Fluctuations from the Semicircle Law Lecture 1

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Women and Math, IAS 2014

May 20, 2014

1. Review
2. Fluctuations
3. Calculation of the Variance

Random (Real) Facts

- Distributions are generalized functions, better understood through the effect they *have* on functions
- Probability distributions define random variables via characteristic functions:

$$X \sim F \quad \text{if } \forall [a,b] \in \mathbb{R}\,, \;\; P[X \in [a,b]] = F([a,b])\,.$$

Random (Real) Facts

- A probability distribution F may be given by a (positive) function (the probability density function) f with $\int_{\mathbb{R}} f(x)dx = 1$. We say $d\mu(x) = f(x)dx$.
- In this case $P[X \in [a, b]] = \int_a^b f(x)dx$.

The Moment Method

- A "nice" distribution is described by the collection of its moments, $\{\mathbb{E}[X^k]\ , \forall k \in \mathbb{Z},\ k \geq 0\}$. This leads to the "moment method".
- More on the relationship between the distribution and its moments in today's Review Session.
- Convergence of moments is weak convergence, i.e., convergence in distribution. Stronger: in probability and almost surely.

Wigner Matrices

- $n \times n$ *real* (or complex, quaternion) matrices W;
- symmetric: $W = W^T$ (or Hermitian $W = W^*$, self-dual $W = W^D$);
- entries are independent up to symmetry $(w_{ij} = w_{ji})$;
- entries are identically distributed up to symmetry (all w_{ij} with $i < j$ are equidistributed, resp. all w_{ii} are equidistributed);

Wigner Matrices

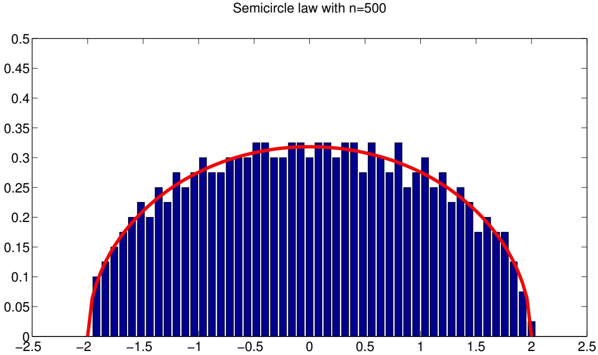
- the distributions have moments of all orders; in particular, $\mathbb{E}(Z^4)$ is the 4th moment;
- all variables are centered (expectation 0) and all variances are 1 (could also consider variance σ^2 on the diagonal).

Wigner Matrices

... actually, we consider the normalized Wigner matrices $W_n = \frac{1}{\sqrt{n}} W$.

The Semicircle Law

$n = 500; A = rand(n); A = (A + A')/\sqrt{n};$
$hist(eig(A))$
$semicircle$



Convergence to the Semicircle

To understand convergence through experiments:

- **Convergence in distribution:** pick an $n \times n$ random Wigner matrix W, pick one of its eigenvalues at random. Repeat many times. Plot histogram.

- **Almost surely:** pick a *single* matrix W, and plot a histogram of *all* its eigenvalues.

LLN and CLT

1−D: let $x_1, x_2, \ldots, x_n, \ldots$ be independent samples from a distribution with mean μ and variance σ^2.

- Law of Large Numbers: $\frac{1}{n}\sum_{i=1}^{n} x_i - \mu \to 0$ as $n \to \infty$.
- Central Limit Theorem:

$$\frac{\sum_{i=1}^{n} x_i - n\mu}{\sqrt{n}\sigma} \sim N(0,1) \ .$$

What about matrices?

LLN for Matrices

The Semicircle Law is akin to a Law of Large Numbers.
Showed:

$$\frac{\mathbf{1}}{\mathbf{n}}\mathbb{E}(\mathbf{tr}(\mathbf{W^k})) = \frac{\mathbf{1}}{\mathbf{n}}\mathbb{E}\left(\sum_{\mathbf{i=1}}^{\mathbf{n}} \lambda_{\mathbf{i}}^{\mathbf{k}}\right) \to \begin{cases} 0, & k \text{ odd}, \\ C_{k/2}, & k \text{ even}. \end{cases}$$

On the right hand side are the moments of the semicircle distribution, with density $\mathbf{s}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}\pi}\sqrt{\mathbf{4} - \mathbf{x^2}}$.

Convergence of moments means that the *expected* distribution of a random eigenvalue converges *in distribution* to the semicircle law.

LLN for Matrices

What this implies immediately is that for all reasonable functions $f : [-2, 2] \to \mathbb{R}$,

$$\frac{1}{n}\mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right) \to \int_{-2}^{2} f(x)s(x)dx \,.$$

Exact expressions for the distribution

The actual expected distributions can be computed, for the GOE/GUE/GSE, for any n. The expressions are not very complicated.

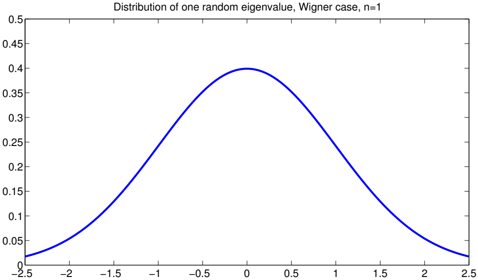


Figure: Distribution of one random eigenvalue, $n = 1$

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Figure: Distribution of one random eigenvalue, $n = 2$

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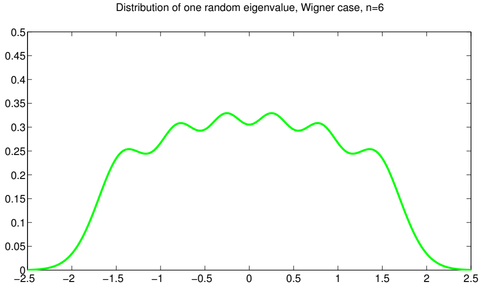

Figure: Distribution of one random eigenvalue, $n = 6$

Exact expressions for the distribution

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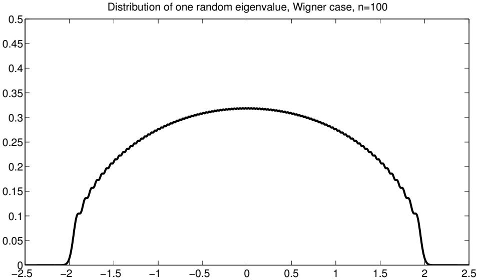

Figure: Distribution of one random eigenvalue, $n = 100$

Bumps = Fluctuations

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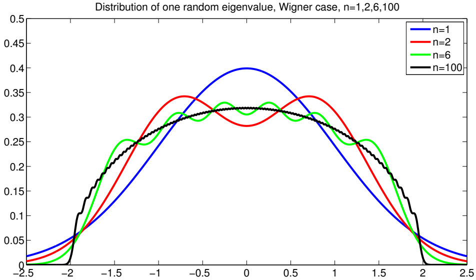

Figure: Distribution of one random eigenvalue, $n = 1, 2, 6, 100$

How can we compute the fluctuation?

Bumps = Fluctuations

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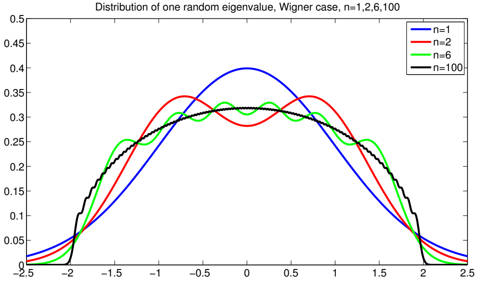

Figure: Distribution of one random eigenvalue, $n = 1, 2, 6, 100$

How can we compute the fluctuation?

Compute moments more carefully.

Format of CLT

- Recall that in 1−D,

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^{n} X_i - \mathbb{E}\left(\sum_{i=1}^{n} X_i\right)}{\sqrt{n}\sigma} \to N(0,1)$$

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- For Wigner matrices we will have something similar:

$$\frac{\sum_{i=1}^{n} f(\lambda_i) - \mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right)}{\sigma_f} \to N(0,1)\ ,$$

provided that f is "smooth enough".

Format of CLT

Start with the simplest smooth functions, $f(x) = x^k$; must then show that for a Wigner *real* matrix W_n

$$X_{n,k} := \text{tr}(W_n^k) - \mathbb{E}(\text{tr}(W_n^k)) ,$$

has the property that

$$\frac{X_{n,k}}{\sqrt{\text{Var}(X_{n,k})}} \rightarrow N(0,1) ,$$

where $N(0,1)$ is the standard normal variable, and convergence is in distribution.

Variance

Need to show that all moments of $X_{n,k}/\sqrt{\text{Var}(X_{n,k})}$ converge to those of the standard normal variable.

The first step is to calculate the variance

$$\text{Var}(X_{n,k}) = \mathbb{E}\left(\left(\text{tr}(W_n^k)\right)^2\right) - \left(\mathbb{E}(\text{tr}(W_n^k))\right)^2 .$$

Expansion of the trace power

Write

$$\mathrm{tr}(W_n^k) = \sum_{I \in \mathcal{I}} w_I \,,$$

where

$$\mathcal{I} := \{I = (i_1, i_2, \ldots, i_k), \mid 1 \leq i_1, i_2, \ldots, i_k \leq n\} \,,$$

that is, ordered k-tuples; we also use the notation

$$w_I = w_{i_1 i_2} w_{i_2 i_3} \ldots w_{i_k i_1} \,.$$

Expansion of the trace power

Then

$$\begin{aligned} \mathrm{Var}(X_{n,k}) &= \mathbb{E}\left(\left(\mathrm{tr}(W_n^k)\right)^2\right) - \left(\mathbb{E}(\mathrm{tr}(W_n^k))\right)^2 \\ &= \sum_{I,J\in\mathcal{I}} \mathbb{E}\left(w_I w_J\right) - \mathbb{E}(w_I)\mathbb{E}(w_J)\ . \end{aligned}$$

To each $I \in \mathcal{I}$ there corresponds a graph G_I, with vertex labels $\{i_1, \ldots, i_k\}$, with **v** vertices and **e** edges, having an edge between vertices i_j and i_l if they occur consecutively in I (loops are ok).

Graphs and labels

Consider the graph which is the union of the two graphs corresponding to I and J (for a given I, J).

- In the walks corresponding to I and J, edges may be repeated; loops are also possible.
- Total # of edges in the walks, with multiplicities, $= 2k$.
- Enough to consider the case when the graph is connected; otherwise w_I, w_J independent and

$$\mathbb{E}(w_I w_J) - \mathbb{E}(w_I)\mathbb{E}(w_J) = 0 \ .$$

Graphs and labels

- If any edge has multiplicity 1 in the union of the walks, $\mathbb{E}(w_I w_J) = 0 = \mathbb{E}(w_I)\mathbb{E}(w_J)$.
- ... therefore only need to consider walks where all edges are repeated.

Graphs and labels

So

- Total # of edges in the union of walks with multiplicities $= 2k$, every edge repeated.
- The graph is connected.
- So total # of *actual* edges $e \leq k$ and $e \geq v - 1$, $\mathbf{v \leq k + 1}$.

- Given $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k$, the total number of such graphs is independent of n.
- Asymptotics are given by those graphs for which v is as large as possible.

First Attempt: $v = k + 1$

No such terms are relevant.

First Attempt: $v = k + 1$

- Must have $v = k + 1$, $e = v - 1$; so join graph is a tree on which each edge is walked on twice.
- Hence the two closed walks that form it are trees on which each edge is walked on twice.
- No edge overlap, so w_I is independent from w_J.
- Term contributes 0 to covariance.