Advanced Lecture Course
Lecture 3: Modulo $p$ Galois representations and beyond

2010 Program for Women and Mathematics

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We have seen in the previous lecture that Colmez-Fontaine equivalence of categories between crystalline representations and admissible filtered $\varphi$-modules gives an explicit classification of crystalline representations of $G_{Q_p}$, and then an explicit $p$-adic local Langlands correspondence in dimension 2.

To take into account all the $p$-adic representations of $G_{Q_p}$, we have to consider a new family of objects called $(\varphi, \Gamma)$-modules. We will only describe here the ideas involved in this theory and its main steps. For further details and proofs, we refer to [Be, Section 9], [BeBr, Section 3] and to M.-F. Vignéras lectures during the next week.
1 What you need to know / Prerequisites

- The classification of 2-dimensional modulo $p$ Galois representations
  \[ \rho : G_{\mathbb{Q}_p} \rightarrow GL_2(\mathbb{F}_p) \]
  will be recalled in the next section and is constructed in Advanced Homework Session 1.

- We have the following result, which is proved in Advanced Homework Sessions 3 and 4:

  **Theorem 1.** Assume that $E$ is endowed with the discrete topology.
  \begin{enumerate}
    \item $H^1(H_{\mathbb{Q}_p}, GL_d(E)) = \{1\}$ for any integer $d \geq 1$.
    \item $H^1(H_{\mathbb{Q}_p}, E) = \{0\}$.
  \end{enumerate}

  Basically, this theorem says that any free $E$-module of rank $d$ endowed with a semi-linear action of $H_{\mathbb{Q}_p}$ is isomorphic to $E^d$. We refer to Advanced Homework Session 3 for more details.

2 Modulo $p$ representations of $G_{\mathbb{Q}_p}$

2.1 The 2-dimensional case

As we mentioned it in the previous section, we have a complete classification of absolutely irreducible representations of $G_{\mathbb{Q}_p}$ on finite fields (Advanced Homework Session 1):

**Theorem 2.** Let $E$ be a finite extension of $\mathbb{Q}_p$ with residue field $k_E$. Any 2-dimensional $k_E$-linear absolutely irreducible representation of $G_{\mathbb{Q}_p}$ is isomorphic to $\rho(r, \chi) = \text{ind}(\omega_2^{r+1}) \otimes \chi$ for some integer $0 \leq r \leq p-1$ and some character $\chi : G_{\mathbb{Q}_p} \rightarrow k_E^\times$.

Here we denote by $\omega_2 : I_{\mathbb{Q}_p} \rightarrow \mathbb{F}_p^\times$ the character sending $y$ to $g \frac{p^{1/2}-1}{p^{1/2}}$ and $\text{ind}(\omega_2^r)$ is defined as the unique semi-simple representation $\rho$ of dimension 2 whose determinant is equal to $\omega_2^r$ and such that $\rho|_{I_{\mathbb{Q}_p}} = \omega_2^r \otimes \omega_2^{-r}$.

The problem is that this classification offers a too naive point of view to be generalized. To become less naive, we have to introduce a new category of objects called $(\varphi, \Gamma)$-modules.

2.2 $(\varphi, \Gamma)$-modules in characteristic $p$

First recall that $\tilde{E}$ is (non-canonically) isomorphic to the algebraic closure of $E_{\mathbb{Q}_p} := \mathbb{F}_p((\epsilon - 1))$. We denote by $E$ the separable closure of $E_{\mathbb{Q}_p}$, which is not equal to its algebraic closure! The following result, which is a particular case of a powerful theorem due to Fontaine-Wintenberger [FW], will be proved in Advanced Homework Session 3:
Theorem 3. There exists a group isomorphism:
\[ H_{Q_p} \simeq \text{Gal}(E/E_{Q_p}). \]
Denote by \( \Gamma = \Gamma_{Q_p} \) the Galois group of \( Q_p^\infty/Q_p \). A \((\phi, \Gamma)\)-module over \( E_{Q_p} \) is a free \( E_{Q_p} \)-module of finite rank \( d \) endowed with a semi-linear Frobenius map \( \phi \) such that \( \text{Mat}(\phi) \in GL_d(E_{Q_p}) \) and a continuous semi-linear action of \( \Gamma \) that commutes with \( \phi \).

We do the two following remarks:

1. First note that the condition on \( \text{Mat}(\phi) \) does not depend on the basis in which the matrix of \( \phi \) is considered.
2. If we choose a basis \( e \) of \( D \), an element \( \gamma \in \Gamma \) and set \( P := \text{Mat}_e(\phi) \) and \( G := \text{Mat}_e(\gamma) \), then requiring that \( \phi \) and \( \gamma \) commute as semi-linear operators is equivalent to require that \( P\phi(G) = G\gamma(P) \).

Assuming the fact that \( E^{H_{Q_p}} = E_{Q_p} \) (Advanced Homework Session 3), we can prove the following result:

**Proposition 1.** Let \( W \) be an \( \mathbb{F}_p \)-representation of \( G_{Q_p} \) of dimension \( d \). Then \( D(W) := (E \otimes_{E_{Q_p}} W)^{H_{Q_p}} \) is a \((\phi, \Gamma)\)-module over \( E_{Q_p} \) of dimension \( d \) such that
\[ E \otimes_{E_{Q_p}} D(W) \simeq E \otimes_{\mathbb{F}_p} W. \]

In particular, we have
\[ W = (E \otimes_{E_{Q_p}} D(W))^{\phi=1}. \]

**Démonstration.** Let \( W : G_{Q_p} \rightarrow GL_d(\mathbb{F}_p) \) be such a representation. Its restriction to \( H_{Q_p} \), defines an element \([W \otimes_{\mathbb{F}_p} E]\) of \( H^1(H_{Q_p}, GL_d(E)) \), which is trivial by the first point of Theorem 1. This means that \( W \otimes_{\mathbb{F}_p} E \) is isomorphic to \( E^d \) as an \( H_{Q_p} \)-representation, so that \( D(W) := (E \otimes_{\mathbb{F}_p} W)^{H_{Q_p}} \) is an \( E_{Q_p} \)-vector space of dimension \( d \) which is stable under \( \phi \) and \( \Gamma \). Moreover, we have an isomorphism of \( E_{Q_p} \)-vector spaces (but not of \((\phi, \Gamma)\)-modules):
\[ D(W) \simeq E^d_{Q_p} \]
so that \( (E \otimes_{E_{Q_p}} D(W))^{\phi=1} = W. \)

**Proposition 2.** Let \( D \) be a \((\phi, \Gamma)\)-module of rank \( d \) over \( E_{Q_p} \). Then \( W(D) := (E \otimes_{E_{Q_p}} D)^{\phi=1} \) is an \( \mathbb{F}_p \)-representation of \( G_{Q_p} \) of dimension \( d \) such that
\[ E \otimes_{\mathbb{F}_p} W(D) \simeq E \otimes_{E_{Q_p}} D. \]

**Démonstration.** [Be, Proposition 9.1.5]

**Corollaire 1.** The map \([W \mapsto D(W)]\) defines an equivalence of categories:
\[ \{ \mathbb{F}_p \text{-linear representations of } G_{Q_p} \} \leftrightarrow \{ (\phi, \Gamma) \text{-modules over } E_{Q_p} \}. \]

**Démonstration.** [Be, Theorem 9.1.8]
To finish this section, note that if we forget about the action of $\Gamma$, we then get the following result:

**Corollaire 2.** There exists an equivalence of categories:

$$\{\mathbb{F}_p\text{-linear representations of } H_{Q_p}\} \leftrightarrow \{\varphi\text{-modules over } E_{Q_p}\}.$$ 

### 3 Application to a mod $p$ Langlands correspondence for $n = 2$

We just saw how to go from Galois representations to $(\varphi, \Gamma)$-modules. We now introduce an operator $\psi$ on these $(\varphi, \Gamma)$-modules in order to define (as Colmez did) a representation of the Borel subgroup $B_2(Q_p) \subset GL_2(Q_p)$.

#### 3.1 The operator $\psi$

Recall that $E_{Q_p} := \mathbb{F}_p((\epsilon - 1))$ is a vector space over $\varphi(E_{Q_p}) = \mathbb{F}_p((\varphi(\epsilon - 1)))$ which admits $\{1, \epsilon, \ldots, \epsilon^{p-1}\}$ as a basis. Any $\alpha \in E_{Q_p}$ can therefore be uniquely written as

$$\alpha = \sum_{j=0}^{p-1} \epsilon^j \alpha_j$$

with $\alpha_j \in \varphi(E_{Q_p})$. We set $\psi(\alpha) := \alpha_0$.

Let now $D$ be a $(\varphi, \Gamma)$-module over $E_{Q_p}$; then $D$ admits a basis $(\varphi(e_1), \ldots, \varphi(e_d))$ made from elements of $\varphi(D)$, so that any $x \in D$ can be uniquely written

$$x = \sum_{j=1}^{d} x_j \varphi(e_j)$$

with $x_j \in E_{Q_p}$. We set $\psi(x) := \sum_{j=1}^{d} \psi(x_j)e_j$.

**Lemma 1.** The map $\psi : D \to D$ defined just above doesn’t depend on the choice of the basis $(\varphi(e_1), \ldots, \varphi(e_d))$ and commutes to the action of $\Gamma$.

#### 3.2 Main steps to a mod $p$ Langlands correspondence

The construction of a $GL_2(Q_p)$-representation starting from a $(\varphi, \Gamma)$-module attached to a $G_{Q_p}$-representation, known as Colmez’ functor, splits into three main steps:

**1st step:** Let $D$ be a $(\varphi, \Gamma)$-module. There exists some $\psi$-stable lattice $N$ in $D$, and we let $(\lim_{\psi} D)^{\dagger}$ be the set of elements $x = (x_n)_{n \in \mathbb{N}} \in D^\mathbb{N}$ satisfying the two following conditions:

$$\begin{cases}
\forall n \in \mathbb{N}, \psi(x_{n+1}) = x_n ; \\
\exists k \in \mathbb{N} \mid \forall n \in \mathbb{N}, x_n \in \pi^{-k}N .
\end{cases}$$

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2nd step: We set \( D^\sharp := \{ x_0, x \in (\lim_{\to \psi} D)^b \} \). One can prove that \( D^\sharp \) is stable under \( \psi \) and \( \Gamma \), and that \( \psi : D^\sharp \to D^\sharp \) is surjective. We can also build \( \lim_{\to \psi} D^\sharp \) starting from \( D^\sharp \) as we built \( (\lim_{\to \psi} D)^b \) starting from \( D \).

3rd step: Let \( \chi : \mathbb{Q}^\times \to E^\times \) be a smooth character. We endow \( \lim_{\to \psi} D^\sharp \) with an action of the Borel subgroup \( B_2(\mathbb{Q}_p) \) as follows:

\[
\begin{cases}
( \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} x)_n = \chi^{-1}(t)x_n; & \left( \begin{pmatrix} 1 & 0 \\ 0 & p^l \end{pmatrix} x \right)_n = x_{n-j} = \psi^j(x_n) \\
( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} x)_n = \gamma_a^{-1}(x_n); & \left( \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} x \right)_n = \pi^{p^n}x_n.
\end{cases}
\]

Here \( x \in \lim_{\to \psi} D^\sharp, a \in \mathbb{Z}_p^\times \) and \( z \in \mathbb{Q}_p \). Moreover, we set \( \gamma_{a^{-1}} \), the element of \( \Gamma \) such that \( \chi_{\text{cycl}}(\gamma_{a^{-1}}) = a^{-1} \) (where \( \chi_{\text{cycl}} \) denotes the cyclotomic character).

The key point of this construction is that if \( D \) is the \((\varphi, \Gamma)\)-module associated to an absolutely irreducible representation of \( G_{\mathbb{Q}_p} \), then this representation of \( B_2(\mathbb{Q}_p) \) lifts in a unique way to an irreducible representation of \( GL_2(\mathbb{Q}_p) \).

4 First steps in characteristic 0

As usual, we want to lift to characteristic 0 what we have done in characteristic \( p \). To do this, we will (as usual) introduce some new rings of period.

Let \( \hat{A} := W(E) \) be the ring of Witt vectors with coefficients in \( E \). Denote by \( A_{\mathbb{Q}_p} \) the \( p \)-adic completion of \( \mathbb{Z}_p[[\pi]][\frac{1}{\pi}] \) inside \( \hat{A} \) and set \( B_{\mathbb{Q}_p} := A_{\mathbb{Q}_p}[\frac{1}{p}] \): this is a local field with residue field equal to \( E_{\mathbb{Q}_p} \).

Let \( \hat{B} := A[\frac{1}{p}] \) and denote by \( B \) the \( p \)-adic completion of the maximal unramified extension of \( B_{\mathbb{Q}_p} \) inside \( \hat{B} \). Finally set \( A := \hat{A} \cap B \).

We then set the following definitions:

- A \((\varphi, \Gamma)\)-module over \( A_{\mathbb{Q}_p} \) is a free \( A_{\mathbb{Q}_p} \)-module \( D \) of finite rank \( d \) equipped with a semi-linear Frobenius \( \varphi \) such that \( \text{Mat}(\varphi) \in GL_d(A_{\mathbb{Q}_p}) \) and a continuous semi-linear action of \( \Gamma \) which commutes to \( \varphi \).

- A \((\varphi, \Gamma)\)-module over \( B_{\mathbb{Q}_p} \) is a free \( B_{\mathbb{Q}_p} \)-module \( D \) of finite rank \( d \) equipped with a semi-linear Frobenius \( \varphi \) such that \( \text{Mat}(\varphi) \in GL_d(B_{\mathbb{Q}_p}) \) and a continuous semi-linear action of \( \Gamma \) which commutes to \( \varphi \).

- We say that a \((\varphi, \Gamma)\)-module over \( B_{\mathbb{Q}_p} \) is étale if there exists a basis \( e \) of \( D \) such that \( \text{Mat}_e(\varphi) \in GL_d(A_{\mathbb{Q}_p}) \).

As in characteristic \( p \), we have \( H^1(H_{\mathbb{Q}_p}, GL_d(A)) = \{1\} \) if \( A \) is endowed with the \( p \)-adic topology (Advanced Homework Session 4). This leads to the following results:
Lemma 2. Let $T$ be a free $\mathbb{Z}_p$-module of finite rank $d$ endowed with a continuous action of $G_{\mathbb{Q}_p}$. Then $D(T) := (A \otimes_{\mathbb{Z}_p} T)^{H_{\mathbb{Q}_p}}$ is a $(\varphi, \Gamma)$-module of rank $d$ over $A_{\mathbb{Q}_p}$ and it satisfies:

$$A \otimes_{A_{\mathbb{Q}_p}} D(T) \simeq A \otimes_{\mathbb{Z}_p} T.$$ 

In particular, we have:

$$(A \otimes_{A_{\mathbb{Q}_p}} D(T))^{\varphi = 1} = T.$$ 

Lemma 3. Let $V$ be a free $\mathbb{Q}_p$-module of finite rank $d$ endowed with a continuous action of $G_{\mathbb{Q}_p}$. Then $D(V) := (B \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$ is an étale $(\varphi, \Gamma)$-module of rank $d$ over $B_{\mathbb{Q}_p}$ and it satisfies:

$$B \otimes_{B_{\mathbb{Q}_p}} D(V) \simeq B \otimes_{\mathbb{Q}_p} V.$$ 

In particular, we have

$$(B \otimes_{B_{\mathbb{Q}_p}} D(V))^{\varphi = 1} = V.$$ 

Theorem 4. The functor $D(.)$ defines equivalences of categories:

$$\{\text{free } \mathbb{Z}_p\text{-representations of } G_{\mathbb{Q}_p}\} \leftrightarrow \{\text{ } (\varphi, \Gamma)\text{-modules over } A_{\mathbb{Q}_p}\}.$$ 

$$\{\mathbb{Q}_p\text{-linear representations of } G_{\mathbb{Q}_p}\} \leftrightarrow \{\text{étale } (\varphi, \Gamma)\text{-modules over } B_{\mathbb{Q}_p}\}.$$ 

Once again, if we forget about the $\Gamma$-action, we have the following result:

Corollaire 3. We have the following equivalences of categories:

$$\{\text{free } \mathbb{Z}_p\text{-representations of } H_{\mathbb{Q}_p}\} \leftrightarrow \{\varphi\text{-modules over } A_{\mathbb{Q}_p}\};$$

$$\{\mathbb{Q}_p\text{-linear representations of } H_{\mathbb{Q}_p}\} \leftrightarrow \{\text{étale } \varphi\text{-modules over } B_{\mathbb{Q}_p}\}.$$ 

It is also possible to define an operator $\psi$ and to make the three-steps construction that has been seen in characteristic $p$, but it is harder to go from $B_2(\mathbb{Q}_p)$-representations to $GL_2(\mathbb{Q}_p)$-representations in the characteristic 0 setting. In fact, one can prove that it works for some big enough family of $p$-adic representations so that we can conclude that it works for any $p$-adic representation by density. For further details, we refer to [C, Section II.3.2].

Références

[Be] L. Berger, Partial notes for the course “Galois representations and $(\varphi, \Gamma)$-modules”, Galois Trimester 2010 at the IHP.

