SPARSITY:
GENERALIZED SPARSITY MEASURES
AND APPLICATIONS

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Generalized Sparsity Measures
**Total Variation**

- Sometimes an image’s gradient is sparse.
- There is not a good orthonormal basis representation of this.
- Total variation approximately measures the image gradient.
**Total Variation**

\[ \hat{f} = \arg \min_f \|y - Rf\|_2^2 + \tau \|f\|_{TV} \]

where

\[ \|f\|_{TV} \triangleq \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2-1} |f_{i_1,i_2} - f_{i_1+1,i_2}| + |f_{i_1,i_2} - f_{i_1,i_2+1}| \]

or

\[ \|f\|_{TV} \triangleq \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2-1} \sqrt{|f_{i_1,i_2} - f_{i_1+1,i_2}|^2 + |f_{i_1,i_2} - f_{i_1,i_2+1}|^2} \]

This approach produces state-of-the-art results in many settings, but there is relatively less theoretical support.

Rudin, Osher, Fatemi, 1992
Until now, we have focused exclusively on sparsity, but have not assumed any additional structure.

However, we often understand more about structure in signals and images than sparsity alone…

Baraniuk, Cevher, Duarte & Hegde, 2010
STRUCTURED SPARSITY

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STRUCTURED SPARSITY

Baraniuk, Cevher, Duarte & Hegde, 2010
**Structured Sparsity Results**

Original image

Standard CoSamP
PSNR = 19.9dB

Structure-aware CoSamP
PSNR = 26.8dB

Baraniuk, Cevher, Duarte & Hegde, 2010
This rich image is not especially sparse in “usual” bases such as a wavelet basis
It does contain quite a bit of structure, however. In particular, many “patches” of pixels appear repeatedly throughout the image…
We can think of these patches as lying on a low-dimensional submanifold…
What if we could find a basis (or, more generally, dictionary) for patches so that all the patches are sparse in that basis?
DICTIONARY LEARNING

Original

Noisy

Dictionary elements
**Dictionary Learning**

First we extract all overlapping patches, \( \{x_i\}_{i=1}^{N} \). Next we solve a matrix factorization problem:

\[
\min_{\alpha_i, D} \sum_{i=1}^{N} \|x_i - D\alpha_i\|_2^2 + \tau \|\alpha_i\|_1
\]

- \( x_i \) is the \( i^{th} \) noisy patch
- \( D \) is the dictionary
- \( \alpha_i \) is the vector of dictionary coefficients for patch \( i \)

Once we have the dictionary \( D \), we can denoise each patch in the dictionary (\( \hat{x}_i = D\alpha_i \)) and re-form the denoised image.

Elad & Aharon, 2006
DENOISING WITH LEARNING DICTIONARY

Wavelet, SNR=23.33dB

Learning, SNR=24.73dB
Dictionary Learning

Denoising result

Mairal, Bach, Ponce, Sapiro & Zisserman, 2009
Dictionary Learning

Image completion example

Mairal, Sapiro & Elad, 2008
Compressive Coded Aperture Imaging
**Key Challenges**

- Fast methods to compute $\hat{f}$
- Good sparsity models for “natural” images
  - Images with boundaries
  - Images with texture
  - Hyperspectral images
- Incorporating real-world constraints
  - Photon noise
  - Non-negative intensities
  - Quantization effects
- Practical systems to measure compressed sensing data
**Challenge: Build Imaging Systems that Exploit CS Theory**

- Projection ($A$) with randomly-drawn elements **not realizable** in most optical systems:
  - Light intensities cannot be **negative**
  - Simultaneous projections require **complex** systems
  - Individual random projections taken at each time step not suitable for **dynamic** scenes
- Verifying the RIP for a particular matrix computationally **intractable**
Coded Apertures

- Simple to build and to incorporate into practical, compact optical designs
- RIP satisfied with high probability using CS theory for Toeplitz matrices
- Weaker theoretical guarantees
APERTURES

Signal
Small pinholes allow little light ⇒ dark observations.
Larger pinholes allow more light but leads to decrease in resolution ⇒ blurry observations.
Multiple small pinholes $\Rightarrow$ overlapping observations.
Coded Aperture Imaging (Modified Uniformly Redundant Arrays)

Resolution restricted to size of detector

Gottesman and Fenimore (1989)
COMPRESSIVE CODED APERTURE IMAGING

Observation model:

\[ x = \underbrace{D(f \ast p)}_{\text{Downsampling Operator}} + \epsilon \]

- Signal
- Coded aperture (pattern of small pinholes)
- Low-resolution coded observations
- Reconstruction

Downsampling Operator  Signal  Coded Aperture  Noise
The sensing matrix $A_p$ is block circulant with circulant blocks:

**Theorem:** A mask pattern $p$ can be designed such that the resulting projection matrix $A_p$ satisfies a (weakened) RIP with high probability.

Marcia and Willett (2008a)
Romberg (2009)
COMPRESSIVE CODED APERTURE: VIDEO EXAMPLE

Ground truth

Uncoded observation (1/16 as many pixels)

Coded observation (1/16 as many pixels)

Reconstruction

CS Reconstruction
RECONSTRUCTIONS OF THE 25TH FRAME

Original Scene  Downsampled  Reconstruction from coded observation
RECONSTRUCTED VIDEO (FROM 2-FRAME METHOD)
Reconstructions of the 25th frame
**RECONSTRUCTED VIDEO (FROM 2-FRAME METHOD)**

Original Scene

Original Scene  |  Downsampling  |  Reconstruction from coded observation
RECONSTRUCTIONS OF THE 25\(^{\text{th}}\) FRAME

Original Scene

Original Scene  Downsamped  Reconstruction from coded observation
RECONSTRUCTED VIDEO (FROM 2-FRAME METHOD)
Reconstructions of the 25th frame

Original Scene

Downsampled

Reconstruction from coded observation
Concluding remarks
Sparsity and Computing

- Sparsity plays a critical role in processing high-dimensional data
  - It increases our robustness to noise
  - It facilitates efficient storage and transmission of data
  - It allows us to fill in values of missing data
  - It helps us circumvent the curse of dimensionality and achieve accurate prediction performance

- Computational methods which exploit sparsity are
  - Fast
  - Sophisticated
  - Fun!
Shameless plugs
Duke Workshop on Sensing and Analysis of High-Dimensional Data (SAHD)

The Duke University Workshop on Sensing and Analysis of High-Dimensional Data is planned for July 26-28, 2011. The meeting will be held at the Washington Duke Inn and Golf Club, adjacent to the Duke campus. The meeting is being organized and hosted by the following Duke faculty: David Brady, Robert Calderbank, Lawrence Carin, Ingrid Daubechies, David Dunson, Mauro Maggioni and Rebecca Willett.

The meeting is co-sponsored by AFOSR, AFRL, ARO, DARPA, NGA and ONR.
I hire students and postdocs from many backgrounds!
Have a nice day
Have a nice day