CAYLEY-LIKE REALIZATIONS OF RAMANUJAN BIGRAPHS AND SIMPLY TRANSITIVE ACTIONS OF LATTICES ON BUILDINGS
REPORT

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We are grateful for the support and hospitality of the IAS during the Summer Collaborators Program. It allowed us to work intensely and make considerable progress on our project.

At the beginning of the program we set out to investigate the following questions.

(1) Find an explicit Cayley-like construction of biregular graphs (with two different degrees) and apply it to the Ramanujan bigraphs constructed via arithmetic unitary group in three variables.
(2) Construct several infinite families of Ramanujan and non-Ramanujan bigraphs by analyzing the automorphic spectrum of the unitary group in three variables.
(3) Study the spectrum of different adjacency operators of bigraphs, such as the classical, non-backtracking and geometric, and compare the different notions of Ramanujan bigraphs that exist today. In particular, we show that being Ramanujan in the sense of Hashimoto [H89], is spectrally stronger then being Ramanujan in the sense of Marcus-Spielman-Srivastava [MSS14].
(4) Prove that Ramanujan bigraphs (in the stronger, Hashimoto sense) display a total-variation cutoff phenomenon, which was proven recently by Lubetzky-Peres [LP16] for regular Ramanujan graphs.
(5) Find other simply-transitive p-arithmetic lattices and prove that they also give Ramanujan bigraphs. Some of the simply transitive lattices we found, relay on the construction of simply-transitive 2-arithmetic lattices, given by Mumford [Mum79] and Cartwright-Mantero-Steger-Zappa [CMSZ93].
(6) Consider other combinatorial results for Ramanujan bigraphs, such as the expander mixing lemma.

Most of the work during the two weeks concentrated on points (1), (2), and (5), but we also clarified the direction of future work regarding (3), (4), and (6).

We found the correct definition of bi-Cayley graphs that is analogous to the definition of Cayley graphs in the regular case. A bi-Cayley (K, k)-biregular graph is determined by a group G and a generating set S (symmetric and $1 \notin S$) of size $K(k-1)$, partitioned into K disjoint subsets of equal size $k-1$, which satisfy a specific axiom. We define an equivalence relation on $G \times \{1, 2, \dots, K\}$. The left vertices of the graph correspond to the elements of G and the right vertices correspond to equivalence classes in $G \times \{1, 2, \dots, K\}$. The edges of the graph are $\{(g, [g, i]) \mid g \in G,\ i \in [K]\}$. We also defined bi-Schreier bigraphs in analogy to Schreier graphs. These bi-Cayley and bi-Schreier graphs can be explicitly and efficiently programmed on a computer.

By analyzing the classification of automorphic spectrum of the unitary group in three variables (by Rogawski [R90, R92]), together with the recent advancement toward the generalized Ramanujan conjecture (by Shin [S11]), we were able to give several criteria for proving that the Ramanujan condition is satisfied for bigraphs constructed via arithmetic unitary groups in three variables. These criteria are then applied to the bigraphs constructions coming from the simply transitive lattices of Evra-Parzachevski [EP18] ($w = \sqrt{-1}$ and $r = 2 + 2\sqrt{-1}$) and our recent constructions ($w = \frac{1-\sqrt{-3}}{2}$ and $r = 3$)

$$\left\{ A \in GU_3\left(\mathbb{Z}\left[w, \tfrac{1}{p}\right]/\mathbb{Z}\right) \,\middle|\, A \equiv \left(\begin{smallmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{smallmatrix}\right) \pmod{r} \right\} /\text{center}.$$

We show how to get from these infinte families of $(p^3+1, p+1)$-biregular Ramanujan graphs, where $p \equiv 3 \pmod 4$ in the EP case and $p \equiv 2 \pmod 3$ in the our second case, and these graphs can be realized as bi-Cayley graphs in the first case and bi-Schreier graphs in the second case. We suspect that in the second case, each graph has a bi-Cayley double cover which is not Ramanujan.

We are also able to construct infinite families of $(p^3+1, p+1)$-biregular Ramanujan graphs and simply-transitive p-arithmetic lattices from the simply transitive 2-arithmetic lattices introduced by Mumford [Mum79] and Cartwright-Mantero-Steger-Zappa [CMSZ93]. By studying the Mumford construction, we were able to answer a question we raised earlier, namely whether in order to construct a simply transitive lattice one always has to take a congruence condition at a ramified prime. The answer is negative. Using the work of CMSZ, we were also able to construct Ramanujan bigraphs and simply-transitive p-arithmetic lattices, defined via the quadratic imaginary field $\mathbb{Q}(\sqrt{-15})$, which is not of class number one, while the resulting unitary group $U_3(\mathbb{Q}(\sqrt{-15})/\mathbb{Q})$ is of class number one. Finally, the work of CMSZ, also yields simply-transitive arithmetic lattices which come from a division algebra. These give rise to further infinite families of Ramanujan bigraphs, since Rogawski showed that no A-packets occur in this case and thus the Ramanujan property holds by the work of Shin.

We intend to finish the project and submit an article for publication by the fall.

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