{st}(C)$ in $\overline{\mathcal{M}}_{g,n}$ where $\mathrm{st}(C)$ is the stable model of C; furthermore, C is obtained from $\mathrm{st}(C)$ by inserting (unstable domain) bubble trees at finitely many points P of $\mathrm{st}(C)$;

(b) $f_n \circ \varphi_n \to f$ uniformly in C^∞ on compacts away from the nodes of C;

(c) $f_n \circ \varphi_n \to f$ in Hausdorff distance thus $f_*[C] = f_{n*}[C_n]$ (so no energy is lost in the limit).

Gromov compactness provides in particular a topology on $\overline{\mathcal{M}}_{g,n,A}(X)$ such that

$$\mathrm{st} : \overline{\mathcal{M}}_{g,n,A}(X) \to \overline{\mathcal{M}}_{g,n}$$

is continuous. Here is a brief outline of the main steps/ingredients in the proof of Gromov compactness (assuming that C_n are smooth for simplicity):

(1) (compactness of $\overline{\mathcal{M}}_{g,n}$): after passing to a subsequence, C_n converge to a limit C_0 in $\overline{\mathcal{M}}_{g,n}$; we can then regard the maps f_n as defined on the fibers C_n of the universal curve $\overline{\mathcal{U}} \to \overline{\mathcal{M}}$ with the induced metric g_n; in particular, outside the union of the balls $B(x,\varepsilon) \subset \overline{\mathcal{U}}$ centered around the nodes x of C_0 we have a (smooth) identification of the curves C_n and C_0 (with varying metrics g_n but uniformly equivalent in this region); in the region $B(x,\varepsilon)$ the injectivity radius of g_n goes to zero as a geodesic pinches to produce the node x of C_0;

(2) bounded energy implies $f_n \to f_0$ uniformly in C^∞ on compacts away from a (a postiori) finite collection of points $P \cup D$ of C_0 which includes all the nodes D of C_0; here P is the collection of points p of C_0 where the energy density $|df_n|^2 dvol_n$ concentrates in any ball $B(p,\varepsilon)$ (some could be smooth points of C_0, but some could also be nodes). In fact, if C_0 is smooth, the energy densities converge $|df_n|^2 dvol_n \to |df_0|^2 dvol_0 + \sum_{p\in P} e_p \delta_p$ where δ_p are delta functions supported at the points $p \in P$ and $e_p > 0$ is the energy lost at p.

(3) around each blow-up point $p \in P$ we can rescale the metrics g_n in the regions $C_n \cap B(p,\varepsilon)$ (reparametrize the domains) to catch a bubble forming; note that there is a slight difference in the topology/analysis of the rescaling depending whether p is a smooth point of C_0 or a node;

(4) (removable singularity) any bounded energy J-holomorphic map defined on a punctured Riemann surface extends as a smooth, J-holomorphic map across the punctures; this applies in particular to both the limit f_0 from step (2) and also to each bubble from step (3);

(5) (lower energy bound) there is a constant $\alpha_X > 0$ such that any non-constant J-holomorphic map $f : S^2 \to X$ has energy at least α; in particular, this implies that the collection of blow-up points P is finite, and that iterating the rescaling process (3) terminates after finitely many steps with a J-holomorphic limit f defined a priori on a smooth, but disconnected curve $\widetilde{C}$; the convergence to f is now without any loss of energy.

(6) (nodes connect) isoperimetric inequality in the neck regions implies that the limit $f : C \to X$ is in fact continuous at the nodes of C.

Exercise 2.11 Construct (by hand) the stable map limit of the following sequences of genus zero maps to $\mathbb{CP}^2$: (a) $f_n(z) = \left[z^2, z, \frac{1}{n}\right]$; (b) $f_n(z) = \left[z(z-\frac{1}{n}), z, \frac{1}{n}\right]$ (c) $f_n(z) = \left[z^2 - \frac{1}{n^2}, z - \frac{1}{n^2}, \frac{1}{n}\right]$; (d) $f_n(z) = [z, z^n, 1]$.
Hint: Look for the points z where $|df_n(z)| \to \infty$; then make a separate change of coordinates (rescale) around each of these points so that $|df_n|$ becomes bounded (with respect to the new coordinates).

Remark 2.12 There are various statements of Gromov compactness, which then induce a topology on the moduli space of stable J-holomorphic maps or its family version as the parameters J lie in a compact region. For example it can be extended to the case when $J_n \to J$ in C^0. The convergence to the limit can also be slightly improved to gain some control on the behavior around the nodes of C (for example we could get C^0 convergence, but not C^1, etc).

2.4 Analytical set-up of the moduli space of stable maps

In general, for transversality purposes we need to consider the perturbed holomorphic map equation:

$$\overline{\partial}_{jJ} f = \nu \tag{2.8}$$

We will set-up the moduli space $\overline{\mathcal{M}}_{g,n,A}(X)$ of solutions of (2.8) as a *family version* of (J,ν)-holomorphic maps $f : C \to X$ defined on a fixed domain C (as we let the domains vary in a finite dimensional family); the parameters (J,ν) will also vary in an infinite dimensional (but contractible) family $\mathcal{J}(X)$. To simplify the discussion, we will make the following:

Assumption 2.13 *Assume that*

(a) all domains are stable thus $Aut(C)$ finite, and

(b) all domains have been further decorated so that they have non nontrivial automorphisms. This means (as discussed in the appendix)

Remark 2.14 There are many ways to locally stabilize the domains and so locally get oneself in the situation described above; understanding how these local choices patch together to give a global picture is at the heart of the construction of the virtual fundamental cycle in GW theories.

The Assumption 2.13 has the following consequences. Recall that we have a smooth, compact, finite dimensional Deligne-Mumford type moduli space $\overline{\mathcal{M}}$ parametrizing all the (decorated) domains C and which comes with a (smooth, projective) universal curve $\pi : \overline{\mathcal{U}} \to \overline{\mathcal{M}}$ whose fiber at $C \in \overline{\mathcal{M}}$ is isomorphic to C. Fix also a particular holomorphic embedding

$$\overline{\mathcal{U}} \to \mathbb{P}^N \tag{2.9}$$

of the universal curve $\overline{\mathcal{U}}$. This embedding (2.9) gives a canonical choice of a complex structure j on each domain C obtained by restricting the complex structure on $\mathbb{P}^N$ to the fiber C of the universal curve at C. So the embedding (2.9) provides a *global slice* to the action of the reparametrization group on the moduli space of maps (under the simplifying Assumption 2.13). This embedding also simultaneously gives us a simple type of global perturbation ν of the holomorphic map equation

$$\overline{\partial}_{jJ} f(z) = \nu(z, f(z)) \tag{2.10}$$

coming from $\overline{\mathcal{U}} \times X$ as it sits inside $\mathbb{P}^N \times X$. This is in fact equivalent to looking at the graph $F(z) = (z, f(z))$ of f and asking that it is J_ν-holomorphic (as was originally suggested by Gromov). Note that this graph F is always simple (under the Assumption 2.13) and thus the universal moduli space is cut transversely!

Exercise 2.15 Consider an almost complex structure J_ν on $\mathbb{P}^N \times X$ of the form:

$$J_\nu = \begin{pmatrix} j & 0 \\ -\nu \circ j & J \end{pmatrix} \tag{2.11}$$

where $\nu \circ j + J \circ \nu = 0$. Show that f is a solution of (2.10) iff its graph $F(z)$ is J_ν-holomorphic into $\overline{\mathcal{U}} \times X \subset \mathbb{P}^N \times X$.

Remark 2.16 In this context, it is more natural to consider an extended notion of the energy of a map $f : C \to X$ defined as the energy of its graph $F : C \to \overline{\mathcal{U}} \times X$ which takes into account the energy of the domain as well. With this definition, the extended energy is also bounded away from 0 (as constant components must have stable domains).

For each fixed topological type Σ, there is a fiber bundle $\mathcal{M}aps_\Sigma(X) \to \mathcal{M}_\Sigma$ whose fiber at a stable curve $C \in \mathcal{M}_\Sigma$ (i.e. homeomorphic to Σ) is the space $\mathcal{M}aps(C, X)$ of maps from C to X (regarded as maps defined on the fiber of the universal curve over C, and completed in the appropriate Sobolev norms). We also have a vector bundle

$$\begin{array}{c} \Lambda^{0,1} \\ \pi \big\downarrow \big\uparrow \sigma \\ \mathcal{M}aps_\Sigma(X) \times \mathcal{J}(X) \end{array} \tag{2.12}$$

whose fiber at (f, J, ν) is $\Lambda^{0,1}(f^*TX)$; the equation (2.8) defines a section

$$s(f, J, \nu) = \overline{\partial}_J f - \nu$$

of this bundle, whose zero locus describes an (open) stratum $\mathcal{UM}_\Sigma(X)$ of the universal moduli space of solutions:

$$\begin{array}{ccc} \mathcal{M}_\Sigma(X) & \longrightarrow & \mathcal{UM}_\Sigma(X) \\ & & \pi \big\downarrow \\ & & \mathcal{J}(X) \end{array} \tag{2.13}$$

Its fiber over a fixed parameter $(J, \nu) \in \mathcal{J}(X)$ is $\mathcal{M}_\Sigma(X)$ the *moduli space of* (J, ν)*-holomorphic maps* $f : C \to X$ where C is homeomorphic to Σ.

To set up the linearization to this problem, we need

- fix local trivializations of the universal curve over $\mathcal{M}_\Sigma$, which then induce local trivializations in the bundle $\mathcal{M}aps_\Sigma(X) \to \mathcal{M}_\Sigma$;
- add to the linearization the variations in the holomorphic structure of the domain (provided for example by the Kodaira-Spencer theory);
- Proposition 2.4 then implies transversality;

Therefore for generic parameter, the (open) stratum $\mathcal{M}_\Sigma(X)$ of the moduli space $\overline{\mathcal{M}}(X)$ is smooth (under Assumption 2.13) and is locally modeled at the point $f \in \mathcal{M}_\Sigma(X)$ by $\mathrm{Ker} L_f$ while $\mathrm{Coker} L_f = 0$. Therefore it is a finite dimensional manifold of dimension:

$$\dim = \mathrm{ind} L_f = \mathrm{ind} D_f + \dim \overline{\mathcal{M}}_{g,n} = 2c_1(TX)A + (\dim X - 6)(1 - g) + 2n;$$

Remark 2.17 The stratum $\mathcal{M}_\sigma(X)$ is not compact because the topological type of Σ is fixed, but the Gromov compactness describes how the various strata fit together. Gluing (done in compact families) provides the local model normal to each stratum of the universal moduli space $\overline{\mathcal{UM}}(X) \to \mathcal{J}(X)$ and thus a postiori provides a manifold structure on the moduli space $\overline{\mathcal{M}}(X)$ (under Assumption 2.13). Note that the perturbations are "coherent" in this case, so the strata fit together nicely.

Remark 2.18 Another consequence of the simplifying Assumption 2.13 is that all the solutions of the perturbed equation (2.10) will converge *without any further bubbling* to their Gromov limit (as any further bubbling would produce an unstable bubble tree). So a postiori we also have uniform convergence of all the maps on compacts away from the nodes of the limit domain, or more precisely away from the singular locus S of $\overline{\mathcal{U}} \to \overline{\mathcal{M}}$ (traced by the nodes). Technically speaking, for analytical purposes, one should first work on compact subsets of each open strata of the moduli space $\overline{\mathcal{M}}(X)$, where we can hope to have uniform estimates: in particular, as long as we work with a *finite dimensional model* for the space of domains, and we stay away from the strata with more nodes, all metrics on the domains will be uniformly equivalent and the injectivity radius uniformly bounded from below.

Remark 2.19 There are several other ways to set-up the universal moduli space. Fix for simplicity Σ a smooth genus g closed oriented surface, and define

$$\mathcal{UM}(\Sigma, X) \quad = \quad \{(f, j, x_1, \ldots, x_n, J, \nu) \mid f : \Sigma \to X,\ \overline{\partial}_{jJ} f = \nu\}/\mathrm{Diff}^+(\Sigma) \tag{2.14}$$

where an element $\varphi \in \mathrm{Diff}^+(\Sigma)$ acts by reparametrizations:

$$\varphi(f, j, x_i, J, \nu) = (f \circ \varphi^{-1},\ (\varphi^{-1})^* j,\ \varphi(x_i),\ J,\ \nu \circ (\varphi^{-1})^*)$$

Equivalently, we could can fix (Σ, x) a smooth genus g, n-pointed closed oriented surface and define

$$\mathcal{UM}_\Sigma(X) \quad = \quad \{(f, j, J, \nu) \mid f : \Sigma \to X,\ \overline{\partial}_{jJ} f = \nu\}/\mathrm{Diff}^+(\Sigma, x) \tag{2.15}$$

where $\mathrm{Diff}^+(\Sigma, x)$ are those diffeomorphisms that preserve the marked points x (pointwise). Either way, we need to choose a slice to the action of the infinite dimensional group Diff^+ and then take the Sobolev completion of that slice, which is what we have in effect done anyway.

2.4.1 Orientations and the Index bundle

The stratum $\mathcal{M}_\Sigma(X)$ of the moduli space is also canonically orientable (assuming that it is cut transversely). The basic reason is that the linearizations L_f and D_f can both be deformed to a complex linear operator see (2.6), and this comes with a canonical orientation. Intrinsically, for each one of the linearizations L_f or D_f we have an index bundle

$$\mathrm{Index} L \longrightarrow \mathcal{M}aps_\Sigma(X)$$

whose fiber at $f : C \to X$ is intrinsically $\mathrm{Ker} L_f - \mathrm{Coker} L_f$.

Because as a Fredholm operator varies both its kernel and coker can jump (by isomorphic pieces) this Index bundle is a virtual bundle of rank equal to the difference in dimensions (the index). This is best set-up in K-theory (see Atiyah); there is both a version for complex operators (K-theory) and also real operators (KO-theory). In any event, the Index bundle comes with an actual line bundle, the determinant line bundle:

$$\det L \longrightarrow \mathcal{M}aps_{\Sigma}(X)$$

whose fiber at $f : C \to X$ is intrinsically $\Lambda^{top}\mathrm{Ker}L_f \otimes (\Lambda^{top}\mathrm{Coker}L_f)^*$; the w_1 of this bundle is the obstruction to (globally and coherently) orienting the moduli spaces (as it agrees with $w_1(\Lambda^{top}T_f\mathcal{M}_{\Sigma}(X))$ when $\mathcal{M}_{\Sigma}(X)$ is cut transversaly at f). Complex line bundle are orientable and come with a canonical orientation. In our case, both the linearization D_f and L_f can be homotoped to a complex operator (see (2.6)) and thus the determinant bundle of L_f is canonically oriented over $\mathcal{M}aps_{\Sigma}(X)$.

Remark 2.20 In the case of an index zero operator L_f, the moduli space is 0 dimensional and we can associate an actual sign to each f for which $\mathrm{Ker}L_f = \mathrm{Coker}L_f = 0$ (therefore L_f invertible):

$$sign(f) = (-1)^{SF(L_f)}$$

where $SF(L_f)$ is the mod 2 spectral flow of L_f. This counts the number of times (mod 2) a generic path of Fredholm operators from L_f to an invertible complex operator L_0 crosses the stratum of $Fred$ where $\mathrm{Ker}L = \mathbb{R}$; it is independent of the path and of the invertible complex operator used (this is because any path between two complex operators can be homotoped to a path of complex operators and therefore generically avoids the complex codimension one stratum of $Fred_{\mathbb{C}}$ consisting of operators with a 1 complex dimensional kernel).

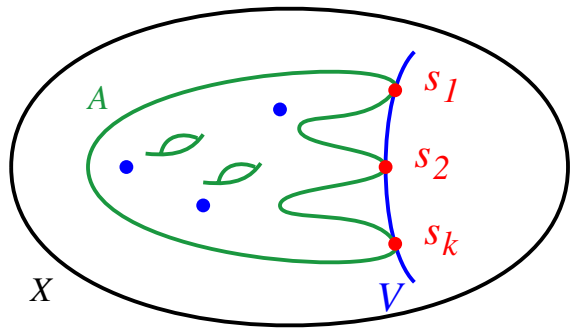
Lecture 3: Other versions of Moduli Spaces

There are many other versions of the moduli spaces of (perturbed) holomorphic curves that are used to obtain other symplectico-topological consequences.

3.1 Relative GW invariants

Assume V is symplectic codimension 2 submanifold of (X, ω) and choose an almost complex structure compatible with both ω and V.

Now we also keep track of how maps behave near V: maps without components in V have well defined multiplicities of intersection with V:



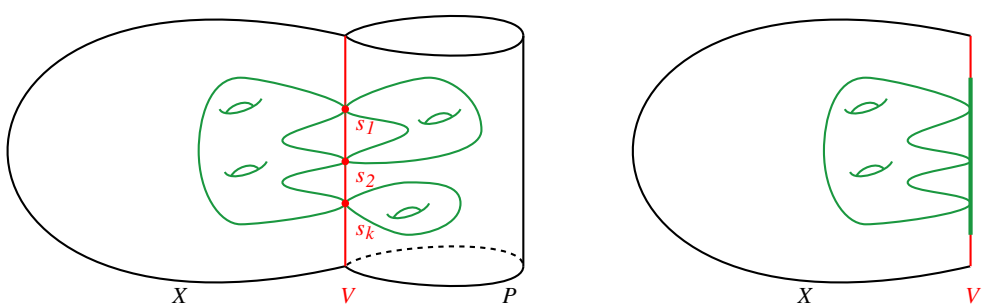
All the points in $f^{-1}(V)$ are now marked and each one is decorated by its intersection multiplicity; they are called contact points to V.

In the stable map compactification $\overline{\mathcal{M}}(X)$, components may "fall into V" (energy concentrates along V) where we loose the contact information to V (and worse dimensions of that boundary stratum could become larger than the expected dimension of the moduli space);

We can construct a refined compactification, by also inductively rescaling the target normal V as well, to prevent this from happening. This way one obtains

$$\overline{\mathcal{M}}_{g,n,A,s}(X, V) = \text{moduli space of } \textit{relatively stable maps}$$

Here we allow not only for the domains to pinch, but also the target may degenerate;



The difference between $\overline{\mathcal{M}}(X, V)$ and $\overline{\mathcal{M}}(X)$

Here $\mathbb{P}_V = \mathbb{P}(N_V \oplus \mathbb{C})$ is the $\mathbb{CP}^1$ bundle over V obtained from the normal bundle N_V of V.

We similarly have two natural, continuous maps

$$\overline{\mathcal{M}}_{g,n+\ell} \xleftarrow{\text{st}} \overline{\mathcal{M}}_{g,n,A,s}(X) \xrightarrow{\text{ev}} X^n \times V^\ell \tag{3.16}$$

where the evaluation map at the extra ℓ contact points x_i maps to V and also keeps track of their contact multiplicity s_i. One can then similarly extract an appropriate relative GW invariant of the pair (X, V). For example:

Theorem 3.1 *For generic perturbation (J, ν) the image of $\overline{\mathcal{M}}_{A,g,n,s}(X, V)$ under* st $\times$ ev *defines a homology class*

$$GW_{A,g,n,s}(X, V) \in H_{\dim}(\overline{\mathcal{M}}_{g,n} \times X^n \times V^\ell)$$

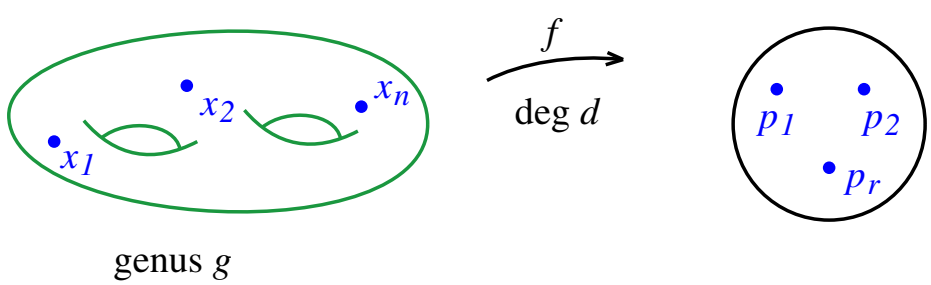
in dimension

$$\dim = \dim_{\mathbb{R}} \overline{\mathcal{M}}_{A,g,n,s}(X, V) = 2c_1(TX)A + (\dim X - 6)(1 - g) + 2n + 2\ell - V \cdot A \tag{3.17}$$

which is invariant under the $S_n \times S_\ell$ action reordering the n ordinary marked points and the ℓ contact points.

The class $GW_{A,g,n,s}(X, V)$ is independent of the perturbation ν and is invariant under smooth deformations of the pair (X, V) and (ω, J) through V-compatible structures; it is called the relative GW cycle of (X, V).

Example 3.2 When $X = \Sigma$ is 2 dimensional (e.g. $\mathbb{CP}^1$) and $V = \{p_1, \ldots, p_r\}$ consists of finitely many points, the relative moduli space $\overline{\mathcal{M}}_s(X, V)$ is the moduli space of (branch) covers of Σ with prescribed ramification pattern s over the points in V.



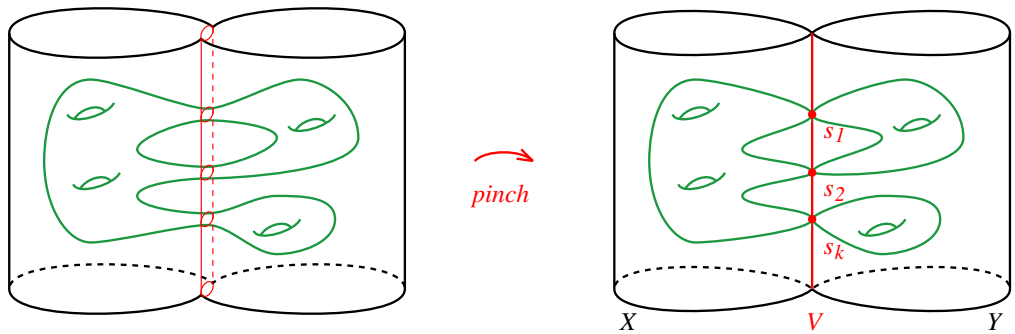
Exercise 3.3 Consider the (relative) moduli space $\overline{\mathcal{X}}$ of degree 2 covers of $\mathbb{CP}^1$ branched at 4 points, together with its two maps natural maps

$$\overline{\mathcal{M}}_{1,4} \xleftarrow{\text{st}} \overline{\mathcal{X}} \xrightarrow{\text{ev}} \overline{\mathcal{M}}_{0,4}$$

Note that now ev records precisely the location of the 4 branch points in the target. Show that (a) $\overline{\mathcal{X}}$ is smooth, but each point has a trivial $\mathbb{Z}_2$ automorphism (a) ev is a finite, positive degree map (b) after further forgetting the marking of 3 of the 4 branch points, the LHS map also becomes a finite positive degree map $\overline{\mathcal{M}}_{1,1} \xleftarrow{\text{st}} \overline{\mathcal{X}}$; (d) calculate the degree of these two maps and describe their branch locus? (e) relate this construction to the "Deligne-Mumford stack $\overline{\mathcal{M}}_{1,1}$" and its degree 24 smooth compact (but branched) cover mentioned in the appendix.

Exercise 3.4 Consider next the (relative) moduli space $\overline{\mathcal{X}}$ of degree 2 covers of $\mathbb{CP}^1$ branched at 6 points. Relate it to the genus two moduli space $\overline{\mathcal{M}}_2$.

The relative version of the GW invariant is used to describe how the GW invariants behave under the symplectic sum $X\#_V Y$;



Pictorial outline of the proof of the symplectic sum formula

One can also define orbifold GW and relative GW relative normal crossings divisors.

Some applications:

- get relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$ from push-pull from relations in $H^*(\overline{\mathcal{M}}_{0,r})$:

$$\overline{\mathcal{M}}_{g,n} \xleftarrow{\text{st}} \overline{\mathcal{X}}_{d,g} \xrightarrow{\text{ev}} \overline{\mathcal{M}}_{0,r}$$

  where $\overline{\mathcal{X}}_{d,g,s}$ are the moduli spaces of degree d holomorphic maps to $\mathbb{CP}^1$ with prescribed ramification pattern r over (moving) points; (see [I1], [I2])

- (Costello) the genus g GW invariants of X are determined by the genus zero orbifold GW of its symmetric product $Sym^{g+1}(X)$.

- construct "fine moduli spaces" (see Abramovich-Vistoli-Corti [ACV] and the appendix);

- Donaldson divisors can be used to globally stabilize all domains (see appendix)

3.2 Open versions of the moduli of curves

There are several motivating example for this, e.g. Floer's approach and the Arnold conjectures; SFT splitting.
There are two types of natural boundary conditions:

(1) asymptotic boundary conditions (e.g. Hamiltonian Floer theory, SFT);

(2) Lagrangian boundary conditions (e.g. Lagrangian Floer theory, real GW invariants);

Could even mix types of boundary values!

*** picture *** (include grad flow lines)

There are several new challenges now:

- more complicated analysis to set-up;
- harder to extract invariants (no "virtual cycle" now) because:
  - the moduli spaces have real codimension one boundary and corners! (at best)
  - for the Lagrangian boundary the moduli spaces may not be orientable
- however, restricting to genus zero often gives well-defined Floer Homology theories;

The Floer theories are now ubiquitous and come in many different flavors. They are modeled on the standard construction of Morse homology, where one makes a complex out of geometric objects (e.g. periodic orbits, Lagrangian intersections, Reeb chords, but also 3-dim SW solutions etc) typically regarded as morally the critical points of some functional $\mathcal{A}$, with a differential counting the number of solutions of the (perturbed) holomorphic map equation (or 4-dim SW equation etc), morally regarded as the gradient flow line equation of $\mathcal{A}$.

In the remaining part of these lectures, we briefly discuss some of these theories.

Lecture 4: Asymptotic boundary conditions

Remark 4.1 In the original Hamiltonian Floer theory the target is a closed manifold, but the solutions have asymptotic boundary conditions because of the special type perturbation used in the holomorphic map equation (coming from a Hamiltonian function); they include as special case the Morse flows; in the symplectic homology version SH one allows the target to be open (with special cylindrical ends) but the Hamiltonian H is required to be quadratic at infinity. This fits as a special case of SFT (Symplectic Field Theory),

Here we discuss only the classical case of the Hamiltonian Floer Theory, see Dietmar Salamon's Park City notes [Sal] for more details. Assume $H_t : M \to \mathbb{R}$ with $t \in S^1$ is a loop of Hamiltonian functions, X_t its Hamiltonian vector field given by $dH_t = \omega(X_t, \cdot)$, and φ_t the flow generated by it.

We are interested in periodic orbits $x : S^1 \to X$ of the flow, or equivalently fixed points of the time 1 map φ_1.

Theorem 4.2 (Arnold Conjecture) *Assume (M, ω) is a closed symplectic manifold and H_t a loop of Hamiltonian functions. Assume all the periodic orbits of the flow are non degenerate. Then*

$$\#\ \textit{contractible orbits}\ \geq \mathrm{rank} H_*(M)$$

Floer's idea: morally do Morse theory for the action functional $\mathcal{A}_H(x) = -\int_{D^2} u^*\omega - \int H_t(x(t))dt$ on (a cover of) the loop space of M. The critical points of $\mathcal{A}_H$ are exactly the periodic orbits of H. This version of Arnold's conjecture should follow from:

- the Floer theory is well defined ($\partial^2 = 0$) and independent of choices;
- it can be calculated e.g. for C^1-small Hamiltonian it agrees with $H_*(M)$;
- Morse inequalities then imply this version of the conjecture;

Brief outline of (classical) Hamiltonian Floer theory setup:

- Pick J_t a loop of compatible almost complex structures;
- $x \in Crit(A_H)$ iff $\dot{x}(t) = X_t(x(t))$ time 1 periodic solution of the flow;
- gradient flow lines correspond to (perturbed) holomorphic cylinders $u : \mathbb{R} \times S^1 \to X$ with

$$\partial_s u + J_t(u)(\partial_t u - X_t(u)) = 0 \tag{4.18}$$

  and asymptotic to periodic orbits $x^\pm$ as $s \to \pm\infty$;

Remark 4.3 If $H_t \equiv 0$ and $J_t = J$ constant in t equation (4.18) becomes $\overline{\partial} u = 0$ and $x^\pm$ are constant loops (removable singularity); if $H_t \equiv H$ constant, and $J_t = J$ constant in t, grad flow lines of H are solutions of the equation (4.18), and these will be all the solutions when H is C^1 small.

Exercise 4.4 Show that $x(t)$ nondeg periodic soln (i.e Hess$\mathcal{A}_H$ nondegenerate) iff $x(0)$ nondegenerate fixed point of φ_1. (hint: $\Gamma_d f \pitchfork \Delta \iff 1$ not an eigenvalue of $d\varphi_1$.)

Key assumption All periodic orbits are non degenerate.

Exercise 4.5 The key assumption corresponds to $\mathcal{A}_H$ being Morse; there is also a Morse-Bott version of it. In any event, this key assumption is used in several crucial places in the construction; can you list some?

Analysis of the moduli space:

- Energy $E(u) = \int_C |\partial_s u|^2 + |\partial_t u - X_t(u)|^2$; restrict to finite energy solutions;
- removable singularity is replaced by asymptotic convergence to the periodic orbits;
- equation (4.18) is invariant under $\mathbb{R}$ action, the moduli space $\mathcal{M}(x, y)$ is the quotient;
- compactification $\overline{\mathcal{M}}(x, y)$: broken cylinders (codim 1, "broken flows") plus holomorphic bubbles (codim 2)
- index still topological: $c_1(TX)$ is now replaced by the Conley-Zehnder index $\mu(x_\pm)$ (coming from the asymptotic operator e.g. via the Atiyah-Patodi-Singer index theorem)
- orientations are OK here;

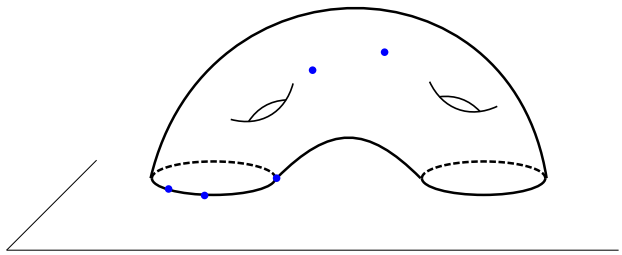
Further properties: there is also a ring structure on HF coming from the pair of pants product (constructed the same way as the quantum product, except now the sphere is replaced by a 3-punctured sphere – pair of pants – asymptotic to 3 periodic orbits α, β, γ).

Remark 4.6 We have seen that periodic orbits of the flow are in 1-1 correspondence with fixed points of the time one map φ_1 and thus with intersection points between two Lagrangians Γ_{df} and Δ in $X \times X^-$. This converts the problem to a Lagrangian intersection problem, discussed in more details in next section.

There are other "open" versions of this theory: for example symplectic Homology $SH(X)$ allows X to have a "cylindrical end" modeled on Y (e.g X is a Liouville manifold), and where one uses a quadratic-at-infinity Hamiltonian H); it also comes with extra structure (a pair of pants product) and enters in several long exact sequences (Seidel). The symplectic homology SH is a special case of SFT; it enters in the recent work of Bourgeois-Ekholm-Eliashberg [BEE] on the behavior of SH under Legendrian surgery, more precisely the attaching of (critical) handles.

Lecture 5: Lagrangian Boundary conditions

Assume next L a Lagrangian in X, or more generally a transverse (or even clean) intersection of Lagrangians in X. Consider the moduli space $\overline{\mathcal{M}}(X, L)$ of stable (perturbed) J-holomorphic maps $f : (\Sigma, \partial\Sigma) \to (X, L)$ with boundary on L. The analytical foundations of this moduli space (and its generalizations that include mixed boundary conditions or Morse flow lines) are not yet available in the full generality needed.



For the genus zero moduli spaces these foundations are described in Fukaya-Ohto-Ono-Oh [FOOO].

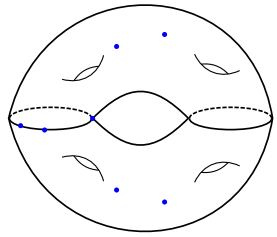
The basic idea is the same as before:

- one starts with a smooth, marked domain with boundary (or more generally boundary and corners);
- energy is topological if L is a Lagrangian: $E(u) = \frac{1}{2}\int_\Sigma |df|^2 = \int_\Sigma f^*\omega$;
- compactify: get the moduli space $\overline{\mathcal{M}}(X, L)$;
- at best, $\overline{\mathcal{M}}(X, L)$ is a stratified manifold (orbifold) with boundary and corners;
- the dimension of the moduli space is still topological, involving now the Maslov index along the boundary.
- the moduli space may not be orientable without extra topological assumptions (e.g. L is spin, or relatively spin, etc); still, its w_1 should be topological;

Exercise 5.1 Consider the space $\Lambda(n)$ of unoriented Lagrangian planes in $\mathbb{C}^n$; show there is an isomorphism (the Maslov index) $\mu : \pi_1(\Lambda(n)) \to \mathbb{Z}$ induced by $\det : U(n) \to S^1$; extend this construction to define a Maslov index for a loop γ in a Lagrangian sub-manifold L of X.

Remark 5.2 Careful: there is now an issue of smoothness of solutions up to boundary; e.g. the higher genus theory currently requires J real analytic along L (uses Schwartz' reflexion principle in a neighborhood of L to describe behaviour of solutions in a collar around the boundary).

Equivalently, this depends on the way the moduli space of domains is set-up: one requires the boundary $\partial\Sigma$ to be a real analytic curve in Σ, so we can reflect across it to get an integrable j on a closed Riemann surface and thus a stable model for it using the Deligne-Mumford moduli space of the double of Σ. There is a nice description of the Deligne-Mumford version of the moduli space of "open" stable curves and its orientability in Melissa Liu's notes [Liu]



One should still be able to extract some Hamiltonian deformation invariants out of it. Two special cases:

- Lagrangian Floer theory [FOOO] (sometimes obstructed);
- real GW invariants;

Lagrangian Floer intersection theory (simplest case) Consider two transversely intersecting Lagrangians L_-, L_+.

- consider a complex generated by the intersection points $L_- \cap L_+$;
- the differential counts holomorphic strips $u : \mathbb{R} \times [-1, 1] \to X$ with boundary on $L_\pm$ and asymptotic to $x_\pm \in L_- \cap L_+$ as $s \to \infty$;
- if well defined, its homology $HF(L_-, L_+)$ is a Hamiltonian deformation invariant;
- sometimes the theory is obstructed ($\partial^2 \neq 0$); in general need a bounding chain and "bulk deformations" (i.e interior marked points and the vanishing of the contribution of a certain type of moduli space), see [FOOO];
- more generally, one can extract an (obstructed, deformed) A_∞-structure using the push-pull (at the chain level) via the evaluation maps:

  $$L^n \xleftarrow{\text{ev}} \overline{\mathcal{M}}(X, L) \xrightarrow{\text{ev}_0} L \times X^k$$

  from the moduli space of disks with $n + 1$ marked points on the boundary (one "outgoing boundary point", n incoming ones) and k interior points ("bulk deformation");
- there are other versions of this, e.g. Wrapped Fukaya-Floer theory $WFH(L)$ that allows for (X, L) to have "cylindrical ends" modeled on (Y, Λ) (where X is Liouville, $\Lambda \subset Y$ is Legendrian and one uses a quadratic-at-infinity Hamiltonian H);

Sample applications:

Theorem 5.3 (Arnold Conjecture) *Assume L is a closed Lagrangian in a closed symplectic manifold; for any Hamiltonian deformation $\varphi(L)$ of L which is transverse to L we have*

$$\#(L \cap \varphi(L)) \geq \text{rank} H_*(L)$$

Basic idea of proof: $HF(L, \varphi(L))$ is well defined, independent of choices, and equal to $H_*(L)$;

Theorem 5.4 (FOOO) *If $H^2(L; \mathbb{Z}_2) = 0$ and L is a compact, embedded Lagrangian in $\mathbb{C}^n$ then $H^1(L, \mathbb{Z}) \neq 0$;*

This extends Gromov's result that there is no embedded Lagrangian sphere in $\mathbb{C}^n$.
Basic idea of proof: Assuming $H^1(L) = 0$, the Lagrangian Floer theory $HF(L)$ is well defined (unobstructed); this implies existence of a non-constant holomorphic curve u with boundary on L. Then $E(u) > 0$; but $\omega = d\lambda$ is exact on $\mathbb{C}^n$ so $E(u) = \int_{D^2} u^*\omega = \int_{\partial D^2} u^*\lambda$ and so $u_*[\partial D^2] \neq 0$ in $H_1(L, \mathbb{Z})$ which gives a contradiction.

Moduli space of stable, real holomorphic maps. Assume ι is an antisymplectic ($\iota^*\omega = -\omega$) involution ι on X;

- first introduced by Welshinger [W] for $\mathbb{CP}^2$, extended in a few other cases;
- the fixed locus of ι (if nonempty) is a Lagrangian $L = X_{\mathbb{R}}$ (the real locus of X);
- this is essentially a $\mathbb{Z}_2$-equivariant theory: ι is a complex conjugation on X ("Real" structure);
- the moduli space of *real* stable holomorphic maps is compact, should be an orbifold (without boundary), but possible nonorientable;
- could extract real GW invariants by using twisted coefficients, so far in special cases (e.g. [PSW])

Harder to set-up and calculate. So far really only done in genus zero under very restrictive topological assumptions on X; still, open mirror symmetry predicts calculations in any genus (technically speaking any Euler characteristic: for $\chi = 0$ it is supposed to count not just tori, but also Klein bottles and Moebius strips).

Topologically, (smooth) curves C with an antiholomorphic involution have been classified by Klein. The classification is in terms of the genus, the number of connected components of the real locus and the orientability index of the quotient C/ι.

Several new issues, e.g.

- to get an invariant, need to allow changing topological type of the domain and this includes the topological type of the involution (in a 1-parameter family the real locus could disappear, change the number of connected components or the quotient could become unorientable);
- understanding how to deal with the automorphisms is crucial now! e.g. if C is a complex curve, there are several inequivalent notions of "real curves" each one with its topological and analytical challenges: for example, in one version, the corresponding Deligne-Mumford space cannot be real analytic at such a curve!

- if $\mathrm{Aut}C = 1$ and C has an antiholomorphic transformation ι, then ι is unique, and must be an involution (a complex conjugation); if $\mathrm{Aut}C \neq 1$ we could have antiholomorphic transformations which are not involutions, or we could have several antiholomorphic involutions, even some which do not commute; it is not at all clear what should be the "right" equivalence relation on these that would produce a "nice" moduli space.

Apendix A. The Deligne-Mumford moduli space of stable curves

Assume we are in the stable range $2g-2+n>0$, and denote by $\mathcal{M}_{g,n}$ the moduli space of smooth genus g complex curves with n marked (distinct) points. An element of $\mathcal{M}_{g,n}$ is an isomorphism class of smooth complex genus g curves $C=(\Sigma, j, x_1, \ldots, x_n)$ where j is complex (integrable) structure on Σ, and $x=\{x_1, \ldots, x_n\}$ are distinct marked points. The condition $2g-2+n>0$ is equivalent to $\mathrm{Aut}C$ is finite.

Example A.1. There is a unique complex structure on $C=\mathbb{CP}^1$ up to automophisms and $\mathrm{Aut}C = PSL(2,\mathbb{C})$, the group of Moebius transformations $\varphi(z)=\frac{az+b}{cz+d}$. Use this to describe $\mathrm{Aut}C$ for $g=0$ and any n; show $\mathcal{M}_{0,3}=pt$ and $\mathcal{M}_{0,4}$ is $\mathbb{CP}^1\setminus\{0,1,\infty\}$; how about $\mathcal{M}_{0,5}$?

There are many descriptions $\mathcal{M}_{g,n}$: it is the classifying space of the mapping class group $\Gamma_{g,n}$:

$$\mathcal{M}_{g,n}=\mathcal{T}_{g,n}/\Gamma_{g,n}$$

where $\mathcal{T}_{g,n} \underset{homeo}{\cong} \mathbb{C}^{3g-3+n}$ is the Teichmuller space of isotopy classes of complex structures on a n-pointed genus g Riemann surface (Σ, x), and the mapping class group $\Gamma_{g,n}=\pi_0(\mathrm{Diff}^+(\Sigma,x))$ is the group of isotopy classes of orientation preserving diffeomorphisms of (Σ, x).

Exercise A.2. Show that $\mathcal{M}_{g,n}$ is also the classifying space of the group $\mathrm{Diff}^+(\Sigma,x)$:

$$\mathcal{M}_{g,n}=\mathcal{J}(\Sigma)/\mathrm{Diff}^+(\Sigma,x)$$

where $\mathcal{J}(\Sigma)$ is the space of almost complex structure on Σ (any almost complex structure is integrable in 2 dim). Show also that

$$\mathcal{M}_{g,n}=\mathcal{J}(\Sigma)\times Conf^n(\Sigma)/\mathrm{Diff}^+(\Sigma)$$

where $Conf^n(\Sigma)$ is the configuration space of n ordered distinct points on Σ.

Exercise A.3. Each complex torus (smooth elliptic curve) with one marked point can be described by a lattice in $\mathbb{C}$ (look at its universal cover), up to automorphisms of $\mathbb{C}$. Use this to describe $\mathcal{M}_{1,1}$ as a topological space; include the automorphism group for each $C\in\mathcal{M}_{1,1}$. Hint: for example $\mathcal{M}_{1,1}=H/SL(2,\mathbb{Z})$ where H is the upper half plane; what it the fundamental domain?

Exercise A.4. More generally, describe $\mathcal{M}_{g,n}$ in terms of hyperbolic geometry using uniformization theorem: in each conformal class of metrics on $\Sigma\setminus x$ there exists a unique hyperbolic metric g on C with constant curvature -1 (and cusps at the marked points x, restrict to $n=0$ if this is confusing). The universal cover of such a genus g curve is the hyperbolic plane H with an action of $Sp(2g,\mathbb{Z})$.

Exercise A.5. Show that the forgetful map $\pi : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ that forgets the last marked point is a fibration, whose fiber at $C = (\Sigma, j, x) \in \mathcal{M}_{g,n}$ is isomorphic (or just homeo, diffeo) to the punctured curve $C \setminus x$ in the case $\mathrm{Aut}C = 1$. What happens when $\mathrm{Aut}C \neq 1$?

Key properties of $\mathcal{M}_{g,n}$:

- it has a Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable genus g curves with n marked points;
- $\overline{\mathcal{M}}_{g,n}$ is a smooth, complex projective (global quotient) orbifold of complex dimension $3g - 3 + n$;
- it has a universal curve $\pi : \overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ whose fiber at $C \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to $C/\mathrm{Aut}C$;
- the universal curve $\overline{\mathcal{U}}_{g,n}$ can be taken to be $\overline{\mathcal{M}}_{g,n+1}$ where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the last marked point.
- it comes with several natural holomorphic bundles:
  - the relative cotangent bundle $\mathcal{L}_{x_i} \to \overline{\mathcal{M}}_{g,n}$ whose fiber at $C = (\Sigma, j, x_1, \dots x_n)$ is $T^*_{x_i}\Sigma$
  - the Hodge bundle $E \to \overline{\mathcal{M}}_{g,n}$, a rank g bundle of holomorphic differentials (whose fiber at C is $H^1(C, \mathbb{C})^*$, related to an Index bundle)

An element of $\overline{\mathcal{M}}_{g,n}$ is an isomorphism class of stable, nodal curves C; a nodal curve is a possibly singular complex algebraic curve C (of algebraic genus g), with finitely many singular points which are all simple double points (nodes) and a collection of n distinct marked points which are all smooth points of C. Such a curve is stable if $\mathrm{Aut}C$ is finite.

The Kodaira-Spencer map $T_C\overline{\mathcal{M}}_{g,n} \cong H^1(C, TC)$ (a natural isomorphism) describes all the defomations of C (here $TC = T\Sigma \otimes O(-\sum x_i)$ whenever $C = (\Sigma, j, x_1, \dots, x_n)$).

Exercise A.6. Show that C is stable iff C has no spherical components with fewer than 3 special points. Hint: An automorphism of C must preserve the n marked points (pointwise), and also preserve the singular points (nodes) but not necessarily pointwise (can permute them); it also could permute the components of C.

Remark A.7. Any stable genus zero curve C has a trivial automorphism group $\mathrm{Aut}C = 1$ (exercise!), so $\overline{\mathcal{M}}_{0,n}$ is a smooth manifold and the fiber of the universal curve at $C \in \overline{\mathcal{M}}_{0,n}$ is C. The generic automorphism group of a stable curve C is trivial unless $(g, n) = (1, 1)$ or $(2, 0)$ in which case the generic automorphism group is $\mathbb{Z}_2$ (coming from the hyperelliptic involution: any such curve can be written as a degree 2 branch cover of the sphere).

Remark A.8. Over the top stratum of smooth curves (with trivial automorphism group) the map $\pi : \mathcal{U} \to \mathcal{M}$ is an actual topological fibration whose fiber at C is C. The monodromy of this fibration around the nodal stratum is a Dehn twist about the curve that pinches to become the node. If the node is "non-separating" (irreducible boundary divisor) this Dehn twist is homotopically nontrivial. See the example of the rational elliptic surface E below; the monodromy around the nodal singularity is precisely a Dehn twist, which is homologically nontrivial (why?); however, its 12'th power is homologically trivial (why?).

Exercise A.9. Discuss boundary strata of $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{0,5}$ and identify these as (blowups) of projective spaces. How many boundary strata does $\overline{\mathcal{M}}_{1,5}$ have? How about $\overline{\mathcal{M}}_{2,3}$?

Exercise A.10. Any $C \in \mathcal{M}_{1,1}$ can be expressed as a degree 2 branch cover of $\mathbb{P}^1$ branched at $0, 1, \lambda, \infty$ where $\lambda \in \mathbb{C} \setminus \{0, 1\}$ (it has Weierstrass normal form $y^2 = x(x-1)(x-\lambda)$ in $\mathbb{CP}^2$). The map $\lambda \mapsto j(\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ is generically a degree 6 map, and two curves C are isomorphic iff they have the same j invariant. Show that $\mathbb{CP}^1 \to \overline{\mathcal{M}}_{1,1}$ given by $\lambda \to j(\lambda)$ is a degree 6 map, branched at the points $j = 0, 1728$ and ∞. For $j(C) \neq 0, 1728$ $\mathrm{Aut}(C) = \mathbb{Z}_2$, while $j(C) = 0$ has autom $\mathbb{Z}_6$ and $j(C) = 1728 = 2^6 \cdot 3^3$ has automorphism $\mathbb{Z}_4$; there are 3 points over $j(C) = \infty$, each with double ramification and automorphism $\mathbb{Z}_2$.

Remark A.11. Technically speaking, $\overline{\mathcal{M}}_{g,n}$ is a Deligne-Mumford stack that is defined in any characteristic (not just over $\mathbb{C}$); in particular, for $\overline{\mathcal{M}}_{1,1}$ in characteristic 2 or 3, the curve with $j = 0 = 1728$ is supersingular, and its automorphism group is $\mathbb{Z}_{24}$ in characteristic 2 and $\mathbb{Z}_{12}$ in characteristic 3. This means that one should really go to a degree 24 cover of $\overline{\mathcal{M}}_{1,1}$ to resolve all the orbifold singularities.

Exercise A.12. Relate this to the (relative) moduli space $\overline{\mathcal{X}}$ of degree 2 covers of $\mathbb{CP}^1$ branched at 4 points (what it the automorphism group of such covers?), together with its two maps $br : \overline{\mathcal{X}} \to \overline{\mathcal{M}}_{0,4}$ that records the location of the branch points in the target (of degree 1) and $\mathrm{st} : \overline{\mathcal{X}} \to \overline{\mathcal{M}}_{1,1}$ is the map that records the domain (after forgetting the marking at 3 of the 4 branch points), and which has degree $4! = 24$.

Exercise A.13. The locus of a generic degree 3 polynomial in $\mathbb{CP}^2$ is an embedded, genus one curve; in a generic 1 parameter family, these curves become nodal. Prove this for the pencil C_λ, the zero locus of the of cubics $Q_1 + \lambda Q_2$, for $\lambda \in \mathbb{CP}^1$ (where Q_1 and Q_2 are two generic degree 3 polynomials). Show that the cubics intersect in 9 (distinct) points; and the generic pencil has 12 nodal curves; after blowing up the 9 intersection points, we get an elliptic fibration E over $\mathbb{CP}^2$ whose generic fiber is smooth elliptic (genus zero); show this is provides a model for the universal curve $\overline{\mathcal{U}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ in genus zero.

Any nodal curve C has a smooth (compact, but possibly disconnected) resolution $\widetilde{C}$ obtained by removing each double point of C and replacing it by an extra pair of marked points of $\widetilde{C}$. If l is the number of nodes of C then $\widetilde{C}$ has $2l$ extra marked points and $\chi(\widetilde{C}) - 2l = 2 - 2g$.

The moduli space $\overline{\mathcal{M}}_{g,n}$ is naturally stratified by the topological type of the curve C: the top dimensional (open) stratum is $\mathcal{M}_{g,n}$ corresponding to smooth curves C, while the rest of strata (called boundary strata) corresponding to curves with $l \geq 1$ nodes are complex codimension l. They are modeled on lower dimensional Deligne-Mumford spaces, and their normal bundle is modeled on the direct sum of the tensor product of the relative cotangent bundles at the two branches of each node.

More precisely, for each fixed nodal, marked and possibly disconnected (topological) surface Σ_0, we can denote by $\mathcal{M}_{\Sigma_0}$ the moduli space of stable complex curves which are homeomorphic

to Σ_0, and by $\overline{\mathcal{M}}_{\Sigma_0}$ its corresponding Deligne-Mumford compactification. Each nodal surface Σ_0 has a smooth resolution $\widetilde{\Sigma}$ (obtained by marking the $2l$ points corresponding to the l nodes of Σ_0) and also a smoothing Σ (obtained by the connect sum construction at each one of the l nodes). The attaching map

$$\xi_S : \overline{\mathcal{M}}_{\widetilde{\Sigma}} \to \overline{\mathcal{M}}_{\Sigma_0} \subset \overline{\mathcal{M}}_{\Sigma}$$

identifies the pairs of marked points of $\widetilde{C} \in \overline{\mathcal{M}}_{\widetilde{\Sigma}}$ indexed by S to produce a curve $C \in \overline{\mathcal{M}}_{\Sigma_0}$ with l more nodes (singular points). The attaching map provides a model for a complex codimension l (closed) stratum of $\overline{\mathcal{M}}_{\Sigma}$: it is a finite cover (of order at most $2^l l!$) over the open stratum, but could be singular over the higher codimension strata.

****** picture: resolution vs deformation for a nodal curve *****

Exercise A.14. The codimension 1 strata of $\overline{\mathcal{M}}_{g,n}$: if we denote the node by $x_- = x_+$ then the codimension one boundary strata are described by the images of the attaching map. The irreducible stratum is the image of

$$\xi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$$

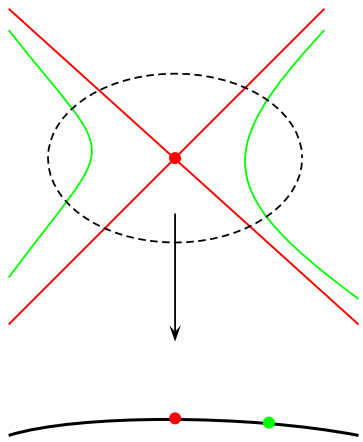
For each partition (g_1, g_2) if the genus g and of the n marked points into two collection of n_1 and n_2 points (with $n = n_1 + n_2$), we also get a reducible boundary stratum as the image of

$$\xi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$$

Show that the attaching map is a degree 2 map, but in general nontrivial a cover of the irreducible stratum; for the reducible strata it give a trivial cover unless $n = 0$ and $g_1 = g_2$. Discuss the origin of (nontrivial) global monodromy in the model of the strata with $l \geq 2$ nodes.

Exercise A.15. Describe all the boundary strata of $\overline{\mathcal{M}}_{2,1}$; are they all submanifolds? (Hint: some of them have self intersections)

The boundary strata form what is called a normal crossing divisor in $\overline{\mathcal{M}}_{g,n}$, i.e. the divisor has a resolution which is smooth, and comes with a holomorphic line bundle N (the 'normal' bundle) and an immersion of it into $\overline{\mathcal{M}}_{g,n}$ such that the local model at each $C \in \overline{\mathcal{M}}_{g,n}$ (with l nodes) is isomorphic to that of l coordinate planes (with their standard normal directions) in $\mathbb{C}^{3h-3+n}$. Informally, ignoring the orbifold issues, this means that the boundary strata are locally modeled on transversely intersecting codimension one complex submanifolds, but globally the strata may have self intersections:



Local picture of the nodal singularity

The forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forgets the marked point x_0 has the following properties (crucial in understanding how the strata of $\overline{\mathcal{M}}_{g,n}$ fit together):

- it is a (singular) fibration whose fiber over $C \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to $C/\mathrm{Aut}C$, traced by the point x_0 as it moves on C;
- it comes with n canonical sections σ_i (corresponding to $x_0 = x_i$);
- the relative cotangent bundles $\mathcal{L}_{x_i} \to \overline{\mathcal{M}}_{g,n}$ are defined as $\sigma_i^* K$ where K is the relative dualizing sheaf of π ("vertical cotangent bundle"); the fiber of $\mathcal{L}_{x_i}$ at C is $T^*_{x_i}C$;
- a non-vanishing holomorphic local section z of $\mathcal{L}_{x_i}$ provides the (germ) of a local coordinate z on C at x_i;
- the singular locus $S \subset \overline{\mathcal{M}}_{g,n+1}$ of π is a smooth, complex codimension two subvariety (orbifold) traced by the nodes (singular points) of C;
- the restriction of π to S is a degree l covering map over the stratum with l nodes;
- S is modeled by the image of the attaching map ξ that identifies l pairs of marked points of a resolution $\widetilde{C}$ of C to produce the l nodes of C;
- at each node $x_- = x_+$ of C (corresponding to the points $x_\pm$ in the resolution $\widetilde{C}$ of C), S is locally the intersection of the divisors $x_0 = x_-$ and $x_0 = x_+$;
- more precisely, we can choose holomorphic coordinates (z, w) normal to S in which the two divisors are given by $z = 0$ and respectively $w = 0$ (the union of the divisors is $zw = 0$ and their intersection S is $z = w = 0$);
- in these coordinates, the two disks traced by x_0 as it moves on C in the neighborhood of the node are cut by the equation $zw = 0$; in particular z and w provide holomorphic local coordinates on each one of the disks centered at the node;

- assume next that C_0 is a curve with one node $x_+ = x_-$. We can further more choose local holomorphic coordinates at this node such that $\pi(u, z, w) = (u, zw)$; here u provides a local coordinate on both S as well as the stratum containing C_0, while (z, w) provide coordinates as described above in which local model of the nodal singularity becomes $zw = 0$;

- furthermore, in these coordinates, all the curves in $\overline{\mathcal{M}}_{g,n}$ near $C_0 \in \partial\overline{\mathcal{M}}_{g,n}$ can be described in terms of a nodal curve $C_0(u)$ and a deformation (gluing) parameter μ; the curve $C_\mu(u)$ is cut by the equation $zw = \mu$ near the node ($C_\mu(u)$ is a deformation of the nodal curve $C_0(u)$ constructed using μ as a gluing parameter).

- intrinsically the gluing parameter μ is a local holomorphic section of the (orbi-)bundle $\mathcal{L}_{x_+} \oplus \mathcal{L}_{x_-}$ over $\partial\overline{\mathcal{M}}_{g,n}$, thus describing the normal direction to this stratum inside $\overline{\mathcal{M}}_{g,n}$;

- moreover, the (orbifold) normal bundle of S and respectively of the nodal stratum $\partial\overline{\mathcal{M}}_{g,n}$ are modeled on $\mathcal{L}_{x_-} \oplus \mathcal{L}_{x_+}$ and respectively $\mathcal{L}_{x_-} \otimes \mathcal{L}_{x_+}$, while the projection π on

$$\begin{array}{rcl} \mathcal{L}_{x_-} \oplus \mathcal{L}_{x_+} & \to & \mathcal{L}_{x_-} \otimes \mathcal{L}_{x_+} \\ (z, w) & \mapsto & zw \end{array}$$

This means in particular that for any nodal curve $C \in \partial\overline{\mathcal{M}}_{g,n}$ with a collection S of l nodes $x_i^+ = x_i^-$ and resolution $\widetilde{C}$ we can find local coordinates z_i and w_i on C at nodes (so that (z_i, w_i) provide normal coordinates to S at the point $x_i^- = x_i^+$), and normal coordinates $(\mu_1, \dots \mu_l)$ to the stratum containing C in $\overline{\mathcal{M}}_{g,n}$ such that all the nearby curves in $\overline{\mathcal{M}}_{g,n}$ are described by the equations

$$z_i w_i = \mu_i \quad \text{at the node } x_i^+ = x_i^- \in S$$

with $\mu_i = 0$ corresponding to a nodal curve.

Intrinsically, the gluing parameters μ are sections of the (orbi)-bundle

$$\bigoplus_{x \in S} \mathcal{L}_{x_-} \otimes \mathcal{L}_{x_+}$$

over the cover of a closed stratum of $\overline{\mathcal{M}}_{g,n}$ consisting of curves with at least l nodes, with precisely l of them indexed by S; the fiber of this bundle at (C_0, S) describes the family of local deformations of C_0 coming from all possible ways of "rounding off" the nodes S of C_0 (but not any of the other nodes); here S determines a (partial) resolution $\widetilde{C}_S$ of C obtained by replacing each node x in S by two points $x_\pm$ in $\widetilde{C}$ (but leaving the other nodes alone); the attaching map ξ_S that describe this stratum identifies the points $x_\pm$ of $\widetilde{C}$ to produce the node $x \in S$ of C.

Remark A.16. In genus zero, the discussion above simplifies somewhat because all stable curves have trivial automorphisms (and all the nodes are separating; in some sense, it is precisely the presence of the non-separating node and the homotopically non-trivial Dehn twist around it that obstructs the smoothness of $\overline{\mathcal{M}}_{g,n}$).

Remark A.17. There are in fact several ways to kill automorphisms in higher genus, by going to a finite (branched) cover of $\overline{\mathcal{M}}_{g,n}$. This is achieved by adding extra finite topological information t to the curves C such that the automorphism group of the decorated curves becomes trivial: $\mathrm{Aut}(C,t) = 1$. The first example (for $n = 0$) was constructed by Looijenga [Lo] using Prym structures (extended later by Pickaart and de Jong); there are more recent examples due to Abramovich-Corti-Vistoli, where one can obtain a cover which is a "fine moduli space" with an actual universal curve (i.e. all $\mathrm{Aut}C = 1$ and the fiber at C is C): the moduli spaces of balanced, twisted G-covers of C (cf. section 7.5 of their paper [ACV]); these are nothing but certain relative moduli spaces of covers of C (ramified only at the special points). In fact, there are (several) finite groups G and moduli spaces $\overline{\mathcal{M}}^G$ of decorated stable curves (C,t) with $\mathrm{Aut}(C,t) = 1$, which come with a G action such that $\overline{\mathcal{M}}^G/G = \overline{\mathcal{M}}$. Furthermore:

$$\overline{\mathcal{M}}^G_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

is a finite $|G|$ cover, branched along the boundary strata; there is a corresponding statement for the universal curve.

Apendix B. Stabilization

A priori, the domain of a stable map $f : C \to X$ is a nodal marked (decorated) curve C (with possibly finitely many unstable components). If we start in the stable range $2g - 2 + n > 0$, all the unstable domain components are genus zero with one or two marked points; after collapsing some rational components, the domain C has a stable model $\mathrm{st}(C) \in \overline{\mathcal{M}}$, so we still have a map $\mathrm{st} : \overline{\mathcal{M}}(X) \to \overline{\mathcal{M}}$. The curve C is obtained from $\mathrm{st}(C)$ by attaching finitely many (unstable) bubble trees at smooth points of $\mathrm{st}(C)$ or in between the two branches of some of the nodes of C. Each such bubble tree has a canonical complex structure j_0 but has infinitely many automorphisms. In fact, unless C is already stable, then we have a residual action of $\mathrm{Aut}C$, which is now a finite dimensional (but not compact!) group. Note that by Gromov compactness, for each fixed homology class $A \in H_2(X)$ (or more generally if the energy is uniformly bounded), then we have a uniform bound on the number of bubble trees and also on their topological type; in particular, there are only finitely many types of $\mathrm{Aut}(C)$ appearing. To set up a model of the moduli space, we would also need to choose a (local) slice to this action as well, and correspondingly enlarge the space of perturbations. Any perturbation ν of (2.10) which is pulled back from the universal curve must vanish on all the unstable components of the domain, so these are J-holomorphic. In the case all the domains are stable to begin with, or more generally when the restriction of f to the unstable part of the domain is a simple J-holomorphic map, this type of perturbation ν is enough to achieve all the required transversality; in general however there is still a problem achieving transversality using this type of perturbation on the unstable part of the domain that is multiply covered.

There many ways to locally stabilize the domains to locally get oneself in the situation where the domain is stable, usually involving adding marked points (using the fact that each unstable domain component must have $\omega(A) > 0$)

Here is one way: starting with an unstable domain, but stable holomorphic map, we can find small pieces of hyper surfaces that are transverse to its image and thus could be used to stabilize

this map and also a sufficiently small neighborhood of it in the Gromov topology, at the expense of going to a finite cover of this open subset of the moduli space; unfortunately, a priori there is no known way to make this covers consistent across the entire moduli space, thus one gets at best only a branch moduli space structure; (as I said before, understanding how these local choices globally patch together *is at the heart of* the construction of the virtual fundamental cycle in GW theories.)

Another way to stabilize is to use Donaldson's theorem: deform ω to have rational coefficients in cohomology and then choose a Donaldson divisor D (a symplectic submanifold) representing the Poincare dual of $k\omega$ where k is sufficiently large. Consider the relative moduli space $\mathcal{M}(X, V)$; then **all** (nontrivial) domain components stable, so this provides a global way to stabilize all the domains simultaneously! The relative moduli space is not exactly a cover of the usual moduli space, but the symplectic sum theorem gives a universal formula expressing the absolute $GW(X)$ in terms of the relative $GW(X, D)$ (they are equal up to some factorial in genus zero) which could be used to define the $GW(X)$.

However, one has to then show independence of the Donaldson divisor used i.e. of k (if k is large, any two Donaldson divisors for the same k are isotopic); this involves the definition of relative GW relative normal crossing divisors and the symplectic sum theorem for those!

Below is a very brief and historically incomplete collection of references for these notes:

References

[A] Abramovich, D. *Lectures on Gromov-Witten invariants of orbifolds* in Enumerative invariants in algebraic geometry and string theory, Lecture Notes in Math., vol 1947, Springer, Berlin, 2008.

[ACV] Abramovich, Dan; Corti, Alessio; Vistoli, Angelo *Twisted bundles and admissible covers*, Special issue in honor of Steven L. Kleiman, Comm. Algebra 31 (2003), 35473618.

[AC] Arbarello, Enrico; Cornalba, Maurizio *Teichmuller space via Kuranishi families*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 1, 89116.

[BEE] Bourgeois, Frederic; Ekholm, Tobias; Eliashberg, Yakov *Symplectic homology product via Legendrian surgery*, Proc. Natl. Acad. Sci. USA 108 (2011), 81148121.

[Do] Donaldson, S. K. *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. 44 (1996), 666-705.

[EGH] Eliashberg, Y.; Givental, A.; Hofer, H.*Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part II, 560673.

[FOOO] Fukaya, Kenji; Oh, Yong-Geun; Ohta, Hiroshi; Ono, Kaoru *Lagrangian intersection Floer theory: anomaly and obstruction*, AMS/IP Studies in Advanced Mathematics, International Press, 2009.

[G] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307-347.

[I2] Ionel, Eleny, *Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$*, Invent. Math. 148 (2002), 627-658.

[I1] Ionel, Eleny-Nicoleta, *Relations in the tautological ring of $\mathcal{M}_g$*, Duke Math. J. 129 (2005), 157186.

[IP1] Ionel, Eleny-Nicoleta; Parker, Thomas H., *Relative Gromov-Witten invariants*, Ann. of Math. 157 (2003), 45-96.

[IP2] Ionel, Eleny-Nicoleta; Parker, Thomas H., *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. 159 (2004), 935-1025.

[Liu] Liu, Chiu-Chu Melissa, *Moduli of J-Holomorphic Curves with Lagrangian Boundary Conditions and Open Gromov-Witten Invariants for an S^1-Equivariant Pair*, arXiv:math/0210257.

[Lo] Looijenga, Eduard, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Algebraic Geom. (1994), 283-293.

[MS1] McDuff, Dusa; Salamon, Dietmar, *J-holomorphic curves and quantum cohomology*, University Lecture Series, AMS, 1994.

[MS2] McDuff, Dusa; Salamon, Dietmar *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, AMS, 2004.

[PSW] Pandharipande, R.; Solomon, J.; Walcher, J. *Disk enumeration on the quintic 3-fold*, J. Amer. Math. Soc. 21 (2008), no. 4, 11691209.

[Sal] Salamon, Dietmar *Lectures on Floer homology*, Symplectic geometry and topology (Park City, UT, 1997), 143229, IAS/Park City Math. Ser., 1999.

[T] Taubes, Clifford Henry, *Seiberg Witten and Gromov invariants for symplectic 4-manifolds*, First International Press Lecture Series, International Press, 2000.

[W] Welschinger, Jean-Yves, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. 162 (2005), no. 1, 195234