Fluctuations from the Semicircle Law
Lecture 4

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Women and Math, IAS 2014

May 23, 2014

1. An Extension to Smooth Functions
   - Approximation Theory

2. Other results; other models

A Simple Approximation

Lemma (Weierstrass)

Let f be a continuous function on $[a, b]$. There exists a sequence of polynomials $\{p_m\}$ which converge to f uniformly. (I.e., $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m \geq N$, $x \in [a, b]$, $|f(x) - p_m(x)| < \epsilon$.)

We know that we have a polynomial CLT. Can we use the above to get continuous, compactly supported functions?

A Simple Approximation

Examine: want to argue that

$$\sum_{i=1}^{n} f(\lambda_i) - \mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right) \approx \sum_{i=1}^{n} p_m(\lambda_i) - \mathbb{E}\left(\sum_{i=1}^{n} p_m(\lambda_i)\right)$$

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Yes. We are *adding* n values of f, thus possibly the error is on the order of $n\epsilon$... which does NOT go to 0.

A Better Idea

So we must find something better; perhaps a better approximating sequence of polynomials, for which $||f - p_m|| = \sup_{x \in [a,b]} |f(x) - p_m(x)| < \frac{1}{m^{1+\epsilon}}$.

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Unfortunately, this is NOT possible for merely continuous functions... but it is possible for, e.g., C_c^2 functions.

Chebyshev Series

Wlog assume that $f \in C_c^2[-1,1]$ for some large 1.

Theorem (Chebyshev Series)

f has a unique representation on $[-1,1]$ as an absolutely and uniformly convergent series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \ ,$$

with coefficients given by the formulae

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx \ ,$$

for all $k \geq 1$; for a_0 the factor $2/\pi$ changes to $1/\pi$.

Bounds and coefficients

More is known and useful.

- Let f_n be the n-th truncation of the series, i.e., $f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$.
- If $f \in C^2$, then for any $n \geq 3$,

$$||f - f_n|| = \sup_{x \in [-1,1]} |f(x) - f_n(x)| \leq \frac{C_f}{n^2},$$

  where C_f is a constant depending on f.

- And finally, $|a_k| \leq \frac{A_f}{k^3}$, where once again A_f is a constant.

Great candidates

So f_n, the truncation of the Chebyshev Series, satisfies the requirement. In fact, it will follow that for any $\lambda_1, \ldots, \lambda_n$,

$$
\begin{aligned}
X_{n,f} - X_{n,f_n} &:= \sum_{i=1}^{n} (f(\lambda_i) - f_n(\lambda_i)) - \mathbb{E}\left(\sum_{i=1}^{n} (f(\lambda_i) - f_n(\lambda_i)) \right) \\
&\approx \sum_{|\lambda_i| \leq 1} (f(\lambda_i) - f_n(\lambda_i)) - \mathbb{E}\left(\sum_{|\lambda_i| \leq 1} (f(\lambda_i) - f_n(\lambda_i)) \right) = O\left(\frac{1}{n}\right).
\end{aligned}
$$

The variance of X_{n,f_n} is $\sum_{k=0}^{n} k a_k^2$, and thus converges to $\sum_{k=0}^{\infty} k a_k^2$ (convergent due to bound $|a_k| \leq A_f / k^3$.)

Great candidates

It seems that it should immediately follow that $X_{n,f}/\sqrt{\sum_{k=0}^{\infty} k a_k^2}$ converges in distribution to $N(0,1)$....

An apparent wrench in the spokes

But in the second line the sum is only over $|\lambda_i| \leq 1$!
As $f \in C_c^2([-1,1])$, this is not a problem for f:

$$X_{n,f} = \sum_{|\lambda_i|\leq 1} f(\lambda_i) - \mathbb{E}\left(\sum_{|\lambda_i|\leq 1} f(\lambda_i)\right) = \sum_{i=1}^{n} f(\lambda_i) - \mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right) \; !$$

The same is not true for f_n. So what to do?...

A Fudge Factor and an Exponential Salvation

Let $\epsilon > 0$.

We split the sum as follows:

$$\sum_{|\lambda_i| \le 1} (f_n(\lambda_i) - \mathbb{E} f_n(\lambda_i)) + \sum_{1 < |\lambda_i| \le 1+\epsilon} (f_n(\lambda_i) - \mathbb{E} f_n(\lambda_i)) + \sum_{|\lambda_i| > 1+\epsilon} (f_n(\lambda_i) - \mathbb{E} f_n(\lambda_i)) =: S_1 + S_2 + S_3 \, .$$

As it turns out, S_3 is a sum which is nonzero with exponentially small probability (since the chance that the largest eigenvalue is larger, asymptotically, than $1+\epsilon$, is minuscule). S_1 is the approximation we want.

Showing that S_2 is small is technical and we skip it.

Other CLTs for Wigner

We have assumed that the entries of W_n have all moments, and shown that one can obtain a CLT for C_c^2 functions. The following are stronger versions.

- Assume the entries' distribution μ satisfies a Poincaré-type inequality, i.e., for all C^1 functions,

$$\mathrm{Var}_\mu(f) \leq c\mathbb{E}_\mu(|f'|^2) ,$$

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- Another CLT proved for Wigner type matrices with a moment condition, for f analytic in a domain including the support of the limiting ESD (Bai-Yao'06).

Other CLTs for Wigner

Consider Wigner centered matrices with moment existence conditions (4th moment). Real or complex. Eigenvalues on $[-1,1]$.

- Sinai-Soshnikov'98 and Soshnikov'99: CLTs for *large powers*; universality for top eigenvalue fluctuations (Tracy-Widom).

$$\frac{\mathrm{tr}((W_n)^{k_n}) - \mathbb{E}(\mathrm{tr}((W_n)^{k_n}))}{\sigma_{k_n}} \to N(0,1)$$

if $k_n << n^{2/3}$;

$$\frac{\mathrm{tr}((W_n)^{k_n}) - \mathbb{E}(\mathrm{tr}((W_n)^{k_n}))}{\sigma_{k_n}} \to F_1$$

if $k_n = n^{2/3}$, for real W_n and F_1 the Tracy-Widom distribution.

Other models: general β-Hermite ensembles

- Seminal paper (Johansson'98) for a very large variety of potential-defined matrix ensembles.

  Recall GOE/GUE/GSE have *joint eigenvalue distribution*

  $$f_\beta(\lambda_1, \ldots, \lambda_n) \propto \prod_{i \neq j} |\lambda_i - \lambda_j|^\beta \, e^{-\sum_{i=1}^n \lambda_i^2/2} \; ;$$

  where $\beta = 1, 2, 4$ for $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

  Let $\beta > 0$ and replace $\sum_{i=1}^n \lambda_i^2/2$ with $\sum_{i=1}^n V(\lambda_i)$ for V a polynomial of even degree.

  Johansson proved a CLT for all such ensembles, for f sufficiently smooth (about 8.5 derivatives).

  Note these are NOT Wigner matrices, except for GOE/GUE/GSE.

Other models: Wishart matrices, β-Laguerre, β-Jacobi

Let $m \leq n$ and X an $m \times n$ iid matrix of variables with mean 0 and variance 1; let $W = X^T X$. Assume $m/n \to c$ as $m, n \to \infty$.

- Marčenko-Pastur'67: found ESD limit.
- Bai-Silverstein'04: moment conditions, CLT for analytic functions.
- Large body of work by Bai, Silverstein, others on more complicated models (with non-trivial covariance).
- Variances, covariances have complicated expressions.
- Free probability methods (Cabanal-Duvillard'01, Capitaine-Casalis'04) showed the Chebyshev (shifted/scaled) polynomials diagonalize covariance matrix if $\Sigma = I_n$.

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- D.-Edelman'06: β-Laguerre models generalizing Wishart real, complex for normal entries ($\Sigma = 1$): monomial CLT.
- D.-Paquette'13: β-Jacobi models generalizing MANOVA ensembles with trivial covariance: C^1, Lipschitz CLT.

Other models: regular graphs

A regular graph has the same degree d for all vertices. The matrix considered is the adjacency matrix, where $A_{ij} = \delta_{i \sim j}$.

- Ben Arous-Dang'11: $d = 2$, random permutation matrices (directed), bounded variation functions (less actually needed); *fluctuation*, not CLT! Fluctuation converges to infinitely divisible distribution.
- D.-Johnson-Pal-Paquette'12: analytic+ conditions for a permutation-model random regular graph; d fixed implies fluctuations converge to infinitely divisible distribution; $d \to \infty$ slowly with n yields CLT.

Other models: ?

The stage is yours.

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Thank you for your attention.