The following notes are a rush transcript of Rachel’s lecture. Any mistakes or typos are mine, and please point these out so that I can post a corrected copy online.

1 Foliations

**Definition 1.** A \( k \)-dimensional foliation of an \( n \)-manifold \( M \) is a decomposition of \( M \) into \( k \)-manifolds with the following property: there exists an atlas of charts with the property that each chart neighborhood \( U \) locally decomposes as \( U = \mathbb{R}^k \times \mathbb{R}^{n-k} \).

That is, there exists a a foliation atlas such that for all \( p \), there exists a neighborhood \( U \) containing \( p \) with \( U \cong \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \).

**Definition 2.** The \( k \)-manifolds of a foliation \( \mathcal{F} \) are called the *leaves* of the foliation.

2 Examples

Eventually we’ll be particularly interested in the case \( n = 3, k = 2 \).

1. \( M^n = A^k \times B^{n-k} \) where \( A, B \) are manifolds.
2. \( T^2 = S^1 \times S^1 \) can by foliated by circles in one of the factors.
3. Consider the universal cover of the torus, \( \mathbb{R}^2 \). Foliate the plane by lines of a fixed slope. Under the covering map from \( \mathbb{R}^2 \to T^2 \),

\[
(s, t) \mapsto (e^{2\pi is}, e^{2\pi it})
\]

This gives a foliation of the torus. If the slope of lines in \( \mathbb{R}^2 \) is rational, then we get a foliation by circles on the torus. (See the last HW problem from the first day.) If the slope is irrational, the curves will never close up; each leaf is a copy of \( \mathbb{R} \) which is dense in \( T^2 \).

4. Begin with the cylinder \( S^1 \times [0,1] \) where \( \phi : S^1 \to S^1 \) is a homeomorphism of the circle. Consider a foliation of the cylinder by vertical segments \( \{x\} \times [0,1] \). \( \phi \) can be chosen to yield a foliation by lines, simple closed curves of any slope, or combinations of both.
Figure 1: There is a foliation of $T^2$ with two $S^1$ leaves and two intervals’ worth of $\mathbb{R}$ leaves. A $\phi$ that produces this foliation is suggested by the diagram on the left, and some example leaves are shown on the right.

Figure 2: Left: This foliation of $S^1 \times [0,1]$ yields a foliation of $T^2$ by circles. Center: A different foliation of $S^1 \times [0,1]$. Right: This foliation of the cylinder maps to the Reeb foliation of the torus.

5. As an example of the previous construction, consider a foliation of $T^2$ with two $S^1$ leaves and two intervals’ worth of $\mathbb{R}$s, each of which “spirals in” on both circles.

Remark: One can find a simple closed curve $\gamma$ in $T^2$ such that $\gamma$ is everywhere transverse to $\mathcal{F}$ and $\gamma$ meets every leaf of $\mathcal{F}$. For example, the meridian $S^1 \times \{pt\}$ on the original cylinder becomes such a curve in $T^2$.

6. Again, begin with a foliation of the cylinder as shown in the middle diagram in Figure 2. In contrast to the example above, there is no simple closed curve on $T^2$ that meets every leaf of $\mathcal{F}$ transversely.

7. Now consider $n = 3, k = 2$. Begin with $S^1 \times D^2$ and foliate it by a singled boundary torus and an infinite family of “cups” and cylinders as indicated in the right-hand diagram of Figure 2. The foliation meets each end of the cylinder in a collection of nested circles, as well as in a single point of $D^2 \times \{0\}$ where one cup is tangent to the disc. Let $\phi$ be an orientation-preserving homeomorphism of the disc which sends circles to circles; the quotient space under the identification $(x, \{0\}) \sim (\phi(x), \{1\})$ is a foliation of the solid torus. This is an analog of Example 5; instead of curves spiralling towards $S^1$ leaves, we see the cups spiralling toward the boundary $T^2$.

Definition 3. The foliation of $S^1 \times D^2$ with a single torus leaf and $S^1$’s worth of $\mathbb{R}^2$s is the Reeb foliation of the solid torus. (See Figure 2.)

8. $S^3 = (S^1 \times D^2) \cup (S^1 \times D^2)$. Foliating each solid torus with the Reeb foliation gives a two-dimensional foliation of $S^3$. In more standard language, we have a codimension-one foliation of $S^3$.

Let $M$ be a closed orientable three manifold.

**Theorem 1.** Any three manifold $M$ has a codimension-one foliation $\mathcal{F}$. 

Proof. Recall the theorem of Alexander, Myers, and González-Acuña that $M$ can be written as the union of a solid torus and a fibered three manifold $M_0$ whose boundary is a single torus. (See John Etnyre’s notes on open book decompositions for a proof of this fact.) Foliate the $S^1 \times D^2$ with the Reeb foliation. For the fibered piece, $M_0 = F \times [0,1]/(x,0) \sim (\phi(x),1)$, so the fibers give a foliation of $M_0$. In a neighborhood of the boundary, this looks like $S^1 \times [0,1] \times S^1$, and we modify the fibration locally by spiralling each copy of $F$ about $T^2$. (See Figure 3.)

A downside of this theorem is that knowing a three manifold has a codimension-one foliation tells us nothing about the manifold. But there’s hope...

Definition 4. A surface $F$ embedded in a three manifold is incompressible if for every simple closed curve $\gamma$ embedded in $F$ which bounds a disc in $M$, $\gamma$ also bounds an embedded disc in $F$. A surface which is not incompressible is called compressible.

Definition 5. A surface $F$ is essential if $F \neq S^2$ and $F$ is incompressible.

Definition 6. Let $F$ be a codimension-one foliation of a three manifold. Then $F$ is Reebless if it contains no Reeb component. (I.e., $F$ has no Reeb foliation of a solid torus.)

Remark: It is equivalent to say that a foliation is Reebless if there is no compressible $T^2$ leaf of $F$.

Theorem 2. (Reeb) Suppose $F$ is a foliation with an $S^2$ leaf. Then $M = S^1 \times S^2$.

Definition 7. $M$ is irreducible if every embedded $S^2$ bounds a $B^3$.

Theorem 3. (Haefliger, Novikov, Rosenberg) Suppose $M \neq S^1 \times S^2$ and $M$ has a Reebless foliation. Then

1. $M$ is irreducible;
2. The leaves of $F$ are incompressible;
3. If $\gamma$ is a simple closed curve in $M$ and $\gamma \pitchfork F$ then $[\gamma] \in \pi_1(M)$ is of infinite order.

Theorem 4. (Palmeira) Suppose $M \neq S^1 \times S^2$ and $M$ has a Reebless foliation. Then the universal cover $\widehat{M}$ of $M$ is homeomorphic to $\mathbb{R}^3$. If $\widehat{F}$ is the lift of $F$ to $\mathbb{R}^3$, $\mathbb{R}^3$ can be decomposed as $\mathbb{R}^2 \times \mathbb{R}$ so that $\widehat{F}$ is a foliation of $\mathbb{R}^2$ by lines.