Monday, May 19

Def**: A knot is a smooth embedding of $S^1$ into $S^3$. Equivalently, one may think of it as a polygonal simple closed curve in $S^3$ or $\mathbb{R}^3$.

Non-Example:

... **Been** ...

An infinite chain of shrinking knots

It is sometimes useful and completely accurate to use a shoelace or other string joined end to end to represent a knot. This leads to a natural definition of when two knots are equivalent.

Def**: Two knots, $k^1$ and $k^2$, are equivalent, $k^1 \cong k^2$, if there is a smooth deformation of $k^1$ onto $k^2$ such that the knot does not
intersect itself.

These two knots are not equivalent

\[
\begin{array}{c}
\text{the unknot} \\
\text{or trivial knot}
\end{array}
\quad
\begin{array}{c}
\text{a trefoil knot}
\end{array}
\]

since the trefoil cannot be continuously deformed in such a way that it no longer has any crossings.

The above pictures are called knot diagrams and are projections of the three-dimensional knot into \( R^2 \) (or \( S^2 \)) that include crossing information. An undercrossing is represented as a break in the curve.

Clearly, two knots that are equivalent should have equivalent diagrams, so we define a way to transform the diagrams without changing the knot depicted. These are known as Reidemeister moves.
Reidemeister moves

Type I: \[\begin{array}{c}
\circlearrowleft \\
\\end{array} \xrightarrow{\text{Type I}} \begin{array}{c}
\circlearrowright \\
\\end{array}\]

Type II:

Type III:

Note: A type III move may be thought of as moving any of the three strands past the intersection of the other two.

Theorem (Reidemeister 1927): Two diagrams represent the same knot if and only if they are related by a finite sequence of Reidemeister moves.

Proof omitted/left to reader.
Example:

The following knot is equivalent to the unknot, demonstrated by Reidemeister moves on the knot diagram:

\[
\begin{align*}
\text{II} & \quad \rightarrow \\
\text{III} & \quad \rightarrow \\
& \quad \rightarrow
\end{align*}
\]

or

\[
\begin{align*}
& \quad \rightarrow \\
& \quad \rightarrow \\
& \quad \rightarrow
\end{align*}
\]

Given two knot diagrams, it is often a nontrivial problem to find whether there is a series of Reidemeister moves from one to another. Generally, one uses other knot invariants to distinguish two knots. Any invariant property of a knot must be left unchanged by Reidemeister moves on the knot diagram.
Building knots

A. Conway (1970)
John Conway introduced the concept and standard notation for a tangle, which, for our purposes, is two arcs embedded in a ball (or disc) such that each begins and ends on the boundary:

![Diagram of tangles]

Tangles are equivalent if you can change one into the other without
- intersecting the tangle,
- moving the endpoints, or
- leaving the ball.

Examples:

- Tangle 3 tangle -3 tangle

Note: One generally omits the disc or ball.
An \( n \)-tangle has \( n \) crossings where the overcrossing has positive slope (a left-handed crossing), while a \( -n \)-tangle has \( n \) crossings where the overcrossing has negative slope. The tangle is horizontal.

To multiply two tangles, one must first reflect one tangle across a diagonal:

\[
\begin{array}{c}
\text{reflection as in a mirror}
\end{array}
\]

and then add 2 - 3 tangle

\[
\begin{array}{c}
\text{2 - 3 tangle}
\end{array}
\]

Example: 3 2-4 tangle

One may begin or end with a vertical tangle by multiplying by a zero tangle.
To each tangle we attach a rational number:

\[ 3 \circ -4 \] corresponds to \[ -4 + \frac{1}{2 + \frac{1}{3}} \]

\[ x_1, x_2 \ldots x_n \] corresponds to \[ x_n + \frac{1}{x_{n-1} + \frac{1}{x_{n-2} + \frac{1}{\ldots + \frac{1}{x_1}}}} \]

**Theorem:** Two rational tangles are equivalent if and only if they correspond to the same rational number.

**Proof:** one direction is an exercise. The other is difficult and omitted.

One gets a knot by connecting the top two and bottom two endpoints:

\[ \text{3 \circ -4 tangle knot} \]

The sum of two tangles, \( T_1 \) and \( T_2 \), is denoted \( T_1 \cup T_2 \):
Prezel knots can be written as sums of tangles:

\[ 4, 3, 2 \]

though not all knots can be.

B. Braids

**Def**: An \( n \)-string braid consists of \( n \) arcs that are fixed at their endpoints to a "ceiling" and "floor" such that the strands do not intersect and are monotone decreasing.

One connects top to bottom to make a knot or link:

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\[ \text{The closure of a 4-string braid.} \]
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**Theorem (Alexander 1923)**: Every knot is the closure of a braid.
Example:

Figure 8 knot:

Deform the knot so that the arc circle a chosen point in a chosen direction (in this case, clockwise).
Crossings in braids are labeled by the following convention:

\[ \sigma_1, \sigma_2, \sigma_3, \sigma_0^{-1}, \text{ etc} \]

so that \( \sigma_i \) denotes the \( i \)th strand crossing over the \((i+1)\)th strand and \( \sigma_i^{-1} \) denotes the \( i \)th strand crossing under the \((i+1)\)th strand.

One multiplies braids on \( n \) strands by stacking them, thus forming a group:

\[
\begin{array}{c}
\sigma_1 \sigma_2 \sigma_0 \\
\sigma_0 \sigma_1 \sigma_2 \\
\sigma_1 \sigma_0^{-1} \sigma_1 \sigma_0^{-1}
\end{array}
\]

With identity:

\[
\begin{array}{c}
\sigma_1 \sigma_2 \sigma_0 \\
\sigma_0 \sigma_1 \sigma_2 \\
\sigma_1 \sigma_0^{-1} \sigma_1 \sigma_0^{-1}
\end{array}
\]

\[ \sigma_0 \sigma_1^{-1} = 1 \]

Inverses:

\[
\begin{array}{c}
\sigma_1 \sigma_2 \sigma_0 \\
\sigma_0 \sigma_1 \sigma_2 \\
\sigma_1 \sigma_0^{-1} \sigma_1 \sigma_0^{-1}
\end{array}
\]

Identity by R.M.I:

\[ \sigma_1 \sigma_0^{-1} \sigma_1 \sigma_0^{-1} = 1 \]
And some other useful properties:

- $T_i T_j = T_{j-i} T_i$ if $|i-j| > 1$

- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

Reidemeister III

Theorem (Markov 1935): The closure of two braids give the same knot if the two braids are related by the following moves:

i) $T_i T_i^{-1} = 1$ (RM II)

ii) $T_i T_j = T_j T_i$ if $|i-j| > 1$

iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

iv) $T_j w T_j^{-1} = w$ where w is any "word" in $T_i$'s

v) "stabilization"

\[ \begin{array}{c}
\text{v} \\
\text{w} \\
\text{v} \\
\end{array} = \begin{array}{c}
\text{v} \\
\text{w} \\
\text{v} \\
\end{array} \]

i.e., $w = w T_{n+1}$ where w is a word in n strands.

The proofs of (iv) and (v) are left as exercises.