## • Linear algebra exercises

0. Prove that that for any $n \times n$ matrix $M$ there exists a factorization of the form:

$$M = UDV^*$$

where $U$ and $V$ are $n \times n$ unitary matrices, and $D$ is diagonal.

1. Show that a positive-defenite hermitian $n \times n$ matrix $A$ has a unique positive-definite square root B. (That is $B^2 = A$. Positive-definite means that for all non-zero complex vectors $v$ we have $v^* A v > 0$.)

2. Let $A$ and $B$ be two symmetric matrices with common eigenbasis, show that $AB$ is symmetric.

3. Let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $n \times n$ matrix. Let $A$ and $B$ be two Hermitian matrices. Prove that

$$\lambda_1(A + B) \leq \lambda_1(A) + \lambda_1(B).$$

(Hint: show that $\lambda(A) = \sup_{|v|=1} v^* A v$)

4. Show that for any Hermitian $n \times n$ matrix $A$ we have

$$\lambda_i(A) = \sup_{dim(V)=i} \inf_{v \in V, |v|=1} v^* A v$$

where $V$ ranges over all subspaces of $\mathbf{C}^n$ with the indicated dimension.

5. Show that for any Hermitian $n \times n$ matrix $A$ with top left $n-1 \times n-1$ minor $A_{n-1}$ the following enequality holds

$$\lambda_{i+1} \leq \lambda_1(A_{n-1}) \leq \lambda_i(A_n)$$

for all $1 \leq i \leq n-1$

6. Show that for any two Hermitian $n \times n$ matrices $A$ and $B$ the following enequality holds

$$\sum_{i=1}^{n} |\lambda_i(A) - \lambda_i(B)|^2 \leq tr(A - B)^2$$

(Hint: observe that this inequality means that the maximum over matrices $A$ and $B$ with a given spectrum of the right hand side is achieved when the two matrices have the same basis of eigenvectors and more precisely the $k$-th eigenvector correspond to the $k$-th largest eigenvalues of the matrices. This fact can be shown by induction over the dimension n of the matrices).

## • Catalan numbers

Prove that the number of triangulations of a convex $(n+2)$-gon is equal to the Catalan number $c_n$

## • Relaxation of the assumptions on the moments of the entries

Let $A^N$ be a sequence of Hermitian matrices with iid entries such that

(1) $\mathbb{E}[A^N_{ij}] = 0$, $1 \leq i, j \leq N$

(2) $\mathbb{E}[(A^N_{ij})^2] = \frac{1}{N}$

Then

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left((A^N)^k\right) = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ C_{\frac{k}{2}}, & \text{otherwise,} \end{cases}$$

where the convergence holds in expectation and almost surely.
(Hint: fix a costant $C$ and consider the approximation of $A^N$ by

$$\hat{A}^N_{ij} = \frac{A^N_{ij} \mathbf{1}_{\sqrt{N}|A^N_{ij}|\le C} - \mathbb{E}[A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\le C}]}{\sqrt{\mathbb{E}[A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\le C} - \mathbb{E}[A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\le C}]]}}.$$

Use the enequality from the exercise 6. )

- **Band matrices**

Take $X_{N,L}$ an Hermitian $N\times N$ matrix such that $X_{N,L}(ij)$ vanishes for $|i-j| \ge L$ and otherwise $X_{N,L}(ij), i \le j \le i+L$ are independent, equidistributed random variables which are centered and with covariance $(2L)^{-1}$. Assume that the $\sqrt{L}X_{N,L}$ have uniformly bounded moments, independent of $L, N$ Show that

$$\lim_{L\to\infty} \lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\operatorname{Tr}(X^k_{N,L})] = C_{k/2}$$

where $C_{k/2} = 0$ if $k$ is odd and otherwise is the Catalan number.

Hint: expand the expectation in terms of the entries, as in the course consider the graph associated with the indices and show that the main contribution is given by indices such that these graphs are trees.

- **Wishart matrices** We consider a $N \times M$ matrix $X_{N,M}$ with independent and equidistributed real entries which are centered and with variance $N^{-1}$. We assume that

$$\sup_N \max_{i,j} \mathbb{E}[(\sqrt{N}X_{N,M}(ij))^{2k}] < \infty \qquad \forall k \ge 0$$

We let $Y_{N,M}$ be the $M \times M$ matrix

$$Y_{N,M} = X^*_{N,M} X_{N,M}$$

with $X^*_{N,M}(ij) = X_{N,M}(ji)$.

(1) Assume $M$ stays finite while $N$ goes to infinity. Show that $Y_{N,M}$ converges almost surely towards the identity in the set of $M \times M$ matrices. (Hint: use the law of large numbers)

(2) Assume that $M$ goes to infinity while $M/N$ goes to zero. Show that for all bounded continuous function $f$,

$$\frac{1}{M}\sum_{i=1}^{M} f(\lambda_i) - f(1)$$

goes to zero in probability. Hint: show that if $I_M(ij) = 1_{i=j}, 1 \le i, j \le M$,

$$\lim_{M\to\infty} \mathbb{E}[\frac{1}{M}\operatorname{tr}(Y_{N,M} - I_M)^2] = 0.$$

(3) Assume $M, N$ goes to infinity so that $M/N$ goes to one.
-Show that for all $k \ge 0$,

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\operatorname{Tr}(Y^k_{N,M})] = C_k$$

with $C_k$ the Catalan number, that is the number of non-crossing pair partition of $2k$ points.

Hint: generalize the proof of Wigner's theorem: say why the limit is the same.

-Deduce that for all bounded continuous function

$$\lim_{N\to\infty}\mathbb{E}[\frac{1}{M}\sum_{i=1}^{M}f(\lambda_i)] = \frac{1}{2\pi}\int_0^4 f(x)\frac{\sqrt{4-x}}{\sqrt{x}}dx$$

(4) Assume $M, N$ goes to infinity so that $M/N$ goes to $\alpha \in (0,1)$.

We want to show that

$$\lim_{N\to\infty}\mathbb{E}[\frac{1}{M}\mathrm{Tr}(Y_{N,M}^r)] = \sum_{k=0}^{r-1}\alpha^k C_{k,r}$$

where $C_{k,r}$ counts certain subsets of the non-crossing pair partitions of $2r$ points. More precisely, consider $2r$ points on the line and checkerboard color the intervals between points by red or blue, starting with blue the interval connected to $-\infty$. Color in blue the faces between the real line and the non-crossing partitions with first interval on the real line which is blue in a blue interval. $C_{k,r}$ is the number of non-crossing pair partitions of $2r$ points with $k+1$ blue faces.

Hint: -Write the expectation

$$m_N(r) := \mathbb{E}[\frac{1}{M}\mathrm{Tr}(Y_{N,M})^r] = \sum_{\substack{1\le i_{2j+1}\le M\\ 1\le i_{2j}\le N}}\mathbb{E}[X_{N,M}^*(i_1i_2)X_{N,M}(i_2i_3)\cdots X_{N,M}(i_{2r}i_1)]$$

and consider the graph $G_i = (V_i, E_i)$ with vertices $V_i = (i_\ell)_{1\le\ell\le 2r}$ and edges $E_i = (i_ki_{k+1}, k \le 2r-1, (i_{2r}i_1))$ as in the course. Show that the leading contribution to $m_N(r)$ corresponds to the case where the skeleton of $G_i$ is a tree. Consider the coloring of the associated non-crossing partition defined above and show that each non crossing pair-partitions with $k+1$, $1 \le k \le r-1$, blue faces will correspond to $M^{k+1}N^{r-k}$ choices of indices. Conclude.

• **CLT for Heavy tailed matrices** Let $X_N = X_N^*$ be a symmetric $N\times N$ matrix with i.i.d centered entries such that for all $k \ge 0$

$$\lim_{N\to\infty} N\mathbb{E}[X_N(ij)^{2k}] = D_k$$

with $\sum_{k\ge 2} D_k > 0$.

Show that

$$\sigma_k := \lim_{N\to\infty}\mathbb{E}[\frac{1}{N}(\mathrm{tr}(X_N^k) - \mathbb{E}[\mathrm{tr}(X_N^k)])^2] = \sum_{\substack{G,G'\in\mathcal{T}_k\\ E\cap E'\neq\emptyset}}\sum_{\substack{P\in P_k(G)\\ P'\in P_k(G')}}\prod_{e\in E\cup E'} D_{m_P(e)+m_{P'}(e)}$$

where the sum runs over

-$G, G'$ are rooted trees with at most $k/2$ edges (that is live in $\mathcal{T}_k$), who share an edge [that is $E\cap E' \neq \emptyset$] (Trees here have edges with multiplicity 1).

-$P, P'$ are closed paths on $G$ and $G'$ respectively, with length equal to $k$, starting at the root.

-2$(m_P(e) + m_{P'}(e))$ is the number of times that the path $P$ or $P'$ went through the edge $e$ (counted as undirected).

Compare to the case of Wigner matrices whose entries have moments of order $k$ of order $\sqrt{N}^{-k}$.