Lecture 1
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The following notes are a rush transcript of Rachel's lecture. Any mistakes or typos are mine, and please point these out so that I can post a corrected copy online.

Outline

1. Three-manifolds: Presentations and basic structures
2. Foliations, especially codimension 1 foliations
3. Non-trivial examples of three-manifolds, generalizations of foliations to laminations

1 Three-manifolds

Unless otherwise noted, let M denote a closed (compact) and orientable three manifold with empty boundary. (Note that these restrictions are largely for convenience.) We'll start by collecting some useful facts about three manifolds.

Definition 1. For our purposes, a *triangulation* of M is a decomposition of M into a finite colleciton of tetrahedra which meet only along shared faces.

Fact 1. *Every M admits a triangulation.*

This allows us to realize any M as three dimensional simplicial complex.

Fact 2. *M admits a C^∞ structure which is unique up to diffeomorphism.*

We'll work in either the C^∞ or PL category, which are equivalent for three manifolds. In partuclar, we'll rule out wild (i.e. pathological) embeddings.

2 Examples

1. S^3 (Of course!)
2. $S^1 \times S^2$
3. More generally, for F a closed orientable surface, $F \times S^1$

Figure 1: A Heegaard diagram for the genus four splitting of S^3.

Figure 2: Left: Genus one Heegaard decomposition of S^3. Right: Genus one Heegaard decomposition of $S^2 \times S^1$.

We'll begin by studying S^3 carefully. If we view S^3 as compactification of $\mathbb{R}^3$, we can take a neighborhood of the origin, which is a solid ball. The complement of this B^3 is also a B^3 –you can think of it as a neighborhood of ∞– and we can view the three sphere as $B^3 \cup B^3$. Alternately, take a solid torus $S^1 \times D^2$ and note that its complement in S^3 is another solid torus:

$$(S^1 \times D^2) \cup_\phi (S^1 \times D^2)$$

Thus S^3 is the quotient space of the disjoint union of these solid tori identified by $x \sim \phi(x)$ where ϕ is a homeomorphism of $T^2 \to T^2$.

3 Four ways to consider three manifolds

1. Heegaard decompositions

   Definition 2. A genus-g *handlebody* is a three manifold with boundary which is homeomorphic to the solid bounded by the standard genus g surface. (The standard genus g surface is the connected sum of g copies of T^2 with the standard (unknotted) embedding in S^3.)

   Definition 3. A *genus g Heegaard decomposition of M* is a decomposition of M as a union of two genus g handlebodies.

   $$H_g \cup_\phi H'_g = M \;\; \phi : F \to F$$

   where $F = \partial H_g = \partial H'_g$.

   We already have two examples: the genus zero and genus one decompositions for S^3 shown above. In fact, for any genus, there is a genus g Heegaard decomposition of S^3.
   To specify ϕ up to isotopy, it suffices to specify which curves on F bound discs in each of the two handlebodies. Such curves are called *meridians*. If we draw the images of both sets of meridians on the same copy of the genus g surface, we get a *Heegaard diagram* for the splitting.

   Gluing two solid tori using the identity map produces the genus one Heegaard splitting for $S^1 \times S^2$.

Figure 3: Left: Two projections of the unknot. RIght: Two projections of the trefoil.

Theorem 1. *Any three-manifold M has a Heegaard decomposition.*

Proof. Choose a triangulation τ of M and consider the one-skeleton τ^1. $H_g = N(\tau^1)$, where N denotes a regular neighborhood. The complement is also a handlebody, $H_g' = M \backslash H_g$. $\square$

2. Surgeries on links in S^3

Definition 4. A link $\lambda \subset M$ with c components is a PL (or C^∞) embedding of $\sqcup_c S^1 \hookrightarrow M$.

Definition 5. If $c = 1$, λ is a *knot*.

Two links are considered equivalent if one can be isotoped to the other. See Figure 3 for projections of equivalent knots.

Definition 6. $\hat{M}$ is obtained by *Dehn surgery* on the link λ in M if

$$\hat{M} = M - N(\lambda) \cup_\phi \sqcup_c (S^1 \times D^2)$$

for some $\phi : \sqcup T^2 \to \sqcup T^2$.

Examples:

(a) Consider the genus 1 Heegaard decomposition of S^3 described above: $S^3 = (S^1 \times D^2) \cup_\phi (S^1 \times D^2)$. Viewing the core of one of these solid tori as the knot λ, we see that S^3 is obtained by Dehn surgery on the unknot in S^3.

(b) $S^1 \times S^2$ is also obtained by Dehn surgery on the unknot in S^3, but this time the gluing map ϕ is the identity map on T^2.

(c) Let K be the trefoil in S^3 and pick a simple closed curve $\nu \subset \partial(N(K))$. Letting $\phi(\mu) = \nu$ specifies the gluing map up to isotopy and determines a new three manifold.

Theorem 2. *Any three manifold can be realized by Dehn surgery on a knot in S^3.*

Definition 7. Let c_i be a simple closed curve embedded on a surface F. A Dehn twist around c is a homeomorphism of F which corresponds to by cutting along c and regluing after a rotation of 2π.

Proof. Let $M = H_g \cup H_g'$ be a Heegaard decomposition of M with gluing homeomorphism ϕ_1. Any such ϕ can be written as a product of Dehn twists $\phi = \tau_{c_n} ... \tau_{c_1}$ where each τ_{c_i} is a Dehn twist around a curve c_i. Let $S^3 = H_g \cup_{\phi_2} H_g'$, where ϕ_2 is the gluing homeomorphism seen above. In the homework you will show that a Dehn twist along a curve $c \subset F = H_g \cap H_g'$ corresponds to a Dehn surgery along an embedded curve in M. Thus, a decomposition of ϕ_2 into product of Dehn twists determines a collection of Dehn surgeries along components of a link. These curves can be chosen to be disjoint, which shows that every manifold can be realized by Dehn surgery on a link in S^3. $\square$

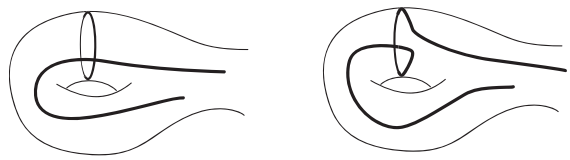

Figure 4: Result of Dehn-twisting along the meridian shown in the left picture.

3. Dehn filling of a fibered three manifold with $\partial M_0 = T^2$.

   Definition 8. M_0 is a fibered manifold if $M_0 = F_0 \times [0,1]/x \sim \phi(x)$ for some closed surface F and some homeomorphism $\phi : F \to F$.

   We can thus express M_0 as a fiber bundle over S^1 with fiber F_0 over every point. If F has an S^1 boundary component, then M_0 will have a T^2 boundary component. As above, we can glue a solid torus to this manifold to get a new manifold M.

   Theorem 3. *Every closed orientable three manifold can be obtained by Dehn filling on some fibered manifold.*

4. There's also a canonical decomposition of any M into pieces with well understood geometry. Thurston's Geometrization Conjecture (proved recently by Perelman) describes this decomposition.