Fluctuations from the Semicircle Law Lecture 2

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Review

- We are examining $\sum_{I,J\in\mathcal{I}} (\mathbb{E}(w_I w_J) - \mathbb{E}(w_I)\mathbb{E}(w_J))$, where $\mathcal{I}$ is the set of all k-tuples.
- To examine this we need the graphs G_I, G_J, and their union, as well as the walks given by I and J on this graph.
- Every edge appears ≥ 2 times, $v - 1 \leq e \leq k$.
- We determined that terms with $v = k + 1$ have 0 contribution.
- We want v as large as possible.

Review

Why must v be large?

$$\sum_{I,J\in\mathcal{I}}(\mathbb{E}(w_I w_J) - \mathbb{E}(w_I)\mathbb{E}(w_J)) = \frac{1}{n^k}\sum_{G_I,G_J,G_I\cup G_J} n(n-1)(n-2)\dots(n-v+1)\cdot Q(k,v,e)\ ,$$
$$= \frac{1}{n^k}\sum_{G_I,G_J,G_I\cup G_J} n^v(1+o(1))\cdot Q(k,v,e)\ .$$

The sum is like a polynomial in n, whose asymptotics is determined by the highest power. Thus, we want the coefficient of that highest power. Established yesterday that $v < k+1$.

Second attempt: $v = k$, k even

Note that we must have *some* edge overlap between the graph induced by I and the one induced by J. Otherwise term contributes 0 to covariance.

As $v = k$, one of these must be true:

- $e = k = v$; graph has a cycle or loop;
- $e = k - 1 = v - 1$; graph is a tree; one edge has multiplicity 4;
- $e = k - 1 = v - 1$; graph is a tree; two edges have multiplicity 3.

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Count for $v = k$, k is even, graph has a cycle

Let $3 \leq r \leq k$ be the length of the cycle. The shape of the graph is a cycle with dangling trees.

The trees divide in two subsets with same number of edges; half, i.e., $(k-r)/2$, belong to the graph induced by I, the others to the graph induced by J.

Count for $v = k$, k is even, graph has a cycle

Let $3 \leq r \leq k$ be the length of the cycle. The shape of the graph is a cycle with dangling trees.

No overlap on the trees! We can think of each vertex of the cycle as having its own pair $(T_{i,I}, T_{i,J})$ of trees, possibly empty, one from I, the other from J. Call their edge sizes $k_{i,I}$ and $k_{i,J}$.

Count for $v = k$, k is even, graph has a cycle

Must have

$$\sum_{i=1}^{r} k_{i,I} = \sum_{i=1}^{r} k_{i,J} = \frac{k-r}{2} .$$

For each pair of partitions of $(k-r)/2$, one has a total number of possible combinations of trees which is

$$\prod_{i=1}^{r} C_{k_{i,I}} \prod_{i=1}^{r} C_{k_{i,J}} .$$

Count for $v = k$, k is even, graph has a cycle

Assume that k vertices have been chosen in $(n)_k = n(n-1)\dots(n-k+1) = n^k(1+o(1))$ ways.

Given a cycle-and-dangling-trees graph, properly partitioned, there are

- a total number k^2 of ways of choosing starting vertices;
- 2 choices on direction (same or opposite);
- a rotational invariance overcount (must divide by r).

Count for $v = k$, k is even, graph has a cycle

Summing over all possible cycle lengths r:

$$n^k(1+o(1))\sum_{r=3}^{k}\frac{2k^2}{r}\left(\sum_{\substack{\kappa\ \vdash\ (k-r)/2\\ length(\kappa)=r}}\ \prod_{i=1}^{r}C_i\right)^2 .$$

Count for $v = k$, k is even, graph has a cycle

Sharing the cycle implies that both $\mathbb{E}(w_I) = \mathbb{E}(w_J) = 0$, and since each edge appears twice, there are k of them, and $\mathbb{E}(w_{ij}^2) = 1/n$, the contribution is asymptotically

$$\sum_{r=3}^{k} \frac{2k^2}{r} \left(\sum_{\substack{\kappa \vdash (k-r)/2 \\ length(\kappa) = r}} \prod_{i=1}^{r} C_i \right)^2 .$$

Count for $v = k$, k is even, graph is a tree

Recall that one edge is overlapped and appears 4 times.

Given a set of k labels, have $C_{k/2}$ possible trees and for each pair, $(k/2)^2$ choices of pair of overlapped edges, and 2 choices of orientation. Total choices:

$$n^k(1 + o(1))2(k/2)^2 C_{k/2}^2 \ ;$$

each choice comes with weight

$$\left(\mathbb{E}(Z^4) - 1\right)(1/n)^k \ ,$$

the contribution is, asymptotically,

$$2(k/2)^2 C_{k/2}^2 \left(\mathbb{E}(Z^4) - 1\right) \ .$$

Asymptotical variance for k even

Putting it all together for k even:

$$\sigma_k^2 := Var\left(W_n^k\right) = \frac{k^2}{2} C_{k/2}^2 \left(\mathbb{E}(Z^4) - 1\right) + \sum_{r=3}^{k} \frac{2k^2}{r} \left(\sum_{\substack{\kappa \,\vdash\, (k-r)/2 \\ length(\kappa) = r}} \prod_{i=1}^{r} C_i \right)^2 .$$

Second attempt: $v = k$, k odd

As $v = k$, one of these must be true:

- $e = k = v$; graph has a cycle or loop.
- $e = k - 1 = v - 1$; graph is a tree; one edge has multiplicity 4;
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Count for $v = k$, k odd, graph has a cycle

Exactly as before; total contribution is

$$\sum_{r=3}^{k} \frac{2k^2}{r} \left(\sum_{\substack{\kappa \ \vdash \ (k-r)/2 \\ length(\kappa) = r}} \prod_{i=1}^{r} C_i \right)^2 .$$

Count for $v = k$, k odd, graph has a loop

The loop must be the overlap. There are k choices where to attach it to a closed walk on a tree with $(k-1)/2$ vertices (at each step). Two trees, so total number of choices is $k^2 C^2_{(k-1)/2}$.

The existence of the loop means that $\mathbb{E}(w_I) = \mathbb{E}(w_J) = 0$.

There are $n^k(1 + o(1))$ choices for vertices, and each edge appears twice for a weight of $(1/n)^k$.

Asymptotical variance for k odd

Putting it all together for k odd:

$$\sigma_k^2 := Var\left(W_n^k\right) = k^2 C^2_{(k-1)/2} + \sum_{r=3}^{k} \frac{2k^2}{r} \left(\sum_{\substack{\kappa \,\vdash\, (k-r)/2 \\ length(\kappa) = r}} \prod_{i=1}^{r} C_i \right)^2 .$$

Convergence in Probability

We have shown that, for any $k \geq 0$,

$$X_{n,k} = \mathrm{tr}(W_n^k) - \mathbb{E}(\mathrm{tr}(W_n^k))$$

has, asymptotically, finite variance. It follows then that

$$\frac{1}{n}\mathrm{tr}(W_n^k) - \frac{1}{n}\mathbb{E}(\mathrm{tr}(W_n^k)) \to 0 \,,$$

in other words

$$\frac{1}{n}\mathrm{tr}(W_n^k) \to \delta_{k\text{ even}} C_{k/2} \,.$$

Convergence in Probability

This means that the average distribution of an eigenvalue converges not just in distribution, but in *probability*.

In fact, we can get more: the variance of $\frac{1}{n}\text{tr}(W_n^k)$ is proportional to $\frac{1}{n^2}$.

Assume we have a sequence of matrices W_n, for $n \geq 1$, on the same probability space (e.g., a doubly infinite, symmetric array of iid variables.)

Almost Sure Convergence

The Chebyshev Inequality together with the Borel-Cantelli Lemma say that, for any $\epsilon > 0$ and $k \geq 1$, the number of matrices for which

$$\left| \frac{1}{n} \mathrm{tr}(W_n^k) - \delta_{k \text{ even}} C_{k/2} \right| > \epsilon$$

is finite.

This means that the convergence to the semicircle happens *almost surely*, and explains the experiment.