Lecture 2
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The following notes are a rush transcript of Rachel's lecture. Any mistakes or typos are mine, and please point these out so that I can post a corrected copy online.

1 Foliations

Definition 1. A *k-dimensional foliation* of an n-manifold M is a decomposition of M into k-manifolds with the following property: there exists an atlas of charts with the property that each chart neighborhood U locally decomposes as $U = \mathbb{R}^k \times \mathbb{R}^{n-k}$

That is, there exists a a foliation atlas such that for all p, there exists a neigbborhood U containing p with $U \approx \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Definition 2. The k-manifolds of a foliation $\mathcal{F}$ are called the *leaves* of the foliation.

2 Examples

Eventually we'll be particularly interested in the case $n = 3, k = 2$.

1. $M^n = A^k \times B^{n-k}$ where A, B are manifolds.
2. $T^2 = S^1 \times S^1$ can by foliated by circles in one of the factors.
3. Consider the universal cover of the torus, $\mathbb{R}^2$. Foliate the plane by lines of a fixed slope. Under the covering map from $\mathbb{R}^2 \to T^2$,

$$(s, t) \mapsto (e^{2\pi is}, e^{2\pi it})$$

   This gives a foliation of the torus. If the slope of lines in $\mathbb{R}^2$ is rational, then we get a foliation by circles on the torus. (See the last HW problem from the first day.) If the slope is irrational, the curves will never close up; each leaf is a copy of $\mathbb{R}$ which is dense in T^2.
4. Begin with the cylinder $S^1 \times [0, 1]$ where $\phi : S^1 \to S^1$ is a homeomorphism of the circle. Consider a foliation of the cylinder by vertical segments $\{x\} \times [0, 1]$. ϕ can be chosen to yield a foliation by lines, simple closed curves of any slope, or combinations of both.

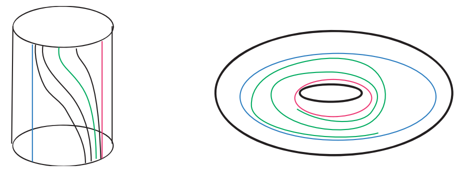

Figure 1: There is a foliation of T^2 with two S^1 leaves and two intervals' worth of $\mathbb{R}$ leaves. A ϕ that produces this foliation is suggested by the diagram on the left, and some example leaves are shown on the right.

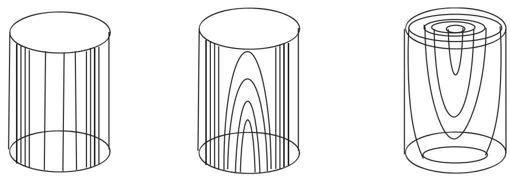

Figure 2: Left: This foliation of $S^1 \times [0, 1]$ yields a foliation of T^2 by circles. .Center: A different foliation of $S^1 \times [0, 1]$. Right: This foliation of the cylinder maps to the Reeb foliation of the torus.

5. As an example of the previous construction, consider a foliation of T^2 with two S^1 leaves and two intervals' worth of $\mathbb{R}$s , each of which "spirals in" on both circles.

   Remark: One can find a simple closed curve γ in T^2 such that γ is everywhere transverse to $\mathcal{F}$ and γ meets every leaf of $\mathcal{F}$. For example, the meridian $S^1 \times \{pt\}$ on the original cylinder becomes such a curve in T^2.

6. Again, begin with a foliation of the cylinder as shown in the middle diagram in Figure 2. In contrast to the example above, there is no simple closed curve on T^2 that meets every leaf of $\mathcal{F}$ transversely.

7. Now consider $n = 3, k = 2$. Begin with $S^1 \times D^2$ and foliate it by a singled boundary torus and an infinite family of "cups" and cylinders as indicated in the right-hand diagram of Figure 2. The foliation meets each end of the cylinder in a collection of nested circles, as well as in a single point of $D^2 \times \{0\}$ where one cup is tangent to the disc. Let ϕ be an orientation-preserving homeomorphism of the disc which sends circles to circles; the quotient space under the identification $(x, \{0\}) \sim (\phi(x), \{1\})$ is a foliation of the solid torus. This is an analog of Example 5; instead of curves spiralling towards S^1 leaves, we see the cups spiralling toward the boundary T^2.

   Definition 3. The foliation of $S^1 \times D^2$ with a single torus leaf and S^1's worth of $\mathbb{R}^2$s is the *Reeb foliation* of the solid torus. (See Figure 2.)

8. $S^3 = (S^1 \times D^2) \cup (S^1 \times D^2)$. Foliating each solid torus with the Reeb foliation gives a two-dimensional foliation of S^3. In more standard language, we have a *codimension-one* foliation of S^3.

Let M be a closed orientable three manifold.

Theorem 1. *Any three manifold M has a codimension-one foliation $\mathcal{F}$.*

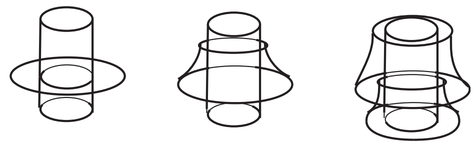

Figure 3: Modifying each fiber in a neigbhorhood of the T^2 yields a foliation of M.

Proof. Recall the theorem of Alexander, Myers, and Gonzalez-Acuña that M can be written as the union of a solid torus and a fibered three manifold M_0 whose boundary is a single torus. (See John Etnyre's notes on open book decompositions for a proof of this fact.) Foliate the $S^1 \times D^2$ with the Reeb foliation. For the fibered piece, $M_0 = F \times [0,1]/(x,0) \sim (\phi(x),1)$, so the fibers give a foliation of M_0. In a neighborhood of the boundary, this looks like $S^1 \times [0,1] \times S^1$, and we modify the fibration locally by spiralling each copy of F about T^2. (See Figure 3.) $\square$

A downside of this theorem is that knowing a three manifold has a codimension-one foliation tells us nothing about the manifold. But there's hope...

Definition 4. A surface F embedded in a three manifold is *incompressible* if for every simple closed curve γ embedded in F which bounds a disc in M, γ also bounds an embedded disc in F. A surface which is not incompressible is called *compressible*.

Definition 5. A surface F is *essential* if $F \neq S^2$ and F is incompressible.

Definition 6. Let $\mathcal{F}$ be a codimension-one foliation of a three manifold. Then $\mathcal{F}$ is *Reebless* if it contains no Reeb component. (I.e., $\mathcal{F}$ has no Reeb foliation of a solid torus.)

Remark: It is equivalent to say that a foliation is Reebless if there is no compressible T^2 leaf of $\mathcal{F}$.

Theorem 2. *(Reeb) Suppose $\mathcal{F}$ is a foliation with an S^2 leaf. Then $M = S^1 \times S^2$.*

Definition 7. M is irreducible if every embedded S^2 bounds a B^3.

Theorem 3. *(Haefliger, Novikov, Rosenberg) Suppose $M \neq S^1 \times S^2$ and M has a Reebless foliation. Then*

1. *M is irreducible;*
2. *The leaves of $\mathcal{F}$ are incompressible;*
3. *If γ is a simple closed curve in M and $\gamma \pitchfork \mathcal{F}$ then $[\gamma] \in \pi_1(M)$ is of infinite order.*

Theorem 4. *(Palmeira) Suppose $M \neq S^1 \times S^2$ and M has a Reebless foliation. Then the universal cover $\tilde{M}$ of M is homeomorphic to $\mathbb{R}^3$. If $\tilde{F}$ is the lift of $\mathcal{F}$ to $\tilde{\mathbb{R}}^2$, $\mathbb{R}^3$ can be decomposed as $\mathbb{R}^2 \times \mathbb{R}$ so that $\tilde{F}$ is a foliation of $\mathbb{R}^2$ by lines.*