AUTOMORPHIC FUNCTIONS, DIFFERENTIAL EQUATIONS, AND MODEL THEORY

DAVID BLÁZQUEZ-SANZ, GUY CASALE, JAMES FREITAG, JOEL NAGLOO

The four authors spend July 21-July 31 at IAS, collaborating on several projects at the intersection of number theory, differential galois theory and number theory. Before explaining what we accomplished at IAS, we will describe the general setting. Let X and Y be algebraic varieties over $\mathbb{C}$ and let $\phi : X^{an} \rightarrow Y^{an}$ be a complex analytic map (which is not algebraic). Then for most algebraic subvarieties $X_0 \subset X$, the image $\phi(X_0)$ is *not* algebraic. The pairs of algebraic subvarieties (X_0, Y_0) with $X_0 \subset X$ and $Y_0 \subset Y$ such that $\phi(X_0) = \phi(Y_0)$ are called *bi-algebraic* for ϕ. Bi-algebraic subvarieties should be rare and revealing of important geometric aspects of the analytic map ϕ.

The condition that X is an algebraic variety is slightly too restrictive for many of the specific interesting examples both here and in the literature, and so generally we will allow X to be an *o-minimally* definable open subset of an algebraic variety.[1] Then an algebraic subvariety of X is a set given by the vanishing of a finite system of polynomial equations on the open set. We will be especially interested in the case that X is the universal cover of Y, where open domains such as $\mathbb{H}$, the complex upper half-plane arise naturally. In fact, though it does not play a significant role in this paper, the maps ϕ which we consider are also o-minimally definable, when restricted to an appropriate fundamental domain. Recent approaches to the problem have relied on o-minimal methods, but our approach, started in (1) is much different. During the IAS visit, the first goal we tackled was generalizing our previous answer to the bi-algebraicity problem, which held only the case that Γ $\mathbb{H}$ is genus zero, to the general case.

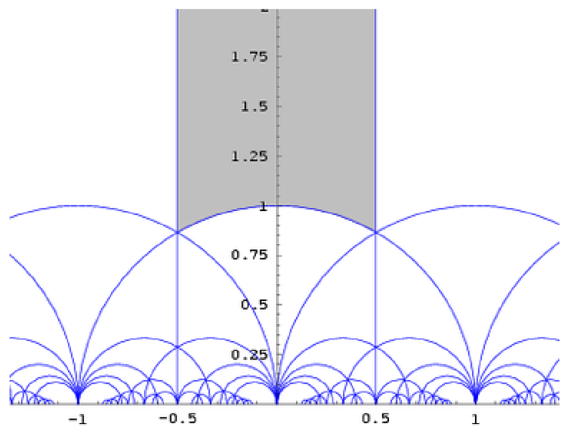

Figure 1. A specific example: the analytic j-function has fundamental domain given by the grey shaded area and is automorphic for $PSL_2(\mathbb{Z})$. The blue lines and semicircles are the geodesics for the hyperbolic metric on $\mathbb{H}$, and every "triangular" region bounded by three of these geodesics is a translate F under $\Gamma = PSL_2(\mathbb{Z})$. *Triangular groups* are a special class of the groups we studied - a case where we obtained a series of very detailed results over the IAS visit, which we are currently writing a manuscript about. We have extended some of these results to the general case.

From there we focused on two lines of results. First, we pursued detailed functional transcendence results for specific classes of Fuchsian groups (we studied triangle groups intensely and have preliminary results for a few other families as well). These results were important because they suggested to us conjectures about general Fuchsian groups which we are actively pursuing. We also had the detailed structure theory of Takeuchi,

[1]There are additional situations in which this condition is too restrictive - see (4, Remark 4.4).

which played a key role (12; 13). We expect to write up these results in a manuscript over the next few weeks.

The second main project we pursued was a so-called Ax-Schanuel type theorem for the covering maps of arbitrary Fuchsian groups of the first kind, a generalization of the work of Pila and Tsimmerman (). We won't formally state the result here, but the Ax-Schanuel result generalizes the bi-algebraicity problem above. It requires one to understand the *analytic* subvarieties of the domain which map to algebraic subvarieties under the analytic map ϕ. Our proof relied on the solution to the bi-algebraicity problem we solved earlier in the visit, and we first tackled the special case of triangle groups, but later realized we could later generalize each part of our proof. We are actively writing this manuscript as well, but the writeup is more involved with a long argument using tools from differential geometry, model theory and number theory.

Our main goals moving forward are to expand to higher-dimensional groups (beyond SL_2). We found that getting together physically at IAS really turbocharged our collaboration - the balance of the group was especially important here. Getting four co-authors together for almost two weeks of research would be extremely challenging without institutional support, so this IAS experience was extremely valuable for us. Casale and Blázquez-Sanz have a long history of collaboration as do Freitag and Nagloo. Having two pairs with similar backgrounds was really optimal for us.

References

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