Ramsey Theory for Metric Spaces

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Abstract

Ultrametrics are special metrics satisfying a strong form of the triangle inequality: For every \( x, y, z \), \( d(x, z) \leq \max\{d(x, y), d(y, z)\} \). We consider Ramsey-type problems for metric spaces of the following flavor:

Every metric space contains a “large” subset having approximate ultrametric structure.

The following theorem implies a variety of Ramsey-type theorems for compact metric spaces with different notions of “size”:

For every \( e > 0 \), every compact metric space \( X \) and every probability measure \( \mu \) on \( X \), there exists a subset \( S \) of \( X \) and a probability measure \( \nu \) supported on \( S \) such that \( S \) is an approximate ultrametric upto distortion \( 9/e \), and for every ball \( B(x, r) \) in \( X \), \( \nu(B(x, r)) \leq \mu(B(x, Cr))^{1-e} \), where \( C = C_e > 1 \) depends only on \( e \).

Those Ramsey-type theorems, besides their intrinsic interest, have applications for algorithms (approximate distance oracles, lower bounds for online problems), metric analysis (Lipschitz surjections onto the \( n \)-dimensional cube), and probability (Talagrand’s majorizing measure theorem).