WORN STONES WITH FLAT SIDES

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In his paper "The shape of Worn Stones", Firey [F] discusses the motion of a compact convex surface $\sum^2$ in space R^3 when each point $\vec{P}$ moves in the inward normal direction $\vec{N}$ with a velocity equal to the Gauss curvature K. The equation models wear under impact at a random angle. For our first result in this paper we show that if the surface $\sum^2$ contains a cylinder at one time, then at a later time it must still contain a smaller cylinder erected on the same base plane. This has the consequence that if the surface $\sum^2$ starts with a flat side, then a little later there will be a smaller flat side, and it takes some positive time for the surface $\sum^2$ to become strictly convex.

Since the equation is invariant under translation and rotation, and dilation where time dilates like distance cubed, it suffices to prove the result for a standard case. Our result then is the following. We work in cylindrical coordinates r, θ, z on R^3.

Main Theorem 1. Suppose a smooth compact convex surface $\sum^2$ evolves under the Gauss Curvature Flow

$$\frac{\partial \vec{P}}{\partial t} = K\vec{N}$$

and that at time $t = 0$ the surface $\sum^2$ contains the cylinder

$$0 \leq r \leq 4, \quad 0 \leq z \leq 2.$$

Then at time $t = 1/3$ the surface $\sum^2$ still contains the cylinder

$$0 \leq r \leq 1, \quad 0 \leq z \leq 1.$$

PROOF. We start with a lemma which shows that we still have something big in $\sum^2$ at $t = 1/3$.

Lemma. *Under the hypotheses, at time $t = 1/3$ the surface $\sum^2$ still contains the disk*

$$0 \leq r \leq 3, \quad z = 1.$$

PROOF. At $t = 0$ the surface contains every sphere of radius $s = 1$ centered at a point with $r = 3$ and $z = 1$, since each such sphere lies in the large cylinder $0 \leq r \leq 4, \quad 0 \leq z \leq 2$. By the maximum principle, if a sphere of radius s lies inside the surface $\sum^2$ at the start, and the sphere also evolves by the Gauss Curvature Flow, it remains inside. Now on the sphere of radius s, the Gauss curvature is $1/s^2$. Thus s evolves by the ordinary differential equation

$$\frac{ds}{dt} = -\frac{1}{s^2}$$

and this makes

$$s = (1 - 3t)^{1/3}$$

so $s = 0$ at $t = 1/3$. Thus the circle of points at $r = 3$ and $z = 1$ remains inside the surface $\sum^2$ at $t = 1/3$. But $\sum^2$ stays convex, so the disk $0 \leq r \leq 1, \quad z = 1$ lies inside also. This proves the lemma.

As a consequence, the surface $\sum^2$cannot turn vertical at any point under the disk. Therefore we can write the lower part of $\sum^2$ as the graph of a function

$$z = f(x, y, t)$$

at least over the disk $0 \leq r \leq 3$ for $0 \leq t \leq 1/3$, we know $z \leq 0$ on the disk at $t = 0$, and we must have $z \leq 1$ on the disk for $0 \leq t \leq 1/3$. Since $\sum^2$ solves the Gauss Curvature Flow, the function f will satisfy the partial differential equation.

$$f_t = \frac{f_{xx}\, f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^{3/2}}$$

by a standard computation. We now obtain an upper bound for z by constructing a supersolution.

Our supersolution will of course be rotationally symmetric, so we write it in polar coordinates

$$z = g(r, t)$$

where g is independent of θ. Then we want g to satisfy the inequality

$$g_t \geq \frac{g_r\, g_{rr}}{r(1 + g_r^2)^{3/2}}$$

where the corresponding equality would be the equation for the Gauss Curvature Flow for a surface of revolution. We look for a supersolution with $z = 0$ on $0 \leq r \leq 1$ for $0 \leq t \leq 1/3$. Since

$$(1 + g_r^2)^{3/2} \geq 1$$

it suffices for $r \geq 1$ to have

$$g_t = g_r\, g_{rr}.$$

We look for a solution to this equation which moves by translation, and find a remarkably simple one

$$z = (r + 2t)^2$$

as is easy to check. This is our basic building block.

Now first we have to move it back in time to make $z = 0$ for $r \leq 1$ and $0 \leq t \leq 1/3$. Therefore we take

$$z = (r + 2t - 2)^2$$

instead. We then define the supersolution g by the formula

$$g(r,t) = \begin{cases} 0 & \text{for} \quad 0 \leq r \leq 2 - 2t \\ (r + 2t - 2)^2 & \text{for} \quad 2 - 2t \leq r \leq 3 \end{cases}$$

on the time interval $0 \leq t \leq 1/3$. We can check that g is continuous and $g \geq 0$ everywhere, while $g = 1$ at $r = 3$ for all t, and $g = 0$ on $0 \leq r \leq 1$ for $0 \leq t \leq 1/3$. Our Main Theorem now follows from the following result.

Theorem. *Given a smooth convex solution $z = f(x, y, t)$ to the Gauss Curvature Flow*

$$f_t = \frac{f_{xx}\, f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^{3/2}}$$

on $0 \leq r \leq 3$ for $0 \leq t \leq 1/3$ with $f \leq 0$ at $t = 0$ and $f \leq 1$ for $0 \leq t \leq 1/3$, we have $f \leq g$ everywhere for the function g constructed above.

PROOF. Of course we use the maximum principle. For any $\delta > 0$ let

$$\tilde{g}(r,t) = g(r,t) + \delta t + \delta.$$

We claim $f < \tilde{g}$ everywhere for all $\delta > 0$. This suffices to show $f \leq g$.

Suppose $f < \tilde{g}$ fails for some δ. By compactness there must be a first time $\hat{t}$ where it fails, and suppose it fails at the point $\widehat{P} = (\hat{x}, \hat{y}, \hat{z})$ at radius $\hat{r}$. Then we have $\hat{t} > 0$ and

$$f(P,t) \leq \tilde{g}(r,t) \quad \text{for all } P \text{ at radius } r \leq 3 \quad \text{for} \quad 0 \leq t \leq \hat{t}$$

while

$$f(\widehat{P},\hat{t}) = \tilde{g}(\hat{r},\hat{t}).$$

Now since $\tilde{g}$ is defined in pieces, we must consider three cases separately. First suppose $0 \leq \hat{r} \leq 3 - 2\hat{t}$. The function $\tilde{g}$ has a continuous first derivative everywhere, and it is zero on this interval. Since the function f is convex, the only way f can touch $\tilde{g}$ on the flat part is if f is also flat there. Then

$$f_t(\widehat{P},\hat{t}) = 0 \quad \text{but} \quad \tilde{g}_t(\hat{r},\hat{t}) = \delta > 0.$$

Consequently f was greater than $\tilde{g}$ at $\hat{r}$ for t a little before $\hat{t}$, which is impossible. Second we note that

$$\tilde{g} \geq 1 + \delta t + \delta > 1 \quad \text{for} \quad r = 3$$

so that $\hat{r}$ cannot lie at the end $r = 3$.

The third and last case is where $2 - 2\hat{t} < \hat{r} < 3$, where

$$\tilde{g} = (r + 2t - 2)^2 + \delta t + \delta.$$

Since $\hat{t} \leq 1/3, \hat{r} > 1$ and we have

$$\tilde{g}_t > \tilde{g}_r \tilde{g}_{rr} \geq \frac{\tilde{g}_r \tilde{g}_{rr}}{r(1 + \tilde{g}_r^2)^{3/2}}$$

at $\hat{r}, \hat{t}$. This is impossible by the maximum principle, which proves the Main Theorem.

2. For our second result we give an estimate in the direction, which shows that if the stone has a side where the Gauss Curvature $K = 0$ but which is not flat, but rather curves like a cylinder, this side will immediately move inward.

Main Theorem 2. Suppose we have a smooth compact convex solution to the Gauss Curvature Flow which is contained in the half-infinite cylinder

$$0 \le r \le 1, \quad 0 \le z < \infty$$

at time $t = 0$. Then for all $t \ge 0$ the solution is contained in the set

$$0 \le r \le erf\frac{z+2}{\sqrt{t}}, \quad 0 \le z < \infty$$

where erf is the error function

$$erf\ v = \frac{2}{\sqrt{\pi}} \int_0^v e^{-u^2} du\ .$$

Since $erf\ v < 1$ for all $v < \infty$, the solution pulls away from the sides of the cylinder.

PROOF. A rotationally symmetric solution of the Gauss Curvature Flow is cylindrical coordinates given as the graph of

$$r = f(z,t)$$

satisfies the equation

$$f_t = \frac{f_{zz}}{f(1+f_z^2)^{3/2}}\ .$$

We look for a supersolution g on $0 \le z < \infty$ with

$$g_t \ge \frac{g_{zz}}{g(1+g_z^2)^{3/2}}\ .$$

To do this we take g to solve the heat equation

$$g_t = \frac{1}{4} g_{zz}$$

on $0 \le t < \infty$, $-2 \le z < \infty$ where we choose g to be a smooth function at $t = 0$ such that $g = 0$ at $z = -2$, $0 \le g \le 1$, $0 \le g_z \le 1$, $g_{zz} \le 0$, and $g = 1$ for $0 \le z < \infty$; and we impose

the boundary condition that $g = 0$ at $z = -2$ for $t \geq 0$, and that g remains bounded as $x \to \infty$. The maximum principle guarantees that for $t \geq 0$ we keep $0 \leq g \leq 1$, $0 \leq g_z \leq 1$ and $g_{zz} \leq 0$, since the condition $g = 0$ at $z = -2$ and the equation $g_t = g_{zz}$ forces $g_{zz} = 0$ at $z = -2$ also. Moreover $g < 1$ for any z for $t > 0$, and in fact we have the following estimate.

Lemma. *The function g above satisfies*

$$g \leq erf\, \frac{z+2}{\sqrt{t}}$$

where erf is the error function.

PROOF. The function $erf[(z+2)/\sqrt{t}]$ satisfies the heat equation and equals 1 for $z > -2$ at $t = 0$ and equals 0 at $z = -2$ for $t > 0$, with a jump discontinuity at the corner. For any $\delta > 0$, it is clear that we can find $\tau > 0$ such that

$$\delta + erf\, \delta / \sqrt{\tau} > 1$$

and hence

$$g < \delta + erf\, \frac{z+2+\delta}{\sqrt{t}}$$

on $-2 \leq z < \infty$ up to $t = \tau > 0$. Moreover the right hand side still satisfies the heat equation, and is at least δ for all $t \geq \tau$ at $z = -2$. Then the maximum principle shows the inequality remains valid for $t \geq \tau$ and $-2 \leq z < \infty$ also. We can then let $\delta \to 0$ to prove the Lemma.

Now since $0 \leq g \leq 1$ and $0 \leq g_z \leq 1$, we have

$$g(1 + g_z^2)^{3/2} \leq 4$$

and since $g_{zz} \leq 0$ we get

$$g_t = \frac{1}{4} g_{zz} \geq \frac{g_{zz}}{g\,(1+g_z^2)^{3/2}}$$

so g is a smooth supersolution to the rotationally symmetric Gauss Curvature Flow. Our Main Theorem 2 now follows from the following result.

Theorem. *Any smooth compact convex solution to the Gauss Curvature Flow which starts in the cylinder*

$$0 \leq r \leq 1, \quad 0 \leq z < \infty$$

remains in the set

$$0 \leq r \leq g(z,t)\ .$$

PROOF. As usual we sneak in $\delta > 0$. Let

$$\tilde{g}(z,t) = g(z,t) + \delta t + \delta.$$

For any finite time T we can choose δ small enough that

$$(1 + \delta T + \delta) 2^{3/2} \leq 4$$

and then we still have $0 \leq \tilde{g}_z \leq 1$ and

$$\tilde{g}(1 + \tilde{g}_z^2)^{3/2} \leq 4.$$

Then

$$\tilde{g}_t = g_t + \delta \quad \text{and} \quad \tilde{g}_{zz} = g_{zz} < 0$$

gives

$$\tilde{g}_t > \frac{\tilde{g}_{zz}}{\tilde{g}(1 + \tilde{g}_z^2)^{3/2}}$$

by at least $\delta > 0$.

We claim our solution which starts in the half-infinite cylinder remains strictly inside the open set

$$0 \leq r < \tilde{g}(z,t), \quad -1 < z < \infty >$$

For if not, there will be a first time $\hat{t}$ when it touches the boundary, and a point $\widehat{P} = (\hat{x}, \hat{y}, \hat{z})$ at radius $\hat{r}$ where it touches. Clearly $\hat{z} \geq 0$ so they must touch at $\hat{r} = \tilde{g}(\hat{z}, \hat{t})$. Rotate the solution so the tangent plane at $\widehat{P}$ at time $\hat{t}$ is parallel to the x-axis. Then we can write the solution as a graph near $\widehat{P}$ and near $\hat{t}$

$$y = f(x, z, t)$$

solving the Gauss Curvature Flow equation

$$f_t = \frac{f_{xx} f_{zz} - f_{xz}^2}{(1 + f_x^2 + f_z^2)^{3/2}}$$

in Cartesian coordinates. The rotationally symmetric supersolution likewise is the graph of

$$y = \tilde{g}(x, z, t) = \sqrt{\tilde{g}(z,t)^2 - x^2}$$

which satisfies the inequality

$$\tilde{g}_t > \frac{\tilde{g}_{xx} \tilde{g}_{zz} - \tilde{g}_{xz}^2}{(1 + \tilde{g}_x^2 + \tilde{g}_z^2)^{3/2}}$$

and $f \leq g$ near $\widehat{P}$ up to time $\hat{t}$ when equality holds. This gives a contradiction by the maximum principle, which proves the theorem.

3. Since we stated our results for smooth solutions, we offer a remark on weak solutions. Given any compact convex body with non-empty interior, we can approximate it by compact smooth strictly convex bodies both inside and out with volumes as close to the volume

of the original body as we please. We can then flow these approximaations by the Gauss Curvature Flow. The intersection of the compact smooth strictly convex bodies outside is the maximal weak solution; the union of these inside is the minimal weak solution. We claim they agree. The reason is that under the Gauss Curvature Flow the volume of any body decreases at a constant rate equal to the volume of the 2-sphere 4π. Hence the small volume between two good approximations remains constant. It only remains to observe that for convex bodies we can estimate the distance between by the volume between.

Lemma. *Given two compact convex bodies one inside the other and both contained in the ball around the origin of radius R and containing the ball around the origin of radius r, if V is the volume of the region between the two convex bodies and s is the greatest distance between them then*

$$V \geq \frac{\pi}{3} \frac{r^2}{R^2} s^3 .$$

PROOF. Look at the point P in the larger body at greatest distance s from the smaller body. Suppose P is at distance R from the origin. The whole ball around the point P of radius s avoids the smaller body. But if the larger body contains the ball around the origin of radius r, it also contains the cone on this ball from P. The cone makes an angle θ with its axis where

$$\theta = \sin \frac{r}{R} .$$

We also have $R \geq r + s$ if the smaller body contains the ball of radius s around the origin. Then the part of the ball of radius s around P which lies inside the cone at P of angle θ lies in the larger body and outside the smaller body. Its volume is

$$V = \frac{2\pi}{3}(1 - \cos\theta)s^3$$

since $2\pi(1-\cos\theta)$ is the surface area on the unit sphere cut off by the cone of angle θ.

Now we use

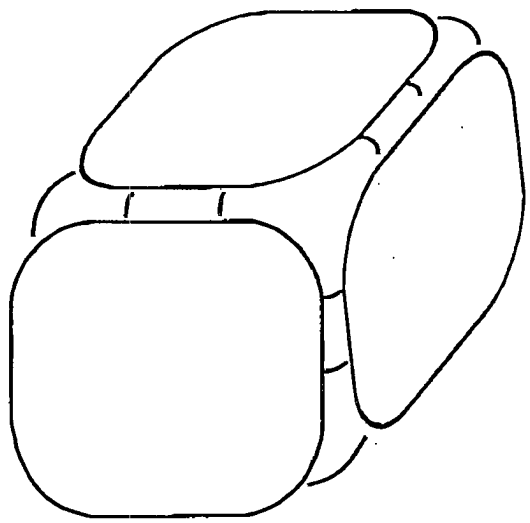
$$1-\cos\theta = 1-\frac{\sqrt{R^2-r^2}}{R} = \frac{r^2}{R(R+\sqrt{R^2-r^2})} \geq \frac{r^2}{2R}$$

to estimate

$$V \geq \frac{\pi}{3}\frac{r^2}{R^2}s^3$$

as desired. If we have a bound on the radius R as in the Theorem we are done.

4. It is now clear that the previous estimates apply to weak solutions also. We end by considering the case where the initial body is a cube. Main Theorem 1 implies that for a short time the center part of the six faces remains stationary and stays flat. Main Theorem 2 implies that the eight vertices and the twelve edges immediately move inward; as we see by circumscribing a cylinder around the cube touching four of the edges at $t = 0$.The resulting figure after a short positive time should look like this.



It would be fun to see a computer picture. Also, can we give a positive lower bound $K > 0$ in the region which moves?

References

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