During our three weeks at the IAS we focused on two questions. The first question is the following:

**Problem 1.** Let $D_5$ denote the dihedral group of order 10, consisting of the symmetries of a regular pentagon. What is the asymptotic number of degree 5 field extensions over $\mathbb{Q}$ whose normal closure has Galois group $D_5$, when ordering such fields by discriminant?

Answering this question will have applications towards the $p = 5$ case of the Cohen–Lenstra heuristics [5], as well as Malle’s conjecture [7] for the group $D_5$.

We use a two step strategy towards answering Problem 1: first, parametrize the set of $D_5$-fields via group orbits on a linear space or an affine homogeneous space, and second, count these group orbits via analytic techniques. We made substantial progress towards the first step of the argument while at the IAS, which we summarize below.

Let $K$ denote a $D_5$-quintic field and let $\overline{K}$ denote its Galois closure. The sextic resolvent of $K$ is simply $\mathbb{Q} \oplus K$, and the sextic resolvent map as defined by Bhargava [3] is an alternating map $(\mathbb{Q} \oplus K)^2 \to K$, which recovers the entire multiplication table of $K$. Using a particular linear operator $\delta \in \mathbb{Z}[\text{Gal}(\overline{K}/\mathbb{Q})]$, we give an extremely explicit description of the sextic resolvent map over any base field $k$ containing $\mathbb{Q}(\sqrt{5})$. This allows us to describe a parametrization of $D_5$-quintic extensions of $\mathbb{Q}(\sqrt{5})$ in terms of certain $\text{SL}_2(k) \times \text{SL}_2(k)$-orbits on a $\text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$-invariant subspace of $k^2 \otimes k^2 \otimes (k^2 \oplus k^2)$. Additionally, we show that this subspace can be interpreted as a (reducible) 8-dimensional representation of $\text{SL}_2(k)$, and over $\mathbb{C}$ breaks up as a direct sum of $\text{Sym}^3(\mathbb{C}^2)$ and two copies of $\text{Sym}^1(\mathbb{C}^2)$.

We have obtained computational and theoretical evidence that the projection of a given $\text{SL}_2(k)$-orbit of the subspace to the direct summand $\text{Sym}^2(k)$ should itself be enough to recover the sextic resolvent map, and hence the $D_5$-quintic extension of $k$. We are next working towards proving this. Afterwards, we will generalize these parametrizations from rational orbits to integral orbits. We will then employ and develop analytic methods to count these integral orbits and obtain asymptotics on the number of $D_5$-extensions of $\mathbb{Q}(\sqrt{5})$. Finally, we will be required to descend our argument to also work over $\mathbb{Q}$.

The second question is concerned with Malle’s conjecture and its modification by Klüners and Tückel [8]. Recall that this conjecture predicts the asymptotic number of degree-$d$ number fields whose normal closure over $\mathbb{Q}$ has a fixed Galois group $G$, when these degree-$d G$-fields are ordered by discriminant. In the case of degree-$d S_d$-number fields, Bhargava [2] predicted a mass formula for the constant of growth. Afterwards, Kedlaya [6] described a framework for analogous mass formulae for more general Galois groups $G$. However, as studied by Kedlaya and Wood [9], these predictions for general $G$ have frequently yielded the wrong asymptotics. In particular, the predicted mass formulae do not coincide with the asymptotics obtained by Cohen-Diaz y Diaz-Olivier for the family of $D_4$-fields ordered by discriminant [4]. Nevertheless, in previous joint work, we proved that the prediction is indeed correct for the same family ordered instead by conductor [1].

This suggests that the discriminant is not always the most natural invariant with which to order degree $d G$-fields when $G \neq S_d$. In our second project, we studied families of degree-$d G$-number fields ordered by “virtual” Artin conductors, arising from virtual characters of $G$. During the IAS, we also studied the following question:
**Problem 2.** If one orders degree-$d$ $G$-number fields by a virtual Artin conductor $C(d,G)$, exactly when does a Hardy–Littlewood local-global principle predict the correct asymptotics?

We give a complete conjectural answer to this question for every group $G$ and virtual conductor $C(d,G)$, simply in terms of the linear representation theory of $G$. In the case when $G$ is abelian, our predictions agree with work of Wood [10], and it furthermore describes a larger class of “good” ordering invariants.

In addition, our conjectures agree with known results in the non-abelian cases when $G = S_3$, $S_4$, $S_5$, and $D_4$. We are currently engaged in verifying the conjecture in other known cases as well as collecting further theoretical and numerical evidence towards this conjecture.

We were very fortunate to have three dedicated weeks via the Summer Collaborators program that allowed us to make decisive progress on two important directions of our long-term collaborative project. Our time at the Institute for Advanced Study will result in multiple forthcoming publications answering Problem 1 and studying Problem 2. Overall, our stay at the IAS has contributed greatly to the future success of our joint research program, and we are grateful to the Institute for its support.

### References


