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There are natural open questions arising from what we have seen in the previous lectures:

- One can wonder what happens if $\mathbb{Q}_p$ is replaced by some finite extension $K/\mathbb{Q}_p$, i.e. if we consider $G_K := \text{Gal}(\overline{\mathbb{Q}}_p/K)$-representations on the Galois side and $GL_2(K)$-representations on what we have called the $GL_n$ side in the first lecture;

- One can also ask what happens for $n$-dimensional representations with $n$ arbitrary. Not much is known for $n \neq 2$: some partial results have been obtained by people like Herzig [H], Ollivier [O1] [O2], Ollivier-Sécherre [OS], Schraen [Sch], Vignéras [V].

These questions are very difficult: for example, one can naturally define $(\varphi, \Gamma)$-modules for any integer $n$ and any finite extension $K$ of $\mathbb{Q}_p$, but Colmez’ functor only gives $B_2(\mathbb{Q}_p)$-representations.

The aim of this talk is to present more naive open questions that arise on the Galois side. More precisely, we focus on the two following topics (inspired by a work in progress of Fontaine-Mézard):

- the reduction modulo $p$ of crystalline representations;

- a generalization of $(\varphi, \Gamma)$-modules.
1 Modulo $p$ reduction for crystalline representations

Let $E$ be a finite extension of $\mathbb{Q}_p$ (with $p \neq 2$) with ring of integers $\mathcal{O}_E$, maximal ideal $\mathfrak{m}_E$, uniformizer $\varpi_E$ and residue field $k_E := \mathcal{O}_E/\mathfrak{m}_E$. For any integer $k \geq 2$ and any $a_p \in \mathfrak{m}_E$, we define a filtered $\varphi$-module $D_{k,a_p} := Ee_1 \oplus Ee_2$ with Frobenius map given by $Mat(\varphi) = \begin{pmatrix} 0 & -1 \\ p k^{-1} & a_p \end{pmatrix}$ and filtration defined by

$$\text{Fil}^i D_{k,a_p} := \begin{cases} D_{k,a_p} & \text{if } i \leq 0 \\ E e_1 & \text{if } 1 \leq i \leq k-1 \\ \{0\} & \text{if } i \geq k. \end{cases}$$

By Colmez-Fontaine Theorem (Lecture 2), this filtered $\varphi$-module is attached to a crystalline representation $V_{k,a_p} : G_{\mathbb{Q}_p} \rightarrow GL_2(E)$ such that

$$D_{\text{cris}}(V^*_{k,a_p}) = D_{k,a_p}.$$

Considering Colmez-Fontaine Theorem, a first naive question could be the following one:

**Question 1.** Can we compute the admissible filtered $(\varphi, N)$-modules for all $n \geq 2$?

Some computations due to Dousmanis [D], Ghat [GM] and Imai [I] give a complete description of admissible filtered $(\varphi, N)$-modules for $n = 2$. For $n \geq 3$, this is still an open problem.

Now let $T$ be a $G_{\mathbb{Q}_p}$-stable lattice of $V_{k,a_p}$ and $\overline{V}_{k,a_p}$ be the semi-simplification of $T/\varpi_E T$. It is known that $\overline{V}_{k,a_p}$ only depends on $V_{k,a_p}$ (and not on the choice of $T$), so that we would like to describe $\overline{V}_{k,a_p}$ in terms of $k$ and $a_p$. We only have partial results when $k \geq 2p+1$ in the following cases:

- if $v_p(a_p) > \left\lceil \frac{k-2}{p-1} \right\rceil$ (very big), then $\overline{V}_{k,a_p} = \text{ind}(\omega_2^{k-1})$;
- if $0 < v_p(a_p) < 1$ (very small), then $\overline{V}_{k,a_p}$ can be described with a parameter $t \in \{1, \ldots, p-1\}$ congruent to $k-1$ modulo $p-1$. The description depends on whether $p-1$ divides $k-3$ or not.

For an explicit description of all the cases where $\overline{V}_{k,a_p}$ is known, we refer to [Be1, Theorem 5.2.1].

We also have the following general result, due to Berger-Breuil [BeBr], that relates what is above to the Langlands correspondence:

**Theorem 1.** If $V$ is an absolutely irreducible 2-dimensional $E$-linear representation of $G_{\mathbb{Q}_p}$, then the semi-simplification $\overline{V}$ corresponds by the local modulo $p$ Langlands correspondence to $\Pi(V)$.

**Conjecture 1** (Buzzard-Gee, [BG]). If $p \neq 2$, if $k$ is even and if $\overline{V}_{k,a_p}$ is reducible, then $v_p(a_p)$ is an integer.

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The strategy is then to find an algorithm able to compute $V_{k,a_p}$ for $n$ large enough starting from the data $(k, a_p \mod \varpi_n^p)$. Two useful tools can be used to reach this goal as they give a way to build lattices and to make modulo $p$ reductions:

- Wach modules [Be2];
- Breuil-Kisin modules [K].

2 Generalization of $(\varphi, \Gamma)$-modules

Let $k$ be a perfect field of characteristic $p > 0$ and let $\sigma : x \mapsto x^p$ be its Frobenius map. Denote by $W = W(k)$ the ring of Witt vectors with coefficients in $k$ and by $K_0 := W[\frac{1}{p}]$ its fraction field. We can extend the Frobenius $\sigma$ to $W$ and then to $K_0$.

Let $K$ be a totally ramified extension of $K_0$ of finite degree $e$: we can then write $K = K_0(\pi_0)$. Fix some algebraic closure $\overline{K}$ of $K$, denote by $m_\overline{K}$ its maximal ideal and by $q_0 \in W[X]$ the minimal polynomial of $\pi_0$ over $K_0$: it's an Eisenstein polynomial satisfying $q_0(\pi_0) = 0$.

We set the following definition: a $\varphi$-data is a data $(\mathcal{F}, (\pi_n)_{n \in \mathbb{N}})$ with:

- $\mathcal{F} = \sum_{i \geq 0} a_i X^i \in W[[X]]$ such that $\mathcal{F}(X) \equiv X^p \mod p$;
- $(\pi_n)_{n \in \mathbb{N}}$ is a compatible system of elements of $m_\overline{K}$ such that:
  $$\forall n \geq 1, \sum_{i \geq 0} \sigma^{-n}(a_i) \pi_n^i = \pi_{n-1}.$$ 

Let $(\mathcal{F}, (\pi_n)_{n \in \mathbb{N}})$ be a $\varphi$-data. For any $n \geq 1$, set $K_n := K[\pi_n]$ and $K_\infty := \bigcup_{n \geq 1} K_n$. Also set $K_{\text{cyc}} := K(\mu_{p^\infty})$ and $L := K_\infty K_{\text{cyc}}$ the composite extension. A naive question that naturally arises is the following:

**Question 2.** For which $\mathcal{F}$ is the extension $L/K$ a Galois extension?

We don’t know so far if there are many such datas or not. We only know two examples of cases where it is actually Galois:

1st example: The cyclotomic tower:

Let $\pi_0 := \zeta_0 - 1$ with $\zeta_0$ a primitive $p$-root of unity and $K := K_0(\zeta_0)$. Consider $\mathcal{F}(X) := (X + 1)^p - 1$ and $\pi_n$ given by $\zeta_n = 1 + \pi_n$ where $\zeta_n$ is a $p$-th root of $\zeta_{n-1}$. Then $(\mathcal{F}, (\pi_n)_{n \in \mathbb{N}})$ is a $\varphi$-data and the fields $K_n$ that it defines are precisely the fields of the cyclotomic tower of $K$.

2nd example: The Kummer extension:

It does correspond to the following $\varphi$-data: $\mathcal{F}(X) := X^p$ and $\pi_n^p = \pi_{n-1}$.

Now return to the general setting and consider a $\varphi$-data $(\mathcal{F}, (\pi_n)_{n \in \mathbb{N}})$: since $\mathcal{F}(X) \equiv X^p \mod p$, we have $\pi_n^p \equiv \pi_{n-1}$ so that $(\pi_n)_{n \in \mathbb{N}}$ defines an element of
\( \mathcal{E} \), which is in fact in \( \mathcal{E}^+ \) as \( \text{val}(\pi_0) \geq 0 \) by definition. We can then consider 
\[ \pi = \{ (\pi_n)_{n \in \mathbb{N}} \} \in \hat{A}^+ := W(\mathbb{E}^+), \]
and we easily have the following result:

**Lemma 1.** There exists a unique continuous injective morphism of \( W \)-algebras
\[ W[[u]] \rightarrow \hat{A}^+ \]
commuting with the Frobenius maps and such that \( u \) has image \( \pi \) through the composite map
\[ W[[u]] \rightarrow \hat{A}^+ \rightarrow \hat{E}^+. \]

Lemma 1 then allows us to see \( W[[u]] \) as a subring of \( \hat{A}^+ \). Remember that we saw in Lecture 2 that there is a map \( \theta : \hat{A}^+ \rightarrow \mathcal{O}_{\mathbb{C}_p} \) that extends to \( \mathbb{B}^+ \rightarrow \mathbb{C}_p \).

If we let \( H := \text{Gal}(\overline{K}/K_\infty) \), we then can prove the following theorem:

**Theorem 2.** We have the following equivalences of categories:

\[
\{ \text{\( \phi \)-representations of} \ H \} \leftrightarrow \{ \text{\( \psi \)-modules over} \ E_{\mathbb{Q}_p} \} ; \\
\{ \text{free} \ \mathbb{Z}_p\text{-representations of} \ H \} \leftrightarrow \{ \text{\( \psi \)-modules over} \ A_{\mathbb{Q}_p} \} ; \\
\{ \mathbb{Q}_p\text{-representations of} \ H \} \leftrightarrow \{ \text{étale} \ \phi\text{-modules over} \ B_{\mathbb{Q}_p} \} .
\]

It could be interesting to compare these categories for different \( \phi \) or different \( H \).

In the case of a Galois extension, we can also define a theory of (\( \phi, \Gamma \))-modules based on these constructions, but this is a whole new story...

**Références**


