Group Theory

Spectral gaps in SU(d)

Trou spectral dans SU(d)

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\begin{abstract}
It is shown that if $g_1, \ldots, g_k$ are algebraic elements in SU(d) generating a dense subgroup, then the corresponding Hecke operator has a spectral gap.
\end{abstract}

\begin{version}
On démontre que si $g_1, \ldots, g_k$ sont des éléments algébriques de SU(d) et le groupe engendré par $g_1, \ldots, g_k$ est dense, alors l'opérateur de Hecke défini par ces éléments a un trou spectral.
\end{version}

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\begin{thm}
Soit $g_1, \ldots, g_k \in SU(d) \cap \text{Mat}_{d \times d}(\mathbb{Q})$ et $\Gamma = \langle g_1, \ldots, g_k \rangle$ le groupe engendré par $g_1, \ldots, g_k$. Supposons $\Gamma$ dense dans SU(d).

\begin{align*}
\text{Théorème.} \quad \text{L'opérateur de Hecke} \\
Tf(x) = \frac{1}{2k} \sum_{1 \leq j \leq k} (f(g_j x) + f(g_j^{-1} x))
\end{align*}

a un trou spectral.

Ceci généralise le résultat antérieur [4] pour SU(2). L'approche suivie ici diffère cependant et elle est essentiellement analogue à celle de [5] pour les groupes SL(d,p\textsuperscript{n}) avec p fixé et n \to \infty. Des techniques d'arithmétique combinatoire, de la théorie des représentations et produits aléatoires de matrices y sont utilisées.

1. We assume $g_1, \ldots, g_k \in SU(d) \cap \text{Mat}_{d \times d}(\mathbb{Q})$ and denote $\Gamma = \langle g_1, \ldots, g_k \rangle$ the generated group. Assume further that $\Gamma$ is Zariski dense in SL\textsubscript{d} or, equivalently, that $\Gamma$ is topologically dense in SU(d).

Denote

$$
(Tf)(x) = \frac{1}{2k} \sum_{j=1}^{k} (f(g_j x) + f(g_j^{-1} x))
$$

the corresponding Hecke operator acting on $L^2(G), G = SU(d)$.

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Theorem 1. $T$ has a spectral gap.

The result for $d = 2$ was obtained in [4]. As in [4], we rely on methods from arithmetic combinatorics. But the approach followed here is significantly different from that of [4] and resembles that of [5] on expansion in groups $SL_d(p^n)$ with $p$ fixed and $n \to \infty$. Similarly to [5], the assumption of Zariski density is exploited through the theory of random matrix products (cf. [1]).

2. By a result of [6], we may take $k = 2$ and assume $(g_1, g_2)$ free generators of the free group $F_2$. Define

$$
\nu = \frac{1}{4} (\delta_{g_1} + \delta_{g_2} + \delta_{g_1^{-1}} + \delta_{g_2^{-1}})
$$

the symmetric probability measure on $G$ and denote $\nu^\ell$ its $\ell$-fold convolution power. Set for $\delta > 0$

$$
P_\delta = \frac{1_{B(1, \delta)}}{|B(1, \delta)|}
$$

providing an approximate identity on $G$.

Proposition 1. There is $\kappa > 0$ such that if $G_1$ is a non-trivial closed subgroup of $G$, then

$$
\nu^\ell(G_1) < e^{-\kappa \ell} \quad \text{for } \ell \to \infty.
$$

The proof of this ‘escape property’ relies on our assumption that $T$ is Zariski dense and results on random matrix products, that are applied in suitable exterior powers of the adjoint representation of $G$. As in [4], we establish the following ‘flattening property’:

Proposition 2. Given $\tau > 0$, there is a positive integer $\ell < C(\tau) \log \frac{1}{\delta}$ such that

$$
\| \nu^\ell * P_\delta \|_\infty < \delta^{-\tau}.
$$

It is derived by straightforward iteration of

Lemma 1. Given $\gamma > 0$, there is $\kappa > 0$ such that for $\delta > 0$ small enough, $\ell \sim \log \frac{1}{\delta}$, if

$$
\| \nu^\ell * P_\delta \|_2 > \delta^{-\gamma}.
$$

Then

$$
\| \nu^{2\ell} * P_\delta \|_2 < \delta^\kappa \| \nu^\ell * P_\delta \|_2.
$$

With Proposition 2 at hand, the proof of a spectral gap may then be completed by considerations from representation theory (the Sarnak-Xue argument, also used in [4], or variants).

3. Returning to Lemma 1, the first step is to apply T. Tao's version of the Balog-Szemerédi-Gowers lemma (cf. [7]) for compact groups. Denoting $\mu = \nu^\ell * P_\delta$ and assuming (4) fails, one obtains a subset $H \subset G$, $H$ a union of $\delta$-balls, and a finite subset $X$ of $G$ such that

(5) $H = H^{-1}$,
(6) $H \cdot H \subset H, X \cap X \cdot H$,
(7) $|X| < \delta^{-\varepsilon}$,
(8) $\mu(aH) > \delta^\varepsilon$ for some $a \in G$,
(9) $|H| < \delta^\varepsilon$

(here $\varepsilon > 0$ is an arbitrary small, fixed number and $| |$ is used in (7) to denote ‘cardinality’ and in (9) for ‘Haar-measure’).

Recall that (5)–(6) mean that $H$ is an ‘approximate group’ (cf. [7]). The goal is to show that properties (5)–(9) are not compatible and get a contradiction.

4. Next we specify some technical ingredients.

Crucial use is made of the ‘discretized ring theorem’ (see [2,3]). The version needed here is the following
**Proposition 3.** Given $\sigma > 0$, there is $\gamma > 0$ such that if $\delta > 0$ is small enough and $A \subset C^d \cap B(0,1)$ satisfies

$$N(A, \delta) > \delta^{-\sigma}$$

then there is $k \in C^d$, $|k| = 1$ such that

$$[0, \delta^2]k \subset A + B(0, \delta^{\sigma+1}).$$

Here $A'$ denotes a 'sum-product' set $s_1A^{(s_2)} - s_1A^{(s_2)}$ of $A$, with $s_1$, $s_2$ bounded in terms of $\sigma$.

In (10), $N(A, \delta)$ refers to the metrical entropy, i.e. the minimum number of $\delta$-balls needed to cover $A$. We used the notations $A = A + \cdots + A$ and $A^{(s)} = A + \cdots + A$ for the $s$-fold sum (resp. product) sets.

Proposition 3 is derived from the following result that generalizes [3]:

**Theorem 2.** Let $A \subset [0, 1]^d$ satisfy

$$N(A, \delta) = \delta^{-\sigma} \quad (0 < \sigma < d)$$

and also a non-concentration property

$$N(A \cap I, \delta) < C \delta^\eta N(A, \delta) \quad \text{if} \ \delta < \delta_1 < 1 \ \text{and} \ I \text{ any \ } \delta_1 \text{-ball}.$$ (13)

Let $\mu$ be a probability measure on $L(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\|b\| \leq 1 \quad \text{for} \ b \in \text{supp} \ \mu,$$

$$\max_{|v| = 1} \mu\left(\left|\langle bv, w \rangle \right| < \delta_1\right) < \delta^\varepsilon \quad \text{if} \ \delta < \delta_1 < 1.$$ (14)

Then, for some $b \in \text{supp} \ \mu$

$$N(A + A, \delta) + N(A + bA, \delta) > \delta^{-\sigma - \tau}$$ (15)

with $\tau = \tau(\sigma, \kappa) > 0$.

In order to apply Proposition 3, we construct 'almost' diagonal sets of matrices, using the following:

**Lemma 2.** Assume $\{g_1, g_2\}$ in $U(d) \cap \text{Mat}_{d \times d}(\mathbb{Q})$ generate a free group and let $H \subset W_{d}(g_1, g_2)$ (= the set of 'words' or length $\leq \ell$) satisfy

$$\log |H| > c \ell.$$ (16)

Then there is a subset $A$ of a product set $H^{(s)}$, $s < C$ and $\delta > 0$ such that

(17) $\log \frac{1}{\delta} \sim \ell$.

(18) The elements of $A$ are $\delta$-separated.

(19) In an appropriate orthonormal basis, we have

$$\text{dist}(x, \Delta) < \delta \quad \text{for} \ x \in A$$

where $\Delta$ denotes the set of diagonal matrices.

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References


