Fluctuations from the Semicircle Law
Lecture 4

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An Extension to Smooth Functions

- Approximation Theory

Other results; other models
Lemma (Weierstrass)

Let \( f \) be a continuous function on \([a, b]\). There exists a sequence of polynomials \( \{p_m\} \) which converge to \( f \) uniformly. (I.e., \( \forall \epsilon > 0, \exists N \in \mathbb{N} \) such that \( \forall m \geq N, x \in [a, b], |f(x) - p_m(x)| < \epsilon \).)

We know that we have a polynomial CLT. Can we use the above to get continuous, compactly supported functions?
Examine: want to argue that

$$\sum_{i=1}^{n} f(\lambda_i) - \mathbb{E} \left( \sum_{i=1}^{n} f(\lambda_i) \right) \approx \sum_{i=1}^{n} p_m(\lambda_i) - \mathbb{E} \left( \sum_{i=1}^{n} p_m(\lambda_i) \right)$$

for some polynomials $p_m$. Anything troublesome?
A Simple Approximation

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for some polynomials $p_m$. Anything troublesome?

Yes. We are adding $n$ values of $f$, thus possibly the error is on the order of $n\epsilon...$ which does NOT go to 0.
A Better Idea

So we must find something better; perhaps a better approximating sequence of polynomials, for which

$$||f - p_m|| = \sup_{x \in [a,b]} |f(x) - p_m(x)| < \frac{1}{m^{1+\epsilon}}.$$
So we must find something better; perhaps a better approximating sequence of polynomials, for which
\[ \| f - p_m \| = \sup_{x \in [a,b]} |f(x) - p_m(x)| < \frac{1}{m^{1+\epsilon}}. \]

Unfortunately, this is NOT possible for merely continuous functions... but it is possible for, e.g., $C^2_c$ functions.
Chebyshev Series

Wlog assume that $f \in C_c^2[-1, 1]$ for some large 1.

**Theorem (Chebyshev Series)**

$f$ has a unique representation on $[-1, 1]$ as an absolutely and uniformly convergent series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

with coefficients given by the formulae

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \, dx,$$

for all $k \geq 1$; for $a_0$ the factor $2/\pi$ changes to $1/\pi$. 
More is known and useful.

- Let $f_n$ be the $n$-th truncation of the series, i.e., $f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$.

- If $f \in C^2$, then for any $n \geq 3$, 
  \[ ||f - f_n|| = \sup_{x \in [-1,1]} |f(x) - f_n(x)| \leq \frac{C_f}{n^2}, \]
  where $C_f$ is a constant depending on $f$.

- And finally, $|a_k| \leq \frac{A_f}{k^3}$, where once again $A_f$ is a constant.
Great candidates

So $f_n$, the truncation of the Chebyshev Series, satisfies the requirement. In fact, it will follow that for any $\lambda_1, \ldots, \lambda_n$,

$$X_{n,f} - X_{n,f_n} := \sum_{i=1}^{n} (f(\lambda_i) - f_n(\lambda_i)) - \mathbb{E} \left( \sum_{i=1}^{n} (f(\lambda_i) - f_n(\lambda_i)) \right)$$

$$\approx \sum_{|\lambda_i| \leq 1} (f(\lambda_i) - f_n(\lambda_i)) - \mathbb{E} \left( \sum_{|\lambda_i| \leq 1} (f(\lambda_i) - f_n(\lambda_i)) \right) = O \left( \frac{1}{n} \right).$$

The variance of $X_{n,f_n}$ is $\sum_{k=0}^{n} ka_k^2$, and thus converges to $\sum_{k=0}^{\infty} ka_k^2$ (convergent due to bound $|a_k| \leq A_f/k^3.$)
It seems that it should immediately follow that $X_{n,f} / \sqrt{\sum_{k=0}^{\infty} ka_k^2}$ converges in distribution to $N(0,1)$. ...
An apparent wrench in the spokes

But in the second line the sum is only over $|\lambda_i| \leq 1$!

As $f \in C^2_c([-1, 1])$, this is not a problem for $f$:

$$X_{n,f} = \sum_{|\lambda_i| \leq 1} f(\lambda_i) - \mathbb{E} \left( \sum_{|\lambda_i| \leq 1} f(\lambda_i) \right)$$

$$= \sum_{i=1}^{n} f(\lambda_i) - \mathbb{E} \left( \sum_{i=1}^{n} f(\lambda_i) \right)$$

The same is not true for $f_n$. So what to do?...
Let $\epsilon > 0$.

We split the sum as follows:

$$
\sum_{|\lambda_i| \leq 1} (f_n(\lambda_i) - \mathbb{E}f_n(\lambda_i)) + \sum_{1 < |\lambda_i| \leq 1 + \epsilon} (f_n(\lambda_i) - \mathbb{E}f_n(\lambda_i)) + \sum_{|\lambda_i| > 1 + \epsilon} (f_n(\lambda_i) - \mathbb{E}f_n(\lambda_i)) =: S_1 + S_2 + S_3.
$$

As it turns out, $S_3$ is a sum which is nonzero with exponentially small probability (since the chance that the largest eigenvalue is larger, asymptotically, than $1 + \epsilon$, is minuscule). $S_1$ is the approximation we want.

Showing that $S_2$ is small is technical and we skip it.
Other CLTs for Wigner

We have assumed that the entries of $W_n$ have all moments, and shown that one can obtain a CLT for $C^2_c$ functions. The following are stronger versions.

- Assume the entries’ distribution $\mu$ satisfies a Poincaré-type inequality, i.e., for all $C^1$ functions,
  \[ \text{Var}_\mu(f) \leq c \mathbb{E}_\mu(|f'|^2), \]
  then one can prove a CLT for $C^1$, Lipschitz functions $f$ (Anderson-Zeitouni’08).

- Assuming symmetry of the distributions, a Lindeberg-condition for the 4th moment of the distributions, one can show a CLT for $f \in C^5$ (Pastur-Lytova’08).

- Another CLT proved for Wigner type matrices with a moment condition, for $f$ analytic in a domain including the support of the limiting ESD (Bai-Yao’06).
Other results; other models

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Other CLTs for Wigner

Consider Wigner centered matrices with moment existence conditions (4th moment). Real or complex. Eigenvalues on $[-1, 1]$.

- Sinai-Soshnikov’98 and Soshnikov’99: CLTs for large powers; universality for top eigenvalue fluctuations (Tracy-Widom).

\[
\frac{\text{tr}((W_n)^{k_n}) - \mathbb{E}(\text{tr}((W_n)^{k_n}))}{\sigma_{k_n}} \rightarrow N(0, 1)
\]

if $k_n \ll n^{2/3}$;

\[
\frac{\text{tr}((W_n)^{k_n}) - \mathbb{E}(\text{tr}((W_n)^{k_n}))}{\sigma_{k_n}} \rightarrow F_1
\]

if $k_n = n^{2/3}$, for real $W_n$ and $F_1$ the Tracy-Widom distribution.
Other results; other models

**Other models: general $\beta$-Hermite ensembles**

- Seminal paper (Johansson’98) for a very large variety of potential-defined matrix ensembles.

Recall GOE/GUE/GSE have *joint eigenvalue distribution*

$$f_\beta(\lambda_1, \ldots, \lambda_n) \propto \prod_{i \neq j} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^{n} \lambda_i^2/2};$$

where $\beta = 1, 2, 4$ for $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Let $\beta > 0$ and replace $\sum_{i=1}^{n} \lambda_i^2/2$ with $\sum_{i=1}^{n} V(\lambda_i)$ for $V$ a polynomial of even degree.

Johansson proved a CLT for all such ensembles, for $f$ sufficiently smooth (about 8.5 derivatives).

Note these are NOT Wigner matrices, except for GOE/GUE/GSE.
Other models: Wishart matrices, $\beta$-Laguerre, $\beta$-Jacobi

Let $m \leq n$ and $X$ an $m \times n$ iid matrix of variables with mean 0 and variance 1; let $W = X^TX$. Assume $m/n \to c$ as $m, n \to \infty$.

- Marčenko-Pastur’67: found ESD limit.
- Bai-Silverstein’04: moment conditions, CLT for analytic functions.
- Large body of work by Bai, Silverstein, others on more complicated models (with non-trivial covariance).
- Variances, covariances have complicated expressions.
- Free probability methods (Cabanal-Duvillard’01, Capitaine-Casalis’04) showed the Chebyshev (shifted/scaled) polynomials diagonalize covariance matrix if $\Sigma = I_n$. 
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- D.-Edelman’06: $\beta$-Laguerre models generalizing Wishart real, complex for normal entries ($\Sigma = 1$): monomial CLT.
- D.-Paquette’13: $\beta$-Jacobi models generalizing MANOVA ensembles with trivial covariance: $C^1$, Lipschitz CLT.
Other models: regular graphs

A regular graph has the same degree $d$ for all vertices. The matrix considered is the adjacency matrix, where $A_{ij} = \delta_{i\sim j}$.

- Ben Arous-Dang’11: $d = 2$, random permutation matrices (directed), bounded variation functions (less actually needed); fluctuation, not CLT! Fluctuation converges to infinitely divisible distribution.

- D.-Johnson-Pal-Paquette’12: analytic+ conditions for a permutation-model random regular graph; $d$ fixed implies fluctuations converge to infinitely divisible distribution; $d \to \infty$ slowly with $n$ yields CLT.
Other models: ?

The stage is yours.
Other models: ?

The stage is yours.

Thank you for your attention.