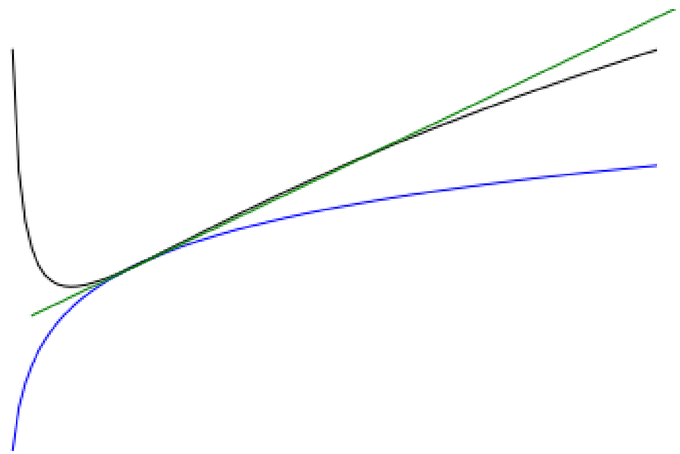
GEODESIC CONVEXITY



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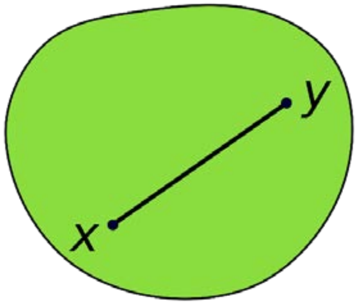
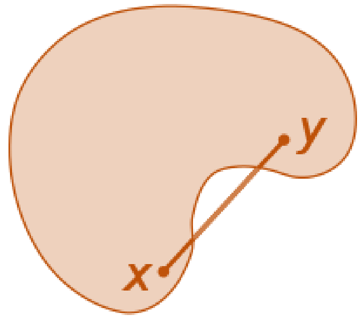
https://nisheethvishnoi.wordpress.com/convex-optimization/

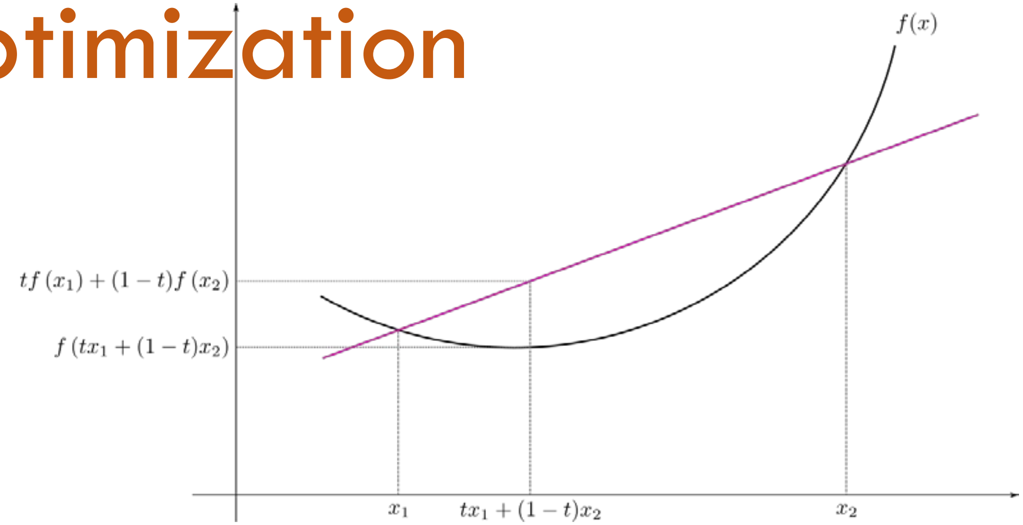
Institute for Advanced Study, June 7, 2018

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Convexity and Optimization







$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y)$ $\quad$ $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$ $\quad$ $\nabla^2 f(x) \succcurlyeq 0$

For a convex function local minimum = global min.		Goal: $f(\hat{x}) \leq f(x^\star) + \varepsilon$
Gradient Descent	$\frac{dx}{dt} = -\nabla f(x)$	$x^{k+1} = x^k - \nabla f(x^k)$
Newton-type methods	$\frac{dx}{dt} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$	$x^{k+1} = x^k - \left(\nabla^2 f(x^k)\right)^{-1} \nabla f(x^k)$
Cutting plane methods	more later …	

Running Times: $T(n, \varepsilon, \|\nabla f\|, \kappa(\nabla^2 f), \|x^0 - x^\star\|, t_{\text{grad}}, t_{\text{Hessian}}, \ldots)$

Ushered in an TCS/ML revolution …

Optimization Problems: Commutative and Non-Commutative

Given evaluation oracle to $p(x) \in \mathbb{R}_+[x_1, \ldots, x_m]$ and $\theta \in \mathbb{R}_+^m$

P1: $\inf_{x \in \mathbb{R}_{>0}^m} \log p(x) - \sum \theta_i \log x_i$

Applications to discrete counting problems [Gurvits '04, SinghV. '14, StraszakV. '17a]

$\log\ p(x)$ is not convex (sometimes concave) –not a convex optimization problem!

Given m $\ell \times n$ real-valued matrices $B_1, B_2, \ldots, B_m$ and a $\theta \in \mathbb{R}_+^m$

P2: $\inf_{X \succ 0} \sum \theta_j \operatorname{logdet}\left(B_j X B_j^\top\right) - \operatorname{logdet} X$

Applications to Brascamp-Lieb const. [SraV.Yildiz '18]; studied by [BCCT '05, GGOW+ '16+]

Not a convex optimization problem either (rank 1 case as above)

Both problems are geodesically convex!

Convexity vs Geodesic Convexity

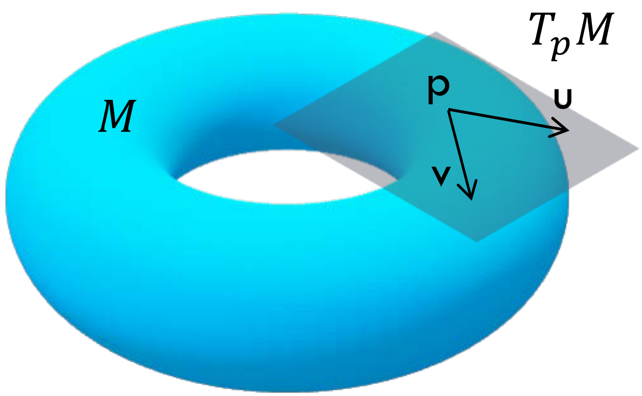
Euclidean space	Smooth manifolds
Calculus (differentiation / integration)	Affine connections
Straight Lines	Geodesics
Convex Sets	Geodesically convex sets
Convex functions	Geodesically convex functions
Local = Global	Local = global
Algorithms for convex optimization	??

Plan for the talk:

a) Manifolds, Geodesics, Geodesic convexity
b) Geodesic convexity of the applications
c) An algorithm for P1

MANIFOLDS, GEODESICS, GEODESIC CONVEXITY

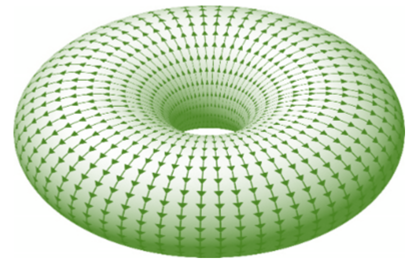
Manifolds, Calculus and Metrics

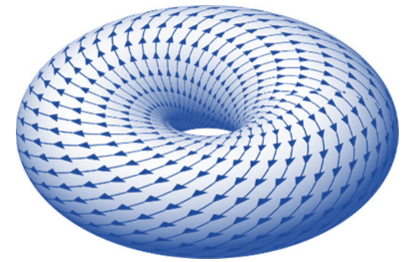


Smooth manifolds

$\mathfrak{X}(M)$: vector fields over M

Curves





Euclidean Space: $D_X(f_1, \dots, f_n)$ is just the directional derivative

Affine Connection: $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$	Riemannian Metric Tensor: $g_p(u, v)$
$\forall X, Y, Z \in \mathfrak{X}(M),\quad \forall f \text{ on } M$	$\forall u, u', v \in T_pM, \forall c \in \mathbb{R}$
Linear in first term: $\nabla_{X+fY} Z = \nabla_X Z + f \nabla_Y Z$	Symmetric: $g_p(u, v) = g_p(v, u)$
Linear in second term: $\nabla_X Y + Z = \nabla_X Y + \nabla_X Z$	Bilinear: $g_p(u + cu', v) = g_p(u, v) + c g_p(u', v)$
Leibniz's rule: $\nabla_X fY = f \nabla_X Y + Y D_X f$	**Positive definite:** $g_p(u, u) > 0, u \neq 0$

Compatibility: $D_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Torsion free: $\nabla_X Y - \nabla_Y X = [X, Y]$

Levi-Civita connection: Unique, torsion-free, affine-connection compatible with metric

Examples

Manifold: Positive Orthant $\mathbb{R}^m_+$

Tangent Space: $\mathbb{R}^m$

Riemannian Metric: For $p \in \mathbb{R}^m_+$, and $u, v \in \mathbb{R}^m$

$$g(u,v) := \langle P^{-1}u, P^{-1}v \rangle = \sum \frac{u_i v_i}{p_i^2}$$

Hessian: Let $f(p) = -\sum \log p_i$

Then $g = \text{Hessian of } f$

Levi-Civita Connection: At a point p

$$\nabla_{e_i} e_i = p_i^{-1} e_i$$
$$= 0 \text{ o.w.}$$

Manifold: Positive Definite Matrices $\mathbb{S}^n_{++}$

Tangent Space: $\mathbb{S}^n$

Riemannian Metric: For $P \in \mathbb{S}^n_{++}$, and $U, V \in \mathbb{S}^n$

$$g(U,V) := \mathrm{Tr}\, P^{-1} U P^{-1} V$$

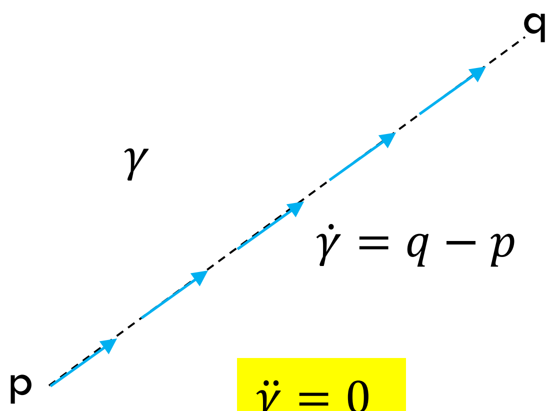
Hessian: Let $f(P) = -\text{logdet}\, P$

Then $g = \text{Hessian of } f$

Calculations of Levi-Civita get more complicated …

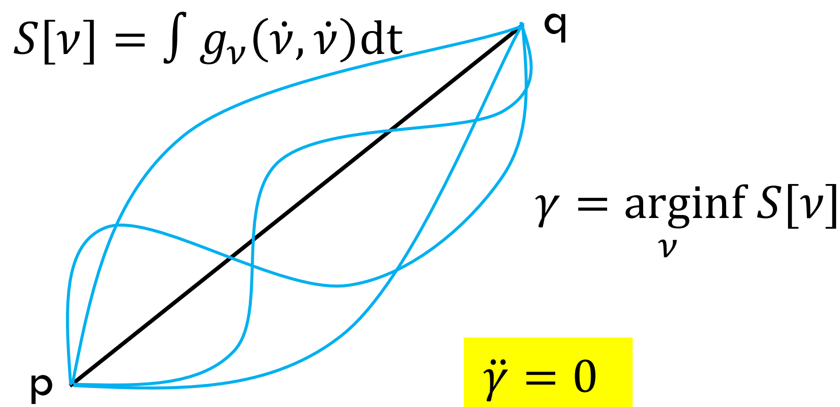
Geodesics: Two Views

Curves that take tangent vectors "parallel" on the curve



$\ddot{\gamma} = 0$

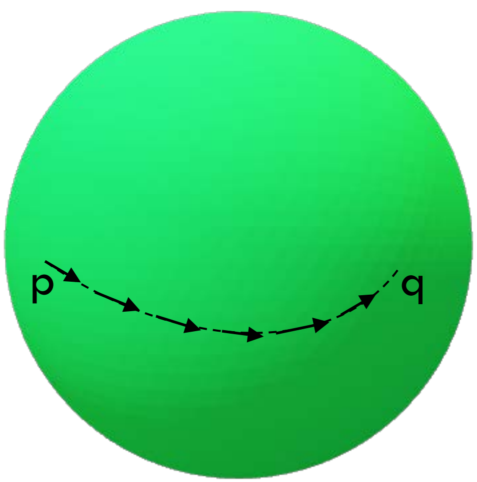
Shortest curves between points

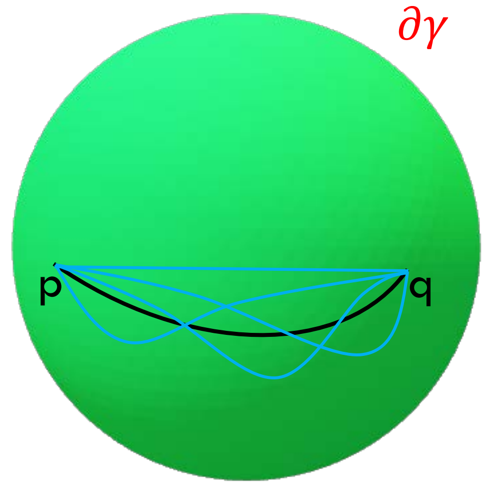


$\ddot{\gamma} = 0$

$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$

$$\frac{\partial g_\gamma(\dot{\gamma}, \dot{\gamma})}{\partial \gamma} = \frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial g_\gamma(\dot{\gamma}, \dot{\gamma})}{\partial \dot{\gamma}}$$





Euler-Lagrange Dynamics

Examples

Manifold: Positive Orthant $\mathbb{R}_+^m$

Levi-Civita Connection: At a point p

$$\nabla_{e_i} e_i = p_i^{-1} e_i$$

Geodesic Equation: $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$

Simplify (exercise):

$$\forall_i \quad \ddot{\gamma}_i \quad = \quad \dot{\gamma_i}^2 \gamma_i^{-1}$$

Solve ODE:

$$\frac{d}{dt} \log \dot{\gamma}_i = \frac{d}{dt} \log \gamma_i$$

$$\dot{\gamma}_i = \alpha_i \gamma_i \text{ for some } \alpha_i$$

$$\gamma_i = \beta_i e^{\alpha_i t} \text{ for some } \beta_i > 0, \alpha_i$$

Manifold: Positive Definite Matrices $\mathbb{S}_{++}^n$

Geodesic Equation:

$$\frac{\partial g_\gamma(\dot{\gamma}, \dot{\gamma})}{\partial \gamma} = \frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial g_\gamma(\dot{\gamma}, \dot{\gamma})}{\partial \dot{\gamma}}$$

Simplify:

$\dot{\gamma}\gamma^{-1} = C$ for some constant matrix C

Solve:

$$\gamma(t) = \exp(tC)\, D$$

Geodesic between $\boldsymbol{P}, \boldsymbol{Q}$:

$$P^{\frac{1}{2}} \left(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}\right)^t P^{\frac{1}{2}}$$

Geodesics Convexity

0th Order Characterization

- $f: M \to \mathbb{R}$ is geodesically convex if for any geodesic $\gamma: [0,1] \to M$ and $\forall t \in [0,1]$
- $f(\gamma(t)) \leq (1-t) f(\gamma(0)) + t\, f(\gamma(1))$

1st Order Characterization

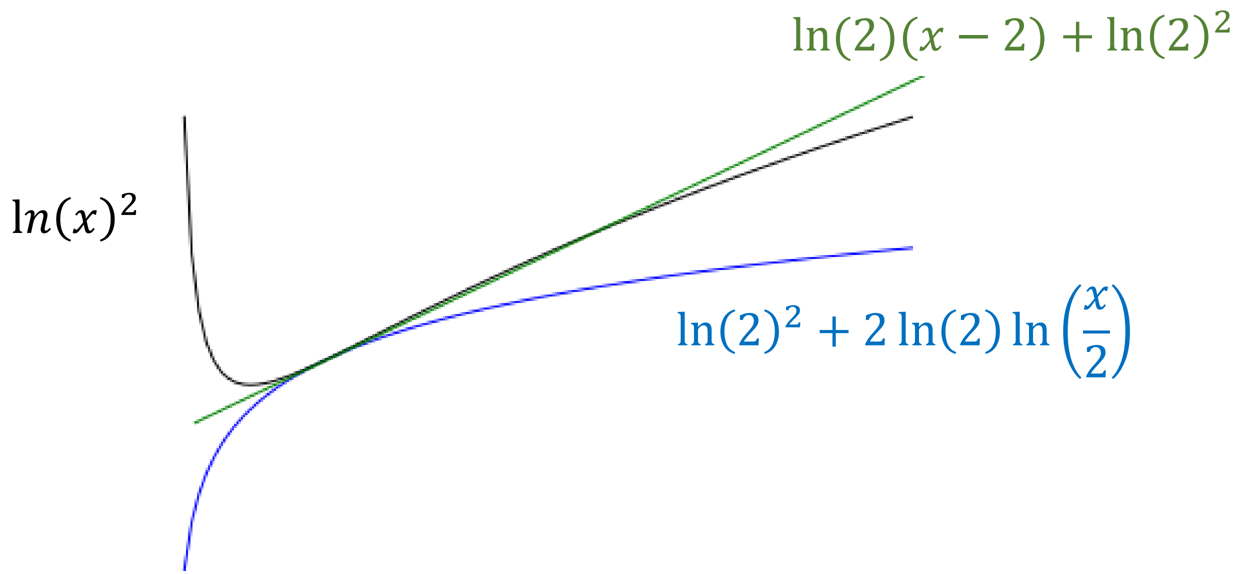
- $f: M \to \mathbb{R}$ is geodesically convex iff for any $p, q \in M$ with geodesic joining them γ_{pq},
- $f(p) + \dot{\gamma}_{pq}(f)(p) \leq f(q)$

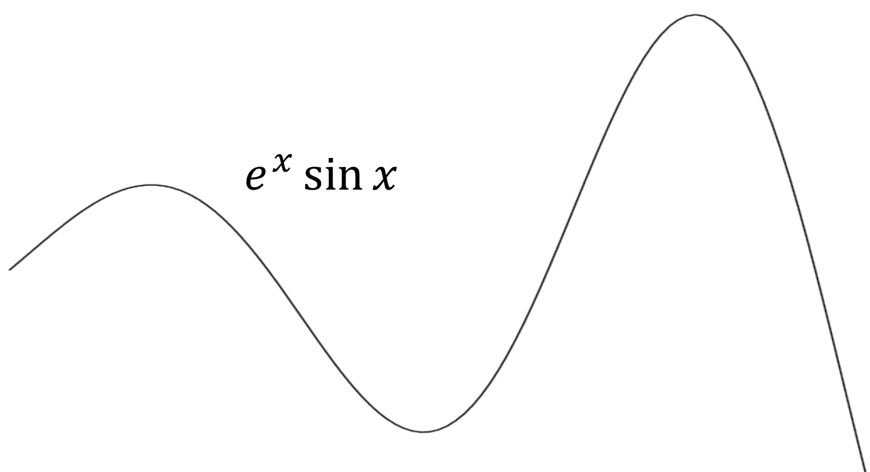
2nd Order Characterization

- $f: M \to \mathbb{R}$ is geodesically convex if for any geodesic $\gamma: [0,1] \to M$ and $\forall t \in [0,1]$
- $\frac{d^2 f(\gamma(t))}{d\, t^2} \geq 0$

If f is geodescially convex, then every local minimum is also a global minimum

Geodesic Convexity vs Non-convexity





Geodesic Convexity of $\log p(x)$

Theorem: Given $p(x) = \sum_{\tau \in \mathcal{F}} c_\tau x^\tau \in \mathbb{R}_+[x_1, \ldots, x_m]$ where $x^\tau = \prod_{j \in [m]} x_j^{\tau_j}$ and $\mathcal{F} \subset \mathbb{Z}_{\geq 0}^m$, $\log p(x)$ is geodesically convex

---

Geodesic: $\gamma(t) \coloneqq (\beta_1 e^{t\alpha_1}, \ldots, \beta_m e^{t\alpha_m})$ for real vectors $\beta \in \mathbb{R}_+^m$ and $\alpha \in \mathbb{R}^m$

Second Order Convexity: $\log p(x)$ is geodesically convex if for any geodesic $\gamma(t)$

$$\forall t \in [0,1], \qquad \frac{d^2 \log p(\gamma(t))}{d\, t^2} \geq 0$$

First derivative:

$$\frac{d \log p\,(\gamma(t))}{d\, t} = \frac{\dot{p}}{p} = \frac{\sum_{\tau \in F} c_\tau \langle \alpha, \tau \rangle \gamma(t)^\tau}{\sum_{\tau \in F} c_\tau \gamma(t)^\tau}$$

Second derivative:

$$\frac{d^2 \log p\,(\gamma(t))}{d\, t^2} = \frac{\ddot{p}}{p} - \left(\frac{\dot{p}}{p}\right)^2 = \frac{\sum_{\tau, \tau' \in F} (\langle \alpha, \tau \rangle - \langle \alpha, \tau' \rangle)^2 c_\tau c_{\tau'} \gamma(t)^\tau \gamma(t)^{\tau'}}{\left(\sum_{\tau \in F} c_\tau \gamma(t)^\tau\right)^2} \geq 0$$

Geodesic Convexity of Brascamp-Lieb

Given m $\ell \times n$ real-valued matrices $B_1, B_2, \ldots, B_m$ and a $\theta \in \mathbb{R}_+^m$

$$\inf_{X \succ 0} \sum \theta_j \operatorname{logdet}\left(B_j X B_j^\top\right) - \operatorname{logdet} X$$

Theorem(s) [SraV.Yildiz '18]: Geodesically convex and computes BL-constant!

Geodesic: Given PD matrices P and Q, the geodesic between them

$$\gamma(t) \coloneqq P^{\frac{1}{2}}\left(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}\right)^t P^{\frac{1}{2}}$$

Simple Fact: $\log\det X$ is geodesically linear

$$\forall t \in [0,1], \quad \log\det(\gamma(t)) = (1-t)\log\det P + t\log\det Q$$

Theorem [AndoKubo '79]: If $T(X)$ is a strictly positive linear operator, then $\log\det T(X)$ is geodesically convex

By taking positive combinations, enough to show: $T_j(X) = B_j X B_j^\top$ is a strictly positive linear map for $j \in [m]$ if $\ell \sum \theta_j = n$ and $\dim(\mathbb{R}^n) = \sum_{j\in[m]} \theta_j \dim(B_j \mathbb{R}^n)$

Proof: Assume $T_i(X)$ is not strictly positive linear. Then for some $X \in \mathbb{S}_{++}^n$, there exists $v \in \mathbb{R}^\ell$ such that $v^\top T_i(X) v \le 0$. Thus, $\left(B_i^\top v\right)^\top X \left(B_i^\top v\right) \le 0$. Thus, $\left(B_i^\top v\right) = 0$ and $\dim(B_i \mathbb{R}^n) < \ell$.

Consequently

$$n = \dim(\mathbb{R}^n) = \sum_{j\in[m]} \theta_j \dim\left(B_j \mathbb{R}^n\right) < \sum_{j\in[m]} \theta_j \ell = n$$ – ***contradiction!***

ALGORITHM FOR P1 (RANK ONE BL)

Ellipsoid Method

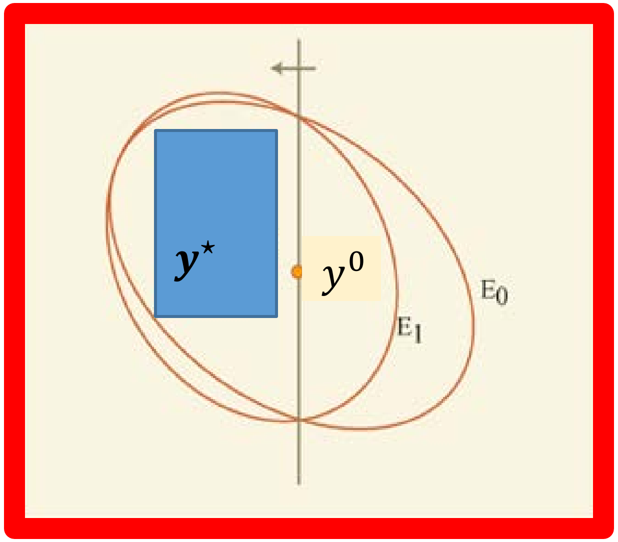
$$\mathbf{OPT} = \inf_{y \in \mathbb{R}^m} \log p(e^y) - \sum \theta_i y_i = \inf_{y \in \mathbb{R}^m} f(y)$$

Reduce to Feasibility: Given A, check if **OPT** is $\leq A + \varepsilon$ or $> A$

Assume $\|y^\star\| \leq R, f \in [-M, M]$

Ellipsoid Algorithm:

- **Start** with an ellipsoid E_0 that contains $y^\star$
- At kth step, let E_k be the ellipsoid centered at y^k
  - **IF** $f(y^k) \leq A$, **DONE**
  - **ELSE**
    - use evaluation oracle for p to get $\nabla f(y^k)$
    - $E_{k+1} \supseteq E_k \cap \{y\colon \langle y - y^k, \nabla f(y^k)\rangle \leq 0\}$
- **Stop** when the radius of the ellipsoid becomes $\leq \varepsilon R/M$



Invariant: If $f(y^\star) \leq A$ then $y^\star \in E_k$ for all k

Proof: Convexity of f implies $\langle y^\star - y^k, \nabla f(y^k)\rangle + f(y^k) \leq f(y^\star) \leq A$

Since $f(y^k) > A$, $\langle y^\star - y^k, \nabla f(y^k)\rangle < 0$

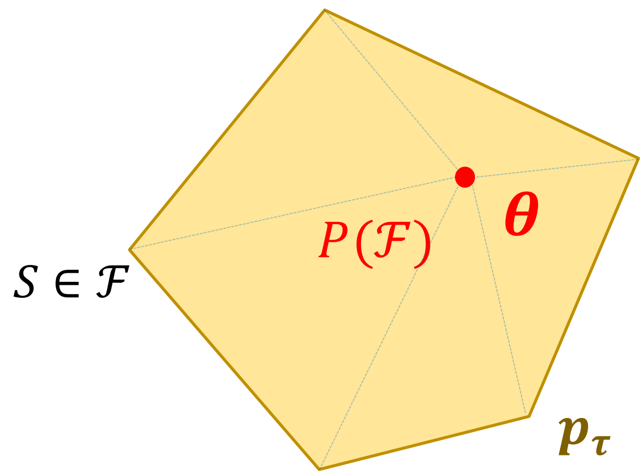
Running Time: $\mathrm{poly}(m, t_f, t_{\nabla f}, \log \frac{RM}{\varepsilon})$

Bounding R and M?

$$\inf_{y \in \mathbb{R}^m} \log p(e^y) - \sum \theta_i y_i$$

$$\sup_{q} \sum_{\tau \in \mathcal{F}} q_\tau \log \frac{p_\tau}{q_\tau}$$

$\Rightarrow M \leq m$



- q – prob. distribution over $\mathcal{F}$
- The expectation of q is θ

Bounding R?: As θ comes close to the boundary, $y^\star$ must blow up. By how much?

Theorem [SinghV. '14, StraszakV. '17b]: If the unary complexity of all facets of the polytope is polynomial in m then, $R \leq \text{poly}(m)$ – includes all combinatorial polytopes

Entropy interpretation seems important to obtain the bit complexity bounds

Summary and Challenges

- Some non-convex problems can be geodesically convex – find a metric!
- Geodesics and their study is a highly developed area in math and physics
- Working with geodesics may comes at additional costs

- Polynomial time algorithm for Brascamp-Lieb constant for rank >1?
- Entropy interpretation of Brascamp-Lieb for rank >1?
- Understanding functions that are geodesically convex?
- Develop more methods for geodesic convex optimization?
- Sampling from geodesically convex densities?

Thanks! Questions?