Dissipative Intermittent Euler Flows

satisfying the local energy inequality

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Abstract

The goal of this thesis is to show the existence of dissipative solutions to the incompressible Euler equations with almost 1/3 of a derivative in L^3 that satisfy the local energy inequality strictly. This proves an intermittent form of the Strong Onsager Conjecture proposed by Philip Isett. The contents of this thesis are joint work with Hyunju Kwon and Matthew Novack.

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Contents

Chapter 1

Introduction

Let's consider the incompressible Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \\ \mathrm{div} u = 0 \end{cases} \tag{1.1}$$

on the periodic domain $\mathbb{T}^3 = [-\pi, \pi]^3$. Here $u : [0, T] \times \mathbb{T}^3 \to \mathbb{R}^3$ is a time-dependant vector field on the torus $\mathbb{T}^3$ called the *fluid velocity* and $p : [0, T] \times \mathbb{T}^3 \to \mathbb{R}$ is a scalar function called the *pressure*. The first equation, called the momentum equation, is derived from Newton's second law of motion with internal force $-\nabla p$. The incompressibility of the fluid is ensured by the divergence-free condition. These equations were introduced by Euler more than 250 years ago to model the flow of an ideal volume-preserving fluid with no internal friction.

We note that the pressure (up to addition of a constant) can be recovered from the fluid velocity by solving the elliptic equation

$$\Delta p = -\mathrm{div}\left[(u \cdot \nabla) u\right] .$$

Indeed, the pressure acts as a Lagrange multiplier that ensures the divergence-free constraint on the vector field u. The lack of internal friction suggests that the *kinetic energy* $|u|^2/2$ cannot

be dissipated as heat and so one expects that the total kinetic energy is conserved. Indeed, if we multiply the first equation by u we see that

$$\begin{aligned} 0 &= (\partial_t u + (u \cdot \nabla)u + \nabla p) \cdot u = \partial_t \frac{|u|^2}{2} + (u \cdot \nabla)\frac{|u|^2}{2} + u \cdot \nabla p \\ &= \partial_t \frac{|u|^2}{2} + \operatorname{div}\left[u\left(\frac{|u|^2}{2} + p\right)\right] . \end{aligned} \tag{1.2}$$

Taking the integral over the whole domain and using the divergence theorem tells us that the total kinetic energy is conserved

$$\frac{d}{dt}\int_{\mathbb{T}^3} \frac{|u|^2}{2} = 0 . \tag{1.3}$$

However, this conserved quantity alone is unable to show the global existence of a solution from a smooth initial data. Indeed, the problem of global existence for the 3D incompressible Euler equations is a major unsolved problem.

The equation (1.2) is a stronger statement than the conservation of total kinetic energy and it is called the *local energy identity*. It implies that the conservation of kinetic energy holds in any local region. In other words, the the rate of change of kinetic energy in an arbitrary local region is balanced with energy flux and work done by pressure through the boundary of the region.

1.1 Anomalous dissipation in the vanishing viscosity limit

Upon adding a dissipative term to the Euler equations, one gets better compactness for approximate solutions. Such a dissipative term could be a Laplacian, a fractional Laplacian, etc. In case of the Laplacian, one obtains the incompressible Navier-Stokes equations. For a

given constant $\nu > 0$, these are given by

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla p^\nu = \nu \Delta u^\nu \\ \operatorname{div} u^\nu = 0 \end{cases} \tag{1.4}$$

The constant ν is called the *viscosity* of the fluid and is a measure of the internal friction in the fluid. Note that one formally gets the Euler equations (1.1) in the *vanishing viscosity limit* $\nu \to 0^+$.

In the case of the 3D incompressible Navier-Stokes equations (1.4), one is able to show the global existence of so-called *suitable weak solutions*. These are weak solutions of the N-S eqns. that in addition satisfy an analogous local-energy inequality:

$$\partial_t \frac{|u^\nu|^2}{2} + \operatorname{div}\left[u^\nu \left(\frac{|u^\nu|^2}{2} + p^\nu\right)\right] \leq \nu \Delta \frac{|u^\nu|^2}{2} - \nu |\nabla u^\nu|^2 \,. \tag{1.5}$$

In fact, such suitable weak solutions exist for a wide class of dissipative terms like the fractional Laplacians Δ^α for $\alpha > \frac{3}{4}$. Also, for $\alpha > \frac{5}{4}$, one has the existence of a unique global smooth solution from smooth initial data. Proving the same statement for the Navier-Stokes eqns. is another major unsolved problem and is one of the Millennium Prize problems.

We now introduce the important concept of the *Duchon-Robert measure*. It will quantify the change in kinetic energy in a fluid due to the possible presence of singularities in the fluid velocity field. For a divergence-free vector field u, the Duchon-Robert measure $D(u)$ is defined as follows: for a smooth, compactly-supported function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that ϕ is even, non-negative, and has unit mean, let $\phi_\varepsilon(x) := \varepsilon^{-3}\phi(x/\varepsilon)$ for $\varepsilon > 0$; now we define

$$D(u) := \mathcal{D} - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \nabla \phi_\varepsilon(\ell) \cdot (u(\cdot + \ell) - u(\cdot)) \, |u(\cdot + \ell) - u(\cdot)|^2 \, d\ell \tag{1.6}$$

where $\mathcal{D} - \lim$ denotes the limit in the sense of distributions. We note that $D(u) \equiv 0$ for a smooth vector field u. Duchon and Robert introduced this concept in [20] and in that

paper they proved that for the weak solutions of Navier-Stokes constructed by Leray in his influential paper [32], one has that $D(u^\nu) \geq 0$ and moreover one has the equality

$$\partial_t \frac{|u^\nu|^2}{2} + \operatorname{div}\left[u^\nu\left(\frac{|u^\nu|^2}{2} + p^\nu\right)\right] = \nu\Delta\frac{|u^\nu|^2}{2} - \nu|\nabla u^\nu|^2 - D(u^\nu)\,.$$

In particular, $D(u^\nu)$ is independent of the choice of ϕ. The above equality implies that dissipation of energy of turbulent Navier-Stokes flows u^ν occurs either from viscosity or from potential singularities of u^ν and these are measured by $\nu|\nabla u^\nu|^2$ and $D(u^\nu)$, respectively.

It is a natural question to ask whether suitable weak solutions to N-S converge to a solution of Euler as the viscosity parameter $\nu \to 0^+$. If we assumed that the solutions u^ν to the corresponding ν-N-S equations converged to a vector field u in $L^3_{t,x}$, then the local energy inequality would imply that

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left[u\left(\frac{|u|^2}{2} + p\right)\right] = -D(u) \leq 0\,,$$

where $D(u)$ is the Duchon-Robert measure for the solution u given by (1.6). This measure exists (and is unique for any choice of regularization by convolution) for any $L^3_{t,x}$ solution of the incompressible Euler equations as proved by Duchon and Robert (c.f. [20]). Thus the content of the above formula is that this measure is non-negative: i.e. energy cannot be created locally through the evolution of the solution. Based on the above reasoning based on vanishing viscocity limits, Duchon and Robert further postulate that physical solutions of the incompressible Euler eqns. must, in addition, satisfy $D(u) \geq 0$ (i.e. the local energy inequality).

In this context, anomalous dissipation is the statement that $D(u) \geq 0$ *and* $D(u) > 0$ somewhere. Upon taking spatial averages $\langle\cdot\rangle$ in the above vanishing viscosity limit, one has that

$$\bar{\varepsilon} := \lim_{\nu\to 0^+} \langle \nu|\nabla u^\nu|^2 + D(u^\nu)\rangle = \langle D(u)\rangle > 0 \tag{1.7}$$

In other words, anomalous dissipation states that the mean total energy dissipation rate remains strictly positive in the inviscid limit. Thus solutions of incompressible Euler (1.1) exhibiting anomalous dissipation will strictly dissipate their total kinetic energy. This phenomenon of anomalous dissipation has been experimentally verified and confirmed by various numerical simulations and often goes by the term *the zeroth law of turbulence*. As already discussed smooth solutions to the incompressible Euler eqns. must conserve total kinetic energy and so, they cannot model real turbulent fluids. On the other hand, considerations based on vanishing viscosity suggests that turbulence is necessarily modelled by the incompressible Euler eqns. (c.f. [36]) So, we conclude that real turbulent fluids can be modelled by weak *but not* strong solutions to the incompressible Euler eqns. Here kinetic energy can transfer from large to small scales and eventually dissipate into thermal energy, even in the absence of viscosity. The mathematical validation of this law remains one of the major open problems in fluid dynamics. (For the mathematical formulation of the zeroth law, see for example [27].) A natural question now is to explore at what regularity solutions can start to dissipate energy. This will be the content of the next section.

1.2 The strong Onsager conjecture

Lars Onsager in his influential study of turbulence ([36]) explored weak solutions of Euler and Navier-Stokes noted that:

" It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

$$|v(r' + r) - v(r')| < (const.)r^n$$

for any order n greater than 1/3; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation in terms of FOURIER series will do. "

It is amusing to note that what Onsager means by spaces satisfying a Lipschitz condition is what we now call Hölder spaces. Thus, in explaining the anomalous dissipation phenomenon, Onsager proposed a threshold Hölder regularity for the conservation of total kinetic energy in Euler flows as 1/3; when a weak solution[1] is in the space $L_t^\infty C_x^\alpha$ with $\alpha > 1/3$, the total kinetic energy is always conserved as in (1.3), while for $\alpha < 1/3$, the conservation may fail. This is now referred to as the Onsager's theorem after the rigidity statement is rigorously proved by Constantin-E-Titi [11] (extended further in [10]), and the flexibility part is resolved by Isett [26] (see also [6]) building upon the serious of developments [14, 15, 25, 1, 2, 4, 3, 5, 12, 28].

Theorem 1.2.1 (The Onsager Theorem). *Let (v, p) be a weak solution of the incompressible Euler eqns. on the periodic 3-dimensional torus $\mathbb{T}^3$ with*

$$|v(x,t) - v(y,t)| \leq C|x-y|^\theta \qquad \forall x, y, t$$

(where C is a constant independent of x, y, t).

- **(Rigidity)** *If $\theta > \frac{1}{3}$, then $E(t) = \int_{\mathbb{T}^3} |v|^2 \, dx$ is necessarily constant;*
- **(Flexibility)** *For $\theta < \frac{1}{3}$ there are solutions for which $E(t)$ is strictly decreasing.*

However, the solutions constructed in the flexible side of the Onsager theorem do *not* satisfy the local energy inequality (1.5) and consequently, these solutions have no chance of arising as vanishing viscosity limits of suitable weak solutions of the Navier-Stokes eqns. Recall that the local energy inequality prevents the local creation of kinetic energy but

[1] u is a weak solution to the Euler equations iff $u \in L^2_{t,x}$ satisfies (1.1) in the distributional sense.

allows its dissipation and, as we have seen, following [20], such dissipation can arise from a possible singularity of the solution u and is captured by Duchon-Robert measure $D(u)$. In the context of a hyperbolic system of conservation laws, the local energy inequality serves as the entropy condition, which plays a crucial role in identifying physically acceptable solutions, particularly in scalar conservation laws. For instance, the Burger's equation, which are regarded as a 1D model of the Euler equations, have the uniqueness of bounded weak solution under the entropy condition analogous to (1.5). Motivated by these considerations, in an attempt to obtain physically-relevant solutions to the incompressible Euler eqns. (1.1), Isett in [27] proposed the following *strong Onsager conjecture*:

Conjecture 1 (**Strong Onsager Conjecture).** *There exists an open interval I, and a weak solution (v, p) to the incompressible Euler equations on $I \times \mathbb{T}^3$ that is of class $v \in L^\infty_t C^{1/3}_x$ and satisfies the local energy inequality with the left hand side $D(u)$ not identically zero.*

Isett in [27] also showed the existence of such solutions that have a 1/15-Hölder regularity. This regularity has been improved to 1/7 by De Lellis and Kwon in [17] where they prove the following theorem which is the state-of-the-art (at the time of writing this) towards Hölder continuous solutions to the above strong Onsager conjecture.

Theorem 1.2.2. *For any $0 \leq \beta < 1/7$ there are* strictly dissipative *weak solutions v to the incompressible Euler equation in $C^\beta([0, T] \times \mathbb{T}^3)$ for which $D(u) \leq 0$.*

1.2.1 Relation to weak-strong uniqueness

We remark that the techniques used to obtain above existence results as well as those used in this thesis are unable to specify the initial data in a Cauchy problem. Indeed, if one were to specify smooth initial data and were able to construct dissipative solutions to (1.1) satisfying (1.5), then one would have proved a blow-up result for the incompressible Euler equations. This is because local well-posedness will imply the existence of a smooth solution for a short time and our constructed solution will have to agree with this smooth solution during its

time of existence by the weak-strong uniqueness principle. The fact that the solution is dissipative implies that the smoothness must break down in finite time by the rigidity part of the Onsager theorem.

Also, it is important to note that the flows exhibiting anomalous dissipation that have been observed in experiments and numerical simulations need not have developed from smooth initial data. Quoting Eyink in [22]: The most common experiments study turbulent flows produced downstream of wire-mesh grids or are generated by flows past an obstacle. In either case, the generation of turbulence is associated to vorticity fed into these flows by viscous boundary layers that detach from the walls. Since the boundary layers get thinner as $\nu \to 0$, the initial data of these experiments cannot be considered to be smooth uniformly in ν. See [22] for more details.

1.3 Fully-developed turbulence

Kolmogorov initiated the modern statistical study of turbulence in his famous 1941 papers, c.f. [30, 31]. The theory initiated by him goes by the name K41 theory and is closely related to the conjectures of Onsager outlined above. We describe the theory and a local formulation of it due to Eyink [22] below.

K41 is a statistical theory by which we mean that we have a good notion of a "random" solution to the fluid equations that can be chosen from some appropriate probability space. We will denote here by $\langle \cdot \rangle_{\text{stat}}$ to denote the statistical average of the random solutions. Note that this notion has not yet been given a fully rigorous mathematical foundation and must therefore be regarded as purely "physical" motivation.

Given a random velocity field that describes the motion of an incompressible fluid on $\mathbb{T}^3$, we define for $x, r \in \mathbb{T}^3$ the two-point velocity increment as

$$\delta u(x, r) := u(x + r) - u(x) \,.$$

We also define the longitudinal and transverse velocity increments respectively as

$$\delta^{\parallel}u(x,r) := \delta u(x,r)\cdot\frac{r}{|r|}\,, \qquad \delta^{\perp}u(x,r) := \delta u(x,r) - \delta^{\parallel}u(x,r)\frac{r}{|r|}\,.$$

The famous 4/5 law of Kolmogorov's K41 theory states that

$$\langle(\delta^{\parallel}(x,\ell))^3\rangle_{\text{stat}} = -\frac{4}{5}\bar{\varepsilon}|\ell|\,. \tag{1.8}$$

This law was derived by Kolmogorov from his assumptions of homogeneity, isotropy, and self-similarity. It has also been verified by numerous experiments and numerical simulations and is regarded as an exact law of turbulence.

1.3.1 A local K41 theory

The Duchon-Robert measure is remarkable as it is connected with an exact law in turbulence: the so-called "Kármán-Howarth-Monin relation"

$$\nabla_\ell\cdot\langle\delta u(x,\ell)|\delta u(x,\ell)|^2\rangle_{\text{stat}} = -4\bar{\varepsilon}\,.$$

More generally, it was noticed by Eyink in [22] that the Duchon-Robert measure is a key player in deriving local versions of the various laws in K41 theory.

Already, Duchon and Robert in [20] prove a rigorous form of the the so-called "4/3-law". We reproduce their argument here: first set

$$S(u)(x,r) := \int_{S^2}|\delta u(x,r\ell)|^2(\delta u(x,r\ell))\cdot\widehat{\ell}\,d\ell$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $\widehat{\ell}$ is the unit vector. Since the Duchon-Robert measure does not depend on the choice of ϕ for the solutions of (1.1) that we consider, we choose $\phi(\ell) = \phi(|\ell|)$ to be a radially-symmetric function. Now a computation gives us that upon

setting

$$D^\varepsilon(u) := \frac{1}{4}\int_0^\infty \phi'(r) r^3 \frac{S(u)(x,\varepsilon r)}{\varepsilon r}\, dr$$

we have that $\mathcal{D} - \lim_{\varepsilon\to 0} D^\varepsilon(u) = D(u)$. Moreover, *assuming* that as $\varepsilon \to 0$, $S(u)(x,\varepsilon)/\varepsilon$ tends to a limit $s(u)(x)$, then

$$D^\varepsilon(u) \to \frac{1}{4} s(u) \int_0^\infty \phi'(r) r^3\, dr = -\frac{3}{4} s(u)$$

and so we get $s(u) = -\frac{4}{3} D(u)$ which is a local and non-random form of Kolmogorov's 4/3 law.

Similarly, the main theorem in [22] is

Theorem 1.3.1 (Theorem 1 of [22]). *Let $u \in L^3([0,T]\times\mathbb{T}^3)$ be a weak solution of* (1.1). *Now let*

$$D_L^\varepsilon(u) := \frac{3}{4}\int_{\mathbb{T}^3} \nabla\phi_\varepsilon(\ell)\cdot\delta u(\ell)|\delta^{\|}u(\ell)|^2 + \frac{2}{|\ell|}\phi_\varepsilon(\ell)\delta^{\|}u(\ell)|\delta^{\perp}u(\ell)|^2\, d\ell$$
$$D_T^\varepsilon(u) := \frac{3}{8}\int_{\mathbb{T}^3} \nabla\phi_\varepsilon(\ell)\cdot\delta u(\ell)|\delta^{\perp}u(\ell)|^2 - \frac{2}{|\ell|}\phi_\varepsilon(\ell)\delta^{\|}u(\ell)|\delta^{\perp}u(\ell)|^2\, d\ell$$

Then, both $D_L^\varepsilon(u)$ and $D_T^\varepsilon(u)$ converge as distributions to $D(u)$ as $\varepsilon \to 0$.

As corollaries of this theorem, under similar assumptions needed in the 4/3 law, Eyink is able to derive rigorously local versions of the 4/5 and 4/15 laws in K41 theory:

$$\mathcal{D} - \lim_{\ell\to 0}\frac{1}{\ell}\int_{S^2}(\delta^{\|}u(x,\ell))^3\, d\ell = -\frac{4}{5}D(u)\,, \quad \mathcal{D} - \lim_{\ell\to 0}\frac{1}{\ell}\int_{S^2}\delta^{\|}u(x,\ell)|\delta^{\perp}u(x,\ell)|^2\, d\ell = -\frac{8}{15}D(u)$$

The sign of $D(u)$ then implies the remarkable fact of K41 theory that the cubic powers of the longitudinal increments have a sign on average.

1.3.2 Anomalous dissipation as a conservation law anomaly

A fruitful way of thinking about anomalous dissipation is to think of the energy in a fluid being transferred from coarse scales to finer scales through non-linear interactions of the fluid. However, our experiments cannot detect arbitrarily small scales and so we are constrained to make measurements above a certain scale $\ell > 0$. So we can only measure a "coarse-grained" u_ℓ flow rather than the actual flow with infinite precision. One way of mathematically-modelling this coarse grained flow is by mollifying u with a smooth, compactly-supported, unit mean bump function at spatial scale ℓ. Another (almost equivalent way) is to model the coarse grained flow as $\mathbb{P}_{\leq \ell^{-1}} u$ where $\mathbb{P}_{\leq \ell^{-1}}$ is a Littlewood-Paley projector to frequencies less than ℓ^{-1}. In fact this is the approach Onsager took in [36] and which we will now briefly explain following Eyink in [23].

This coarse grained flow $\mathbb{P}_{\leq \ell^{-1}} u$ does not solve the incompressible Euler eqns. (1.1) as the projector $\mathbb{P}_{\leq \ell^{-1}}$ does not commute with the quadratic non-linearity in (1.1). However, applying $\mathbb{P}_{\leq \ell^{-1}}$ to (1.1) gives us

$$\begin{cases} \partial_t \mathbb{P}_{\leq \ell^{-1}} u + \operatorname{div}(\mathbb{P}_{\leq \ell^{-1}} u \otimes \mathbb{P}_{\leq \ell^{-1}} u) + \nabla \mathbb{P}_{\leq \ell^{-1}} p = \operatorname{div}\left(\mathbb{P}_{\leq \ell^{-1}} u \otimes \mathbb{P}_{\leq \ell^{-1}} u - \mathbb{P}_{\leq \ell^{-1}}(u \otimes u)\right) \\ \operatorname{div} \mathbb{P}_{\leq \ell^{-1}} u = 0 \end{cases} \tag{1.9}$$

The 2-tensor in the divergence on the right hand side of the first equation is called the Reynolds stress and we denote it by $-R_\ell$. Now taking the dot product of the first eqn. with $\mathbb{P}_{\leq \ell^{-1}} u$, we get the local energy identity for the coarse-grained system

$$\partial_t \frac{|\mathbb{P}_{\leq \ell^{-1}} u|^2}{2} + \operatorname{div}\left[\mathbb{P}_{\leq \ell^{-1}} u \left(\frac{|\mathbb{P}_{\leq \ell^{-1}} u|^2}{2} + \mathbb{P}_{\leq \ell^{-1}} p + R_\ell\right)\right] = \nabla \mathbb{P}_{\leq \ell^{-1}} u : R_\ell \tag{1.10}$$

The quantity on the right hand side of the above eqn. is the so-called "deformation work" of the large-scale strain against the small-scale Reynolds stress. It can also be interpreted as the "energy flux" from the resolved scales $\geq \ell$ to the unresolved scales $< \ell$.

The work of Constantin-E-Titi [11] shows that for $|\delta u(x,r)| \lesssim r^\beta$, we have

$$R_\ell = O(\ell^{2\beta})\,, \qquad \nabla \mathbb{P}_{\leq \ell^{-1}} u : R_\ell = O(\ell^{3\beta-1})$$

from which we can easily see that $\beta > 1/3$ implies conservation of energy.

The Reynolds stress tensor R_ℓ is not a simple functional of the resolved/coarse-grained velocity $\mathbb{P}_{\leq \ell^{-1}} u$. Indeed, path integral approaches combined with renormalization group techniques compute R_ℓ to be a highly complicated functional of $\mathbb{P}_{\leq \ell^{-1}} u$ with transcendental non-linearity, long-term memory, and intrinsic stocasticity. See Eyink [21] for more details. This lack of a simple expression for R_ℓ in terms of the resolved velocity is what is referred to as the "closure problem" in turbulence. One can guess that this might lead to potential non-uniqueness and indeed, the convex integration techniques used in this thesis exploit this.

From the renormalization group point of view, the weak solutions of (1.1) proposed by Onsager correspond to taking the UV limit $\ell \to 0$, so that

$$R_\ell \to 0\,, \qquad \mathcal{D} - \lim \nabla \mathbb{P}_{\leq \ell^{-1}} u : R_\ell \to -D(u)$$

Thus we see that the local energy identity (1.2) has to be modified by the "anomaly term" $D(u)$ due to the non-linear energy flux $\nabla \mathbb{P}_{\leq \ell^{-1}}$ which persists even as the length scale $\ell \to 0$. Quoting Eyink [23]: *As first noted by Polyakov [37, 38], there is a striking analogy to conservation-law anomalies in quantum field theory, where terms similar to $D(u)$ appear that vitiate conservation laws which hold classically.*

1.3.3 Intermittency

The moments of the velocity increments $\delta u, \delta^\parallel u, \delta^\perp u$ encode encode valuable information about the fine structure of turbulent flow and are referred to as *structure functions*. Let us define

$$S_{m,n}(|r|) := \langle\langle (\delta^\parallel u(x,r))^m |\delta^\perp u(x,r)|^n \rangle\rangle_{\text{stat}} \tag{1.11}$$

where the inner angle brackets denote an average in space, time, and the angle $r/|r| \in S^2$. The reason for this averaging is though K41 theory assumes statistical homogeneity in space and time and isotropy of our random field, numerical simulations and experiments will always have residual inhomogeneity and anisotropy due to the setup which we'd like to remove. See [29] for more details.

In the preceding discussion, we have already come across various structure functions. For instance, the 4/5 law is simply the statement $S_{3,0}(|\ell|) = -{}^4\!/_5\bar{\varepsilon}|\ell|$. However, notice that we have only discussed these functions in cases when $p := m + n = 3$. Indeed, the approach of Duchon and Robert [20] and Eyink [22] are unable to say anything regarding the case $p \neq 3$. For any m, n let us define the *structure function exponents* as

$$\zeta_{(m,n)} := \lim_{h \to 0} \frac{\log S_{m,n}(h)}{\log(\bar{\varepsilon}h)} .$$

Note that the self-similar scaling in K41 theory predicts that $\zeta_{(m,n)} = {}^{(m+n)}\!/_3$.

The case $p = 3$ correspond to exact laws of turbulence and is supported by all the experimental evidence. However, in the case $p \neq 3$: experiments and numerical simulations have confirmed that the real world fluid does not align with Kolmogorov's prediction $\zeta_{(p,0)} \approx {}^p\!/_3$, rather it shows the following deviation; when $p < 3$, ${}^{\zeta_{(p,0)}}\!/_p > 1/3$ while when $p > 3$, ${}^{\zeta_{(p,0)}}\!/_p < 1/3$. Such deviation is attributed to the intermittency of turbulent flows. To put down a definition for intermittency, we quote Buckmaster and Vicol in [8]:

" In a broad sense, intermittency is characterized as a deviation from the Kolmogorov 1941 laws. Already in 1942 Landau remarked that the rate of energy dissipation in a fully developed turbulent flow is observed to be spatially and temporally inhomogeneous, and thus Kolmogorov's homogeneity and isotropy assumptions need not be valid ... The main feature seems to be the presence of sporadic dramatic events, during which there are large excursions away from the average. "

The results of numerical simulations performed by Iyer, Sreenivasan and Yeong [29]

suggest that as n grows, the transverse structure function exponents $\zeta_{(0,n)}$ appear to saturate towards a value $\zeta_\infty^T \approx 2$. This indicates that the flows have very large jumps in the transverse gradient and that they are merely bounded with no Hölder regularity!

Now similar to $S_{m,n}$, the pth order *absolute* structure function $S_p(\ell)$ satisfies a scaling relation of the form

$$S_p(\ell) := \langle |\delta u(x,\ell)|^p \rangle_{\text{stat}} \sim |\ell|^{\zeta_p} ,$$

where the exponent ζ_p depending on p is given as a positive number. It's worth noting that the implicit constant in this relation is independent of viscosity. For $p = m + n$, this indicates the uniform boundedness of turbulent flows in the Besov space $B^{\zeta_p/p}_{p,\infty}$, considering the equivalence

$$\|v\|_{B^s_{p,\infty}(\mathbb{T}^3)} \sim \|v\|_{L^p(\mathbb{T}^3)} + \sup_{|z|>0} \frac{\|v(\cdot + z) - v\|_{L^p(\mathbb{T}^3)}}{|z|^s} .$$

The experimental and numerical observations as briefly outlined above suggest that $B^{1/3}_{3,\infty} \cap L^\infty$ is a more physically reasonable space for turbulent flows than Hölder space $C^{1/3}$, where the Onsager theorem was proven. In this direction, the intermittent Onsager theorem was recently obtained in [35] where a constructed non-conservative solution to (1.1) in the class $C^0_t(H^{1/2-} \cap L^{\infty-}) \subset C^0_t B^{1/3-}_{3,\infty}$ and so accommodates the intermittent nature of observed turbulence.

1.4 An L^3-based strong Onsager conjecture

With the significance of the local energy inequality in mind and adapting to the intermittent nature of turbulence, we now introduce an L^3-based version of the strong Onsager conjecture.

Conjecture 2 (L^3**-based strong Onsager conjecture).** *Let* $\beta \in (0,1)$ *and* $T \in (0,\infty)$.

- **(Rigidity)** *For any* $\beta > 1/3$, *if a weak solution to the Euler equations is in* $C^0([0,T]; B^\beta_{3,\infty}(\mathbb{T}^3))$,

then it satisfies the local energy identity $D(u) \equiv 0$ in distribution sense.

- **(Flexibility)** *For any $\beta < 1/3$, there exists a weak solution to the Euler equations in $C^0([0,T]; B^{\beta}_{3,\infty}(\mathbb{T}^3))$ which satisfies the local energy inequality* (1.5) *in distribution sense but not identity $D(u) \equiv 0$.*

The rigidity part has been established by Duchon-Robert [20]. In this thesis, we will give a proof of the flexible side when β is in the remaining region $[1/7, 1/3)$, leading to the full resolution of the L^3-based strong Onsager conjecture. We note that the critical case $\beta = 1/3$ still remains open.

Chapter 2

The Main theorem

In our main theorem, we solve the flexible side of our L^3-based version of the strong Onsager conjecture.

Theorem 2.0.1 (**Main theorem**). *For any fixed* $\beta \in (0, 1/3)$ *and* $T > 0$*, we can find a weak solution* u *in* $C_t^0(B_{3,1}^{\beta} \cap L^{\frac{27-80\beta}{9(1-3\beta)}})([0,T] \times \mathbb{T}^3)$ *to the Euler equations which dissipates the total kinetic energy and satisfies the local energy inequality. In particular, the solution is in* $C_t^0 B_{3,\infty}^{\beta}$.

The proof of the main theorem will follow from the inductive proposition 5.6.1 and will be given in section 5.6. We now outline the organization of the chapters of this thesis:

Chapter 3 explains the main new ideas and difficulties in the proof of the inductive Proposition 5.6.1.

Chapter 4 specifies parameter choices and provides useful inequalities resulting from these choices.

Chapter 5 outlines the proof of Theorem 2.0.1 through an inductive argument based on convex integration. We list the inductive hypotheses, introduce an inductive proposition 5.6.1, and present a proof of the theorem assuming the proposition.

Chapter 6 is a short technical chapter that provides definitions and estimates on various mollified objects in the scheme. This is to overcome the "loss of derivatives" phenomenon

that is common in convex integration schemes.

Chapter 7 introduces the main "building blocks" for our wavelet-based scheme: the intermittent Mikado bundles. We discuss dodging between straight pipes first and prove the disjoint support property. We then introduce a general wavelet decomposition for our intermittent objects: the synthetic Littlewood-Paley decompositions.

Chapter 8 is a technical chapter that constructs various partitions of unity.

Chapter 9 constructs the velocity perturbation increment that will be used to correct our errors.

Chapter 10 considers the stress error generated by adding the new velocity increment to the relaxed Euler-Reynolds system at qth step. We define the stress errors and associated pressure increments, providing estimates for both.

Chapter 11 considers the current error generated by adding the new velocity increment to the relaxed local energy inequality at qth step. We define the current errors and associated pressure increments, providing estimates for both.

Chapter 12 constructs a partition of unity so that the different regions provide a sharp Lipschitz control of our background intermittent valocity field.

Chapter 13 gathers all pressure increments constructed in previous chapters and defines a new pressure increment and a new intermittent pressure, along with their estimates and verification of relevant inductive assumptions. These new pressures generates new current errors. We estimate them and finalize the definition of stress/current errors at $(q+1)$th step.

Appendix A contains various tools that are needed to run our convex integration scheme. These include a L^p decoupling lemma, an inverse divergence operator that (essentially) preserves the support of its input, and lemma that constructs a high-frequency "pressure increment" that will be used to pointwise dominate the various errors.

Chapter 3

Idea of Proof

Our approach to constructing solutions to (1.1) that satisfy (1.5) will be a convex integration or Nash iterative scheme. The idea is to solve the equation approximately by solving a "relaxed" equation with an error term. We run an iterative procedure where at each step of the iteration we cancel the error by adding in a velocity perturbation to our original velocity. The error is cancelled by the *non-linear* interactions with the perturbation with itself: where the interactions are from the quadratic non-linearity in (1.1) and from the cubic non-linearity in (1.5). We refer to the excellent surveys [16] and [8] for more information surrounding convex integration schemes for equations in fluid dynamics.

The first step then is to write down the relaxed equations that we will use in our scheme. In order to motivate these equations, we should think of these equations as modelling a coarse-grained solution at some scale ℓ as was discussed in subsection 1.3.2. Over there, we already derived the relaxed equations for (1.1). Indeed, they are exactly the equations (1.9):

$$\begin{cases} \partial_t \overline{u} + \operatorname{div}(\overline{u} \otimes \overline{u}) + \nabla \overline{p} = \operatorname{div}\left(\overline{u} \otimes \overline{u} - \overline{u \otimes u}\right) = \operatorname{div} R \\ \operatorname{div} \overline{u} = 0 \end{cases}$$

where we have used the notation $\overline{(\cdot)} = \mathbb{P}_{\leq \ell^{-1}}(\cdot)$.

Now to write the relaxed equations to (1.5), rather than following Onsager and writing

(1.10), it turns out to be more helpful to coarse-grain (1.5) directly, i.e., we apply the coarse-graining operator (say, $\mathbb{P}_{\leq \ell^{-1}}$) to (1.5) directly to derive an equation that is satisfied by $\overline{u}$ and $\overline{p}$. So we have

$$\partial_t \frac{|\overline{u}|^2}{2} + \mathrm{div}\left[\overline{u}\left(\frac{|\overline{u}|^2}{2} + \overline{p}\right)\right] \\ = (\partial_t + \overline{u}\cdot\nabla)\left[\frac{|\overline{u}|^2}{2} - \frac{\overline{|u|^2}}{2}\right] - \mathrm{div}(R_\ell \overline{u}) - \mathrm{div}\overline{(p-\overline{p})u} - \mathrm{div}\left(\frac{1}{2}\overline{|w|^2 w}\right)$$

where we have used $R = \overline{u}\otimes\overline{u} - \overline{u\otimes u}$ and set $w = u - \overline{u}$. We set $\kappa := \frac{|\overline{u}|^2}{2} - \frac{\overline{|u|^2}}{2} = \frac{1}{2}\mathrm{tr}\, R$. We have also used that

$$\frac{1}{2}\overline{|u|^2 u} - \frac{1}{2}|\overline{u}|^2\overline{u} = \frac{1}{2}\overline{|w|^2 w} - R\overline{u} - \kappa\overline{u}$$

So, we get the relaxed equations

$$\partial_t \frac{|\overline{u}|^2}{2} + \mathrm{div}\left[\overline{u}\left(\frac{|\overline{u}|^2}{2} + \overline{p}\right)\right] = (\partial_t + \overline{u}\cdot\nabla)\kappa + \mathrm{div}(R_\ell\overline{u}) + \mathrm{div}\varphi_\ell$$

where φ_ℓ is the *unresolved energy flux* that scales like the cubic power of w. Using these relaxed equations, one can now derive the new error terms obtained by adding in a specially chosen perturbation w such that the "low-frequency" part of $w\otimes w$ cancels R_ℓ and the "low-frequency" part of $\frac{1}{2}w|w|^2$ cancels φ_ℓ. A detailed derivation of these new error terms for the relaxed Euler-Reynolds system and the relaxed local energy inequality will be found at the beginning of Chapters 10 and 11 respectively.

3.1 Heuristic computations

We now provide heuristic estimates that indicate the choices of the sizes of the various parameters that will lead to the solutions to the relaxed equations converging an actual solution of (1.1) and (1.5) having our desired regularity.

3.1.1 Difficulties in a homogeneous scheme

In order to motivate an L^3 iteration, we must identify the main difficulties in a hypothetical $C^{1/3-}$ iteration. We recall that in [27], Isett constructed $C^{1/15-}$ weak solutions of 3D Euler as a limit of subsolutions u_q to the following system:

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \operatorname{div} R_q \\ \partial_t \left(\frac{1}{2}|u_q|^2\right) + \operatorname{div}\left(\left(\frac{1}{2}|u_q|^2 + p_q\right) u_q\right) \leq (\partial_t + u_q \cdot \nabla)\kappa_q + \operatorname{div}\varphi_q + \operatorname{div}(R_q u_q) \\ \operatorname{div} u_q = 0\,. \end{cases} \tag{3.1}$$

Here R_q is a negative definite symmetric tensor known as the Reynolds stress error, $\kappa_q = {}^1\!/\!_2 \operatorname{tr} R_q$, and φ_q is a vector field called the current error; all three terms converge to zero in the sense of distributions, thus producing in the limit a weak solution u to 3D Euler which satisfies the local energy inequality. The functions u_q, R_q, φ_q are assumed to oscillate at spatial frequencies no larger than $\lambda_q \approx a^{(b^q)}$, where a is sufficiently large and $b > 1$ is as small as possible. Abbreviating the mixed $L^\infty_t L^p_x$ norms with simply $\|\cdot\|_p$, the natural inductive estimates for a C^β scheme are

$$\|u_q\|_\infty \lesssim 1\,, \qquad \left\|\nabla_x^N \nabla u_q\right\|_\infty \leq \lambda_q^{-\beta+N+1}\,, \qquad \left\|\nabla_x^N R_q\right\|_\infty \leq \lambda_{q+1}^{-2\beta}\lambda_q^N\,, \qquad \left\|\nabla_x^N \varphi_q\right\|_\infty \leq \lambda_{q+1}^{-3\beta}\lambda_q^N\,.$$

Note that interpolating the first two bounds shows that $\{u_q\}_{q=1}^\infty$ is uniformly bounded in C^β norms. Then $w_{q+1} = u_{q+1} - u_q = w_{q+1,R} + w_{q+1,\varphi}$ is constructed to oscillate at frequency $\lambda_{q+1} = \lambda_q^b$. Eliding for the moment the fact that R_q and φ_q are not scalar-valued functions, we define

$$w_{q+1,\varphi} \approx (\sin(\lambda_{q+1}(x_2+x_3)), 0, 0)^T (-2\varphi_q)^{1/3}\,, \quad w_{q+1,R} \approx (\sin(\lambda_{q+1}(x_2+x_3)), 0, 0)^T \left((-2\varphi_q)^{1/3} + (-R_q)^{1/2}\right)$$

using products of high-frequency shear flows and low-frequency functions. Notice that both terms in the low-frequency portion of $w_{q+1,R}$ have the same size, and that the low-

frequency components of the quadratic and cubic nonlinear terms $\mathbb{P}_{\leq\lambda_q}(w_{q+1,R}\otimes w_{q+1,R})$ and $\mathbb{P}_{\leq\lambda_q}(|w_{q+1,\varphi}|^2 w_{q+1,\varphi})$, respectively, cancel $R_q+\mathbb{P}_{\leq\lambda_q}\left(w_{q+1,\varphi}\otimes w_{q+1,\varphi}\right)$ and $2\varphi_q$, respectively. Then w_{q+1} satisfies

$$\left\|\nabla_x^N w_{q+1}\right\|_\infty \lesssim \|R_q\|_\infty^{\frac{1}{2}}\lambda_{q+1}^N + \|\varphi_q\|_\infty^{1/3}\lambda_{q+1}^N \lesssim \lambda_{q+1}^{-\beta+N}\,.$$

Interpolating the bounds for $N=0,1$, we find that w_{q+1} has unit C^β norm, as did u_q. The new Reynolds stress will then include the error term $R_{q+1,\text{Nash}}=\operatorname{div}^{-1}\left(w_{q+1}\cdot\nabla u_q\right)$ (named after the analogous error term in Nash's original isometric embedding iteration [34]), which can be estimated by

$$\begin{aligned}
\Big|\underbrace{\operatorname{div}^{-1}}_{\text{gains }\lambda_{q+1}}\big(\underbrace{w_{q+1}}_{\lambda_{q+1}^{-\beta}}\cdot\underbrace{\nabla u_q}_{\approx\lambda_q^{1-\beta}}\big)\Big| &\leq \lambda_{q+2}^{-2\beta}\\
\iff \lambda_{q+1}^{-1-\beta}\lambda_q^{1-\beta} &\leq \lambda_{q+2}^{-2\beta}\\
\iff \lambda_q^{b(-1-\beta)+1-\beta+2\beta b^2} &\leq 1\\
\iff \beta(2b^2-b-1) &\leq b-1\\
\iff \beta &\leq \frac{1}{2b+1}\,.
\end{aligned}$$

Thus as $b\to 1$, $\beta\to 1/3$, as desired. However, the analogous error term in the local energy inequality, called the Nash current error, only satisfies the estimate

$$\begin{aligned}
\Big|\underbrace{\operatorname{div}^{-1}}_{\text{gains }\lambda_{q+1}}\big(\underbrace{\mathbb{P}_{=\lambda_{q+1}}(w_{q+1}\otimes w_{q+1})}_{\lambda_{q+1}^{-2\beta}}:\underbrace{\nabla u_q}_{\approx\lambda_q^{1-\beta}}\big)\Big| &\leq \lambda_{q+2}^{-3\beta}\\
\iff \lambda_{q+1}^{-1-2\beta}\lambda_q^{1-\beta} &\leq \lambda_{q+2}^{-3\beta}\\
\iff \lambda_q^{b(-1-2\beta)+1-\beta+3\beta b^2} &\leq 1\\
\iff \beta(3b^2-2b-1) &\leq b-1\\
\iff \beta &\leq \frac{1}{3b+1}\,.
\end{aligned}\tag{3.2}$$

This evident $1/4$ regularity ceiling is also imposed by several similar current error terms. All the evidence from existing Nash iteration schemes indicates that the above heuristics cannot be improved. Furthermore, the best C^α result to date suffers from further complications which limit the regularity to $C^{1/7-}$ [17], suggesting that *even in the most optimistic scenario, Nash iterations are incapable of reaching the $C^{1/3-}$ threshold for the strong Onsager conjecture.*

3.1.2 Heuristics for an intermittent scheme

The first difference between an intermittent Nash iteration and a C^α iteration is that the high-frequency shear flow $\sin(\lambda_{q+1}(x_2+x_3))\vec{e}_1$ is replaced by a pair of *intermittent* shear flows $\varrho_{q+1,R}(x_2,x_3)\vec{e}_1$ and $\varrho_{q+1,\varphi}(x_2,x_3)\vec{e}_1$ (described in detail in the next section) which satisfy

$$\left\|\nabla_x^N \varrho_{q+1,R}\right\|_p \lesssim r_q^{\frac{2}{p}-1}\lambda_{q+1}^N\,, \qquad \left\|\nabla_x^N \varrho_{q+1,\varphi}\right\|_p \lesssim r_q^{\frac{2}{p}-\frac{2}{3}}\lambda_{q+1}^N\,, \qquad \text{for some } 0 < r_q \ll 1\,. \quad (3.3)$$

Then we approximately define $w_{q+1} = w_{q+1,R} + w_{q+1,\varphi}$ by

$$w_{q+1,\varphi} \approx \varrho_{q+1,\varphi}(-2\varphi_q)^{1/3}\,, \qquad w_{q+1,R} \approx \varrho_{q+1,R}\left(r_q^{1/3}(-2\varphi_q)^{1/3} + (-R_q)^{1/2}\right)\,.$$

We shall explain below that the flexibility afforded by the extra parameter r_q allows our solutions to exceed the $1/4$ threshold described above. To see this, we must first recall that the iteration in [35] required a "Goldilocks amount" of intermittency $r_q = (\lambda_q\lambda_{q+1}^{-1})^{1/2}$ in order to produce a solution in $B^{1/3-}_{3,\infty}$; any larger or smaller choice of r_q causes the size of ∇w_{q+1} to grow too quickly as $q \to \infty$. Rather remarkably, we shall see below that the Goldilocks amount of intermittency is *precisely* the minimum amount required in order to make the estimate for the current Nash error consistent with $B^{1/3-}_{3,\infty}$ regularity.

We first interpolate the L^1 and L^∞ inductive estimates for R_q and L^2 and L^∞ inductive

estimates for ∇u_q from [35] to posit that

$$\|u_q\|_3 \lesssim 1 \ , \qquad \left\|\nabla^N \nabla u_q\right\|_3 \leq \lambda_q^{-\beta+N+1} r_{q-1}^{-1/3} \ , \qquad \left\|\nabla_x^N R_q\right\|_{3/2} \leq \lambda_{q+1}^{-2\beta} \lambda_q^N \ ,$$

where $\beta < 1/3$ and $u_q \to u$ in the $B^{\beta-}_{3,\infty}$ topology; we write $\beta-$ to emphasize that $r_{q-1}^{-1/3}$ incurs a small power loss $\lambda_{q-1}^{\frac{b-1}{6}}$ which disappears as $b \to 1$. We claim that the matching inductive bound for φ_q is

$$\left\|\nabla^N \varphi_q\right\|_1 \leq \lambda_{q+1}^{-3\beta} r_q^{-1} \lambda_q^N \ .$$

Combining this bound with the sharp L^p decoupling estimate proved in the appendix A and the extra factor of $r_q^{1/3}$ in the definition of $w_{q+1,R}$ above yields the balanced estimates

$$\begin{aligned}
&\left\|\nabla^N w_{q+1,\varphi}\right\|_3 \lesssim \|\varphi_q\|_1^{1/3} \left\|\nabla^N \varrho_{q+1,\varphi}\right\|_3 \approx \lambda_{q+1}^{-\beta+N} r_q^{-1/3} \ , \\
&\left\|\nabla^N w_{q+1,R}\right\|_3 \lesssim \left(\|R_q\|_{3/2}^{1/2} + r_q^{1/3} \|\varphi_q\|_1^{1/3}\right) \left\|\nabla^N \varrho_{q+1,R}\right\|_3 \approx \lambda_{q+1}^{-\beta+N} r_q^{-1/3} \ .
\end{aligned}$$

Now recalling the structure of $w_{q+1,\varphi} \approx \varrho_{q+1,\varphi}(-2\varphi_q)^{1/3}$ and using decoupling, Hölder's inequality, and our estimates on $\varrho_{q+1,\varphi}$, φ_q, and ∇u_q, we may estimate the Nash current error term corresponding to $w_{q+1,\varphi}$ by

$$\begin{aligned}
\|\mathrm{div}^{-1}\mathbb{P}_{=\lambda_{q+1}}(w_{q+1,\varphi} \otimes w_{q+1,\varphi}) \, : \, \nabla u_q\|_1 &\lesssim \left\|\mathrm{div}^{-1}\mathbb{P}_{=\lambda_{q+1}}(\varrho_{q+1,\varphi}^2)\right\|_1 \left\||\varphi_q|^{2/3} \nabla u_q\right\|_1 \\
&\lesssim \lambda_{q+1}^{-1} r_q^{2/3} \|\varphi_q\|_1^{2/3} \|\nabla u_q\|_3 \\
&\lesssim \lambda_{q+1}^{-1} r_q^{2/3} \lambda_{q+1}^{-2\beta} r_q^{-2/3} \lambda_q^{-\beta+1} r_{q-1}^{-1/3} \ .
\end{aligned}$$

In order for this estimate to meet the desired inductive bound of $\lambda_{q+2}^{-3\beta} r_{q+1}^{-1}$, we see that we need

$$\lambda_{q+1}^{-1-2\beta} \lambda_q^{1-\beta} r_{q-1}^{-1/3} \leq \lambda_{q+2}^{-3\beta} r_{q+1}^{-1} \qquad \underbrace{\Longleftarrow}_{r_{q-1}^{1/3} r_{q+1}^{-1/3} > 1} \qquad \lambda_{q+1}^{-1-2\beta} \lambda_q^{1-\beta} \leq \lambda_{q+2}^{-3\beta} r_{q+1}^{-2/3}$$

Note crucially that the inequality on the right has gained $r_{q+1}^{-2/3}$ compared with (3.2). Then using that $r_{q+1}^{-1} = \lambda_q^{\frac{b(b-1)}{2}}$, the inequality on the right is equivalent to

$$\beta\left(3b^2 - 2b - b\right) \leq (b-1)\left(1 + \frac{b}{3}\right) \qquad \Longleftrightarrow \qquad \beta \leq \frac{1 + \frac{b}{3}}{3b+1},$$

so that $\beta \to 1/3$ as $b \to 1$. Similar estimates hold for the Nash current error from $w_{q+1,R}$, as well as for a number of current error terms which faced $C^{1/4}$ regularity limitations in the C^α iteration.

We conclude by noting that while the basic scaling considerations above indicate that an L^3 iteration inspired by [35] has some hope, the techniques from [35] would suffer from a number of significant shortcomings if one were to attempt to use them in a proof of the strong L^3 Onsager conjecture. We explain the most immediate of these shortcomings in the next two sections.

3.2 The continuous scheme

In order to understand the need for partial wavelet sums in our iteration, we must examine the consequences of replacing sinusoidal shear flows with intermittent shear flows. Intermittency in Nash iterations dates back to the work of Buckmaster and Vicol [9] for the 3D Navier-Stokes equations. The intermittent Mikado flows used in [35] were later introduced by Modena and Székelyhidi in [33] (see also the homogeneous Mikado flows due to Daneri and Székelyhidi [12]). One should visualize the intermittent Mikado flows $\varrho_{q+1,\varphi}\vec{e}_1$ or $\varrho_{q+1,R}\vec{e}_1$ as shear flows supported in thin tubes of diameter λ_{q+1}^{-1} around lines in the $\vec{e}_1$ direction, which have been periodized to scale $\lambda_{q+1}^{-1} r_q$. The parameter $r_q = \lambda_q^{1/2}\lambda_{q+1}^{-1/2}$ thus quantifies both the measure of the support and the L^p norms, and the effective frequencies are contained in the range $[\lambda_{q+1} r_q, \lambda_{q+1}] = [(\lambda_q\lambda_{q+1})^{1/2}, \lambda_{q+1}]$. Thus we see that intermittency *smears out* the frequency support of w_{q+1}.

This smearing of frequencies greatly affects nonlinear errors such as the current oscillation

error

$$\mathrm{div}^{-1} \circ \mathrm{div}\left(\varphi_q + (\mathbb{P}_{\leq \lambda_q} + \mathbb{P}_{>\lambda_q})(1/2|w_{q+1,\varphi}|^2 w_{q+1,\varphi})\right) \approx -\mathrm{div}^{-1}\mathbb{P}_{\geq \lambda_q^{1/2}\lambda_{q+1}^{1/2}}\left(1/2|\varrho_{q+1,\varphi}|^2 \varrho_{q+1,\varphi}\right)\vec{e}_1 \cdot \nabla\varphi_q\,.$$

In the above approximate equality we have used the form of $w_{q+1,\varphi} = (-2\varphi_q)^{1/3}\varrho_{q+1,\varphi}\vec{e}_1$, the identity $\vec{e}_1 \cdot \nabla\varrho_{q+1,\varphi} \equiv 0$, and the heuristic that the leading order behavior of the operator div^{-1} on a product of high and low frequency terms can be understood by simply applying it to the high frequency term. The maximum frequency of the leftovers is λ_{q+1}, and minimum frequency is $\lambda_q^{1/2}\lambda_{q+1}^{1/2}$. Then if we attempt to absorb this error term into φ_{q+1}, we see that

$$\begin{aligned}
\Big\| -\underbrace{\mathrm{div}^{-1}\mathbb{P}_{\geq \lambda_q^{1/2}\lambda_{q+1}^{1/2}}}_{\text{gains } \lambda_q^{-1/2}\lambda_{q+1}^{-1/2}}\underbrace{\left(1/2|\varrho_{q+1,\varphi}|^2 \varrho_{q+1,\varphi}\right)\vec{e}_1}_{\text{unit } L^1 \text{ norm}} \cdot \underbrace{\nabla\varphi_q}_{\substack{L^1 \text{ size} \\ \lambda_{q+1}^{-3\beta} r_q^{-1}}} \Big\|_{L^1} &\leq \lambda_{q+2}^{-3\beta} r_{q+1}^{-1} \\
\iff \lambda_q^{3\beta b^2 - 3\beta b + \frac{1}{2}(1-b) + \frac{1}{2}(1-b)(b-1)} &\leq 1 \\
\iff \beta &\leq \frac{1}{6}\,.
\end{aligned}$$

Thus intermittency has the effect of creating errors at frequencies *lower* than λ_{q+1} which are too large to be absorbed into φ_{q+1}.

3.2.1 Necessity of a continuous scheme

In [35], the analogue of this issue in the Euler-Reynolds system was rectified by performing a further frequency decomposition of $[(\lambda_q\lambda_{q+1})^{1/2}, \lambda_{q+1}]$ into pieces and adding further velocity increments to handle the errors at frequencies lower than λ_{q+1}. Attempting such a strategy here leads one to define the *higher order error* $\varphi_{q,\alpha}$ at frequency $\lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}$ for $\alpha \in [1/2, 1]$ by

$$\varphi_{q,\alpha} := -\mathrm{div}^{-1}\mathbb{P}_{\approx \lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}}\left(1/2|\varrho_{q+1,\varphi}|^2 \varrho_{q+1,\varphi}\right)\vec{e}_1 \cdot \nabla\varphi_q\,.$$

This higher order error then satisfies the estimate

$$\|\varphi_{q,\alpha}\|_1 = \left\|\mathrm{div}^{-1}\mathbb{P}_{\approx\lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}}\left(1/2|\varrho_{q+1,\varphi}|^2\varrho_{q+1,\varphi}\right)\vec{e}_1\cdot\nabla\varphi_q\right\|_1 \leq \lambda_q^{\alpha-1}\lambda_{q+1}^{-\alpha}\lambda_{q+1}^{-3\beta}r_q^{-1}\lambda_q = \lambda_{q+1}^{-3\beta}r_q^{-1}\frac{\lambda_q^{\alpha}}{\lambda_{q+1}^{\alpha}}$$

and would be corrected by a *higher order velocity increment* $w_{q+1,\alpha,\varphi}$. In order to keep the maximum frequency of $w_{q+1,\alpha,\varphi}$ no larger than λ_{q+1}, the strategy of [35] was to define $w_{q+1,\alpha,\varphi} = (-2\varphi_{q,\alpha})^{1/3}\varrho_{q+1,\alpha,\varphi}$ using intermittent Mikado flows $\varrho_{q+1,\alpha,\varphi}$ with frequency support $[\lambda_{q+1}r_{q,\alpha}, \lambda_{q+1}]$. The question then becomes "which values of $r_{q,\alpha}$ will work for the local energy inequality?"

First, we note that we must have $\lambda_{q+1}r_{q,\alpha} \geq \lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}$. If not, then $|w_{q+1,\alpha,\varphi}|^2 w_{q+1,\alpha,\varphi}$ will create a *current* oscillation error at a frequency *below* that of $\varphi_{q,\alpha}$, which was $\lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}$. This however stands in contradiction with the fundamental ansatz that Nash iterations use high-frequency perturbations to correct low-frequency errors. Next, we note that $w_{q+1,\alpha,\varphi}$ will also create a *Reynolds* oscillation error at frequency $\lambda_{q+1}r_{q,\alpha}$ given by

$$(-2\varphi_{q,\alpha})^{2/3}\int_{\mathbb{T}^3}\varrho_{q+1,\alpha,\varphi}\otimes\varrho_{q+1,\alpha,\varphi}\,.$$

The $L^{3/2}$ norm of this Reynolds stress error is

$$\|\varphi_{q,\alpha}\|_1^{2/3}\,\|\varrho_{q+1,\alpha,\varphi}\|_2^2 = \left(\lambda_{q+1}^{-3\beta}r_q^{-1}\frac{\lambda_q^{\alpha}}{\lambda_{q+1}^{\alpha}}\right)^{2/3} r_{q,\alpha}^{2/3}\,. \tag{3.4}$$

The first constraint ($\lambda_{q+1}r_{q,\alpha} \geq \lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}$) implies that $r_{q,\alpha} \to 1$ as $\alpha \to 1$, and so this Reynolds stress error lives at frequency $\lambda_{q+1}r_{q,\alpha} \to \lambda_{q+1}$. We expect the size of the Reynolds error R_{q+1} at frequency λ_{q+1} to be $\lambda_{q+2}^{-2\beta}$. Plugging in $\alpha = 1$ to (3.4), we see that

$$\lambda_{q+1}^{-2\beta}r_q^{-2/3}\frac{\lambda_q^{2/3}}{\lambda_{q+1}^{2/3}}r_{q,\alpha}^{2/3} \leq \lambda_{q+2}^{-2\beta} \quad \underbrace{\Longleftarrow}_{r_{q,\alpha}\leq r_q} \quad \lambda_{q+1}^{-2\beta}\frac{\lambda_q^{2/3}}{\lambda_{q+1}^{2/3}} \leq \lambda_{q+2}^{-2\beta} \quad \Longleftrightarrow \quad \beta \leq \frac{1}{3b}\,.$$

Thus we see that we need both $r_{q,\alpha} \leq r_q = \lambda_q^{1/2}\lambda_{q+1}^{-1/2}$ *and* $r_{q,\alpha} \to 1$ as $\alpha \to 1$, implying that

there are *no* satisfactory choices of $r_{q,\alpha}$. We emphasize that the constraint $r_{q,\alpha} \leq r_q$ only arises due to the interplay of cubic and quadratic error terms. In [35], which constructed solutions that certainly *do not* satisfy the local energy inequality, one could indeed choose $r_{q,\alpha}$ such that $r_{q,\alpha} \to 1$ as $\alpha \to 1$.

After a bit of thought, one can identify the culprit in the failure of the above analysis; namely, we insisted that the maximum frequency of $w_{q+1,\alpha,\varphi}$ was λ_{q+1}. Why not construct $w_{q+1,\alpha,\varphi}$ to correct $\varphi_{q,\alpha}$ in a manner completely analogous to how w_{q+1} was constructed to correct φ_q? Recalling that $\varphi_{q,\alpha}$ lives at frequency $\lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}$ and is corrected using the intermittent pipe flow $\varrho_{q+1,\alpha,\varphi}$, we should set the maximum frequency of the intermittent pipe flow $\varrho_{q+1,\alpha,\varphi}$ to be $\lambda_{q+1}^{1-\alpha}\lambda_{q+2}^{\alpha}$, and the minimum frequency to be in accord with the Goldilocks ratio, i.e. $\left(\lambda_q^{1-\alpha}\lambda_{q+1}^{\alpha}\lambda_{q+1}^{1-\alpha}\lambda_{q+2}^{\alpha}\right)^{1/2}$. Interestingly, one may view this choice as a restoration of self-similarity which had been broken by the scheme in [35]. Indeed the choice of $r_{q,\alpha}$ from [35] implies that $w_{q+1,\alpha,R}$ was *much* less intermittent than $w_{q+1,R}$ as $\alpha \to 1$, thus breaking the intermittent self-similarity of the different components of the velocity field. The natural conclusion of these observations, which in some sense is validated by our analysis in this paper, is that *the local energy inequality imposes intermittent self-similarity by fixing the Goldilocks parameter of intermittency throughout the iteration.*

The discussion in the previous paragraph is far from a complete prescription for an intermittent, wavelet-inspired L^3 scheme. Several ideas outlined in the remainder of the introduction are needed in order to fully justify our modifications to the original Euler-Reynolds system and relaxed local energy inequality given in (3.1). Nonetheless, we present the basic form of our iteration here in order to set ideas. We assume the existence of a velocity field $u_q = \widehat{u}_q + (u_q - \widehat{u}_q)$ (where the "hat" notation is used to encode frequency information described below), a Reynolds stress R_q, a current error φ_q, a pressure p_q, and

an intermittent pressure $-\pi_q$ which satisfy

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \operatorname{div}(R_q - \pi_q \mathrm{Id}) \\ \partial_t \left(\frac{1}{2}|u_q|^2\right) + \operatorname{div}\left(\left(\frac{1}{2}|u_q|^2 + p_q\right) u_q\right) \leq (\partial_t + \widehat{u}_q \cdot \nabla) \underbrace{{}^1\!/_2 \mathrm{tr}\ (R_q - \pi_q \mathrm{Id})}_{:=\kappa_q} + \operatorname{div}\varphi_q + \operatorname{div}((R_q - \pi_q \mathrm{Id})\widehat{u}_q) \\ \operatorname{div} u_q = 0\,. \end{cases} \tag{3.5}$$

We assume the existence of a large parameter $\bar{n}$ (fixed throughout the iteration) such that $\widehat{u}_q$ oscillates at spatial frequencies no larger than λ_q and $u_q - \widehat{u}_q$ oscillates at spatial frequencies in between λ_{q+1} and $\lambda_{q+\bar{n}-1}$. In general, the subscript q' with a "hat" (as in $u_{q'}$) denotes a velocity field with maximum frequency $\lambda_{q'}$, while the subscript q' and no "hat" (as in $u_{q'}$) denotes a velocity field with maximum frequency $\lambda_{q'+\bar{n}-1}$. Choosing β close to ${}^1\!/_3$, we then inductively assume that

$$\|u_q\|_3 \lesssim 1\,, \quad \left\|\nabla_x^N \nabla u_q\right\|_3 \lesssim \lambda_{q+\bar{n}-1}^{-\beta+1+N} \quad \Longleftrightarrow \quad \|\widehat{u}_{q+\bar{n}-1}\|_3 \lesssim 1\,, \qquad \left\|\nabla_x^N \nabla \widehat{u}_{q+\bar{n}-1}\right\|_3 \lesssim \lambda_{q+\bar{n}-1}^{-\beta+1+N}\,.$$

Next, the Reynolds stress R_q may decomposed as $R_q = \sum_{q'=q}^{q+\bar{n}-1} R_q^{q'}$, the intermittent pressure π_q may be decomposed as $\pi_q = \sum_{q'=q}^{\infty} \pi_q^{q'}$, and the current error φ_q may be decomposed as $\varphi_q = \sum_{q'}^{q+\bar{n}-1} \varphi_q^{q'}$. The parameter q' encodes the frequency $\lambda_{q'}$ at which $R_q^{q'}$, $\varphi_q^{q'}$, and $\pi_q^{q'}$ oscillate. We therefore assume that

$$\left\|\nabla_x^N R_q^{q'}\right\|_{3/2} + \left\|\nabla_x^N \pi_q^{q'}\right\|_{3/2} \leq \lambda_{q'+\bar{n}}^{-2\beta} \lambda_{q'}^N\,, \qquad \left\|\nabla_x^N \varphi_q^{q'}\right\|_1 \leq \lambda_{q'+\bar{n}}^{-3\beta} r_{q'}^{-1} \lambda_{q'}^N\,,$$

where $r_{q'} = \lambda_{q'+\bar{n}/2}^{1/2} \lambda_{q'+\bar{n}}^{-1/2}$. One should conceive of φ_q^q as identical to the previous φ_q, $\varphi_q^{q'}$ for $q' > q$ as analogous to the previous $\varphi_{q,\alpha}$, and similarly for $R_q^{q'}$. We shall require that $|\pi_q^{q'}| > R_q^{q'}$ so that the tensor on the right-hand side of (3.5) is negative definite (see subsubsection 3.3).

We then construct $w_{q+1} = \widehat{w}_{q+\bar{n}} = u_{q+1} - u_q$ using intermittent Mikado flows $\varrho_{q+\bar{n},R}$ and $\varrho_{q+\bar{n},\varphi}$ which have minimum frequency $\lambda_{q+\bar{n}/2}$ and maximum frequency $\lambda_{q+\bar{n}}$. Since w_{q+1}

is used to correct errors at frequency λ_q, these choices adhere to the Goldilocks ratio of intermittency. Furthermore, w_{q+1} is used to correct R_q^q and φ_q^q while leaving $R_q^{q'}$ and $\varphi_q^{q'}$ intact for $q' > q$. The net result of adding w_{q+1} will be the creation of new stress and current errors, which will get sorted into bins between λ_{q+1} and $\lambda_{q+\bar{n}}$ and added to $R_q^{q'}$ and $\varphi_q^{q'}$ to form R_{q+1} and φ_{q+1}. We emphasize that the terms in the partial sum $u_{q+1} = w_{q+1} + w_q + w_{q-1} + \dots$ have overlap in frequency when $|q' - q''| \leq \bar{n}/2$, so that u_{q+1} should be thought of as a partial wavelet decomposition of the limiting solution rather than a partial Fourier decomposition.

3.2.2 An obstruction

The inductive set-up described above needs to be complemented with assumptions on spatial support, as well as a methodology for propagating such information throughout the iteration. To give an example of the kind of support properties we require, let us define the velocity increment $w_{q+1} = w_{q+1,R} + w_{q+1,\varphi}$ by

$$w_{q+1,\varphi} = (-2\varphi_q^q)^{1/3} \varrho_{q+\bar{n},\varphi}\,, \qquad w_{q+1,R} = \left(r_q^{1/3}(-2\varphi_q^q)^{1/3} + (R_q^q)^{1/2}\right) \varrho_{q+\bar{n},R}\,, \tag{3.6}$$

where $\varrho_{q+\bar{n},\varphi}$ and $\varrho_{q+\bar{n},R}$ satisfy estimates identical to (3.3) after replacing λ_{q+1} with $\lambda_{q+\bar{n}}$ and using the new definition of $r_q = \lambda_{q+\bar{n}/2}^{1/2} \lambda_{q+\bar{n}}^{-1/2}$. Then w_{q+1} satisfies the balanced estimates

$$\begin{aligned}
\left\|\nabla^N w_{q+1,\varphi}\right\|_3 &\lesssim \|\varphi_q\|_1^{1/3} \left\|\nabla^N \varrho_{q+\bar{n},\varphi}\right\|_3 \approx \lambda_{q+\bar{n}}^{-\beta+N} r_q^{-1/3}\,, \\
\left\|\nabla^N w_{q+\bar{n},R}\right\|_3 &\lesssim \left(\left\|R_q^q\right\|_{3/2}^{1/2} + r_q^{1/3} \left\|\varphi_q^q\right\|_1^{1/3}\right) \left\|\nabla^N \varrho_{q+\bar{n},R}\right\|_3 \approx \lambda_{q+\bar{n}}^{-\beta+N} r_q^{-1/3}\,.
\end{aligned}$$

Now consider the Nash error obtained from adding $w_{q+1,\varphi}$, which we may estimate[1] by

$$\left\|\mathrm{div}^{-1}(\varrho_{q+\bar{n},\varphi}(\varphi_q^q)^{1/3}\nabla u_q)\right\|_{3/2} \lesssim \Big\| \underbrace{\mathrm{div}^{-1}\varrho_{q+\bar{n},\varphi}}_{L^{3/2}\text{ size }\lambda_{q+\bar{n}}^{-1}r_q^{2/3}} \underbrace{(\varphi_q^q)^{1/3}}_{L^3\text{ size }\lambda_{q+\bar{n}}^{-\beta}r_q^{-1/3}} \cdot \underbrace{\nabla\widehat{u}_q}_{L^3\text{ size }\lambda_q^{-\beta+1}r_{q-\bar{n}}^{-1/3}} \Big\|_{3/2}$$
$$+ \Big\| \underbrace{\mathrm{div}^{-1}\varrho_{q+\bar{n},\varphi}}_{L^{3/2}\text{ size }\lambda_{q+\bar{n}}^{-1}r_q^{2/3}} \underbrace{(\varphi_q^q)^{1/3}}_{L^3\text{ size }\lambda_{q+\bar{n}}^{-\beta}r_q^{-1/3}} \cdot \underbrace{(\nabla u_q - \nabla\widehat{u}_q)}_{L^3\text{ size }\lambda_{q+\bar{n}-1}^{-\beta+1}r_{q-1}^{-1/3}} \Big\|_{3/2}.$$

Since this error term oscillates at frequency $\lambda_{q+\bar{n}}$, we expect its size to be $\lambda_{q+2\bar{n}}^{-2\beta}$ (the analogue of δ_{q+2} from [35], for example). After a bit of arithmetic, one may check that the first term satisfies a sharp estimate when $\beta \to 1/3$ (analogous to $\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{-1} \leq \delta_{q+2}$ from a $C^{1/3-}$, which is the size of the Nash error). The second term, however, is far too large, due to the fact that $\nabla u_q - \nabla\widehat{u}_q$ has much larger L^3 norm than $\widehat{\nabla} u_q$. The only way to close the estimate for the Nash error is then if

$$\mathrm{supp}\, w_{q+1} \cap \mathrm{supp}\ (u_q - \widehat{u}_q) = \emptyset \quad \Longleftarrow \quad \mathrm{supp}\, w_{q+1} \cap (\mathrm{supp}\, \widehat{w}_{q+1} \cup \mathrm{supp}\, \widehat{w}_{q+2} \cdots \cup \mathrm{supp}\, \widehat{w}_{q+\bar{n}-1}) = \emptyset\,,$$

where we have recalled that our "hat" notation gives that $u_q - \widehat{u}_q = \widehat{w}_{q+1} + \widehat{w}_{q+2} + \cdots + \widehat{w}_{q+\bar{n}-1}$.

There is however a clear obstruction to this assertion. Consider the velocity increments $\widehat{w}_{q'}$ defined analogously to (3.6) for $q+1 \leq q' \leq q + \bar{n}/2$. These velocity increments are constructed using intermittent Mikado flows $\varrho_{q',R}$ and $\varrho_{q',\varphi}$ which have pipe spacing $\lambda_{q'-\bar{n}/2}^{-1}$ and pipe thickness $\lambda_{q'}^{-1}$. Since the thickness of these pipes is *larger* than the spacing of the pipes we plan to use at step q, namely $\lambda_{q+\bar{n}/2}^{-1}$, there is no way we can arrange the support of $\widehat{w}_{q+\bar{n}}$ to be disjoint from the support of $\widehat{w}_{q+1}, \ldots \widehat{w}_{q+\bar{n}/2}$. We have solved this issue through the creation of *intermittent Mikado bundles*. The details of this construction will be found in sections 7.2 and 9.2.

[1] Note the inverse divergence gain of $\lambda_{q+\bar{n}}$, which is larger than the minimum frequency $\lambda_{q+\bar{n}/2}$ of w_{q+1}. One can test the validity of this estimate by computing the one-dimensional version, where div^{-1} is simply integration.

3.3 The intermittent pressure

As common to many convex integration schemes for the Euler equations, in order to get sharp regularity solutions, one needs to transport the "high-frequenccy" building blocks (here, the intermittent Mikado bundles) along the flow of the background, coarse-grained velocity field. However, such a flow will generally mix anything it transports after a certain time. This timescale is given by the inverse of the local Lipschitz constant of the background flow and one can only obtain good estimates for the flow for times smaller this timescale. This necessitates needing to "switch-off" and "switch-on" the flow by time cut-off functions, so that on the support of a given time cut-off function, one has good control on the flow map.

But now, when one switches-on a time cut-off function, the intermittent bundle that is contained in its support adds local energy to the system along trajectories of the background flow that intersect its support. The only way one is still able to close the scheme is if this energy added is small enough *in a weak sense* to be absorbed into the next unresolved current error.

In our scheme, the way the above phenomenon manifests is by the "dynamic pressure" $\frac{1}{2}|w_{q+1}|^2$ that shows up inside a time/material derivative and one needs to gain from inverting the divergence on this in order to put this into φ_{q+1}. But note that $|w_{q+1}|^2$ is positive and so has a bunch of low frequencies in it that prevents us from gaining by inverting the divergence. The only way to make this error term manageable is to handcraft a positive function, the intermittent pressure, to *subtract* from $|w_q|^2$ in order to make it high-frequency. In order to do this, we have to do a wavelet decomposition of the dynamic pressure, and then add pressure terms to beat the individual parts of it. The key here is that other than the highest frequency terms in this wavelet decomposition, all other pressure terms depend on objects that have been constructed in the scheme at previous steps and thus these pressure terms can be *anticipated* in advance and added into the scheme at these previous steps where we can get away with the worse estimates. Note that these anticipated pressure terms end up inside

a material derivative in the relaxed local energy inequality, which then has to be estimated. So we need an "abstract machine" which takes every error term, makes it positive, does a wavelet decomposition, applies a material derivative, and then inverts the divergence on each individual piece. This intermittent and anticipated pressure is one of the main new ideas in this thesis. Indeed, in all previous convex integration schemes, the pressure has played a secondary role. The importance of the appearance of the dynamic pressure in our scheme might be an indication that the ideas of long-term memory and the backwards cascade from turbulence theory are now entering the mathematical realm of convex integration.

Chapter 4

Parameters

4.1 Definitions and inequalities

In this section, we choose the values of the parameters and list important consequences. The choices in items (i)–(viii) are rather delicate, while all the choices in items (ix)–(xix) follow the plan of "choosing a giant parameter which dwarfs all the preceding parameters." It is imperative that each inequality below depends *only* on parameters which have already been chosen, and that none depend on q. We point out that in item (iv), we define two parameters λ_q and δ_q in terms of an undetermined large natural number a. This is merely for ease of notation and computation. Indeed one can check that none of the inequalities below require a precise choice of a, nor depend on q; rather, any sufficiently large choice of a which may be used to absorb implicit constants will do. Therefore the precise choice of a is made at the very end in item (xix).

(i) Choose an L^3 regularity index $\beta \in (0, 1/3)$. In light of [17], there is no reason to take $\beta < 1/7$.

(ii) Choose $\bar{n}$ a large positive multiple of 6 such that

$$\beta < \frac{1}{3} \cdot \frac{\bar{n}/3}{\bar{n}/3 + 2} - \frac{2}{\bar{n}/3 + 2}\,, \qquad \beta < \frac{2}{3} \cdot \frac{\bar{n}/2 - 1}{\bar{n}}\,, \tag{4.1}$$

which is possible since $\beta < 1/3$.

(iii) Choose $b \in (1, 25/24)$ such that

$$\beta < \frac{1}{3b^{\bar{n}}} \cdot \frac{1+b+\cdots+b^{\bar{n}/3-1}}{1+b+\cdots+b^{\bar{n}/3+1}} - \frac{2\left(1+(b-1)(1+\cdots+b^{\bar{n}/2-1})^2\right)}{1+b+\cdots+b^{\bar{n}/3+1}}, \qquad \beta < \frac{2}{3b^{\bar{n}/2}} \cdot \frac{1+\cdots+b^{\bar{n}/2-2}}{1+\cdots+b^{\bar{n}-1}} \tag{4.2a}$$

$$b^{\bar{n}} < 2, \qquad \frac{(b^{\bar{n}/2-1}+\cdots+b+1)^2}{b^{\bar{n}/2-1}+\cdots+b+1}(b-1) < (b-1)^{1/2}. \tag{4.2b}$$

The inequalities in (4.2a) are possible since (4.1) is just (4.2a) evaluated at $b = 1$, and both expressions in (4.2a) are continuous in b in a neighborhood of $b = 1$. The first inequality in (4.2b) is trivial, and the second is possible since the fraction in the expression is continuous at $b = 1$ and equal to $\bar{n}/2$ if $b = 1$. It is clear that as $\beta \to 1/3$, we are forced to choose $\bar{n} \to \infty$ and $b \to 1$.

(iv) For an undetermined natural number a, define

$$\lambda_q = 2^{\lceil (b^q) \log_2 a \rceil}, \qquad \delta_q = \lambda_q^{-2\beta}. \tag{4.3}$$

Note that with the above definition of λ_q, we have that

$$a^{(b^q)} \le \lambda_q \le 2a^{(b^q)} \qquad \text{and} \qquad \frac{1}{3}\lambda_q^b \le \lambda_{q+1} \le 2\lambda_q^b. \tag{4.4}$$

As a consequence of these definitions, we shall deduce a number of inequalities, each of which is independent of the choice of a and of q once a is sufficiently large. At the end we will thus choose a sufficiently large to absorb a number of implicit constants, including those in (4.4). Therefore, in many of the following computations, we may make the slightly incorrect assumption that λ_q is *actually equal* to $a^{(b^q)}$ in order to streamline the arithmetic.

(a) An immediate consequence of these definitions and of the first inequality in (4.2a)

is that

$$
\begin{aligned}
&\delta_{q+\bar{n}} \left(\lambda_q \lambda_{q+\bar{n}/3}^{-1}\right)^{2/3} \lambda_{q+\bar{n}+1}^4 \lambda_{q+\bar{n}}^{-4} \frac{\lambda_q^4 \lambda_{q+\bar{n}}^4}{\lambda_{q+\bar{n}/2}^8} < \delta_{q+4\bar{n}/3+2} \\
&\quad \iff 2\beta b^{4\bar{n}/3+2} - 2\beta b^{\bar{n}} < \frac{2}{3} b^{\bar{n}/3} - \frac{2}{3} - 4b^{\bar{n}+1} + 4b^{\bar{n}} - 4b^{\bar{n}} + 8b^{\bar{n}/2} - 4 \\
&\quad \iff 2\beta b^{\bar{n}}(b-1)(1+b+\cdots+b^{\bar{n}/3+1}) < \frac{2}{3}(b-1)(1+b+\cdots+b^{\bar{n}/3-1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 4b^{\bar{n}}(b-1) - 4(1+\cdots+b^{\bar{n}/2-1})^2(b-1)^2 \\
&\quad \iff \beta < \frac{1}{3b^{\bar{n}}} \cdot \frac{1+b+\cdots+b^{\bar{n}/3-1}}{1+b+\cdots+b^{\bar{n}/3+1}} - \frac{2}{1+b+\cdots+b^{\bar{n}/3+1}} - \frac{2(b-1)(1+\cdots+b^{\bar{n}/2-1})^2}{(1+b+\cdots+b^{\bar{n}/3+1})b^{\bar{n}}}\,,
\end{aligned}
$$

where we have written out the quantity at the beginning in terms of $\lambda_q \approx a^{(b^q)}$ and then compared exponents on both sides. It is easy to generalize the above to

$$
\delta_{q+\bar{n}} \left(\lambda_q \lambda_{q+k}^{-1}\right)^{2/3} \lambda_{q+\bar{n}+1}^4 \lambda_{q+\bar{n}}^{-4} \frac{\lambda_q^4 \lambda_{q+\bar{n}}^4}{\lambda_{q+\bar{n}/2}^8} < \delta_{q+\bar{n}+k+2} \qquad \forall k \geq \bar{n}/3\,. \tag{4.5}
$$

(b) A consequence of the second inequality in (4.2a) is that

$$
\begin{aligned}
&\frac{\delta_{q+\bar{n}}}{\delta_{q+\bar{n}-1}} \left(\frac{\lambda_{q+\bar{n}/2}/\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}/2-1}/\lambda_{q+\bar{n}-1}}\right)^{4/3} < \frac{\delta_{q+2\bar{n}}}{\delta_{q+2\bar{n}-1}} \\
&\quad \iff -2\beta b^{\bar{n}} + 2\beta b^{\bar{n}-1} + (b^{\bar{n}/2} - b^{\bar{n}})(b-1)\frac{4}{3b} < -2\beta b^{2\bar{n}} + 2\beta b^{2\bar{n}-1} \\
&\quad \iff 2\beta b^{\bar{n}-1}\left(b^{\bar{n}+1} - b - b^{\bar{n}} + 1\right) < (b^{\bar{n}} - b^{\bar{n}/2})(b-1)\frac{4}{3b} \\
&\quad \iff \beta < \frac{2}{3b^{\bar{n}/2}} \frac{1+\cdots+b^{\bar{n}/2-1}}{1+\cdots+b^{\bar{n}-1}}\,.
\end{aligned}
$$

(c) A consequence of the definition of λ_q is that for $q' \geq q - \bar{n}/2 + 1$,

$$
\frac{\lambda_{q'+\bar{n}/2}\lambda_{q+\bar{n}/2}}{\lambda_q \lambda_{q'+\bar{n}}} < 1\,. \tag{4.6}
$$

Indeed when $q' = q - \bar{n}/2 + 1$, the inequality reduces to $\lambda_{q+1}\lambda_q^{-1}\lambda_{q+\bar{n}/2}\lambda_{q+\bar{n}/2+1}^{-1} < 1$, which is an immediate consequence of the super-exponential growth; larger q' are similar.

(d) We have that $\delta_q \lambda_q^{2/3} < \delta_{q'} \lambda_{q'}^{2/3}$ for all $q' > q$. A stronger inequality is that for all $k \geq 1$, $\delta_{q+\bar{n}} \lambda_q^{2/3} < \delta_{q+k+\bar{n}} \lambda_{q+k}^{2/3}$, which is in fact equivalent to $\beta < 1/3b^{\bar{n}}$, which is implied by the first inequality in (4.2). A final consequence of both inequalities is

$$\delta_{q+\bar{n}} \frac{\lambda_q^{2/3}}{\lambda_{q+\bar{n}}^{2/3}} < \delta_{q+2\bar{n}} \implies \delta_{q+\bar{n}}^{1/2} \delta_q^{1/2} \frac{\lambda_q}{\lambda_{q+\bar{n}}} < \delta_{q+2\bar{n}} \implies \delta_{q+\bar{n}}^{1/2} \delta_q^{1/2} \frac{\lambda_q}{\lambda_{q+\bar{n}}} \frac{\lambda_{q+\bar{n}/2}^{1/3} \lambda_{q+\bar{n}}^{-1/3}}{\lambda_{q+\bar{n}/2-1}^{1/3} \lambda_{q+\bar{n}-1}^{-1/3}} < \delta_{q+2\bar{n}} \,. \tag{4.7}$$

(e) From the second inequality in (4.2a), we have that

$$\beta < \frac{2}{3b^{\bar{n}/2}} \cdot \frac{1 + \cdots + b^{\bar{n}/2-1}}{1 + \cdots + b^{\bar{n}-1}} \implies \delta_{q+\bar{n}} \lambda_{q+\bar{n}/2}^{4/3} < \delta_{q+2\bar{n}} \lambda_{q+\bar{n}}^{4/3} \,.$$

(v) Choose $\mathsf{C}_b = \frac{6+b}{b-1}$.

(vi) Define Γ_q, r_q, τ_q, and Λ_q by[1]

$$\Gamma_q = 2^{\left\lceil \varepsilon_\Gamma \log_2\left(\frac{\lambda_{q+1}}{\lambda_q}\right)\right\rceil} \approx \left(\frac{\lambda_{q+1}}{\lambda_q}\right)^{\varepsilon_\Gamma} \approx \lambda_q^{(b-1)\varepsilon_\Gamma} \,, \qquad r_q = \frac{\lambda_{q+\bar{n}/2} \Gamma_q}{\lambda_{q+\bar{n}}} \tag{4.8}$$

$$\tau_q^{-1} = \delta_q^{1/2} \lambda_q r_{q-\bar{n}}^{-1/3} \Gamma_q^{35} \,, \qquad \Lambda_q = \lambda_q \Gamma_q^{10} \,, \tag{4.9}$$

[1]The same type of comparability that we have in (4.4) holds for Γ_q as defined in (4.8).

where we choose $0 < \varepsilon_\Gamma \ll (b-1)^2 < 1$ such that

$$(\delta_{q-\bar{n}}\delta_{q-\bar{n}-1}^{-1})^{1/10}\Gamma_{q+\bar{n}}^{1000} \leq 1\,, \tag{4.10a}$$

$$\Gamma_q^{25}\lambda_q\delta_q^{1/2}r_{q-\bar{n}}^{-1/3} \leq \tau_q^{-1} \leq \Gamma_q^{50}\lambda_q\delta_q^{1/2}r_{q-\bar{n}}^{-1/3}\,, \qquad \Gamma_{q+\bar{n}}^{300}\tau_{q+\bar{n}-1}^{-1} \leq \tau_{q+\bar{n}}^{-1}\,, \tag{4.10b}$$

$$\Gamma_{q+\bar{n}}^{25}\frac{\delta_{q+\bar{n}}}{\delta_{q+\bar{n}-1}}\left(\frac{r_q}{r_{q-1}}\right)^{4/3} < \frac{\delta_{q+2\bar{n}}}{\delta_{q+2\bar{n}-1}} \tag{4.10c}$$

$$\lambda_{n-1}^{-2}\lambda_n\lambda_{q+\bar{n}/2} \leq \Gamma_q^{-1} \quad \text{for } q+\bar{n}/2+3 \leq n \leq q+\bar{n}+2\,, \tag{4.10d}$$

$$\Gamma_{q+\bar{n}}^{3}\Gamma_q^{-2}\frac{\lambda_{q'+\bar{n}/2}\lambda_{q+\bar{n}/2}}{\lambda_q\lambda_{q'+\bar{n}}} \leq 1 \qquad \text{for all } q' \text{ such that } q+\bar{n}/2+1-\bar{n} \leq q' \leq q\,, \tag{4.10e}$$

$$\left(\frac{\lambda_q}{\lambda_{q'}}\right)^{2/3}\Gamma_{q+\bar{n}}^{2000+10\mathsf{C}_b} < \left(\frac{\delta_q}{\delta_{q'}}\right)^{-1} \tag{4.10f}$$

$$r_q^{4/3}\delta_{q+\bar{n}}\Gamma_q^{600} \leq \delta_{q+2\bar{n}} \implies r_q^{4/3}\Gamma_q^{600}\delta_q \leq \delta_{q+\bar{n}} \tag{4.10g}$$

$$\left(\frac{r_{q+1}}{r_q}\right)\Gamma_{q+\bar{n}}^{1000+10\mathsf{C}_b} \leq 1 \tag{4.10h}$$

$$\Gamma_q^{5\mathsf{C}_b+300}\delta_{q+\bar{n}}^{1/2}r_q^{1/3}\lambda_{q+\bar{n}}^{-1}\tau_q^{-1} \leq \Gamma_{q+\bar{n}}^{-10}\delta_{q+2\bar{n}}\,, \tag{4.10i}$$

$$\Gamma_{q+\bar{n}}\delta_{q+\bar{n}-1}^{-1/2}r_{q-1}^{-2/3} \leq \delta_{q+\bar{n}}^{-1/2}r_q^{-2/3}\,, \tag{4.10j}$$

$$\Gamma_{q+\bar{n}}^{1000} < \min\left(\lambda_q\lambda_{q+\bar{n}}^{-1}r_q^{-2}, \lambda_q^{-1/10}\lambda_{q+1}^{1/10}, \delta_q^{1/10}\delta_{q+1}^{-1/10}\right) \tag{4.10k}$$

$$\left\lceil\frac{(b^{\bar{n}/2-1}+\cdots+b+1)^2}{\varepsilon_\Gamma(b^{\bar{n}-1}+\cdots+b+1)}\right\rceil \geq 20\,, \qquad 2000\varepsilon_\Gamma b^{\bar{n}} < 1\,. \tag{4.10l}$$

Indeed we have that the first inequality in (4.10b) is immediate, the second is possible since τ_q^{-1} is increasing in q, (4.10c) is possible due to item (ivb), (4.10d) and (4.10l) are possible from immediate computation, (4.10e) is possible due to item (ivc), (4.10f), (4.10g), and (4.10i) are possible due to item (ivd), (4.10h), (4.10j), and (4.10a) are possible since r_q and δ_q are decreasing in q, and (4.10k) is possible due to (4.8) and the super-exponential growth, which shows that $\lambda_q\lambda_{q+\bar{n}}^{-1}\lambda_{q+\bar{n}/2}^{-2}\lambda_{q+\bar{n}}^2, \lambda_q\lambda_{q+2}\lambda_{q+1}^{-2} > 1$.

(vii) Choose C_∞ as

$$\mathsf{C}_\infty = 3\left\lceil \frac{(b^{\bar{n}/2}-1)^2}{(b-1)^2\varepsilon_\Gamma(b^{\bar{n}/2-1}+\cdots+b+1)} + \frac{2000b^{\bar{n}}}{b^{\bar{n}/2}-1} + \frac{4b^{\bar{n}-1}}{(b-1)\varepsilon_\Gamma(1+\cdots+b^{\bar{n}/2-1})}\right\rceil . \tag{4.11}$$

As a consequence of this definition and (4.10l), we have that

$$10 \leq \mathsf{C}_\infty . \tag{4.12}$$

We furthermore have that for all $\bar{n}/2 \leq k \leq \bar{n}$,

$$\begin{aligned}
&\Gamma_q^{\mathsf{C}_\infty}\lambda_q^2\lambda_{q+k}^4\lambda_{q+\bar{n}/2}^{-4}\lambda_{q+k}^2\lambda_{q+k-1}^{-4} < \Gamma_{q+\bar{n}/2}^{\mathsf{C}_\infty} \\
&\quad\iff 2\left(1-2b^{\bar{n}/2}+b^k+2b^k-2b^{k-1}\right) < \mathsf{C}_\infty(b-1)\varepsilon_\Gamma(b^{\bar{n}/2}-1) \\
&\quad\impliedby 2\left(1-2b^{k/2}+b^k+2b^{\bar{n}}-2b^{\bar{n}-1}\right) < \mathsf{C}_\infty(b-1)\varepsilon_\Gamma(b^{\bar{n}/2}-1) \\
&\quad\iff 2\left(b^{k/2}-1\right)^2 + 4b^{\bar{n}-1}(b-1) < \mathsf{C}_\infty(b-1)^2\varepsilon_\Gamma(1+\cdots+b^{\bar{n}/2-1}) \\
&\quad\iff \frac{2\left(b^{k/2}-1\right)^2}{(b-1)^2\varepsilon_\Gamma(1+\cdots+b^{\bar{n}/2-1})} + \frac{4b^{\bar{n}-1}}{(b-1)\varepsilon_\Gamma(1+\cdots+b^{\bar{n}/2-1})} < \mathsf{C}_\infty ,
\end{aligned}$$

which is implied by (4.11). As a consequence of the above inequality, (4.10l), (4.10k), and (4.11), we have that for all $\bar{n}/2 \leq k \leq \bar{n}$,

$$\Gamma_q^{\mathsf{C}_\infty} \leq \Gamma_{q+\bar{n}/2}^{\mathsf{C}_\infty}\Gamma_{q+\bar{n}}^{-2000} , \qquad \Gamma_q^{\mathsf{C}_\infty+500}\Lambda_q\left(\frac{\lambda_{q+k}}{\lambda_{q+\bar{n}/2}}\right)^2\lambda_{q+k-1}^{-2}\lambda_{q+k} \leq \Gamma_{q+\bar{n}/2}^{\mathsf{C}_\infty}\Gamma_{q+\bar{n}}^{-200} . \tag{4.13a}$$

(viii) Choose $\alpha = \alpha(q) \in (0,1)$ such that

$$\lambda_{q+\bar{n}}^{\alpha} = \Gamma_q^{1/10} . \tag{4.14}$$

(ix) Choose T_q according to the formula

$$\frac{1}{2}\mathrm{T}_{q-1}^{-1} = \tau_q^{-1}\Gamma_q^{\mathsf{C}_\infty+100}\delta_q^{-1/2}r_q^{-2/3} + \Gamma_q^{\mathsf{C}_\infty+100}\delta_q^{-1/2}r_q^{-1}\Lambda_q^3 . \tag{4.15}$$

(x) Choose $\mathsf{N}_{\rm pr}$ such that

$$\Gamma_{q+\mathsf{N}_{\rm pr}}\Lambda^4_{q+\bar{n}} \le \Gamma_{q+\mathsf{N}_{\rm pr}+1}\,. \tag{4.16}$$

(xi) Choose $\mathsf{N}_{\rm cut,t}$ and $\mathsf{N}_{\rm cut,x}$ such that

$$\mathsf{N}_{\rm cut,t} \le \mathsf{N}_{\rm cut,x}\,, \tag{4.17a}$$

$$\lambda^{200}_{q+\bar{n}}\left(\frac{\Gamma_{q-1}}{\Gamma_q}\right)^{\frac{\mathsf{N}_{\rm cut,t}}{5}} \le \min\left(\lambda^{-4}_{q+\bar{n}}\delta^2_{q+3\bar{n}}, \Gamma^{-\mathsf{C}_\infty-17-\mathsf{C}_b}_{q+\bar{n}}\delta^2_{q+3\bar{n}}r_q\right)\,, \tag{4.17b}$$

$$\delta^{-1/2}_{q+\bar{n}}r_q^{-1}\Gamma^{\mathsf{C}_\infty/2+16+\mathsf{C}_b}_{q+\bar{n}}\left(\frac{\Gamma_{q+\bar{n}-1}}{\Gamma_{q+\bar{n}}}\right)^{\mathsf{N}_{\rm cut,x}} \le \Gamma^{-1}_{q+\bar{n}}\,. \tag{4.17c}$$

(xii) Choose $\mathsf{N}_{\rm ind,t}$ such that

$$\mathsf{N}_{\rm ind,t} \ge \mathsf{N}_{\rm cut,t}, \quad \Gamma_q^{-\mathsf{N}_{\rm ind,t}}(\tau_q^{-1}\Gamma_q^{i+40})^{-\mathsf{N}_{\rm cut,t}-1}(\mathrm{T}_q^{-1}\Gamma_q)^{\mathsf{N}_{\rm cut,t}+1} \le 1\,. \tag{4.18}$$

(xiii) Choose $N_{\rm g}, N_{\rm c}$ so that

$$\Gamma^{-N_{\rm g}}_{q-1}\Gamma^2_q \le \Gamma_{q+1}\mathrm{T}^{50\mathsf{N}_{\rm ind,t}}_{q+1}\delta^3_{q+3\bar{n}}\,, \tag{4.19a}$$

$$2(\mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^{10}_{q+\bar{n}-1})^{5\mathsf{N}_{\rm ind,t}}\Gamma^{2\mathsf{C}_\infty+\mathsf{C}_b+100}_{q+\bar{n}}r_q^{-2}\Gamma^{-N_{\rm c}/2}_{q-1} \le \Gamma^{-N_{\rm g}}_{q+\bar{n}}\delta^3_{q+3\bar{n}}\tau^{50\mathsf{N}_{\rm ind,t}}_{q+\bar{n}-1}\,, \tag{4.19b}$$

$$N_{\rm g} \le N_{\rm c} \le \frac{\mathsf{N}_{\rm ind}}{40}\,.^{2} \tag{4.19c}$$

[2]This inequality is independent from the first two, and can be ensured by a large choice of $\mathsf{N}_{\rm ind}$ in the next step. Since all the inequalities in (4.19) are used together, we break the order slightly and include (4.19c) in this bullet point.

(xiv) Choose $\mathsf{N}_{\rm ind}$ such that (4.19c) is satisfied and

$$\mathsf{N}_{\rm ind,t} \leq \mathsf{N}_{\rm ind}\,, \tag{4.20a}$$

$$\left(\Gamma_{q-1}^{\mathsf{N}_{\rm ind}}\Gamma_q^{-\mathsf{N}_{\rm ind}}\right)^{1/10} \leq \delta_{q+5\bar{n}}^3\Gamma_q^{-2\mathsf{C}_\infty-3}r_q\,. \tag{4.20b}$$

(xv) Choose $\mathsf{N}_{\rm dec}$ such that

$$(\lambda_{q+\bar{n}+2}\Gamma_q)^4 \leq \left(\frac{\Gamma_q^{1/10}}{4\pi}\right)^{\mathsf{N}_{\rm dec}}, \qquad \mathsf{N}_{\rm ind} \leq \mathsf{N}_{\rm dec}\,. \tag{4.21}$$

(xvi) Choose $K_\circ$ large enough so that

$$\lambda_q^{-K_\circ} \leq \delta_{q+3\bar{n}}^3\mathrm{T}_{q+\bar{n}}^{5\mathsf{N}_{\rm ind}}\lambda_{q+\bar{n}+2}^{-100}\,. \tag{4.22}$$

(xvii) Choose d and N_{**} such that

$$2\mathsf{d}+3 \leq N_{**}\,, \tag{4.23a}$$

$$\lambda_{q+\bar{n}}^{100}\Gamma_q^{-\mathsf{d}/200}\Lambda_{q+\bar{n}+2}^{5+K_\circ}\left(1+\frac{\max(\lambda_{q+\bar{n}}^2\mathrm{T}_q^{-1},\Lambda_q^{1/2}\Lambda_{q+\bar{n}})}{\tau_q^{-1}}\right)^{20\mathsf{N}_{\rm ind}} \leq \mathrm{T}_{q+\bar{n}}^{200\mathsf{N}_{\rm ind,t}}\,, \tag{4.23b}$$

$$\lambda_{q+\bar{n}}^{100}\Gamma_q^{-N_{**}/20}\Lambda_{q+\bar{n}+2}^{5+K_\circ}\left(1+\frac{\max(\lambda_{q+\bar{n}}^2\mathrm{T}_q^{-1},\Lambda_q^{1/2}\Lambda_{q+\bar{n}})}{\tau_q^{-1}}\right)^{20\mathsf{N}_{\rm ind}} \leq \mathrm{T}_{q+\bar{n}}^{20\mathsf{N}_{\rm ind,t}}\,. \tag{4.23c}$$

(xviii) Choose $\mathsf{N}_{\rm fin}$ such that

$$2\mathsf{N}_{\rm dec}+4+10\mathsf{N}_{\rm ind} \leq {}^{\mathsf{N}_{\rm fin}}/_{40000} - \mathsf{d}^2 - 10\mathsf{N}_{\rm cut,x} - 10\mathsf{N}_{\rm cut,t} - N_{**} - 300\,. \tag{4.24a}$$

(xix) Having chosen all the parameters mentioned in items (i)–(xviii) except for a, there exists a sufficiently large parameter a_* such that $a_*^{(b-1)\varepsilon_\Gamma b^{-2\bar{n}}}$ is at least fives times larger than *all* the implicit constants throughout the paper, as well as those which have been suppressed in the computations in this section. Choose a to be any natural

number larger than a_*.

4.2 A few more inequalities

For all $q + {}^{\bar{n}}/_2 - 1 \leq m \leq m' \leq q + \bar{n}$, we have that

$$\Gamma_q^{500+5\mathsf{C}_b} \lambda_q \left(\frac{\delta_{q+\bar{n}}}{\delta_{m+\bar{n}}} \right)^{3/2} \Lambda_q^{2/3} \left(\lambda_{m'-1}^{-2} \lambda_{m'} \right)^{2/3} \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}}) \Gamma_q}{\lambda_{q+{}^{\bar{n}}/_2}} \right)^{4/3} \lambda_{m-1}^{-2} \lambda_m \leq \Gamma_q^{-250} \,, \tag{4.25}$$

and

$$\Gamma_q^{500+5\mathsf{C}_b} \Lambda_q \left(\frac{\min(\lambda_{m'}, \lambda_{q+\bar{n}})}{\lambda_{q+{}^{\bar{n}}/_2}} \right)^{2/3} \left(\frac{\delta_{q+\bar{n}}}{\delta_{m+\bar{n}}} \right)^{3/2} \Lambda_q \lambda_{m'-1}^{-2} \lambda_{m'} \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}}) \Gamma_q}{\lambda_{q+{}^{\bar{n}}/_2}} \right)^{4/3} \lambda_{m-1}^{-2} \lambda_m \leq \Gamma_q^{-250} \,. \tag{4.26}$$

We claim the first inequality is morally equivalent to

$$\lambda_q \left(\frac{\delta_{q+\bar{n}}}{\delta_{m+\bar{n}}} \right)^{3/2} \lambda_q^{2/3} \left(\min(\lambda_m, \lambda_{q+\bar{n}}) \right)^{2/3} \lambda_{q+{}^{\bar{n}}/_2}^{-4/3} \lambda_m^{-1} \leq 1 \,.$$

This equivalence is due to (4.5) (used to absorb a feq meaningless losses of $\lambda_k \lambda_{k-1}^{-1}$) and (4.10f) (used to absorb $\Gamma_{q+\bar{n}}^{2000+10\mathsf{C}_b}$, which itself can be absorbed in on meaningless loss of $\lambda_k \lambda_{k-1}^{-1}$ from (4.10k)). Checking the simplified inequality then boils down to applying (4.5). We leave further details to the reader. The second inequality is morally equivalent to

$$\lambda_q \left(\frac{\lambda_{m'}}{\lambda_{q+{}^{\bar{n}}/_2}} \right)^{2/3} \left(\frac{\delta_{q+\bar{n}}}{\delta_{m+\bar{n}}} \right)^{3/2} \lambda_q \lambda_{m'}^{-1} \lambda_m^{-1} \left(\frac{\lambda_m}{\lambda_{q+{}^{\bar{n}}/_2}} \right)^{4/3} \leq 1 \,,$$

which can be checked by again using similar reasoning.

At this point, we list a number of additional inequalities, each of which can be checked

by similar reasoning as the two inequalities above. We leave further details to the reader.

$$\lambda_q \Gamma_q^{250} \Lambda_q^{2/3} \left(\frac{r_{q+\bar{n}/2+1}}{r_q} \right)^{2/3} \lambda_{q+\bar{n}/2}^{-2/3} \left(\frac{\lambda_{q+\bar{n}/2+1} \Gamma_q}{\lambda_{q+\bar{n}/2}} \right)^{4/3} \lambda_{q+\bar{n}/2}^{-1} \delta_{q+\bar{n}}^{3/2} \leq \delta_{q+\bar{n}+\bar{n}/2+1}^{3/2} \,, \tag{4.27a}$$

$$\lambda_q \Gamma_q^{250+5\mathsf{C}_b} \Lambda_q \lambda_{q+\bar{n}/2}^{-1} \left(\frac{\lambda_{q+\bar{n}/2+1} \Gamma_q}{\lambda_{q+\bar{n}/2}} \right)^{2} \lambda_{q+\bar{n}/2}^{-1} \delta_{q+\bar{n}}^{3/2} \leq \delta_{q+\bar{n}+\bar{n}/2+1}^{3/2} \,, \tag{4.27b}$$

$$\delta_{q+\bar{n}} \Gamma_q^{500} \Lambda_q^{2/3} \left(\lambda_{m-1}^2 \lambda_m^{-1} \right)^{-2/3} \leq \delta_{m+\bar{n}} \quad \text{for} \quad q + \bar{n}/2 - 5 \leq m \leq q + \bar{n} + 5 \,, \tag{4.27c}$$

$$\delta_{q+\bar{n}} \Lambda_q \Gamma_q^{400+5\mathsf{C}_b} \left(\frac{\lambda_m}{\lambda_{q+\bar{n}} r_q} \right)^{2/3} \lambda_{m-1}^{-2} \lambda_m \leq \Gamma_m^{-9} \delta_{m+\bar{n}} \,, \tag{4.27d}$$

$$\frac{\delta_{q+\bar{n}}}{\delta_{m+\bar{n}}} \Gamma_q^{200+5\mathsf{C}_b} \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q} \right)^{2/3} \Lambda_q \lambda_{m-1}^{-2} \min(\lambda_m, \lambda_{q+\bar{n}}) \leq \Gamma_{q+\bar{n}/2}^{-100} \,. \tag{4.27e}$$

Chapter 5

Inductive assumptions

We begin by fixing a few notational conventions that will be used throughout this thesis.

Remark 5.0.1 (Geometric upper bounds with two bases). For all $n \geq 0$, we define

$$\mathcal{M}\left(n, N_*, \lambda, \Lambda\right) := \lambda^{\min\{n, N_*\}} \Lambda^{\max\{n - N_*, 0\}} .$$

Remark 5.0.2 (Space-time norms). In the remainder of the paper, we shall always measure objects using uniform-in-time norms $\sup_{t \in [T_1, T_2]} \| \cdot (t) \|$, where $\| \cdot (t) \|$ is any of a variety of norms used to measure functions defined on $\mathbb{T}^3 \times [T_1, T_2]$ but restricted to time t. In a slight abuse of notation, we shall always abbreviate these space-time norms with simply $\| \cdot \|$.

Remark 5.0.3 (Space-time balls). For any set $\Omega \subseteq \mathbb{T}^3 \times \mathbb{R}$, we shall use the notations

$$B(\Omega, \lambda^{-1}) := \left\{ (x, t) \, : \, \exists \, (x_0, t) \in \Omega \text{ with } |x - x_0| \leq \lambda^{-1} \right\} \tag{5.1a}$$

$$B(\Omega, \lambda^{-1}, \tau) := \left\{ (x, t) \, : \, \exists \, (x_0, t_0) \in \Omega \text{ with } |x - x_0| \leq \lambda^{-1} \, , |t - t_0| \leq \tau \right\} \tag{5.1b}$$

for space and space-time neighborhoods of Ω of radius λ^{-1} in space and τ in time, respectively.

5.1 Relaxed equations

We shall assume that all inductive assumptions at the qth step hold on the domain $[-\tau_{q-1}, T+\tau_{q-1}] \times \mathbb{T}^3$.

We assume that there exists a given q-independent Radon measure $E = E(t,x) \geq 0$ such that the approximate solution $(u_q, p_q, R_q, \varphi_q, -\pi_q)$ at the qth step satisfies the Euler-Reynolds system

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \operatorname{div}(R_q - \pi_q \mathrm{Id}) \\ \operatorname{div} u_q = 0 \end{cases} \tag{5.2}$$

and the relaxed local energy identity with dissipation measure E

$$\partial_t \left(\frac{1}{2}|u_q|^2\right) + \operatorname{div}\left(\left(\frac{1}{2}|u_q|^2 + p_q\right) u_q\right) = (\partial_t + \widehat{u}_q \cdot \nabla)\kappa_q + \operatorname{div}((R_q - \pi_q \mathrm{Id})\widehat{u}_q) + \operatorname{div}\varphi_q - E\,. \tag{5.3}$$

In the above equation, we have set $\kappa_q = {}^{\operatorname{tr}(R_q - \pi_q \mathrm{Id})}/_2$, and we use the decomposition and notations

$$u_q = \underbrace{\widehat{u}_{q-1} + \widehat{w}_q}_{=:\widehat{u}_q} + \widehat{w}_{q+1} + \cdots + \widehat{w}_{q+\bar{n}-1} =: \widehat{u}_{q+\bar{n}-1} \tag{5.4}$$

for the velocity field. The stress error R_q has a decomposition

$$R_q = \sum_{k=q}^{q+\bar{n}-1} R_q^k\,. \tag{5.5}$$

where R_q^k are symmetric matrices. The pressure π_q has a decomposition

$$\pi_q = \sum_{k=q}^{\infty} \pi_q^k\,. \tag{5.6}$$

Similarly, the current error φ_q has a decomposition

$$\varphi_q = \sum_{k=q}^{q+\bar{n}-1} \varphi_q^k \,. \tag{5.7}$$

In the continuous scheme, the Reynolds stress R_q and current error φ_q at stage q of the iteration will have frequency support in frequencies less than $\lambda_{q+\bar{n}-1}$ (effectively speaking). We correct the portions of both which live at frequencies no higher than λ_q. We denote these portion by R_q^q and φ_q^q, respectively. More generally, we denote the portions of R_q and φ_q with spatial derivative cost λ_k by R_q^k and φ_q^k, respectively.

5.2 Inductive assumptions for velocity cutoff functions

Given the intermittent nature of the velocity vector field $\widehat{u}_{q'}$, the cost of its associated material derivative $D_{t,q'}$ of errors can vary significantly across different level sets of the velocity. To address this issue, we introduce a velocity cutoff function $\psi_{i,q'}$ defined inductively. By applying these cutoffs, we partition the domain into distinct level sets of the velocity, which allows us to analyze the material derivative cost of errors on each support in the following subsections. We first record its key properties useful for later analysis in this subsection, and the L^∞ estimates for velocity increment $\widehat{w}_{q'}$ and velocity $\widehat{u}_{q'}$, obtained as a consequence of the definition of $\psi_{i,q'}$, can be found in subsection 5.5.

All assumptions in subsection 5.2 are assumed to hold for all $q-1 \leq q' \leq q+\bar{n}-1$. First, we assume that the velocity cutoff functions form a partition of unity:

$$\sum_{i\geq 0} \psi_{i,q'}^6 \equiv 1, \qquad \text{and} \qquad \psi_{i,q'}\psi_{i',q'} = 0 \quad \text{for} \quad |i-i'| \geq 2\,. \tag{5.8}$$

Second, we assume that there exists an $i_{\max} = i_{\max}(q') \geq 0$, which is bounded uniformly in

q' by

$$i_{\max}(q') \leq \frac{\mathsf{C}_\infty + 12}{(b-1)\varepsilon_\Gamma}\,, \tag{5.9}$$

such that

$$\psi_{i,q'} \equiv 0 \quad \text{for all} \quad i > i_{\max}(q')\,, \qquad \text{and} \qquad \Gamma_{q'}^{i_{\max}(q')} \leq \Gamma_{q'-\bar{n}}^{\mathsf{C}_\infty/2+18}\delta_{q'}^{-1/2} r_{q'-\bar{n}}^{-2/3}\,. \tag{5.10}$$

For all $0 \leq i \leq i_{\max}$, we assume the following pointwise derivative bounds for the cutoff functions $\psi_{i,q'}$. First, for mixed space and material derivatives and multi-indices $\alpha, \beta \in \mathbb{N}^k$, $k \geq 0$, $0 \leq |\alpha| + |\beta| \leq \mathsf{N}_{\text{fin}}$, we assume that

$$\frac{\mathbf{1}_{\operatorname{supp}\psi_{i,q'}}}{\psi_{i,q'}^{1-(K+M)/\mathsf{N}_{\text{fin}}}} \left| \left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q'-1}^{\beta_l} \right) \psi_{i,q'} \right| \leq \Gamma_{q'}(\Gamma_{q'}\lambda_{q'})^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}}, \Gamma_{q'-1}^{i+3}\tau_{q'-1}^{-1}, \Gamma_{q'-1}\mathrm{T}_{q'-1}^{-1}\right)\,. \tag{5.11}$$

Next, with α, β, k as above, $N \geq 0$ and $D_{q'} := \widehat{w}_{q'} \cdot \nabla$, we assume that

$$\begin{aligned} &\frac{\mathbf{1}_{\operatorname{supp}\psi_{i,q'}}}{\psi_{i,q'}^{1-(N+K+M)/\mathsf{N}_{\text{fin}}}} \left| D^N \left(\prod_{l=1}^{k} D_{q'}^{\alpha_l} D_{t,q'-1}^{\beta_l} \right) \psi_{i,q'} \right| \\ &\qquad \leq \Gamma_{q'}(\Gamma_{q'}\lambda_{q'})^N (\Gamma_{q'}^{i-5}\tau_{q'}^{-1})^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}}, \Gamma_{q'-1}^{i+3}\tau_{q'-1}^{-1}, \Gamma_{q'-1}\mathrm{T}_{q'-1}^{-1}\right) \end{aligned} \tag{5.12}$$

for $0 \leq N + |\alpha| + |\beta| \leq \mathsf{N}_{\text{fin}}$. Moreover, for $0 \leq i \leq i_{\max}(q')$, we assume the L^1 bound

$$\|\psi_{i,q'}\|_1 \leq \Gamma_{q'}^{-3i+\mathsf{C}_b} \qquad \text{where} \qquad \mathsf{C}_b = \frac{6+b}{b-1}\,. \tag{5.13}$$

Lastly, we assume that local timescales dictated by velocity cutoffs at a fixed point in spacetime are decreasing in q. More precisely, for all $q' \leq q + \bar{n} - 1$ and all $q'' \leq q' - 1$, we assume

$$\psi_{i',q'}\psi_{i'',q''} \not\equiv 0 \quad \Longrightarrow \quad \tau_{q'}\Gamma_{q'}^{-i'} \leq \tau_{q''}\Gamma_{q''}^{-i''-25}\,. \tag{5.14}$$

This will be useful when we upgrade material derivative from $D_{t,q''}$ to $D_{t,q'}$.

The concrete construction of $\psi_{i,q+\bar{n}}$ and the verification of (5.8)–(5.13) for $q \mapsto q+1$ (i.e., $q' = q + \bar{n}$) is given in chapter 12.

5.3 Inductive bounds on the intermittent pressure

The intermittent pressure π_q is designed to majorize errors and velocity increments pointwise. Thus, we introduce estimates for this function in subsection 5.3.1 and establish precise relations between the intermittent pressure and errors/velocity increments in subsection 5.3.3. (The L^p estimates of the errors will follow consequently.) Furthermore, the intermittent pressure has been constructed to anticipate the low-frequency part of the future pressure increments. We record the relevant properties in subsection 5.3.2. All inductive assumptions appearing in subsection 5.3 will be verified for $q \mapsto q+1$ in Section 13.

5.3.1 $L^{3/2}$, L^∞, and pointwise bounds for π_q^k

We assume that for $q \le k \le q + \bar{n} - 1$ and $N + M \le 2\mathsf{N}_{\text{ind}}$, π_q^k satisfies

$$\left\|\psi_{i,k-1} D^N D_{t,k-1}^M \pi_q^k\right\|_{3/2} \le \Gamma_q \Gamma_k \delta_{k+\bar{n}} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^i \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\right) . \tag{5.15a}$$

$$\left\|\psi_{i,k-1} D^N D_{t,k-1}^M \pi_q^k\right\|_{\infty} \le \Gamma_q \Gamma_k^{\mathsf{C}_\infty+1} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^i \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\right) , \tag{5.15b}$$

$$\left|\psi_{i,k-1} D^N D_{t,k-1}^M \pi_q^k\right| \le \Gamma_q \Gamma_k \pi_q^k \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^i \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\right) . \tag{5.15c}$$

For $q + \bar{n} \le k \le q + \mathsf{N}_{\text{pr}} - 1$ and $N + M \le 2\mathsf{N}_{\text{ind}}$, we assume that π_q^k satisfies

$$\left\|\psi_{i,q+\bar{n}-1} D^N D_{t,q+\bar{n}-1}^M \pi_q^k\right\|_{3/2} \le \Gamma_q \Gamma_k \delta_{k+\bar{n}} \Lambda_{q+\bar{n}-1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}-1}^i \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{5.16a}$$

$$\left\|\psi_{i,q+\bar{n}-1} D^N D_{t,q+\bar{n}-1}^M \pi_q^k\right\|_{\infty} \le \Gamma_q \Gamma_{q+\bar{n}-1}^{\mathsf{C}_\infty+1} \Lambda_{q+\bar{n}-1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}-1}^i \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\right) , \tag{5.16b}$$

$$\left|\psi_{i,q+\bar{n}-1} D^N D_{t,q+\bar{n}-1}^M \pi_q^k\right| \le \Gamma_q \pi_q^k \Lambda_{q+\bar{n}-1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}-1}^i \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\right) . \tag{5.16c}$$

Throughout the paper, we shall use the phrase "pointwise estimates" to refer to bounds on stress errors, current errors, or velocities in terms of various π's which resemble the third bound in either of the above displays.

5.3.2 Lower and upper bounds for π_q^k

For $k \geq q$, we assume that π_q^k has the lower bound

$$\pi_q^k \geq \delta_{k+\bar{n}} \,. \tag{5.17}$$

For all $q + \bar{n} - 1 \leq k' < k \leq q + \mathsf{N}_{\mathrm{pr}} - 1$, we assume that π_q^k has the upper bound

$$\pi_q^k \leq \pi_q^{k'} \,. \tag{5.18}$$

For all $k \geq q + \mathsf{N}_{\mathrm{pr}}$, we assume that

$$\pi_q^k \equiv \Gamma_k \delta_{k+\bar{n}} \,. \tag{5.19}$$

We finally assume that for all $q \leq q' < q'' < \infty$,

$$\frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}} \pi_q^{q'} < 2^{q'-q''} \pi_q^{q''} \,, \qquad \text{if}\, q + {}^{\bar{n}}\!/_2 \leq q'' \tag{5.20a}$$

$$\frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}} \pi_q^{q'} < \pi_q^{q''} \,, \qquad \text{otherwise}\,. \tag{5.20b}$$

This final bound says that the π_q^k's obey a scaling law which may be roughly translated as "any π_q^{k+m} for $m > 0$ can be bounded from below by an appropriately rescaled π_q^k."

5.3.3 Pointwise bounds for errors, velocities, and velocity cutoffs

We assume that we have the pointwise estimates

$$\left|\psi_{i,k-1} D^N D^M_{t,k-1} R^k_q\right| < \Gamma_q \Gamma_k^{-8} \pi_q^k \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+20} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{10}\right) , \tag{5.21a}$$

$$\left|\psi_{i,k-1} D^N D^M_{t,k-1} \varphi^k_q\right| < \Gamma_q \Gamma_k^{-12} (\pi_q^k)^{\frac{3}{2}} r_k^{-1} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+20} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{10}\right) , \tag{5.21b}$$

$$\left|\psi_{i,k-1} D^N D^M_{t,k-1} \widehat{w}_k\right| < \Gamma_q r_{k-\bar{n}}^{-1} (\pi_q^k)^{1/2} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{2}\right) \tag{5.21c}$$

for $q \leq k \leq q + \bar{n} - 1$, where the first bound holds for $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$, the second bound holds for $N + M \leq {}^{\mathsf{N}_{\mathrm{ind}}}/_4$, and the third bound holds for $N + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2$.

While the main L^p estimates on the Reynolds stress follow from the pointwise estimates in terms of the pressure, we are forced to assume that R_q^k has a decomposition $R_q^k = R_q^{k,l} + R_q^{k,*}$, where $R_q^{k,*}$ satisfies the stronger bound

$$\left\|D^N D^M_{t,k-1} R_q^{k,*}\right\|_\infty \leq \Gamma_q^2 \mathrm{T}_k^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{k+2\bar{n}} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\right) \tag{5.22}$$

for all $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$. The extra superscript l stands for "local," in the sense that $R_q^{k,l}$ is a stress error over which we maintain control of the spatial support, whereas $*$ refers to non-local terms which are negligibly small. The reader can safely ignore such non-local error terms.

Finally, we assume that for all $q \leq q' \leq q + \bar{n} - 1$,

$$\sum_{i=0}^{i_{\max}} \psi_{i,q'}^2 \delta_{q'} r_{q'-\bar{n}}^{-2/3} \Gamma_{q'}^{2i} \leq 2^{q-q'} \Gamma_{q'} r_{q'-\bar{n}}^{-2} \pi_q^{q'} . \tag{5.23}$$

Remark 5.3.1 (L^p **estimates on Reynolds errors from pointwise estimates).** The estimates on R_q^k in (5.21a) and the estimates on π_q^k in (5.15) imply that for $q \leq k \leq q + \bar{n} - 1$

and $N+M \leq 2\mathsf{N}_{\text{ind}}$, R_q^k satisfies

$$\left\|\psi_{i,k-1} D^N D_{t,k-1}^M R_q^k\right\|_{3/2} \leq \Gamma_q^2 \Gamma_k^{-7} \delta_{k+\bar{n}} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+20} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_q^{10}\right), \tag{5.24a}$$

$$\left\|\psi_{i,k-1} D^N D_{t,k-1}^M R_q^k\right\|_{\infty} \leq \Gamma_q^2 \Gamma_k^{-7} \Gamma_k^{\mathsf{C}_\infty} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+20} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_q^{10}\right). \tag{5.24b}$$

Remark 5.3.2 (Velocity cutoffs, timescales, and pressure). Using the timescale parameter $\tau_q^{-1} \approx \delta_q^{1/2} \lambda_q r_{q-\bar{n}}^{-1/3}$ defined precisely in (vi), we may now record the following version of (5.23) for $q' = q$;

$$\psi_{i,q} \tau_q^{-1} \Gamma_q^i \leq \lambda_q \Gamma_q \left(\pi_q^q\right)^{1/2} r_q^{-1}. \tag{5.25}$$

5.4 Dodging principle ingredients

In this subsection, we list "dodging" inductive hypotheses. As discussed in the introduction, one of the crucial elements for the continuous scheme is dodging between velocity increments, which is elaborated as Hypothesis 5.4.1. To construct a new velocity increment with such dodging, it is necessary to keep a record of the density of previous velocity increments as stated in Hypothesis 5.4.2. These two hypotheses can be seen as improved and inductive versions of the "pipe dodging" technique used in [7] or [35]. As byproducts of a special construction of velocity increments, we also have Hypothesis 5.4.4 and Hypothesis 5.4.5, which explain dodging properties between velocity increments and stress error/intermittent pressure. This will be utilized later in the stress current estimate discussed in subsection 11.2.4.

Hypothesis 5.4.1 (Effective dodging). For $q', q'' \leq q + \bar{n} - 1$ that satisfy $0 < |q'' - q'| \leq \bar{n} - 1$, we have that[1]

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap B\left(\operatorname{supp} \widehat{w}_{q''}, \lambda_{q''}^{-1} \Gamma_{q''+1}\right) = \emptyset. \tag{5.26}$$

[1]Here we are considering the support of $\widehat{w}_q$ in time and space, then expanding to a ball of radius $\lambda_q^{-1}\Gamma_{q+1}$ in space only; see (5.1).

Hypothesis 5.4.2 (Density of old pipe bundles). There exists a q-independent constant $\mathcal{C}_D$ such that the following holds. Let $\bar{q}', \bar{q}''$ satisfy $q \leq \bar{q}'' < \bar{q}' \leq q + \bar{n} - 1$, and set[2]

$$d(\bar{q}', \bar{q}'') := \min\left[(\lambda_{\bar{q}''}\Gamma^7_{\bar{q}''})^{-1}, (\lambda_{\bar{q}'-\bar{n}/2}\Gamma_{\bar{q}'-\bar{n}})^{-1}\right] . \tag{5.27}$$

Let $t_0 \in \mathbb{R}$ be any time and $\Omega \subset \mathbb{T}^3$ be a convex set of diameter at most $d(\bar{q}', \bar{q}'')$. Let i be such that $\Omega \times \{t_0\} \cap \operatorname{supp} \psi_{i,\bar{q}''} \neq \emptyset$. Let $\Phi_{\bar{q}''}$ be the flow map such that

$$\begin{cases} \partial_t \Phi_{\bar{q}''} + (\widehat{u}_{\bar{q}''} \cdot \nabla)\, \Phi_{\bar{q}''} = 0 \\ \Phi_{\bar{q}''}(t_0, x) = x\,. \end{cases}$$

We define $\Omega(t) = \Phi_{\bar{q}''}(t)^{-1}(\Omega)$.[3] Then there exists a set[4] $L = L(\bar{q}', \bar{q}'', \Omega, t_0) \subseteq \mathbb{T}^3 \times \mathbb{R}$ such that for all $t \in (t_0 - \tau_{\bar{q}''}\Gamma^{-i+2}_{\bar{q}''}, t_0 + \tau_{\bar{q}''}\Gamma^{-i+2}_{\bar{q}''})$,

$$(\partial_t + \widehat{u}_{\bar{q}''} \cdot \nabla)\mathbf{1}_L(t, \cdot) \equiv 0 \qquad \text{and} \qquad \operatorname{supp}_x \widehat{w}_{\bar{q}'}(x,t) \cap \Omega(t) \subseteq L \cap \{t\}\,. \tag{5.28}$$

Furthermore, there exists a finite family of Lipschitz curves $\{\ell_{j,L}\}_{j=1}^{\mathcal{C}_D}$ of length at most $2d(\bar{q}', \bar{q}'')$ which satisfy

$$L \cap \{t = t_0\} \subseteq \bigcup_{j=1}^{\mathcal{C}_D} B\left(\ell_{j,L}, 3\lambda^{-1}_{\bar{q}'}\right) . \tag{5.29}$$

Remark 5.4.3 (Segments of deformed pipes of thickness $\lambda^{-1}_{\bar{q}'}$). We will refer to a $3\lambda^{-1}_{\bar{q}'}$ neighborhood of a Lipschitz curve of length at most $2(\lambda_{\bar{q}'-\bar{n}/2}\Gamma_{\bar{q}'-\bar{n}})^{-1}$ as a "segment of deformed pipe;" see Definition 7.1.8. Since $(\lambda_{\bar{q}'-\bar{n}/2}\Gamma_{\bar{q}'-\bar{n}})^{-1}$ will be the scale to which our high-frequency pipes will be periodized, Hypothesis 5.4.2 then asserts that at each step of

[2]The reasoning behind the choice of $d(\bar{q}', \bar{q}'')$ is as follows. The set should be small enough that it can be contained in the support of a single $\bar{q}''$ velocity cutoff. Since these functions oscillate at frequencies no larger than $\approx \lambda_{q''}$, the first number inside the minimum ensures that this is the case. The set should also be no larger than the size of a periodic cell for pipes of thickness $\bar{q}'$, which is ensured by the second number inside the minimum.

[3]For any set $\Omega' \subset \mathbb{T}^3$, $\Phi_{\bar{q}''}(t)^{-1}(\Omega') = \{x \in \mathbb{T}^3 : \Phi_{\bar{q}''}(t,x) \in \Omega'\}$. We shall also sometimes use the notation $\Omega \circ \Phi_{\bar{q}''}(t)$.

[4]Heuristically this set is $\cup_t \operatorname{supp}_x \widehat{w}_{\bar{q}'}(\cdot, t) \cap \Omega(t)$, but in order to ensure that $(\partial_t + \widehat{u}_{\bar{q}''} \cdot \nabla)\mathbf{1}_L \equiv 0$, L does not include any "time cutoffs" which turn pipes on and off.

the iteration, our algorithm can use at most a finite number of high-frequency pipe segments inside any single periodic cell.

Hypothesis 5.4.4 (Stress dodging). For all k, q'' such that $q \leq q'' \leq k-1$ and $q \leq k \leq q+\bar{n}-1$, we assume that

$$B\left(\operatorname{supp} \widehat{w}_{q''}, \lambda_{q''}^{-1}\Gamma_{q''+1}\right) \cap \operatorname{supp} R_q^{k,l} = \emptyset\,. \tag{5.30}$$

Hypothesis 5.4.5 (Pressure dodging). We assume that for all $q < k \leq q+\bar{n}-1$, $k \leq k'$, and $N+M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\left|\psi_{i,k-1} D^N D_{t,k-1}^M \left(\widehat{w}_k \pi_q^{k'}\right)\right| < \Gamma_q \Gamma_k^{-100} \left(\pi_q^k\right)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right)\,. \tag{5.31a}$$

Hypothesis 5.4.4 for $q \mapsto q+1$ will be verified in section 10.3. A stronger statement than Hypothesis 5.4.1 and Hypothesis 5.4.2 for $q \mapsto q+1$ will be proved in Section 9.2 (see Lemma 9.2.2). Lastly, Hypothesis 5.4.5 for $q \mapsto q+1$ will be verified in subsection 13.2.

5.5 Inductive velocity bounds

In this section, we present inductive L^∞-bounds for velocity increments and velocity, which are derived from the construction of velocity cutoffs. Additionally, we also introduce velocity increment potentials, which express velocity increments as dth divergence of the velocity increment potential with small homogeneous error. This representation will be useful to deal with a pressure current error (see subsection 13.4 for more details). All inductive assumptions in subsection 5.5 except for (5.40) at $q \mapsto q+1$ will be verified in section 9.4, and we prove (5.40) for $q \mapsto q+1$ in Section 13.

5.5.1 Velocities and velocity increments

In this subsection, we assume that $0 \le q' \le q + \bar{n} - 1$. First, for $0 \le i \le i_{\max}$, $k \ge 1$, $\alpha, \beta \in \mathbb{N}^k$, we assume that

$$\left\| \left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q'-1}^{\beta_l} \right) \widehat{w}_{q'} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q'})} \le \Gamma_{q'}^{i+2} \delta_{q'}^{1/2} r_{q'-\bar{n}}^{-1/3} (\lambda_{q'} \Gamma_{q'})^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i+3} \tau_{q'-1}^{-1}, \Gamma_{q'-1} \mathrm{T}_{q'-1}^{-1} \right) \tag{5.32}$$

for $|\alpha| + |\beta| \le {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2} + 1$. We also assume that for $N \ge 0$,

$$\begin{aligned} &\left\| D^N \Big(\prod_{l=1}^{k} D_{q'}^{\alpha_l} D_{t,q'-1}^{\beta_l} \Big) \widehat{w}_{q'} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q'})} \\ &\quad \le (\Gamma_{q'}^{i+2} \delta_{q'}^{1/2} r_{q'-\bar{n}}^{-1/3})^{|\alpha|+1} (\lambda_{q'} \Gamma_{q'})^{N+|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i+3} \tau_{q'-1}^{-1}, \Gamma_{q'-1} \mathrm{T}_{q'-1}^{-1} \right) \qquad (5.33a) \\ &\quad \le \Gamma_{q'}^{i+2} \delta_{q'}^{1/2} r_{q'-\bar{n}}^{-1/3} (\lambda_{q'} \Gamma_{q'})^{N} (\Gamma_{q'}^{i-5} \tau_{q'}^{-1})^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i+3} \tau_{q'-1}^{-1}, \Gamma_{q'-1} \mathrm{T}_{q'-1}^{-1} \right) \qquad (5.33b) \end{aligned}$$

whenever $N + |\alpha| + |\beta| \le {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2} + 1$. Next, we assume

$$\left\| \left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q'}^{\beta_l} \right) D \widehat{u}_{q'} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q'})} \le \tau_{q'}^{-1} \Gamma_{q'}^{i-4} (\lambda_{q'} \Gamma_{q'})^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i-5} \tau_{q'}^{-1}, \Gamma_{q'-1} \mathrm{T}_{q'-1}^{-1} \right) \tag{5.34}$$

for $|\alpha| + |\beta| \le {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. In addition, we assume the lossy bounds

$$\left\| \left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q'}^{\beta_l} \right) \widehat{u}_{q'} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q'})} \le \tau_{q'}^{-1} \Gamma_{q'}^{i+2} \lambda_{q'} (\lambda_{q'} \Gamma_{q'})^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i-5} \tau_{q'}^{-1}, \Gamma_{q'-1} \mathrm{T}_{q'-1}^{-1} \right) \tag{5.35a}$$

$$\left\| D^{|\alpha|} \partial_t^{|\beta|} \widehat{u}_{q'} \right\|_{L^\infty} \le \Lambda_{q'}^{1/2} \Lambda_q^{|\alpha|} \mathrm{T}_{q'}^{-|\beta|}, \tag{5.35b}$$

hold, where the first bounds holds for $|\alpha| + |\beta| \le {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2} + 1$, and the second bound holds for $|\alpha| + |\beta| \le 2\mathsf{N}_{\mathrm{fin}}$.

Remark 5.5.1 (Upgrading material derivatives for velocity and velocity cutoffs).

We have the bound

$$\left\|D^N D^M_{t,q'}\widehat{w}_{q'}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q'})} \lesssim \Gamma_{q'}^{i+2}\delta_{q'}^{1/2} r_{q'-\bar{n}}^{-1/3}(\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i-5}\tau_{q'}^{-1}, \Gamma_{q'-1}\mathrm{T}_{q'-1}^{-1}\right) \quad (5.36)$$

for all $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}+1$. Specifically, we set $B = D_{t,q'-1}$ and $A = D_{q'}$, so that $A+B = D_{t,q'}$. Then the estimate (5.36) follows from the aforementioned Lemma and (4.10b). We also have that (5.12) and (4.18) imply that for all $N+M \leq \mathsf{N}_{\mathrm{fin}}$,

$$\begin{aligned}\frac{\mathbf{1}_{\operatorname{supp}\psi_{i,q'}}}{\psi_{i,q'}^{1-(N+M)/\mathsf{N}_{\mathrm{fin}}}}\left|D^N D^M_{t,q'}\psi_{i,q'}\right| &\leq \Gamma_{q'}(\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}} - \mathsf{N}_{\mathrm{cut,t}}, \Gamma_{q'}^{i-5}\tau_{q'}^{-1}, \Gamma_{q'-1}\mathrm{T}_{q'-1}^{-1}\right)\\ &\lesssim \Gamma_{q'}(\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i-4}\tau_{q'}^{-1}, \Gamma_{q'-1}^{2}\mathrm{T}_{q'-1}^{-1}\right). \quad (5.37)\end{aligned}$$

5.5.2 Velocity increment potentials

We assume that for all $q-1 < q' \leq q+\bar{n}-1$ and $\widehat{w}_{q'}$ as in (5.4), there exists a velocity increment tensor potential $\widehat{\upsilon}_{q'}$ and an error $\widehat{e}_{q'}$ such that $\widehat{w}_{q'}$ can be decomposed as

$$\widehat{w}_{q'} = \mathrm{div}^{\mathsf{d}}\widehat{\upsilon}_{q'} + \widehat{e}_{q'}\,, \quad (5.38)$$

which written component-wise gives $\widehat{w}^{\bullet}_{q'} = \partial_{i_1}\cdots\partial_{i_{\mathsf{d}}}\widehat{\upsilon}_{q'}^{(\bullet,i_1,\cdots,i_{\mathsf{d}})} + \widehat{e}^{\bullet}_{q'}$. Next, we assume that $\widehat{\upsilon}_{q'}$ and $\widehat{e}_{q'}$ satisfy

$$B\left(\operatorname{supp}(\widehat{w}_{q''}), \frac{1}{4}\lambda_{q''}\Gamma_{q''}^2\right) \cap \left(\operatorname{supp}(\widehat{\upsilon}_{q'}) \cup \operatorname{supp}(\widehat{e}_{q'})\right) = \emptyset \quad (5.39)$$

for any $q+1 \leq q'' < q'$. In addition, we assume that $\widehat{\upsilon}^{\bullet}_{q',k} := \lambda_{q'}^{\mathsf{d}-k}\partial_{i_1}\cdots\partial_{i_k}\widehat{\upsilon}_{q'}^{(\bullet,i_1,\ldots,i_{\mathsf{d}})}$, $0 \leq k \leq \mathsf{d}$, satisfies the pointwise estimates

$$\left|\psi_{i,q'-1}D^N D^M_{t,q'-1}\widehat{\upsilon}_{q',k}\right| < \Gamma_q\Gamma_{q'}\left(\pi_q^{q'}\right)^{1/2} r_{q'-\bar{n}}^{-1}(\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'-1}^{i}\tau_{q'-1}^{-1}, \mathrm{T}_{q'-1}^{-1}\Gamma_{q'-1}^{2}\right) \quad (5.40)$$

for $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. Finally, we assume that for $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$, $\widehat{e}_{q'}$ satisfies the estimates

$$\left\|D^N D^M_{t,q'-1}\widehat{e}_{q'}\right\|_\infty \leq \delta^3_{q'+2\bar{n}}\mathrm{T}_{q'}^{5\mathsf{N}_{\mathrm{ind,t}}}\lambda_{q'}^{-10}(\lambda_{q'}\Gamma_{q'})^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_{q'-1}^{-1},\mathrm{T}_{q'-1}^{-1}\Gamma^2_{q'-1}\right) . \tag{5.41}$$

5.6 Inductive proposition and the proof of the main theorem

In this section, we introduce the inductive proposition and give a proof of the main theorem.

Proposition 5.6.1 (Main inductive proposition). *Fix $\beta \in (0, {}^1/_3)$, $b \in (1, {}^{25}/_{24})$ and $\bar{n} \in \mathbb{N}$ satisfying $b^{\bar{n}} < (3\beta)^{-1}$, $T > 0$, and fix a Radon measure $E \geq 0$. There exist parameters ε_Γ, C_∞, d, N_{pr}, $\mathsf{N}_{\mathrm{cut,t}}$, $\mathsf{N}_{\mathrm{ind,t}}$, $\mathsf{N}_{\mathrm{ind}}$, $\mathsf{N}_{\mathrm{fin}}$, depending only on β, b, and $\bar{n}$ (see Section 4.1) such that we can find sufficiently large $a_* = a_*(b,\beta,\bar{n},T)$ such that for $a \geq a_*(b,\beta,\bar{n},T)$, the following statements hold for any $q \geq 0$. Suppose that we have an approximate solution $(u_q, p_q, R_q, \varphi_q, -\pi_q)$ which satisfies the Euler-Reynold system (5.2) and the relaxed local energy identity (5.3) with dissipation measure E on the time interval $[0,T]+\tau_{q-1}$, and suppose we have a partition of unity $\{\psi^6_{i,q'}\}_{i\geq 0}$ of $([0,T]+\tau_{q-1})\times\mathbb{T}^3$ for $q-1 \leq q' \leq q+\bar{n}-1$ such that*

- *$\psi_{i,q'}$ satisfies (5.8)–(5.14).*
- *The velocity u_q and the errors R_q, φ_q, and π_q may be decomposed as in (5.4)–(5.7) so that (5.15)–(5.23), Hypotheses 5.4.1–Hypothesis 5.4.5, (5.32)–(5.35), and (5.38)–(5.41) hold.*

Then there exist a new partition of unity $\{\psi^6_{i,q+\bar{n}}\}_{i\geq 0}$ of $([0,T]+\tau_q)\times\mathbb{T}^3$ satisfying (5.8)–(5.14) for $q' = q+\bar{n}$, and a new approximate solution $(u_{q+1}, p_{q+1}, R_{q+1}, \varphi_{q+1}, -\pi_{q+1})$ satisfying (5.2) and (5.3) with dissipation measure E for $q \mapsto q+1$ on $[0,T]+\tau_q$ satisfying the following. The approximate solution may be decomposed as in (5.4)–(5.7) for $q \mapsto q+1$ so that (5.15)–

(5.23) *and Hypothesis 5.4.1–Hypothesis 5.4.5 hold for* $q \mapsto q+1$*, and* (5.32)–(5.35) *and* (5.38)–(5.41) *hold for* $q' = q + \bar{n}$*.*

Assuming for the moment that the main proposition holds, we give a proof of Theorem 2.0.1.

Proof of Theorem 2.0.1. To avoid confusion between β in the statement of the theorem and β in the statement of Proposition 5.6.1, we set $\overline{\beta} := \beta$. Since $\overline{\beta} \in (0, 1/3)$, we can choose $\beta \in (\overline{\beta}, 1/3)$. Let $a_* = a_*(\beta, b, \bar{n})$ be as in Proposition 5.6.1, fix $a \geq a_*$, and define additional parameters by

$$\Gamma_q = 1 \quad \text{for } -\bar{n} \leq q \leq -1\,, \qquad r_q = \begin{cases} \lambda_{q+\bar{n}/2}\lambda_{q+\bar{n}}^{-1} & \text{for } -\bar{n}/2 \leq q < 0 \\ \lambda_0^{1/2}\lambda_{\bar{n}}^{-1/2} & \text{for } -\bar{n} \leq q < -\bar{n}/2\,. \end{cases}$$

Step 1: Construction of initial approximate solution $(u_0, p_0, R_0, \varphi_0, -\pi_0)$ on the time interval $[-1, T+1]$. We first define

$$\widehat{u}_{-1} = 0\,, \quad \widehat{w}_{q'} = 0 \quad \text{for } 0 \leq q' \leq \bar{n} - 2\,, \qquad p_0 = \varphi_0 = 0\,, \qquad \pi_0^k = \Gamma_k \delta_{k+\bar{n}}\,,$$

$$\psi_{i,q'} = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 0 \leq q' \leq \bar{n} - 1\,,$$

so that

$$\widehat{u}_{q'} = 0 \quad \text{for } 0 \leq q' \leq \bar{n} - 2\,, \qquad \pi_0 = \sum_{k=0}^{\infty} \Gamma_k \delta_{k+\bar{n}}\,.$$

Given a smooth $\mathbb{R}^{(2^{\mathrm{d}})}$-tensor field $\bar{\vartheta}(x) = \bar{\vartheta}(x_1, x_2)$ with $\operatorname{supp}(\bar{\vartheta}) \subset B(0, 1/4)$. Then applying Proposition 7.1.5 we have $\bar{\vartheta}_{e_3,\lambda_{\bar{n}-1},r_{-1}}$ and $\bar{\varrho}_{e_3,\lambda_{\bar{n}-1},r_{-1}}$, depending only on the x_1, x_2 variables, which are $(\mathbb{T}^3/\lambda_{\bar{n}/2-1}\Gamma_{-1})$)-periodic, have support contained in pipes of thickness $\lambda_{\bar{n}-1}^{-1}$, and

satisfy

$$\bar{\varrho}_{e_3,\lambda_{\bar{n}-1},r_{-1}} = \lambda_{\bar{n}-1}^{-\mathsf{d}} \mathrm{div}^{\mathsf{d}} \bar{\vartheta}_{e_3,\lambda_{\bar{n}-1},r_{-1}}, \quad \left\|\nabla^n \bar{\vartheta}_{e_3,\lambda_{\bar{n}-1},r_{-1}}\right\|_{L^\infty} + \left\|\nabla^n \bar{\varrho}_{e_3,\lambda_{\bar{n}-1},r_{-1}}\right\|_{L^\infty} \leq C_0 \lambda_{\bar{n}-1}^n r_{-1}^{-1}$$

for all $n \leq 2\mathsf{N}_{\mathrm{fin}} + \mathsf{d}$ and some positive constant $C_0 = C_0(2\mathsf{N}_{\mathrm{fin}}, \mathsf{d})$. For simplicity, we denote $\bar{\varrho}_{e_3,\lambda_{\bar{n}-1},r_{-1}}$ and $\bar{\vartheta}_{e_3,\lambda_{\bar{n}-1},r_{-1}}$ by ρ_0 and θ_0, respectively. We then define

$$u_0 = \widehat{w}_{\bar{n}-1} = e(t)\rho_0(x_1, x_2)e_3, \quad e(t) := \Gamma_{\bar{n}}^{-100} \delta_{2\bar{n}} r_{-1} \exp(-\tau_0^{-1}(t+1))$$

$$R_0 = e'(t) \begin{pmatrix} 0 & 0 & (\lambda_{\bar{n}-1}^{-\mathsf{d}} \mathrm{div}^{\mathsf{d}-1}\theta_0)_1 \\ 0 & 0 & (\lambda_{\bar{n}-1}^{-\mathsf{d}} \mathrm{div}^{\mathsf{d}-1}\theta_0)_2 \\ (\lambda_{\bar{n}-1}^{-\mathsf{d}} \mathrm{div}^{\mathsf{d}-1}\theta_0)_1 & (\lambda_{\bar{n}-1}^{-\mathsf{d}} \mathrm{div}^{\mathsf{d}-1}\theta_0)_2 & 0 \end{pmatrix} = R_0^{\bar{n}-1,l},$$

where $(g)_k$ denotes the kth component of the vector g. Here, R_0 is constructed to have $\partial_t u_0 = \mathrm{div} R_0$.

From the construction, we have $\mathrm{div} u_0 = \mathrm{div}(|u_0|^2 u_0) = 0$ and $\mathrm{div}(u_0 \otimes u_0) = 0$, so that one can easily see that $(u_0, p_0, R_0, \varphi_0, -\pi_0)$ satisfies the Euler-Reynold system (5.2) and the relaxed local energy identity (5.3) with $E(t,x) := -e(t)e'(t)|\rho_0|^2$ on the time interval $[-1, T+1]$. We will now check that the constructed approximate solution pair and the partitions of unity satisfy the inductive assumptions appearing in Section (5.1)–(5.5) on $[-1, T+1]$. For $0 \leq q' \leq \bar{n}-1$, $\psi_{i,q'}$ satisfies (5.8)–(5.14).

Letting $R_0^k = 0$ for $0 \leq k \leq \bar{n}-2$, $R_0^{\bar{n}-1} = R_0^{\bar{n}-1,l}$, and $\varphi_0^k = 0$ for $0 \leq k \leq \bar{n}-1$, we have the decompositions (5.4)–(5.7). Using the convention $B(A,r) = \emptyset$ for the empty set A and setting $\widehat{v}_{\bar{n}-1} = e(t)\theta_0 e_3$, $\widehat{v}_{q'} = e_{q''} = 0$ for $0 \leq q' \leq \bar{n}-2$ and $0 \leq q'' \leq \bar{n}-1$, we can easily verify (5.15)–(5.23), Hypothesis 5.4.1, 5.4.4, 5.4.5, (5.32), (5.33), (5.38)–(5.41) for $q = 0$. As for Hypothesis 5.4.2, it is enough to consider $q' = \bar{n}-1$. Since $q'' < q = \bar{n}-1$ and i needs to be 0, recalling that $\widehat{u}_{q''} = 0$, we have $\Phi_{q''}(t,x) = x$ and hence $\Omega(t) = \Omega$. Then, (5.28) is

equivalent to

$$\operatorname{supp} \widehat{w}_{\bar{n}-1} \cap \Omega \subset L \cap \Omega \,.$$

Therefore, we choose L as the collection of the $(\mathbb{T}/\lambda_{\bar{n}/2-1}\Gamma_{\bar{n}/2-1})^3$-periodic pipes of thickness $\lambda_{\bar{n}-1}^{-1}$ containing the support of $\widehat{w}_{\bar{n}-1}$, which verifies the hypothesis. Lastly, considering (5.34) and (5.35), it is enough to prove it when $q' = \bar{n} - 1$ and $i = 0$. Since we have $\widehat{u}_{\bar{n}-1} = \widehat{w}_{\bar{n}-1}$ and

$$\left\| \left(\prod_{l=1}^{k} D^{\alpha_l} \partial_t^{\beta_l} \right) \widehat{w}_{\bar{n}-1} \right\|_{L^\infty(\mathbb{T}^3)} \leq C_0 \Gamma_{\bar{n}}^{-100} \delta_{2\bar{n}} \lambda_{\bar{n}-1}^{|\alpha|} \tau_0^{-|\beta|} \,,$$

applying Remark A.2.6 to $p = \infty$, $v = 0$, $w = \widehat{w}_{\bar{n}-1}$, $\Omega = \mathbb{T}^3$, $N_* = {}^{7\mathsf{N}_{\text{fin}}}/_4$, we get the desired estimates (5.34) and (5.35).

Step 2: From Proposition 5.6.1 to Theorem 2.0.1. In **Step 1**, we checked that the inductive assumptions hold at the base case of the induction $q = 0$, and we may inductively apply Proposition 5.6.1 with $E(t,x) = -e(t)e'(t)\rho_0^2$, to produce a sequence of approximate solutions $(u_q, p_q, R_q, \varphi_q, -\pi_q)$ such that all inductive assumptions hold for all $q \geq 0$. Then, by construction, we have that for any $\overline{\beta} < \beta' < \beta$, the series $\sum_{q \geq \bar{n}} \widehat{w}_q$ is absolutely summable in $C_t^0 W^{\beta',3}$ and hence in $C_t^0 B_{3,1}^{\overline{\beta}}$, which justifies the definition of the limiting velocity field $u = u_0 + \sum_{q \geq \bar{n}} \widehat{w}_q \in C_t^0 B_{3,1}^{\overline{\beta}}$. As $R_q, \pi_q \to 0$ in $C_t^0 L^{3/2}$, from the equation $-\Delta p_q = \operatorname{div}\operatorname{div}(u_q \otimes u_q + \pi_q \mathrm{Id} - R_q)$, we get the limiting pressure $p \in C_t^0 L^{3/2}$. In addition to this, as $\varphi_q \to 0$ in $C_t^0 L^1$, the limiting pair (u, p) solves the 3D Euler system and satisfies the local energy inequality,

$$\partial_t \left(\frac{1}{2}|u|^2 \right) + \operatorname{div} \left(\left(\frac{1}{2}|u|^2 + p \right) u \right) = e(t)e'(t)\rho_0^2 \leq 0$$

in the sense of distributions. In particular, the strict inequality holds in the interior of $\operatorname{supp}(\rho_0)$, which leads to the total kinetic energy dissipation.

In order to conclude the proof of the theorem, we only need to show that $u \in C_t^0 L^{\frac{27-80\overline{\beta}}{9(1-3\overline{\beta})}}$.

For this purpose, note that we have the identity $u = \lim_{q\to\infty} u_q = u_0 + \sum_{q\geq\bar{n}} \widehat{w}_q$. Using the bounds on $\widehat{w}_q$ provided by (5.21c) and (5.15), we may sum over $0 \leq i \leq i_{\max}(q)$ using the partition of unity property (5.8), to arrive at

$$\|\widehat{w}_q\|_{L^3} \leq C\delta_{q+\bar{n}}^{1/2} r_{q-\bar{n}}^{-1}\Gamma_q \qquad \text{and} \qquad \|\widehat{w}_q\|_{L^\infty} \leq C\Gamma_q^{\mathsf{C}_\infty/2} r_{q-\bar{n}}^{-1}$$

where the constant C depends only on our upper bound for $i_{\max}(q)$, and so only on β and b through (5.9). Using Lebesgue interpolation, the definition (4.11) of C_∞, and the above established bounds, for $p \in [3,\infty)$ we obtain

$$\begin{aligned}\|\widehat{w}_q\|_{L^p} \leq \|\widehat{w}_q\|_{L^3}^{\frac{3}{p}} \|\widehat{w}_q\|_{L^\infty}^{1-\frac{3}{p}} &\leq C(\delta_{q+\bar{n}}\Gamma_q)^{\frac{3}{2p}}(\Gamma_q^{\mathsf{C}_\infty})^{\left(\frac{1}{2}-\frac{3}{2p}\right)} r_{q-\bar{n}}^{-1} \\ &\approx \lambda_{q-\bar{n}}^{\frac{3}{2p}(-2\beta b^{2\bar{n}})+\left(\frac{1}{2}-\frac{3}{2p}\right)\left(\frac{3b^{\bar{n}}(1+4b^{\bar{n}-1})}{1+\cdots+b^{\frac{\bar{n}}{2}-1}}+3b^{\bar{n}}(b-1)^{\frac{1}{2}}\right)+b^{\frac{\bar{n}}{2}}(b^{\frac{\bar{n}}{2}}-1)+O(\varepsilon_\Gamma(b-1))}\end{aligned} \tag{5.42}$$

where the constant $C = C(\beta, b) \geq 1$ and we have used $(b-1)\mathsf{C}_\infty\varepsilon_\Gamma \leq \frac{3(1+4b^{\bar{n}-1})}{1+\cdots b^{\frac{\bar{n}}{2}-1}} + 3(b-1)^{\frac{1}{2}} + O(\varepsilon_\Gamma(b-1))$ from (4.11), (4.2b), and (4.10l). Thus, in order to ensure the absolute summability of $\{\widehat{w}_q\}_{q\geq\bar{n}}$ in L^p, the exponent of $\lambda_{q-\bar{n}}$ appearing on the right side of (5.42) must be strictly negative. After a short computation, we deduce that we must have

$$p < p_*(\beta, b) =: \frac{3\left(2\beta b^{2\bar{n}} + \left(\frac{3b^{\bar{n}}(1+4b^{\bar{n}-1})}{1+\cdots+b^{\frac{\bar{n}}{2}-1}} + 3b^{\bar{n}}(b-1)^{\frac{1}{2}}\right)\right)}{\left(\frac{3b^{\bar{n}}(1+4b^{\bar{n}-1})}{1+\cdots+b^{\frac{\bar{n}}{2}-1}} + 3b^{\bar{n}}(b-1)^{\frac{1}{2}}\right) + b^{\frac{\bar{n}}{2}}(b^{\frac{\bar{n}}{2}}-1)} + O(\bar{n}\varepsilon_\Gamma(b-1))\,. \tag{5.43}$$

Since the second and last term in the denominator is $O((b-1)^{1/2})$, we use $(1+x)^{-1} \geq 1-x$ when $x \geq 0$, $\beta \in (0, 1/3)$, and $b^{\bar{n}} \in (1,2)$ to simplify $p_*(\beta, b)$. Then, it is enough to show that

$$3 < 3 + \frac{\beta}{9(1-3\beta)} < 3 + \frac{6\beta b^{2\bar{n}}(1+\cdots+b^{\frac{\bar{n}}{2}-1})}{3b^{\bar{n}}(1+4b^{\bar{n}-1})} + O((b-1)^{\frac{1}{2}}\bar{n}) - O((b-1)^{\frac{1}{2}}\bar{n}^2)\,.$$

This can be verified using $2/\bar{n} < 1-3\beta$ from the second inequality of (4.1) and $b^{\bar{n}} \in (1,2]$,

and adjusting the choice of $\bar{n} = \bar{n}(\beta)$ and $b = (\bar{n}, \beta)$ if necessary. Since $3 + \frac{\beta}{9(1-3\beta)} = \frac{27-80\beta}{9(1-3\beta)}$ is increasing in β, we get the desired conclusion $u \in C_t^0 L^{\frac{27-80\overline{\beta}}{9(1-3\overline{\beta})}}$. $\square$

Chapter 6

Mollification and upgrading material derivatives

In this section, we introduce suitable mollifications of π_q^k, R_q^k, κ_q^q, and φ_q^q in preparation of later analysis. The following lemma says that the mollified functions satisfy the same estimates essentially as the unmollified ones, ignoring extra Γ_k costs. The difference between the mollified function and the original function, on the other hand, can be made small.

Lemma 6.0.1 (**Mollification and upgrading material derivative estimates).** *Assume that* all *inductive assumptions listed in subsections 5.1-5.5 hold. Let $\mathcal{P}_{q,x,t}$ be a space-time mollifier for which the kernel is a product of $\mathcal{P}_{q,x}(x)$, which is compactly supported in space at scale $\Lambda_q^{-1}\Gamma_{q-1}^{-1/2}$, and $\mathcal{P}_{q,t}(t)$, which is compactly supported in time at scale $\mathrm{T}_{q-1}\Gamma_{q-1}^{1/2}$; we further assume that both kernels have vanishing moments up to $10\mathsf{N}_{\mathrm{fin}}$ and are $C^{10\mathsf{N}_{\mathrm{fin}}}$-differentiable. Define*

$$R_\ell = \mathcal{P}_{q,x,t}R_q^q\,, \qquad \varphi_\ell = \mathcal{P}_{q,x,t}\varphi_q^q\,, \qquad \pi_\ell = \mathcal{P}_{q,x,t}\pi_q^q, \qquad \kappa_\ell = \frac{1}{2}\mathrm{tr}\,\left(R_\ell - \pi_\ell \mathrm{Id}\right)\,. \tag{6.1}$$

on the space-time domain $[-\tau_{q-1}/2, T + \tau_{q-1}/2] \times \mathbb{T}^3$. For q' such that $q < q' \leq q + \bar{n} - 1$, we define $\mathcal{P}_{q',x,t}$ in an analogous way after making the appropriate parameter substitutions,

and we set $R_\ell^{q'} = \mathcal{P}_{q',x,t} R_q^{q'}$ *and* $\pi_\ell^{q'} = \mathcal{P}_{q',x,t}\pi_q^{q'}$*. For* q' *with* $q+\bar{n} \le q' < q+\mathsf{N}_{\mathrm{pr}}$*, we define* $\overline{\mathcal{P}}_{q+\bar{n}-1,x,t}$ *analogously at the spatial scale* $\Lambda_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{-1/2}$ *and temporal scale* $\mathrm{T}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}^{-1/2}$ *and set* $\pi_\ell^{q'} = \overline{\mathcal{P}}_{q+\bar{n}-1,x,t}\pi_q^{q'}$*. Then the following hold.*

(i) The following relaxed equations (replacing (5.2) *and* (5.3)*) are satisfied:*

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q \\ \qquad = \operatorname{div}\left(R_\ell + \sum_{k=q+1}^{q+\bar{n}-1} R_q^k - \left(\pi_\ell + \sum_{k=q+1}^{q+\mathsf{N}_{\mathrm{pr}}-1} \pi_q^k\right)\mathrm{Id}\right) + \operatorname{div}\left(R_q^q - R_\ell + \left(\pi_\ell - \pi_q^q\right)\mathrm{Id}\right) \\ \partial_t\left(\frac{1}{2}|u_q|^2\right) + \operatorname{div}\left(\left(\frac{1}{2}|u_q|^2 + p_q\right)u_q\right) \\ \qquad = (\partial_t + \widehat{u}_q\cdot\nabla)\kappa_q + \operatorname{div}\left(\left(R_\ell + \sum_{k=q+1}^{q+\bar{n}-1} R_q^k - \left(\pi_\ell + \sum_{k=q+1}^{q+\mathsf{N}_{\mathrm{pr}}-1}\pi_q^k\right)\mathrm{Id}\right)\widehat{u}_q\right) \\ \qquad\quad + \operatorname{div}\left(\left(R_q^q - R_\ell + (\pi_\ell - \pi_q^q)\mathrm{Id}\right)\widehat{u}_q\right) + \operatorname{div}\left(\varphi_\ell + \sum_{k=q+1}^{q+\bar{n}-1}\varphi_q^k\right) + \operatorname{div}\left(\varphi_q^q - \varphi_\ell\right) - E(t)\,. \end{cases} \tag{6.2}$$

(ii) The inductive assumptions for π_q^q *in* (5.15) *are replaced with the following upgraded bounds for* π_ℓ *for all* $N+M \le \mathsf{N}_{\mathrm{fin}}$*:*

$$\left\|\psi_{i,q} D^N D_{t,q}^M \pi_\ell\right\|_{3/2} \lesssim \Gamma_q^2 \delta_{q+\bar{n}} \left(\Lambda_q\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right)\,, \tag{6.3a}$$

$$\left\|\psi_{i,q} D^N D_{t,q}^M \pi_\ell\right\|_{\infty} \lesssim \Gamma_q^{2+\mathsf{C}_\infty} \left(\Lambda_q\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right)\,, \tag{6.3b}$$

$$\left|\psi_{i,q} D^N D_{t,q}^M \pi_\ell\right| \lesssim \Gamma_q^3 \pi_\ell \left(\Lambda_q\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right)\,. \tag{6.3c}$$

While we do not replace the inductive bounds in (5.15) *and* (5.16) *for* $k \ne q$*, we do record the following additional bounds for* π_ℓ^k *with* $q < k \le q+\bar{n}-1$ *and* $N+M \le \mathsf{N}_{\mathrm{fin}}$*,*

$$\left\|\psi_{i,k-1}D^N D^M_{t,k-1}\pi^k_\ell\right\|_{3/2} \lesssim \Gamma_k^2\delta_{k+\bar{n}}\left(\Lambda_k\Gamma_{k-1}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+2}\tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}\right) , \tag{6.4a}$$

$$\left\|\psi_{i,k-1}D^N D^M_{t,k-1}\pi^k_\ell\right\|_{\infty} \lesssim \Gamma_k^{2+\mathsf{C}_\infty}\left(\Lambda_k\Gamma_{k-1}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+2}\tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}\right) , \tag{6.4b}$$

$$\left|\psi_{i,k-1}D^N D^M_{t,k-1}\pi^k_\ell\right| \leq 2\Gamma_k^3\pi_\ell^k\left(\Lambda_k\Gamma_k\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+3}\tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}^2\right) . \tag{6.4c}$$

and for π_ℓ^k *with* $q+\bar{n} \leq k < q+\mathsf{N}_{\mathrm{pr}}$ *and* $N+M \leq \mathsf{N}_{\mathrm{fin}}$,

$$\begin{aligned}\left\|\psi_{i,q+\bar{n}-1}D^N D^M_{t,q+\bar{n}-1}\pi^k_\ell\right\|_{3/2} &\lesssim \Gamma_k^2\delta_{k+\bar{n}}\left(\Lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}\right)^N \\ &\quad\times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}-1}^{i+2}\tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right) ,\end{aligned} \tag{6.5a}$$

$$\begin{aligned}\left\|\psi_{i,q+\bar{n}-1}D^N D^M_{t,q+\bar{n}-1}\pi^k_\ell\right\|_{\infty} &\lesssim \Gamma_{q+\bar{n}-1}^{2+\mathsf{C}_\infty}\left(\Lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}\right)^N \\ &\quad\times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}-1}^{i+2}\tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right) ,\end{aligned} \tag{6.5b}$$

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1}D^N D^M_{t,q+\bar{n}-1}\pi^k_\ell\right| &\leq 2\Gamma_k^3\pi_\ell^k\left(\Lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}^2\right)^N \\ &\quad\times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}-1}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^2\right) .\end{aligned} \tag{6.5c}$$

The inductive assumptions (5.19) *and subsection 5.2 remain unchanged. While we do not discard the estimate in* (5.17)*, we however record the additional estimate*

$$\frac{1}{2}\delta_{q+\bar{n}} \leq \pi_\ell \leq 2\pi_q^q \leq 4\pi_\ell\,, \qquad \frac{1}{2}\delta_{k+\bar{n}} \leq \pi_\ell^k \leq 2\pi_q^k \leq 4\pi_\ell^k\,. \tag{6.6}$$

(iii) The inductive assumptions in (5.21a)–(5.21c) *for* $k=q$ *are replaced with the following upgraded bounds for all* $N+M \leq \mathsf{N}_{\mathrm{fin}}$ *in the first two inequalities, and* $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$

in the third:

$$\left|\psi_{i,q} D^N D^M_{t,q} R_\ell\right| \lesssim \Gamma_q^{-7} \pi_\ell \left(\Lambda_q \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\right) \tag{6.7a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \varphi_\ell\right| \lesssim \Gamma_q^{-11} \pi_\ell^{3/2} r_q^{-1} \left(\Lambda_q \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\right) \tag{6.7b}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \widehat{w}_k\right| \lesssim r_{k-\bar{n}}^{-1} \pi_\ell^{1/2} \left(\Lambda_q \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\right) . \tag{6.7c}$$

For k such that $q < k \leq q + \bar{n} - 1$, we have for $N + M \leq \mathsf{N}_{\mathrm{fin}}$ the additional bound

$$\left|\psi_{i,k-1} D^N D^M_{t,k-1} R_\ell^k\right| \lesssim \Gamma_q^{-7} \pi_\ell^k \left(\Lambda_k \Gamma_k\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i+23} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{12}\right) . \tag{6.8}$$

(iv) The symmetric tensor $R_\ell - R_q^q$ and the pressure $\pi_q^q - \pi_\ell$ satisfy

$$\begin{aligned} &\left\|D^N D^M_{t,q}\left(\pi_\ell - \pi_q^q\right)\right\|_\infty + \left\|D^N D^M_{t,q}\left(R_\ell - R_q^q\right)\right\|_\infty \\ &\qquad \lesssim \Gamma_{q+1} \mathrm{T}_{q+1}^{4\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^2 \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \Gamma_q^{-1} \mathrm{T}_q^{-1}\right) \end{aligned} \tag{6.9}$$

for all $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$, while $\varphi_\ell - \varphi_q^q$ satisfies

$$\left\|D^N D^M_{t,q}\left(\varphi_\ell - \varphi_q^q\right)\right\|_\infty \leq \delta_{q+3\bar{n}}^{3/2} \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \Gamma_q^{-1} \mathrm{T}_q^{-1}\right) \tag{6.10}$$

for all $N + M \leq \mathsf{N}_{\mathrm{ind}}/4$. For k such that $q < k \leq q + \bar{n} - 1$ and $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$, we have that

$$\begin{aligned} &\left\|D^N D^M_{t,k-1}\left(\pi_q^k - \pi_\ell^k\right)\right\|_\infty + \left\|D^N D^M_{t,k-1}\left(R_q^k - R_\ell^k\right)\right\|_\infty \\ &\qquad \lesssim \Gamma_{k+1} \mathrm{T}_{k+1}^{4\mathsf{N}_{\mathrm{ind,t}}} \delta_{k+3\bar{n}}^2 (\Lambda_k \Gamma_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{k-1}^{-1} \Gamma_{k-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{11}\right) . \end{aligned} \tag{6.11}$$

and for k with $q+\bar{n} \leq k < q+\mathsf{N}_{\mathrm{pr}}$ and $N+M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\left\|D^N D^M_{t,q+\bar{n}-1}\left(\pi^k_q - \pi^k_\ell\right)\right\|_\infty \lesssim \Gamma_{q+\bar{n}+1}\mathrm{T}^{4\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}+1}\delta^2_{q+4\bar{n}}(\Lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^N \\ \times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}\right) . \quad (6.12)$$

Proof. We first note that (6.2) is immediate from (5.2)–(5.3) and the definitions in (6.1). At this point, we split the proof into steps, in which we first carry out the mollifications, and then upgrade the material derivatives.

Step 1: Mollifying the pressure π^k_q. We first consider the case $k=q$ and apply the abstract mollification Proposition A.6.1 with the following choices:

$$p = 3/2, \infty\,, \quad N_{\mathrm{g}}, N_{\mathrm{c}} \text{ as in (xiii)}\,, \quad M_t = \mathsf{N}_{\mathrm{ind,t}}\,, \quad N_* = 2\mathsf{N}_{\mathrm{ind}}\,,$$
$$N_\gamma = \mathsf{N}_{\mathrm{fin}}\,, \quad \Omega = \operatorname{supp}\psi_{i,q-1}\,, \quad v = \widehat{u}_{q-1}\,, \quad i = i\,,$$
$$\lambda = \Lambda_{q-1}\,, \quad \Lambda = \Lambda_q\Gamma_{q-1}\,, \quad \Gamma = \Gamma_{q-1}\,, \quad \tau = \tau_{q-1}\Gamma_{q-1}\,, \quad \mathrm{T} = \mathrm{T}_{q-1},$$
$$f = \pi^q_q, \quad \mathcal{C}_{f,3/2} = \Gamma^2_q\delta_{q+\bar{n}}, \quad \mathcal{C}_{f,\infty} = \widetilde{\mathcal{C}}_f = \Gamma^{\mathsf{C}_\infty+2}_q \quad \mathcal{C}_v = \Lambda^{1/2}_{q-1}\,.$$

First, we have that the assumptions on the parameters in (A.225a) are satisfied by (4.19c), (4.20a),(4.24a), (4.15) and (5.10). The assumptions in (A.225b) are satisfied from (4.19b), and the assumptions in (A.226) are satisfied from (5.35b). Next, the assumptions in (A.227a) are satisfied from (5.15) (where we apply the bound with $\psi_{i\pm,q-1}$ in order to obtain a bound for $L^p(\operatorname{supp}\psi_{i,q-1})$). Finally, in order to verify (A.227b), we apply Remark A.2.6 with the following choices. We set $p=\infty$, $N_x = N_t = \infty$, $N_* = 2\mathsf{N}_{\mathrm{ind}}$, $\Omega = \mathbb{T}^3\times\mathbb{R}$, $v = -w = \widehat{u}_{q-1}$, $\mathcal{C}_w = \Gamma^{i_{\max}+2}_{q-1}\delta^{1/2}_{q-1}\lambda^2_{q-1}$, $\lambda_w = \widetilde{\lambda}_w = \Lambda_{q-1}$, $\mu_w = \widetilde{\mu}_w = \Gamma^{-1}_{q-1}\mathrm{T}^{-1}_{q-1}$ in (A.34), while in (A.27) and (A.28) we set $v = \widehat{u}_{q-1}$, $\mathcal{C}_v = \mathcal{C}_w$, $\lambda_v = \widetilde{\lambda}_v = \Lambda_{q-1}$, $\mu_v = \widetilde{\mu}_v = \Gamma^{-1}_{q-1}\mathrm{T}^{-1}_{q-1}$, $f = \pi^q_q$, $\mathcal{C}_f = \Gamma^{\mathsf{C}_\infty+2}_q$, $\lambda_f = \widetilde{\lambda}_f = \Lambda_q$, $\mu_f = \widetilde{\mu}_f = \mathrm{T}^{-1}_{q-1}$. Then (A.27) and (A.28) are satisfied from (5.34) at level $q-1$, (5.15), (5.10), and (4.15). Next, (A.34) is satisfied from (5.35a) at level $q-1$. Thus

from (A.35) and (4.15), we obtain that

$$\left\|D^N \partial_t^M \pi_q^q\right\|_\infty \lesssim \Gamma_q^{\mathsf{C}_\infty+2} \Lambda_q^N \mathrm{T}_{q-1}^{-M} \tag{6.13}$$

for $N + M \leq 2\mathsf{N}_{\text{ind}}$, thus verifying the final assumption (A.227b) from Lemma A.6.1.

We first apply (A.228) to conclude that for $N + M \leq \mathsf{N}_{\text{fin}}$,

$$\left\|\psi_{i,q-1} D^N D_{t,q-1}^M \pi_\ell\right\|_{3/2} \lesssim \Gamma_q^2 \delta_{q+\bar{n}} \left(\Lambda_q \Gamma_{q-1}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^{i+2} \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}\right) \tag{6.14a}$$

$$\left\|\psi_{i,q-1} D^N D_{t,q-1}^M \pi_\ell\right\|_\infty \lesssim \Gamma_q^{\mathsf{C}_\infty+2} \left(\Lambda_q \Gamma_{q-1}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^{i+2} \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}\right) . \tag{6.14b}$$

Next, we have from (A.229) and (4.19a) that the difference $\pi_q^q - \pi_\ell$ satisfies

$$\left\|D^N D_{t,q-1}^M \left(\pi_q^q - \pi_\ell\right)\right\|_\infty \lesssim \Gamma_{q+1} \mathrm{T}_{q+1}^{4\mathsf{N}_{\text{ind,t}}} \delta_{q+3\bar{n}}^2 (\Lambda_q \Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q-1}^{-1} \Gamma_{q-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}\right) \tag{6.15}$$

for $N + M \leq 2\mathsf{N}_{\text{ind}}$. Note also that since we have a lower bound on π_q^q given by (5.17), the above estimate implies that (after a sufficiently large choice of λ_0 so that the implicit constant is absorbed)

$$\pi_\ell \geq \pi_q^q - \delta_{q+2\bar{n}} \geq \frac{1}{2} \delta_{q+\bar{n}} ,$$

which is the first inequality for π_ℓ and π_q^q in (6.6). The other two inequalities there follow similarly. Finally, we note that by (5.15c) and (6.6),

$$\begin{aligned}
\left|\psi_{i,q-1} D^N D_{t,q-1}^M \pi_\ell\right| &\leq \left|\psi_{i,q-1} D^N D_{t,q-1}^M \pi_q^q\right| + \left|D^N D_{t,q-1}^M \left(\pi_q^q - \pi_\ell\right)\right| \\
&\leq \Gamma_q^2 \pi_q^q \Lambda_q^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^i \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\right) \\
&\qquad + \delta_{q+3\bar{n}}^2 (\Lambda_q \Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}\right) \\
&\leq \Gamma_q^3 \pi_\ell (\Lambda_q \Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^i \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}\right)
\end{aligned}$$

for $N+M \leq 2\mathsf{N}_{\text{ind}}$. For $2\mathsf{N}_{\text{ind}} < N+M \leq \mathsf{N}_{\text{fin}}$, we have from (6.14b) and (4.20b) that

$$\left|D^N D^M_{t,q-1}\pi_\ell\right| \leq \delta^2_{q+\bar{n}}(\Lambda_q \Gamma^{1/2}_{q-1}\Gamma^{1/2}_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{q-1}\tau^{-1}_{q-1}, \mathrm{T}^{-1}_{q-1}\Gamma^2_{q-1}\right) .$$

In the case $k \neq q$, we may obtain the bounds (6.4a), (6.4b), (6.5a), (6.5b), and the second inequality of (6.6), via an argument identical to the proof of (6.3) and the first inequality of (6.6). We additionally have the pointwise bound for $q+1 \leq k \leq q+\bar{n}-1$ and $N+M \leq \mathsf{N}_{\text{fin}}$

$$\begin{aligned}\left|\psi_{i,k-1} D^N D^M_{t,k-1}\pi^k_\ell\right| &\leq (\Gamma^3_k \pi^k_q + \delta^2_{k+\bar{n}})(\Lambda_k \Gamma^{1/2}_{k-1}\Gamma^{1/2}_k)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{k-1}\tau^{-1}_{k-1}, \mathrm{T}^{-1}_{k-1}\Gamma^2_{k-1}\right) \\ &\leq 2\Gamma^3_k \pi^k_\ell (\Lambda_k \Gamma^{1/2}_{k-1}\Gamma^{1/2}_k)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{k-1}\tau^{-1}_{k-1}, \mathrm{T}^{-1}_{k-1}\Gamma^2_{k-1}\right) , \end{aligned} \tag{6.16}$$

and for $q+\bar{n} \leq k < q+\mathsf{N}_{\text{pr}}$ and $N+M \leq \mathsf{N}_{\text{fin}}$

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1}\pi^k_\ell\right| &\leq (\Gamma^3_k \pi^k_q + \delta^2_{k+\bar{n}})(\Lambda_{q+\bar{n}-1} \Gamma^2_{q+\bar{n}-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}-1}\right) \\ &\leq 2\Gamma^3_k \pi^k_\ell (\Lambda_{q+\bar{n}-1} \Gamma^2_{q+\bar{n}-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}-1}\right) , \end{aligned} \tag{6.17}$$

which again follows from a similar argument as in the proof of the corresponding bounds for $q = k$ and (6.6). Furthermore, we have that the difference $\pi^k_q - \pi^k_\ell$ satisfies (6.11) and (6.12), which follows directly from the mollification lemma and (4.19a) with q replaced by $k-1$ or $q+\bar{n}$, as in the case $k = q$. Finally, the bounds in (6.6) for π^m_ℓ follow similarly as before. At this point we have completed the proofs of the required estimates in (6.4)–(6.6) and (6.11)–(6.12) for π^k_ℓ.

Step 2: Mollifying the stress and current errors. We apply the abstract mollification Proposition A.6.1 with the same choices as before, except for the stress error we choose

$$f = R^k_q , \quad q \leq k \leq q+\bar{n}-1 , \quad p = \infty , \quad \mathcal{C}_{f,\infty} = \Gamma^{\mathsf{C}_\infty+2}_k , \quad \tau = \tau_{k-1} , \quad c = 20 , \quad \mathrm{T} = \mathrm{T}_{k-1}\Gamma^{-10}_q .$$

We then have that (A.225a)–(A.225b) are satisfied as in the previous step, as is (A.226). In order to verify (A.227a), we appeal to (5.21a) and (5.15b). In order to verify (A.227b), we use Remark A.2.6 exactly as in the previous step, but with R_q^k replacing π_q^k. Thus from (A.228)–(A.229) and (4.19a), we have that for $q \leq k \leq q + \bar{n} - 1$ (we denote R_ℓ by R_ℓ^q for concision here)

$$\left|\psi_{i,k-1} D^N D_{t,k-1} R_\ell^k\right| \lesssim \Gamma_k^{\mathsf{C}_\infty+2} (\Lambda_k \Gamma_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+22} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{11}\right) \tag{6.18a}$$

$$\left|D^N D_{t,k-1}^M \left(R_\ell^k - R_q^k\right)\right| \lesssim \Gamma_{k+1} \mathrm{T}_{k+1}^{4\mathsf{N}_{\text{ind,t}}} \delta_{k+3\bar{n}}^2 (\Lambda_k \Gamma_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{11}\right) , \tag{6.18b}$$

where the first bound holds for $N+M \leq \mathsf{N}_{\text{fin}}$, and the second bound holds for $N+M \leq 2\mathsf{N}_{\text{ind}}$. The second bound verifies (6.11) for the difference $R_q^k - R_\ell^k$. Appealing to (5.21a), (6.18b), and (6.6), we then may write that in the case $k = q$,

$$\begin{aligned}
\left|\psi_{i,q-1} D^N D_{t,q-1}^M R_\ell\right| &\leq \left|\psi_{i,q-1} D^N D_{t,q-1}^M R_q^q\right| + \left|D^N D_{t,q-1}^M \left(R_q^q - R_\ell\right)\right| \\
&\leq \Gamma_q^{-7} \pi_q^q \Lambda_q^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^{i+20} \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_q^{11}\right) \\
&\qquad + \delta_{q+3\bar{n}}^2 (\Lambda_q \Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}^{11}\right) \\
&\lesssim \Gamma_q^{-7} \pi_\ell (\Lambda_q \Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^{i} \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}^{11}\right)
\end{aligned}$$

for $N + M \leq 2\mathsf{N}_{\text{ind}}$. For $2\mathsf{N}_{\text{ind}} < N + M \leq \mathsf{N}_{\text{fin}}$, we have from (6.18a) and (4.20b) that

$$\left|D^N D_{t,q-1}^M R_\ell\right| \leq \delta_{q+\bar{n}}^2 (\Lambda_q \Gamma_{q-1}^{1/2} \Gamma_q^{1/2})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^{i+23} \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1} \Gamma_{q-1}^{12}\right) .$$

In the case $q \neq k$, we have that for $N + M \leq \mathsf{N}_{\text{fin}}$,

$$\left|\psi_{i,k-1} D^N D_{t,k-1}^M R_\ell^k\right| \lesssim (\Gamma_k^{-7} \pi_\ell^k + \delta_{k+\bar{n}}^2)(\Lambda_k \Gamma_{k-1}^{1/2} \Gamma_k^{1/2})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+23} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{12}\right) ,$$

giving the desired bound in (6.8) after using (5.20a) again.

In the case of the current error, we again apply Proposition A.6.1 with the same choices as in the first portion of this step, except we choose

$$f = \varphi_q^q\,, \quad \mathcal{C}_{f,\infty} = \Gamma_q^{\frac{3\mathsf{C}_\infty}{2}+3} r_q^{-1} \quad c = 20\,, \quad \mathrm{T} = \mathrm{T}_{q-1}\Gamma_q^{10}\,, \quad N_* = {}^{\mathsf{N}_{\mathrm{ind}}}/_4.$$

We then have that (A.225a)–(A.225b) are satisfied exactly as in the previous step, as is (A.226). In order to verify (A.227a), we appeal to (5.21b) and (5.15b). In order to verify (A.227b), we use Remark A.2.6 exactly as in the first part of this step, but with φ_q^q replacing R_q^q. We conclude that (A.227b) is satisfied with $\widetilde{\mathcal{C}}_f = \mathcal{C}_{f,\infty}$. Thus from (A.228)–(A.229), we have that

$$\left|\psi_{i,q-1} D^N D_{t,q-1}\varphi_\ell\right| \lesssim \Gamma_q^{\frac{3\mathsf{C}_\infty}{2}+3} r_q^{-1} (\Lambda_q \Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q-1}^{i+22}\tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^{11}\right) \tag{6.19a}$$

$$\left|D^N D_{t,q-1}\left(\varphi_\ell - \varphi_q^q\right)\right| \lesssim \Gamma_{q+1}\mathrm{T}_{q+1}^{4\mathsf{N}_{\mathrm{ind,t}}}\delta_{q+3\bar{n}}^2 (\Lambda_q\Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^{11}\right)\,, \tag{6.19b}$$

where the first bound holds for $N+M \le \mathsf{N}_{\mathrm{fin}}$, and the second bound holds for $N+M \le {}^{\mathsf{N}_{\mathrm{ind}}}/_4$. Appealing to (5.21b), (6.19b), and (6.6), we then may write that

$$\begin{aligned}
\left|\psi_{i,q-1} D^N D_{t,q-1}^M \varphi_\ell\right| &\le \left|\psi_{i,q-1} D^N D_{t,q-1}^M \varphi_q^q\right| + \left|D^N D_{t,q-1}^M\left(\varphi_q^q - \varphi_\ell\right)\right| \\
&\le \Gamma_q^{-11}(\pi_q^q)^{3/2} r_q^{-1}\Lambda_q^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q-1}^{i+20}\tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_q^{10}\right) \\
&\qquad + \delta_{q+2\bar{n}}^2 (\Lambda_q\Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^{11}\right) \\
&\lesssim \Gamma_q^{-11}\pi_\ell^{3/2} r_q^{-1}(\Lambda_q\Gamma_{q-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q-1}^{i+20}\tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^{11}\right)
\end{aligned}$$

for $N + M \le {}^{\mathsf{N}_{\mathrm{ind}}}/_4$. For ${}^{\mathsf{N}_{\mathrm{ind}}}/_4 < N + M \le \mathsf{N}_{\mathrm{fin}}$, we have from (6.19a) and (4.20b) that

$$\left|D^N D_{t,q-1}^M \varphi_\ell\right| \le \delta_{q+\bar{n}}^2 (\Lambda_q \Gamma_{q-1}^{1/2}\Gamma_q^{1/2})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q-1}^{i+23}\tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^{12}\right)\,.$$

Step 3: Upgrading material derivatives for $k = q$. We begin with the pointwise bounds

for π_ℓ. Combining the bounds from Step 1 with (5.14) with $q' = q$ and $q'' = q-1$, we have that for $N + M \leq \mathsf{N}_{\text{fin}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q-1} \pi_\ell\right| \leq 2\Gamma_q^3 \pi_\ell \left(\Lambda_q \Gamma_{q-1}^{1/2} \Gamma_q^{1/2}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i-2}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^2\right) . \tag{6.20}$$

We shall apply Remark A.2.6 (with the adjustment in Remark A.2.4 for derivative bounds) with the following choices, at a point $(t,x) \in \operatorname{int}(\operatorname{supp}\psi_{i,q})$ for which the neighborhood $\Omega_{t,x} \subset \operatorname{supp}\psi_{i,q}$:

(A.34) choices: $p = \infty\,, \quad N_x = \infty\,, \quad N_t = \mathsf{N}_{\text{ind,t}}\,, \quad N_* = \mathsf{N}_{\text{fin}}\,, \quad w = \widehat{w}_q\,,$

$\Omega = \Omega_{t,x}\,, \quad v = \widehat{u}_{q-1}\,, \quad \mathcal{C}_w = \Gamma_q^{i+2}\delta_q^{1/2} r_{q-\bar{n}}^{-1/3}\,,$

$\lambda_w = \widetilde{\lambda}_w = \Lambda_q\,, \quad \mu_w = \Gamma_{q-1}^{i+3}\tau_{q-1}^{-1}\,, \quad \widetilde{\mu}_w = \Gamma_q^{-1}\mathrm{T}_q^{-1}\,,$

(A.27) choices: $\mathcal{C}_v = \Gamma_q^{i+2}\delta_q^{1/2} r_{q-\bar{n}}^{-1/3}\,, \quad \lambda_v = \widetilde{\lambda}_v = \Lambda_q\,, \quad \mu_v = \Gamma_q^i \tau_q^{-1}\,, \quad \widetilde{\mu}_v = \mathrm{T}_q^{-1}\Gamma_q^{-1}\,, \quad \Omega = \Omega_{t,x}\,,$

(A.28) choices: $f = \pi_\ell\,, \quad \mathcal{C}_f = \sup_{\Omega_{t,x}} \pi_\ell\,, \quad \lambda_f = \widetilde{\lambda}_f = \Lambda_q(\Gamma_{q-1}\Gamma_q)^{1/2}\,, \quad \mu_f = \mu_v\,, \quad \widetilde{\mu}_f = \widetilde{\mu}_v\,, \quad \Omega = \Omega_{t,x}\,.$

Then we have that (A.34) holds from (5.32) at level q, (A.27) holds from (5.34) at level q, and (A.28) holds from (6.20). Taking $\Omega_{t,x}$ to be arbitrary and using the continuity of π_ℓ, we thus have from (A.35) that for $N + M \leq \mathsf{N}_{\text{fin}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \pi_\ell\right| \lesssim \Gamma_q^3 \pi_\ell \left(\Lambda_q(\Gamma_{q-1}\Gamma_q)^{1/2}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^i, \mathrm{T}_q^{-1}\Gamma_q^{-1}\right) ,$$

matching (6.3c). In order to obtain (6.3a) and (6.3b), we use the $L^{3/2}$ and L^∞ bounds on π_ℓ shown in (6.3). Combined with Step 1, this concludes the proof of (ii).

In order to prove (6.7a), we argue in a manner very similar to the proof of (6.3c) carried out just previously. The only difference is that from Step 2, we have the bound

$$\left|D^N D_{t,q-1} R_\ell\right| \lesssim \Gamma_q^{-7} \pi_\ell \left(\Lambda_q(\Gamma_{q-1}\Gamma_q)^{1/2}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q-1}^{i+23}\tau_{q-1}^{-1}, \mathrm{T}_{q-1}^{-1}\Gamma_{q-1}^{12}\right) . \tag{6.21}$$

Carrying out the same steps with the obvious modifications, we deduce that (6.7a) holds as desired. The proof of (6.7b) is again quite similar, and we omit the details. To conclude the proof of (iii), we must show (6.7c). Following the exact same steps as before but beginning instead with the bound (5.21c) and appealing to (6.6), we obtain the desired estimate, concluding the proof of item (iii).

Finally, we must upgrade the material derivatives to $D_{t,q}$ on the differences in order to conclude the proofs of (6.9)–(6.10) from item (iv). Arguing in a similar fashion as in the first part of this step but applying Remark A.2.6 to the differences, choosing $\mathcal{C}_w = \mu_w = \widetilde{\mu}_w = \mathcal{C}_v = \mu_v = \widetilde{\mu}_v = \mathrm{T}_{q+1}^{-1}$ and using the extra prefactors from $\mathrm{T}_{q+1}^{4\mathsf{N}_{\mathrm{ind,t}}}$ to absorb the lossy material derivative cost yields the desired estimates in (6.9)–(6.10).

□

Chapter 7

Intermittent Mikado bundles and synthetic Littlewood-Paley decompositions

7.1 Definition of intermittent Mikado flows and basic properties

We shall require the following lemmas regarding decompositions of symmetric positive definite tensor fields. Typically such lemmas are stated and applied for tensors in a neighborhood of the identity. Since it will be convenient for us to decompose tensors for which some rescaling of the original tensors belongs to a neighborhood of the identity, and later estimates (see Lemma 9.3.1) will depend on the rescaling factor, we include a slightly altered statement with full proof.

Proposition 7.1.1 (Geometric lemma I). *Let $\Xi \subset \mathbb{Q}^3 \cap \mathbb{S}^2$ denote the set $\{{}^3\!/_5 e_i \pm {}^4\!/_5 e_j\}_{1\leq i<j\leq 3}$, and for every ξ in Ξ. Then there exists $\epsilon > 0$ such that every symmetric 2-tensor in $B(\mathrm{Id}, \epsilon)$ can be written as a unique, positive linear combination of $\xi \otimes \xi$ for $\xi \in \Xi$. Furthermore, for*

a given large number $K > 1$, let C_K denote the set

$$C_K := \bigcup_{1 \le k \le K} B(k\mathrm{Id}, k\epsilon)\,, \tag{7.1}$$

which we note is contained in the set of positive definite, symmetric 2-tensors for ϵ sufficiently small. Then there exist functions $\gamma_{\xi,K}$ for $\xi \in \Xi$ such that every element $R \in C_K$ can also be written as a unique, positive linear combination

$$R = \sum_{\xi \in \Xi} \left(\gamma_{\xi,K}(R)\right)^2 \xi \otimes \xi\,. \tag{7.2}$$

Additionally, we have that for all $1 \le N \le 3\mathsf{N}_{\mathrm{fin}}$,

$$1 \lesssim |\gamma_{\xi,K}| \lesssim K^{1/2}, \quad \left|D^N \gamma_{\xi,K}\right| \lesssim 1\,, \tag{7.3}$$

where the implicit constants above depend on Ξ and $\mathsf{N}_{\mathrm{fin}}$ but not K.

Proof. By direct computation, we have that the identity matrix can be written as a strictly positive linear combination of $\xi \otimes \xi$ for $\xi \in \Xi$, and that the set of simple tensors $\{\xi \otimes \xi\}_{\xi \in \Xi}$ is linearly independent in the set of symmetric matrices. Therefore, there exists $\epsilon < 1$ and linear functions $(\gamma_\xi)^2$ for $\xi \in \Xi$ such that for all $R \in B(\mathrm{Id}, \epsilon)$,

$$R = \sum_{\xi \in \Xi} \gamma_\xi^2(R) \xi \otimes \xi\,,$$

and there exist implicit constants depending only on Ξ such that

$$1 \lesssim \gamma_\xi(R) \lesssim 1\,, \quad \left|D[\gamma_\xi^2(R)]\right| \lesssim 1\,, \qquad D^N[\gamma_\xi^2(R)] \equiv 0 \quad \forall N \ge 2\,. \tag{7.4}$$

Now let K be given, and let $1 \leq k \leq K$ and R be such that ${}^{R}/_{k} \in B(\mathrm{Id}, \epsilon)$. We define[1]

$$\gamma_{\xi,K}^2(R) = k\gamma_\xi^2\left(\frac{R}{k}\right), \tag{7.5}$$

so that

$$\sum_{\xi\in\Xi} \gamma_{\xi,K}^2(R)\xi\otimes\xi = \sum_{\xi\in\Xi} \gamma_\xi^2\left(\frac{R}{k}\right) k\xi\otimes\xi = R\,,$$

and (7.2) is satisfied. We then have that

$$1 \lesssim \gamma_{\xi,K}(R) \lesssim K^{1/2}\,, \quad \left|D[\gamma_{\xi,K}^2(R)]\right| \lesssim 1\,, \qquad D^N[\gamma_{\xi,K}^2(R)] \equiv 0 \quad \forall N \geq 2\,,$$

where the implicit constants are those from (7.4) and depend only on Ξ. We immediately deduce from the lower bound for $\gamma_{\xi,K}(R)$ that

$$|D\gamma_{\xi,K}(R)| \leq \frac{\left|D[\gamma_{\xi,K}^2(R)]\right|}{|\gamma_{\xi,K}(R)|} \lesssim 1\,.$$

Now for $N \geq 1$, we may write that

$$2\gamma_{\xi,K}(R)D^{N+1}\gamma_{\xi,K}(R) = D^{N+1}\left(\gamma_{\xi,K}^2(R)\right) + \sum_{0<N'<N+1} c_{N,N'}D^{N'}\left(\gamma_{\xi,K}(R)\right)D^{N+1-N'}\left(\gamma_{\xi,K}(R)\right)\,.$$

Assuming by induction that $|D^{N''}\gamma_{\xi,K}(R)| \lesssim 1$ for $1 \leq N'' \leq N$, we use the lower bound for $\gamma_{\xi,K}(R)$ to divide both sides by $\gamma_{\xi,K}(R)$ and deduce that $|D^{N+1}\gamma_{\xi,K}(R)| \lesssim 1$, concluding the proof of (7.3). □

The following lemma appears in [17].

Proposition 7.1.2 (Geometric lemma II). *Let $\{\xi_1, \xi_2, \xi_3, \xi_4\} \subset \mathbb{Z}^3$ be a set of nonzero*

[1] This is well-defined since $(\gamma_\xi)^2$ is linear, and so the choice of K is irrelevant.

vectors satisfying

$$\{\xi_1, \xi_2, \xi_3\} \text{ is an orthogonal basis of } \mathbb{R}^3 \text{ and } \xi_4 := -(\xi_1 + \xi_2 + \xi_3).$$

Fix $C_0 > 0$ and let $B_{C_0} := \{\phi \in \mathbb{R}^3 : |\phi| \leq C_0\}$. Then, there exist positive functions $\{\widetilde{\gamma}_{\xi_i}\}_{i=1}^4 \subset C^\infty(B_{C_0})$ such that for each $\phi \in B_{C_0}$, we have

$$\phi = \frac{1}{2}\sum_{i=1}^{4}(\widetilde{\gamma}_{\xi_i}(\phi))^3\xi_i\,.$$

In particular, the set $\{e_1, 2e_2, 2e_3, -(e_1+2e_2+2e_3)\}$ satisfies the assumption. We denote the set of their normalized vectors by $\Xi' := \{e_1, e_2, e_3, -1/3(e_1 + 2e_2 + 2e_3)\} \subset \mathbb{Q}^3 \cap \mathbb{S}^2$, and with slight abuse of the notation we redefine $\widetilde{\gamma}_\xi$ to have

$$2\phi = \sum_{\xi\in\Xi'}(\widetilde{\gamma}_\xi(\phi))^3\xi\,. \tag{7.6}$$

Definition 7.1.3. *For any $\xi \in \Xi \cup \Xi'$, we choose $\xi', \xi'' \in \mathbb{Q}^3 \cap \mathbb{S}^2$ such that $\{\xi, \xi', \xi''\}$ is an orthonormal basis of $\mathbb{R}^3$. We then denote by n_* the least positive integer such that $n_*\xi, n_*\xi' n_*\xi'' \in \mathbb{Z}^3$ for all $\xi \in \Xi \cup \Xi'$.*

We now recall [7, Proposition 4.3], which details the choices for shifts enjoyed by a function with sparse support. In our setting, such functions will be pipe densities, or equivalently the densities associated to their potentials.

Proposition 7.1.4 (Rotating, Shifting, and Periodizing). *Fix $\xi \in \Xi$ (or $\in \Xi'$), where Ξ is as in Proposition 7.1.1 (or as in Proposition 7.1.2). Let $r^{-1}, \lambda \in \mathbb{N}$ be given such that $\lambda r \in \mathbb{N}$. Let $\varkappa : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with support contained inside a ball of radius $1/4$. Then for $k \in \{0, ..., r^{-1} - 1\}^2$, there exist functions $\varkappa^k_{\lambda,r,\xi} : \mathbb{R}^3 \to \mathbb{R}$ defined in terms of $\varkappa$, satisfying the following additional properties:*

(1) We have that $\varkappa^k_{\lambda,r,\xi}$ is simultaneously $\left(\frac{\mathbb{T}^3}{\lambda r}\right)$-periodic and $\left(\frac{\mathbb{T}^3_\xi}{\lambda r n_*}\right)$-periodic. Here, by $\mathbb{T}^3_\xi$

we refer to a rotation of the standard torus such that $\mathbb{T}^3_\xi$ has a face perpendicular to ξ.

(2) *Let F_ξ be one of the two faces of the cube $\frac{\mathbb{T}^3_\xi}{\lambda r n_*}$ which is perpendicular to ξ. Let $\mathbb{G}_{\lambda,r} \subset F_\xi \cap 2\pi\mathbb{Q}^3$ be the grid consisting of r^{-2}-many points spaced evenly at distance $2\pi(\lambda n_*)^{-1}$ on F_ξ and containing the origin. Then each grid point g_k for $k \in \{0, ..., r^{-1}-1\}^2$ satisfies*

$$\left(\operatorname{supp} \varkappa^k_{\lambda,r,\xi} \cap F_\xi\right) \subset \left\{x : |x - g_k| \leq 2\pi \left(4\lambda n_*\right)^{-1}\right\}. \tag{7.7}$$

(3) *The support of $\varkappa^k_{\lambda,r,\xi}$ is a pipe (cylinder) centered around a $\left(\frac{\mathbb{T}^3}{\lambda r}\right)$-periodic and $\left(\frac{\mathbb{T}^3_\xi}{\lambda r n_*}\right)$-periodic line parallel to ξ, which passes through the point g_k. The radius of the cylinder's cross-section is as in* (7.7).

(4) *We have that $\xi \cdot \nabla \varkappa^k_{\lambda,r,\xi} = 0$.*

(5) *For $k \neq k'$, $\operatorname{supp} \varkappa^k_{\lambda,r,\xi} \cap \operatorname{supp} \varkappa^{k'}_{\lambda,r,\xi} = \emptyset$.*

We now state a slightly modified version of [7, Proposition 4.4] or equivalently [35, Proposition 3.3], which rigorously constructs the L^2-normalized intermittent pipe flows and enumerates the necessary properties.

Proposition 7.1.5 (Intermittent pipe flows for Reynolds corrector). *Fix a vector ξ belonging to the set of rational vectors $\Xi \subset \mathbb{Q}^3 \cap \mathbb{S}^2$ from Proposition 7.1.1, $r^{-1}, \lambda \in \mathbb{N}$ with $\lambda r \in \mathbb{N}$, and large integers N_{fin} and D. There exist vector fields $\mathcal{W}^k_{\xi,\lambda,r} : \mathbb{T}^3 \to \mathbb{R}^3$ for $k \in \{0, ..., r^{-1} - 1\}^2$ and implicit constants depending on N_{fin} and D but not on λ or r such that:*

(1) *There exists $\varrho : \mathbb{R}^2 \to \mathbb{R}$ given by the iterated divergence $\operatorname{div}^{\mathsf{D}} \vartheta =: \varrho$ of a pairwise symmetric tensor potential $\vartheta : \mathbb{R}^2 \to \mathbb{R}^{2^{\mathsf{D}}}$ with compact support in a ball of radius $\frac{1}{4}$ such that the following holds. Let $\varrho^k_{\xi,\lambda,r}$ and $\vartheta^k_{\xi,\lambda,r}$ be defined as in Proposition 7.1.4, in terms of ϱ and ϑ (instead of $\varkappa$). Then there exists $\mathcal{U}^k_{\xi,\lambda,r} : \mathbb{T}^3 \to \mathbb{R}^3$ such that if*

$\{\xi, \xi', \xi''\} \subset \mathbb{Q}^3 \cap \mathbb{S}^2$ *form an orthonormal basis of* $\mathbb{R}^3$ *with* $\xi \times \xi' = \xi''$*, then we have*[2]

$$\mathcal{U}^k_{\xi,\lambda,r} = -\frac{1}{3}\xi' \underbrace{\lambda^{-\mathsf{D}}\xi'' \cdot \nabla \left(\mathrm{div}^{\mathsf{D}-2}\left(\vartheta^k_{\xi,\lambda,r}\right)\right)^{ii}}_{=:\varphi''^k_{\xi,\lambda,r}} + \frac{1}{3}\xi'' \underbrace{\lambda^{-\mathsf{D}}\xi' \cdot \nabla \left(\mathrm{div}^{\mathsf{D}-2}\left(\vartheta^k_{\xi,\lambda,r}\right)\right)^{ii}}_{=:\varphi'^k_{\xi,\lambda,r}}, \tag{7.8}$$

and thus

$$\mathrm{curl}\,\mathcal{U}^k_{\xi,\lambda,r} = \xi\lambda^{-\mathsf{D}}\mathrm{div}^{\mathsf{D}}\left(\vartheta^k_{\xi,\lambda,r}\right) = \xi\varrho^k_{\xi,\lambda,r} =: \mathcal{W}^k_{\xi,\lambda,r}\,, \tag{7.9}$$

and

$$\xi \cdot \nabla\vartheta_{\xi,\lambda,r} = (\xi\cdot\nabla)\mathcal{W}^k_{\xi,\lambda,r} = (\xi\cdot\nabla)\mathcal{U}^k_{\xi,\lambda,r} = 0\,. \tag{7.10}$$

(2) The sets of functions $\{\mathcal{U}^k_{\xi,\lambda,r}\}_k$*,* $\{\varrho^k_{\xi,\lambda,r}\}_k$*,* $\{\vartheta^k_{\xi,\lambda,r}\}_k$*, and* $\{\mathcal{W}^k_{\xi,\lambda,r}\}_k$ *satisfy items 1–5 in Proposition 7.1.4.*

(3) $\mathcal{W}^k_{\xi,\lambda,r}$ *is a stationary, pressureless solution to the Euler equations.*

(4) $\fint_{\mathbb{T}^3} \mathcal{W}^k_{\xi,\lambda,r} \otimes \mathcal{W}^k_{\xi,\lambda,r} = \xi \otimes \xi$*.*

(5) $\fint_{\mathbb{T}^3} |\mathcal{W}^k_{\xi,\lambda,r}|^2\mathcal{W}^k_{\xi,\lambda,r} = \fint_{\mathbb{T}^3} (\varrho^k_{\xi,\lambda,r})^2\mathcal{U}^k_{\xi,\lambda,r} = \int_{\mathbb{T}^3} \varrho^k_{\xi,\lambda,r}\mathcal{U}^k_{\xi,\lambda,r} = 0\,.$

(6) For all $n \leq 3\mathsf{N}_{\mathrm{fin}}$*,*

$$\left\|\nabla^n\vartheta^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^n r^{\left(\frac{2}{p}-1\right)}, \qquad \left\|\nabla^n\varrho^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^n r^{\left(\frac{2}{p}-1\right)} \tag{7.11}$$

and

$$\left\|\nabla^n\mathcal{U}^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^{n-1} r^{\left(\frac{2}{p}-1\right)}, \qquad \left\|\nabla^n\mathcal{W}^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^n r^{\left(\frac{2}{p}-1\right)}. \tag{7.12}$$

(7) We have that $\mathrm{supp}\,\vartheta^k_{\xi,\lambda,r} \subseteq B\left(\mathrm{supp}\,\varrho_{\xi,\lambda,r}, 2\lambda^{-1}\right)$*.*

[2]The double index ii indicates that $\mathrm{div}^{\mathsf{D}-2}\left(\vartheta^k_{\xi,\lambda,r}\right)$ is a 2-tensor, and we are summing over the diagonal components. The factor of $1/3$ appears because each component on the diagonal of this 3×3 matrix is $\Delta^{-1}\varrho^k_{\xi,\lambda,r}$. The formula then follows from the identity $\mathrm{curl}\,\mathrm{curl} = -\Delta$ for divergence-free vector fields.

(8) Let $\Phi : \mathbb{T}^3 \times [0,T] \to \mathbb{T}^3$ be the periodic solution to the transport equation

$$\partial_t \Phi + v \cdot \nabla \Phi = 0\,, \qquad \Phi|_{t=t_0} = x\,, \tag{7.13}$$

with a smooth, divergence-free, periodic velocity field v. Then

$$\nabla \Phi^{-1} \cdot \left(\mathcal{W}^k_{\xi,\lambda,r} \circ \Phi\right) = \operatorname{curl}\left(\nabla \Phi^T \cdot \left(\mathcal{U}^k_{\xi,\lambda,r} \circ \Phi\right)\right). \tag{7.14}$$

(9) For any convolution kernel K, Φ as in (7.13), $A = (\nabla\Phi)^{-1}$, and for $i = 1,2,3$,

$$\begin{aligned}\left[\nabla \cdot \left(A\, K * \left(\mathcal{W}^k_{\xi,\lambda,r} \otimes \mathcal{W}^k_{\xi,\lambda,r}\right)(\Phi) A^T\right)\right]_i &= A^j_m K * \left((\mathcal{W}^k_{\xi,\lambda,r})^m (\mathcal{W}^k_{\xi,\lambda,r})^l(\Phi)\right) \partial_j A^i_l \\ &= A^j_m \xi^m \xi^l \partial_j A^i_l\, K * \left(\left(\varrho^k_{\xi,\lambda,r}\right)^2(\Phi)\right). \end{aligned} \tag{7.15}$$

In the above display, k indicates the choice of placement, i is the component of the vector field on either side of the equality, and m, l, and j are repeated indices over which summation is implicitly encoded.

Proof. The only small changes relative to the cited Propositions are as follows. First, we write the pipe density ϱ as the iterated *divergence* of a pairwise symmetric vector potential $\operatorname{div}^{\mathsf{D}}\vartheta = \varrho$ to match the form required for our inverse divergence operator (cf. Proposition A.3.3). By "pairwise symmetric," we mean that permuting the $2n-1$ and $2n$ components for $1 \le n \le {}^{\mathsf{D}}/_2$ leaves ϑ unchanged. Since one can always rewrite the identity $\Delta f = g$ as $\partial_i \partial_j \delta_{ij} f = g$, it is easy to convert the equality $\Delta^{\mathsf{D}/2}\widetilde{\vartheta} = \varrho$ into $\operatorname{div}^{\mathsf{D}}\vartheta = \varrho$ where ϑ is a pairwise symmetric tensor (see (7.38)).

Second, (5) is new. We first show that the second integral vanishes for any radial pipe density. This is a simple computation in polar coordinates which is a consequence of the fact that $\mathcal{U}_{\xi,\lambda,r}$ can be written as the 2-dimensional perpendicular gradient of a radial scalar potential $\mathcal{V}$ (see the computation below). Then the only θ dependence in the integrand will

come from $\mathcal{U} = \nabla^{\perp}_{x,y}\mathcal{V} = (-\partial_y \mathcal{V}, \partial_x \mathcal{V}) = (-\sin\theta \mathcal{V}'(r), \cos\theta \mathcal{V}'(r))$, which will then integrate to zero against the other radial functions in the integrand. In order to show that the first integral vanishes, we can take any radial pipe density which defines a specific $\mathcal{W}^k_{\xi,\lambda,r}$ and set $\widetilde{\mathcal{W}} = \mathcal{W}^k_{\xi,\lambda,r} - \mathcal{W}^{k+1}_{\xi,\lambda,r}$, where k and $k+1$ are adjacent choices of placement. Then it is clear that the second now integral vanishes for $\widetilde{\mathcal{W}}$, and the other properties are unchanged up to renormalizations and adjustments of implicit constants. In order to show that the third integrand vanishes, we recall (7.8), the fact that ξ, ξ', ξ'' forms an orthonormal basis with $\xi \times \xi' = \xi''$, and the fact that $\xi \cdot \nabla \vartheta_{\xi,\lambda,r} = 0$. From these facts, we deduce the existence of a scalar potential $\mathcal{V}$ such that

$$(\mathcal{U}_{\xi,\lambda,r})_i = \xi'_i \xi''_l \partial_l \mathcal{V} - \xi''_i \xi'_l \partial_l \mathcal{V} = \epsilon_{ijk}\xi_k \left(\xi''_l \xi''_j + \xi'_l \xi'_j\right) \partial_l \mathcal{V} = \epsilon_{ijk}\partial_j \mathcal{V} \xi_k \,.$$

From the fact that $\mathrm{curl}\,\mathcal{U}_{\xi,\lambda,r} = \mathcal{W}_{\xi,\lambda,r}$ and the fact that $\xi \cdot \nabla \vartheta_{\xi,\lambda,r} = 0$, we deduce that

$$(\mathcal{W}_{\xi,\lambda,r})_i = \epsilon_{ilm}\partial_l \epsilon_{mjk}\partial_j \mathcal{V}\xi_k = (\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj})\partial_l \partial_j \mathcal{V}\xi_k = -\xi_i \partial_{jj}\mathcal{V} \,.$$

Therefore we have that

$$\int_{\mathbb{T}^3} \xi_i \partial_{jj}\mathcal{V}\epsilon_{lmk}\partial_m \mathcal{V}\xi_k = -\frac{1}{2}\int_{\mathbb{T}^3} \epsilon_{lmk}\xi_i\xi_k \partial_m \left(\partial_j \mathcal{V}\partial_j \mathcal{V}\right) = 0 \,,$$

which implies that the third integrand in (5) vanishes.

Finally, (7) is new, but it follows immediately from definitions and (7.7). □

We shall require a set of intermittent pipe flows which possess nearly the same properties as above, but which are however normalized in L^3, and have non-vanishing cubic mean.

Proposition 7.1.6 (Intermittent pipe flows for current corrector). *Fix a vector ξ belonging to the set of rational vectors $\Xi' \subset \mathbb{Z}^3$ from Proposition 7.1.2. The statement is same as in Proposition 7.1.5, but item 4 is not imposed, and items 5–6 are replaced by*

(5) $\fint_{\mathbb{T}^3} |\mathcal{W}^k_{\xi,\lambda,r}|^2 \mathcal{W}^k_{\xi,\lambda,r} = |\xi|^2\xi\,, \quad \fint_{\mathbb{T}^3} (\varrho^k_{\xi,\lambda,r})^2 \mathcal{U}^k_{\xi,\lambda,r} = \fint_{\mathbb{T}^3} \varrho^k_{\xi,\lambda,r}\mathcal{U}^k_{\xi,\lambda,r} = 0.$

(6) For all $n \leq 3\mathsf{N}_{\text{fin}}$,

$$\left\|\nabla^n \vartheta^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^n r^{\left(\frac{2}{p}-\frac{2}{3}\right)}, \qquad \left\|\nabla^n \varrho^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^n r^{\left(\frac{2}{p}-\frac{2}{3}\right)} \tag{7.16}$$

and

$$\left\|\nabla^n \mathcal{U}^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^{n-1} r^{\left(\frac{2}{p}-\frac{2}{3}\right)}, \qquad \left\|\nabla^n \mathcal{W}^k_{\xi,\lambda,r}\right\|_{L^p(\mathbb{T}^3)} \lesssim \lambda^n r^{\left(\frac{2}{p}-\frac{2}{3}\right)}\,. \tag{7.17}$$

Proof. The differences in (6) relative to (6) from the preceding proposition are simply a result of the L^3 normalization and require no further justification. In order to ensure (5), it remains to show that one can construct a radial pipe density $\varrho_{\xi,\lambda,r}$ which has non-vanishing cubic mean and is the iterated Laplacian of a scalar potential, and then convert the scalar potential to a pairwise symmetric tensor potential. As the latter task has already been carried out in the previous proposition, we can focus on the former. One can start with a smooth function $f : (1/2, 1) \to \mathbb{R}$ for which $\int_0^{2\pi} (f^{(\mathsf{D})})^3(x)\,dx \neq 0$, and then define $F(r) = f(\lambda_1 r + \lambda_2)$, where λ_1 and λ_2 are chosen to ensure that to leading order, $\Delta_r^{\mathsf{D}/2} F \approx \lambda_1^{\mathsf{D}} f^{(\mathsf{D})}(\lambda_1 r + \lambda_2)$. Then periodizing concludes the proof. □

In order to control the geometry of pipes which are deformed by a velocity field on a local Lipschitz timescale, we recall [35, Lemma 3.7].

Lemma 7.1.7 (Control on Axes, Support, and Spacing). *Consider a convex neighborhood of space* $\Omega \subset \mathbb{T}^3$. *Let* v *be an incompressible velocity field, and define the flow* $X(x,t)$ *and inverse* $\Phi(x,t) = X^{-1}(x,t)$, *which solves*

$$\partial_t \Phi + v \cdot \nabla \Phi = 0\,, \qquad \Phi|_{t=t_0} = x\,.$$

Define $\Omega(t) := \{x \in \mathbb{T}^3 : \Phi(x,t) \in \Omega\} = X(\Omega, t)$. *For an arbitrary* $C > 0$, *let* $\tau > 0$ *be a*

timescale parameter and $\Gamma > 3$ a large multiplicative prefactor such that the vector field v satisfies the Lipschitz bound

$$\sup_{t\in[t_0-\tau,t_0+\tau]} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega(t))} \lesssim \tau^{-1}\Gamma^{-2}\,.$$

Let $\mathcal{W}^k_{\xi,\lambda,r} : \mathbb{T}^3 \to \mathbb{R}^3$ be a set of straight pipe flows constructed as in Proposition 7.1.4, Proposition 7.1.5, and Proposition 7.1.6 which are $(\mathbb{T}/\lambda r)^3$-periodic and concentrated around axes $\{A_i\}_{i\in\mathcal{I}}$ oriented in the vector direction ξ for $\xi \in \Xi, \Xi'$, passing through the grid-points in item 2 of Proposition 7.1.4. Then $\mathcal{W} := \mathcal{W}^k_{\xi,\lambda,r}(\Phi(x,t)) : \Omega(t) \times [t_0-\tau, t_0+\tau]$ satisfies the following conditions:

(1) We have the inequality

$$\operatorname{diam}(\Omega(t)) \le \left(1+\Gamma^{-1}\right)\operatorname{diam}(\Omega)\,. \tag{7.18}$$

(2) If x and y with $x \neq y$ belong to a particular axis $A_i \subset \Omega$, then

$$\frac{X(x,t)-X(y,t)}{|X(x,t)-X(y,t)|} = \frac{x-y}{|x-y|} + \delta_i(x,y,t) \tag{7.19}$$

where $|\delta_i(x,y,t)| < \Gamma^{-1}$.

(3) Let x and y belong to $A_i \cap \Omega$ for some i, where the axes A_i are defined above. Denote the length of the axis $A_i(t) := X(A_i \cap \Omega, t)$ in between $X(x,t)$ and $X(y,t)$ by $L(x,y,t)$. Then

$$L(x,y,t) \le \left(1+\Gamma^{-1}\right)|x-y|\,. \tag{7.20}$$

(4) The support of $\mathcal{W}$ is contained in a $\left(1+\Gamma^{-1}\right)2\pi(4n_*\lambda)^{-1}$-neighborhood of the set

$$\bigcup_i A_i(t)\,. \tag{7.21}$$

(5) $\mathcal{W}$ is "approximately periodic" in the sense that for distinct axes A_i, A_j with $i \neq j$, we have

$$\left(1-\Gamma^{-1}\right) \operatorname{dist}\left(A_i \cap \Omega, A_j \cap \Omega\right) \leq \operatorname{dist}\left(A_i(t), A_j(t)\right) \leq \left(1+\Gamma^{-1}\right) \operatorname{dist}\left(A_i \cap \Omega, A_j \cap \Omega\right). \tag{7.22}$$

A consequence of Lemma 7.1.7 is that a set of $(\mathbb{T}/\lambda r)^3$-periodic intermittent pipe flows which are flowed by a locally Lipschitz vector field on the Lipschitz timescale can be decomposed into "segments of deformed pipe" in the sense of Remark 5.4.3. Furthermore, any neighborhood of diameter $\approx (\lambda r)^{-1}$ contains at most a finite number of such segments of deformed pipe.

Definition 7.1.8 (Segments of deformed pipes). *A single "segment of deformed pipe with thickness λ^{-1} and spacing $(\lambda r)^{-1}$" is defined as a $3\lambda^{-1}$ neighborhood of a Lipschitz curve of length at most $2(\lambda r)^{-1}$.*

7.2 Pipe dodging and intermittent Mikado bundles

In the continuous scheme, the building block flows are *intermittent Mikado bundles*, which are bundles of pipes carefully designed to dodge previously placed intermittent Mikado bundles. To give the idea, suppose that intermittent Mikado bundles comprised of deformed pipes of thickness $\lambda_{q+1}^{-1}, \cdots \lambda_{q+\bar{n}}^{-1}$ are given in a rectangular prism Ω_0 of particular dimensions. If certain conditions are satisfied with respect to the spacing of the new bundles and the dimensions of the prism Ω_0, we can successfully place new bundles of thickness $\lambda_{q+\bar{n}}^{-1}$ that dodge all given bundles. Furthermore, the pipes in each new bundles will be placed to be at least at a distance $\lambda_{q+i}^{-1}\Gamma_{q+i}$ away from a given deformed pipe of thickness λ_{q+i}^{-1}. We call this additional property *effective dodging*, and it will play a crucial role throughout our scheme.

The key observation is that the intermittency alone need not dictate the spacing of the pipes in a bundle. For example, consider a set of pipes of thickness $\lambda_{q+\bar{n}}^{-1}$ and spacing $\lambda_{q+\bar{n}/2}^{-1}$

restricted to the support of a set of a small number of pipes of thickness and spacing λ_{q+1}^{-1}. An intermittent Mikado bundle is precisely such an object; a *low* frequency, *small* number of *homogeneous* pipes on which *high* frequency, *large* numbers of *intermittent* pipes live. Placing new bundles made up of pipes of thickness $\lambda_{q+\bar{n}}^{-1}$ consists of two steps. We divide $\mathbb{T}^3$ into the rectangular prisms of dimensions $\lambda_{q+1}^{-1}\Gamma_q^5 \times \lambda_{q+1}^{-1}\Gamma_q^5 \times \lambda_q^{-1}\Gamma_q^{-8}$, and first construct the low frequency, homogeneous (bundling) pipes to effectively dodge all given pipes of thicknesses $\lambda_{q+1}^{-1}, \cdots \lambda_{q+\lfloor \bar{n}/2 \rfloor}^{-1}$ in each prism. The pipes of thickness $\lambda_{q+\bar{n}}^{-1}$ will then be placed in the support of the low frequency, homogeneous (bundling) pipes.

Proposition 7.2.1 (**"Bundling" pipe flows $\boldsymbol{\rho}_{\xi,\diamond}^k$ for Reynolds and current correctors).** *Fix a vector ξ belonging to either of the sets of rational vectors from Propositions 7.1.1 or 7.1.2. Then for $k \in \{1, \ldots, \Gamma_q^6\}$, there exist master scalar functions $\overline{\boldsymbol{\rho}}_{\xi,k}$ and subsidiary bundling pipe flows $\boldsymbol{\rho}_{\xi,R}^k := \overline{\boldsymbol{\rho}}_{\xi,k}^3$ for Reynolds correctors and $\boldsymbol{\rho}_{\xi,\varphi}^k := \overline{\boldsymbol{\rho}}_{\xi,k}^2$ for current correctors satisfying the following.*

(i) $\boldsymbol{\rho}_{\xi,\diamond}^k$ *is* $\left(\mathbb{T}/\lambda_{q+1}\Gamma_q^{-4}\right)^3$*-periodic and satisfies* $\xi \cdot \nabla \boldsymbol{\rho}_{\xi,\diamond}^k \equiv 0$*, where either* $\diamond = R$ *or* $\diamond = \varphi$*.*

(ii) The set of functions $\{\boldsymbol{\rho}_{\xi,\diamond}^k\}_k$ *satisfies the conclusions of Proposition 7.1.4 with* $r^{-1} = \Gamma_q^3$*,* $\lambda = \lambda_{q+1}\Gamma_q^{-1}$*. In particular,* $\operatorname{supp} \boldsymbol{\rho}_{\xi,\diamond}^k \cap \operatorname{supp} \boldsymbol{\rho}_{\xi,\diamond}^{k'} = \emptyset$ *for* $k \neq k'$*, and there are* Γ_q^6 *disjoint choices of placement.*

(iii) $\displaystyle\int_{\mathbb{T}^3} \overline{\boldsymbol{\rho}}_{\xi,k}^6 = 1.$

(iv) For all $n \leq 3\mathsf{N}_{\text{fin}}$ *and* $p \in [1, \infty]$*,*

$$\left\|\nabla^n \boldsymbol{\rho}_{\xi,R}^k\right\|_{L^p(\mathbb{T}^3)} \lesssim \left(\Gamma_q^{-1}\lambda_{q+1}\right)^n \Gamma_q^{-3\left(\frac{2}{p}-1\right)}, \qquad \left\|\nabla^n \boldsymbol{\rho}_{\xi,\varphi}^k\right\|_{L^p(\mathbb{T}^3)} \lesssim \left(\Gamma_q^{-1}\lambda_{q+1}\right)^n \Gamma_q^{-3\left(\frac{2}{p}-\frac{2}{3}\right)}. \tag{7.23}$$

Proof. The proof is a straightforward adaptation of the proofs of Propositions 7.1.5 or 7.1.6 after construction of an L^6 normalized master function $\overline{\boldsymbol{\rho}}_{\xi,k}$ which satisfies the shift and support properties from Proposition 7.1.4. We omit further details. □

With the bundling pipe flows defined, we record our first dodging proposition, which uses the bundling pipes to dodge pipes with thickness *at least* $\lambda_{q+\bar{n}/2}^{-1}$ and *at most* λ_{q+1}^{-1}. We record and prove a statement for $\xi = e_3$ and leave the case for general direction vectors to the reader.

Lemma 7.2.2 (**Using bundling pipes to dodge very old, thick pipes).** *Let Ω_0 be a rectangular prism of dimensions $\lambda_{q+1}^{-1}\Gamma_q^5 \times \lambda_{q+1}^{-1}\Gamma_q^5 \times \lambda_q^{-1}\Gamma_q^{-8}$. Suppose that there exists a q-independent constant $\mathcal{C}_P$ such that at most $\mathcal{C}_P$ segments of their deformed segments with thickness $\lambda_{q'+\bar{n}}^{-1}$ and spacing $(\lambda_{q'+\bar{n}/2}\Gamma_{q'})^{-1}$ for some $q - \bar{n} < q' \leq q - \bar{n}/2$ (in the sense of Definition 7.1.8) have non-empty intersection with Ω_0. Let $E_0 \subset \Omega_0$ denote the support of such deformed segments inside Ω_0. Then there exists $k \in \{1, \ldots, \Gamma_q^6\}$ and a bundling pipe flow $\boldsymbol{\rho}_{e_3,\diamond} := \boldsymbol{\rho}_{e_3,\diamond}^k$ defined as in Proposition 7.2.1 such that*

$$B\left(\operatorname{supp} \boldsymbol{\rho}_{e_3,\diamond}^k, \lambda_{q+1}^{-1}\Gamma_q^2\right) \cap E_0 = \emptyset \quad \textit{i.e.,} \quad B\left(E_0, \lambda_{q+1}^{-1}\Gamma_q^2\right) \cap \operatorname{supp} \boldsymbol{\rho}_{e_3,\diamond}^k = \emptyset\,. \tag{7.24}$$

Proof. We first divide the face $[0, \lambda_{q+1}^{-1}\Gamma_q^5]^2$ of the prism into the grid of squares of sidelength $\approx \lambda_{q+1}^{-1}\Gamma_q$, and we will find a set of squares in which we can place a new bundling pipe flow $\boldsymbol{\rho}_{e_3,\diamond}^k$. Since the set of squares will be placed $(\mathbb{T}/\lambda_{q+1}\Gamma_q^{-4})^2$-periodically, we have from (ii) that

$$(\text{the possible number of placement of a set of squares}) = \left(\frac{\text{spacing}}{\text{thickness}}\right)^2 = \left(\frac{\lambda_{q+1}^{-1}\Gamma_q^4}{\lambda_{q+1}^{-1}\Gamma_q}\right)^2 = \Gamma_q^6\,.$$

By assumption there exist at most $\mathcal{C}_P$ number of deformed pipe segments in the prism. When we enlarge these segments by a factor of $\lambda_{q+1}^{-1}\Gamma_q^2$ and project the enlarged neighborhood onto the face $[0, \lambda_{q+1}^{-1}\Gamma_q^5]^2$, each projection will be contained in a $\approx \lambda_{q+1}^{-1}\Gamma_q^2$-neighborhood of a curve

of length at most $\approx \lambda_{q+1}^{-1}\Gamma_q^5$ by (7.20) and (7.21). It then follows that[3]

$$\begin{aligned}
&\text{(the number of grid squares occupied by given enlarged segments)}\\
&\quad\lesssim \text{number of segments} \times \frac{\text{area occupied by an enlarged segment}}{\text{area of a grid square}}\\
&\quad\lesssim \mathcal{C}_P \times \frac{\lambda_{q+1}^{-2}\Gamma_q^7}{\Gamma_q^2\lambda_{q+1}^{-2}}\\
&\quad= \mathcal{C}_P\Gamma_q^5\,,
\end{aligned}$$

which is less than Γ_q^6 for sufficiently large λ_0. Therefore, from the pigeonhole principle, there exists a set of squares in which we can place the pipe $\boldsymbol{\rho}_{e_3,\diamond}^k$ satisfying (7.24). □

We now use the intermittent pipe flows from Propositions 7.1.5 or 7.1.6 to dodge pipes with thickness *at least* $\lambda_{q+\bar{n}}^{-1}$ and *at most* than $\lambda_{q+\bar{n}/2+1}^{-1}$. Combined with the previous proposition, we will have successfully dodged pipes of thicknesses in between λ_{q+1}^{-1} and $\lambda_{q+\bar{n}}^{-1}$. As before, we present the statement for $\xi = e_3$ and omit further details.

Lemma 7.2.3 (**Using very intermittent pipes to dodge newer, less thick pipes).** *Let Ω_1 be a rectangular prism of dimensions $\lambda_{q+\bar{n}/2}^{-1} \times \lambda_{q+\bar{n}/2}^{-1} \times \lambda_q^{-1}\Gamma_q^{-8}$ with the long side in the e_3 direction. Suppose that a finite number of sets of $(\mathbb{T}/\lambda_{q''+\bar{n}/2}\Gamma_{q''})^3$-periodic pipes of thickness $\lambda_{q''+\bar{n}}^{-1}$ are given for all $q - \bar{n} + \bar{n}/2 < q'' \leq q$, constructed as in Propositions 7.1.5 or 7.1.6. Furthermore, suppose that for each such q'' and any convex subset $\Omega' \subset \Omega_1$ with $\operatorname{diam}(\Omega') \lesssim \lambda_{q''+\bar{n}/2}^{-1}\Gamma_{q''}^{-1}$, there exists a q-independent constant $\mathcal{C}_P$ such that at most $\mathcal{C}_P\Gamma_{q''}$ segments of the deformed pipes of thickness $\lambda_{q''+\bar{n}}^{-1}$ have non-empty intersection with Ω'. For fixed q'', let $E_{q''}$ denote the support of such segments inside Ω_1. Then for either $\diamond = R$ or $\diamond = \varphi$, there exists k and a corresponding intermittent pipe flow $\mathbb{W}_{e_3,\diamond} := \mathcal{W}^k_{\xi,\lambda_{q+\bar{n}},\lambda_{q+\bar{n}/2}\Gamma_q/\lambda_{q+\bar{n}}}$*

[3] A fully rigorous version of this estimate would utilize a standard covering argument which is predicated on the geometric constraints imposed by Lemma 7.1.7, or even Definition 7.1.8; we however content ourselves with a slightly heuristic version and refer the reader to [7, Proposition 4.8] for further details.

constructed as in Propositions 7.1.5 or 7.1.6 such that for all $q - {}^{\bar{n}}/_{2} < q'' \leq q$,

$$B\left(\operatorname{supp} \mathbb{W}_{e_3,\diamond}, \Gamma^2_{q''+\bar{n}}\lambda^{-1}_{q''+\bar{n}}\right) \cap E_{q''} = \emptyset \quad i.e., \quad B\left(E_{q''}, \Gamma^2_{q''+\bar{n}}\lambda^{-1}_{q''+\bar{n}}\right) \cap \operatorname{supp} \mathbb{W}_{e_3,\diamond} = \emptyset\,.$$

Proof. As in the previous lemma, since we want to place a new pipe which enjoys *effective dodging* with previously placed deformed pipes, instead of considering the previously placed pipes themselves, we consider a thickened neighborhood of them. More precisely, for a deformed pipe of thickness $2\lambda^{-1}_{q+i}$, we consider instead a neighborhood of it of thickness $\Gamma^2_{q+i}\lambda^{-1}_{q+i}$ and call these new objects 'thickened pipes'. Then, it is enough to place a new pipe that dodges these thickened pipes, so that a new pipe effectively dodges all previously placed deformed pipes.

We divide the face of Ω_1 into a grid of squares of sidelength $\lambda^{-1}_{q+\bar{n}}$. Since a new pipe will be placed $({}^{\mathbb{T}}/_{\lambda_{q+\bar{n}/2}\Gamma_q})^3$-periodically, we have from Proposition 7.1.5 or 7.1.6 that

$$(\text{the possible number of placement of a set of squares}) = \left(\frac{\text{spacing}}{\text{thickness}}\right)^2 = \left(\frac{\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^2. \tag{7.25}$$

Now, we count the number of grid squares occupied by given enlarged segments and compare it to this number. From the assumption that there exists $\mathcal{C}_P$ which controls the density of thickened, deformed pipe segments of thickness $\lambda^{-1}_{q''+\bar{n}}$ that can intersect a ball Ω' of volume $\approx (\lambda_{q''+\bar{n}/2}\Gamma_{q''})^{-3}$, we have that the total number of thickened pipe segments that can intersect Ω_1 is at most

$$\mathcal{C}_P\Gamma_{q''} \times \frac{\text{length of } \Omega_1}{\lambda^{-1}_{q''+\bar{n}/2}\Gamma^{-1}_{q''}} \times \frac{(\text{width of } \Omega_1)^2}{\min\left(\lambda^{-1}_{q''+\bar{n}/2}\Gamma^{-1}_{q''}, \text{width of } \Omega_1\right)^2} \leq \mathcal{C}_P\Gamma_{q''} \times \frac{\lambda_{q''+\bar{n}/2}\Gamma^3_{q''}}{\Gamma^8_q\lambda_q}\,.$$

When we project all these thickened segments onto the face of Ω_1, each projection will be contained in a $\approx \lambda^{-1}_{q''+\bar{n}}\Gamma^2_{q''+\bar{n}}$-neighborhood of a curve of length at most $\approx \lambda^{-1}_{q+\bar{n}/2}$ from (7.20) and (7.21). Therefore, the number of grid squares occupied by each enlarged pipe projection

is

$$\frac{\text{area occupied by an enlarged segment}}{\text{area of a grid square}} \approx \frac{\lambda_{q+\bar{n}/2}^{-1}\lambda_{q''+\bar{n}}^{-1}\Gamma_{q''+\bar{n}}^{2}}{\lambda_{q+\bar{n}}^{-2}}\,.$$

Thus the total number of grid squares covered by the union of all projections is

$$\sim \sum_{q''=q-\bar{n}/2+1}^{q} \mathcal{C}_P\Gamma_{q''} \times \frac{\lambda_{q''+\bar{n}/2}\Gamma_{q''}^{3}}{\lambda_{q''+\bar{n}}\Gamma_{q''+\bar{n}}^{-2}}\frac{\lambda_{q+\bar{n}}^{2}\lambda_{q+\bar{n}/2}^{-1}}{\Gamma_{q}^{8}\lambda_{q}}\,, \tag{7.26}$$

or the product of the two numbers computed above and summed over q''. This number will be less than the the number in (7.25) if

$$\bar{n}\mathcal{C}_p\Gamma_{q+\bar{n}}^{2}\Gamma_{q}^{-2}\frac{\lambda_{q''+\bar{n}/2}\lambda_{q+\bar{n}/2}}{\lambda_{q}\lambda_{q''+\bar{n}}} \leq 1$$

for $q-\bar{n}/2+1 \leq q'' \leq q$, which is precisely (4.10e). □

Considering the dimensions of the prism in Lemma 7.2.3, we further divide the support of the bundling pipes using the following anisotropic cut-offs and assign different pipes on the support of different cut-offs.

Definition 7.2.4 (Strongly anisotropic cut-off)**.** *To each $\xi \in \Xi$, we associate a partition of the orthogonal space $\xi^{\perp} \in \mathbb{T}^3$ into a grid*[4] *of squares of sidelength $\approx \lambda_{q+\bar{n}/2}^{-1}$. We index the squares $\mathcal{S}$ in this partition by I_ξ which we will also denote by simply I. To this grid, we associate a partition of unity $\boldsymbol{\zeta}_\xi^I$, i.e.,*

$$\boldsymbol{\zeta}_\xi^I = \begin{cases} 1 & \text{on } \frac{3}{4}\mathcal{S}_I \\ 0 & \text{outside } \frac{5}{4}\mathcal{S}_I \end{cases}, \qquad \sum_I (\boldsymbol{\zeta}_\xi^I)^6 = 1\,, \tag{7.27}$$

which in addition satisfies $(\xi\cdot\nabla)\boldsymbol{\zeta}_\xi = 0$ and $\left\|\nabla^N\boldsymbol{\zeta}_\xi^I\right\|_\infty \lesssim \lambda_{q+\bar{n}/2}^{N}$ for all $N \leq 3\mathsf{N}_{\text{fin}}$ and all I, where the implicit constants depend only on Ξ.

[4]One can use some version of the grid from Proposition 7.1.4, as the periodicity issues have been avoided there.

Remark 7.2.5. We note that the number of grid squares of sidelength $\lambda_{q+\bar{n}/2}^{-1}$ partitioning the orthogonal space $\xi^{\perp} \subset \mathbb{T}^3$ is $\lesssim \lambda_{q+\bar{n}/2}^2$. Consequently, we bound the cardinality of the index set I as

$$|\{I \in \mathcal{S}\}| \lesssim \lambda_{q+\bar{n}/2}^2 \,.$$

We now introduce *intermittent pipe bundles*, which are defined on the support of a broad rectangular prism at scale close to λ_q^{-1}. These objects are *multi-scale* and consist of nearly homogeneous bundling pipes at scale λ_{q+1}^{-1}, upon which various intermittent pipes are placed on the support of much finer cutoffs. We write the following definition under the assumptions of Lemmas 7.2.2 and 7.2.3, which demand that the broad rectangular prism is inhabited by a limited number of deformed pipes at various scales which avoid the support of the newly constructed pipes. In our inductive argument, this assumption corresponds to Hypothesis 5.4.2 and will be verified in subsection 9.2.

Definition 7.2.6 (Intermittent pipe bundles). *For rectangular prisms Ω_0 as in Lemma 7.2.2, the intermittent pipe bundle associated to them is given by*

$$\mathbb{B}_{(\xi),R} = \boldsymbol{\rho}_{(\xi),R} \sum_I (\boldsymbol{\zeta}_\xi^I)^3 \mathbb{W}_{(\xi),R}^I \quad \textit{and} \quad \mathbb{B}_{(\xi),\varphi} = \boldsymbol{\rho}_{(\xi),\varphi} \sum_I (\boldsymbol{\zeta}_\xi^I)^2 \mathbb{W}_{(\xi),\varphi}^I$$

where $\boldsymbol{\rho}_{(\xi),\diamond}$ and $\mathbb{W}_{(\xi),\diamond}^I$ are chosen in Lemmas 7.2.2 and 7.2.3, respectively.

Remark 7.2.7 (Notational conventions). We shall frequently denote the intermittent pipe bundles defined above as follows:

$$\mathbb{B}_{(\xi),\diamond} = \boldsymbol{\rho}_{(\xi)}^{\diamond} \sum_I \zeta_\xi^{I,\diamond} \mathbb{W}_{(\xi),\diamond}^I \,. \tag{7.28}$$

The meaning of this notation is as follows:

(i) The choice of placements for each bundle $\mathbb{B}_{(\xi),\diamond}$ will depend on which of the various

mildly anisotropic checkerboard cutoff functions $\zeta_{q,\diamond,i,k,\xi,\vec{l}}$ (these are defined in Definition 8.4.1 and correspond to the set Ω_0 in Lemma 7.2.2) we are trying to construct the bundle on. Thus each bundle will depend on all the indices for $\zeta_{q,\diamond,i,k,\xi,\vec{l}}$, as well as the index j for the pressure cutoffs defined in Definition 8.3.2. We will suppress these indices most of the time and simply write (ξ) in parentheses, where the parentheses is a stand-in for the omitted indices $q, i, k, \vec{l}, j, \diamond$.

(ii) The subscript "$\diamond$" in $\mathbb{B}_{(\xi),\diamond}$ will be equal to either φ or R, corresponding to velocity increments designed to correct current errors or stress errors, respectively.

(iii) We abbreviate the bundling pipes $\boldsymbol{\rho}_{(\xi),\diamond}$ by $\boldsymbol{\rho}^{\diamond}_{(\xi)}$. We write the $\diamond$ in the exponent to emphasize that the only difference between $\diamond = \varphi$ and $\diamond = R$ is the power of the scalar function $\overline{\boldsymbol{\rho}}_{\xi,k}$ used to define them.

(iv) We abbreviate the very anisotropic cutoff functions by $\boldsymbol{\zeta}^{I,\diamond}_{\xi}$. We do *not* write ξ in parentheses, since $\boldsymbol{\zeta}^{I,\diamond}_{\xi}$ does not depend on anything besides the vector direction ξ and the index I used to index the partition of unity. Also, the only difference between $\diamond = \varphi$ and $\diamond = R$ is the power, so we write $\diamond$ in the exponent.

(v) We write $\mathbb{W}^{I}_{(\xi),\diamond}$ for the following reasons: first, the pipe flow depends on more indices than just ξ, so we write (ξ) to denote the omitted indices; we include the index I to emphasize that the placement of the intermittent pipe flow depends not just on the omitted indices in (ξ), but on the index I as well. Finally, we leave $\diamond$ in the subscript since the difference between $\mathbb{W}^{I}_{(\xi),R}$ and $\mathbb{W}^{I}_{(\xi),\varphi}$ is more than just a power; the former has vanishing cubic mean, while the latter does not.

7.3 Synthetic Littlewood-Paley decomposition

When we estimate material derivatives of oscillation stress and current errors, we need dodging in order to handle the differential operator $(\widehat{u}_{k-1} - \widehat{u}_q) \cdot \nabla$ in the material derivative

applied to the error. To ensure a spatial support property even after taking the inverse divergence operator and a frequency projection operator on a squared pipe density, we introduce a *synthetic Littlewood-Paley projector* $\widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}$. This operator is defined using convolution with a compactly supported kernel, and thus behaves like the original projection operator $\mathbb{P}_{(\lambda_1,\lambda_2]}$ in estimates but has an additional spatial support property.

Definition 7.3.1 (Synthetic Littlewood-Paley projector). *Let $\bar{\varphi} \in C_c^\infty(\mathbb{R})$ satisfy*

$$\operatorname{supp}(\bar{\varphi}) \subset (-1/\sqrt{2}, 1/\sqrt{2}), \qquad \int_{\mathbb{R}} \bar{\varphi} ds = 1, \qquad \int_{\mathbb{R}} s^n \bar{\varphi} ds = 0$$

for $n = 1, \ldots, 10\mathsf{N}_{\text{fin}}$. Define $\bar{\varphi}_\lambda(\cdot) = \lambda\bar{\varphi}(\lambda\cdot)$, and set $\varphi_\lambda(x) = \bar{\varphi}_\lambda(x_1)\bar{\varphi}_\lambda(x_2)$. For $f \in C^\infty(\mathbb{T}^2)$, we define the synthetic Littlewood-Paley projectors *by*

$$\widetilde{\mathbb{P}}_\lambda f(x) := \int_{\mathbb{R}^2} \varphi_\lambda(y) f(x-y) dy, \qquad \widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]} f := (\widetilde{\mathbb{P}}_{\lambda_2} - \widetilde{\mathbb{P}}_{\lambda_1}) f, \tag{7.29}$$

where in the convolution we consider f as a periodic function defined on $\mathbb{R}^2$.

From the definition, it is easy to see that $\operatorname{supp}(\varphi_{\lambda_2} - \varphi_{\lambda_1}) \subseteq \operatorname{supp}(\varphi_{\lambda_1})$ and hence $\operatorname{supp}(\widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]} f) \subset B(\operatorname{supp}(f), \lambda_1^{-1})$. With a bit of care, this property persists even after inverting the divergence.

Lemma 7.3.2 (Inverse divergence with spatial support property). *For given $f \in C^\infty(\mathbb{T}^2)$ and $\mathsf{D} \geq 1$, there exists a symmetric tensor field $\Theta_f^{\lambda_1,\lambda_2} : \mathbb{T}^2 \to \mathbb{R}^{(2^{\mathsf{D}})}$ such that*

$$\widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}(f) = \widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}(f - \langle f \rangle) = \left(\lambda_1^{-1}\mathrm{div}\right)^{(\mathsf{D})} \Theta_f^{\lambda_1,\lambda_2}, \quad \operatorname{supp}\left(\Theta_f^{\lambda_1,\lambda_2}\right) \subset B(\operatorname{supp}(f), \lambda_1^{-1}). \tag{7.30}$$

Proof. By a simple computation, we have

$$\varphi_{\lambda_2}(x) - \varphi_{\lambda_1}(x) = (\bar{\varphi}_{\lambda_2}(x_1) - \bar{\varphi}_{\lambda_1}(x_1))\bar{\varphi}_{\lambda_2}(x_2) + \bar{\varphi}_{\lambda_1}(x_1)(\bar{\varphi}_{\lambda_2}(x_2) - \bar{\varphi}_{\lambda_1}(x_2)). \tag{7.31}$$

Now define $g_0(z) = \bar{\varphi}_{\lambda_2}(z) - \bar{\varphi}_{\lambda_1}(z)$. We first construct a function $g_{\mathsf{D}}(z) : \mathbb{R} \to \mathbb{R}$ with zero mean such that upon differentiating D many times,

$$g_{\mathsf{D}}^{(\mathsf{D})} = g_0 \,, \qquad \operatorname{supp}(g_{\mathsf{D}}) \subset (-(\sqrt{2}\lambda_1)^{-1}, (\sqrt{2}\lambda_1)^{-1}) \,.$$

The construction follows from applying the following claim iteratively: if $g_i \in C_c^\infty(\mathbb{R})$ for some $i \in \{0, \dots, \mathsf{D}-1\}$ satisfies $\int s^n g_i ds = 0$ for all $n = 0, \cdots, \mathsf{D}-i$, then we can find g_{i+1} such that

$$g_{i+1}' = g_i \,, \qquad \operatorname{supp}(g_{i+1}) \subset (-(\sqrt{2}\lambda_1)^{-1}, (\sqrt{2}\lambda_1)^{-1}) \,, \qquad \int_{\mathbb{R}} s^n g_{i+1} ds = 0 \text{ for } n = 0, \dots, \mathsf{D}-i-1 \,.$$

Assuming the claim, then g_0 satisfies $\int_{\mathbb{R}} s^n g_0(s) ds = 0$ for $n = 0, \cdots, \mathsf{D}$, so we can find g_{D} with zero-mean such that

$$g_{\mathsf{D}}^{(\mathsf{D})} = g_{\mathsf{D}-1}^{(\mathsf{D}-1)} = \cdots = g_0 \,, \qquad \operatorname{supp}(g_{\mathsf{D}}) \subset (-(\sqrt{2}\lambda_1)^{-1}, (\sqrt{2}\lambda_1)^{-1}) \,.$$

To prove the claim, we define g_{i+1} by $g_{i+1}(t) := \int_{-a}^{t} g_i ds$, where a is chosen so that $\operatorname{supp}(g_i) \subset (-a, a)$. Since g_i has zero-mean, we can easily see that $\operatorname{supp}(g_{i+1}) \subset (-a, a)$, and $g_{i+1}(a) = g_{i+1}(-a) = 0$. Using the latter, the vanishing moment condition follows from

$$\int_{\mathbb{R}} s^n g_{i+1} ds = \frac{1}{n+1} \int_{-a}^{a} (s^{n+1})' g_{i+1} ds = -\frac{1}{n+1} \int_{-a}^{a} s^{n+1} g_i ds = 0 \,.$$

Now, we set $\theta_1^{(1,\dots,1)}(x_1, x_2) = g_{\mathsf{D}}(x_1)\bar{\varphi}_{\lambda_2}(x_2)$, and otherwise $\theta_1^{(i_1,\dots,i_{\mathsf{D}})}$ is zero, and $\theta_2^{(2,\dots,2)}(x_1, x_2) = \bar{\varphi}_{\lambda_1}(x_1) g_{\mathsf{D}}(x_2)$, and otherwise $\theta_2^{(i_1,\dots,i_{\mathsf{D}})}$ is zero. Then

$$\begin{aligned} \partial_{i_1 \cdots i_{\mathsf{D}}} \theta_1^{(i_1,\cdots,i_{\mathsf{D}})} &= g_0(x_1)\bar{\varphi}_{\lambda_2}(x_2) \,, \quad \operatorname{supp}(\theta_1^{(i_1,\cdots,i_{\mathsf{D}})}) \subset B(0, \lambda_1^{-1}) \\ \partial_{i_1 \cdots i_{\mathsf{D}}} \theta_2^{(i_1,\cdots,i_{\mathsf{D}})} &= \bar{\varphi}_{\lambda_1}(x_1) g_0(x_2) \,, \quad \operatorname{supp}(\theta_2^{(i_1,\cdots,i_{\mathsf{D}})}) \subset B(0, \lambda_1^{-1}) \,. \end{aligned} \tag{7.32}$$

Lastly, we define the desired tensor function $\Theta_f^{\lambda_1,\lambda_2}$ by

$$(\Theta_f^{\lambda_1,\lambda_2})^{(i_1,\dots,i_{\mathsf{D}})}(x_1,x_2) := \Theta * f(x_1,x_2) := \lambda_1^{\mathsf{D}}[(\theta_1+\theta_2)^{(i_1,\cdots,i_{\mathsf{D}})}] * f(x_1,x_2)\,, \tag{7.33}$$

which by (7.31) and direct computation satisfies $\left(\lambda_1^{-1}\mathrm{div}\right)^{(\mathsf{D})}\Theta_f^{\lambda_1,\lambda_2} = \widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}f$. The desired spatial support property follows from (7.33) and (7.32). We note that since $\varphi_{\lambda_2} - \varphi_{\lambda_1}$ has zero mean, $\widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}\langle f\rangle = 0$. □

With the previous Lemma in hand, we aim to apply various synthetic Littlewood-Paley projectors to smooth functions (such as squared pipe densities) and derive estimates for the projected function, and its "inverse divergence potentials." We shall generally decompose a smooth, $(\mathbb{T}/\lambda r)^3$-periodic function ρ which has derivative cost λ as a sum of the form

$$\widetilde{\mathbb{P}}_{\lambda_0}(\rho) + \left(\sum_{k=1}^{K}\widetilde{\mathbb{P}}_{(\lambda_{k-1},\lambda_k]}(\rho)\right) + \left(\mathrm{Id} - \widetilde{\mathbb{P}}_{\lambda_K}\right)(\rho)\,, \tag{7.34}$$

where λ_0 is slightly larger than λr, and λ_K is slightly larger than λ. The terms in the sum are precisely of the form to which the previous lemma applies, and we estimate these in Lemma 7.3.4. The bottom and top shells which correspond to the two terms not in the summand are slightly unique cases; for these we record the following Lemma. Note that spatial localization is not relevant for these unique cases, as the lowest shell will have no spatial localization properties at all, and the highest shell will be vanishingly small.

Lemma 7.3.3 (Inverse divergence, special cases). *Fix $q \in [1,\infty]$. Let N a positive integer, $N_{**} \le \mathsf{N}/2$ a positive integer, r,λ such that $\lambda r, \lambda \in \mathbb{N}$, and $\rho : (\mathbb{T}/\lambda r)^2 \to \mathbb{R}$ a smooth function such that there exists a constant $\mathcal{C}_{\rho,q}$ with*

$$\left\|D^N\rho\right\|_{L^q(\mathbb{T}^2)} \lesssim \mathcal{C}_{\rho,q}\lambda^N\,. \tag{7.35}$$

for $N \le \mathsf{N}$. Let λ_0, λ_K be given with $\lambda r < \lambda_0 < \lambda < \lambda_K$. If the kernel $\overline{\varphi}$ used in Defini-

tion 7.3.1 has N_{**} vanishing moments, then for $p \in [q, \infty]$ we have that

$$\left\| D^N \left(\widetilde{\mathbb{P}}_{\lambda_0} \rho \right) \right\|_{L^p} \lesssim \mathcal{C}_{\rho,q} \left(\frac{\lambda_0}{\lambda r} \right)^{2/q - 2/p} \lambda_0^N \qquad \forall N \leq \mathsf{N}\,, \tag{7.36a}$$

$$\left\| D^N \left(\left(\mathrm{Id} - \widetilde{\mathbb{P}}_{\lambda_K} \right) \rho \right) \right\|_{L^\infty} \lesssim \left(\frac{\lambda}{\lambda_K} \right)^{N_{**}} \mathcal{C}_{\rho,q} \lambda^{N+3} \qquad \forall N \leq \mathsf{N} - N_{**} - 3\,. \tag{7.36b}$$

Furthermore, for any chosen positive even integer D and any small positive number α, there exist adjacent-pairwise symmetric[5] *rank-D tensor potentials ϑ_0 and ϑ_K such that for $0 \leq k \leq \mathsf{D}$ and N in the same range as above,*

$$\mathrm{div}^{\mathsf{D}} \vartheta_0 = \widetilde{\mathbb{P}}_{\lambda_0} \mathbb{P}_{\neq 0} \rho\,, \qquad \left\| D^N \mathrm{div}^k \vartheta_0 \right\|_{L^p} \lesssim \lambda_0^\alpha \mathcal{C}_{\rho,q} \left(\frac{\lambda_0}{\lambda r} \right)^{2/q - 2/p} (\lambda r)^{k - \mathsf{D}} \mathcal{M}\left(N, \mathsf{D} - k, \lambda r, \lambda_0 \right)\,, \tag{7.37a}$$

$$\mathrm{div}^{\mathsf{D}} \vartheta_K = (\mathrm{Id} - \widetilde{\mathbb{P}}_{\lambda_K}) \rho\,, \qquad \left\| D^N \mathrm{div}^k \vartheta_K \right\|_{L^\infty} \lesssim \left(\frac{\lambda}{\lambda_K} \right)^{N_{**}} \mathcal{C}_{\rho,q} \lambda^3 (\lambda r)^{k - \mathsf{D}} \mathcal{M}\left(N, \mathsf{D} - k, \lambda r, \lambda \right)\,. \tag{7.37b}$$

The implicit constants above depend on α but do not depend on λ, λ_0, λ_K, or r.

Proof. For the proof of (7.36a), we first define $F(x) = (\widetilde{\mathbb{P}}_{\lambda r} \rho)(x/\lambda r)$ to be the 1-periodic rescaling of $\widetilde{\mathbb{P}}_{\lambda r} \rho$. Then we can write that

$$\begin{aligned}
\sup_{x \in \mathbb{T}^2} \left| D^N \left(\widetilde{\mathbb{P}}_{\lambda r} \rho \right) \right| (x) &= (\lambda r)^N \sup_{x \in \mathbb{T}^2} \left| D^N F \right| (x) \\
&= (\lambda r)^N \sup_{x \in \mathbb{T}^2} \left| D_x^N \int_{\mathbb{R}^2} \rho(x/\lambda r - y) \varphi_{\lambda_0}(y)\, dy \right| \\
&= (\lambda r)^N \sup_{x \in \mathbb{T}^2} \left| D_x^N \int_{\mathbb{R}^2} \rho\left(\frac{x - z}{\lambda r} \right) \varphi_{\frac{\lambda_0}{\lambda r}}(z)\, dz \right| \\
&= (\lambda r)^N \sup_{x \in \mathbb{T}^2} \left| \int_{\mathbb{R}^2} \rho\left(\frac{x - z}{\lambda r} \right) (D_z^N \varphi_{\frac{\lambda_0}{\lambda r}})(z)\, dz \right| \\
&\lesssim (\lambda r)^N \left(\frac{\lambda_0}{\lambda r} \right)^N \left(\frac{\lambda_0}{\lambda r} \right)^{2/q} \mathcal{C}_{\rho,q} = \lambda_0^N \left(\frac{\lambda_0}{\lambda r} \right)^{2/q} \mathcal{C}_{\rho,q}
\end{aligned}$$

for all N, and in particular for all $N \leq \mathsf{N}$. This proves (7.36a) for $p = \infty$, and the full estimate

[5] By "adjacent-pairwise symmetric," we mean that permuting the $2n - 1$ and $2n$ components for $1 \leq n \leq \mathsf{D}/2$ leaves ϑ unchanged.

follows from interpolation with the trivial L^q estimate. To prove the second estimate, we use the vanishing moments condition to expand ρ as a Taylor series and eliminate the first $N_{**}-1$ terms; in particular, we have that

$$\begin{aligned}
&\left|D^N\left(\left(\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_K}\right)\rho\right)\right|(x)\\
&\quad=\left|\int_{\mathbb{R}^2}\varphi_{\lambda_K}(x-y)\left(\sum_{|\beta|=N_{**}}\frac{|\beta|(y-x)^\beta}{\beta!}\int_0^1(1-\eta)^{N_{**}-1}D^\beta D^N\rho(x+\eta(y-x))\,,d\eta\right)dy\right|\\
&\quad\lesssim\left\|D^{N+N_{**}}\varrho\right\|_{L^\infty}(\lambda_K)^{-N_{**}}\\
&\quad\lesssim\left(\frac{\lambda}{\lambda_K}\right)^{N_{**}}\lambda^{N+3}\mathcal{C}_{\rho,q}\,.
\end{aligned}$$

The above computation holds for $N+N_{**}+3\leq\mathsf{N}$, concluding the proof of the second estimate.

To prove the estimates for the tensor potentials, for $k=0,K$ we first define

$$\vartheta_0^{i_1i_2\dots i_{\mathsf{D}-1}i_{\mathsf{D}}}=\delta^{i_1i_2}\cdots\delta^{i_{\mathsf{D}-1}i_{\mathsf{D}}}\Delta^{-\frac{\mathsf{D}}{2}}\widetilde{\mathbb{P}}_{\lambda_0}\mathbb{P}_{\neq0}\rho\,, \tag{7.38a}$$

$$\vartheta_K^{i_1i_2\dots i_{\mathsf{D}-1}i_{\mathsf{D}}}=\delta^{i_1i_2}\cdots\delta^{i_{\mathsf{D}-1}i_{\mathsf{D}}}(\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_K})\Delta^{-\frac{\mathsf{D}}{2}}\mathbb{P}_{\neq0}\rho \tag{7.38b}$$

where δ^{jl} is the usual Kronecker delta. Then by direct computation and standard Littlewood-Paley analysis, (7.37a) and (7.37b) hold. The α loss in the first estimate is due to the failure of the Calderon-Zygmund inequality in endpoint cases. □

We now move to the middle cases from (7.34), for which the spatial localization will be important.

Lemma 7.3.4 (General localized inverse divergence). *Fix $q\in[1,\infty]$. Let $\rho:\mathbb{T}^2\to\mathbb{R}$ be a smooth function which is $(\mathbb{T}/\lambda_r)^2$-periodic and for $N\leq2\mathsf{N}_{\mathrm{fin}}$ satisfies*

$$\left\|D^N\rho\right\|_{L^q(\mathbb{T}^2)}\lesssim\mathcal{C}_{\rho,q}\lambda^N\,. \tag{7.39}$$

For $\lambda r < \lambda_1 < \lambda_2$, define $\Theta_\rho^{\lambda_1,\lambda_2}$ using Lemma 7.3.2. Then for $p \in [q,\infty]$, $0 \le k \le \mathsf{D}$, $0 < \alpha \ll 1$, and $N \le \mathsf{N}_{\text{fin}}$, we have

$$\left(\lambda_1^{-1}\text{div}\right)^{(\mathsf{D})} \Theta_\rho^{\lambda_1,\lambda_2} = \widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}(\rho) = \widetilde{\mathbb{P}}_{(\lambda_1,\lambda_2]}(\rho - \langle\rho\rangle) \tag{7.40a}$$

$$\left\|D^N \partial_{i_1\cdots i_{\mathsf{D}-k}}(\lambda_1^{-\mathsf{D}}\Theta_\rho^{\lambda_1,\lambda_2})^{(i_1,\cdots,i_{\mathsf{D}})}\right\|_{L^p(\mathbb{T}^2)} \lesssim_{\mathsf{D},\alpha} \mathcal{C}_{\rho,q}\left(\frac{\min(\lambda,\lambda_2)}{\lambda r}\right)^{\frac{2}{q}-\frac{2}{p}+\alpha} \lambda_1^{-k}\min(\lambda,\lambda_2)^N\,, \tag{7.40b}$$

$$\text{supp}\,(\Theta_\rho^{\lambda_1,\lambda_2}) \subset B(\text{supp}\,(\rho),\lambda_1^{-1})\,. \tag{7.40c}$$

The implicit constants above depend on α but do not depend on λ, λ_1, λ_2, or r.

Proof. The spatial property immediately follows from Lemma 7.3.2. To obtain L^p-norm estimates, we will obtain L^q and L^∞ norm estimates and then interpolate them. We first rescale by setting

$$\widetilde{\rho}(\cdot) = \rho\left(\frac{\cdot}{\lambda r}\right)\,, \qquad \widetilde{\lambda}_1 = \frac{\lambda_1}{\lambda r}\,, \qquad \widetilde{\lambda}_2 = \frac{\lambda_2}{\lambda r}\,, \qquad \widetilde{\lambda} = \frac{\lambda}{\lambda r} = r^{-1}\,, \tag{7.41}$$

so that $\widetilde{\rho}$ is $\mathbb{T}^2$ periodic and satisfies

$$\left\|D^N\widetilde{\rho}\right\|_{L^q(\mathbb{T}^2)} \lesssim \mathcal{C}_{\rho,q}\widetilde{\lambda}^N\,.$$

Constructing θ_1 and θ_2 as in the previous lemma but for the choices in (7.41), we have

$$\partial_1^{\mathsf{D}-k}\theta_1^{(1,\ldots,1)}(x_1,x_2) = g_k(x_1)\bar{\varphi}_{\widetilde{\lambda}_2}(x_2)\,, \quad \partial_2^{\mathsf{D}-k}\theta_2^{(2,\ldots,2)}(x_1,x_2) = \bar{\varphi}_{\widetilde{\lambda}_1}(x_1)g_k(x_2)\,.$$

By direction computation, i.e. simply integrating a difference of mollifiers, we have that $\widetilde{g}_k$

satisfies

$$\left\|D^N g_k\right\|_{L^1(\mathbb{R})} \lesssim_{\mathsf{D}} \widetilde{\lambda}_1^{-k}\mathcal{M}\left(N, k-1, \widetilde{\lambda}_1, \widetilde{\lambda}_2\right), \qquad \left\|D^N g_k\right\|_{L^\infty(\mathbb{R})} \lesssim_{\mathsf{D}} \widetilde{\lambda}_1^{1-k}\mathcal{M}\left(N, k-1, \widetilde{\lambda}_1, \widetilde{\lambda}_2\right), \qquad k \geq 1,$$
$$\left\|D^N g_0\right\|_{L^1(\mathbb{R})} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N}, \qquad \left\|D^N g_0\right\|_{L^\infty(\mathbb{R})} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N+1}.$$

Then we have the bounds

$$\left\|D^N \partial_1^{\mathsf{D}-k}\theta_1^{(1,\ldots,1)}\right\|_{L^1(\mathbb{R}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N}\widetilde{\lambda}_1^{-k}, \qquad \left\|D^N \partial_1^{\mathsf{D}-k}\theta_1^{(1,\ldots,1)}\right\|_{L^\infty(\mathbb{R}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N+2}\widetilde{\lambda}_1^{-k},$$
$$\left\|D^N \partial_2^{\mathsf{D}-k}\theta_2^{(2,\ldots,2)}\right\|_{L^1(\mathbb{R}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N}\widetilde{\lambda}_1^{-k}, \qquad \left\|D^N \partial_2^{\mathsf{D}-k}\theta_2^{(2,\ldots,2)}\right\|_{L^\infty(\mathbb{R}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N+1}\widetilde{\lambda}_1^{-k+1}.$$

Thus it follows by interpolation for $1/q' = 1 - 1/q$ that

$$\left\|D^N \partial_1^{\mathsf{D}-k}\theta_1^{(1,\ldots,1)}\right\|_{L^{q'}(\mathbb{R}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N+2/q}\widetilde{\lambda}_1^{-k}, \qquad \left\|D^N \partial_2^{\mathsf{D}-k}\theta_2^{(2,\ldots,2)}\right\|_{L^{q'}(\mathbb{R}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_2^{N+1/q}\widetilde{\lambda}_1^{-k+1}.$$

We therefore have that for $k = 0, \ldots, \mathsf{D}$,

$$\left\|D^N \partial_{i_1\cdots i_{\mathsf{D}-k}}(\Theta_{\widetilde{\rho}}^{\widetilde{\lambda}_1,\widetilde{\lambda}_2})^{(i_1,\cdots,i_{\mathsf{D}})}\right\|_{L^q(\mathbb{T}^2)} \lesssim \widetilde{\lambda}_1^{\mathsf{D}-k}\min\left(\widetilde{\lambda}, \widetilde{\lambda}_2\right)^N \mathcal{C}_{\rho,q}$$
$$\left\|D^N \partial_{i_1\cdots i_{\mathsf{D}-k}}(\Theta_{\widetilde{\rho}}^{\widetilde{\lambda}_1,\widetilde{\lambda}_2})^{(i_1,\cdots,i_{\mathsf{D}})}\right\|_{L^\infty(\mathbb{T}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_1^{\mathsf{D}-k}\min\left(\widetilde{\lambda}, \widetilde{\lambda}_2\right)^{N+2/q+\alpha} \mathcal{C}_{\rho,q},$$

where if $\widetilde{\lambda}_2 \leq \widetilde{\lambda}$, we let the derivatives fall on θ_i, and if $\widetilde{\lambda}_2 > \widetilde{\lambda}$, we let the derivatives fall on $\widetilde{\rho}$. Using the interpolation inequality, we obtain

$$\left\|D^N \partial_{i_1\cdots i_{\mathsf{D}-k}}(\Theta_{\widetilde{\rho}}^{\widetilde{\lambda}_1,\widetilde{\lambda}_2})^{(i_1,\cdots,i_{\mathsf{D}})}\right\|_{L^p(\mathbb{T}^2)} \lesssim_{\mathsf{D}} \widetilde{\lambda}_1^{\mathsf{D}-k}\min(\widetilde{\lambda}, \widetilde{\lambda}_2)^{N+2/q-2/p+\alpha}\mathcal{C}_{\rho,q}.$$

Undoing our original rescaling, we find that

$$\begin{aligned}\left\|D^N \partial_{i_1\cdots i_{\mathsf{D}-k}}(\Theta_\rho^{\lambda_1,\lambda_2})^{(i_1,\cdots,i_{\mathsf{D}})}\right\|_{L^p(\mathbb{T}^2)} &\lesssim_{\mathsf{D}} (\lambda r)^{N+\mathsf{D}-k}\left\|D^N\left[\partial_{i_1\cdots i_{\mathsf{D}-k}}(\Theta_{\widetilde{\rho}}^{\widetilde{\lambda}_1,\widetilde{\lambda}_2})^{(i_1,\cdots,i_{\mathsf{D}})}\right]\right\|_{L^p(\mathbb{T}^2)} \\ &\leq \left(\frac{\min(\lambda,\lambda_2)}{\lambda r}\right)^{\frac{2}{q}-\frac{2}{p}+\alpha}\mathcal{C}_{\rho,q}\lambda_1^{\mathsf{D}-k}\min(\lambda,\lambda_2)^N .\end{aligned}$$

□

Chapter 8

Non-inductive cutoffs

8.1 Time cutoffs

Let $\chi : (-1,1) \to [0,1]$ be a C^∞ function which induces a partition of unity according to

$$\sum_{k\in\mathbb{Z}} \chi^6(\cdot - k) \equiv 1\,. \tag{8.1}$$

Consider the translated and rescaled function

$$\chi\left(2t\tau_q^{-1}\Gamma_q^{i+2} - k\right)\,,$$

which is supported in the set of times t satisfying

$$\left|t - {}^1\!/\!_2\tau_q\Gamma_q^{-i-2}k\right| \leq {}^1\!/\!_2\tau_q\Gamma_q^{-i-2} \quad \iff t \in \left[(k-1){}^1\!/\!_2\tau_q\Gamma_q^{-i-2}, (k+1){}^1\!/\!_2\tau_q\Gamma_q^{-i-2}\right]\,. \tag{8.2}$$

We then define temporal cut-off functions

$$\chi_{i,k,q}(t) = \chi\left(2t\tau_q^{-1}\Gamma_q^{i+2} - k\right)\,. \tag{8.3}$$

It is then clear that

$$|\partial_t^m \chi_{i,k,q}| \lesssim (\Gamma_q^{i+2}\tau_q^{-1})^m \tag{8.4}$$

for $m \geq 0$ and

$$\chi_{i,k_1,q}(t)\chi_{i,k_2,q}(t) = 0 \tag{8.5}$$

for all $t \in \mathbb{R}$ unless $|k_1 - k_2| \leq 1$. In analogy to $\psi_{i\pm,q}$, we define

$$\chi_{i,k\pm,q}(t) := \left(\chi_{i,k-1,q}^6(t) + \chi_{i,k,q}^6(t) + \chi_{i,k+1,q}^6(t)\right)^{\frac{1}{6}} , \tag{8.6}$$

which are cutoffs with the property that

$$\chi_{i,k\pm,q} \equiv 1 \text{ on } \operatorname{supp}\left(\chi_{i,k,q}\right). \tag{8.7}$$

Next, we define the cutoffs $\widetilde{\chi}_{i,k,q}$ by

$$\widetilde{\chi}_{i,k,q}(t) = \chi\left(t\tau_q^{-1}\Gamma_q^i - k\Gamma_q^{-2}\right) . \tag{8.8}$$

For comparison with (8.2), we have that $\widetilde{\chi}_{i,k,q}$ is supported in the set of times t satisfying

$$\left|t - \tau_q\Gamma_q^{-i-2}k\right| \leq \tau_q\Gamma_q^{-i}. \tag{8.9}$$

Let (i,k) and (i^*,k^*) be such that $\operatorname{supp}\chi_{i,k,q} \cap \operatorname{supp}\chi_{i^*,k^*,q} \neq \emptyset$ and $i^* \in \{i-1, i, i+1\}$. Then as a consequence of these definitions and a sufficiently large choice of λ_0,

$$\operatorname{supp}\chi_{i,k,q} \subset \operatorname{supp}\widetilde{\chi}_{i^*,k^*,q}. \tag{8.10}$$

8.2 Estimates on flow maps

We can now make estimates regarding the flows of the vector field $\widehat{u}_{q'}$ for $q' \leq q + \bar{n} - 1$ on the support of a velocity and time cutoff function. This section is completely analogous to [7, Section 6.4], and we omit the proofs.

Lemma 8.2.1 (Lagrangian paths don't jump many supports). *Let $q' \leq q + \bar{n} - 1$ and (x_0, t_0) be given. Assume that the index i is such that $\psi^2_{i,q'}(x_0, t_0) \geq \kappa^2$, where $\kappa \in \left[\frac{1}{16}, 1\right]$. Then the forward flow $(X(t), t) := (X(x_0, t_0; t), t)$ of the velocity field $\widehat{u}_{q'}$ originating at (x_0, t_0) has the property that $\psi^2_{i,q'}(X(t), t) \geq \kappa^2/2$ for all t such that $|t - t_0| \leq \tau_{q'} \Gamma_{q'}^{-i+4}$.*

Corollary 8.2.2 (Backwards Lagrangian paths don't jump many supports). *Suppose (x_0, t_0) is such that $\psi^2_{i,q'}(x_0, t_0) \geq \kappa^2$, where $\kappa \in [1/16, 1]$. For $|t - t_0| \leq \tau_{q'} \Gamma_{q'}^{-i+3}$, define x to satisfy*

$$x_0 = X(x, t; t_0)\,.$$

That is, the forward flow X of the velocity field $\widehat{u}_{q'}$, originating at x at time t, reaches the point x_0 at time t_0. Then we have

$$\psi_{i,q'}(x, t) \neq 0\,.$$

Definition 8.2.3 (Flow maps). *We define $\Phi_{i,k,q'}(x, t) = \Phi_{(i,k)}(x, t)$ to be the flows induced by $\widehat{u}_{q'}$ with initial datum at time $k\tau_{q'}\Gamma_q^{-i-2}$ given by the identity, i.e.*

$$\begin{cases} (\partial_t + \widehat{u}_{q'} \cdot \nabla)\Phi_{i,k,q'} = 0 \\ \Phi_{i,k,q'}(x, k\tau_{q'}\Gamma_{q'}^{-i-2}) = x\,. \end{cases} \tag{8.11}$$

We will use $D\Phi_{(i,k)}$ to denote the gradient of $\Phi_{(i,k)}$ (which is a thus matrix-valued function). The inverse of the matrix $D\Phi_{(i,k)}$ is denoted by $\left(D\Phi_{(i,k)}\right)^{-1}$, in contrast to $D\Phi^{-1}_{(i,k)}$, which is the gradient of the inverse map $\Phi^{-1}_{(i,k)}$.

Corollary 8.2.4 (Deformation bounds). *For $k \in \mathbb{Z}$, $0 \le i \le i_{\max}$, $q' \le q + \bar{n} - 1$, and $2 \le N \le {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2} + 1$, we have the following bounds on the support of $\psi_{i,q'}(x,t)\widetilde{\chi}_{i,k,q'}(t)$.*

$$\left\| D\Phi_{(i,k)} - \mathrm{Id} \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \lesssim \Gamma_{q'}^{-1} \tag{8.12a}$$

$$\left\| D^N \Phi_{(i,k)} \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \lesssim \Gamma_{q'}^{-1} (\lambda_{q'}\Gamma_{q'})^{N-1} \tag{8.12b}$$

$$\left\| (D\Phi_{(i,k)})^{-1} - \mathrm{Id} \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \lesssim \Gamma_{q'}^{-1} \tag{8.12c}$$

$$\left\| D^{N-1} \left((D\Phi_{(i,k)})^{-1} \right) \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \lesssim \Gamma_{q'}^{-1} (\lambda_{q'}\Gamma_{q'})^{N-1} \tag{8.12d}$$

$$\left\| D^N \Phi_{(i,k)}^{-1} \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \lesssim \Gamma_{q'}^{-1} (\lambda_{q'}\Gamma_{q'})^{N-1} \tag{8.12e}$$

Furthermore, we have the following bounds for $1 \le N + M \le {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$ and $0 \le N' \le N$:

$$\left\| D^{N-N'} D_{t,q'}^{M} D^{N'+1} \Phi_{(i,k)} \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \le (\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i} \tau_{q'}^{-1}, \mathrm{T}_{q'-1}^{-1} \Gamma_{q'-1} \right) \tag{8.13a}$$

$$\left\| D^{N-N'} D_{t,q'}^{M} D^{N'} (D\Phi_{(i,k)})^{-1} \right\|_{L^\infty(\mathrm{supp}\,(\psi_{i,q'}\widetilde{\chi}_{i,k,q'}))} \le (\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'}^{i} \tau_{q'}^{-1}, \mathrm{T}_{q'-1}^{-1} \Gamma_{q'-1} \right) . \tag{8.13b}$$

8.3 Intermittent pressure cutoffs

In this section, we introduce cutoff functions for the level sets of π_ℓ. Estimates for π_ℓ are provided by (6.3a)–(6.3c).

8.3.1 Definition of the intermittent pressure cutoffs

We first introduce a partition of unity which is slightly more general than is needed at the moment; however, the generality will prove useful in the construction of the velocity cutoffs. The statement is almost identical to [7, Lemma 6.2]. The only slight difference is that (8.14) holds for the sixth power (the least common multiple of two and three, corresponding to cubic and quadratic error terms, respectively), and the estimates in (5) hold for arbitrary

integer powers of the cutoff functions. The more general bounds follow from the fact that since the cutoff functions are defined by gluing together exponential functions, raising to a power is (locally) equivalent to dilation.

Lemma 8.3.1. *For all $q \geq 1$ and $0 \leq m \leq \mathsf{N}_{\mathrm{cut,t}}$, there exist smooth cutoff functions $\widetilde{\gamma}_{m,q}, \gamma_{m,q} : [0,\infty) \to [0,1]$ which satisfy the following.*

(1) The function $\widetilde{\gamma}_{m,q}$ satisfies $\mathbf{1}_{[0,\frac{1}{4}\Gamma_q^{2(m+1)}]} \leq \widetilde{\gamma}_{m,q} \leq \mathbf{1}_{[0,\Gamma_q^{2(m+1)}]}$.

(2) The function $\gamma_{m,q}$ satisfies $\mathbf{1}_{[1,\frac{1}{4}\Gamma_q^{2(m+1)}]} \leq \gamma_{m,q} \leq \mathbf{1}_{[\frac{1}{4},\Gamma_q^{2(m+1)}]}$.

(3) For all $y \geq 0$, a partition of unity is formed as

$$\widetilde{\gamma}_{m,q}^6(y) + \sum_{i\geq 1} \gamma_{m,q}^6\big(\Gamma_q^{-2i(m+1)} y\big) = 1\,. \tag{8.14}$$

(4) $\widetilde{\gamma}_{m,q}$ and $\gamma_{m,q}(\Gamma_q^{-2i(m+1)}\cdot)$ satisfy

$$\begin{aligned} \operatorname{supp} \widetilde{\gamma}_{m,q}(\cdot) \cap \operatorname{supp} \gamma_{m,q}\big(\Gamma_q^{-2i(m+1)}\cdot\big) &= \emptyset \quad \text{if} \quad i \geq 2, \\ \operatorname{supp} \gamma_{m,q}\big(\Gamma_q^{-2i(m+1)}\cdot\big) \cap \operatorname{supp} \gamma_{m,q}\big(\Gamma_q^{-2i'(m+1)}\cdot\big) &= \emptyset \quad \text{if} \quad |i-i'| \geq 2\,. \end{aligned} \tag{8.15}$$

(5) For $0 \leq N \leq \mathsf{N}_{\mathrm{fin}}$, when $0 \leq y < \Gamma_q^{2(m+1)}$ we have

$$|D^N \widetilde{\gamma}_{m,q}(y)| \lesssim (\widetilde{\gamma}_{m,q}(y))^{1-N/\mathsf{N}_{\mathrm{fin}}} \Gamma_q^{-2N(m+1)}. \tag{8.16}$$

For $\frac{1}{4} < y < 1$ we have

$$|D^N \gamma_{m,q}(y)| \lesssim (\gamma_{m,q}(y))^{1-N/\mathsf{N}_{\mathrm{fin}}}\,, \tag{8.17}$$

while for $\frac{1}{4}\Gamma_q^{2(m+1)} < y < \Gamma_q^{2(m+1)}$ we have

$$|D^N \gamma_{m,q}(y)| \lesssim \Gamma_q^{-2N(m+1)} (\gamma_{m,q}(y))^{1-N/\mathsf{N}_{\mathrm{fin}}}\,. \tag{8.18}$$

In each of the above inequalities, the implicit constants depend on N but not m or q. If $\gamma_{m,q}$ or $\widetilde{\gamma}_{m,q}$ is replaced on the left hand side with $\gamma_{m,q}^p$, respectively $\widetilde{\gamma}_{m,q}^p$ for $p \in \mathbb{N}$, then a similar inequality holds after substituting the same power on the right-hand side and changing implicit constants.

We now introduce the intermittent pressure cut-off functions.

Definition 8.3.2 (Intermittent pressure cutoff functions). *For $j \geq 1$ the cut-off functions are defined by*

$$\omega_{j,q}(x,t) = \gamma_0\Big(\Gamma_q^{-2j}\,(\delta_{q+\bar{n}})^{-1}\pi_\ell(x,t)\Big)\,, \tag{8.19}$$

while for $j = 0$ we let

$$\omega_{0,q}(x,t) = \widetilde{\gamma}_0\Big((\delta_{q+\bar{n}})^{-1}\pi_\ell(x,t)\Big)\,, \tag{8.20}$$

where $\gamma_0 := \gamma_{0,q}$ and $\widetilde{\gamma}_0 := \widetilde{\gamma}_{0,q}$.

An immediate consequence of (8.14) with $m = 0$ is that $\{\omega_{j,q}^6\}_{j\geq 0}$ satisfies

$$\sum_{j\geq 0}\omega_{j,q}^6 = 1\,, \qquad \omega_{j,q}\omega_{j',q} \equiv 0 \quad \text{if} \quad |j-j'| > 1 \tag{8.21}$$

on $\mathbb{T}^3 \times \mathbb{R}$.

8.3.2 Estimates for intermittent pressure cutoffs

Lemma 8.3.3. *For all $m + k \leq \mathsf{N}_{\text{fin}}$ and $j \geq 0$, we have that*

$$\mathbf{1}_{\text{supp}\,(\omega_{j,q}\psi_{i,q})}|D^k D^m_{t,q}\pi_\ell(x,t)| \leq \Gamma_q^{2j+6}\delta_{q+\bar{n}}(\Gamma_q\Lambda_q)^k\mathcal{M}\left(m, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right) , \tag{8.22a}$$

$$1/4\delta_{q+\bar{n}}\Gamma_q^{2j} \leq \mathbf{1}_{\text{supp}\,(\omega_{j,q})}\pi_\ell \tag{8.22b}$$

$$1/8\sum_j \omega_{j,q}\delta_{q+\bar{n}}\Gamma_q^{2j} \leq \pi_\ell , \tag{8.22c}$$

$$\mathbf{1}_{\text{supp}\,(\omega_{j,q}\psi_{i,q})}|D^k D^m_{t,q}R_\ell(x,t)| \leq \Gamma_q^{2j-4}\delta_{q+\bar{n}}(\Gamma_q\Lambda_q)^k\mathcal{M}\left(m, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right) , \tag{8.22d}$$

$$\mathbf{1}_{\text{supp}\,(\omega_{j,q}\psi_{i,q})}|D^k D^m_{t,q}\varphi_\ell(x,t)| \leq \Gamma_q^{3j-7}\delta_{q+\bar{n}}^{\frac{3}{2}}r_q^{-1}(\Gamma_q\Lambda_q)^k\mathcal{M}\left(m, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right) . \tag{8.22e}$$

Proof. First, observe that by the construction of $\omega_{j,q}$, we have that for all $j \geq 0$,

$$\mathbf{1}_{\text{supp}\,(\omega_{j,q})}|\pi_\ell| = \mathbf{1}_{\text{supp}\,(\omega_{j,q})}\pi_\ell \leq \Gamma_q^{2(j+1)}\delta_{q+\bar{n}} . \tag{8.23}$$

Then, recalling the pointwise estimate (6.3c) and using (8.23), we have that

$$\begin{aligned}\mathbf{1}_{\text{supp}\,(\omega_{j,q})}|\psi_{i,q}D^k D^m_{t,q}\pi_\ell(x,t)| &\lesssim \mathbf{1}_{\text{supp}\,(\omega_{j,q})}\Gamma_q^3\pi_\ell(\Gamma_q\Lambda_q)^k\mathcal{M}\left(m, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right) \\ &\leq \Gamma_q^{2(j+3)}\delta_{q+\bar{n}}(\Gamma_q\Lambda_q)^k\mathcal{M}\left(m, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i\tau_q^{-1}, \mathrm{T}_q^{-1}\right) .\end{aligned}$$

To obtain the lower bounds on π_ℓ on the support of $\omega_{j,q}$, we appeal to (6.6) in the case $j = 0$, and the definition of $\omega_{j,q}$ in the case $j \geq 1$. Summing over j and appealing to (8.21) yields (8.22c). Next, we can obtain the pointwise estimates (8.22d) and (8.22e) for R_q^q and φ_q^q in a similar way by using (6.7a) and (6.7b), respectively. Finally, we obtain (8.22c) from (6.6), the definition of $\omega_{j,q}$ for $j \geq 0$, and . □

Corollary 8.3.4. *For $q \geq 0$, $0 \leq i \leq i_{\max}$, and $\alpha, \beta \in \mathbb{N}_0^k$ with $|\alpha| + |\beta| \leq \mathsf{N}_{\text{fin}}$, we have*

$$\left\| \left(\prod_{\ell=1}^{k} D^{\alpha_\ell} D_{t,q}^{\beta_\ell} \right) \pi_\ell \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q}\omega_{j,q}))} \lesssim \Gamma_q^{2j+6} \delta_{q+\bar{n}} (\Gamma_q \Lambda_q)^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\right) \tag{8.24a}$$

$$\left\| \left(\prod_{\ell=1}^{k} D^{\alpha_\ell} D_{t,q}^{\beta_\ell} \right) R_\ell \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q}\omega_{j,q}))} \lesssim \Gamma_q^{2j-4} \delta_{q+\bar{n}} (\Gamma_q \Lambda_q)^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\right) \tag{8.24b}$$

$$\left\| \left(\prod_{\ell=1}^{k} D^{\alpha_\ell} D_{t,q}^{\beta_\ell} \right) \varphi_\ell \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q}\omega_{j,q}))} \lesssim \Gamma_q^{3j-7} \delta_{q+\bar{n}}^{\frac{3}{2}} r_q^{-1} (\Gamma_q \Lambda_q)^{|\alpha|} \mathcal{M}\left(|\beta|, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\right) . \tag{8.24c}$$

Proof of Corollary 8.3.4. We only work on the estimate for π_ℓ because the estimates for R_q^q and φ_q^q can be obtained in a completely analogous way from Lemma 8.3.3 and Lemma A.2.3, Remark A.2.4. We then apply Lemma A.2.3 with $v = \widehat{u}_q$, $f = \pi_\ell$, $\Omega = \operatorname{supp} \psi_{i,q} \cap \operatorname{supp} \omega_{j,q}$, and $p = \infty$. In view of estimate (5.34) at level q, the assumption (A.27) holds with $\mathcal{C}_v = \tau_q^{-1} \Gamma_q^i \Lambda_q^{-1}$, $\lambda_v = \widetilde{\lambda}_v = \Lambda_q$, $N_x = \infty$, $\mu_v = \Gamma_q^i \tau_q^{-1}$, $\widetilde{\mu}_v = \Gamma_q^{-1} \mathrm{T}_q^{-1}$, and $N_t = \mathsf{N}_{\text{ind,t}}$. On the other hand, the bound (8.22a) implies assumption (A.28) with $\mathcal{C}_f = \Gamma_{q+1}^{2j+6} \delta_{q+\bar{n}}$, $\lambda_f = \widetilde{\lambda}_f = \Gamma_q \Lambda_q$, $\mu_f = \Gamma_q^i \tau_q^{-1}$, $\widetilde{\mu}_f = \mathrm{T}_q^{-1}$, and $N_t = \mathsf{N}_{\text{ind,t}}$. We then deduce from the bound (A.31) that (8.24a) holds, thereby concluding the proof. □

Lemma 8.3.5 (Maximal j index). *Fix $q \geq 0$. There exists a $j_{\max} = j_{\max}(q) \geq 1$, determined by the formula*

$$j_{\max} = \inf \left\{ j \, : \, \frac{1}{4} \Gamma_q^{2j} \delta_{q+\bar{n}} \geq \Gamma_q^{3+\mathsf{C}_\infty} \right\} \tag{8.25}$$

and which is bounded independently of q, such that

$$\omega_{j,q} \equiv 0 \qquad \textit{for all} \qquad j > j_{\max} . \tag{8.26}$$

Moreover, we have the bound

$$\Gamma_q^{2j_{\max}} \leq \delta_{q+\bar{n}}^{-1} \Gamma_q^{\mathsf{C}_\infty+6} . \tag{8.27}$$

Proof of Lemma 8.3.5. The proof of (8.26) follows immediately from the definition in (8.25), the bound (8.22a), and the bound (6.3b), where the extra factor of Γ_q absorbs the implicit constant in (6.3b). Checking that $j_{\max}$ is independent of q is a simple calculation, as is the bound in (8.27). □

Lemma 8.3.6 (Derivative bounds). *For $q \geq 0$, $0 \leq i \leq i_{\max}$, $0 \leq j \leq j_{\max}$, and $N + M \leq \mathsf{N}_{\text{fin}}$, we have*

$$\frac{\mathbf{1}_{\operatorname{supp}\psi_{i,q}} |D^N D_{t,q}^M \omega_{j,q}|}{\omega_{j,q}^{1-(N+M)/\mathsf{N}_{\text{fin}}}} \lesssim (\Gamma_q^5 \Lambda_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_q^{i+4} \tau_q^{-1}, \mathrm{T}_q^{-1}\right) . \tag{8.28}$$

Proof of Lemma 8.3.6. We shall apply the mixed-derivative Fa'a di Bruno formula from [7, Lemma A.5] with the following choices, where we use the parameter names from there:

$$\begin{aligned}
&\psi = \gamma_0 \text{ or } \widetilde{\gamma}_0 , \quad \Gamma_\psi = \Gamma_q , \quad v = \widehat{u}_q , \\
&\Gamma = \delta_{q+\bar{n}}^{1/2} \Gamma_q^{-j} , \quad \lambda = \widetilde{\lambda} = \Lambda_q \Gamma_q , \quad \mu = \tau_q^{-1} \Gamma_q^i , \quad \widetilde{\mu} = \mathrm{T}_q^{-1} , \\
&N_x = \infty , \quad N_t = \mathsf{N}_{\text{ind,t}} , \quad h = \pi_\ell , \quad \mathcal{C}_h = \delta_{q+\bar{n}} \Gamma_q^{2j+6} .
\end{aligned}$$

The assumption [7, A.24] is verified due to (8.16)–(8.18), and [7, (A.25)] is verified due to (8.24a), which holds on the support of $\omega_{j,q}\psi_{i,q}$. From conclusion [7, (A.26)] and the equality $(\Gamma_\psi \Gamma)^{-2} \mathcal{C}_h = \Gamma_q^4$, we find that (8.28) holds; note that for the $N = M = 0$ case, we just use the fact that $\omega_{j,q} \leq 1$ rather than incur the loss $\mathcal{C}_h \Gamma^{-2}$ from [7, (A.26)]. □

Lemma 8.3.7 (Support bounds). *For any $r \geq 3/2$ and $0 \leq j \leq j_{\max}$, we have that*

$$\|\omega_{j,q}\|_{L^r} \lesssim \Gamma_q^{\frac{3(1-j)}{r}} . \tag{8.29}$$

Proof of Lemma 8.3.7. We prove only the case $r = 3/2$, at which point the remaining estimates follow from Lebesgue interpolation and the fact that $\omega_{j,q} \leq 1$ for all j, q. For $j = 0, 1$ the estimate is trivial from the pointwise bound for $\omega_{j,q}$, and so we consider now $j \geq 2$. Using Chebyshev's inequality, (6.3a), and (8.22b), we have that

$$\begin{aligned}
\|\omega_{j,q}\|_{3/2}^{3/2} &\leq \sup_{t\in\mathbb{R}} \int_{\mathbb{T}^3} \mathbf{1}_{\{\pi_\ell(t,\cdot)\geq 1/4\delta_{q+\bar{n}}\Gamma_q^{2j}\}} \\
&\lesssim \frac{\|\pi_\ell\|_{3/2}^{3/2}}{\delta_{q+\bar{n}}^{3/2}\Gamma_q^{3j}} \\
&\lesssim \Gamma_q^{3(1-j)} .
\end{aligned}$$

□

8.4 Mildly and strongly anisotropic checkerboard cutoffs

We first construct mildly anisotropic checkerboard cutoff functions which are well-suited for intermittent pipe flows with axes parallel to e_1. The construction for general $\xi \in \Xi$ follows by rotation. We include all the details since the power for which the partition is summable to 1 is absolutely crucial for the definition of the perturbation in (9.8) and its estimates in Lemma 9.3.1, and the Reynolds and current oscillation errors in subsections 10.2.1 and 11.2.1, respectively.

`Step 1: Partitioning the space perpendicular to` x_1. Consider a partition of $\mathbb{T}^2_{x_2,x_3}$ into the squares defined using the periodized base square

$$\left\{(x_2, x_3) \in \mathbb{T}^2 \,:\, 0 \leq x_2, x_3 \leq \frac{\pi}{8}\Gamma_q^5 \left(\lambda_{q+1}\right)^{-1}\right\} \tag{8.30}$$

and its periodized translations by

$$\left(l_2 \cdot {}^{\pi}\!/_{8} \cdot \Gamma_q^5 (\lambda_{q+1})^{-1}, l_3 \cdot {}^{\pi}\!/_{8} \cdot \Gamma_q^5 (\lambda_{q+1})^{-1}\right)$$

for

$$l_2, l_3 \in \{0, \ldots, 16\Gamma_q^{-5}\lambda_{q+1} - 1\}\,.$$

Note that the periodized squares evenly partition $[-\pi, \pi]^2$. We let $l^\perp := (l_2, l_3)$ be an ordered pair using the indices defined above, and choose $\{\mathcal{X}_{q,e_1,l^\perp}\}_{l^\perp}$ to be a C^∞ partition of unity adapted to these periodized squares such that

$$\sum_{l^\perp} \mathcal{X}^2_{q,e_1,l^\perp}(x_2, x_3) \equiv 1, \quad \forall (x_2, x_3) \in \mathbb{T}^2_{x_2,x_3}\,, \quad \mathcal{X}_{q,e_1,l^\perp}\mathcal{X}_{q,e_1,\widetilde{l}^\perp} \equiv 0 \quad \text{if } |l_2 - \widetilde{l}_2| > 1 \ |l_3 - \widetilde{l}_3| > 1\,, \tag{8.31a}$$

$$\operatorname{supp} \mathcal{X}_{q,e_1,l_0^\perp} = [-{}^{1}\!/_{8}\Gamma_q^5\lambda_{q+1}^{-1}, {}^{5}\!/_{8}\Gamma_q^5\lambda_{q+1}^{-1}]^2 \qquad \text{for } l_0^\perp = (0,0)\,. \tag{8.31b}$$

We shall later need that

$$\left\langle \sum_{l^\perp} \chi^3_{q,e_1,l^\perp}(x_2, x_3) \right\rangle = c_3\,, \tag{8.32}$$

where the constant c_3 is geometric and bounded independently of q.

`Step 2: Partitioning the space parallel to` x_1. Next, consider a partition of $\mathbb{T}_{x_1}$ into the line segments defined using the base line segment

$$\left\{x_1 \in \mathbb{T} \,:\, 0 \le x_1 \le \frac{\pi}{8}\lambda_q^{-1}\Gamma_q^{-8}\right\} \tag{8.33}$$

and its translations by

$$l \cdot {}^{1}\!/_{2} \cdot \lambda_q^{-1}\Gamma_q^{-8}\,, \qquad l \in \{0, \ldots, 16\lambda_q^{-1}\Gamma_q^{-8} - 1\}\,.$$

Note that the segments evenly partition $[-\pi, \pi]$. Choose $\{\mathcal{X}_{q,e_1,l}\}_l$ to be a C^∞ partition of

unity adapted to these segments such that for $N \leq 3\mathsf{N}_{\text{fin}}$,

$$\sum_{l} \mathcal{X}^6_{q,e_1,l}(x_1) \equiv 1 \quad \forall (x_1) \in \mathbb{T}_{x_1}\,, \quad \mathcal{X}_{q,e_1,l}\mathcal{X}_{q,e_1,\widetilde{l}} \equiv 0 \quad \text{if } |l-\widetilde{l}| > 1\,, \qquad \left|D^N \mathcal{X}_{q,\xi',l'}\right| \lesssim (\lambda_q \Gamma_q^8)^N\,, \tag{8.34a}$$

$$\operatorname{supp}(\mathcal{X}_{q,e_1,0}) = [-{}^1\!/\!_8 \lambda_q^{-1}\Gamma_q^{-8}, {}^5\!/\!_8 \lambda_q^{-1}\Gamma_q^{-8}]\,. \tag{8.34b}$$

`Step 3: Reynolds cutoffs.` Combining $l, l^\perp$ into integer triples $\vec{l} = (l, l_2, l_3) = (l, l^\perp)$, we now have a division of $\mathbb{T}^3$ into rectangular prisms indexed by $\vec{l}$. We define

$$\mathcal{X}_{q,e_1,\vec{l},R}(x_1,x_2,x_3) = \mathcal{X}^3_{q,e_1,l}(x_1)\mathcal{X}_{q,e_1,l^\perp}(x_2,x_3)$$

and note that

$$\sum_{\vec{l}} \mathcal{X}^2_{q,e_1,\vec{l},R}(x_1,x_2,x_3) \equiv 1 \qquad \forall\,(x_1,x_2,x_3) \in \mathbb{T}^3\,.$$

`Step 4: Current cutoffs.` We combine $l, l^\perp$ into integer triples $\vec{l}$ as above but now define

$$\mathcal{X}_{q,e_1,\vec{l},\varphi}(x_1,x_2,x_3) = \mathcal{X}^2_{q,e_1,l}(x_1)\mathcal{X}_{q,e_1,l^\perp}(x_2,x_3)$$

and note that for each fixed value of $l = l_0$,

$$\sum_{\vec{l}\,:\,l=l_0} \mathcal{X}^2_{q,e_1,\vec{l},\varphi}(x_1,x_2,x_3) \equiv \mathcal{X}^4_{q,e_1,l_0}(x_1) \qquad \forall\,(x_1,x_2,x_3) \in \mathbb{T}^3\,.$$

Conversely, for each fixed value of $l^\perp = l_0^\perp$, we have that

$$\sum_{\vec{l}\,:\,l^\perp=l_0^\perp} \mathcal{X}^3_{q,e_1,\vec{l},\varphi}(x_1,x_2,x_3) \equiv \mathcal{X}^3_{q,e_1,l_0^\perp}(x_2,x_3)\,.$$

With the time-independent cutoffs in hand, we define the time-dependent cutoff which is adapted to the flows of the velocity field $\widehat{u}_q$.

Definition 8.4.1 (Mildly anisotropic checkerboard cutoff functions). *Given q, $\xi \in \Xi$, $i \leq i_{\max}$, and $k \in \mathbb{Z}$, we define*

$$\zeta_{q,\diamond,i,k,\xi,\vec{l}}(x,t) = \mathcal{X}_{q,\xi,\vec{l},\diamond}\left(\Phi_{i,k,q}(x,t)\right) . \tag{8.35}$$

These cutoff functions satisfy properties which we enumerate in the following lemma.

Lemma 8.4.2. *The cutoff functions $\{\zeta_{q,\diamond,i,k,\xi,\vec{l}}\}_{\vec{l}}$ satisfy the following properties.*

(i) The material derivative $D_{t,q}(\zeta_{q,\diamond,i,k,\xi,\vec{l}})$ vanishes.

(ii) We have the summability properties for all $(x,t) \in \mathbb{T}^3 \times \mathbb{R}$;

$$\sum_{\vec{l}} \left(\zeta_{q,R,i,k,\xi,\vec{l}}(x,t)\right)^2 \equiv 1 , \tag{8.36a}$$

$$\sum_{\vec{l}\,:\, l = l_0} \zeta^2_{q,\varphi,i,k,\xi,\vec{l}}(x,t) \equiv \mathcal{X}^4_{q,\xi,l_0}(\Phi_{i,k,q}(x,t)) , \tag{8.36b}$$

$$\sum_{\vec{l}\,:\, l^\perp = l_0^\perp} \zeta^3_{q,\varphi,i,k,\xi,\vec{l}}(x_1,x_2,x_3) = \mathcal{X}^3_{q,\xi,l_0^\perp}(\Phi_{i,k,q}(x,t)) . \tag{8.36c}$$

(iii) Let $A = (\nabla \Phi_{(i,k)})^{-1}$. Then we have the spatial derivative estimate

$$\begin{aligned} \left\| D^{N_1} D^M_{t,q} (\xi^\ell A^j_\ell \partial_j)^{N_2} \zeta_{q,\diamond,i,k,\xi,\vec{l}} \right\|_{L^\infty\left(\operatorname{supp} \psi_{i,q} \widetilde{\chi}_{i,k,q}\right)} &\lesssim \left(\Gamma_q^{-5}\lambda_{q+1}\right)^{N_1} \left(\Gamma_q^{8}\lambda_q\right)^{N_2} \\ &\quad \times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^{-1}\right) . \end{aligned} \tag{8.37}$$

for all $N_1 + N_2 + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2 + 1$.

(iv) There exists an implicit dimensional constant $\mathcal{C}_\chi$ independent of q, k, i, and $\vec{l}$ such that for all $(x,t) \in \operatorname{supp} \psi_{i,q} \widetilde{\chi}_{i,k,q}$, the support of $\zeta_{q,\diamond,i,k,\xi,\vec{l}}(\cdot,t)$ satisfies

$$\operatorname{diam}(\operatorname{supp}\left(\zeta_{q,\diamond,i,k,\xi,\vec{l}}(\cdot,t)\right)) \lesssim \Gamma_q^{-8}\lambda_q^{-1} . \tag{8.38}$$

Proof of Lemma 8.4.2. The proof of (i) is immediate from (8.35). The first equality in (8.36) follows from (i) and the definition of the Reynolds cutoffs in Step 3 above. The second and third equalities follow from (i) and the definition of the current cutoffs in Step 4 above. To verify (iii), the only nontrivial calculations are those including the differential operator $\xi^\ell A^j_\ell \partial_j$. Using the Leibniz rule, the contraction

$$\xi^\ell A^j_\ell \partial_j \zeta_{q,\diamond,i,k,\xi,\vec{l}} = \xi^\ell A^j_\ell (\partial_m \mathcal{X}_{q,\xi,\vec{l},\diamond})(\Phi_{i,k,q}) \partial_j \Phi^m_{i,k,q} = \xi^m (\partial_m \mathcal{X}_{q,\xi,\vec{l},\diamond})(\Phi_{i,k,q}) \,,$$

the diameter of the cutoffs defined in Steps 1 and 2 above, and (8.13a)–(8.13b) gives the desired estimate. The proof of (8.38) follows from the construction of $\mathcal{X}_{q,\xi,\vec{l},\diamond}$ and the Lipschitz bound obeyed by $\widehat{u}_q$ on the support of $\psi_{i,q}$; see for example (7.18). □

We may similarly obtain estimates on the flowed cutoff functions $\boldsymbol{\zeta}^I_\xi$ which come from Definition 7.2.4. The proof is quite similar to the one above, and we omit the details.

Lemma 8.4.3 (**Strongly anisotropic checkerboard cutoff function).** *The cutoff functions $\boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)}$ satisfy the following properties:*

(1) The material derivative $D_{t,q}(\boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)})$ vanishes.

(2) For all fixed values of q, i, k, ξ, each $t \in \mathbb{R}$, and all $x = (x_1, x_2, x_3) \in \mathbb{T}^3$,

$$\sum_I (\boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)})^6(x,t) = 1 \,. \tag{8.39}$$

(3) Let $A = (\nabla \Phi_{(i,k)})^{-1}$. Then we have the spatial derivative estimate

$$\left\| D^{N_1} D^M_{t,q} (\xi^\ell A^j_\ell \partial_j)^{N_2} \boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)} \right\|_{L^\infty\left(\operatorname{supp} \psi_{i,q} \widetilde{\chi}_{i,k,q}\right)} \lesssim \lambda^{N_1}_{q+\lfloor \bar{n}/2 \rfloor} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^i_q \tau_q^{-1}, \mathrm{T}_q^{-1} \Gamma_q^{-1}\right) \,. \tag{8.40}$$

for all $N_1 + N_2 + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2 + 1$.

(4) There exists an implicit dimensional constant $\mathcal{C}_\chi$ independent of q, k, i, and ξ such that for all $(x,t) \in \operatorname{supp} \psi_{i,q}\widetilde{\chi}_{i,k,q}$, the support of $\boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)}(\cdot,t)$ satisfies

$$\operatorname{diam}(\operatorname{supp}(\boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)}(\cdot,t))) \lesssim \Gamma_q^{-8}\lambda_q^{-1}\,. \tag{8.41}$$

We also need the following lemma that bounds the cardinality of these anisotropic cut-offs.

Lemma 8.4.4. *For fixed q, i, k, ξ, we have that*

$$\#\left\{(\vec{l}, I) : \operatorname{supp}\left(\zeta_{q,i,k,\xi,\vec{l}}\boldsymbol{\zeta}^I_\xi \circ \Phi_{(i,k)}\right) \neq \emptyset\right\} \lesssim \Gamma_q^8 \lambda_q \lambda_{q+\bar{n}/2}^2\,. \tag{8.42}$$

Proof. Note first that for a fixed I, there are at most 4 values of $l_0^\perp$ such that $\operatorname{supp}(\mathcal{X}_{q,\xi,l_0^\perp}\boldsymbol{\zeta}^I_\xi) \neq \emptyset$. Also note that for a fixed $l_0^\perp$, we have $\#\{\vec{l} : l^\perp = l_0^\perp\} \lesssim \lambda_q\Gamma_q^8$. Putting these together along with the bound on the number of I given by Remark 7.2.5, we get that

$$\#\{(\vec{l}, I) : \operatorname{supp}(\mathcal{X}_{q,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^I_\xi) \neq \emptyset\} \lesssim \Gamma_q^8 \lambda_q \lambda_{q+\bar{n}/2}^2\,.$$

Now the desired conclusion follows as all these cut-offs are flowed by the same $\Phi_{(i,k)}$. □

8.5 Definition of the cumulative cutoff function

Finally, combining the cutoff functions defined in Definition 12.1.4, Definition 8.3.2, (8.3), and the previous subsection, we define the cumulative cutoff functions by

$$\eta_{i,j,k,\xi,\vec{l},\diamond}(x,t) = \psi^\diamond_{i,q}(x,t)\omega^\diamond_{j,q}(x,t)\chi^\diamond_{i,k,q}(t)\zeta_{q,\diamond,i,k,\xi,\vec{l}}(x,t)\,, \tag{8.43}$$

where the $\diamond$ in the superscript of the first three functions is equal to 2 if $\diamond = \varphi$ (so that they are cubic-summable to 1) and 3 if $\diamond = R$ (so that they are square-summable to 1). We

conclude this section with estimates on the L^p norms of the cumulative cutoff functions.

Lemma 8.5.1 (Cumulative support bounds for cutoff functions). *For $r_1, r_2 \in [1,\infty]$ with $\frac{1}{r_1}+\frac{1}{r_2}=1$ and any $0 \le i \le i_{\max}$, $0 \le j, \le j_{\max}$, $\xi \in \Xi, \Xi'$, and $\diamond = \varphi, R$, we have that for each t,*

$$\sum_{\vec{l}} \left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond}(t,x) \right) \right| \lesssim \Gamma_q^{\frac{-3i+\mathsf{C}_b}{r_1}+\frac{-3j}{r_2}+3} . \tag{8.44}$$

We furthermore have that

$$\sum_{i,j,k,\xi,\vec{l},I,\diamond} \mathbf{1}_{\operatorname{supp}\eta_{i,j,k,\xi,\vec{l},\diamond}} \boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I}_{\xi} \approx \sum_{i,j,k,\xi,\vec{l},\diamond} \mathbf{1}_{\operatorname{supp}\eta_{i,j,k,\xi,\vec{l},\diamond}} \boldsymbol{\rho}^{\diamond}_{(\xi)} \lesssim 1 . \tag{8.45}$$

Proof of Lemma 8.5.1. We shall prove the first bound for $\diamond = \varphi$. Then from (8.43), the only differences between $\diamond = R$ and $\diamond = \varphi$ are the powers to which various cutoff functions are raised, and so we shall omit the proof for $\diamond = R$. To prove the bound for $\diamond = \varphi$, we have that

$$\begin{aligned}
\sum_{\vec{l}} \left| \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\varphi} \right| &\lesssim \left\| (\psi^6_{i-1,q}+\psi^6_{i,q}+\psi^6_{i+1,q})^{1/6} (\omega^6_{j-1,q}+\omega^6_{j,q}+\omega^6_{j+1,q})^{1/6} \right\|_{L^1} \\
&\lesssim \left\| (\psi^6_{i-1,q}+\psi^6_{i,q}+\psi^6_{i+1,q})^{1/6} \right\|_{L^{r_1}} \left\| (\omega^6_{j-1,q}+\omega^6_{j,q}+\omega^6_{j+1,q})^{1/6} \right\|_{L^{r_2}} \\
&\lesssim \Gamma_q^{\frac{-3(i-1)+\mathsf{C}_b}{r_1}} \Gamma_q^{\frac{-3(j-1)}{r_2}} .
\end{aligned}$$

To achieve the final inequality, we have used interpolation, (5.13) at level q, and (8.29). Using that $\frac{1}{r_1}+\frac{1}{r_2}=1$ gives the desired estimate. Finally, to prove (8.45), we appeal to (5.8) at level q, (8.1) and (8.5), (8.21), item (ii) from Proposition 7.2.1, Definition 7.2.4, and Lemma 8.4.2. □

8.6 Cutoff aggregation lemmas

Corollary 8.6.1 (**Aggregated L^p estimates**). *Let $\theta \in (0,3]$, and $\theta_1, \theta_2 \geq 0$ with $\theta_1 + \theta_2 = \theta$. Let $H = H_{i,j,k,\xi,\vec{l},\diamond}$ or $H = H_{i,j,k,\xi,\vec{l},I,\diamond}$ be a function with*

$$\operatorname{supp} H_{i,j,k,\xi,\vec{l},\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \qquad \text{or} \qquad \operatorname{supp} H_{i,j,k,\xi,\vec{l},I,\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \circ \Phi_{(i,k)} \,. \tag{8.46}$$

Let $p \in [1,\infty)$ and let $\theta_1, \theta_2 \in [0,3]$ be such that $\theta_1 + \theta_2 = {}^3\!/_p$. Assume that there exists $\mathcal{C}_H, N_*, M_*, N_x, M_t$ and $\lambda, \Lambda, \tau, \mathrm{T}$ such that

$$\begin{aligned} \left\| D^N D^M_{t,q} H_{i,j,k,\xi,\vec{l},\diamond} \right\|_{L^p} \lesssim \sup_{t\in\mathbb{R}} \left(\left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond}(t,x) \right) \right|^{1/p} \right) \\ \times \mathcal{C}_H \Gamma_q^{\theta_1 i + \theta_2 j} \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) \end{aligned} \tag{8.47a}$$

$$\begin{aligned} \left\| D^N D^M_{t,q} H_{i,j,k,\xi,\vec{l},I,\diamond} \right\|_{L^p} \lesssim \sup_{t\in\mathbb{R}} \left(\left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \circ \Phi_{(i,k)}(t,x) \right) \right|^{1/p} \right) \\ \times \mathcal{C}_H \Gamma_q^{\theta_1 i + \theta_2 j} \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) \end{aligned} \tag{8.47b}$$

for $N \leq N_*, M \leq M_*$. Then in the same range of N and M,

$$\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},\diamond} \right\|_{L^p} \lesssim \Gamma_q^{3+\theta_1 \mathsf{C}_b} \mathcal{C}_H \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^{i+1}, \mathrm{T}^{-1}\right) \tag{8.48a}$$

$$\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right\|_{L^p} \lesssim \Gamma_q^{3+\theta_1 \mathsf{C}_b} \mathcal{C}_H \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^{i+1}, \mathrm{T}^{-1}\right) . \tag{8.48b}$$

Proof. We prove only (8.48b), as (8.48a) is slightly easier and follows the same method. Using (8.46), (5.8) at level q, (8.47b), Lemma 8.5.1 with $r_1 = \frac{3}{p\theta_1}, r_2 = \frac{3}{p\theta_2}$, $\theta_1 + \theta_2 = {}^3\!/_p$, we

may write that

$$
\begin{aligned}
&\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right\|_p^p \\
&\quad \leq \sup_{t\in\mathbb{R}} \int_{\mathbb{T}^3} \psi_{i,q} \left| \sum_{\substack{i-1\leq i'\leq i+1 \\ j,k,\xi,\vec{l},I,\diamond}} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right|^p (t,x)\, dx \\
&\quad \leq \sup_{t\in\mathbb{R}} \sum_{\substack{i-1\leq i'\leq i+1 \\ j,k,\xi,\vec{l},I,\diamond}} \left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \circ \Phi_{(i,k)}(t,x) \right) \right| \mathcal{C}_H^p \Gamma_q^{p\theta_1 i + p\theta_2 j} \\
&\qquad\qquad \times \left(\mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right)\right)^p \\
&\quad \lesssim \sup_{t\in\mathbb{R}} \sum_{\substack{i-1\leq i'\leq i+1 \\ j,k,\xi,\vec{l},\diamond}} \left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond}(t,x) \right) \right| \mathcal{C}_H^p \Gamma_q^{p\theta_1 i + p\theta_2 j} \left(\mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(N, N_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right)\right)^p \\
&\quad \leq \mathcal{C}_H^p \Gamma_q^{p\theta_1 \mathsf{C}_b + 3p} \left(\mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right)\right)^p ,
\end{aligned}
$$

concluding the proof. □

Remark 8.6.2 (**Aggregated L^1 estimates with Γ_q^i**). Assume that (8.46)–(8.47b) hold for $p = 3/2$, but with $\mathcal{C}_H = \Gamma_q^i \widetilde{\mathcal{C}}_H$. Then we can obtain the L^1 estimates

$$
\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},\diamond} \right\|_1 \lesssim \widetilde{\mathcal{C}}_H \Gamma_q^{2\mathsf{C}_b+3} \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) \tag{8.49a}
$$

$$
\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right\|_1 \lesssim \widetilde{\mathcal{C}}_H \Gamma_q^{2\mathsf{C}_b+3} \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) . \tag{8.49b}
$$

Indeed, considering (8.49b), we have

$$
\begin{aligned}
&\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right\|_1 \\
&\quad \leq \sup_{t\in\mathbb{R}} \sum_{\substack{i-1\leq i'\leq i+1 \\ j,k,\xi,\vec{l},I,\diamond}} \int_{\mathbb{T}^3} \psi_{i,q} \mathbf{1}_{\operatorname{supp}_x\left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond} \circ \Phi_{(i,k)}\right)} \left| D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right| (t,x)\, dx \\
&\quad \leq \sup_{t\in\mathbb{R}} \left[\sum_{\substack{i-1\leq i'\leq i+1 \\ j,k,\xi,\vec{l},I,\diamond}} \Gamma_q^{3i} \left\| \psi_{i,q} \mathbf{1}_{\operatorname{supp}_x\left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond} \circ \Phi_{(i,k)}\right)} \right\|_3^3 \right]^{1/3} \left[\sum_{\substack{i-1\leq i'\leq i+1 \\ j,k,\xi,\vec{l},I,\diamond}} \Gamma_q^{-3/2 i} \left\| D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond} \right\|_{3/2}^{3/2} \right]^{2/3} \\
&\quad \lesssim \sup_{t\in\mathbb{R}} \left[\sum_{j,k,\xi,\vec{l},\diamond} \left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond}(t,x) \right) \right| \Gamma_q^{3i} \right]^{1/3} \left[\sum_{j,k,\xi,\vec{l},\diamond} \left| \operatorname{supp}_x \left(\eta_{i,j,k,\xi,\vec{l},\diamond}(t,x) \right) \right| \Gamma_q^{3/2(\theta_1 i + \theta_2 j)} \right]^{2/3} \\
&\qquad\qquad \cdot \widetilde{\mathcal{C}}_H \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) \\
&\quad \leq \widetilde{\mathcal{C}}_H \Gamma_q^{2\mathsf{C}_b+3} \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) .
\end{aligned}
$$

In the last inequality, we used Lemma 8.5.1 with $r_1 = 1, r_2 = \infty$ and with $r_1 = \frac{3}{p\theta_1}, r_2 = \frac{3}{p\theta_2}$, and $\theta_1 + \theta_2 = {}^3\!/_p$.

We now state two similar corollaries which allow us to aggregate pointwise estimates.

Corollary 8.6.3 (**Aggregated pointwise estimates**). *Let $H = H_{i,j,k,\xi,\vec{l},\diamond}$ or $H = H_{i,j,k,\xi,\vec{l},I,\diamond}$ be a function with*

$$
\operatorname{supp} H_{i,j,k,\xi,\vec{l},\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \qquad \text{or} \qquad \operatorname{supp} H_{i,j,k,\xi,\vec{l},I,\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond} \circ \Phi_{(i,k)} \tag{8.50}
$$

and let $\varpi = \varpi_{i,j,k,\xi,\vec{l},\diamond}$ or $\varpi = \theta_{i,j,k,\xi,\vec{l},I,\diamond}$ be a non-negative function such that

$$
\operatorname{supp} \varpi_{i,j,k,\xi,\vec{l},\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \qquad \text{or} \qquad \operatorname{supp} \varpi_{i,j,k,\xi,\vec{l},I,\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond} \circ \Phi_{(i,k)} \tag{8.51}
$$

Let $p \in (0,\infty)$ and assume that there exists λ, Λ, τ such that

$$|D^N D_{t,q} H_{i,j,k,\xi,\vec{l},\diamond}| \lesssim \varpi^p_{i,j,k,\xi,\vec{l},\diamond} \mathcal{M}(N, N_x, \lambda, \Lambda) \mathcal{M}\left(N, N_t, \tau^{-1}\Gamma^i_q, \mathrm{T}^{-1}\right) \tag{8.52a}$$

$$|D^N D_{t,q} H_{i,j,k,\xi,\vec{l},I,\diamond}| \lesssim \varpi^p_{i,j,k,\xi,\vec{l},I,\diamond} \mathcal{M}(N, N_x, \lambda, \Lambda) \mathcal{M}\left(N, N_t, \tau^{-1}\Gamma^i_q, \mathrm{T}^{-1}\right) \tag{8.52b}$$

for $N \leq N_*, M \leq M_*$. Then in the same range of N and M,

$$\left|\psi_{i,q} \sum_{i',j,k,\xi,\vec{l},\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},\diamond}\right| \lesssim \left(\sum_{i,j,k,\xi,\vec{l},\diamond} \varpi_{i,j,k,\xi,\vec{l},\diamond}\right)^p \mathcal{M}(N, N_x, \lambda, \Lambda) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma^{i+1}_q, \mathrm{T}^{-1}\right) \tag{8.53a}$$

$$\left|\psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond}\right| \lesssim \left(\sum_{i,j,k,\xi,\vec{l},I,\diamond} \varpi_{i,j,k,\xi,\vec{l},I,\diamond}\right)^p \mathcal{M}(N, N_x, \lambda, \Lambda) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma^{i+1}_q, \mathrm{T}^{-1}\right). \tag{8.53b}$$

Corollary 8.6.4 (Aggregated pointwise estimates with Γ^i_q). *Let $H = H_{i,j,k,\xi,\vec{l},I,\diamond}$ be a function with*

$$\operatorname{supp} H_{i,j,k,\xi,\vec{l},\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \qquad \text{or} \qquad \operatorname{supp} H_{i,j,k,\xi,\vec{l},I,\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_\xi \circ \Phi_{(i,k)} \tag{8.54}$$

and let ϖ be a non-negative function and assume that there exists $\lambda, \Lambda, \tau, \mathrm{T}$ such that for $H = H_{i,j,k,\xi,\vec{l},\diamond}$ or $H_{i,j,k,\xi,\vec{l},I,\diamond}$

$$\left|D^N D^M_{t,q} H\right| \lesssim \tau_q^{-1}\Gamma^i_q \psi_{i,q} \varpi \mathcal{M}(N, N_x, \lambda, \Lambda) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma^i_q, \mathrm{T}^{-1}\right) \tag{8.55a}$$

for $N \leq N_*, M \leq M_*$. Then in the same range of N and M,

$$\left|\psi_{i,q} \sum_{i',j,k,\xi,\vec{l},\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},\diamond}\right| \lesssim \Gamma_q r_q^{-1} \lambda_q \left(\pi_q^q\right)^{1/2} \varpi \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^{i+1}, \mathrm{T}^{-1}\right) \tag{8.56a}$$

$$\left|\psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond}\right| \lesssim \Gamma_q r_q^{-1} \lambda_q \left(\pi_q^q\right)^{1/2} \varpi \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^{i+1}, \mathrm{T}^{-1}\right) . \tag{8.56b}$$

Proofs of Corollaries 8.6.3 and 8.6.4. We will give the full details for estimate (8.56b) from Corollary 8.6.4, since the proofs of all the other estimates are slightly easier and follow the same method. We first note that summing the estimate in (8.55a) over $j, k, \xi, \vec{l}, I, \diamond$ and using (8.21), (8.5), (8.31a), (8.34a), and (7.27), we find that

$$\left|\sum_{j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i,j,k,\xi,\vec{l},I,\diamond}\right| \lesssim \psi_{i\pm,q} \tau_q^{-1} \Gamma_q^i \varpi \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right)$$

since $\operatorname{supp} H_{i,j,k,\xi,\vec{l},I,\diamond} \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond} \circ \Phi_{(i,k)} \subseteq \operatorname{supp} \psi_{i,q}$ and $\psi_{i\pm,q} = (\psi_{i-1,q}^6 + \psi_{i,q}^6 + \psi_{i+1,q}^6)^{1/6}$. Now summing on i and using (5.8) and Remark 5.3.2, we find that

$$\begin{aligned}\left|\psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} H_{i',j,k,\xi,\vec{l},I,\diamond}\right| &\lesssim \left(\sum_i \Gamma_q^i \tau_q^{-1} \psi_{i\pm,q}\right) \varpi \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^i, \mathrm{T}^{-1}\right) \\ &\lesssim \Gamma_q r_q^{-1} (\pi_q^q)^{1/2} \lambda_q \varpi \mathcal{M}\left(N, N_x, \lambda, \Lambda\right) \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma_q^{i+1}, \mathrm{T}^{-1}\right) .\end{aligned}$$

□

Chapter 9

The velocity increment

9.1 Definition of the corrector

In this subsection, we define the premollified velocity increment w_{q+1}, *except* for the choice of placement, which we handle in the next subsection and which requires the application of Lemmas 7.2.3 and 7.2.2. None of the discussion or properties in this subsection depend on the choice of placement.

9.1.1 Definition of the current corrector

For any fixed values of i, k, we recall the constant c_3 from (8.32) and define

$$\varphi_{q,i,k} = {}^{-1}\!/_{c_3} \nabla \Phi_{(i,k)} \varphi_\ell \,. \tag{9.1}$$

Let $\xi \in \Xi'$, cf. Proposition 7.1.2. For all $\xi \in \Xi'$, we define the coefficient function $a_{\xi,i,j,k,\vec{l},\varphi}$ by

$$a_{\xi,i,j,k,\vec{l},\varphi} = a_{(\xi),\varphi} = \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} \Gamma_q^{j-1} \psi_{i,q}^{\varphi} \omega_{j,q}^{\varphi} \chi_{i,k,q}^{\varphi} \zeta_{q,\varphi,i,k,\xi,\vec{l}} |\nabla \Phi_{(i,k)}^{-1} \xi|^{-2/3} \widetilde{\gamma}_\xi \left(\frac{\varphi_{q,i,k}}{\delta_{q+\bar{n}}^{3/2} r_q^{-1} \Gamma_q^{3j-3}} \right) , \tag{9.2}$$

where $\widetilde{\gamma}_\xi$ is defined in Proposition 7.1.2, $\zeta_{q,\varphi,i,k,\xi,\vec{l}}$ is defined in Definition 8.4.1, and

$$\psi^{\varphi}_{i,q} := \psi^2_{i,q}\,, \qquad \omega^{\varphi}_{j,q} := \omega^2_{j,q}\,, \qquad \chi^{\varphi}_{i,k,q} := \chi^2_{i,k,q}\,. \tag{9.3}$$

From Corollary 8.3.4 and estimate (8.12a) from Corollary 8.2.4, we have that $|\varphi_\ell| \lesssim \Gamma_q^{3j-7}\delta_{q+\bar{n}}^{3/2} r_q^{-1}$, and so $\varphi_{q,i,k}$ is well-defined on the support of $\psi^{\varphi}_{i,q}\omega^{\varphi}_{j,q}$ once λ_0 is sufficiently large.

The coefficient function $a_{(\xi),\varphi}$ is then multiplied by an intermittent pipe bundle $\nabla\Phi^{-1}_{(i,k)}\mathbb{B}_{(\xi),\varphi}\circ\Phi_{(i,k)}$, where we have used Proposition 7.1.6 (with $\lambda = \lambda_{q+\bar{n}}$ and $r = r_q$), Definition 7.2.6, and the shorthand notation

$$\mathbb{B}_{(\xi),\varphi} = \boldsymbol{\rho}^{\varphi}_{(\xi)} \sum_I \boldsymbol{\zeta}^{I,\varphi}_{\xi} \mathbb{W}^I_{(\xi),\varphi} \tag{9.4}$$

to refer to the pipe bundle associated with the region $\Omega_0 = \operatorname{supp}\zeta_{q,\varphi,i,k,\xi,\vec{l}} \cap \{t = k\tau_q\Gamma_q^{-i}\}$ and the index j. The choice of placement of this pipe bundle will be detailed in subsection 9.2. We will use $\mathbb{U}^I_{(\xi),\varphi}$ to denote the potential satisfying $\operatorname{curl}\mathbb{U}^I_{(\xi),\varphi} = \mathbb{W}^I_{(\xi),\varphi}$. Applying the algebraic identity (7.14) from Proposition 7.1.5, we define the principal part of the current corrector by

$$w^{(p)}_{q+1,\varphi} = \sum_{i,j,k,\xi,\vec{l},I} \underbrace{a_{(\xi),\varphi}\left(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi}\right)\circ\Phi_{(i,k)}\operatorname{curl}\left(\nabla\Phi^T_{(i,k)}\mathbb{U}^I_{(\xi),\varphi}\circ\Phi_{(i,k)}\right)}_{=:w^{(p),I}_{(\xi),\varphi}}\,. \tag{9.5}$$

The notation $w^{(p),I}_{(\xi),\varphi}$ refers to *fixed* values of the indices $i, j, k, \xi, \vec{l}, I$. We add the divergence corrector

$$w^{(c)}_{q+1,\varphi} = \sum_{i,j,k,\xi,\vec{l},I} \underbrace{\nabla\left(a_{(\xi),\varphi}\left(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi}\right)\circ\Phi_{(i,k)}\right)\times\left(\nabla\Phi^T_{(i,k)}\mathbb{U}^I_{(\xi),\varphi}\circ\Phi_{(i,k)}\right)}_{=:w^{(c),I}_{(\xi),\varphi}}\,, \tag{9.6}$$

so that the mean-zero, divergence-free total current corrector is given by

$$w_{q+1,\varphi} = w^{(p)}_{q+1,\varphi} + w^{(c)}_{q+1,\varphi} = \sum_{i,j,k,\xi,\vec{l},I} \underbrace{\operatorname{curl}\left(a_{(\xi),\varphi}\left(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi}\right)\circ\Phi_{(i,k)}\nabla\Phi^{T}_{(i,k)}\mathbb{U}^{I}_{(\xi),\varphi}\circ\Phi_{(i,k)}\right)}_{=:w^{I}_{(\xi),\varphi}} . \quad (9.7)$$

9.1.2 Definition of the Euler-Reynolds corrector

For any fixed values of i, k, we recall (8.34a) and define

$$\begin{aligned} R_{q,i,k} &= -\nabla\Phi_{(i,k)}\Bigg(R_{\ell} - \pi_{\ell}\mathrm{Id} \\ &+ \sum_{\substack{\xi',i',j'\\ k',l'}} \frac{\delta_{q+\bar{n}}\Gamma_q^{2j'-2}C\Gamma_q^{-2}}{\left|\nabla\Phi^{-1}_{(i',k')}\xi'\right|^{4/3}}\psi^4_{i',q}\omega^4_{j',q}\chi^4_{i',k',q}\mathcal{X}^4_{q,\xi',l'}\circ\Phi_{i',k',q}\widetilde{\gamma}^2_{\xi'}\nabla\Phi^{-1}_{(i',k')}\xi'\otimes\xi'\left(\nabla\Phi^{-T}_{(i',k')}\right)\Bigg)\nabla\Phi^{T}_{(i,k)} , \end{aligned} \quad (9.8)$$

where the constant $C = c_0c_1c_2$ is geometric and bounded independently of q; see (10.5b). For all $\xi \in \Xi_R$, we define the coefficient function $a_{\xi,i,j,k,\vec{l},R}$ by

$$a_{\xi,i,j,k,\vec{l},R} = a_{(\xi),R} = \delta^{1/2}_{q+\bar{n}}\Gamma_q^{j-1}\psi^R_{i,q}\omega^R_{j,q}\chi^R_{i,k,q}\zeta_{q,R,i,k,\xi,\vec{l}}\gamma_{\xi,\Gamma_q^9}\left(\frac{R_{q,i,k}}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}}\right) \quad (9.9)$$

where γ_{ξ,Γ_q^9} is defined in Proposition 7.1.1 with the parameter choice $K = \Gamma_q^9$, and

$$\psi^R_{i,q} := \psi^3_{i,q} , \qquad \omega^R_{j,q} := \omega^3_{j,q} , \qquad \chi^R_{i,k,q} := \chi^3_{i,k,q} . \quad (9.10)$$

In order to show that (9.9) is well-defined, we first recall (8.22b) from Lemma 8.3.3, which gives that $\pi_{\ell}|_{\operatorname{supp}\omega_{j,q}} \geq {}^1\!/_4\Gamma_q^{2j}\delta_{q+\bar{n}}$. Using this in combination with Corollary 8.3.4, we find that for all j,

$$\Gamma_q \leq \frac{\pi_{\ell}|_{\operatorname{supp}\omega_{j,q}}}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}} \leq \Gamma_q^9 . \quad (9.11)$$

Furthermore, from (9.8), (8.21), and Corollary 8.2.4, we have that the second term in (9.8) is pointwise bounded by $2C\delta_{q+\bar{n}}\Gamma_q^{2j-2}$, or upon division by $\delta_{q+\bar{n}}\Gamma_q^{2j-2}$ is bounded above by $2C$.

Finally, from (8.22d), we have that $\nabla\Phi_{(i,k)}R_\ell\nabla\Phi_{(i,k)}^T$ is pointwise bounded by $\delta_{q+\bar{n}}\Gamma_q^{2j-3}$, or upon division by $\delta_{q+\bar{n}}\Gamma_q^{2j-2}$ is pointwise bounded by Γ_q^{-1}. Combining the above arguments, we find that

$$\left|\frac{R_{q,i,k}}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}} - \frac{\pi_\ell}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}}\mathrm{Id}\right| \leq \Gamma_q\,,$$

and so Proposition 7.1.1 may be applied with $K = \Gamma_q^9$ since $\frac{R_{q,i,k}}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}}$ belongs to the ball of radius Γ_q around $\frac{\pi_\ell \mathrm{Id}}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}}$, which itself is a multiply of the identity bounded between 1 and Γ_q^9 from (9.11).

The coefficient function $a_{(\xi),R}$ is then multiplied by an intermittent pipe bundle $\nabla\Phi_{(i,k)}^{-1}\mathbb{B}_{(\xi),R}\circ\Phi_{(i,k)}$, where we have used Proposition 7.1.5 (with $\lambda = \lambda_{q+\bar{n}}$ and $r = r_q$), Definition 7.2.6, and the shorthand notation

$$\mathbb{B}_{(\xi),R} = \boldsymbol{\rho}_{(\xi)}^R \sum_I \boldsymbol{\zeta}_\xi^{I,R}\mathbb{W}_{(\xi),R}^I \tag{9.12}$$

to refer to the pipe bundle associated with the region $\Omega_0 = \operatorname{supp}\zeta_{q,R,i,k,\xi,\vec{l}} \cap \{t = k\tau_q\Gamma_q^{-i}\}$ and the index j. We will use $\mathbb{U}_{(\xi),R}^I$ to denote the potential satisfying $\operatorname{curl}\mathbb{U}_{(\xi),R}^I = \mathbb{W}_{(\xi),R}^I$. Applying (7.14) from Proposition 7.1.5, we define the principal part of the Reynolds corrector by

$$w_{q+1,R}^{(p)} = \sum_{i,j,k,\xi,\vec{l},I} \underbrace{a_{(\xi),R}\left(\boldsymbol{\rho}_{(\xi)}^R\boldsymbol{\zeta}_\xi^{I,R}\right)\circ\Phi_{(i,k)}\mathrm{curl}\left(\nabla\Phi_{(i,k)}^T\mathbb{U}_{(\xi),R}^I\circ\Phi_{(i,k)}\right)}_{=:w_{(\xi),R}^{(p),I}}\,. \tag{9.13}$$

The notation $w_{(\xi),R}^{(p),I}$ refers to *fixed* values of $i, j, k, \xi, \vec{l}, I$. We add the divergence corrector

$$w_{q+1,R}^{(c)} = \sum_{i,j,k,\xi,\vec{l},I} \underbrace{\nabla\left(a_{(\xi),R}\left(\boldsymbol{\rho}_{(\xi)}^R\boldsymbol{\zeta}_\xi^{I,R}\right)\circ\Phi_{(i,k)}\right)\times\left(\nabla\Phi_{(i,k)}^T\mathbb{U}_{(\xi),R}^I\circ\Phi_{(i,k)}\right)}_{=:w_{(\xi),R}^{(c),I}}\,, \tag{9.14}$$

so that the mean-zero, divergence-free total Reynolds corrector is given by

$$w_{q+1,R} = \sum_{i,j,k,\xi,\vec{l},I} \underbrace{\operatorname{curl}\left(a_{(\xi),R}\left(\boldsymbol{\rho}^{R}_{(\xi)}\boldsymbol{\zeta}^{I,R}_{\xi}\right)\circ\Phi_{(i,k)}\nabla\Phi^{T}_{(i,k)}\mathbb{U}^{I}_{(\xi),R}\circ\Phi_{(i,k)}\right)}_{=:w^{I}_{(\xi),R}} . \tag{9.15}$$

9.1.3 Definition of the complete corrector

We shall sometimes want to aggregate pieces of the Reynolds and current velocity correctors as

$$w_{q+1} = w_{q+1,R} + w_{q+1,\varphi}\,, \qquad w^{(p)}_{q+1} := w^{(p)}_{q+1,R} + w^{(p)}_{q+1,\varphi}\,, \qquad w^{(c)}_{q+1} := w^{(c)}_{q+1,R} + w^{(c)}_{q+1,\varphi}\,. \tag{9.16}$$

9.2 Dodging for new velocity increment

Definition 9.2.1 (Definition of $\widehat{w}_{q+\bar{n}}$ and u_{q+1}). *Let $\widetilde{\mathcal{P}}_{q+\bar{n},x,t}$ denote a space-time mollifier which is a product of compactly supported kernels at spatial scale $\lambda^{-1}_{q+\bar{n}}\Gamma^{-1/2}_{q+\bar{n}-1}$ and temporal scale T^{-1}_{q+1}. We again assume that both kernels have vanishing moments up to $10\mathsf{N}_{\mathrm{fin}}$ and are $C^{10\mathsf{N}_{\mathrm{fin}}}$ differentiable and define*

$$\widehat{w}_{q+\bar{n}} := \widetilde{\mathcal{P}}_{q+\bar{n},x,t} w_{q+1}\,, \qquad u_{q+1} = u_q + \widehat{w}_{q+\bar{n}}\,. \tag{9.17}$$

We also recall from (5.1) the notations $B(\Omega,\lambda^{-1})$ and $B(\Omega,\lambda^{-1},\tau)$ for space and space-time balls, respectively, around a space-time set Ω. Using these notations, we may write that

$$\operatorname{supp}\widehat{w}_{q+\bar{n}} \subseteq B\left(\operatorname{supp} w_{q+1}, {}^{1}\!/_{2}\lambda^{-1}_{q+\bar{n}}, {}^{1}\!/_{2}\mathrm{T}_{q}\right)\,. \tag{9.18}$$

Now recalling the formula in (7.9) for an intermittent Mikado flow, (9.4), and (9.12), we set

$$\varrho^{I}_{(\xi),\diamond} := \xi\cdot\mathbb{W}^{I}_{(\xi),\diamond}\,. \tag{9.19}$$

Next, in slight conflict with (5.1), we shall also use the notation

$$B\left(\operatorname{supp} \varrho^I_{(\xi),\diamond}, \lambda^{-1}\right) := \left\{x \in \mathbb{T}^3 \,:\, \exists y \in \operatorname{supp} \varrho^I_{(\xi),\diamond}\,, |x-y| \leq \lambda^{-1}\right\} \tag{9.20}$$

throughout this section, despite the fact that $\operatorname{supp} \varrho^I_{(\xi),\diamond}$ is not a set in space-time, but merely a set in space. We shall also use the same notation but with $\varrho^I_{(\xi),\diamond}$ replaced by $\boldsymbol{\rho}^\diamond_\xi$. Finally, for any smooth set $\Omega \subseteq \mathbb{T}^3$ and any flow map Φ defined in Definition 8.2.3, we use the notation

$$\Omega \circ \Phi := \{(y,t) : t \in \mathbb{R}, \Phi(y,t) \in \Omega\} = \operatorname{supp}\left(\mathbf{1}_\Omega \circ \Phi\right)\,. \tag{9.21}$$

In other words, for any smooth set $\Omega \subseteq \mathbb{T}^3$, $\Omega \circ \Phi$ is a space-time set whose characteristic function is annihilated by $D_{t,q}$.

We can now verify Hypotheses 5.4.1–5.4.2, as well as several related useful dodging results.

Lemma 9.2.2 (**Dodging and preventing self-intersections for w_{q+1} and $\widehat{w}_{q+\bar{n}}$**). *We construct w_{q+1} so that the following hold.*

(i) Let $q+1 \leq q' \leq q + \bar{n}/2$ and fix indices $\diamond, i, j, k, \xi, \vec{l}$, which we abbreviate by $((\xi), \diamond)$, for a coefficient function $a_{(\xi),\diamond}$ (cf. (9.2), (9.9)). Then

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \frac{1}{2}\lambda_{q+1}^{-1}\Gamma_q^2, 2\mathrm{T}_q\right) \cap \operatorname{supp}\left(\widetilde{\chi}_{i,k,q}\zeta_{q,\diamond,i,k,\xi,\vec{l}}\boldsymbol{\rho}^\diamond_{(\xi)} \circ \Phi_{(i,k)}\right) = \emptyset\,. \tag{9.22}$$

(ii) Let q' satisfy $q+1 \leq q' \leq q+\bar{n}-1$, fix indices $((\xi), \diamond, I)$, and assume that $\Phi_{(i,k)}$ is the identity at time $t_{(\xi)}$, cf. Definition 8.2.3. Then we have that

$$\begin{aligned} B\left(\operatorname{supp} \widehat{w}_{q'}, \frac{1}{4}\lambda_{q'}^{-1}\Gamma_{q'}^2, 2\mathrm{T}_q\right) \cap \operatorname{supp}&\left(\widetilde{\chi}_{i,k,q}\zeta_{q,\diamond,i,k,\xi,\vec{l}}\left(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\right) \circ \Phi_{(i,k)}\right) \\ &\cap B\left(\operatorname{supp} \varrho^I_{(\xi),\diamond}, \frac{1}{2}\lambda_{q'}^{-1}\Gamma_{q'}^2\right) \circ \Phi_{(i,k)} = \emptyset\,. \end{aligned} \tag{9.23}$$

As a consequence we have

$$B\left(\operatorname{supp}\widehat{w}_{q'}, \frac{1}{4}\lambda_{q'}^{-1}\Gamma_{q'}^2, 2\mathrm{T}_q\right) \cap \operatorname{supp} w_{q+1} = \emptyset\,. \tag{9.24}$$

(iii) Consider the set of indices $\{((\xi), \diamond, I)\}$*, whose elements we use to index the correctors constructed in* (9.7) *and* (9.15)*, and let* $\mathtt{l}, \overline{\mathtt{l}} \in \{p, c\}$ *denote either principal or divergence corrector parts. Then if* $(\overline{\diamond}, (\overline{\xi}), \overline{I}) \neq (\diamond, (\xi), I)$*, we have that for any* $\mathtt{l}, \overline{\mathtt{l}}$*,*

$$\operatorname{supp} w_{(\xi),\diamond}^{(\mathtt{l}),I} \cap \operatorname{supp} w_{(\overline{\xi}),\overline{\diamond}}^{(\overline{\mathtt{l}}),\overline{I}} = \emptyset\,. \tag{9.25}$$

(iv) $\widehat{w}_{q+\bar{n}}$ *satisfies Hypothesis 5.4.2 with* q *replaced by* $q+1$*.*

Remark 9.2.3 (Verifying Hypothesis 5.4.1). We claim that (9.24) and (9.18) imply that Hypothesis 5.4.1 holds with $q+1$ replacing all instances of q. To check this, we must show that (5.26) holds for $q', q'' \leq q + \bar{n}$ and $0 < |q' - q''| \leq \bar{n} - 1$. By induction on q and the symmetry of q'' and q', the only case we must check is the case that $q + \bar{n} = q''$ and $0 < q + \bar{n} - q' \leq \bar{n} - 1$. But it is a simple exercise in set theory to check that for $q + 1 \leq q' \leq q + \bar{n} - 1$, (9.24) is equivalent to $\operatorname{supp}\widehat{w}_{q'} \cap B(\operatorname{supp} w_{q+1}, {}^1\!/\!_4\lambda_{q'}^{-1}\Gamma_{q'}^2, 2\mathrm{T}_q) = \emptyset$. Then using (9.18) and the inequalities $\lambda_{q'}^{-1}\Gamma_{q'}^2 \geq \lambda_{q+\bar{n}}^{-1}$, $b < 2 \implies \Gamma_{q'+1} \ll \Gamma_{q'}^2$ implies that (5.26) holds.

Proof of Lemma 9.2.2. We split the proof up into steps, in which we first carry out some preliminary set-up before verifying item (i), items (ii)–(iii), and finally item (iv).

`Step 0: Ordering of cutoff functions and set-up.` Consider all coefficient functions $a_{\xi,i,j,k,\vec{l},\diamond}$ utilized at stage $q+1$, cf. (9.9) and (9.2). Using natural numbers $z \in \mathbb{N}$ as indices, we choose an ordering of the tuples $(i, j, k, \xi, \vec{l}, \diamond)$ such that for any choice of $(i, j, k, \xi, \vec{l}, \diamond)$ and $(i^*, j^*, k^*, \xi^*, \vec{l}^*, \diamond^*)$, we have

$$i < i^* \implies (i, j, k, \xi, \vec{l}, \diamond) <_{\text{ordering}} (i^*, j^*, k^*, \vec{l}^*, \diamond^*)\,, \tag{9.26}$$

where the implied inequality holds for the natural numbers assigned to each tuple in our chosen ordering. This automatically provides an ordering for the coefficient functions $a_{\xi,i,j,k,\vec{l},\diamond}$ and associated pipe bundles $\mathbb{B}_{(\xi),\diamond} \circ \Phi_{(i,k)}$. We will place pipe bundles inductively according to this ordering so that all the conclusions in the statement of Lemma 9.2.2 hold. (9.26) ensures that timescales are decreasing with respect to this ordering and mitigates the fact that the number of possible overlaps between $\psi_{i,q}\chi_{i,k,q}$ and $\psi_{i+1,q}\chi_{i+1,k',q}$ could be of order Γ_q (see 8.3).[1] To lighten the notation, we will abbreviate the newly ordered coefficient and cutoff functions and associated intermittent pipe bundles (cf. (9.2), (9.3), (9.4), (9.9), (9.10), and (9.12)) as

$$a_z\,, \psi_z\,, \omega_z\,, \chi_z\,, \zeta_z\,, \qquad (\mathbb{B} \circ \Phi)_z = \boldsymbol{\rho}_z \circ \Phi_z \sum_I (\boldsymbol{\zeta}_z \mathbb{W}_z^I) \circ \Phi_z\,,$$

respectively, where $z \in \mathbb{N}$ corresponds to the ordering. Now for fixed z and a_z, we will place $\boldsymbol{\rho}_z$ and $\mathbb{W}_z^I$ with two goals in mind. First, we must dodge the velocity increments $\widehat{w}_{q'}$ for $q+1 \leq q' \leq q + \bar{n}/2$ and $\widehat{w}_{q''}$ for $q + \bar{n}/2 + 1 \leq q'' \leq q + \bar{n} - 1$. Second, we must dodge all pipe bundles $(\mathbb{B} \circ \Phi)_{\widehat{z}}$ with coefficient functions $a_{\widehat{z}}$ such that $\widehat{z} < z$ in the aforementioned ordering.

`Step 1: Proof of item` (i). We will apply Lemma 7.2.2 with the following choices. We recall that at the time t_z at which Φ_z is the identity, the cutoff function η_z contains a checkerboard cutoff function ζ_z which from (8.31b) and (8.34b) is contained in a rectangular prism of dimensions no larger than $3/4\lambda_q^{-1}\Gamma_q^{-8}$ in the direction of ξ_z, and $3/4\Gamma_q^5(\lambda_{q+1})^{-1}$ in the directions perpendicular to ξ_z. Thus we set

$$\Omega_0 = \operatorname{supp} \zeta_z \cap \{t = t_z\}\,.$$

Notice that $\operatorname{diam}(\Omega_0) \leq \lambda_q^{-1}\Gamma_q^{-8}$, which satisfies (5.27) for $\bar{q}', \bar{q}''$ chosen as $\bar{q}'' = q$ and $\bar{q}' = q'$

[1] This is not strictly necessary– one can always adust the choice of parameters to accommodate a spare Γ_q.

as in (i) so that $q+1 \le q' = \bar{q}' \le q + {}^{\bar{n}}\!/_{2}$. Then by applying Hypothesis 5.4.2 at level q with $\bar{q}' = q'$, $\bar{q}'' = q$, $\Omega = \Omega_0$ as defined above, $t_0 = t_z$, and $\Phi_{\bar{q}''} = \Phi_z$, we have that for each $q+1 \le q' \le q + {}^{\bar{n}}\!/_{2}$, there exists a set $L(q', q, \Omega_0, t_z)$ such that (5.28) and (5.29) hold. Now we set

$$E_0 := \bigcup_{q'=q+1}^{q+{}^{\bar{n}}/_{2}} L(q', q, \Omega_0, t_z) \cap \{t = t_z\}\,, \qquad \mathcal{C}_P = \mathcal{C}_D \bar{n}\,.$$

We now appeal to the conclusion of Lemma 7.2.2 to choose a placement for $\boldsymbol{\rho}_z$ such that

$$B\left(\operatorname{supp} \boldsymbol{\rho}_z, \lambda_{q+1}^{-1}\Gamma_q^2\right) \cap E_0 = \emptyset\,. \tag{9.27}$$

An immediate consequence of (9.27), Hypothesis 5.4.2, and (8.11) is that for t such that $|t - t_z| \le \tau_q \Gamma_q^{-i_z+2}$,

$$D_{t,q}\left(\mathbf{1}_{B\left(\operatorname{supp} \boldsymbol{\rho}_z, \lambda_{q+1}^{-1}\Gamma_q^2\right)\circ\Phi_z} \mathbf{1}_{L(q',q,\Omega_0,t_z)}\right)(t,x) \equiv 0\,. \tag{9.28}$$

This in turn implies that in the same range of t,

$$B\left(\operatorname{supp} \boldsymbol{\rho}_z, \lambda_{q+1}^{-1}\Gamma_q^2\right) \circ \Phi_z(t) \cap L(q', q, \Omega_0, t_z) \cap \{t = t_z\} = \emptyset\,. \tag{9.29}$$

Next, we claim that (9.29) implies that

$$B\left(\operatorname{supp} \boldsymbol{\rho}_z \circ \Phi_z \cap (\mathbb{T}^3 \times \{t\}), {}^{3}\!/_{4}\lambda_{q+1}^{-1}\Gamma_q^2\right) \cap L(q', q, \Omega_0, t_z) = \emptyset \quad \text{for } |t - t_z| \le \tau_q \Gamma_q^{-i_z+2}\,, \tag{9.30}$$

which we now prove. Indeed, this follows from the fact that on the Lipschitz timescale $\tau_q \Gamma_q^{-i_z+2}$, spatial distances can change by at most a multiplicative factor of $(1 \pm \Gamma_q^{-1})$ due to

(5.34) (see also Lemma 7.1.7, which contains similar assertions). Finally, we claim that

$$B\left(\bigcup_{|t-t_z|\leq\frac{1}{2}\tau_q\Gamma_q^{-i_z+2}} \operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z\cap(\mathbb{T}^3\times\{t\}),\frac{1}{2}\lambda_{q+1}^{-1}\Gamma_q^2,2\mathrm{T}_q\right)$$
$$\subseteq\bigcup_{|t-t_z|\leq\tau_q\Gamma_q^{-i_z+2}} B\left(\operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z\cap(\mathbb{T}^3\times\{t\}),{}^3\!/\!_4\lambda_{q+1}^{-1}\Gamma_q^2\right). \tag{9.31}$$

Assuming that (9.31) holds, we have then from (9.30), (8.9), and Hypothesis 5.4.2 that

$$B\left(\operatorname{supp}\left(\widetilde{\chi}_z\zeta_z\boldsymbol{\rho}_z\circ\Phi_z\right),{}^1\!/\!_2\lambda_{q+1}^{-1}\Gamma_q^2,2\mathrm{T}_q\right)\cap\operatorname{supp}\widehat{w}_{q'}=\emptyset,$$

which is equivalent to (9.22) after using the same sort of set-theoretic reasoning as in Remark 9.2.3. To prove (9.31), suppose that $(\widetilde{x},\widetilde{t})$ belongs to the set on the left-hand side of the inclusion in (9.31). Then by definition, there exists (t_0,x_0) such that $|\widetilde{t}-t_0|\leq 2\mathrm{T}_q$, $|\widetilde{x}-x_0|\leq{}^1\!/\!_2\lambda_{q+1}^{-1}\Gamma_q^2$, $|t_0-t_z|\leq{}^1\!/\!_2\tau_q\Gamma_q^{-i_z+2}$, and $(x_0,t_0)\in\operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z\cap(\mathbb{T}^3\times\{t_0\})$. Then from (5.10) and (4.15), we have that $|\widetilde{t}-t_z|\leq\tau_q\Gamma_q^{-i_z+2}$. So we need to find x' such that $(x',\widetilde{t})\in\operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z\cap(\mathbb{T}^3\times\{\widetilde{t}\})$ and $|x'-\widetilde{x}|<{}^3\!/\!_4\lambda_{q+1}^{-1}\Gamma_q^2$. Now from (8.11), (5.35b), Corollary 8.2.4, and (4.15), we have that

$$\begin{aligned}\left\|\Phi_z(\widetilde{t},\cdot)-\Phi_z(t_0,\cdot)\right\|_{L^\infty(\mathbb{T}^3)}&\lesssim\mathrm{T}_q\left\|\partial_t\Phi_z\right\|_{L^\infty\left(\mathbb{T}^3\times(t_z-\tau_q\Gamma_q^{-i_z+2},t_z+\tau_q\Gamma_q^{-i_z+2})\right)}\\&\lesssim\mathrm{T}_q\left\|\widehat{u}_q\right\|_\infty\left\|\nabla\Phi_z\right\|_{L^\infty\left(\mathbb{T}^3\times(t_z-\tau_q\Gamma_q^{-i_z+2},t_z+\tau_q\Gamma_q^{-i_z+2})\right)}\\&\lesssim\Gamma_q^{-1}\lambda_{q+1}^{-1}.\end{aligned}$$

Therefore, although it may not be the case that $(x_0,\widetilde{t})\in\operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z\cap(\mathbb{T}^3\times\{\widetilde{t}\})$, there must exist x' such that $|x'-x_0|\leq\Gamma_q^{-1/2}\lambda_{q+1}^{-1}$ and $(x',\widetilde{t})\in\operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z\cap(\mathbb{T}^3\times\{\widetilde{t}\})$.[2] Since $|x'-\widetilde{x}|\leq|x'-x_0|+|x_0-\widetilde{x}|\leq\Gamma_q^{-1/2}\lambda_{q+1}^{-1}+{}^1\!/\!_2\lambda_{q+1}\Gamma_q^2<{}^3\!/\!_4\lambda_{q+1}^{-1}\Gamma_q^2$, we have thus concluded the proof of (9.31).

[2]We have that $(x_0,t_0)\in\operatorname{supp}\boldsymbol{\rho}_z\circ\Phi_z$ if and only if $\Phi_z(x_0,t_0)\in\operatorname{supp}\boldsymbol{\rho}_z$. Using the bound on the difference between $\Phi_z(\widetilde{t})$ and $\Phi_z(t_0)$, we may say that although $\Phi_z(x_0,\widetilde{t})$ is not necessarily in the support of $\boldsymbol{\rho}_z$, it is very close.

`Step 2: Proofs of items` (ii) `and` (iii)`.` In order to proceed with this portion of the proof, we assume inductively that a version of Hypothesis 5.4.2 holds for the portion of w_{q+1} already constructed. More precisely, we extend the ordering on $z \in \mathbb{N}$ from `Step 0` to ordered pairs $(z, I) \in \mathbb{N}^2$ such that $\widehat{z} < z \implies (\widehat{z}, \widehat{I}) < (z, I)$ for any $\widehat{I}, I$ (that is, we fix z and finish placing an entire bundle for all its various values of I before moving to different $\widehat{z}$). We thus assume the following inductive hypothesis.

Hypothesis 9.2.4 (Density of already placed pipe bundles in w_{q+1}). There exists a geometric constant $\mathcal{C}_{\text{pipe}}$ such that the following holds. Fix z and set

$$\mathfrak{s}_{\widehat{z},\widehat{I}}(t) := \operatorname{supp}\left[\chi_{\widehat{i},\widehat{k},q}\zeta_{(\widehat{\xi})}\left(\boldsymbol{\rho}^{\diamond}_{(\widehat{\xi})}\boldsymbol{\zeta}^{\widehat{I},\diamond}_{\widehat{\xi}}\right)\circ\Phi_{(\widehat{i},\widehat{k})}\right]\cap B\left(\varrho^{\widehat{I}}_{(\widehat{\xi}),\diamond}, \frac{1}{2}\lambda^{-1}_{q+\bar{n}}\Gamma_q^2\right)\circ\Phi_{(\widehat{i},\widehat{k})}\cap(\mathbb{T}^3\times\{t\}),$$

$$w_{z,I} := \sum_{(\widehat{z},\widehat{I})<(z,I)} a_{\widehat{z}}(\boldsymbol{\rho}_{\widehat{z}}\boldsymbol{\zeta}^{\widehat{I}}_{\widehat{z}}\mathbb{W}^{\widehat{I}}_{\widehat{z}})\circ\Phi^{\widehat{I}}_{\widehat{z}}, \qquad \mathfrak{S}_{z,I}(t) := \bigcup_{(\widehat{z},\widehat{I})<(z,I)} \mathfrak{s}_{\widehat{z},\widehat{I}}(t).$$

Let i_z be the value of i corresponding to z and a_z, and let t_0 be any time and $\Omega \subset \mathbb{T}^3$ be a convex set with diameter at most $\left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^{-1}$ such that $\Omega \times \{t_0\} \cap \operatorname{supp}\psi_{i_z,q} \neq \emptyset$. Let Φ solve $D_{t,q}\Phi = 0$ with initial data $\Phi|_{t=t_0} = \mathrm{Id}$. We set $\Omega(t) = \Phi(t)^{-1}(\Omega)$ and

$$\mathcal{N}_{\Omega,z,I} = \#\left\{(\widehat{z},\widehat{I}) < (z,I) \,:\, \exists t \in [t_0 - \tau_q\Gamma_q^{-i_z-2}, t_0 + \tau_q\Gamma_q^{-i_z-2}] \text{ with } \mathfrak{s}_{\widehat{z},\widehat{I}}(t)\cap\Omega(t) \neq \emptyset\right\}.$$

Then there exists an Ω-dependent set $L_{(z,I)} \subseteq \Omega$ consisting of at most $\mathcal{N}_{\Omega,z,I}\mathcal{C}_{\text{pipe}}$ segments of deformed pipe segments with thickness $\lambda^{-1}_{q+\bar{n}}$ such that for all $t \in [t_0 - \tau_q\Gamma_q^{-i_z-2}, t_0 + \tau_q\Gamma_q^{-i_z-2}]$,

$$[\operatorname{supp} w_{z,I}(\cdot,t)\cap\Omega(t)] \subseteq [\mathfrak{S}_{z,I}(t)\cap\Omega(t)] \subseteq \left[\Phi(t)^{-1}(L_{(z,I)})\cap\Omega(t)\right]. \tag{9.32}$$

One should understand this hypothesis as asserting that at all steps in the construction of w_{q+1}, there is no more than a finite number of pipe segments of thickness $\lambda^{-1}_{q+\bar{n}}$ in any set of diameter proportional to the size of a periodic cell. Indeed, $\mathcal{N}_{\Omega,z,I}$ is bounded independently of z, I, and hence q. From the finite maximal cardinality of the indices $(\widehat{i},\widehat{j},\widehat{\xi},\widehat{\diamond})$ and

decreasing time scale with respect to the ordering (9.26), the indices $(\widehat{i}, \widehat{j}, \widehat{k}, \widehat{\xi}, \widehat{\diamond})$ takes a finite number, independent of z, I, and q. Fix these indices and we now count the remaining indices $(\widehat{\vec{l}}, \widehat{I})$. Since $\Phi_{(i,k)}$ and Φ are advected by the same velocity field $\widehat{u}_q$, recalling Definition 8.35, it is enough to count the indices to have $\mathfrak{s}_{\widehat{z},\widehat{I}}(t) \cap \Omega(t) \neq \emptyset$ at some fixed time $\bar{t} \in \operatorname{supp} \chi_{\widehat{i},\widehat{k},q} \cap [t_0 - \tau_q \Gamma_q^{-i_z-2}, t_0 + \tau_q \Gamma_q^{-i_z-2}]$. From the diameter bound on Ω and Lemma 7.1.7, we have $\operatorname{diam}(\Omega(\bar{t})) \leq 2(\lambda_{q+\bar{n}/2}\Gamma_q)^{-1}$, while the spatial derivative costs of $\zeta_{(\widehat{\xi})}$ and $\boldsymbol{\zeta}_{\widehat{\xi}}^{\widehat{I},\widehat{\diamond}} \circ \Phi_{(\widehat{i},\widehat{k})}$ are $\lambda_{q+1}\Gamma_q^{-5}$ and $\lambda_{q+\bar{n}/2}$, respectively, from (8.37) and (8.40). Since the inverse of the derivative costs are much greater than $2(\lambda_{q+\bar{n}/2}\Gamma_q)^{-1}$, only for finite number of indices $\widehat{\vec{l}}$ and $\widehat{I}$, the intersection $\mathfrak{s}_{\widehat{z},\widehat{I}}(\bar{t}) \cap \Omega(\bar{t}) \neq \emptyset$ occurs, where the number is independent of z, I, and q. Therefore, we can set an upper bound of $\mathcal{N}_{\Omega,z,I}$ as a geometric constant $\mathcal{C}$. Lastly, we note that Hypothesis 9.2.4 is vacuously true in the base case where (z, I) is the smallest element in our ordering.

We will now justify the application of Lemma 7.2.3. We recall from (9.2) and (9.9) that at the time t_z at which Φ_z is the identity, $a_z \boldsymbol{\rho}_z \boldsymbol{\zeta}_z^I$ contains both a strongly anisotropic checkerboard cutoff function $\boldsymbol{\zeta}_z^I$ and a mildly anistropic checkerboard cutoff function ζ_z. The support of the product of these cutoff functions is contained in a rectangular prism of dimensions no larger than $3/4\lambda_q^{-1}\Gamma_q^{-8}$ in the direction of ξ_z from (8.34b), and $\lambda_{q+\lfloor \bar{n}/2 \rfloor}^{-1}$ in the directions perpendicular to ξ_z from Definition 7.2.4. Thus we can contain the support of $a_z \boldsymbol{\rho}_z \boldsymbol{\zeta}_z^I$ at time t_z inside a prism of dimensions $3/4\lambda_q^{-1}\Gamma_q^{-8}$ and $\lambda_{q+\lfloor \bar{n}/2 \rfloor}^{-1}$, and so we set

$$\Omega_1 = \operatorname{supp} \zeta_z \boldsymbol{\zeta}_z^I \cap \{t = t_z\}.$$

By applying Hypothesis 5.4.2 at level q with $\bar{q}' = q''$ for each $q + \bar{n}/2 \leq q'' \leq q + \bar{n} - 1$, $\bar{q}'' = q$, $\Omega' \subset \Omega_1$ any convex subset of diameter at most $(\lambda_{q''-\bar{n}+\bar{n}/2}\Gamma_{q''-\bar{n}})^{-1}$ (which satisfies (5.27)), $t_0 = t_z$, and $\Phi_{\bar{q}''}$ as defined in Hypothesis 5.4.2, we have that there exists a set $L(q'', q, \Omega', t_z)$ such that (5.28) and (5.29) hold. We therefore see that the density condition of Lemma 7.2.3

is verified with

$$\mathcal{C}_P = \mathcal{C}_{\text{pipe}}\mathcal{C} + \bar{n}\mathcal{C}_D$$

from Hypothesis 5.4.2, which contributes the second term counting the number of old pipe segments belonging to $\widehat{w}_{q''}$ for $q+{}^{\bar{n}}/_{2}+1 \leq q'' \leq q+\bar{n}-1$, and our inductive Hypothesis 9.2.4, which contributes the first term counting the number of current pipe segments belong to $w_{z,I}$. We define $E_{q''}$ (including the endpoint case $q'' = q + \bar{n}$ which contains already placed pipes from $w_{z,I}$) as in Lemma 7.2.3 so that it contains the support of $\widehat{w}_{q''}$ inside Ω_1 if $q'' < q + \bar{n}$, and $w_{z,I}$ inside Ω_1 in the endpoint case.

Now appealing to the conclusion of Lemma 7.2.3, we may choose the support of $\mathbb{W}_z^I = (\mathbb{W} \circ \Phi)_z^I|_{t=t_z}$ so that for $q + {}^{\bar{n}}/_{2} + 1 \leq q'' \leq q + \bar{n} - 1$,

$$\operatorname{supp}_x (\mathbb{W} \circ \Phi)_z^I|_{t=t_z} \cap B\left(\operatorname{supp} \widehat{w}_{q''}(\cdot, t_z), \lambda_{q''}\Gamma_{q''}^2\right) \cap \Omega = \emptyset\,,$$
$$\operatorname{supp}_x (\mathbb{W} \circ \Phi)_z^I|_{t=t_z} \cap B\left(\mathfrak{S}_{z,I}(t_z), \lambda_{q+\bar{n}}^{-1}\Gamma_{q+\bar{n}}^2\right) \cap \Omega = \emptyset\,.$$

Reasoning as in the final portion of `Step 1` and assuming for the moment that Hypothesis 9.2.4 can be propagated throughout the construction of w_{q+1}, we have that the first assertion implies (9.23) and therefore also the weaker assertion (9.24). In addition, the second assertion verifies (9.25) at $t = t_z$, and (9.25) at all times follows from a similar type of argument (in fact simpler since no expansion by $2\mathrm{T}_q$ in time is required), but with Hypothesis 9.2.4 replacing Hypothesis 5.4.2.

We now verify that Hypothesis 9.2.4 has been preserved by the placement of $\mathbb{W}_z^I$. Note that we have placed $\left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^{-1}$-periodic straight pipes at time $t = t_z$ and have composed them with a diffeomorphism Φ_z. Furthermore, this diffeomorphism and the vector field $\widehat{u}_q$ obey the conditions and conclusions of Lemma 7.1.7 on the support of $a_z\boldsymbol{\rho}_z\boldsymbol{\zeta}_z^I$. Thus we have that there exists $\mathcal{C}_{\text{pipe}}$ such that for any convex set Ω' of diameter at most $\left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^{-1}$ and

for any time t,

$$\operatorname{supp}(\mathbb{W}\circ\Phi)^I_z(\cdot,t)\cap\Omega'(t)\cap\operatorname{supp}((a_z(\boldsymbol{\rho}_z\boldsymbol{\zeta}^I_z)\circ\Phi_z)(\cdot,t)$$

is contained in at most $\mathcal{C}_{\text{pipe}}$ deformed pipe segments. Now fix a convex set Ω and a time t_0 as in Hypothesis 9.2.4, and let $(z,I)^+$ denote the next element after (z,I) in our ordering. Then if $\Omega(t)$ has empty intersection with the spatial support of $a_z(\boldsymbol{\rho}_z\boldsymbol{\zeta}^I_z)\circ\Phi_z$ for all times t, we define $L_{(z,I)^+}=L_{(z,I)}$. If not, we set

$$L_{(z,I)^+}=L_{(z,I)}\cup\operatorname{supp}_x(\mathbb{W}\circ\Phi)^I_z(t_0)\cap\Omega\,.$$

Then to see that we have verified Hypothesis 9.2.4, in particular (9.32), we use that all flow maps $\Phi_{\widehat{i},\widehat{k}}$ and $\Phi_{(i,k)}$ are advected by the same velocity field $\widehat{u}_q$ and the observation above concerning the support of $(\mathbb{W}\circ\Phi)^I_z$.

`Step 3: Proof of item` (iv)`, or Hypothesis 5.4.2 for` $q+1$. In order to verify Hypothesis 5.4.2 at level $q+1$, we must consider $\bar{q}'=q+\bar{n}$ and any $\bar{q}''$ such that $q+1\le\bar{q}''<q+\bar{n}$, any convex set Ω of diameter[3]

$$d(q+\bar{n},\bar{q}'')=\min\left[(\lambda_{\bar{q}''}\Gamma^7_{\bar{q}''})^{-1},(\lambda_{q+\bar{n}/2}\Gamma_q)^{-1}\right]\,,$$

any time t_0, and any i'' such that $\Omega\times\{t_0\}\cap\operatorname{supp}\psi_{i'',\bar{q}''}\neq\emptyset$. Given these choices and the flow map $\Phi_{\bar{q}''}$ as defined in Hypothesis 5.4.2, we must define $L(q+\bar{n},\bar{q}'',\Omega,t_0)$ satisfying (5.28) and (5.29). We divide the proof into substeps, in which we first count the number of cutoff functions which can overlap with $\Omega(t)$, before defining $L(q+\bar{n},\bar{q}'',\Omega,t_0)$ and verifying (5.28) and (5.29) in the second substep.

[3]Hypothesis 5.4.2 in fact allows for sets of smaller diameter, for which the results follow trivially by embedding a smaller set into a set of the largest possible diameter.

`Step 3a: Counting overlap.` Let us pick any fixed but arbitrary $x \in \Omega$ and set

$$\widetilde{\Omega} = B\left(x, 3d(q+\bar{n}, \bar{q}'') + 3\lambda_{q+\bar{n}}^{-1}\right) . \tag{9.33}$$

Note that $\widetilde{\Omega}$ contains a ball of radius $3\lambda_{q+\bar{n}}^{-1}$ around Ω. We claim that the cardinality of the set of indices $(\xi, i, j, k, \vec{l}, \diamond, I)$ such that

$$\begin{aligned}\operatorname{supp}\left[\chi_{i,k,q}\zeta_{q,\diamond,i,k,\xi,\vec{l}}\left(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\circ\Phi_{(i,k)}\right] \cap B\left(\varrho^{I}_{(\xi),\diamond}, 3\lambda_{q+\bar{n}}^{-1}\right)\circ\Phi_{(i,k)} \\ \cap\,\widetilde{\Omega}\circ\Phi_{\bar{q}''}\cap\left[\mathbb{T}^3\times\{|t-t_0|\le 2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}\}\right]\neq\emptyset\end{aligned} \tag{9.34}$$

is bounded by a finite, q-independent constant $\mathcal{C}_{\text{counting}}$. We first count the possible values for i and k. From the diameter bound on Ω, (5.11), Lemma 8.2.1, and Corollary 8.2.2, we have that $\widetilde{\Omega}\circ\Phi_{\bar{q}''}\subseteq \operatorname{supp}\psi_{i''\pm,\bar{q}''}$ restricted to $\mathbb{T}^3\times\{|t-t_0|\le 2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}\}$. From eqn. (5.14), we have that if t is in the same range as above, $(x,t)\in\widetilde{\Omega}\circ\Phi_{\bar{q}''}$, and $\psi_{i,q}(x,t)\neq 0$, then it must be the case that $2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}\le\tau_q\Gamma_q^{-i-7}$. Thus if i is such that $\psi_{i,q}(x,t)\neq 0$ at some $(x,t)\in\widetilde{\Omega}\circ\Phi_{\bar{q}''}$ for t in the same range as above, from (8.2) there exist at most two values of k such that $\chi_{i,k,q}$ satisfies

$$\operatorname{supp}\left(\psi_{i,q}\chi_{i,k,q}\right)\cap\widetilde{\Omega}\circ\Phi_{\bar{q}''}\cap\left[\mathbb{T}^3\times\{|t-t_0|\le 2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}\}\right]\neq\emptyset$$

Next, recall that we have bounds $i_{\max}$ and $j_{\max}$ for the number of values of i and j, and ξ and $\diamond$ take a finite number of q-independent values.

In order to conclude the proof of the claim concerning intersections with (9.34), it only remains to count $\vec{l}$ and I for fixed $(\xi, i, j, k, \diamond)$. Since i and k are fixed, we can drop the time cutoff $\chi_{i,k,q}$ and consider in the intersection in the time interval $[t_0 - 2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}, t_0 +$

$2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}] \cap \operatorname{supp} \chi_{i,k,q}$. We then observe that from (8.9) and (9.23),

$$\begin{aligned} &D_{t,\bar{q}''}\mathbf{1}_{\left\{\zeta_{\mathfrak{i}}(\boldsymbol{\rho}_{\mathfrak{i}}\boldsymbol{\zeta}_{\mathfrak{i}})\circ\Phi_{\mathfrak{i}}\cap B\left(\varrho_{\mathfrak{i}},3\lambda_{q+\bar{n}}^{-1}\right)\circ\Phi_{\mathfrak{i}}\right\}} \\ &\quad = D_{t,q}\mathbf{1}_{\left\{\zeta_{\mathfrak{i}}(\boldsymbol{\rho}_{\mathfrak{i}}\boldsymbol{\zeta}_{\mathfrak{i}})\circ\Phi_{\mathfrak{i}}\cap B\left(\varrho_{\mathfrak{i}},3\lambda_{q+\bar{n}}^{-1}\right)\circ\Phi_{\mathfrak{i}}\right\}} + \left(D_{t,\bar{q}''} - D_{t,q}\right)\mathbf{1}_{\left\{\zeta_{\mathfrak{i}}(\boldsymbol{\rho}_{\mathfrak{i}}\boldsymbol{\zeta}_{\mathfrak{i}})\circ\Phi_{\mathfrak{i}}\cap B\left(\varrho_{\mathfrak{i}},3\lambda_{q+\bar{n}}^{-1}\right)\circ\Phi_{\mathfrak{i}}\right\}} = 0\,. \end{aligned} \tag{9.35}$$

on $\mathbb{T}^3 \times \{|t-t_0| \leq 2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}\}$. Here, recalling Definition 8.4.1, the first term vanishes since $D_{t,q}\Phi_{\mathfrak{i}} \equiv 0$, while the second term vanishes due to dodging. It follows that the set $\left(\operatorname{supp}\left(\mathcal{X}_{q,\xi,\vec{l},\diamond}\boldsymbol{\rho}_{(\xi)}^{\diamond}\boldsymbol{\zeta}_{\xi}^{I,\diamond}\right) \cap B\left(\varrho_{(\xi),\diamond}^{I}, 3\lambda_{q+\bar{n}}^{-1}\right)\right) \circ \Phi_{(i,k)}$ remains the same even though the deformation is replaced by the one induced by the vector field $\widehat{u}_{\bar{q}''}$. Therefore, applying the same argument to get the upper bound of $\mathcal{N}_{\Omega,z,I}$ in `Step 2`, we can count the remaining indices at some fixed time and conclude the proof the claim concerning the cardinality of the set of indices satisfying (9.34).

We define

$$\mathfrak{I}_{\Omega,t_0} = \left\{(\xi,i,j,k,\vec{l},\diamond,I) \text{ such that (9.34) holds}\right\}$$

and note that its cardinality is bounded by the q-independent constant $\mathcal{C}_{\text{counting}}$. In the remainder of the proof we shall abbreviate a tuple of indices $(\xi,i,j,k,\vec{l},\diamond,I)$ with $\mathfrak{i}$ and use $\mathfrak{i}$ as a subscript/superscript on any cutoff functions or flow maps which are part of w_{q+1}.

`Step 3b: Defining` L `and checking` (5.28) `and` (5.29)`.` We now define

$$L(q+\bar{n},\bar{q}'',\Omega,t_0) = \widetilde{\Omega}\circ\Phi_{\bar{q}''}\bigcap\left[\bigcup_{\mathfrak{I}_{\Omega,t_0}}\operatorname{supp}\left[\zeta_{\mathfrak{i}}\left(\boldsymbol{\rho}_{\mathfrak{i}}\boldsymbol{\zeta}_{\mathfrak{i}}\right)\circ\Phi_{\mathfrak{i}}\right]\cap B\left(\varrho_{\mathfrak{i}},3\lambda_{q+\bar{n}}^{-1}\right)\circ\Phi_{\mathfrak{i}}\right].$$

The first claim easily follows from (9.35) and $D_{t,\bar{q}''}\Phi_{\bar{q}''} = 0$.

In order to prove the second claim in (5.28), we first note that by the definition of $\mathfrak{I}_{\Omega,t_0}$, $L(q+\bar{n},\bar{q}'',\Omega,t_0)$ contains $\operatorname{supp} w_{q+1} \cap \Omega\circ\Phi_{\bar{q}''}$. Then due to the fact that in the definition of L we have enlarged the support of each $\varrho_{\mathfrak{i}}$, the fact that $\widetilde{\Omega}(t) := \Phi_{\bar{q}''}(t)^{-1}(\widetilde{\Omega})$ contains a ball of radius $2\lambda_{q+\bar{n}}^{-1}$ around $\Omega(t)$ for all $|t-t_0| \leq \tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2}$, and the fact that (9.34) has

doubled the timescale over which overlap is being considered, we have that the second claim in (5.28) follows from Definition 9.2.1.

Finally, we must check (5.29). Note that at time t_0, L is defined using intermittent pipe bundles which have been deformed on the Lipschitz timescale $2\tau_{\bar{q}''}\Gamma_{\bar{q}''}^{-i''+2} \leq \tau_q \Gamma_q^{-i-7}$. Note furthermore that due to `Step 3a`, we are only considering $\mathcal{C}_{\text{counting}}$ many such bundles, and that due to the fact that the diameter of $\widetilde{\Omega}(t)$ is bounded on the Lipschitz timescale by a constant times the size $\lambda_{q+\bar{n}/2}^{-1}\Gamma_q^{-1}$ of a periodic cell, each bundle may only contribute a q-independent number $\mathcal{C}_\xi$ of deformed pipe segments. We set $\mathcal{C}_D = \mathcal{C}_\xi \mathcal{C}_{\text{counting}}$, which we emphasize is independent of q, concluding the proof of (5.29). □

9.3 Estimates for w_{q+1}

Lemma 9.3.1 (Coefficient function estimates). *For N, N', N'', M with $N'', N' \in \{0,1\}$ and $N, M \leq \mathsf{N}_{\text{fin}}/3$, we have the following estimates.*

$$\begin{aligned}
&\left\| D^{N-N''} D_{t,q}^M (\xi^\ell A_\ell^h \partial_h)^{N'} D^{N''} a_{\xi,i,j,k,\vec{l},\varphi} \right\|_r \\
&\quad \lesssim \left| \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\varphi} \right|^{1/r} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} \Gamma_q^{j-1} \left(\Gamma_q^{-5} \lambda_{q+1} \right)^N \left(\Gamma_q^5 \Lambda_q \right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+4}, \mathrm{T}_q^{-1} \right) ,
\end{aligned} \tag{9.36a}$$

$$\begin{aligned}
&\left\| D^{N-N''} D_{t,q}^M (\xi^\ell A_\ell^h \partial_h)^{N'} D^{N''} \left(a_{\xi,i,j,k,\vec{l},\varphi} \left(\boldsymbol{\rho}_{(\xi)}^{\varphi} \boldsymbol{\zeta}_\xi^{I,\varphi} \right) \circ \Phi_{(i,k)} \right) \right\|_r \\
&\quad \lesssim \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},\varphi} \boldsymbol{\zeta}_\xi^{I,\varphi} \right) \right|^{1/r} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} \Gamma_q^{j+1} \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \right)^N \left(\Gamma_q^5 \Lambda_q \right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+4}, \mathrm{T}_q^{-1} \right) ,
\end{aligned} \tag{9.36b}$$

$$\begin{aligned}
&\left\| D^{N-N''} D_{t,q}^M (\xi^\ell A_\ell^h \partial_h)^{N'} D^{N''} a_{\xi,i,j,k,\vec{l},R} \right\|_r \\
&\quad \lesssim \left| \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},R} \right|^{1/r} \delta_{q+\bar{n}}^{1/2} \Gamma_q^{j+4} \left(\Gamma_q^{-5} \lambda_{q+1} \right)^N \left(\Gamma_q^{13} \Lambda_q \right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) ,
\end{aligned} \tag{9.36c}$$

$$\begin{aligned}
&\left\| D^{N-N''} D_{t,q}^M (\xi^\ell A_\ell^h \partial_h)^{N'} D^{N''} \left(a_{\xi,i,j,k,\vec{l},R} \left(\boldsymbol{\rho}_{(\xi)}^{R} \boldsymbol{\zeta}_\xi^{I,R} \right) \circ \Phi_{(i,k)} \right) \right\|_r \\
&\quad \lesssim \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},R} \boldsymbol{\zeta}_\xi^{I,R} \right) \right|^{1/r} \delta_{q+\bar{n}}^{1/2} \Gamma_q^{j+7} \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \right)^N \left(\Gamma_q^{13} \Lambda_q \right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) .
\end{aligned} \tag{9.36d}$$

In the case that $r = \infty$, the above estimates give that

$$\left\| D^{N-N''} D^M_{t,q} (\xi^\ell A^h_\ell \partial_h)^{N'} D^{N''} a_{\xi,i,j,k,\vec{l},R} \right\|_\infty \lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+7} \left(\Gamma_q^{-5}\lambda_{q+1}\right)^N \times \left(\Gamma_q^{13}\Lambda_q\right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) . \tag{9.37a}$$

$$\left\| D^{N-N''} D^M_{t,q} (\xi^\ell A^h_\ell \partial_h)^{N'} D^{N''} a_{\xi,i,j,k,\vec{l},\varphi} \right\|_\infty \lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+2} r_q^{-1/3} \left(\Gamma_q^{-5}\lambda_{q+1}\right)^N \times \left(\Gamma_q^{8}\Lambda_q\right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+4}, \mathrm{T}_q^{-1}\right) , \tag{9.37b}$$

with analogous estimates (incorporating a loss of Γ_q^3 for $\diamond = R$ and Γ_q^2 for $\diamond = \varphi$) holding for the product $a_{(\xi),\diamond}\boldsymbol{\zeta}_\xi^{I,\diamond}\boldsymbol{\rho}^\diamond_{(\xi)}$. Finally, we have the pointwise estimates

$$\left| D^{N-N''} D^M_{t,q} (\xi^\ell A^h_\ell \partial_h)^{N'} D^{N''} a_{\xi,i,j,k,\vec{l},R} \right| \lesssim \Gamma_q^{12} \pi_\ell^{1/2} \left(\Gamma_q^{-5}\lambda_{q+1}\right)^N \left(\Gamma_q^{13}\Lambda_q\right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \tag{9.38a}$$

$$\left| D^{N-N''} D^M_{t,q} (\xi^\ell A^h_\ell \partial_h)^{N'} D^{N''} a_{\xi,i,j,k,\vec{l},\varphi} \right| \lesssim \Gamma_q^{12} \pi_\ell^{1/2} r_q^{-1/3} \left(\Gamma_q^{-5}\lambda_{q+1}\right)^N \left(\Gamma_q^{5}\Lambda_q\right)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+4}, \mathrm{T}_q^{-1}\right) . \tag{9.38b}$$

Proof of Lemma 9.3.1. We first prove (9.36a) and (9.36b), since a portion of $a_{(\xi),\varphi}$ appears in the definition of the Reynolds corrector in (9.8). We further simplify by computing (9.36a) for the case $r = \infty$ first. Recalling estimate (8.24c), we have that for all $N, M \leq \mathsf{N}_{\mathrm{fin}}/2$,

$$\left\| D^N D^M_{t,q} \varphi_\ell \right\|_{L^\infty(\operatorname{supp}\psi_{i,q}\omega_{j,q})} \lesssim \delta_{q+\bar{n}}^{3/2} r_q^{-1} \Gamma_q^{3j-7} \left(\Gamma_q\Lambda_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i}, \mathrm{T}_q^{-1}\right) .$$

Thus from definition (9.1), the Leibniz rule, and Corollary 8.2.4, and the fact that $\operatorname{supp}\eta_{i,j,k,\xi,\vec{l},\varphi}$ is contained in $\operatorname{supp}\psi_{i,q}\omega_{j,q}\chi_{i,k,q}$ we have that for $N, M \leq \mathsf{N}_{\mathrm{fin}}/2$,

$$\left\| D^N D^M_{t,q} \varphi_{q,i,k} \right\|_{L^\infty(\operatorname{supp}\eta_{i,j,k,\xi,\vec{l},\varphi})} \lesssim \delta_{q+\bar{n}}^{3/2} r_q^{-1} \Gamma_q^{3j-7} \left(\Gamma_q\Lambda_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i}, \mathrm{T}_q^{-1}\right) . \tag{9.39}$$

The above estimates allow us to apply [7, Lemma A.5] with $N' = M' = \mathsf{N}_{\mathrm{fin}}/2$, $\psi = \widetilde{\gamma}_{\xi,}$,

$\Gamma_\psi = 1$, $v = \widehat{u}_q$, $D_t = D_{t,q}$, $h(x,t) = \varphi_{q,i,k}(x,t)$, $C_h = \delta_{q+\bar{n}}^{3/2} r_q^{-1} \Gamma_q^{3j-6} = \Gamma^2$, $\lambda = \widetilde{\lambda} = \Lambda_q \Gamma_q$, $\mu = \tau_q^{-1}\Gamma_q^i$, $\widetilde{\mu} = \mathrm{T}_q^{-1}$, and $N_t = \mathsf{N}_{\mathrm{ind,t}}$. We obtain that for all $N, M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_4$,

$$\left\| D^N D_{t,q}^M \widetilde{\gamma}_\xi \left(\frac{\varphi_{q,i,k}}{\delta_{q+\bar{n}}^{3/2} r_q^{-1} \Gamma_q^{3j-3}} \right) \right\|_{L^\infty(\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\varphi})} \lesssim (\Gamma_q \Lambda_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^i, \mathrm{T}_q^{-1}\right) . \quad (9.40)$$

Finally, from Corollary 8.2.4 and an application of the mixed derivative Fa'a di Bruno formula from [7, Lemma A.5] with $\psi(\cdot) : B_{1/2}(\xi) \to \mathbb{R}$ defined by $\psi(\cdot) = |\cdot|^{-4/3}$, $\Gamma_\psi = 1$, $v = \widehat{u}_q$, $\Gamma = 1$, $\lambda = \widetilde{\lambda} = \Lambda_q$, $\mu = \tau_q^{-1}\Gamma_q^i$, $\widetilde{\mu} = \Gamma_q^{-1}\mathrm{T}_q^{-1}$, $N_x = 0$, $N_t = \mathsf{N}_{\mathrm{ind,t}}$, $h = \nabla \Phi_{(i,k)}^{-1} \xi$, and $\mathcal{C}_h = 1$, we have that for all $N + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2$,

$$\left\| D^N D_{t,q}^M \left(\left| \nabla \Phi_{(i,k)}^{-1} \xi \right|^{-4/3} \right) \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q}^\varphi \chi_{i,k,q}^\varphi))} \lesssim \Lambda_q^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^{-1}\right) .$$

From the above three bounds, definition (9.2), the Leibniz rule, estimate (5.37) at level q, (8.4), (8.28), and (8.37), we obtain that for $N' = 0, 1$ and $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_2$,

$$\left\| D^N D_{t,q}^M (\xi^\ell A_\ell^j \partial_j)^{N'} a_{\xi,i,j,k,\vec{l},\varphi} \right\|_\infty \lesssim \delta_{q+\bar{n}}^{1/2} \Gamma_q^{j-1} r_q^{-1/3} (\Gamma_q^{-5} \lambda_{q+1})^N (\Gamma_q^5 \Lambda_q)^{N'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+4}, \mathrm{T}_q^{-1}\right) . \quad (9.41)$$

Using (8.27), we obtain (9.37b). When $r \neq \infty$, we use $\|f\|_{L^r} \leq \|f\|_{L^\infty} |\{\operatorname{supp} f\}|^{1/r}$ and the demonstrated bound for $r = \infty$ to obtain (9.36a) for the full range of r and for $N'' = 0$. The estimate in (9.36b) for $N'' = 0$ follows in the same way using (7.23) for $p = \infty$ and (8.40). Similar estimates for $N'' = 1$ in both cases are nearly identical, and we omit the details

We now compute (9.36c) for the case $r = \infty$, from which the remaining bounds in (9.36d) and (9.37a) will follow as before. Recalling estimates (8.24a) and (8.24b), we have that for all $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_2$,

$$\begin{aligned} &\left\| D^N D_{t,q}^M R_\ell \right\|_{L^\infty(\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},R})} + \left\| D^N D_{t,q}^M \pi_\ell \right\|_{L^\infty(\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},R})} \\ &\lesssim \delta_{q+\bar{n}} \Gamma_q^{2j+6} (\Gamma_q \Lambda_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^i, \mathrm{T}_q^{-1}\right) . \end{aligned}$$

From (5.37) and (5.8) at level q, (8.21), (8.28), (8.4), (8.34a), (8.2.4), and (9.40), we find that

$$\left\| D^N D^M_{t,q} \sum_{i',j',k',\xi',\vec{l}'} \frac{\delta_{q+\bar{n}} \Gamma_q^{2j'-4} C}{|\nabla \Phi_{i',k'} \xi'|^{4/3}} \psi^4_{i',q} \omega^4_{j',q} \chi^4_{i',k',q} \mathcal{X}^4_{q,\xi',l'} \circ \Phi_{i',k',q} \widetilde{\gamma}^2_\xi \nabla \Phi^{-1}_{(i',k')} \xi' \otimes \xi' \nabla \Phi^{-T}_{(i',k')} \right\|_{L^\infty(\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},R})}$$
$$\lesssim \delta_{q+\bar{n}} \Gamma_q^{2j-4} \left(\Gamma_q^5 \Lambda_q \right)^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+5}, \mathrm{T}_q^{-1} \right) .$$

Thus from the Leibniz rule and definition (9.8), we find that for $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_2$,

$$\left\| D^N D^M_{t,q} R_{q,i,k} \right\|_{L^\infty(\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},R})} \lesssim \delta_{q+\bar{n}} \Gamma_q^{2j+6} \left(\Gamma_q^5 \Lambda_q \right)^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+5}, \mathrm{T}_q^{-1} \right) ; \quad (9.42)$$

the loss of Γ_q in the sharp material derivative cost comes from the fact that the sum includes $\psi_{i',q}$ and is estimated on the supported of $\psi_{i,q}$. The above estimates allow us to apply [7, Lemma A.5] with $N' = M' = {}^{\mathsf{N}_{\mathrm{fin}}}/_2$, $\psi = \Gamma_q^{-5} \gamma_{\xi,\Gamma_q^9}$ as in (7.5),[4] $\Gamma_\psi = 1$, $v = \widehat{u}_q$, $D_t = D_{t,q}$, $h(x,t) = R_{q,i,k}(x,t)$, $C_h = \delta_{q+\bar{n}} \Gamma_q^{2j+6}$, $\Gamma^2 = \delta_{q+\bar{n}} \Gamma_q^{2j-2}$, $\lambda = \widetilde{\lambda} = \Lambda_q \Gamma_q^5$, $\mu = \tau_q^{-1} \Gamma_q^{i+5}$, $\widetilde{\mu} = \mathrm{T}_q^{-1}$, and $N_t = \mathsf{N}_{\mathrm{ind,t}}$. We obtain that for all $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_2$,

$$\left\| D^N D^M_{t,q} \gamma_{\xi,\Gamma_q^9} \left(\frac{R_{q,i,k}}{\delta_{q+\bar{n}} \Gamma_q^{2j-2}} \right) \right\|_{L^\infty(\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},R})} \lesssim \Gamma_q^5 \left(\Gamma_q^{13} \Lambda_q \right)^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) .$$

From the above bound, definition (9.9), the Leibniz rule, estimate (5.37) at level q, (8.13b), (8.4), (8.28), and (8.37), we obtain that for $N' = 0, 1$ and $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_2$,

$$\left\| D^N D^M_{t,q} (\xi^\ell A^j_\ell \partial_j)^{N'} a_{\xi,i,j,k,\vec{l},R} \right\|_{L^\infty} \lesssim \delta^{1/2}_{q+\bar{n}} \Gamma_q^{j+4} (\Gamma_q^{-5} \lambda_{q+1})^N (\Gamma_q^{13} \Lambda_q)^{N'} \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) .$$

Using (8.27), we obtain (9.37a) for $N'' = 0$. When $r \neq \infty$, we use $\|f\|_{L^r} \leq \|f\|_{L^\infty} |\{\operatorname{supp} f\}|^{1/r}$ and the demonstrated bound for $r = \infty$ to obtain (9.36c) for the full range of r and $N'' = 0$. The estimate in (9.36d) follows in the same way using (7.23) for $p = \infty$ and (8.40) and the

[4] Since γ_{ξ,Γ_q^9} and all its derivatives are bounded by Γ_q^5 from (7.3), we first rescale by Γ_q^{-5} on the outside and then apply the Faa di Bruno lemma, which requires ψ to be bounded in between 0 and 1. Rescaling back then produces the desired bound.

fact that $\boldsymbol{\zeta}_{\xi}^{I,R} \leq 1$. Estimates for $N'' = 1$ are again nearly identical, and we omit further details.

Finally, we prove the pointwise estimates. Recalling that the left-hand side of (9.41) is supported inside the support of $\omega_{j,q}$ and using (8.21) and (8.22c) proves the claim for $\diamond = \varphi$. Arguing analogously for $\diamond = R$ concludes the proof. $\square$

Corollary 9.3.2 (**Full velocity increment estimates**). *For $N, M \leq \mathsf{N}_{\text{fin}}/4$, we have the estimates*

$$\left\| D^N D_{t,q}^M w_{(\xi),\diamond}^{(p),I} \right\|_{L^r} \lesssim \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \right|^{1/r} \delta_{q+\bar{n}}^{1/2} \Gamma_q^{j+7} r_q^{\frac{2}{r}-1} \lambda_{q+\bar{n}}^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) \tag{9.43a}$$

$$\left\| D^N D_{t,q}^M w_{(\xi),\diamond}^{(p),I} \right\|_{L^\infty} \lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+10} r_q^{-1} \lambda_{q+\bar{n}}^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) . \tag{9.43b}$$

Also, for $N, M \leq \mathsf{N}_{\text{fin}}/4$, we have that

$$\left\| D^N D_{t,q}^M w_{(\xi),\diamond}^{(c),I} \right\|_{L^r} \lesssim r_q \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \right|^{1/r} \delta_{q+\bar{n}}^{1/2} \Gamma_q^{j+7} r_q^{\frac{2}{r}-1} \lambda_{q+\bar{n}}^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) \tag{9.44a}$$

$$\left\| D^N D_{t,q}^M w_{(\xi),\diamond}^{(c),I} \right\|_{L^\infty} \lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+10} \lambda_{q+\bar{n}}^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) . \tag{9.44b}$$

Proof of Corollary 9.3.2. Recalling the definition of $w_{(\xi),\diamond}^{(p),I}$ from (9.5) and (9.13), we shall prove (9.43a) by applying Lemma A.1.3 with

$$N_* = M_* = \mathsf{N}_{\text{fin}}/4 , \qquad f = a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \nabla \Phi_{(i,k)}^{-1} , \qquad \Phi = \Phi_{(i,k)} ,$$

$$\lambda = \lambda_{q+\lfloor \bar{n}/2 \rfloor} , \qquad \tau^{-1} = \tau_q^{-1} \Gamma_q^{i+13} , \qquad \mathrm{T} = \mathrm{T}_q \Gamma_q^{-8} , \qquad \mathcal{C}_{f,R} = \left| \operatorname{supp} \eta_{(\xi),R} \boldsymbol{\zeta}_{\xi}^{I,R} \right|^{1/r} \delta_{q+\bar{n}}^{1/2} \Gamma_q^{j+7}$$

$$\mathcal{C}_{f,\varphi} = \left| \operatorname{supp} \eta_{(\xi),\varphi} \boldsymbol{\zeta}_{\xi}^{I,\varphi} \right|^{1/r} \delta_{q+1}^{1/2} r_q^{-1/3} \Gamma_q^{j+7} , \qquad v = \widehat{u}_q , \qquad \varphi = \mathbb{W}_{(\xi),\diamond}^I , \qquad \mu = \lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q ,$$

$$\Upsilon = \Lambda = \lambda_{q+\bar{n}} , \qquad \mathcal{C}_{\varrho,R} = r_q^{\frac{2}{r}-1} , \qquad \mathcal{C}_{\varrho,\varphi} = r_q^{\frac{2}{r}-\frac{2}{3}} , \qquad N_t = \mathsf{N}_{\text{ind,t}} .$$

From (9.36), Corollary 8.2.4, and (8.40), we have that for $N, M \leq \mathsf{N}_{\text{fin}}/4$,

$$\begin{aligned}
&\left\| D^N D_{t,q}^M \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \right) \right\|_r \\
&\qquad \lesssim \left| \operatorname{supp} \eta_{(\xi),\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right|^{1/r} \delta_{q+1}^{1/2} \Gamma_q^{j+7} \lambda_{q+\lfloor \bar{n}/2 \rfloor}^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+13}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) \qquad (9.45) \\
&\left\| D^N D_{t,q}^M (D\Phi_{(i,k)})^{-1} \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q} \widetilde{\chi}_{i,k,q}))} \leq \Lambda_q^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_q^i \tau_q^{-1}, \mathrm{T}_q^{-1} \Gamma_q^{-1} \right), \qquad (9.46) \\
&\left\| D^N \Phi_{(i,k)} \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q} \widetilde{\chi}_{i,k,q}))} + \left\| D^N \Phi_{(i,k)}^{-1} \right\|_{L^\infty(\operatorname{supp}(\psi_{i,q} \widetilde{\chi}_{i,k,q}))} \lesssim \Gamma_q^{-1} \Lambda_q^{N-1}, \qquad (9.47)
\end{aligned}$$

showing that (A.12), (A.13), and (A.14) are satisfied. From Proposition 7.1.5 and 7.1.6, we have that from $\mathbb{W}_{(\xi),\diamond}^I$ is periodic to scale $\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q$, in addition to the estimates (7.12) and (7.17), and so (A.15) is satisfied for $\diamond = R, \varphi$. Next, from (4.21) and (4.24a), the assumptions (A.16) and (A.17) are satisfied. We may thus apply Lemma A.1.3 to obtain that for $N, M \leq \mathsf{N}_{\text{fin}}/4$, (9.43a) is satisfied. Applying (8.27) then gives (9.43b).

The argument for the corrector is similar, save for the fact that $D_{t,q}$ will land on $\nabla a_{(\xi)}$, and so we require an extra commutator estimate from Lemma A.2.3, specifically Remark A.2.4. We omit the details of this commutator estimates and refer the reader to [7, Corollary 8.2]. However, we note that the gain in amplitude comes from the quotient of a spatial derivative cost of $\lambda_{q+\lfloor \bar{n}/2 \rfloor}$ on the low-frequency function, and a gain of $\lambda_{q+\bar{n}}$ from (7.12) or (7.17). Using the definition of r_q gives a net gain of $r_q \Gamma_q^{-1}$, concluding the proof. □

9.4 Velocity increment potential

In this section, we define a potential for w_{q+1} along with an error term, construct its pressure increment and the associated current errors, and investigate their properties.

Lemma 9.4.1 (Velocity increment potential). *For a given $w_{q+1}^{(\mathfrak{l})}$, $\mathfrak{l} = p, c$, as in* (9.16), *there exists a tensor $\upsilon_{q+1}^{(\mathfrak{l})}$ and an error $e_{q+1}^{(\mathfrak{l})}$ such that the following hold.*

(i) Let d *be as in* (xvii) *of section 4.1. Then $w_{q+1}^{(\mathfrak{l})}$ can be written in terms of $\upsilon_{q+1}^{(\mathfrak{l})}$ and $e_{q+1}^{(\mathfrak{l})}$*

as

$$\begin{aligned} w_{q+1}^{(p)} &= \operatorname{div}^{\mathsf{d}} \upsilon_{q+1}^{(p)} + e_{q+1}^{(p)} \\ w_{q+1}^{(c)} &= \operatorname{div}^{\mathsf{d}} (r_q \Gamma_q^{-1} \upsilon_{q+1}^{(c)}) + r_q \Gamma_q^{-1} e_{q+1}^{(c)} \,, \end{aligned} \tag{9.48}$$

or equivalently notated component-wise as $(w_{q+1}^{(p)})^{\bullet} = \partial_{i_1} \dots \partial_{i_{\mathsf{d}}} \upsilon_{q+1}^{(p,\bullet,i_1,\dots,i_{\mathsf{d}})} + e_{q+1}^{\bullet}$.

(ii) $\upsilon_{q+1}^{(l)}$ *and* $e_{q+1}^{(l)}$ *have the support property*[5]

$$\begin{aligned} &\operatorname{supp}(\upsilon_{q+1}^{(l)}),\ \operatorname{supp}(e_{q+1}^{(l)}) \\ &\qquad \subseteq \bigcup_{\xi,i,j,k,\vec{l},I,\diamond} \operatorname{supp}\left(\chi_{i,k,q} \zeta_{q,\diamond,i,k,\xi,\vec{l}} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond}\right) \circ \Phi_{(i,k)}\right) \cap B\left(\operatorname{supp} \varrho_{(\xi),\diamond}^{I}, 2\lambda_{q+\bar{n}}^{-1}\right) \circ \Phi_{(i,k)} \,. \end{aligned} \tag{9.49}$$

(iii) For $0 \le k \le \mathsf{d}$, $(\upsilon_{q+1,k}^{(l)})^{\bullet} := \lambda_{q+\bar{n}}^{\mathsf{d}-k} \partial_{i_1} \cdots \partial_{i_k} \upsilon_{q+1}^{(l,\bullet,i_1,\dots,i_{\mathsf{d}})}$,[6] *satisfies the estimates*

$$\left\| \psi_{i,q} D^N D_{t,q}^M \upsilon_{q+1,k}^{(l)} \right\|_3 \le \Gamma_q^{10} \delta_{q+\bar{n}}^{\frac{1}{2}} r_q^{-\frac{1}{3}} \lambda_{q+\bar{n}}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+14} \tau_q^{-1}, \Gamma_q^8 \mathrm{T}_q^{-1}\right) \tag{9.50a}$$

$$\left\| \psi_{i,q} D^N D_{t,q}^M \upsilon_{q+1,k}^{(l)} \right\|_\infty \le \Gamma_q^{\frac{\mathsf{c}_\infty}{2}+10} r_q^{-1} \lambda_{q+\bar{n}}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+14} \tau_q^{-1}, \Gamma_q^8 \mathrm{T}_q^{-1}\right) \tag{9.50b}$$

for $N \le \mathsf{N}_{\mathrm{fin}}/4 - 2\mathsf{d}^2$ *and* $M \le \mathsf{N}_{\mathrm{fin}}/5$.

(iv) $e_{q+1}^{(l)}$ *satisfies*

$$\left\| D^N D_{t,q}^M e_{q+1}^{(l)} \right\|_\infty \le \delta_{q+3\bar{n}}^3 \mathrm{T}_{q+\bar{n}}^{20\mathsf{N}_{\mathrm{ind,t}}} \lambda_{q+\bar{n}}^{-10} \lambda_{q+\bar{n}}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \Gamma_q^8 \mathrm{T}_q^{-1}\right) \,. \tag{9.51}$$

for $N \le \mathsf{N}_{\mathrm{fin}}/4 - 2\mathsf{d}^2$ *and* $M \le \mathsf{N}_{\mathrm{fin}}/5$.

Remark 9.4.2 (Notation for cumulative velocity increment potential). We let $\upsilon_{q+1} := \upsilon_{q+1}^{(p)} + r_q \Gamma_q^{-1} \upsilon_{q+1}^{(c)}$ and $\upsilon_{q+1,k}^{\bullet} := \lambda_{q+\bar{n}}^{\mathsf{d}-k} \partial_{i_1} \cdots \partial_{i_k} \upsilon_{q+1}^{(\bullet,i_1,\dots,i_{\mathsf{d}})}$. As a corollary of Lemma

[5] For any smooth set $\Omega \subset \mathbb{T}^3$, we use $\Omega \circ \Phi_{(i,k)}$ to denote the set $\Phi_{(i,k)}^{-1}(\Omega) \subset \mathbb{T}^3 \times \mathbb{R}$, i.e. the space-time set whose characteristic function is annihilated by $D_{t,q}$.

[6] If $k = 0$, we adopt the convention that $\partial_{i_1} \cdots \partial_{i_k}$ is the identity operator.

9.4.1, we have that

$$w_{q+1} = \mathrm{div}^{\mathsf{d}} \upsilon_{q+1} + e_{q+1}\,,$$

where υ_{q+1} and e_{q+1} share the properties (9.49)–(9.51) with $\upsilon^{(\mathtt{1})}_{q+1}$ and $e^{(\mathtt{1})}_{q+1}$ after adjusting the inequalities to include an implicit constant.

Proof. Recall from subsection 9.1 that $w_{q+1} = w_{q+1,R} + w_{q+1,\varphi}$ where

$$w_{q+1,\diamond} = \sum_{i,j,k,\xi,\vec{l},I} a_{(\xi),\diamond} \nabla \Phi^{-1}_{(i,k)} (\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} \mathbb{W}^{I}_{(\xi),\diamond} \circ \Phi_{(i,k)} \tag{9.52}$$

$$+ \sum_{i,j,k,\xi,\vec{l},I} \nabla \left((\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} a_{(\xi),\diamond} \right) \times \left(\nabla \Phi_{(i,k)} \mathbb{U}^{I}_{(\xi),\diamond} \circ \Phi_{(i,k)} \right) \tag{9.53}$$

for $\diamond = R, \varphi$. To construct υ_{q+1} and e_{q+1}, we will apply Corollary A.3.11 to the right hand side terms. We shall adhere to the convention set out in Remark A.3.8 and treat each component separately, so that the resulting tensor potential does not have any special symmetry properties.

Fix values for all indexes $i, j, k, \xi, \vec{l}, I$, set $\diamond = R$, and consider one component, indexed by $\bullet$, of the vector field in (9.52). Set

$$\begin{aligned}
&p = 3, \infty\,, \quad N_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_{4}, \quad M_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_{5}, \quad M_t = \mathsf{N}_{\mathrm{ind,t}},\\
&G = a_{(\xi),R} \nabla \Phi^{-1}_{(i,k)} (\boldsymbol{\rho}^{R}_{(\xi)} \boldsymbol{\zeta}^{I,R}_{\xi}) \circ \Phi_{(i,k)} \xi^{\bullet} r_q^{-1/3}\,, \quad \Phi = \Phi_{(i,k)}, \quad \pi = \pi_{\ell} \Gamma_q^{30}, \quad r_G = r_q\\
&\mathcal{C}_{G,p} = \left| \mathrm{supp}\, \left(\eta_{i,j,k,\xi,\vec{l},R} \boldsymbol{\zeta}^{I,R}_{\xi} \right) \right|^{1/p} \delta^{\frac{1}{2}}_{q+\bar{n}} \Gamma_q^{j+7}, \quad \lambda = \lambda_{q+\bar{n}/2}, \quad \lambda' = \lambda_q \Gamma_q, \quad \nu = \tau_q^{-1} \Gamma_q^{i+13}, \quad \nu' = \mathrm{T}_q^{-1} \Gamma_q^{8},\\
&\varrho = r_q^{1/3} \varrho_{\xi, \lambda_{q+\bar{n}}, \frac{\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q}{\lambda_{q+\bar{n}}}}\,, \quad \widetilde{\vartheta} = \widetilde{\vartheta}_{\xi, \lambda_{q+\bar{n}}, \frac{\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q}{\lambda_{q+\bar{n}}}, R}\\
&\mathcal{C}_{*,3} = 1\,, \quad \mathcal{C}_{*,\infty} = r_q^{-2/3}, \quad \mu = \lambda_{q+\bar{n}/2} \Gamma_q, \quad \Upsilon = \Upsilon' = \Lambda = \lambda_{q+\bar{n}},
\end{aligned}$$

where $\widetilde{\vartheta}$ is constructed from Proposition 7.1.5 with $\mathsf{D} = \mathsf{d}^2$. Then, all assumptions of Corollary A.3.11 hold by (4.24a), (9.36d), (9.38a), (5.34), Corollary 8.2.4, (8.11), and Proposi-

tion 7.1.5. Then from (A.89), there exist $R =: \upsilon^{(p)}_{(\xi),I,R}$ and $E =: e^{(p)}_{(\xi),I,R}$ such that

$$a_{(\xi),R}\nabla\Phi^{-1}_{(i,k)}(\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^{I,R}_{\xi})\circ\Phi_{(i,k)}\mathbb{W}^I_{(\xi),R}\circ\Phi_{(i,k)} = \operatorname{div}^{\mathsf{d}}\upsilon^{(p)}_{(\xi),I,R} + e^{(p)}_{(\xi),I,R}\,.$$

From (A.92), we have that

$$\begin{aligned}\left\|D^N D^M_{t,q}\partial_{i_1}\cdots\partial_{i_l}\upsilon^{(p)}_{(\xi),I,R}\right\|_3 &\le \left\|D^N D^M_{t,q} D^l \upsilon^{(p)}_{(\xi),I,R}\right\|_3 \\ &\lesssim \left|\operatorname{supp}\left(\eta_{(\xi),R}\boldsymbol{\zeta}^{I,R}_{\xi}\right)\right|^{1/3}\delta^{\frac{1}{2}}_{q+\bar{n}}\Gamma_q^{j+7}r_q^{-1/3}\lambda^{l-\mathsf{d}+N}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\,, \end{aligned} \tag{9.54a}$$

$$\begin{aligned}\left\|D^N D^M_{t,q}\partial_{i_1}\cdots\partial_{i_l}\upsilon^{(p)}_{(\xi),I,R}\right\|_\infty &\le \left\|D^N D^M_{t,q} D^l \upsilon^{(p)}_{(\xi),I,R}\right\|_\infty \\ &\lesssim \delta^{\frac{1}{2}}_{q+\bar{n}}\Gamma_q^{j+7}r_q^{-1}\lambda^{l-\mathsf{d}}_{q+\bar{n}}\lambda^{N+\alpha}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\ &\lesssim \Gamma_q^{(\mathsf{C}_\infty+20)/2}r_q^{-1}\lambda^{l-\mathsf{d}}_{q+\bar{n}}\lambda^{N+\alpha}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+3}, \mathrm{T}_q^{-1}\right)\,, \end{aligned} \tag{9.54b}$$

for $0 \le l \le \mathsf{d}$, $N + l \le {}^{\mathsf{N}_{\text{fin}}}/_4 - \mathsf{d}^2$, and $M \le {}^{\mathsf{N}_{\text{fin}}}/_5$, where we used (8.27) in the last inequality. From (A.93), we have that

$$\left\|D^N D^M_{t,q} e^{(p)}_{(\xi),I,R}\right\|_\infty \lesssim \delta^{\frac{1}{2}}_{q+\bar{n}}\Gamma_q^{j+7}r_q^{-1}\left({}^{\lambda_{q+\bar{n}/2}}/_{\lambda_{q+\bar{n}}}\right)^{\mathsf{d}}\lambda^{N+\alpha}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\,. \tag{9.55}$$

for $N \le {}^{\mathsf{N}_{\text{fin}}}/_4 - \mathsf{d}^2$, and $M \le {}^{\mathsf{N}_{\text{fin}}}/_5$. Furthermore, from (A.90) and (7) from Proposition 7.1.5, we have that the supports of $\upsilon^{(p)}_{(\xi),I,R}$ and $e^{(p)}_{(\xi),I,R}$ are contained in the set on the right-hand side of (9.49).

We now sum over indexes $i, j, k, \xi, \vec{l}, I$ and set

$$\upsilon^{(p)}_{q+1,R} = \sum_{i,j,k,\xi,\vec{l},I}\upsilon^{(p)}_{(\xi),I,R}\,, \qquad e^{(p)}_{q+1,R} = \sum_{i,j,k,\xi,\vec{l},I} e^{(p)}_{(\xi),I,R}\,, \tag{9.56}$$

which verifies the first equality in (9.48) and (9.49). Using (8.45) to obtain an L^∞ bound for the sum and Corollary 8.6.1 with $H_{i,j,k,\xi,\vec{l},R} = \upsilon^{(p)}_{(\xi),I,R}, \theta_2 = \theta = 1, p = 3, \mathcal{C}_H = \delta^{\frac{1}{2}}_{q+\bar{n}}\Gamma_q^7 r_q^{-1}, N_x = N_* = {}^{\mathsf{N}_{\text{fin}}}/_4 - \mathsf{d}^2$, the obvious choices for the other parameters, (9.54a), (9.54b), (9.55), and

(4.23b), we have that $\upsilon^{(p)}_{q+1,R}$ and $e^{(p)}_{q+1,R}$ satisfy

$$\begin{aligned}
\left\|\psi_{i,q}D^N D^M_{t,q}\partial_{i_1}\dots\partial_{i_k}(\upsilon^{(p)}_{q+1,R})^{(i_1,\dots,i_{\mathsf{d}})}\right\|_3 &\lesssim \Gamma_q^{10}\delta^{\frac{1}{2}}_{q+\bar{n}}r_q^{-1/3}\lambda^{k-\mathsf{d}}_{q+\bar{n}}\lambda^{N+\alpha}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\\
\left\|\psi_{i,q}D^N D^M_{t,q}\partial_{i_1}\dots\partial_{i_k}(\upsilon^{(p)}_{q+1,R})^{(i_1,\dots,i_{\mathsf{d}})}\right\|_\infty &\lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+10}r_q^{-1}\lambda^{k-\mathsf{d}}_{q+\bar{n}}\lambda^{N+\alpha}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right),\\
\left\|D^N D^M_{t,q}e^{(p)}_{q+1,R}\right\|_\infty &\lesssim \delta^3_{q+3\bar{n}}\mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\lambda^{-10}_{q+\bar{n}}\lambda^{N}_{q+\bar{n}}\mathcal{M}\left(M, M_t, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right)
\end{aligned}$$

for $N \leq \mathsf{N}_{\mathrm{fin}}/4 - \mathsf{d}^2$, and $M \leq \mathsf{N}_{\mathrm{fin}}/5$. The first inequality follows from Lemma (8.5.1) and Remark 8.6.1, and the second and the last inequalities use the support property noted earlier.

In a similar way, we work on (9.52) with φ and (9.53) with R, φ and generate $(\upsilon^{(p)}_{q+1,\varphi}, e^{(p)}_{q+1,\varphi})$, $(\upsilon^{(c)}_{q+1,R}, e^{(c)}_{q+1,R})$, and $(\upsilon^{(c)}_{q+1,\varphi}, e^{(c)}_{q+1,\varphi})$, respectively. Indeed, for (9.52) with φ, we set

$$G = a_{(\xi),\varphi}\nabla\Phi^{-1}_{(i,k)}(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi})\circ\Phi_{(i,k)}\xi, \quad \varrho = \varrho_{\xi,\lambda_{q+\bar{n}},\frac{\lambda_{q+\lfloor\bar{n}/2\rfloor\Gamma q}}{\lambda_{q+\bar{n}}},\varphi}, \quad \widetilde{\vartheta} = r_q^{-1/3}\widetilde{\vartheta}_{\xi,\lambda_{q+\bar{n}},\frac{\lambda_{q+\lfloor\bar{n}/2\rfloor\Gamma q}}{\lambda_{q+\bar{n}}},\varphi}$$

where $\widetilde{\vartheta}$ is constructed from Proposition 7.1.6 with $\mathsf{D} = \mathsf{d}^2$, and choose the rest of parameters and functions as in the case $\diamond = R$. The rest of the conclusions follow analogously to the case $\diamond = R$, and we omit further details. In the case of (9.53), we write

$$(w^{(c),I}_{(\xi),\diamond})^{\bullet} = r_q\Gamma_q^{-1}G_\diamond(\varrho_\diamond\circ\Phi),$$

where $G_\diamond$ and $\varrho_\diamond$ are defined by

$$\begin{aligned}
G_R &= \lambda^{-1}_{q+\bar{n}/2}\epsilon_{\bullet pr}\partial_p\left(a_{(\xi),R}\left(\boldsymbol{\rho}^{R}_{(\xi)}\boldsymbol{\zeta}^{I,R}_{\xi}\right)\circ\Phi_{(i,k)}\right)\partial_r\Phi^s_{(i,k)}, \quad \varrho_R = \lambda_{q+\bar{n}}(\mathbb{U}^I_{(\xi),R})^s, \quad \Phi = \Phi_{(i,k)}\\
G_\varphi &= r_q^{1/3}\lambda^{-1}_{q+\bar{n}/2}\epsilon_{\bullet pr}\partial_p\left(a_{(\xi),\varphi}\left(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi}\right)\circ\Phi_{(i,k)}\right)\partial_r\Phi^s_{(i,k)}, \quad \varrho_\varphi = r_q^{-1/3}\lambda_{q+\bar{n}}(\mathbb{U}^I_{(\xi),\varphi})^s, \quad \Phi = \Phi_{(i,k)}.
\end{aligned}$$

Due to the rescaling by $r_q\Gamma_q^{-1}$, we may apply Corollary A.3.11 to $(r_q\Gamma_q^{-1})^{-1}(w^{(c),I}_{(\xi),\diamond})^{\bullet}$ with the same choice of parameters as in the case $\mathtt{l} = p$. As a consequence, we obtain $(\upsilon^{(c)}_{q+1,\diamond}, e^{(c)}_{q+1,\diamond})$, defined as in (9.56), which enjoy the same properties as $(\upsilon^{(p)}_{q+1,R}, e^{(p)}_{q+1,R})$. Note that from the

construction, the velocity increment potential associated to the correctors satisfies

$$w^{(c)}_{q+1,\diamond} = \mathrm{div}^{\mathsf{d}}(r_q\Gamma_q^{-1}\upsilon^{(c)}_{q+1,\diamond}) + r_q\Gamma_q^{-1}e^{(c)}_{q+1,\diamond}\,.$$

We may now set

$$\begin{aligned}
\upsilon_{q+1} &= \sum_{\diamond=R,\varphi} \upsilon^{(p)}_{q+1,\diamond} + r_q\Gamma_q^{-1}\upsilon^{(c)}_{q+1,\diamond} =: \upsilon^{(p)}_{q+1} + r_q\Gamma_q^{-1}\upsilon^{(c)}_{q+1}\\
e_{q+1} &= \sum_{\diamond=R,\varphi} e^{(p)}_{q+1,\diamond} + r_q\Gamma_q^{-1}e^{(c)}_{q+1,\diamond} =: e^{(p)}_{q+1} + r_q\Gamma_q^{-1}e^{(c)}_{q+1}\,.
\end{aligned}$$

which leads to (9.49), (9.50a), (9.50b), and (9.51). □

Remark 9.4.3 (Decompositions of potentials into pieces to facilitate pressure creation). From the proof of Lemma 9.4.1, the velocity increment potentials $\upsilon^{(\mathtt{l})}_{q+1,k}$, $\mathtt{l} = p, c$, $k = 0, \cdots, \mathsf{d}$, have the additional properties listed below.

(i) Using Corollary A.3.11, (ii), we have that $\upsilon^{(\mathtt{l})}_{q+1,\mathsf{d}} = \lambda^{\mathsf{d}}_{q+\bar{n}}\upsilon^{(\mathtt{l})}_{q+1}$ can be decomposed as

$$\begin{aligned}
\upsilon^{(\mathtt{l})}_{q+1,\mathsf{d}} &= \lambda^{\mathsf{d}}_{q+\bar{n}} \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sum_{\bar{j}=0}^{\bar{\mathcal{C}}_{\mathcal{H}}} H^{\alpha(\bar{j})}_{(\xi),I,\diamond}(\rho^{\beta(\bar{j})}_{(\xi),I,\diamond} \circ \Phi_{(i,k)})\\
&=: \sum_{(\xi),I,\diamond} H_{(\xi),I,\diamond}\rho_{(\xi),I,\diamond} \circ \Phi_{(i,k)} \qquad (9.57)
\end{aligned}$$

where we abuse notation slightly by using (ξ) to include the indices $i, j, k, \xi, \vec{l}, \bar{j}$ as well as the indices in $\alpha(\bar{j})$ or $\beta(\bar{j})$ in the final expression, which take a finite number of values independent of q.

(ii) Let $p = 3$ or ∞. $H_{(\xi),I,\diamond}$ satisfies

$$\operatorname{supp} H_{(\xi),I,\diamond} \subseteq \operatorname{supp}\left((\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)}\right), \tag{9.58a}$$

$$\left\|\prod_{i=1}^{k} D^{\alpha_i} D^{\beta_i}_{t,q} H_{(\xi),I,\diamond}\right\|_p \lesssim \left|\operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{1/p} \delta^{1/2}_{q+\bar{n}}\Gamma_q^{j+7} r_q^{-1/3} \times \lambda^{|\alpha|}_{q+\bar{n}/2}\mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^{8}\right), \tag{9.58b}$$

$$\left|\prod_{i=1}^{k} D^{\alpha_i} D^{\beta_i}_{t,q} H_{(\xi),I,\diamond}\right| \lesssim (\pi_\ell \Gamma_q^{30})^{1/2} r_q^{-1/3} \lambda^{|\alpha|}_{q+\bar{n}/2}\mathcal{M}\left(|\beta|, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^{8}\right), \tag{9.58c}$$

for all integer $k \geq 1$ and multi-indices $\alpha, \beta \in \mathbb{N}^k$ with $|\alpha| \leq \mathsf{N}_{\mathrm{fin}}/4 - \mathsf{d}^2$ and $|\beta| \leq \mathsf{N}_{\mathrm{fin}}/5$.

(iii) $\rho_{(\xi),I,\diamond}$ is $(\mathbb{T}/\lambda_{q+\bar{n}/2}\Gamma_q)^3$-periodic and satisfies

$$\operatorname{supp} \rho_{(\xi),I,\diamond} \subseteq \operatorname{supp}\left(\widetilde{\vartheta}_{\xi,\lambda_{q+\bar{n}},\frac{\lambda_{q+\lfloor \bar{n}/2\rfloor}\Gamma_q}{\lambda_{q+\bar{n}}},\diamond}\right) \tag{9.59a}$$

$$\left\|D^N \rho_{(\xi),I,\diamond}\right\|_{L^p} \lesssim r_q^{\frac{2}{p}-\frac{2}{3}} \lambda^N_{q+\bar{n}} \tag{9.59b}$$

for all $N \leq \mathsf{N}_{\mathrm{fin}}/4 - \mathsf{d}^2$ and $((\xi), I, \diamond)$.

These properties of $H_{(\xi),I,\diamond}$ and $\rho_{(\xi),I,\diamond}$ follow from items (i)–(iv).

From the above properties, we may derive similar formulae and properties for *all* of the various velocity increment potentials $\upsilon^{(1)}_{q+1,h}$ defined in item (iii) for $0 \leq h \leq \mathsf{d}$. Specifically,

we have that $\upsilon^{(\mathtt{1})}_{q+1,h}$ can be decomposed using (9.57) and the Leibniz rule[7] as

$$\begin{aligned}
\upsilon^{(\mathtt{1},\bullet,i_{h+1},\cdots,i_{\mathsf{d}})}_{q+1,h} &= \lambda^{\mathsf{d}-h}_{q+\bar{n}}\partial_{i_1}\cdots\partial_{i_h}\upsilon^{(\mathtt{1},\bullet,i_1,\ldots,i_{\mathsf{d}})}_{q+1} \\
&= \lambda^{\mathsf{d}-h}_{q+\bar{n}}\sum_{\vec{a}_h,\vec{b}_h}\mathcal{C}_{\vec{a}_h,\vec{b}_h}\sum_{i,j,k,\xi,\vec{l},I,\diamond}\sum_{\bar{j}=0}^{\bar{\mathcal{C}}_{\mathcal{H}}}\partial_{\vec{a}_h}H^{\alpha(\bar{j})}_{(\xi),I,\diamond}\partial_{\vec{b}_h}\left(\rho^{\beta(\bar{j})}_{(\xi),I,\diamond}\circ\Phi_{(i,k)}\right) \\
&=: \sum_{(\xi),I,\diamond,h'}H^{h,h'}_{(\xi),I,\diamond}\rho^{h,h'}_{(\xi),I,\diamond}\circ\Phi_{(i,k)} \\
&=: \sum_{(\xi),I,\diamond,h'}\Upsilon^{h,h'}_{(\xi),I,\diamond}\,,
\end{aligned} \tag{9.60}$$

where $H^{h,h'}_{(\xi),I,\diamond}$, $\rho^{h,h'}_{(\xi),I,\diamond}$, and $\Upsilon^{h,h'}_{(\xi),I,\diamond}$ satisfy the following, and we again abuse notation slightly by letting (ξ) denote all indices $i,j,k,\xi,\vec{l},\bar{j}$, as well as those indices needed for the application of the Faa di Bruno formula from (A.9) to $\partial_{\vec{b}_h}\left(\rho^{\beta(\bar{j})}_{(\xi),I,\diamond}\circ\Phi_{(i,k)}\right)$. We again have that (ξ) includes $i,j,k,\xi,\vec{l},\xi$, as well as the a finite, q-independent number of indices.

(i) Let $p=3$ or ∞. $H^{h,h'}_{(\xi),I,\diamond}$ satisfies

$$\operatorname{supp} H^{h,h'}_{(\xi),I,\diamond}\subseteq\operatorname{supp}\left((\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi})\circ\Phi_{(i,k)}\right)\,, \tag{9.61a}$$

$$\begin{aligned}
\left\|\prod_{i=1}^{k}D^{\alpha_i}D^{\beta_i}_{t,q}H^{h,h'}_{(\xi),I,\diamond}\right\|_p &\lesssim \left|\operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{1/p}\delta^{1/2}_{q+\bar{n}}\Gamma^{j+7}_q r_q^{-1/3} \\
&\quad\times\lambda^{|\alpha|}_{q+\bar{n}/2}\mathcal{M}\left(|\beta|,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+13},\mathrm{T}_q^{-1}\Gamma_q^{8}\right)\,,
\end{aligned} \tag{9.61b}$$

$$\left|\prod_{i=1}^{k}D^{\alpha_i}D^{\beta_i}_{t,q}H^{h,h'}_{(\xi),I,\diamond}\right|\lesssim(\pi_\ell\Gamma_q^{30})^{1/2}\lambda^{|\alpha|}_{q+\bar{n}/2}\mathcal{M}\left(|\beta|,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+13},\mathrm{T}_q^{-1}\Gamma_q^{8}\right)\,, \tag{9.61c}$$

for all integer $k\geq 1$ and multi-indices $\alpha,\beta\in\mathbb{N}^k$ with $|\alpha|\leq{}^{\mathsf{N}_{\mathrm{fin}}}/_4-2\mathsf{d}^2$ and $|\beta|\leq{}^{\mathsf{N}_{\mathrm{fin}}}/_5$.

[7] We use the notation

$$\partial_{i_1}\cdots\partial_{i_h}(fg)=\sum_{\substack{\vec{a}_h=(a_1,\ldots,a_A),\\ \vec{b}_h=(b_1,\ldots,b_B)}}C_{\vec{a}_h,\vec{b}_h}\partial_{i_{a_1}}\cdots\partial_{i_{a_A}}f\,\partial_{i_{b_1}}\cdots\partial_{i_{b_A}}g=\sum_{\vec{a}_h,\vec{b}_h}C_{\vec{a}_h,\vec{b}_h}\partial_{\vec{a}_h}f\partial_{\vec{b}_h}g\,,$$

where $\vec{a}_h,\vec{b}_h$ are multi-indices with A, respectively B distinct components for which the union of all indices belonging to either $\vec{a}_h$ or $\vec{b}_h$ is $\{i_1,\ldots,i_h\}$.

(ii) $\rho^{h,h'}_{(\xi),I,\diamond}$ is $(\mathbb{T}/\lambda_{q+\bar{n}/2}\Gamma_q)^3$-periodic and satisfies

$$\operatorname{supp} \rho^{h,h'}_{(\xi),I,\diamond} \subseteq \operatorname{supp} \left(\widetilde{\vartheta}_{\xi,\lambda_{q+\bar{n}},\frac{\lambda_{q+\lfloor \bar{n}/2 \rfloor}\Gamma_q}{\lambda_{q+\bar{n}}},\diamond} \right) \tag{9.62a}$$

$$\left\| D^N \rho^{h,h'}_{(\xi),I,\diamond} \right\|_{L^p} \lesssim r_q^{\frac{2}{p}-\frac{2}{3}} \lambda^N_{q+\bar{n}} \tag{9.62b}$$

for all $N \leq \mathsf{N}_{\text{fin}}/4 - 2\mathsf{d}^2$ and $((\xi), I, \diamond)$.

(iii) For $p = 3, \infty$, we have that

$$\left\| \Upsilon^{h,h'}_{(\xi),I,\diamond} \right\|_p \lesssim \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \right) \right|^{1/p} \delta^{1/2}_{q+\bar{n}} \Gamma_q^{j+7} r_q^{2/p-1} . \tag{9.63}$$

The proofs of these properties follows from backwards induction on the index h. Indeed, the case $h = \mathsf{d}$ has already been shown in the beginning of the remark. The subsequent cases follow from application of the Faa di Bruno formula to (9.57) to derive (9.60), (9.58a)–(9.59b), Corollary 8.2.4, and Lemma A.1.3.

Lemma 9.4.4 (Pressure increment). *Define $\upsilon^{(\mathfrak{l})}_{q+1,k}$, $0 \leq k \leq \mathsf{d}$, $\mathfrak{l} = p, c$, as in Lemma 9.4.1. Then there exists a pressure increment $\sigma_{\upsilon^{(\mathfrak{l})}} = \sigma^+_{\upsilon^{(\mathfrak{l})}} - \sigma^-_{\upsilon^{(\mathfrak{l})}}$ associated to the sum $\sum_{k=0}^{\mathsf{d}} \upsilon^{(\mathfrak{l})}_{q+1,k}$ of velocity increment potentials such that the following properties hold.*

(i) We have that for all $k = 0, 1, \ldots, \mathsf{d}$,

$$\left| \psi_{i,q} D^N D^M_{t,q} \upsilon^{(\mathfrak{l})}_{q+1,k} \right| \lesssim (\sigma^+_{\upsilon^{(\mathfrak{l})}} + \delta_{q+3\bar{n}})^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+16}, \mathrm{T}_q^{-1} \Gamma_q^9 \right) \tag{9.64}$$

for any $0 \leq k \leq \mathsf{d}$ and $N, M \leq \mathsf{N}_{\text{fin}}/5$.

(ii) Set

$$\sigma^{\pm}_{\upsilon} := \sigma^{\pm}_{\upsilon^{(p)}} + \sigma^{\pm}_{\upsilon^{(c)}} , \qquad \sigma_{\upsilon} = \sigma^+_{\upsilon} - \sigma^-_{\upsilon} . \tag{9.65}$$

Then we have that

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^+_\upsilon\right| \lesssim (\sigma^+_\upsilon + \delta_{q+3\bar{n}})(\lambda_{q+\bar{n}}\Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{9.66a}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_\upsilon\right\|_{3/2} \leq \Gamma^{-9}_{q+\bar{n}}\delta_{q+2\bar{n}}(\lambda_{q+\bar{n}}\Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{9.66b}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_\upsilon\right\|_{\infty} \leq \Gamma^{\mathsf{C}_\infty - 9}_{q+\bar{n}}(\lambda_{q+\bar{n}}\Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{9.66c}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^-_\upsilon\right\|_{3/2} \leq \Gamma^{-9}_{q+\bar{n}}\delta_{q+2\bar{n}}(\lambda_{q+\bar{n}/2}\Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{9.66d}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^-_\upsilon\right\|_{\infty} \leq \Gamma^{\mathsf{C}_\infty - 9}_{q+\bar{n}}(\lambda_{q+\bar{n}/2}\Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{9.66e}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^-_\upsilon\right| \lesssim \pi_\ell \Gamma_q^{30} r_q^{4/3}(\lambda_{q+\bar{n}/2}\Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{9.66f}$$

for all $N \leq \mathsf{N}_{\mathrm{fin}}/5$ *and* $M \leq \mathsf{N}_{\mathrm{fin}}/5 - \mathsf{N}_{\mathrm{cut,t}}$.

(iii) We have that

$$\mathrm{supp}\,(\sigma^+_\upsilon) \cap B(\widehat{w}_{q''}, \lambda_{q''}^{-1}\Gamma_{q''+1}) , \quad \mathrm{supp}\,(\sigma^-_\upsilon) \cap B(\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}) = \emptyset \tag{9.67}$$

for $q+1 \leq q'' \leq q+\bar{n}-1$ *and* $q+1 \leq q' \leq q+\bar{n}/2$.

(iv) Define

$$\mathfrak{m}_{\sigma_\upsilon}(t) = \int_0^t \langle D_{t,q}\sigma_\upsilon\rangle\,(s)\,ds\,. \tag{9.68}$$

Then we have that

$$\left|\frac{d^{M+1}}{dt^{M+1}}\mathfrak{m}_{\sigma_\upsilon}\right| \leq (\max(1,T))^{-1}\delta^2_{q+3\bar{n}}\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right) \tag{9.69}$$

for $0 \leq M \leq 2\mathsf{N}_{\mathrm{ind}}$.

Remark 9.4.5 (Pointwise bounds for principal and corrector parts). From (9.48)–

(9.51), (9.64), and (4.24a), we have that

$$\left|\psi_{i,q} D^N D^M_{t,q} w^{(p)}_{q+1}\right| \lesssim (\sigma^+_{\upsilon^{(p)}} + \delta_{q+3\bar{n}})^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{9.70a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} w^{(c)}_{q+1}\right| \lesssim (\sigma^+_{\upsilon^{(c)}} + \delta_{q+3\bar{n}})^{1/2} \Gamma_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{9.70b}$$

for $N, M \leq \mathsf{N}_{\text{fin}}/5$. Note that thanks to the factor $r_q\Gamma_q^{-1}$ in (9.48), the bound in (9.70b) has extra gain of $r_q\Gamma_q^{-1}$ compared to (9.70a). This gain will be useful when we deal with the divergence corrector stress errors in subsection 10.2.3 and divergence corrector current errors in 11.2.5. We also record an upgraded version of (9.70), which states that in the same range of N and M, we have that

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} w^{(p)}_{q+1}\right| &\lesssim (\sigma^+_{\upsilon^{(p)}} + \delta_{q+3\bar{n}})^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \\ &\quad \times \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1}\Gamma^{i-5}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^{-1}_{q+\bar{n}}\right) , \end{aligned} \tag{9.71a}$$

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} w^{(c)}_{q+1}\right| &\lesssim (\sigma^+_{\upsilon^{(c)}} + \delta_{q+3\bar{n}})^{1/2} \Gamma_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \\ &\quad \times \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1}\Gamma^{i-5}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^{-1}_{q+\bar{n}}\right) . \end{aligned} \tag{9.71b}$$

The proof of (9.71) is immediate from Hypothesis 5.14 at level q and Remark 9.2.3, which asserts that Hypothesis 5.4.1 is verified at level $q+1$ with $q' = q + \bar{n}$.

Before giving the proof of Lemma 9.4.4, we record the following lemma, which investigates the current error generated by the pressure increment σ_υ. The proof of both lemmas will proceed using Lemma A.4.3.

Lemma 9.4.6 (Current error from the pressure increment). *There exists a current error ϕ_υ generated by σ_υ such that the following hold.*

(i) We have the decomposition and equalities

$$\phi_\upsilon = \underbrace{\phi_\upsilon^*}_{\text{nonlocal}} + \underbrace{\sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \phi_\upsilon^{m'}}_{\text{local}}$$

$$= \underbrace{(\mathcal{H}+\mathcal{R}^*)(D_{t,q}\sigma_\upsilon^*) + \sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \mathcal{R}^*(D_{t,q}\sigma_\upsilon^{m'})}_{\text{nonlocal}} + \underbrace{\sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \mathcal{H}(D_{t,q}\sigma_\upsilon^{m'})}_{\text{local}}, \tag{9.72a}$$

$$\operatorname{div}\left(\phi_\upsilon^{m'}(t,x) + \mathcal{R}^*(D_{t,q}\sigma_\upsilon^{m'}(t,x))\right) = D_{t,q}\sigma_\upsilon^{m'}(t,x) - \int_{\mathbb{T}^3} D_{t,q}\sigma_\upsilon^{m'}(t,x')\,dx', \tag{9.72b}$$

$$\operatorname{div}\left(\phi_\upsilon^*(t,x) - \sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \mathcal{R}^*(D_{t,q}\sigma_\upsilon^{m'})(t,x)\right) = D_{t,q}\sigma_\upsilon^*(t,x) - \int_{\mathbb{T}^3} D_{t,q}\sigma_\upsilon^*(t,x')\,dx'. \tag{9.72c}$$

(ii) For all $N \le \mathsf{N}_{\text{fin}}/5$ and $M \le \mathsf{N}_{\text{fin}}/5 - \mathsf{N}_{\text{cut,t}} - 1$ and $q + \bar{n}/2 + 1 \le m' \le q + \bar{n}$,

$$\left|\psi_{i,q} D^N D_{t,q}^M \phi_\upsilon^{m'}\right| \lesssim \Gamma_m^{-100} (\pi_q^{m'})^{3/2} r_m^{-1} (\lambda_m \Gamma_{m'})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right). \tag{9.73}$$

(iii) For all $N \le 3\mathsf{N}_{\text{ind}}$ and $M \le 3\mathsf{N}_{\text{ind}}$,

$$\left\|D^N D_{t,q}^M \phi_\upsilon^*\right\|_\infty \lesssim \delta_{q+3\bar{n}}^{3/2} \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}} \lambda_{q+\bar{n}+2}^{-10} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right). \tag{9.74}$$

(iv) For all $q+1 \le q' \le q + \bar{n}/2$, $q + \bar{n}/2 + 2 \le m \le q + \bar{n}$, and $q+1 \le q'' \le m-1$, we have the support properties

$$\operatorname{supp}(\phi_\upsilon^{q+\bar{n}/2+1}) \cap B(\widehat{w}_{q'}, \lambda_{q+1}^{-1}\Gamma_q^2) = \emptyset, \qquad \operatorname{supp}(\phi_\upsilon^m) \cap \operatorname{supp}\widehat{w}_{q''} = \emptyset. \tag{9.75}$$

Proofs of Lemma 9.4.4 and Lemma 9.4.6. `Step 1: Setup and Assumptions from Lemma A.4.3.` In order to create a pressure increment which dominates *all* of the various velocity increment potentials $\upsilon_{q+1,h}^{(1)}$ defined in item (iii), we shall create pressure increments which dominate each separate piece, and then sum at the end. We fix all indices $(\xi), I, \diamond, h, h'$ from the

formula in (9.60) and apply Proposition A.4.3 with the following choices:

$$
\begin{aligned}
&N_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_4 - 2\mathsf{d}^2, \quad M_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_5, \quad M_t = \mathsf{N}_{\mathrm{ind,t}}, \quad N_\circ = M_\circ = 3\mathsf{N}_{\mathrm{ind}}\,,\\
&\widehat{\upsilon} = \Upsilon^{h,h'}_{(\xi),I,\diamond}\,, \quad G = H^{h,h'}_{(\xi),I,\diamond}\,, \quad \rho = \rho^{h,h'}_{(\xi),I,\diamond}\,, \quad \pi = \pi_\ell \Gamma_q^{30}\,, \quad K_\circ \text{ as in}\\
&\mathcal{C}_{G,p} = \left|\mathrm{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{1/p} \Gamma_q^{j+7}\delta_{q+\bar{n}}^{\frac{1}{2}} r_q^{-1/3} + \lambda_{q+\bar{n}}^{-10}\,, \quad K_\circ \text{ as in item (xvi)}\\
&\mathcal{C}_{\rho,p} = r_q^{\frac{2}{p}-\frac{2}{3}}\,, \quad \lambda = \lambda_{q+{}^{\bar{n}}/_2}, \quad \lambda' = \Lambda_q, \quad \nu = \tau_q^{-1}\Gamma_q^{i+13}, \quad \nu' = \mathrm{T}_q^{-1}\Gamma_q^{8}\,, \quad \Lambda = \lambda_{q+\bar{n}},\\
&r_G = r_{\widehat{\upsilon}} = r_q\,, \quad \mu = \lambda_{q+{}^{\bar{n}}/_2}\Gamma_q\,, \quad \Gamma = \Gamma_q^{1/10}\,, \quad \Phi = \Phi_{(i,k)}\,, \quad \upsilon = \widehat{u}_q\,, \quad \mathcal{C}_\upsilon = \Lambda_q^{1/2}\,,\\
&\mu_0 = \lambda_{q+{}^{\bar{n}}/_2+1}, \quad \mu_1 = \lambda_{q+{}^{\bar{n}}/_2+3/2}, \quad \mu_m = \lambda_{q+{}^{\bar{n}}/_2+m}, \quad \mu_{\bar{m}} = \lambda_{q+\bar{n}+1}, \quad \delta_{\mathrm{tiny}} = \delta_{q+3\bar{n}}\,,
\end{aligned}
$$

where $\mu_m = \lambda_{q+{}^{\bar{n}}/_2+m}$ above is defined for $2 \le m \le \bar{m}$. Then we have that (A.158a)–(A.158d) are verified from (9.61a)–(9.63), (A.159a) holds by definition and by (4.21), (A.160a)–(A.160c) hold from (5.34), Corollary 8.2.4, (5.35b), and (4.15), (A.161a) holds from (4.17a), (A.161b) holds due to (4.17b), (A.161c) holds due to (4.24a), (A.162) holds from direct computation, and (A.163a)–(A.163c) hold due to (xvii).

`Step 2: Part 2 from Lemma A.4.3 and proof of Lemma 9.4.4.` We now apply the conclusions from Part 2 of Lemma A.4.3. We first have from (A.164) and (A.165) the existence of a pressure increment $\sigma_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} = \sigma^+_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} - \sigma^-_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}$ such that

$$
\left|D^N D^M_{t,q}\Upsilon^{h,h'}_{(\xi),I,\diamond}\right| \lesssim \left(\sigma^+_{\upsilon^{h,h'}_{(\xi),I,\diamond}} + \delta_{q+3\bar{n}}\right)^{1/2} r_q^{-1}(\lambda_{q+\bar{n}}\Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^{9}\right) \tag{9.76}
$$

for all $N \leq {}^{\mathsf{N}_{\text{fin}}}/_4 - 2\mathsf{d}^2$ and $M \leq {}^{\mathsf{N}_{\text{fin}}}/_5$. Then using items (ii)–(iii) and (4.18), we have that

$$\left| D^N D^M_{t,q} \sigma^+_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right| \lesssim \left(\sigma^+_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} + \delta_{q+3\bar{n}} \right) (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.77a}$$

$$\begin{aligned}
\left\| D^N D^M_{t,q} \sigma^+_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_{3/2} &\lesssim \left| \operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_\xi \right) \right|^{2/3} \Gamma_q^{2j+14} \delta_{q+\bar{n}} r_q^{4/3} \\
&\quad \times (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right),
\end{aligned} \tag{9.77b}$$

$$\left\| D^N D^M_{t,q} \sigma^+_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+20} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.77c}$$

$$\begin{aligned}
\left\| D^N D^M_{t,q} \sigma^-_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_{3/2} &\lesssim \left| \operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_\xi \right) \right|^{2/3} \Gamma_q^{2j+14} \delta_{q+\bar{n}} r_q^{4/3} \\
&\quad \times (\lambda_{q+\bar{n}/2} \Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right),
\end{aligned} \tag{9.77d}$$

$$\left\| D^N D^M_{t,q} \sigma^-_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+20} (\lambda_{q+\bar{n}/2} \Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.77e}$$

$$\left| D^N D^M_{t,q} \sigma^-_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right| \lesssim \pi_\ell \Gamma_q^{30} r_q^{4/3} (\lambda_{q+\bar{n}/2} \Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.77f}$$

for all $N \leq {}^{\mathsf{N}_{\text{fin}}}/_4 - 2\mathsf{d}^2 - \mathsf{N}_{\text{cut,x}}$ and $M \leq {}^{\mathsf{N}_{\text{fin}}}/_5 - \mathsf{N}_{\text{cut,t}}$. In (9.77c) and (9.77e), we used (8.27). Finally, from (A.169), (9.61a), (9.62a), (9.49), and Lemma 9.2.2, we get the support properties

$$\begin{aligned}
\operatorname{supp}\left(\sigma^+_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right) &\subseteq \operatorname{supp}\left(\Upsilon^{h,h'}_{(\xi),I,\diamond} \right) \\
&\subseteq \operatorname{supp}\left(\chi_{i,k,q} \zeta_{q,\diamond,i,k,\xi,\vec{l}} \left(\boldsymbol{\rho}^\diamond_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_\xi \right) \circ \Phi_{(i,k)} \right) \cap B\left(\operatorname{supp} \varrho^I_{(\xi),\diamond}, 2\lambda^{-1}_{q+\bar{n}} \right) \circ \Phi_{(i,k)}, \\
\operatorname{supp}\left(\sigma^-_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right) \cap B(\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'}) &\subseteq \operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_\xi \right) \cap B(\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'}) = \emptyset,
\end{aligned}$$

for $q+1 \leq q' \leq q + {}^{\bar{n}}/_2$.

We now sum over $h, h', (\xi), i, \diamond$ (while recalling from (9.60) that summation over (ξ) includes summation over $i, j, k, \xi, \vec{l}, \vec{j}$ as well as any indices needed for the application of the

Faa di Bruno formula) and set

$$\sigma_v^{\pm} := \sum_{(\xi),I,\diamond,h',h} \sigma^{\pm}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} .$$

From (9.76), (9.60), (8.45), and Corollary 8.6.3 with $H = \Upsilon^{h,h'}_{(\xi),I,\diamond}$ and $\varpi = \sigma^{+}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} + \mathbf{1}_{\operatorname{supp}\Upsilon^{h,h'}_{(\xi),I,\diamond}} \delta_{q+3\bar{n}}$, we have that (9.64) holds. We have (9.65) from the formula above. In order to verify (9.66a)–(9.66f), we appeal to (9.77a)–(9.77f) and Corollaries 8.6.1 and 8.6.3. Specifically, the $L^{3/2}$ estimates in (9.66b) and (9.66d) use (4.10g) and Corollary 8.6.1 with $\theta_2 = \theta = 2$, $H = \sigma^{\pm}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}$, and $\mathcal{C}_H = \delta_{q+\bar{n}} r_q^{4/3} \Gamma_q^{14}$. The L^∞ estimates in (9.66c) and (9.66e) follow from (8.45), (4.13a), and Corollary 8.6.3 and with the same choice of H and $\varpi = \Gamma_q^{\mathsf{C}_\infty + 20} \mathbf{1}_{\operatorname{supp}\Upsilon^{h,h'}_{(\xi),I,\diamond}}$. Finally, the pointwise estimates in (9.66a) and (9.66f) follow from Corollary 8.6.3 in much the same manner as the L^∞ estimates just derived, and we omit further details.

`Step 3: Part 3 from Lemma A.4.3 and proof of Lemma 9.4.6.` We now apply the conclusions from Part 3 of Lemma A.4.3. From item (i), there exist current errors $\phi_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}$ such that we have the decompositions and equalities

$$\begin{aligned}
\phi_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} &= \underbrace{\phi^*_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}}_{\text{nonlocal}} + \underbrace{\sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \phi^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}}_{\text{local}} \qquad (9.78a)\\
&= \underbrace{(\mathcal{H}+\mathcal{R}^*)\left(D_t \sigma^*_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right) + \sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \mathcal{R}^*\left(D_t \sigma^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right)}_{\text{nonlocal}} + \underbrace{\sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \mathcal{H}\left(D_t \sigma^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right)}_{\text{local}},
\end{aligned}$$

$$\operatorname{div}\left(\phi^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}(t,x) + \mathcal{R}^*\left(D_t \sigma^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right)(t,x)\right) = D_t \sigma^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}(t,x) - \int_{\mathbb{T}^3} D_t \sigma^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}(t,x')\,dx' ,$$

$$\operatorname{div}\left(\phi^{*}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}(t,x) - \sum_{m=0}^{\bar{m}} \mathcal{R}^*\left(D_t \sigma^{m'}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right)(t,x)\right) = D_t \sigma^{*}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}(t,x) - \int_{\mathbb{T}^3} D_t \sigma^{*}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}(t,x')\,dx' .$$

Next, from (ii) in Proposition A.4.3, (4.18), and (4.24a), we have that for $(p,p') = (3, 3/2)$ or

(∞,∞) and $2 \le m \le \bar{m}$,

$$\left\| D^N D_t^M \phi^0_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_{p'} \lesssim \tau_q^{-1}\Gamma_q^{i+14}\left(\delta_{q+\bar{n}} r_q^{-2/3}\Gamma_q^{2j+14}\left|\text{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}_\xi^{I,\diamond}\right)\right|^{2/p} + \lambda_{q+\bar{n}}^{-20}\right)$$
$$\times\left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}}\right)^{\frac{4}{3}-\frac{2}{p'}} r_q^2\lambda_{q+\bar{n}/2}^{-1}(\lambda_{q+\bar{n}/2+1})^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.79a}$$

$$\left| D^N D_t^M \phi^0_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right| \lesssim \tau_q^{-1}\Gamma_q^{i+50}\pi_\ell r_q^{4/3}\left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}}\right)^{4/3}\lambda_{q+\bar{n}/2}^{-1}$$
$$\times(\lambda_{q+\bar{n}/2+1})^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.79b}$$

$$\left\| D^N D_t^M \phi^m_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_{p'} \lesssim \tau_q^{-1}\Gamma_q^{i+16}\left(\delta_{q+\bar{n}} r_q^{-2/3}\Gamma_q^{2j+14}\left|\text{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}_\xi^{I,\diamond}\right)\right|^{2/p} + \lambda_{q+\bar{n}}^{-20}\right)$$
$$\times\left(\frac{\min\left(\lambda_{q+\bar{n}/2+m},\lambda_{q+\bar{n}}\right)}{\lambda_{q+\bar{n}/2}}\right)^{\frac{4}{3}-\frac{2}{p'}} r_q^2(\lambda_{q+\bar{n}/2+m-1}^{-2}\lambda_{q+\bar{n}/2+m})$$
$$\times(\min(\lambda_{q+\bar{n}/2+m},\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}))^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.79c}$$

$$\left| D^N D_t^M \phi^m_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right| \lesssim \tau_q^{-1}\Gamma_q^{i+50}\pi_\ell r_q^{4/3}\left(\frac{\min(\lambda_{q+\bar{n}/2+m},\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^{4/3}(\lambda_{q+\bar{n}/2+m-1}^{-2}\lambda_{q+\bar{n}/2+m})$$
$$\times(\min(\lambda_{q+\bar{n}/2+m},\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}))^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{9.79d}$$

for $N \le \mathsf{N}_{\text{fin}}/5$ and $M \le \mathsf{N}_{\text{fin}}/5 - \mathsf{N}_{\text{cut,x}} - 1$. In the case $m = 1$, we have bounds which match the bounds for $m = 2$ above, except that the inverse divergence gain of $\lambda_{q+\bar{n}/2+m-1}^{-2}\lambda_{q+\bar{n}/2+m}$ is replaced with $\lambda_{q+\bar{n}/2+3/2}^{-2}\lambda_{q+\bar{n}/2+1}$. Furthermore, we have from (A.172) and item (xvi) that

$$\left\| D^N D_t^M \phi^*_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \right\|_\infty \lesssim \delta_{q+3\bar{n}}^{3/2}\mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}}\lambda_{q+\bar{n}}^{-60}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N(\tau_q^{-1}\Gamma_q^{i+14})^M \tag{9.80}$$

for $N, M \le 3\mathsf{N}_{\text{ind}}$. Finally, (iii) from Lemma A.4.3, (9.62a), (9.61a), and Lemma 9.2.2 give

that for each $1 \leq m \leq \bar{m}$ and any $q+1 \leq q' \leq q + {}^{\bar{n}}/_{2}$ and $q+1 \leq q'' \leq q + {}^{\bar{n}}/_{2} + m - 1$

$$\operatorname{supp}\left(\phi^{0}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right) \cap B(\widehat{w}_{q'}, \lambda_{q+1}^{-1}\Gamma_q^2) = \emptyset\,, \qquad \operatorname{supp}\left(\phi^{m}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right) \cap \operatorname{supp}\widehat{w}_{q''}\,,$$
$$\operatorname{supp}\left(\phi^{0}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right), \operatorname{supp}\left(\phi^{m}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\right) \subseteq \operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\,. \tag{9.81}$$

We now sum over $h, h', (\xi), i, \diamond$ (while recalling from (9.60) that summation over (ξ) includes summation over $i, j, k, \xi, \vec{l}, \vec{j}$ as well as any indices needed for the application of the Faa di Bruno formula) and set

$$\phi_{\upsilon}^{q+\bar{n}/2+1} := \sum_{(\xi),I,\diamond,h',h} \phi^{0}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\,, \qquad \phi_{\upsilon}^{q+\bar{n}/2+2} := \sum_{(\xi),I,\diamond,h',h} \sum_{m=1}^{2} \phi^{m}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}} \tag{9.82}$$
$$\phi_{\upsilon}^{q+\bar{n}/2+m} := \sum_{(\xi),I,\diamond,h',h} \phi^{m}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\,, \qquad \phi_{\upsilon}^{q+\bar{n}} := \sum_{(\xi),I,\diamond,h',h} \sum_{m=\bar{m}-1}^{\bar{m}} \phi^{m}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}\,, \qquad \phi_{\upsilon}^{*} := \sum_{(\xi),I,\diamond,h',h} \phi^{*}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}$$

for $3 \leq m \leq \bar{m} - 2$.

We can now conclude the proof of Lemma 9.4.6. First, we have that item (i) follows from the definitions in (9.82) and (9.78a). Next, we have that (9.75) follows from the same definitions, (9.81), and Lemma 9.2.2. We can achieve the nonlocal bounds in (9.74) from (9.80) and summation over all indices $(\xi), I, \diamond, h', h$, which from Lemma 8.4.4, (5.9), Lemma 8.3.5, and the discussion following (9.60) is bounded by $\lambda_{q+\bar{n}}^4$. The bound for $\mathfrak{m}_{\sigma_\upsilon}$ in item (iv) follows similarly from (A.174) (4.22), and a large choice of a_* in (xix) to ensure that we can put the prefactor of $\max(1, T)^{-1}$ in the amplitude. Finally, we may conclude (9.73) from an application of Corollary 8.6.4 with $H = \phi^{\bullet}_{\Upsilon^{h,h'}_{(\xi),I,\diamond}}$ (with the value of $\bullet$ according to the divisions in (9.82)) and

$$\varpi = \Gamma_q^{50} \pi_\ell r_q^{4/3} \left(\frac{\min(\lambda_{q+\bar{n}/2+m}, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}}\right)^{4/3} \lambda_{q+\bar{n}/2+m-1}^{-2} \lambda_{q+\bar{n}/2+m}\,.$$

Indeed appealing to (8.56b), (6.6), (5.20), (4.27c), and the fact that

$$r_q^{4/3}\left(\frac{\min(\lambda_{q+\bar{n}/2+m},\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}}\right)^{4/3}\leq\Gamma_q^{10}$$

from the definition of r_q, we conclude the proof. □

9.5 Estimates for new velocity increments and their potentials

Recall the definition of the mollified velocity increment $\widehat{w}_{q+\bar{n}}$ in Definition 9.2.1.

Lemma 9.5.1 (Estimates on $\widehat{w}_{q+\bar{n}}$). *We have that $\widehat{w}_{q+\bar{n}}$ satisfies the following properties.*

(i) For all $N+M\leq 2\mathsf{N}_{\text{fin}}$, we have that

$$\begin{aligned}&\left\|D^N D^M_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^3(\operatorname{supp}\psi_{i,q+\bar{n}-1})}\\&\qquad\lesssim\Gamma_q^{20}\delta_{q+\bar{n}}^{1/2}r_q^{-1/3}\left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}\right)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\Gamma_{q+\bar{n}-1}^{i-1}\tau_{q+\bar{n}-1}^{-1},\mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right)\end{aligned}\tag{9.83a}$$

$$\begin{aligned}&\left\|D^N D^M_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}-1})}\\&\qquad\lesssim\Gamma_q^{\mathsf{C}_\infty/2+16}r_q^{-1}\left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}\right)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\Gamma_{q+\bar{n}-1}^{i-1}\tau_{q+\bar{n}-1}^{-1},\mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right).\end{aligned}\tag{9.83b}$$

(ii) For all $N+M\leq\mathsf{N}_{\text{fin}}/4$, we have that

$$\begin{aligned}\left\|D^N D^M_{t,q+\bar{n}-1}\left(w_{q+1}-\widehat{w}_{q+\bar{n}}\right)\right\|_\infty&\lesssim\delta_{q+3\bar{n}}^3\mathrm{T}_{q+\bar{n}}^{25\mathsf{N}_{\text{ind,t}}}\left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}\right)^N\\&\qquad\times\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_{q+\bar{n}-1}^{-1},\mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right).\end{aligned}\tag{9.84}$$

Proof of Lemma 9.5.1. We prove items (i)–(ii) in steps. First, we apply Corollary (8.6.1) with $\theta=1$, $\theta_1=0$, $\theta_2=1$, $H_{i,j,k,\xi,\vec{l},I,\diamond}=w^{(\bullet),I}_{(\xi),\diamond}$ with $\bullet=p,c$, $p=3$, $\mathcal{C}_H=\delta_{q+\bar{n}}^{1/2}\Gamma_q^{12}r_q^{-1/3}$,

$N_* = M_* = \mathsf{N}_{\text{fin}}/4$, $M_t = \mathsf{N}_{\text{ind,t}}$, $N_x = \infty$, $\lambda = \Lambda = \lambda_{q+\bar{n}}$, $\tau^{-1} = \tau_q^{-1}\Gamma_q^4$, $\mathrm{T} = \mathrm{T}_q$. From the definition of $w_{(\xi),\diamond}^{(\bullet),I}$ and Corollary 9.3.2, we have that(8.46)–(8.47b) are satisfied, and so from (8.48b), we conclude that for $N, M \leq \mathsf{N}_{\text{fin}}/4$

$$\left\|\psi_{i,q} D^N D_{t,q}^M w_{q+1}\right\|_3 \lesssim \Gamma_q^{20} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} \lambda_{q+\bar{n}}^N \mathcal{M}\left(N, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) . \tag{9.85}$$

In the case $p = \infty$, we may aggregate estimates from Corollary 9.3.2 using the fact that only a finite, q-independent number of terms $w_{(\xi),\diamond}^{(\bullet),I}$ are non-zero at any fixed point in space-time to give the bound

$$\left\|\psi_{i,q} D^N D_{t,q}^M w_{q+1}\right\|_\infty \lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+16} r_q^{-1} \lambda_{q+\bar{n}}^N \mathcal{M}\left(N, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\right) . \tag{9.86}$$

Next, from (9.24), which asserts that $\operatorname{supp} w_{q+1} \cap \operatorname{supp} \widehat{w}_{q'} = \emptyset$ for $q+1 \leq q' \leq q+\bar{n}-1$, and from (5.14) applied with $q' = q+\bar{n}-1$ and $q'' = q$, we may upgrade (9.85)–(9.86) to

$$\left\|D^N D_{t,q+\bar{n}-1}^M w_{q+1}\right\|_{L^3(\operatorname{supp}\psi_{i,q+\bar{n}-1})} \lesssim \Gamma_q^{20} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} \lambda_{q+\bar{n}}^N \mathcal{M}\left(N, \mathsf{N}_{\text{ind,t}}, \tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{i-2}, \mathrm{T}_q^{-1}\right) \tag{9.87a}$$

$$\left\|D^N D_{t,q+\bar{n}-1}^M w_{q+1}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}-1})} \lesssim \Gamma_q^{\frac{\mathsf{C}_\infty}{2}+16} r_q^{-1} \lambda_{q+\bar{n}}^N \mathcal{M}\left(N, \mathsf{N}_{\text{ind,t}}, \tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{i-2}, \mathrm{T}_q^{-1}\right) . \tag{9.87b}$$

We now apply Proposition A.6.1 with the choices

$$\begin{aligned}
&p = 3, \infty\,, \quad N_{\mathrm{g}}, N_{\mathrm{c}} \text{ as in (xiii)}\,, \quad M_t = \mathsf{N}_{\text{ind,t}}\,, \quad N_* = \mathsf{N}_{\text{fin}}/4\,,\\
&N_\gamma = 2\mathsf{N}_{\text{fin}}\,, \quad \Omega = \operatorname{supp}\psi_{i,q+\bar{n}-1}\,, \quad v = \widehat{u}_{q+\bar{n}-1}\,, \quad i = i\,,\\
&\lambda = \lambda_{q+\bar{n}}\,, \quad \Lambda = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}\,, \quad \Gamma = \Gamma_{q+\bar{n}-1}, \quad \tau = \tau_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}^{-2}\,, \quad \mathrm{T} = \mathrm{T}_{q+\bar{n}-1}\,,\\
&f = w_{q+1}\,, \quad \mathcal{C}_{f,3} = \Gamma_q^{20}\delta_{q+\bar{n}}^{1/2} r_q^{-1/3}\,, \quad \mathcal{C}_{f,\infty} = \widetilde{\mathcal{C}}_f = \Gamma_q^{\mathsf{C}_\infty/2+16} r_q^{-1}\,, \quad \mathcal{C}_v = \Lambda_{q+\bar{n}-1}^{1/2}\,.
\end{aligned}$$

From (xiii) and (4.15), we have that (A.225) is satisfied. From (5.35b), we have that (A.226) is satisfied. From (9.87), we have that (A.227a) is satisfied. In order to verify (A.227b), we

apply Remark A.2.6 with the following choices. We set $p = \infty$, $N_x = N_t = \infty$, $N_* = \mathsf{N}_{\text{fin}}/4$, $\Omega = \mathbb{T}^3 \times \mathbb{R}$, $v = w = \widehat{u}_{q+\bar{n}-1}$, $\mathcal{C}_w = \Gamma_{q+\bar{n}-1}^{i_{\max}+2}\delta_{q+\bar{n}-1}^{1/2}\lambda_{q+\bar{n}-1}^2$, $\lambda_w = \widetilde{\lambda}_w = \Lambda_{q+\bar{n}-1}$, $\mu_w = \widetilde{\mu}_w = \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}$ in (A.34), while in (A.27) and (A.28) we set $v = \widehat{u}_{q+\bar{n}-1}$, $\mathcal{C}_v = \mathcal{C}_w$, $\lambda_v = \widetilde{\lambda}_v = \Lambda_{q+\bar{n}-1}$, $\mu_v = \widetilde{\mu}_v = \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}$, $f = w_{q+1}$, $\mathcal{C}_f = \Gamma_q^{\mathsf{C}_\infty/2+16}r_q^{-1}$, $\lambda_f = \widetilde{\lambda}_f = \lambda_{q+\bar{n}}$, $\mu_f = \widetilde{\mu}_f = \mathrm{T}_q^{-1}$. Then (A.27) and (A.28) are satisfied from (5.34) at level $q+\bar{n}-1$, (9.87), (5.10), and (4.15). Next, (A.34) is satisfied from (5.35a) at level $q+\bar{n}-1$. Thus from (A.35) and (4.15), we obtain that

$$\left\|D^N\partial_t^M w_{q+1}\right\|_\infty \lesssim \Gamma_q^{\mathsf{C}_\infty/2+16}r_q^{-1}\lambda_{q+\bar{n}}^N\mathrm{T}_{q+\bar{n}-1}^{-M} \tag{9.88}$$

for $N + M \leq \mathsf{N}_{\text{fin}}/4$, thus verifying the final assumption (A.227b) from Lemma A.6.1.

We first apply (A.228) to conclude that (9.83) holds. Finally, we have from (A.229) and (4.19a) that the difference $w_{q+1} - \widehat{w}_{q+\bar{n}}$ satisfies (9.84). □

In a similar fashion, we will now verify the inductive assumptions of subsubsection 5.5.2 in the following proposition. We first recall the definitions of υ_{q+1} and e_{q+1} from Remark 9.4.2 and the mollifier $\widetilde{\mathcal{P}}_{q+\bar{n},x,t}$ from Definition 9.2.1 and define

$$\widehat{\upsilon}_{q+\bar{n}} := \widetilde{\mathcal{P}}_{q+\bar{n},x,t}\upsilon_{q+1}, \qquad \widehat{e}_{q+\bar{n}} := \widetilde{\mathcal{P}}_{q+\bar{n},x,t}e_{q+1}\,. \tag{9.89}$$

Proposition 9.5.2 (**Verifying** (5.38), (5.39), **and** (5.41) **and setting up** (5.40) **at level** $q+1$)**.** *The velocity increment and velocity increment potentials satisfy the following.*

(i) $\widehat{w}_{q+\bar{n}}$ *can be decomposed as*

$$\widehat{w}_{q+\bar{n}} = \operatorname{div}^{\mathsf{d}}\widehat{\upsilon}_{q+\bar{n}} + \widehat{e}_{q+\bar{n}}\,, \tag{9.90}$$

which written component-wise gives $\widehat{w}^{\bullet}_{q+\bar{n}} = \partial_{i_1}\cdots\partial_{i_{\mathsf{d}}}\widehat{\upsilon}^{(\bullet,i_1,\cdots,i_{\mathsf{d}})}_{q+\bar{n}} + \widehat{e}^{\bullet}_{q+\bar{n}}$.

(ii) For all $q+1 \leq q' \leq q+\bar{n}-1$, the supports of $\widehat{v}_{q+\bar{n}}$ and $\widehat{e}_{q+\bar{n}}$ satisfy

$$B\left(\operatorname{supp}(\widehat{w}_{q'}), \frac{1}{4}\lambda_{q'}\Gamma_{q'}^2\right) \cap \left(\operatorname{supp}(\widehat{v}_{q+\bar{n}}) \cup \operatorname{supp}(\widehat{e}_{q+\bar{n}})\right) = \emptyset\,. \tag{9.91}$$

(iii) For $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$, we have that $\widehat{v}^{\bullet}_{q+\bar{n},k} := \lambda_{q+\bar{n}}^{\mathsf{d}-k}\partial_{i_1}\cdots\partial_{i_k}\widehat{v}^{(\bullet,i_1,\ldots,i_{\mathsf{d}})}_{q+\bar{n}}$, $0 \leq k \leq \mathsf{d}$, satisfies the estimates

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1}D^N D^M_{t,q+\bar{n}-1}\widehat{v}_{q+\bar{n},k}\right| < \Gamma_{q+\bar{n}}\left(\sigma^+_{\upsilon^{(p)}} + \sigma^+_{\upsilon^{(c)}} + 2\delta_{q+3\bar{n}}\right)^{1/2} r_q^{-1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \\ \times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^i_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}-1}\right)\,.\end{aligned} \tag{9.92}$$

(iv) For $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$, $\widehat{e}_{q+\bar{n}}$ satisfies

$$\begin{aligned}\left\|D^N D^M_{t,q+\bar{n}-1}\widehat{e}_{q+\bar{n}}\right\|_\infty \leq \delta^3_{q+3\bar{n}}\mathrm{T}^{10\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\lambda^{-10}_{q+\bar{n}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \\ \times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}-1}\right)\,.\end{aligned} \tag{9.93}$$

Proof of Proposition 9.5.2. We first note that (9.90) follows immediately from the definition of $\widehat{v}_{q+\bar{n}}$ and $\widehat{e}_{q+\bar{n}}$ in (9.89) and the identity in Remark 9.4.2.

Next, an immediate consequence of (9.49) and (9.23) is that

$$B\left(\operatorname{supp}(\widehat{w}_{q'}), \frac{1}{2}\lambda_{q'}\Gamma^2_{q'}, 2\mathrm{T}_q\right) \cap \left(\operatorname{supp}(v_{q+1}) \cup \operatorname{supp}(e_{q+1})\right) = \emptyset\,.$$

for all $q+1 \leq q' \leq q+\bar{n}-1$. Now notice that by properties of the mollification, we have that

$$\operatorname{supp}(\widehat{v}_{q+\bar{n}}) \subseteq B\left(\operatorname{supp}(v_{q+1}), \left(\lambda_{q+\bar{n}}\Gamma^{1/2}_{q+\bar{n}-1}\right)^{-1}, \mathrm{T}^{-1}_{q+1}\right)\,,$$

and similarly

$$\operatorname{supp}(\widehat{e}_{q+\bar{n}}) \subseteq B\left(\operatorname{supp}(e_{q+1}), \left(\lambda_{q+\bar{n}}\Gamma^{1/2}_{q+\bar{n}-1}\right)^{-1}, \mathrm{T}^{-1}_{q+1}\right)\,.$$

With this we now see that (9.91) is satisfied.

Note that from (9.91) and (5.14) applied to $q' = q + \bar{n} - 1$ and $q'' = q$, we see that (9.64) implies that for all $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_5$, $0 \leq k \leq \mathsf{d}$ and $\mathtt{l} = p, c$,

$$\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} \widehat{\upsilon}^{(1)}_{q+\bar{n},k}\right| \lesssim (\sigma^+_{\upsilon^{(p)}} + \delta_{q+3\bar{n}})^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1} \Gamma^{i-4}_{q+\bar{n}-1}, \mathrm{T}_q^{-1} \Gamma_q^9\right) . \tag{9.94}$$

Now we apply Proposition A.6.1 with the parameter choices

$$\begin{aligned}
&p = 3, \infty\,, \quad N_{\text{g}}, N_{\text{c}} \text{ as in (xiii)}\,, \quad M_t = \mathsf{N}_{\text{ind,t}}\,, \quad N_* = {}^{\mathsf{N}_{\text{fin}}}/_5\,,\\
&N_\gamma = 2\mathsf{N}_{\text{fin}}\,, \quad \Omega = \operatorname{supp} \psi_{i,q+\bar{n}-1}\,, \quad v = \widehat{u}_{q+\bar{n}-1}\,, \quad i = i\,, \quad c = -1\,,\\
&\lambda = \lambda_{q+\bar{n}}\,, \quad \Lambda = \lambda_{q+\bar{n}} \Gamma_{q+\bar{n}-1}\,, \quad \Gamma = \Gamma_{q+\bar{n}-1}, \quad \tau = \tau_{q+\bar{n}-1} \Gamma^{-2}_{q+\bar{n}-1}\,, \quad \mathrm{T} = \mathrm{T}_{q+\bar{n}-1}\,,\\
&f = \upsilon^{(l)}_{q+1,k}\,, \quad \mathcal{C}_{f,3} = \Gamma_q^{20} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3}\,, \quad \mathcal{C}_{f,\infty} = \widetilde{\mathcal{C}}_f = \Gamma_q^{\mathsf{C}_\infty/2+16} r_q^{-1}\,, \quad \mathcal{C}_v = \Lambda^{1/2}_{q+\bar{n}-1}\,.
\end{aligned}$$

In a similar way to the proof of Lemma 9.5.1, we see that all the assumptions of the proposition are satisfied. Therefore, conclusion (A.229) implies that $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_5$, $0 \leq k \leq \mathsf{d}$ and $l = p, c$,

$$\left\| D^N D^M_{t,q+\bar{n}-1} \left(\widehat{\upsilon}^{(l)}_{q+\bar{n},k} - \upsilon^{(l)}_{q+1,k}\right)\right\|_\infty \lesssim \delta^3_{q+3\bar{n}} \mathrm{T}^{25\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right) .$$

Combining this estimate with the pointwise estimate (9.94) implies (9.92) for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_5$. The case when ${}^{\mathsf{N}_{\text{fin}}}/_5 \leq N + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2$ follows from first noticing that conclusion (A.228) implies that for all $N, M \leq 2\mathsf{N}_{\text{fin}}$, $0 \leq k \leq \mathsf{d}$ and $\mathtt{l} = p, c$, we have

$$\left\|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} \widehat{\upsilon}^{(l)}_{q+\bar{n},k}\right\|_\infty \lesssim \Gamma_q^{\mathsf{C}_\infty/2+16} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma^{1/10}_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1} \Gamma^{i-4}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right) .$$

Then combining this estimate with (4.20b) implies estimate (9.92) in this case.

Finally, to prove (9.93), we must upgrade the nonlocal derivative bound in (9.51). This is trivial using the extra prefactors of $\mathrm{T}^{20\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}}$, and so we omit the details.

$\square$

Chapter 10

Convex integration in the Euler-Reynolds system

10.1 Defining new error terms

We define S_{q+1} by adding $\widehat{w}_{q+\bar{n}}$ to the Euler-Reynolds system for $(u_q, p_q, R_q, -\pi_q)$ in (6.2) (recall also (5.2)) and collecting various error terms, which we shall show are well-defined in the remainder of this section.

$$
\begin{aligned}
\operatorname{div}(S_{q+1}) &= \partial_t \widehat{w}_{q+\bar{n}} + (u_q \cdot \nabla)\widehat{w}_{q+\bar{n}} + (\widehat{w}_{q+\bar{n}} \cdot \nabla)u_q + \operatorname{div}(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} + R_\ell - \pi_\ell \mathrm{Id}) \\
&\quad + \operatorname{div}\left(R_q^q - R_\ell + \left(\pi_\ell - \pi_q^q\right)\mathrm{Id}\right) \\
&= \underbrace{(\partial_t + \widehat{u}_q \cdot \nabla)w_{q+1} + w_{q+1} \cdot \nabla \widehat{u}_q}_{=:\operatorname{div} S_{TN}} + \underbrace{\operatorname{div}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + R_\ell - \pi_\ell \mathrm{Id}\right)}_{=:\operatorname{div} S_O} \\
&\quad + \underbrace{\operatorname{div}\left(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)} + w_{q+1}^{(c)} \otimes w_{q+1}^{(c)}\right)}_{=:\operatorname{div} S_C} + \underbrace{\operatorname{div}\left(R_q^q - R_\ell + \left(\pi_\ell - \pi_q^q\right)\mathrm{Id}\right)}_{=:\operatorname{div} S_{M1}} \qquad (10.1) \\
&\quad + \underbrace{(\partial_t + \widehat{u}_q \cdot \nabla)(\widehat{w}_{q+\bar{n}} - w_{q+1}) + ((\widehat{w}_{q+\bar{n}} - w_{q+1}) \cdot \nabla)\widehat{u}_q + \operatorname{div}(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1})}_{=:\operatorname{div} S_{M2}}
\end{aligned}
$$

In the second equality, we used (9.24) to exchange u_q and $\widehat{u}_q$. (Recall also (5.4).) We note that the symmetric stresses S_O and S_C are not simply the quantities inside parentheses and take some care to construct; see subsubsections 10.2.1, 10.2.3. Also, we note that $\partial_t w_{q+1} + (\widehat{u}_q \cdot \nabla) w_{q+1} + w_{q+1} \cdot \nabla \widehat{u}_q$ has mean-zero, so that it can be written in divergence form $\mathrm{div} S_{TN}$; see subsection 10.2.2. This is because the second and third terms can be written in divergence form, and w_{q+1} is given by the curl of a vector-valued function (see (9.7) and (9.15).) The same reasoning works for the terms in $\mathrm{div} S_{M2}$.

With the above definitions, we set

$$\overline{R}_{q+1} := R_q - R_q^q + S_{q+1} \,. \tag{10.2}$$

We can now see that $(u_{q+1}, p_q, \overline{R}_{q+1}, -(\pi_q - \pi_q^q))$ solves the Euler-Reynolds system (recall from (9.17) that $u_{q+1} = u_q + \widehat{w}_{q+\bar{n}}$)

$$\partial_t u_{q+1} + \mathrm{div}\,(u_{q+1} \otimes u_{q+1}) + \nabla p_q = \mathrm{div}(-(\pi_q - \pi_q^q)\mathrm{Id} + \overline{R}_{q+1}) \,, \qquad \mathrm{div}\, u_{q+1} = 0 \,. \tag{10.3}$$

We will show in the remainder of this section that the new stress error S_{q+1} can be decomposed into components S_{q+1}^k as

$$S_{q+1} = \sum_{k=q+1}^{q+\bar{n}} S_{q+1}^k \,.$$

10.2 Error estimates

10.2.1 Oscillation stress error S_O

In order to define and analyze S_O, cf. (10.1), we first consider

$$\operatorname{div}\left(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}\right)^{\bullet}=\sum_{\xi,i,j,k,\vec{l},\diamond}\partial_\alpha\left(a_{(\xi),\diamond}(\nabla\Phi_{(i,k)}^{-1})_\theta^\alpha\mathbb{B}_{(\xi),\diamond}^\theta(\Phi_{(i,k)})\,a_{(\xi),\diamond}(\nabla\Phi_{(i,k)}^{-1})_\gamma^\bullet\mathbb{B}_{(\xi),\diamond}^\gamma(\Phi_{(i,k)})\right)\,, \tag{10.4}$$

where $\bullet$ denotes the unspecified components of a vector field and we have used (9.25) from Lemma 9.2.2 to eliminate all cross terms. Recalling from (9.4) and (9.12) that $\mathbb{B}_{(\xi),\diamond}=\boldsymbol{\rho}_{(\xi)}^\diamond\sum_I\boldsymbol{\zeta}_\xi^{I,\diamond}\mathbb{W}_{(\xi),\diamond}^I$, that the $\mathbb{W}_{(\xi),\diamond}^I$'s are identical up to a shift, and the notational convention for $\boldsymbol{\rho}_{(\xi)}^\diamond$ from Remark 7.2.7, we decompose

$$\begin{aligned}(\mathbb{B}\otimes\mathbb{B})_{(\xi),\diamond}&=\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\sum_I(\boldsymbol{\zeta}_\xi^{I,\diamond})^2\mathbb{P}_{\neq0}(\mathbb{W}_{(\xi),\diamond}^I\otimes\mathbb{W}_{(\xi),\diamond}^I)+\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\mathbb{P}_{\neq0}\left(\sum_I(\boldsymbol{\zeta}_\xi^{I,\diamond})^2\right)\left\langle\mathbb{W}_{(\xi),\diamond}^I\otimes\mathbb{W}_{(\xi),\diamond}^I\right\rangle\\&\quad+\mathbb{P}_{\neq0}\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\left\langle\sum_I(\boldsymbol{\zeta}_\xi^{I,\diamond})^2\right\rangle\left\langle\mathbb{W}_{(\xi),\diamond}^I\otimes\mathbb{W}_{(\xi),\diamond}^I\right\rangle+\left\langle\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\right\rangle\left\langle\sum_I(\boldsymbol{\zeta}_\xi^{I,\diamond})^2\right\rangle\left\langle\mathbb{W}_{(\xi),\diamond}^I\otimes\mathbb{W}_{(\xi),\diamond}^I\right\rangle\,.\end{aligned}$$

In particular, using (iii) and the definitions of $\boldsymbol{\rho}_{(\xi)}^\diamond$ and $\overline{\boldsymbol{\rho}}_{(\xi)}:=\overline{\boldsymbol{\rho}}_{\xi,k}$ from Proposition 7.2.1, (4) from Proposition 7.1.5, (6) from Proposition 7.1.6, Definition 7.2.6, and (7.27), we obtain that

$$(\mathbb{B}\otimes\mathbb{B})_{(\xi),R}=\left(\overline{\boldsymbol{\rho}}_{(\xi)}\right)^6\sum_I(\boldsymbol{\zeta}_\xi^I)^6\mathbb{P}_{\neq0}(\mathbb{W}_{(\xi),R}^I\otimes\mathbb{W}_{(\xi),R}^I)+\left(\left(\overline{\boldsymbol{\rho}}_{(\xi)}\right)^6-1\right)\xi\otimes\xi+\xi\otimes\xi\,, \tag{10.5a}$$

$$\begin{aligned}(\mathbb{B}\otimes\mathbb{B})_{(\xi),\varphi}&=\left(\overline{\boldsymbol{\rho}}_{(\xi)}\right)^4\sum_I(\boldsymbol{\zeta}_\xi^I)^4\mathbb{P}_{\neq0}(\mathbb{W}_{(\xi),\varphi}^I\otimes\mathbb{W}_{(\xi),\varphi}^I)+c_0\left(\overline{\boldsymbol{\rho}}_{(\xi)}\right)^4r_q^{\frac{2}{3}}\xi\otimes\xi\,\mathbb{P}_{\neq0}\left(\sum_I(\boldsymbol{\zeta}_\xi^I)^4\right)\\&\quad+c_0c_1\mathbb{P}_{\neq0}\left(\left(\overline{\boldsymbol{\rho}}_{(\xi)}\right)^4\right)r_q^{\frac{2}{3}}\xi\otimes\xi+c_0c_1c_2r_q^{\frac{2}{3}}\Gamma_q^{-2}\xi\otimes\xi\,,\end{aligned} \tag{10.5b}$$

for dimensional constants c_0, c_1, and c_2 which are bounded independently of q and depend only on the dimensional constants in (7.23) and (7.16) and the mean of $\sum_I(\boldsymbol{\zeta}_\xi^I)^4$. Since

each vector field used to define the simple symmetric tensors in (10.5a) and (10.5b) does not vary in the ξ-direction (see, (7.10), (i), and Definition 7.2.4), each simple symmetric tensor satisfies $\xi \cdot \nabla(\mathbb{B} \otimes \mathbb{B})_{(\xi),\diamond} = 0$. Then using that each vector field in (10.5a) and (10.5b) has been composed with $\Phi_{(i,k)}$ and the identity $\partial_\alpha \left((\nabla \Phi^{-1}_{(i,k)})^\alpha_\theta (\mathbb{B} \otimes \mathbb{B})_{(\xi),\diamond} \circ \Phi_{(i,k)} \xi^\theta \right) = \xi^\theta (\partial_\theta (\mathbb{B} \otimes \mathbb{B})_{(\xi),\diamond}) \circ \Phi_{(i,k)} = 0$, we have that (10.4) can be expanded as

$$\operatorname{div}\left(w^{(p)}_{q+1} \otimes w^{(p)}_{q+1}\right)^\bullet = \sum_{\xi,i,j,k,\vec{l}} \partial_\alpha \left(a^2_{(\xi),R} (\nabla \Phi^{-1}_{(i,k)})^\alpha_\theta (\nabla \Phi^{-1}_{(i,k)})^\bullet_\gamma (\xi^\theta \xi^\gamma) \right) \tag{10.6a}$$

$$+ \sum_{\xi,i,j,k,\vec{l}} \partial_\alpha \left(a^2_{(\xi),\varphi} (\nabla \Phi^{-1}_{(i,k)})^\alpha_\theta (\nabla \Phi^{-1}_{(i,k)})^\bullet_\gamma c_0 c_1 c_2 \Gamma_q^{-2} r_q^{\frac{2}{3}} (\xi^\theta \xi^\gamma) \right) \tag{10.6b}$$

$$+ \sum_{\xi,i,j,k,\vec{l}} B^\bullet_{(\xi),R} \left(\mathbb{P}_{\neq 0} \overline{\boldsymbol{\rho}}^6_{(\xi)} \right) \circ \Phi_{(i,k)} \tag{10.6c}$$

$$+ \sum_{\xi,i,j,k,\vec{l}} B^\bullet_{(\xi),\varphi} \left(\mathbb{P}_{\neq 0} \overline{\boldsymbol{\rho}}^4_{\xi} \right) \circ (\Phi_{(i,k)}) c_0 c_1 r_q^{\frac{2}{3}} \tag{10.6d}$$

$$+ c_0 \sum_{\xi,i,j,k,\vec{l}} B^\bullet_{(\xi),\varphi} r_q^{\frac{2}{3}} \left(\overline{\boldsymbol{\rho}}^4_{(\xi)} \mathbb{P}_{\neq 0} \sum_I (\boldsymbol{\zeta}^I_\xi)^4 \right) \circ \Phi_{(i,k)} \tag{10.6e}$$

$$+ \sum_{\xi,i,j,k,\vec{l},\diamond} B^\bullet_{(\xi),\diamond} \left(\left(\boldsymbol{\rho}^\diamond_{(\xi)} \right)^2 \sum_I (\boldsymbol{\zeta}^{I,\diamond}_\xi)^2 \mathbb{P}_{\neq 0} (\varrho^I_{(\xi),\diamond})^2 \right) \circ \Phi_{(i,k)} \tag{10.6f}$$

where for convenience we set

$$B^\bullet_{(\xi),\diamond} := \xi^\theta \xi^\gamma \partial_\alpha \left(a^2_{(\xi),\diamond} (\nabla \Phi^{-1}_{(i,k)})^\alpha_\theta (\nabla \Phi^{-1}_{(i,k)})^\bullet_\gamma \right), \qquad \varrho^I_{(\xi),\diamond} := \xi \cdot \mathbb{W}^I_{(\xi),\diamond}. \tag{10.7}$$

The first and second terms above in (10.6a) and (10.6b) cancel out $-R_\ell + \pi_\ell \mathrm{Id}$ from (10.1)

as follows:

$$
\begin{aligned}
&\sum_{\xi,i,j,k,\vec{l}} a^2_{(\xi),R} \nabla\Phi^{-1}_{(i,k)} \left(\xi\otimes\xi\right) \nabla\Phi^{-\top}_{(i,k)} \\
&\underset{(9.9)}{=} \sum_{\xi,i,j,k,\vec{l}} \delta_{q+\bar{n}}\Gamma_q^{2j-2}\psi^6_{i,q}\omega^6_{j,q}\chi^6_{i,k,q}\zeta^2_{q,R,i,k,\xi,\vec{l}}\gamma^2_{\xi,\Gamma^9_q}\left(\frac{R_{q,i,k}}{\delta_{q+\bar{n}}\Gamma_q^{2j-2}}\right)\nabla\Phi^{-1}_{(i,k)}\left(\xi\otimes\xi\right)\nabla\Phi^{-\top}_{(i,k)} \\
&\underset{(7.2),(9.8),(8.36a)}{=} -\sum_{i,j,k}\psi^6_{i,q}\omega^6_{j,q}\chi^6_{i,k,q}\Bigg(R_\ell-\pi_\ell\mathrm{Id} \\
&\quad+\sum_{\substack{\xi',i',j'\\k',l'}}\frac{\delta_{q+\bar{n}}\Gamma_q^{2j'-2}C\Gamma_q^{-2}}{\left|\nabla\Phi^{-1}_{(i',k')}\xi'\right|^{4/3}}\psi^4_{i',q}\omega^4_{j',q}\chi^4_{i',k',q}\mathcal{X}^4_{q,\xi',l'}\circ\Phi_{i',k',q}\widetilde{\gamma}^2_{\xi'}\nabla\Phi^{-1}_{(i',k')}\xi'\otimes\xi'\left(\nabla\Phi^{-T}_{(i',k')}\right)\Bigg) \\
&\underset{(5.8),(8.21),(8.1)}{=} \pi_\ell\mathrm{Id}-R_\ell \\
&\quad-\sum_{\substack{\xi',i',j'\\k',l'}}\frac{\delta_{q+\bar{n}}\Gamma_q^{2j'-2}C\Gamma_q^{-2}}{\left|\nabla\Phi^{-1}_{(i',k')}\xi'\right|^{4/3}}\psi^4_{i',q}\omega^4_{j',q}\chi^4_{i',k',q}\mathcal{X}^4_{q,\xi',l'}\circ\Phi_{i',k',q}\widetilde{\gamma}^2_{\xi'}\nabla\Phi^{-1}_{(i',k')}\xi'\otimes\xi'\left(\nabla\Phi^{-T}_{(i',k')}\right) \\
&\underset{(8.36b)}{=} \pi_\ell\mathrm{Id}-R_\ell-\sum_{\substack{\xi',i',j'\\k',\vec{l}}}\frac{\delta_{q+\bar{n}}\Gamma_q^{2j'-2}C\Gamma_q^{-2}}{\left|\nabla\Phi^{-1}_{(i',k')}\xi'\right|^{4/3}}\psi^4_{i',q}\omega^4_{j',q}\chi^4_{i',k',q}\zeta^2_{q,\varphi,i',k',\xi',\vec{l}}\widetilde{\gamma}^2_{\xi'}\nabla\Phi^{-1}_{(i',k')}\xi'\otimes\xi'\left(\nabla\Phi^{-T}_{(i',k')}\right) \\
&\underset{(9.2)}{=} \pi_\ell\mathrm{Id}-R_\ell-\underbrace{\sum_{\substack{\xi',i',j'\\k',\vec{l}}}a^2_{(\xi),\varphi}c_0c_1c_2\Gamma_q^{-2}r_q^{2/3}\nabla\Phi^{-1}_{(i',k')}\xi'\otimes\xi'\left(\nabla\Phi^{-T}_{(i',k')}\right)}_{=(10.6b)}. \qquad (10.8)
\end{aligned}
$$

The inverse divergence of the remaining terms (10.6c)-(10.6f) will therefore form the oscillation stress errors.

Lemma 10.2.1 (**Applying inverse divergence**). *There exist symmetric stresses S_O^m for $m=1,\dots,q+\bar{n}$ such that the following hold.*

(i) $\mathrm{div}\left(w^{(p)}_{q+1}\otimes w^{(p)}_{q+1}+R_\ell-\pi_\ell\mathrm{Id}\right)=\sum_{m=q+1}^{q+\bar{n}}\mathrm{div}S_O^m$, *where S_O^m can be split into local and non-local errors as $S_O^m=S_O^{m,l}+S_O^{m,*}$.*

(ii) For $m = q+1, \dots, q+\bar{n}$ and $N, M \leq \mathsf{N}_{\text{fin}}/10$, the local parts $S_O^{m,l}$ satisfy

$$\left\| \psi_{i,q} D^N D_{t,q}^M S_O^{m,l} \right\|_{3/2} \lesssim \Gamma_m^{-9} \delta_{m+\bar{n}} \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.9a}$$

$$\left\| \psi_{i,q} D^N D_{t,q}^M S_O^{m,l} \right\|_{\infty} \lesssim \Gamma_m^{\mathsf{C}_\infty - 9} \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{10.9b}$$

When $m = q+2, \dots, q+\bar{n}$ and $q+1 \leq q' \leq m-1$, the local parts satisfy

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} S_O^{m,l} = \emptyset . \tag{10.10}$$

(iii) For $m = q+1, \dots, q+\bar{n}$ and $N, M \leq 2\mathsf{N}_{\text{ind}}$, the non-local parts $S_O^{m,*}$ satisfy

$$\left\| D^N D_{t,q}^M S_O^{m,*} \right\|_{L^\infty} \leq \mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\text{ind,t}}} \delta_{q+3\bar{n}} \lambda_m^N \tau_q^{-M} . \tag{10.11}$$

Remark 10.2.2 (Abstract formulation of the oscillation stress error). For the purposes of analyzing the transport and Nash current errors in subsubsection 11.2.2 and streamlining the creation of pressure increments, it will be useful to abstract the properties of these error terms. First, there exists a q-independent constant $\mathcal{C}_{\mathcal{H}}$ such that

$$S_O^{m,l} = \sum_{i,j,k,\xi,\vec{l},\diamond} \sum_{j'=0}^{\mathcal{C}_{\mathcal{H}}} H_{i,j,k,\xi,\vec{l},\diamond}^{\alpha(j')} \rho_{i,j,k,\xi,\vec{l},\diamond}^{\beta(j')} \circ \Phi_{(i,k)} \qquad \text{if} \qquad m = q+1,\ q+\bar{n}/2 , \tag{10.12a}$$

$$S_O^{m,l} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sum_{j'=0}^{\mathcal{C}_{\mathcal{H}}} H_{i,j,k,\xi,\vec{l},I,\diamond}^{\alpha(j')} \rho_{i,j,k,\xi,\vec{l},I,\diamond}^{\beta(j')} \circ \Phi_{(i,k)} \qquad \text{if } q+\bar{n}/2+1 \leq m \leq q+\bar{n} . \tag{10.12b}$$

These equalities will be proven in the course of proving Lemma 10.2.1. Next, the functions H and ρ (with subscripts and superscripts suppressed for convenience) defined above satisfy the following.

(i) For all $N, M \leq \mathsf{N}_{\text{fin}}/10$,

$$\left| D^N D_{t,q}^M H \right| \lesssim \pi_\ell \Gamma_q^{100} \Lambda_q \bar{\lambda}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \tag{10.13}$$

where $\bar{\lambda} = \lambda_{q+1}\Gamma_q^{-5}$ for $m = q+1, q+{}^{\bar{n}}/_2$ while $\bar{\lambda} = \lambda_{q+{}^{\bar{n}}/_2}$ for $m \geq q + {}^{\bar{n}}/_2 + 1$. For the remaining values of m, $S_O^{m,l}$ is zero. We will prove (10.13) in Lemmas 10.2.3 and 10.2.4.

(ii) We have that

$$\operatorname{supp} H \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \quad \text{if} \quad m = q+1, q+{}^{\bar{n}}/_2 \tag{10.14a}$$

$$\operatorname{supp} H \subseteq \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond} \quad \text{if} \quad q + {}^{\bar{n}}/_2 + 1 \leq m \leq q + \bar{n} \tag{10.14b}$$

We will prove these claims in the course of proving Lemma 10.2.1.

(iii) For d as in (xvii) of section 4.1, there exist a tensor potential ϑ (we suppress the indices at the moment for convenience) such that $\rho = \partial_{i_1 \ldots i_{\mathsf{d}}} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})}$. Furthermore, ϑ is $(\mathbb{T}/\lambda_{q+1}\Gamma_q^{-4})^3$-periodic in the case $m = q+1$, $(\mathbb{T}/\lambda_{q+{}^{\bar{n}}/_2})^3$-periodic in the case $m = q + {}^{\bar{n}}/_2$, and $(\mathbb{T}/\lambda_{q+{}^{\bar{n}}/_2}\Gamma_q)^3$-periodic in the remaining cases. Finally, ϑ satisfies the estimates

$$\left\| D^N \partial_{i_1} \ldots \partial_{i_k} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})} \right\|_{L^p} \lesssim \Gamma_q^{12} (\lambda_{q+1}\Gamma_q^{-4})^{k-\mathsf{d}-1} \mathcal{M}\left(N, \mathsf{d}-k, \lambda_{q+1}\Gamma_q^{-4}, \lambda_{q+1}\Gamma_q^{-1}\right) \quad \text{if} \quad m = q+1 \tag{10.15a}$$

$$\left\| D^N \partial_{i_1} \ldots \partial_{i_k} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})} \right\|_{L^p} \lesssim \Gamma_q^5 \lambda_{q+{}^{\bar{n}}/_2}^{k-\mathsf{d}-1} \lambda_{q+{}^{\bar{n}}/_2}^N \quad \text{if} \quad m = q + {}^{\bar{n}}/_2 \tag{10.15b}$$

$$\begin{aligned} \left\| D^N \partial_{i_1} \ldots \partial_{i_k} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})} \right\|_{L^p} \lesssim{}& \left(\frac{\lambda_{q+{}^{\bar{n}}/_2+1}}{\lambda_{q+\bar{n}} r_q} \right)^{2-2/p} \Gamma_q^2 \lambda_{q+{}^{\bar{n}}/_2}^{-1} (\lambda_{q+{}^{\bar{n}}/_2}\Gamma_q)^{k-\mathsf{d}} \\ &\times \mathcal{M}\left(N, \mathsf{d}-k, \lambda_{q+{}^{\bar{n}}/_2}\Gamma_q, \lambda_{q+{}^{\bar{n}}/_2+1}\right) \quad \text{if} \quad m = q + {}^{\bar{n}}/_2 + 1 \end{aligned} \tag{10.15c}$$

$$\begin{aligned} \left\| D^N \partial_{i_1} \ldots \partial_{i_k} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})} \right\|_{L^p} \lesssim{}& \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q} \right)^{2-2/p} \Gamma_q^2 \lambda_{m-1}^{-2} \lambda_m \lambda_{m-1}^{k-\mathsf{d}} \lambda_m^N \\ &\text{if} \quad q + {}^{\bar{n}}/_2 + 2 \leq m \leq q + \bar{n} \end{aligned} \tag{10.15d}$$

for $p = {}^3/_2, \infty$, all $N \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_5$, and $0 \leq k \leq \mathsf{d}$. We will prove these estimates in the course of proving Lemma 10.2.1 with the help of Remark A.3.6.

(iv) In the cases $m = q+1, q+{}^{\bar{n}}/_2, q+{}^{\bar{n}}/_2+1$, we claim no special support properties for the potential ϑ. In the cases $q+{}^{\bar{n}}/_2+2 \leq m \leq q+\bar{n}$, we have that

$$\operatorname{supp}\,(H\rho \circ \Phi) \cap B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) = \emptyset \tag{10.16}$$

for all $q+1 \leq q' \leq m-1$ (where m refers to the index in $S_O^{m,l}$ from (10.12a)). We will prove this claim in the course of proving Lemma 10.2.1.

Proof of Lemma 10.2.1. To define S_O, we recall the synthetic Littlewood-Paley decomposition (cf. Section 7.3). Indeed, since $\varrho^I_{(\xi),\diamond}$ depends only on the variables in the plane $\xi^\perp$ from (7.10) and is periodized to scale $(\lambda_{q+\bar{n}} r_q)^{-1} = (\lambda_{q+{}^{\bar{n}}/_2}\Gamma_q)^{-1}$, we can decompose $\mathbb{P}_{\neq 0}$ in front of $(\varrho^I_{(\xi),\diamond})^2$ in (10.6f) into

$$\begin{aligned}\mathbb{P}_{\neq 0} &= \widetilde{\mathbb{P}}^{\xi}_{\lambda_{q+{}^{\bar{n}}/_2+1}} \mathbb{P}_{\neq 0} + \sum_{m=q+{}^{\bar{n}}/_2+2}^{q+\bar{n}+1} \widetilde{\mathbb{P}}^{\xi}_{(\lambda_{m-1},\lambda_m]} + (\mathrm{Id} - \widetilde{\mathbb{P}}^{\xi}_{\lambda_{q+\bar{n}+1}}) \\ &=: \widetilde{\mathbb{P}}^{\xi}_{q+{}^{\bar{n}}/_2+1} + \sum_{m=q+{}^{\bar{n}}/_2+2}^{q+\bar{n}+1} \widetilde{\mathbb{P}}^{\xi}_{(m-1,m]} + (\mathrm{Id} - \widetilde{\mathbb{P}}^{\xi}_{q+\bar{n}+1})\,. \end{aligned} \tag{10.17}$$

Assuming we can apply the inverse divergence from Proposition A.3.3, we define

$$S_O^{q+1} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l}} B_{(\xi),R}\left(\mathbb{P}_{\neq 0}\overline{\boldsymbol{\rho}}_\xi^6\right)\circ\Phi_{(i,k)} + \sum_{\xi,i,j,k,\vec{l}} B_{(\xi),\varphi} c_0 c_1 r_q^{\frac{2}{3}}\left(\mathbb{P}_{\neq 0}\overline{\boldsymbol{\rho}}_\xi^4\right)\circ\Phi_{(i,k)}\right] \tag{10.18a}$$

$$S_O^{q+\bar{n}/2} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l}} B_{(\xi),\varphi} c_0 r_q^{\frac{2}{3}}\left(\overline{\boldsymbol{\rho}}_\xi^4 \mathbb{P}_{\neq 0}\left(\sum_I (\boldsymbol{\zeta}_\xi^I)^4\right)\right)\circ\Phi_{(i,k)}\right] \tag{10.18b}$$

$$S_O^{q+\bar{n}/2+1} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} B_{(\xi),\diamond}\left(\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\left(\boldsymbol{\zeta}_\xi^{I,\diamond}\right)^2 \widetilde{\mathbb{P}}^\xi_{q+\bar{n}/2+1}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\diamond})^2\right)\circ\Phi_{(i,k)}\right] \tag{10.18c}$$

$$S_O^{m} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I} B_{(\xi),\diamond}\left(\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\left(\boldsymbol{\zeta}_\xi^{I,\diamond}\right)^2 \widetilde{\mathbb{P}}^\xi_{(m-1,m]}(\varrho^I_{(\xi),\diamond})^2\right)\circ\Phi_{(i,k)}\right] \tag{10.18d}$$

$$S_O^{q+\bar{n}} := \sum_{m=q+\bar{n}}^{q+\bar{n}+1}(\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} B_{(\xi),\diamond}\left(\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\left(\boldsymbol{\zeta}_\xi^{I,\diamond}\right)^2 \widetilde{\mathbb{P}}^\xi_{(m-1,m]}(\varrho^I_{(\xi),\diamond})^2\right)\circ\Phi_{(i,k)}\right] \tag{10.18e}$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} B_{(\xi),\diamond}\left(\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\left(\boldsymbol{\zeta}_\xi^{I,\diamond}\right)^2 (\mathrm{Id}-\widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1})(\varrho^I_{(\xi),\diamond})^2\right)\circ\Phi_{(i,k)}\right] \tag{10.18f}$$

for $m = q + \bar{n}/2 + 2, \cdots, q+\bar{n}-1$. For $q+1 \le m < q+\bar{n}$, we decompose S_O^m into the local part $S_O^{m,l}$ which involves the operator $\mathcal{H}$ and the nonlocal part $S_O^{m,*}$ containing the remaining terms. In the case of $m = q+\bar{n}$, we set

$$S_O^{q+\bar{n},l} := \sum_{m=q+\bar{n}}^{q+\bar{n}+1}\mathcal{H}\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} B_{(\xi),\diamond}\left(\left(\boldsymbol{\rho}_{(\xi)}^\diamond\right)^2\left(\boldsymbol{\zeta}_\xi^{I,\diamond}\right)^2 \widetilde{\mathbb{P}}^\xi_{(m-1,m]}(\varrho^I_{(\xi),\diamond})^2\right)\circ\Phi_{(i,k)}\right] \tag{10.19}$$

and absorb the $\mathcal{R}^*$ terms in (10.18e) and all the terms in (10.18f) into $S_O^{q+\bar{n},*}$. For the undefined S_O^m corresponding to $m = q+2, \cdots, q+\bar{n}/2-1$, we set them as identically zero.

The desired estimates will follow from applying Proposition A.3.3. While many of the

parameter choices will vary depending on the case, we fix the following choices throughout the proof:

$$p = {}^3\!/_2, \infty\,, \quad v = \widehat{u}_q\,, \quad D_t = D_{t,q}\,, \quad N_* = {}^{\mathsf{N}_{\text{fin}}}\!/_4\,, \quad M_* = {}^{\mathsf{N}_{\text{fin}}}\!/_5\,, \tag{10.20a}$$

$$\lambda' = \Lambda_q\,, \quad M_t = \mathsf{N}_{\text{ind,t}}\,, \quad \nu' = \mathrm{T}_q^{-1}\Gamma_q^{8}\,, \quad \mathsf{N}_{\text{dec}} \text{ as in (xv)}\,, \tag{10.20b}$$

$$M_\circ = N_\circ = 2\mathsf{N}_{\text{ind}}\,, \quad K_\circ \text{ as in (xvi)}\,, \quad \mathcal{C}_v = \Lambda_q^{1/2}\,. \tag{10.20c}$$

`Case 1:` Estimates for (10.18a). Fix values of $i, j, k, \xi, \vec{l}$ and consider the term which includes $B_{(\xi),R}$, where we have abbreviated $B^{\bullet}_{(\xi),R} = B^{\bullet}_{(\xi,i,j,k,\vec{l}),R}$. We apply Proposition A.3.3 with the low-frequency choices

$$G^{\bullet} = B^{\bullet}_{(\xi),R}\,, \quad \mathcal{C}_{G,3/2} = \left|\operatorname{supp}\left(\eta^2_{i,j,k,\xi,\vec{l},R}\right)\right|^{2/3} \delta_{q+\bar{n}}\Gamma_q^{2j+21}\Lambda_q\,, \quad \mathcal{C}_{G,\infty} = \Gamma_q^{\mathsf{C}_\infty+30}\Lambda_q\,,$$
$$\lambda = \lambda_{q+1}\Gamma_q^{-5}\,, \quad \nu = \tau_q^{-1}\Gamma_q^{i+13}\,, \quad \Phi = \Phi_{(i,k)}\,,$$

and the choices from (10.20). We have that (A.39) is satisfied by definition. Next, to check (A.40), we observe that in $B^{\bullet}_{(\xi),R}$, the differential operator on a^2_ξ is $\xi^\theta(\nabla\Phi^{-1}_{(i,k)})^\alpha_\theta\partial_\theta$. Therefore G satisfies (A.40) for $p = {}^3\!/_2$ from (9.36c) and for $p = \infty$ from the same inequality and (8.27). By Corollary 8.2.4, $\Phi_{(i,k)}$ satisfies (A.41) and (A.42a) for $\lambda' = \Lambda_q$, and by (5.34) at level q, we have that (A.42b) is satisfied.

To check the high-frequency assumptions, we set

$$\varrho = \left(\mathbb{P}_{\neq 0}\overline{\boldsymbol{\rho}}^6_\xi\right)\,, \quad \mathsf{d} \text{ as in (xvii)}\,, \quad \vartheta = \delta_{i_1 i_2}\delta_{i_3 i_4}\ldots\delta_{i_{\mathsf{d}-1} i_{\mathsf{d}}}\Delta^{-\mathsf{d}/2}\varrho\,, \tag{10.21a}$$

$$\mu = \Upsilon = \Upsilon' = \lambda_{q+1}\Gamma_q^{-4}\,, \quad \overline{\Lambda} = \lambda_{q+1}\Gamma_q^{-1}\,, \quad \mathcal{C}_{*,p} = \Gamma_q^6\lambda_{q+1}^\alpha\,, \tag{10.21b}$$

where α is chosen as in (4.14). Then from Proposition 7.2.1 and standard Littlewood-Paley theory, we have that (A.43) is satisfied. Next, we have that (A.44) is satisfied by definition and from (4.24a). In addition, we have that (A.45) is satisfied from (4.21). In order to check the nonlocal assumptions in Part 4, we first appeal to (4.24a), which gives (A.52). We have

that (A.53) is satisfied from (5.35b), and (A.54) is satisfied from (4.15) and (5.10). Finally, we have that (A.55) is satisfied from (4.23b).

We therefore may appeal to the local conclusions (i)–(vi) of Proposition A.3.3 and the nonlocal outputs from (A.56)–(A.57), from which we have the following. First, we note that from (iii), we have that (10.12a) is satisfied. Next, abbreviating $G\varrho \circ \Phi$ as $T_{i,j,k,\xi,\vec{l},R}$, we have from (A.46) and (A.50) that for $N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$,

$$\begin{aligned}
\left\| D^N D^M_{t,q} \mathcal{H} T_{i,j,k,\xi,\vec{l},R} \right\|_{3/2} &\lesssim \left| \operatorname{supp}\left(\eta^2_{i,j,k,\xi,\vec{l},R}\right) \right|^{2/3} \delta_{q+\bar{n}} \Gamma_q^{2j-2} \Lambda_q \Gamma_q^{42} \\
&\qquad \times \lambda_{q+1}^{-1} \lambda_{q+1}^{\alpha+N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
\left\| D^N D^M_{t,q} \mathcal{H} T_{i,j,k,\xi,\vec{l},R} \right\|_{\infty} &\lesssim \Gamma_q^{\mathsf{C}_\infty+48} \Lambda_q \lambda_{q+1}^{-1} \lambda_{q+1}^{\alpha+N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \\
&\lesssim \Gamma_{q+1}^{\mathsf{C}_\infty-9} \lambda_{q+1}^{N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) ,
\end{aligned}$$

where we have used (4.10k) to achieve the last inequality. Notice that from (ii), the support of $\operatorname{div}\mathcal{H}T_{i,j,k,\xi,\vec{l},R}$ is contained in the support of $T_{i,j,k,\xi,\vec{l},R}$, which itself is contained in the support of $\eta_{i,j,k,\xi,\vec{l},R}$. From this observation, we have that (10.14a) is satisfied. Finally, we have that (10.15a) holds after defining a potential ϑ as in (10.21a) and appealing to standard Littlewood-Paley estimates and (A.49a).

Now we may apply the aggregation Corollaries 8.6.1 and 8.6.3 with $H = \mathcal{H}T_{i,j,k,\xi,\vec{l},R}$ and $\theta = \theta_2 = 2$, $p = 3/2$ in the first case, or $\varpi = \Gamma_{q+1}^{\mathsf{C}_\infty-9}$ in the second case, to estimate

$$S^{q+1,l}_{O,R} := \sum_{i,j,k,\xi,\vec{l}} \mathcal{H}T_{i,j,k,\xi,\vec{l},R} \, .$$

From (8.48a) and (8.48b) in the case $p = 3/2$, and (8.53a) in the case $p = \infty$, we thus have that for N, M in the same range as above,

$$\begin{aligned}
\left\| \psi_{i,q} D^N D^M_{t,q} S^{q+1,l}_{O,R} \right\|_{3/2} &\lesssim \delta_{q+\bar{n}} \Lambda_q \Gamma_q^{50} \lambda_{q+1}^{-1} \lambda_{q+1}^{\alpha+N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
\left\| \psi_{i,q} D^N D^M_{t,q} S^{q+1,l}_{O,R} \right\|_{\infty} &\lesssim \Gamma_{q+1}^{\mathsf{C}_\infty-9} \lambda_{q+1}^{N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) ,
\end{aligned}$$

and so (10.9a) and (10.9b) follow for this term from (4.10f) and (4.24a).

For the nonlocal term, we first note that the left-hand side of the equality in (i) has zero mean, and so we may ignore the means of individual terms that get plugged into the inverse divergence since their sum will vanish. Then from (A.56), (A.57), Remark A.3.4, and Lemma 8.4.4, we have that for $N, M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left\| D^N D^M_{t,q} \sum_{i,j,k,\xi,\vec{l}} \mathcal{R}^* T_{i,j,k,\xi,\vec{l},R} \right\|_\infty \leq \lambda_{q+\bar{n}}^{-5} \delta_{q+3\bar{n}}^{3/2} \mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\text{ind,t}}} \lambda_{q+1}^N \tau_q^{-M} ,$$

matching the desired estimate in (10.11).

Finally, we must estimate the terms which include $B_{(\xi),\varphi}$ from (10.18a). However, we note that from Lemma 9.3.1 $a^2_{(\xi),\varphi}$, differs in size relative to $a^2_{(\xi),R}$ by a factor of $r_q^{-2/3}$, which is exactly balanced out by the factor of $r_q^{2/3}$ in (10.18a); the other differences in size actually make the estimates for $a^2_{(\xi),\varphi}$ stronger than for $a^2_{(\xi),R}$. We therefore may argue exactly as above (in fact the estimates are slightly better since $\overline{\boldsymbol{\rho}}^4_\xi < \overline{\boldsymbol{\rho}}^6_\xi$ and the power on Γ_q is smaller), and we omit further details.

`Case 2:` Estimates for (10.18b). As before, we fix $i, j, k, \xi, \vec{l}$. We apply Proposition A.3.3 with the low-frequency choices

$$G^\bullet = B^\bullet_{(\xi),\varphi} c_0 r_q^{\frac{2}{3}} \overline{\boldsymbol{\rho}}^4_\xi(\Phi_{(i,k)}) , \quad \mathcal{C}_{G,3/2} = \left| \operatorname{supp} \eta^2_{i,j,k,\xi,\vec{l},\varphi} \right|^{2/3} \delta_{q+\bar{n}} \Gamma_q^{2j+25} \Lambda_q , \quad \mathcal{C}_{G,\infty} = \Gamma_q^{\mathsf{C}_\infty+35} \Lambda_q , \tag{10.22a}$$

$$\lambda = \lambda_{q+1} \Gamma_q^{-1} , \quad \nu = \tau_q^{-1} \Gamma_q^{i+13} , \quad \Phi = \Phi_{(i,k)} , \tag{10.22b}$$

as well as the choices from (10.20). The estimates in (A.40) and the assumption in (A.39) hold due to Proposition 7.2.1 and the estimates for $B_{(\xi),\varphi} r_q^{2/3}$ from `Case 1`. (A.41), (A.42a), and (A.42b) are satisfied as in the previous substep.

To check the high-frequency assumptions, we set

$$\varrho = \mathbb{P}_{\neq 0}\left(\sum_I (\boldsymbol{\zeta}_\xi^I)^4\right), \quad \mathsf{d} \text{ as in (xviii)}, \quad \vartheta = \delta_{i_1 i_2}\delta_{i_3 i_4}\dots\delta_{i_{\mathsf{d}-1} i_{\mathsf{d}}}\Delta^{-\mathsf{d}/2}\varrho\,, \tag{10.23a}$$

$$\mu = \Upsilon = \Upsilon' = \Lambda = \lambda_{q+\bar{n}/2}\,, \quad \mathcal{C}_{*,3/2} = \mathcal{C}_{*,\infty} = \lambda_{q+\bar{n}/2}^{\alpha}\,, \tag{10.23b}$$

where α is chosen as in (4.14). Then from Definition 7.2.4, standard Littlewood-Paley theory, and the same inequalities involving N_{dec} as in `Case 1`, we have that (A.43) is satisfied, as well as the other high-frequency assumptions in (i)–(iv). The nonlocal assumptions are identical to those of `Case 1`, and are satisfied trivially.

We therefore may appeal to the local conclusions (i)–(vi) of Proposition A.3.3 and (A.56)–(A.57), from which we have the following. First, we note that from (iii), we have that (10.12a) is satisfied. Next, abbreviating $G\varrho\circ\Phi$ as $T_{i,j,k,\xi,\vec{l},\varphi}$, we have from (A.46) and (A.50) that for $N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$,

$$\begin{aligned}
\left\|D^N D_{t,q}^M \mathcal{H}T_{i,j,k,\xi,\vec{l},\varphi}\right\|_{3/2} &\lesssim \left|\text{supp}\,(\eta^2_{i,j,k,\xi,\vec{l},\varphi})\right|^{2/3} \delta_{q+\bar{n}}\Gamma_q^{2j-2}\Lambda_q\Gamma_q^{50} \\
&\qquad \times \lambda_{q+\bar{n}/2}^{-1}\lambda_{q+\bar{n}/2}^{N+\alpha}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
\left\|D^N D_{t,q}^M \mathcal{H}T_{i,j,k,\xi,\vec{l},\varphi}\right\|_{\infty} &\lesssim \Gamma_q^{\mathsf{C}_\infty+60}\Lambda_q\lambda_{q+\bar{n}/2}^{-1}\lambda_{q+\bar{n}/2}^{N+\alpha}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\,, \\
&\lesssim \Gamma_{q+\bar{n}/2}^{\mathsf{C}_\infty-9}\lambda_{q+\bar{n}/2}^{N}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\,,
\end{aligned}$$

where we have used (4.10k) to achieve the last inequality. Notice that from (ii), the support of $\text{div}\mathcal{H}T_{i,j,k,\xi,\vec{l},\varphi}$ is contained in the support of $T_{i,j,k,\xi,\vec{l},\varphi}$, which itself is contained in the support of $\eta_{i,j,k,\xi,\vec{l},\varphi}$. From this observation, we have that (10.14a) is satisfied. Finally, we have that (10.15b) is satisfied from (A.49a) after arguing in a manner similar to that in `Case 1`.

Now we may apply the aggregation Corollaries 8.6.1 and 8.6.3 as in `Case 1` to estimate

$$S_O^{q+\bar{n}/2,l} := \sum_{i,j,k,\xi,\vec{l}} \mathcal{H}T_{i,j,k,\xi,\vec{l},\varphi}\,.$$

We find that for N, M in the same range as above,

$$\left\|\psi_{i,q} D^N D^M_{t,q} S_O^{q+\bar{n}/2,l}\right\|_{3/2} \lesssim \delta_{q+\bar{n}} \Lambda_q \Gamma_q^{60} \lambda_{q+\bar{n}/2}^{-1} \lambda_{q+\bar{n}/2}^{N+\alpha} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right)$$
$$\left\|\psi_{i,q} D^N D^M_{t,q} S_O^{q+\bar{n}/2,l}\right\|_{\infty} \lesssim \Gamma_{q+1}^{\mathsf{C}_\infty - 9} \lambda_{q+\bar{n}/2}^{N} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) ,$$

and so (10.9a) and (10.9b) follow for this term from (4.10f) and (4.24a). Finally, we must verify (10.10) for $S_O^{q+\bar{n}/2,l}$. This however follows from (iii), which asserts that the support of $S_O^{q+\bar{n}/2,l}$ is contained in the support of $\cup_{(\xi)} a_{(\xi),\varphi} \boldsymbol{\rho}^{\varphi}_{(\xi)} \circ \Phi_{(i,k)}$, and (i) of Lemma 9.2.2. Finally, the nonlocal conclusions for $S_O^{q+\bar{n}/2,l}$ follow in much the same way as in `Case 1`, and we omit further details.

`Case 3:` Estimates for (10.18c), (10.18d), and (10.18e) and $\diamond = R$. Fix $i, j, k, \xi, \vec{l}, I$ and set

$$G^\bullet = B^\bullet_{\xi,i,j,k,\vec{l},R}\left((\boldsymbol{\rho}^R_{(\xi)})^2 (\boldsymbol{\zeta}^{I,R}_\xi)^2\right) \circ \Phi_{(i,k)} , \quad \Phi = \Phi_{(i,k)} , \quad \nu = \tau_q^{-1}\Gamma_q^{i+13} ,$$
$$\mathcal{C}_{G,3/2} = \left|\mathrm{supp}\,(\eta^2_{i,j,k,\xi,\vec{l},R}(\boldsymbol{\zeta}^{I,R}_\xi)^2)\right|^{2/3} \delta_{q+\bar{n}} \Gamma_q^{2j+38} \Lambda_q + \lambda_{q+\bar{n}}^{-10} , \quad \mathcal{C}_{G,\infty} = \Gamma_q^{\mathsf{C}_\infty + 40} \Lambda_q , \quad \lambda = \lambda_{q+\bar{n}/2} , \tag{10.24}$$

as well as the choices from (10.20). We then have that (A.39) is satisfied as in the last step. Next, we have that (A.40) is satisfied by combining the corresponding bounds for $G^\bullet$ from the last step with the bounds for $\boldsymbol{\zeta}^{I,R}_\xi$ from Definition 7.2.4.[1] The bounds in (A.41)–(A.42b) hold as before without any modifications. Finally, we have that the nonlocal assumptions in (A.52)–(A.55) are satisfied for the same reasons as the previous cases. At this point, we split the argument into subcases based on the differing synthetic Littlewood-Paley projectors in (10.18d)–(10.18f).

`Case 3a:` Estimates for (10.18c) and $\diamond = R$. In order to set up the high-frequency

[1] We have added the extra $\lambda_{q+\bar{n}}^{-10}$ in the $\mathcal{C}_{G,3/2}$ bound in order to facilitate the creation of a pressure increment later.

assumptions for this case, we set

$$\mu = \lambda_{q+\bar{n}/2}\Gamma_q = \lambda_{q+\bar{n}} r_q\,, \quad \varrho = \widetilde{\mathbb{P}}^{\xi}_{q+\bar{n}/2+1}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),R})^2\,, \quad \vartheta \text{ as in Lemma 7.3.3}\,, \quad \mathsf{d} \text{ as in item (xvii)}$$
$$\mathcal{C}_{*,3/2} = \lambda^{\alpha}_{q+\bar{n}/2+1}\,, \quad \mathcal{C}_{*,\infty} = \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}} r_q}\right)^2 \lambda^{\alpha}_{q+\bar{n}/2+1}\,, \quad \Upsilon = \Upsilon' = \mu, \quad \Lambda = \lambda_{q+\bar{n}/2+1}\,,$$

where α is chosen as in (4.14). We then have that (A.43) is satisfied by appealing to estimate (7.37a) from Lemma 7.3.3 with $q = 1$ and $p = 3/2$, where we note that the assumption in (7.35) is satisfied with $\mathcal{C}_{\rho,q} = 1$ and $\lambda = \lambda_{q+\bar{n}}$ from Proposition 7.1.5. We have in addition that (A.44) and (A.45) are satisfied by definition and by appealing to the same parameter inequalities as the previous steps. Finally, we have that the nonlocal assumption in (A.55) is satisfied from (4.23b).

We therefore may appeal to the local conclusions (i)–(vi) of Proposition A.3.3 and (A.56)–(A.57), from which we have the following. First, we note that from item (iv), (10.12b) is satisfied. Next, abbreviating $G\varrho \circ \Phi$ as $T_{i,j,k,\xi,\vec{l},I,R}$, we have from (A.46) and (A.50) that for $N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$,

$$\begin{aligned}
\left\| D^N D^M_{t,q} \mathcal{H} T_{i,j,k,\xi,\vec{l},I,R} \right\|_{3/2} &\lesssim \left(\left| \text{supp} \left(\eta^2_{i,j,k,\xi,\vec{l},R} (\boldsymbol{\zeta}^{I,R}_{\xi})^2 \right) \right|^{2/3} \delta_{q+\bar{n}} \Gamma_q^{2j+39} \Lambda_q + \lambda^{-10}_{q+\bar{n}} \right) \\
&\quad \times \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \lambda^{-1}_{q+\bar{n}/2} \lambda^{N+\alpha}_{q+\bar{n}/2+1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
\left\| D^N D^M_{t,q} \mathcal{H} T_{i,j,k,\xi,\vec{l},I,R} \right\|_{\infty} &\lesssim \Gamma_q^{\mathsf{C}_\infty + 40} \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}} \right)^2 \Lambda_q \lambda^{-1}_{q+\bar{n}/2} \lambda^{N+\alpha}_{q+\bar{n}/2+1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\,, \\
&\leq \Gamma^{\mathsf{C}_\infty - 9}_{q+\bar{n}/2} \lambda^N_{q+\bar{n}/2+1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right)\,.
\end{aligned}$$

We have used (4.13a) to simplify the second inequality. Notice that from (ii), the support of $\text{div}\mathcal{H}T_{i,j,k,\xi,\vec{l},I,R}$ is contained in the support of $T_{i,j,k,\xi,\vec{l},I,R}$, which itself is contained in the support of $\eta_{i,j,k,\xi,\vec{l},R}\boldsymbol{\zeta}^{I,R}_{\xi}$. From this observation, we have that (10.14b) is satisfied. Finally, we have that (10.15c) is satisfied from (A.49a) and Lemma 7.3.3 applied with $q = p = 3/2, \infty$.

Now we may again apply the aggregation Corollaries 8.6.1 and 8.6.3 to estimate

$$S_{O,R}^{q+\bar{n}/2+1,l} := \sum_{i,j,k,\xi,\vec{l},I} \mathcal{H}T_{i,j,k,\xi,\vec{l},I,R}\,.$$

From (8.48b) and (8.53b), we then have that for N, M in the same range as above,

$$\begin{aligned}
\left\|\psi_{i,q} D^N D_{t,q}^M S_{O,R}^{q+\bar{n}/2+1,l}\right\|_{3/2} &\lesssim \delta_{q+\bar{n}} \Lambda_q \Gamma_q^{50} \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}}\right)^{2/3} (\lambda_{q+\bar{n}} r_q)^{-1} \\
&\qquad \times \lambda_{q+\bar{n}/2+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,, \\
&\leq \Gamma_{q+\bar{n}/2+1}^{-10} \delta_{q+\bar{n}/2+1+\bar{n}} \lambda_{q+\bar{n}/2+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,, \\
\left\|\psi_{i,q} D^N D_{t,q}^M S_{O,R}^{q+\bar{n}/2+1,l}\right\|_{\infty} &\lesssim \Gamma_{q+\bar{n}/2+1}^{\mathsf{C}_\infty - 9} \lambda_{q+\bar{n}/2+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right),
\end{aligned}$$

where we have used (4.27d) to simplify the first inequality. Finally, the nonlocal conclusions follow in much the same way as in the previous cases, and so we omit further details.

`Case 3b:` Estimates for (10.18d) and (10.18e) and $\diamond = R$. In order to set up the high-frequency assumptions for this case, we consider for the moment the cases when $m > q + \bar{n}/2 + 2$ and set

$$\begin{aligned}
&\mu = \lambda_{q+\bar{n}/2}\Gamma_q = \lambda_{q+\bar{n}} r_q\,, \quad \varrho = \widetilde{\mathbb{P}}_{(m-1,m]}^{\xi}(\varrho_{(\xi),R}^I)^2\,, \quad \vartheta \text{ as in Lemma 7.3.4}\,, \quad \mathsf{d} \text{ as in item (xvii)} \\
&\mathcal{C}_{*,3/2} = \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q}\right)^{2/3}, \quad \mathcal{C}_{*,\infty} = \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q}\right)^2 \lambda_{q+\bar{n}/2+1}^{\alpha}\,, \\
&\Upsilon = \lambda_{m-1}\,, \quad \Upsilon' = \lambda_m\,, \quad \Lambda = \min(\lambda_m, \lambda_{q+\bar{n}})\,.
\end{aligned} \tag{10.25}$$

We then have that (A.43) is satisfied by appealing to (7.40b) with $q = 1$ and $p = 3/2, \infty$; we note that (7.39) is satisfied for $q = 1$ and $\mathcal{C}_{\rho,q} = 1$ and $\lambda = \lambda_{q+\bar{n}}$ as in the last step. Next, we have that (A.44)–(A.45) are satisfied by definition and immediate computation and the same inequalities as in the previous steps. Finally, we have that the nonlocal assumption in (A.55) is satisfied from (4.23b).

In the case of $m = q + \bar{n}/2 + 2$, we have to take an extra step to minimize the gap between

Υ and Υ' in order to ensure that the second inequality in (A.44) is satisfied. Towards this end, we decompose the synthetic Littlewood-Paley operator further as

$$\widetilde{\mathbb{P}}^{\xi}_{(q+\bar{n}/2+1,q+\bar{n}/2+2]} := \widetilde{\mathbb{P}}^{\xi}_{(q+\bar{n}/2+1,q+\bar{n}/2+3/2]} + \widetilde{\mathbb{P}}^{\xi}_{(q+\bar{n}/2+3/2,q+\bar{n}/2+2]}\,, \tag{10.26}$$

where the $q + \bar{n}/2 + 3/2$ portion of the projector correponds to the frequency which is the geometric means of $\lambda_{q+\bar{n}/2+1}$ and $\lambda_{q+\bar{n}/2+2}$. This extra division helps us minimize the gap between Υ and Υ'. Then we can set

$$\begin{aligned}
&\mu = \lambda_{q+\bar{n}/2}\Gamma_q = \lambda_{q+\bar{n}} r_q\,, \quad \varrho = \widetilde{\mathbb{P}}^{\xi}_{\bullet}(\varrho^{I}_{(\xi),R})^2\,, \quad \vartheta \text{ as in Lemma 7.3.4}\,, \quad \mathsf{d} \text{ as in item (xvii)}\\
&\mathcal{C}_{*,3/2} = \left(\frac{\lambda_{q+\bar{n}/2+2}}{\lambda_{q+\bar{n}} r_q}\right)^{2/3}, \quad \mathcal{C}_{*,\infty} = \left(\frac{\lambda_{q+\bar{n}/2+2}}{\lambda_{q+\bar{n}} r_q}\right)^{2} \lambda^{\alpha}_{q+\bar{n}/2+1}\,,\\
&\Upsilon = \lambda_{q+\bar{n}/2+1}\,, \quad \Upsilon' = \lambda_{q+\bar{n}/2+3/2} \text{ if } \bullet \text{ corresponds to the first projector}\,,\\
&\Upsilon = \lambda_{q+\bar{n}/2+3/2}\,, \quad \Upsilon' = \lambda_{q+\bar{n}/2+2} \text{ if } \bullet \text{ corresponds to the second projector}\,.
\end{aligned}$$

We then have that (A.43) is satisfied by appealing to (7.40b) with $q = 1$ and $p = 3/2, \infty$ as before. Next, we have that (A.44)–(A.45) are satisfied by definition and immediate computation (here we crucially use the extra subdivision to ensure that the second inequality in (A.44) holds) and the same inequalities as in the previous steps. Finally, we again have that the nonlocal assumption in (A.55) is satisfied from (4.23b).

We therefore may appeal to the local conclusions (i)–(vi) of Proposition A.3.3 and (A.56)–(A.57), from which we have the following. First, we note that from item (iv), (10.12b) is satisfied. Next, abbreviating $G\varrho \circ \Phi$ as $T_{i,j,k,\xi,\vec{l},I,R}$, we have from (A.46) and (A.50) that for

$N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$,

$$\begin{aligned}
\left\| D^N D^M_{t,q} \mathcal{H} T_{i,j,k,\xi,\vec{l},I,R} \right\|_{3/2} &\lesssim \left(\left| \text{supp}\left(\eta^2_{i,j,k,\xi,\vec{l},R}\right) \right|^{2/3} \delta_{q+\bar{n}} \Gamma_q^{2j+39} \Lambda_q + \lambda_{q+\bar{n}}^{-10} \right) \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q} \right)^{2/3} \lambda_{m-1}^{-2} \\
&\qquad \times \left(\min(\lambda_m, \lambda_{q+\bar{n}})\right)^{N+1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \\
\left\| D^N D^M_{t,q} \mathcal{H} T_{i,j,k,\xi,\vec{l},I,R} \right\|_{\infty} &\lesssim \Gamma_q^{\mathsf{C}_\infty + 40} \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q} \right)^2 \Lambda_q \lambda_{m-1}^{-2} \\
&\qquad \times \left(\min(\lambda_m, \lambda_{q+\bar{n}})\right)^{N+1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \\
&\lesssim \Gamma_{q+\bar{n}/2}^{\mathsf{C}_\infty - 9} (\min(\lambda_m, \lambda_{q+\bar{n}}))^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) ,
\end{aligned}$$

where we have used (4.13a) to achieve the last inequality. Notice that from (ii), the support of $\text{div}\mathcal{H}T_{i,j,k,\xi,\vec{l},I,R}$ is contained in the support of $T_{i,j,k,\xi,\vec{l},I,R}$, which itself is contained in the support of $\eta_{i,j,k,\xi,\vec{l},R}\boldsymbol{\zeta}_\xi^{I,R}$. From this observation, we have that (10.14b) is satisfied. Furthermore, we have that (10.15d) is satisfied from (A.49a) and Lemma 7.3.3 applied with $q = p = 3/2, \infty$. Finally, we have that (10.16) is satisfied due to item (ii) and (7.40c). We note also that (10.10) follows from (10.16) and (9.24).

Now we may again apply the aggregation Corollaries 8.6.1 and 8.6.3 to estimate

$$S^{m,l}_{O,R} := \sum_{i,j,k,\xi,\vec{l},I} \mathcal{H}T_{i,j,k,\xi,\vec{l},I,R} \, .$$

From (8.48b) and (8.53b), we then have that for N, M in the same range as above,

$$\begin{aligned}
\left\| \psi_{i,q} D^N D^M_{t,q} S^{m,l}_{O,R} \right\|_{3/2} &\lesssim \delta_{q+\bar{n}} \Lambda_q \Gamma_q^{50} \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q} \right)^{2/3} \lambda_{m-1}^{-2} \\
&\qquad \times \min(\lambda_m, \lambda_{q+\bar{n}})^{N+1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\lesssim \Gamma_m^{-10} \delta_{m+\bar{n}} (\min(\lambda_m, \lambda_{q+\bar{n}}))^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \\
\left\| \psi_{i,q} D^N D^M_{t,q} S^{m,l}_{O,R} \right\|_{\infty} &\lesssim \Gamma_m^{\mathsf{C}_\infty - 9} (\min(\lambda_m, \lambda_{q+\bar{n}}))^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)
\end{aligned}$$

where we have used (4.27d) to simplify the first inequality. Finally, the nonlocal conclusions follow in much the same way as in the previous cases, and so we omit further details.

`Case 4:` Estimates for (10.18c), (10.18d), and (10.18e) and $\diamond = \varphi$. Estimates for these follow from similar arguments as in the cases when $\diamond = R$. Indeed, the only significant differences are that the estimates for $a^2_{(\xi),\varphi}$ than those of $a^2_{(\xi),R}$ are *worse* by a factor of $r_q^{-2/3}$ from Lemma 9.3.1, while the estimates for ϱ encoded in the constants $\mathcal{C}_{*,3/2}$ and $\mathcal{C}_{*,\infty}$ are *better* by a factor of $r_q^{2/3}$ from Proposition 7.1.6. Therefore, to compensate such loss or gain, we define $G^\bullet = B^\bullet_{\xi,i,j,k,\vec{l},\varphi}\left((\boldsymbol{\rho}^\varphi_{(\xi)})^2(\boldsymbol{\zeta}^{I,\varphi}_\xi)^2\right) \circ \Phi_{(i,k)} r_q^{2/3}$ with the extra factor $r_q^{2/3}$ and define ϱ analogous to the case $\diamond = R$ but with the extra factor $r_q^{-2/3}$. Then, the same choice of parameters and functions as in the case of $\diamond = R$ will lead to the desired estimates. We omit further details.

`Case 5:` Estimates for (10.18f). Here we apply Proposition A.3.3 with $p = \infty$ and the following choices. The low-frequency assumptions in Part 1 are exactly the same as the L^∞ low-frequency assumptions in `Case 3` and `Case 4`. For the high-frequency assumptions, we recall the choice of N_{**} from (xvii) and set

$$\varrho_R = (\mathrm{Id} - \widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1})\mathbb{P}_{\neq 0}\left(\varrho^I_{(\xi),R}\right)^2, \quad \varrho_\varphi = (\mathrm{Id} - \widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1})\mathbb{P}_{\neq 0}\left(\varrho^I_{(\xi),\varphi}\right)^2 r_q^{-2/3}, \quad \vartheta^{i_1 i_2 \dots i_{\mathsf{d}-1} i_{\mathsf{d}}}_\diamond = \delta^{i_1 i_2 \dots i_{\mathsf{d}-1} i_{\mathsf{d}}} \Delta^{-\mathsf{d}/2} \varrho_\diamond,$$
$$\Lambda = \lambda_{q+\bar{n}}\,, \quad \mu = \Upsilon = \Upsilon' = \lambda_{q+\bar{n}/2}\Gamma_q\,, \,, \quad \mathcal{C}_{*,\infty} = \left(\frac{\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}+1}}\right)^{N_{**}} \lambda^3_{q+\bar{n}}\,, \quad \mathsf{N}_{\mathrm{dec}} \text{ as in (xv)}\,, \quad \mathsf{d} = 0\,.$$

Then we have that item (i) is satisfied by definition, item (ii) is satisfied as in the previous steps, (A.43) is satisfied using Propositions 7.1.5 and 7.1.6 and (7.37b) from Lemma 7.3.3, (A.44) is satisfied by definition and as in the previous steps, and (A.45) is satisfied by (4.21). For the nonlocal assumptions, we choose $M_\circ, N_\circ = 2\mathsf{N}_{\mathrm{ind}}$ so that (A.52)–(A.54) are satisfied as in `Case 1`, and (A.55) is satisfied from (4.23c). We have thus satisfied all the requisite assumptions, and we therefore obtain nonlocal bounds very similar to those from the previous steps, which are consistent with (10.11) at level $q + \bar{n}$. We omit further details. □

Lemma 10.2.3 (Low shells have no pressure increment). *The errors S_O^{q+1} and $S_O^{q+\bar{n}/2}$ require no pressure increment as they are already dominated by anticipated pressure. More*

precisely, we have that for $N, M \leq \mathsf{N}_{\text{fin}}/10$,

$$\left|\psi_{i,q} D^N D^M_{t,q} S_O^{q+1,l}\right| \leq \Gamma_{q+1}^{-100} \pi_q^{q+1} \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \tag{10.27a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} S_O^{q+\bar{n}/2,l}\right| \leq \Gamma_{q+\bar{n}/2}^{-100} \pi_q^{q+\bar{n}/2} \lambda_{q+\bar{n}/2}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) . \tag{10.27b}$$

Proof. We first note that the application of Proposition A.3.3 in `Case 1` of the proof of Lemma 10.2.1 can be supplemented with Remark A.3.9. Specifically, we may set

$$\overline{\pi} = \pi_\ell \Gamma_q^{40} \Lambda_q \, , \tag{10.28}$$

so that (A.59) follows from the definition of $B_{(\xi),R}$ in (10.7) and (9.38a). Then from (A.47), (A.49a), and (A.60), we have that

$$\left|D^N D^M_{t,q} \mathcal{H}T_{i,j,k,\xi,\vec{l},R}\right| \lesssim \pi_\ell \Gamma_q^{50} \Lambda_q \lambda_{q+1}^{-1} \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) .$$

We pause also to note that (10.13) in this case follows from (A.47) and (A.60). Now applying the aggregation Corollary 8.6.3 with $H = \mathcal{H}T_{i,j,k,\xi,\vec{l},R}$, $\varpi = \pi_\ell \Gamma_q^{50} \Lambda_q$, and $p = 1$ along with (5.20), (6.6), and (4.10f) gives (10.27a).

The proof of (10.27b) follows similarly from supplementing `Case 2` of the proof of Lemma 10.2.1 with pointwise assumptions. We omit further details. □

Lemma 10.2.4 (**Pressure increment**). *For every* $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$, *there exists a function* $\sigma_{S_O^m} = \sigma^+_{S_O^m} - \sigma^-_{S_O^m}$ *such that the following hold.*

(i) We have that

$$\left|\psi_{i,q} D^N D^M_{t,q} S^{m,l}_O\right| < \left(\sigma^+_{S^m_O} + \delta_{q+3\bar{n}}\right)(\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.29a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S^m_O}\right| < \left(\sigma^+_{S^m_O} + \delta_{q+3\bar{n}}\right)(\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.29b}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S^m_O}\right\|_{3/2} \leq \Gamma_m^{-9}\delta_{m+\bar{n}}(\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.29c}$$

$$\left\|D^N D^M_t \sigma^+_{S^m_O}\right\|_{\infty} \leq \Gamma_{q+1}^{\mathsf{C}_\infty - 9}(\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.29d}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^-_{S^m_O}\right| \lesssim \Gamma_{q+\bar{n}/2}^{-100}\pi_q^{q+\bar{n}/2}\left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.29e}$$

for all $N, M < \mathsf{N}_{\text{fin}}/100$.

(ii) For $m \geq q + \bar{n}/2 + 2$, *we have that*

$$\begin{aligned} &B\left(\operatorname{supp}\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp}\left(\sigma^+_{S^m_O}\right) = \emptyset \qquad \forall q+1 \leq q' \leq m-1 \\ &B\left(\operatorname{supp}\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp}\left(\sigma^-_{S^m_O}\right) = \emptyset \qquad \forall q+1 \leq q' \leq q + \bar{n}/2\,. \end{aligned} \tag{10.30}$$

(iii) Define

$$\mathfrak{m}_{\sigma_{S^m_O}}(t) = \int_0^t \left\langle D_{t,q}\sigma_{S^m_O}\right\rangle(s)\, ds\,. \tag{10.31}$$

Then we have that

$$\left|\frac{d^{M+1}}{dt^{M+1}}\mathfrak{m}_{\sigma_{S^m_O}}\right| \leq (\max(1,T))^{-1}\delta^2_{q+3\bar{n}}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right) \tag{10.32}$$

for $0 \leq M \leq 2\mathsf{N}_{\text{ind}}$.

Proof of Lemma 10.2.4. We follow the case numbering from Lemma 10.2.1. Since we have shown in Lemma 10.2.3 that the low shells have no pressure increment, we only need to analyze `Cases 3` and `4`. Since the only difference between `Case 3` and `Case 4` is the rebalancing of $r_q^{2/3}$, we shall only hint at the proofs in `Case 4` and focus on the case $\diamond = R$. We divide

into subcases `3a` and `3b` and apply Proposition A.4.4.

`Case 3a:` pressure increment for (10.18c) and $\diamond = R$. Recall that Part 1 of Proposition A.4.4 requires preliminary assumptions which are the same as those from the inverse divergence, along with pointwise bounds corresponding to Remark A.3.9. Since we have already chosen parameters corresponding to the inverse divergence, we simply set $\overline{\pi} = \pi_\ell \Gamma_q^{50} \Lambda_q$, which verifies (10.13) in this case. Then the assumption in (A.59) follows from the pointwise estimates for $B_{(\xi),R}$ used in Lemma 10.2.3 along with Proposition 7.2.1, Lemma 8.4.3, and Corollary 8.2.4 to estimate $\left((\boldsymbol{\rho}^R_{(\xi)})^2 (\boldsymbol{\zeta}^{I,R}_\xi)^2 \right) \circ \Phi_{(i,k)}$.

In order to check the additional assumptions from Part 2, we set

$$\begin{aligned} &N_{**} \text{ as in (xvii)}\,, \quad \mathsf{N}_{\text{cut,x}}, \mathsf{N}_{\text{cut,t}} \text{ as in (xi)}\,, \quad \Gamma = \Gamma_q^{1/2}\,, \quad \delta_{\text{tiny}} = \delta^2_{q+3\bar{n}}\,, \\ &\bar{m} = 1\,, \quad \mu_0 = \lambda_{q+\bar{n}/2+1}\Gamma_q^{-1}\,, \quad \mu_{\bar{m}} = \mu_1 = \lambda_{q+\bar{n}/2+1}\Gamma_q^2\,. \end{aligned} \tag{10.33}$$

Then (A.179a)–(A.179b) hold from (4.24a), (A.179c) holds from (4.23a), (A.180a) holds from (4.17a), (A.180b) holds from (4.17b), (A.180c) holds from (4.24a), (A.180d) holds from (4.21), (A.181a) holds by definition, (A.181b) holds by definition and immediate computation, (A.181c) holds due to (4.23b), and (A.181d) holds due to (4.23c).

At this point, we appeal to the conclusions from Part 3 to construct a pressure increment and delineate its properties. First, from (A.182)–(A.183) and (4.24a), we have that there exists a pressure increment $\sigma_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} = \sigma^+_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} - \sigma^-_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}$ such that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left| D^N D^M_{t,q} \mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R} \right| \lesssim \left(\sigma^+_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} + \delta^2_{q+3\bar{n}} \right) (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,. \tag{10.34}$$

From (A.48) and (A.187), we have that

$$\operatorname{supp}\left(\sigma^+_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} \right) \subseteq \operatorname{supp}\left(\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R} \right) \subseteq \operatorname{supp}\left(a_{(\xi),R} \left(\boldsymbol{\rho}^R_{(\xi)} \boldsymbol{\zeta}^I_\xi \right) \circ \Phi_{(i,k)} \right)\,. \tag{10.35}$$

Now define

$$\sigma^{\pm}_{S^{q+\bar{n}/2+1}_{O,R}} = \sum_{i,j,k,\xi,\vec{l},I} \sigma^{\pm}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} \,. \tag{10.36}$$

Then (9.22) gives that (10.30) is satisfied for $m = q + \bar{n}/2 + 1$. From (10.34), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R} \,, \qquad \varpi = \left[\sigma^{+}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} + \delta^2_{q+3\bar{n}}\right] \mathbf{1}_{\operatorname{supp} a_{(\xi),R}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_\xi} \,, \qquad p = 1 \,,$$

we have that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \sum_{i',j,k,\xi,\vec{l},I} \mathcal{H}T^{q+\bar{n}/2+1}_{i',j,k,\xi,\vec{l},I,R}\right| \lesssim \left(\sigma^{+}_{S^{q+\bar{n}/2+1}_{O,R}} + \delta^2_{q+3\bar{n}}\right) \times (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \,. \tag{10.37}$$

We therefore have that (10.29a) is satisfied for $m = q + \bar{n}/2 + 1$. From (A.184), (4.24a), and (4.18), we have that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left|D^N D^M_{t,q} \sigma^{+}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}\right| \lesssim \left(\sigma^{+}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} + \delta^2_{q+3\bar{n}}\right) (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \,. \tag{10.38}$$

From (10.38), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \sigma^{+}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}} \,, \qquad \varpi = \left[H + \delta^2_{q+3\bar{n}}\right] \mathbf{1}_{\operatorname{supp} a_{(\xi),R}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_\xi} \,, \qquad p = 1 \,,$$

we have that (10.29b) is satisfied for $m = q + \bar{n}/2 + 1$.

Next, from (A.185), we have that

$$\left\|\sigma^{\pm}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}\right\|_{3/2} \lesssim \left(\left|\operatorname{supp}\left(\eta^2_{i,j,k,\xi,\vec{l},R}(\boldsymbol{\zeta}^{I,R}_{\xi})^2\right)\right|^{2/3}\delta_{q+\bar{n}}\Gamma_q^{2j+38}\Lambda_q + \lambda_{q+\bar{n}}^{-10}\right)\left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}}r_q}\right)^{2/3}\lambda^{\alpha}_{q+\bar{n}/2+1}\lambda^{-1}_{q+\bar{n}/2}\,.$$

Now from (10.36), (4.27d), and Corollary 8.6.1 with $\theta = 2$, $\theta_1 = 0$, $\theta_2 = 2$, $H = \sigma^{\pm}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}$, and $p = 3/2$, we have that

$$\left\|\psi_{i,q}\sigma^{\pm}_{S^{q+\bar{n}/2+1}_{O,R}}\right\|_{3/2} \lesssim \delta_{q+\bar{n}+\bar{n}/2+1}\Gamma^{-10}_{q+\bar{n}/2+1}\,.$$

Combined with (10.29b), this verifies (10.29c) at level $q + \bar{n}/2 + 1$. Arguing now for $p = \infty$ from (A.185), we have that

$$\left\|\sigma^{\pm}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}\right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+40}\Lambda_q\left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}}r_q}\right)^2\lambda^{\alpha}_{q+\bar{n}/2+1}\lambda^{-1}_{q+\bar{n}/2}\,.$$

Now from (10.36), (4.13a), and Corollary 8.6.3 with $H = \sigma^{\pm}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}$, $\varpi = \mathbf{1}_{\operatorname{supp} a_{(\xi),R}}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_{\xi}$ and $p = 1$, we have that

$$\left\|\psi_{i,q}\sigma^{\pm}_{S^{q+\bar{n}/2+1}_{O,R}}\right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+40}\Lambda_q\left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}}r_q}\right)^2\lambda^{\alpha}_{q+\bar{n}/2+1}\lambda^{-1}_{q+\bar{n}/2} \le \Gamma^{\mathsf{C}_\infty-100}_{q+\bar{n}/2+1}\,.$$

Combined again with (10.29b), this verifies (10.29d) at level $q + \bar{n}/2 + 1$.

Finally, from (A.186), (4.18), (4.24a), (4.27e), (6.6), and (5.20), we have that for $N, M \le \mathsf{N}_{\text{fin}}/7$,

$$\begin{aligned}\left|D^N D^M_{t,q}\sigma^{-}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}\right| &\lesssim \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}}r_q}\right)^{2/3}\lambda^{\alpha}_{q+\bar{n}/2+1}\lambda^{-1}_{q+\bar{n}/2}\pi_\ell\Gamma_q^{50}\Lambda_q \\ &\qquad\times\left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^N\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \\ &\le \Gamma_q^{-100}\pi_q^{q+\bar{n}/2}\left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^N\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,.\end{aligned}$$

Applying (10.36), Corollary 8.6.3 with $H = \sigma^{-}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}$, $\varpi = \Gamma_q^{-100}\pi_q^{q+\bar{n}/2}\mathbf{1}_{\operatorname{supp} a_{(\xi),R}}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_{\xi}$ and $p = 1$, and (6.6), we have that (10.29e) is verified at level $m = q + \bar{n}/2 + 1$. The estimate

for $\mathfrak{m}_{\sigma_{S_O^m}}$ in item (iii) in these cases follows from (A.193), (xvi), and a large choice of a_* in item (xix) to ensure that we can gain the advantageous prefactor of $\max(1,T)^{-1}$.

`Case 3b:` pressure increment for (10.18d) and (10.18e) and $\diamond = R$. We set $\overline{\pi} = \pi_\ell \Gamma_q^{50} \Lambda_q$ as in the previous case since the low-frequency portion of the error term is identical. Since all the preliminary assumptions in Part 1 are now satisfied, we need to check the additional assumptions from Part 2. In order to do so, we set

$$
\begin{aligned}
&N_{**} \text{ as in (xvii)}\,, \quad \mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}} \text{ as in (xi)}\,, \quad \Gamma = \Gamma_q^{1/2}\,, \quad \delta_{\mathrm{tiny}} = \delta_{q+3\bar{n}}^2\,, \quad \mu = \lambda_{q+\bar{n}/2}\Gamma_q\,,\\
&\mu_0 = \lambda_{q+\bar{n}/2+1}\,, \quad \mu_1 = \lambda_{q+\bar{n}/2+3/2}\Gamma_q^2\,,\\
&\mu_{m'} = \lambda_{q+\bar{n}/2+m'}\Gamma_q^2 \quad \text{if} \quad 2 \le m' \le \bar{n}/2\,,\\
&\bar{m} = 1 \quad \text{for the first projector in (10.26) if} \quad m = q + \bar{n}/2 + 2\,,\\
&\bar{m} = 2 \quad \text{for the second projector in (10.26) if} \quad m = q + \bar{n}/2 + 2\,,\\
&\bar{m} = m - q - \bar{n}/2 \quad \text{if} \quad m > q + \bar{n}/2 + 2\,.
\end{aligned}
\tag{10.39}
$$

Then (A.179a)–(A.180a) hold as in the previous case, (A.180b) holds from (4.17b), (A.180c)–(A.180d) hold as in the previous case, (A.181a) holds by definition, (A.181b) holds by definition and immediate computation, (A.181c) holds due to (4.23b), and (A.181d) holds due to (4.23c).

At this point, we appeal to the conclusions from Part 3 to construct a pressure increment and delineate its properties. First, from (A.182)–(A.183) and (4.24a), we have that for $q + \bar{n}/2 + 2 \le m \le q + \bar{n} + 1$, there exists a pressure increment $\sigma_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}} = \sigma^+_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}} - \sigma^-_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}$ such that for $N, M \le \mathsf{N}_{\mathrm{fin}}/7$,

$$
\left| D^N D^M_{t,q} \mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R} \right| \lesssim \left(\sigma^+_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,R}} + \delta^2_{q+3\bar{n}} \right) (\min(\lambda_m, \lambda_{q+\bar{n}})\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,.
\tag{10.40}
$$

From (A.48), (A.187), and (7.40c), we have that

$$\operatorname{supp}\left(\sigma^{+}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}\right) \subseteq \operatorname{supp}\left(\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}\right) \subseteq \operatorname{supp}\left(a_{(\xi),R}\left(\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_\xi\right)\circ\Phi_{(i,k)}\right)\cap B\left(\operatorname{supp}\varrho^I_{(\xi),R},\lambda^{-1}_{m-1}\right) . \tag{10.41}$$

Now define

$$\sigma^{\pm}_{S^m_{O,R}} = \sum_{i,j,k,\xi,\vec{l},I}\sigma^{\pm}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}} \quad \text{if} \quad m \neq q+\bar{n}\,, \tag{10.42a}$$

$$\sigma^{\pm}_{S^m_{O,R}} = \sum_{\widetilde{m}=q+\bar{n}}^{q+\bar{n}+1}\sum_{i,j,k,\xi,\vec{l},I}\sigma^{\pm}_{\mathcal{H}T^{m'}_{i,j,k,\xi,\vec{l},I,R}} \quad \text{if} \quad m = q+\bar{n}\,. \tag{10.42b}$$

Then (9.22) and (9.24) give that (10.30) is satisfied for $q+{}^{\bar{n}}/_2+2 \leq m \leq q+\bar{n}$. From (10.40), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}\,, \qquad \varpi = \left[\sigma^{+}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}} + \delta^2_{q+3\bar{n}}\right]\mathbf{1}_{\operatorname{supp} a_{(\xi),R}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_\xi}\,, \qquad p=1\,,$$

we have that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_7$,

$$\left|\psi_{i,q}D^N D^M_{t,q}\sum_{i',j,k,\xi,\vec{l},I}\mathcal{H}T^m_{i',j,k,\xi,\vec{l},I,R}\right| \lesssim \left(\sigma^{+}_{S^m_{O,R}} + \delta^2_{q+3\bar{n}}\right) \times \left(\min(\lambda_m,\lambda_{q+\bar{n}})\Gamma_q\right)^N \mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{10.43}$$

We therefore have that (10.29a) is satisfied for $q+{}^{\bar{n}}/_2+2 \leq m \leq q+\bar{n}$. From (A.184), (4.24a), and (4.18), we have that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_7$,

$$\left|D^N D^M_{t,q}\sigma^{+}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}\right| \lesssim \left(\sigma^{+}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}} + \delta^2_{q+3\bar{n}}\right)\left(\min(\lambda_m,\lambda_{q+\bar{n}})\Gamma_q\right)^N \mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{10.44}$$

From (10.44), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \sigma^{+}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}\,, \qquad \varpi = \left[H + \delta^2_{q+3\bar{n}}\right] \mathbf{1}_{\operatorname{supp} a_{(\xi),R}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_\xi}\,, \qquad p = 1\,,$$

we have that (10.29b) is satisfied for $q + {}^{\bar{n}}\!/_2 + 2 \leq m \leq q + \bar{n}$.

Next, from (A.185), we have that

$$\left\|\sigma^{\pm}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}\right\|_{3/2} \lesssim \left(\left|\operatorname{supp}\left(\eta^2_{i,j,k,\xi,\vec{l},R}(\boldsymbol{\zeta}^{I,R}_\xi)^2\right)\right|^{2/3} \delta_{q+\bar{n}}\Gamma_q^{2j+38}\Lambda_q + \lambda^{-10}_{q+\bar{n}}\right)\left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q}\right)^{2/3} \lambda^{-2}_{m-1}\lambda_m\,.$$

Now from (10.42), (4.27d), and Corollary 8.6.1 with $\theta = 2$, $\theta_1 = 0$, $\theta_2 = 2$, $H = \sigma^{\pm}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}$, and $p = {}^3\!/_2$, we have that

$$\left\|\psi_{i,q}\sigma^{\pm}_{S^m_{O,R}}\right\|_{3/2} \lesssim \delta_{m+\bar{n}}\Gamma_m^{-10}\,.$$

Combined with (10.29b), this verifies (10.29c) at level m. Arguing now for $p = \infty$ from (A.185), we have that

$$\left\|\sigma^{\pm}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}\right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+40}\Lambda_q\left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q}\right)^2 \lambda_m^{\alpha}\lambda^{-2}_{m-1}\lambda_m\,.$$

Now from (10.42), (4.13a), and Corollary 8.6.3 with $H = \sigma^{\pm}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}$, $\varpi = \mathbf{1}_{\operatorname{supp} a_{(\xi),R}\boldsymbol{\rho}^R_{(\xi)}\boldsymbol{\zeta}^I_\xi}$ and $p = 1$, we have that

$$\left\|\psi_{i,q}\sigma^{\pm}_{S^m_{O,R}}\right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+40}\Lambda_q\left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q}\right)^2 \lambda_m^{\alpha}\lambda^{-1}_{q+\bar{n}/2} \leq \Gamma_m^{\mathsf{C}_\infty-100}\,.$$

Combined again with (10.29b), this verifies (10.29d) at level m.

Finally, from (A.186), (4.18), (4.24a), (4.27e), (6.6), and (5.20), we have that for $N, M \leq$

$\mathsf{N}_{\text{fin}}/7$,

$$\left| D^N D^M_{t,q} \sigma^-_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}} \right| \lesssim \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}} r_q} \right)^{2/3} \lambda_{m-1}^{-2} \lambda_m \pi_\ell \Gamma_q^{50} \Lambda_q$$
$$\times \min(\lambda_m, \lambda_{q+\bar{n}}) \Gamma_q)^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9 \right)$$
$$\leq \Gamma_q^{-100} \pi_q^{q+\bar{n}/2} (\lambda_{q+\bar{n}/2} \Gamma_q)^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9 \right) .$$

Applying (10.42), Corollary 8.6.3 with $H = \sigma^-_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,R}}$, $\varpi = \Gamma_q^{-100} \pi_q^{q+\bar{n}/2} \mathbf{1}_{\operatorname{supp} a_{(\xi),R}} \boldsymbol{\rho}^R_{(\xi)} \boldsymbol{\zeta}^I_\xi$ and $p = 1$, and (6.6), we have that (10.29e) is verified at levels $q + \bar{n}/2 + 2 \leq m \leq q + \bar{n}$. The bounds in item (iii) follow much as in the previous case, and we omit further details.

`Case 4:` pressure increment for $\diamond = \varphi$. As we noted in the beginning of the proof, the only differences between $\diamond = \varphi$ and $\diamond = R$ arise from the redistribution of $r_q^{2/3}$. We may therefore define $\sigma_{S^m_{O,\varphi}}$ for $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$ and set

$$\sigma^\pm_{S^m_O} = \sigma^\pm_{S^m_{O,R}} + \sigma^\pm_{S^m_{O,\varphi}} ,$$

from which (10.29a)–(10.32) follow. □

Lemma 10.2.5 (Pressure current). *For every $m \in \{q + \bar{n}/2 + 1, \ldots, q + \bar{n}\}$, there exists a current error $\phi_{S^m_O}$ associated to the pressure increment $\sigma_{S^m_O}$ defined by Lemma 10.2.4 which satisfies the following properties.*

(i) We have the decompositions and equalities

$$\phi_{S^m_O} = \phi^*_{S^m_O} + \sum_{m'=q+\bar{n}/2+1}^{m} \phi^{m'}_{S^m_O} , \quad \phi^{m'}_{S^m_O} = \phi^{m',l}_{S^m_O} + \phi^{m',*}_{S^m_O} \tag{10.45a}$$
$$\operatorname{div} \phi_{S^m_O} = D_{t,q} \sigma_{S^m_O} - \langle D_{t,q} \sigma_{S^m_O} \rangle . \tag{10.45b}$$

(ii) For $q + {}^{\bar{n}}/_{2} + 1 \leq m' \leq m$ and $N, M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \phi^{m',l}_{S^m_O}\right| < \Gamma^{-100}_{m'} \left(\pi^{m'}_q\right)^{3/2} r^{-1}_{m'} (\lambda_{m'} \Gamma^2_{m'})^M \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+17}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{10.46a}$$

$$\left\|D^N D^M_{t,q} \phi^{m',*}_{S^m_O}\right\|_\infty + \left\|D^N D^M_{t,q} \phi^{*}_{S^m_O}\right\|_\infty < \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} (\lambda_m \Gamma^2_m)^N \tau_q^{-M} . \tag{10.46b}$$

(iii) For all $q + {}^{\bar{n}}/_{2} + 1 \leq m' \leq m$ and all $q + 1 \leq q' \leq m' - 1$,

$$B\left(\operatorname{supp} \widehat{w}_{q'}, {}^1/_2 \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap \operatorname{supp}\left(\phi^{m',l}_{S^m_O}\right) = \emptyset . \tag{10.47}$$

Proof. We utilize the case numbering from Lemma 10.2.4. Note that the only cases which require a pressure increments were `Cases 3a` and `3b`, which correspond to the analysis of (10.18c)–(10.18e) and $\diamond = R$, and `Case 4`, which corresponds to the same terms but with $\diamond = \varphi$. We combine the analysis for $\diamond = R$ and $\diamond = \varphi$ into a single argument, since as explained in the previous lemmas, the estimates are essentially the same.

`Case 3a/4a:` pressure current error from (10.18c) and $\diamond = R, \varphi$. In this case, we recall from (10.33) that we have chosen $\bar{m} = 1$ in item (iii), $\mu_0 = \lambda_{q+\bar{n}/2+1} \Gamma_q^{-1}$, and $\mu_{\bar{m}} = \mu_1 = \lambda_{q+\bar{n}/2+1} \Gamma_q^2$. We therefore have from (A.182) that

$$\sigma_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} = \sigma^+_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} - \sigma^-_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} = \sigma^*_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} + \sigma^0_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} + \sigma^1_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} .$$

We then define

$$\sigma^*_{S^{q+\bar{n}/2+1}_O} := \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^*_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} , \qquad \sigma^{q+\bar{n}/2+1}_{S^{q+\bar{n}/2+1}_O} := \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond \\ \bullet=0,1}} \sigma^\bullet_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} ,$$

so that then using (A.188), we may define the current errors

$$\begin{aligned}
\phi^{*}_{S_O^{q+\bar{n}/2+1}} &:= \sum_{i,j,k,\xi,\vec{l},I,\diamond} \phi^{*}_{S^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} := \sum_{i,j,k,\xi,\vec{l},I,\diamond} (\mathcal{H}+\mathcal{R}^{*}) \left(D_{t,q}\sigma^{*}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right) , \\
\phi^{q+\bar{n}/2+1}_{S_O^{q+\bar{n}/2+1}} &:= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond \\ \bullet=0,1}} \phi^{\bullet}_{S^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} := \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond \\ \bullet=0,1}} (\mathcal{H}+\mathcal{R}^{*}) \left(D_{t,q}\sigma^{\bullet}_{\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right) \\
&= \underbrace{\phi^{q+\bar{n}/2+1,l}_{S_O^{m}}}_{\text{all the } \mathcal{H} \text{ terms}} + \underbrace{\phi^{q+\bar{n}/2+1,*}_{S_O^{m}}}_{\text{all the } \mathcal{R}^{*} \text{ terms}} ,
\end{aligned}$$

which satisfy

$$\begin{aligned}
\mathrm{div}\phi^{*}_{S_O^{q+\bar{n}/2+1}} &= D_{t,q}\sigma^{*}_{S_O^{q+\bar{n}/2+1}} - \int_{\mathbb{T}^3} D_{t,q}\sigma^{*}_{S_O^{q+\bar{n}/2+1}}(t,x')\,dx' , \\
\mathrm{div}\phi^{q+\bar{n}/2+1}_{S_O^{q+\bar{n}/2+1}} &= D_{t,q}\sigma^{q+\bar{n}/2+1}_{S_O^{q+\bar{n}/2+1}} - \int_{\mathbb{T}^3} D_{t,q}\sigma^{q+\bar{n}/2+1}_{S_O^{q+\bar{n}/2+1}}(t,x')\,dx' .
\end{aligned}$$

We decompose the current error further into $\phi^{q+\bar{n}/2+1}_{S_O^{q+\bar{n}/2+1}} = \phi^{q+\bar{n}/2+1,l}_{S_O^{q+\bar{n}/2+1}} + \phi^{q+\bar{n}/2+1,*}_{S_O^{q+\bar{n}/2+1}}$ using item ii.

In order to check (10.46a), we recall the parameter choices from `Case 3a` of Lemma 10.2.1 and the choice of $\overline{\pi} = \pi_\ell \Gamma_q^{50}\Lambda_q$ from Lemma 10.2.4 apply Part 4 of Proposition A.4.4, specifically (A.189c). We then have from (4.24a) that for each $i,j,k,\xi,\vec{l},I,\diamond,\bullet$ and $M,N \leq 2\mathsf{N}_{\mathrm{ind}}$ (after appending a superscript l to refer to the local portion),

$$\begin{aligned}
\left| D^N D^M_{t,q} \phi^{\bullet,l}_{S^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right| &\leq \tau_q^{-1}\Gamma_q^{i+70}\pi_\ell\Lambda_q \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}}r_q} \right)^2 \lambda_{q+\bar{n}/2}^{-1} \\
&\quad \times (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}} - \mathsf{N}_{\mathrm{cut,t}} - 1, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^{9} \right) .
\end{aligned} \tag{10.49}$$

Next, from (A.192), we have that

$$\begin{aligned}
\mathrm{supp}\left(\phi^{\bullet,l}_{S^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right) &\subseteq B\left(\mathcal{H}T^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}, 2\lambda_{q+\bar{n}/2+1}\Gamma_q^{-1} \right) \\
&\subseteq B\left(\mathrm{supp}\left(a_{(\xi),\diamond}(\varrho^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi}) \circ \Phi_{(i,k)} \right), 2\lambda_{q+\bar{n}/2+1}\Gamma_q^{-1} \right) .
\end{aligned}$$

Then applying (9.22), we have that (10.47) is verified for $m = m' = q + \bar{n}/2 + 1$. Returning

to the proof of (10.46a), we can now apply Corollary 8.6.4 with

$$H = \phi^{\bullet,l}_{S^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}\,, \qquad \varpi = \Gamma_q^{70}\pi_\ell\Lambda_q\left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}}r_q}\right)^2\lambda^{-1}_{q+\bar{n}/2}\,.$$

From (8.56b), (4.18), (6.6), (5.20), (4.10h), and (4.27b), we have that

$$\begin{aligned}
&\left|\psi_{i,q}\sum_{i',j,k,\xi,\vec{l},I,R,\bullet}\mathcal{H}\left(D_{t,q}\sigma^{\bullet}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,R}}\right)\right| \\
&\underset{(8.56b)}{\lesssim}\underbrace{r_q^{-1}\lambda_q\left(\pi_q^q\right)^{1/2}}_{\text{cost of } D_{t,q}}\underbrace{\pi_\ell}_{\substack{\text{dominates}\\ \text{low-freq. coeff's}}}\underbrace{\Lambda_q\lambda^{-1}_{q+\bar{n}/2}}_{\text{freq. gain}}\underbrace{\Gamma_q^{76}}_{\text{lower order}}\underbrace{\left(\frac{\lambda_{q+\bar{n}/2+1}\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^2}_{\text{intermittency losses}}\underbrace{\lambda^{-1}_{q+\bar{n}/2}}_{\text{inv. div. gain}} \\
&\qquad\times\left(\lambda_{q+\bar{n}/2+1}\Gamma_q\right)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}}-\mathsf{N}_{\text{cut,t}}-1,\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\underset{(6.6),(5.20)}{\lesssim} r_q^{-1}\Gamma_q^{100}\left(\pi_q^{q+\bar{n}/2+1}\frac{\delta_{q+\bar{n}}}{\delta_{q+\bar{n}/2+1+\bar{n}}}\right)^{3/2}\Lambda_q^2\left(\frac{\lambda_{q+\bar{n}/2+1}\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^2\lambda^{-2}_{q+\bar{n}/2} \\
&\qquad\times\left(\lambda_{q+\bar{n}/2+1}\Gamma_q\right)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}}-\mathsf{N}_{\text{cut,t}}-1,\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\underset{(4.18),(4.27b),(4.10h)}{\leq}\Gamma_q^{-150}r^{-1}_{q+\bar{n}/2+1}\left(\pi_q^{q+\bar{n}/2+1}\right)^{3/2}\left(\lambda_{q+\bar{n}/2+1}\Gamma_q\right)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right)
\end{aligned} \tag{10.50}$$

for $N, M \leq 2\mathsf{N}_{\text{ind}}$ from (4.24a), which verifies (10.46a) at level $q + \bar{n}/2 + 1$. In order to achieve (10.46b), we appeal to (A.190)–(A.191), the choice of $K_\circ$ in item (xvi), (4.24a), and an aggregation quite similar to previous nonlocal aggregations.

`Case 3b/4b:` pressure current error from (10.18d) and (10.18e) and $\diamond = R, \varphi$. In this case we consider the higher shells from the oscillation error. The general principle is that the estimate will only be sharp in the $m = m' = q + \bar{n}$ double endpoint case, for which the intermittency loss is most severe. We now explain why this is the case by parsing estimates (10.49) and (10.50). We incur a material derivative cost of $\tau_q^{-1}\Gamma_q^{i+70}$, which is converted into $r_q^{-1}\lambda_q(\pi_q^q)^{1/2}$ using (5.23) and the rough definition of $\tau_q^{-1} = \delta_q^{1/2}\lambda_q r_q^{-1/3}$, or equivalently Corollary 8.6.4. The $L^{3/2}$ size of the high-frequency coefficients from the oscillation error is $(\lambda_m\lambda^{-1}_{q+\bar{n}/2})^{2/3}$; this encodes the intermittency loss from L^1 to $L^{3/2}$ of a squared, $\leq \lambda_m$ frequency projected, L^2 normalized pipe flow with minimum frequency $\lambda_{q+\bar{n}/2}$ – see also the choices of

$\mathcal{C}_{*,3/2}$ from Lemma 10.2.1. This accounts for $2/3$ of the squared power in the intermittency losses. The low-frequency coefficient function from a quadratic oscillation error incurs a derivative cost of Λ_q (which we have grouped with "frequency gain") and is dominated by π_ℓ. The negative power in the frequency gain will be λ_m and is determined by which shell (indexed by m) of the oscillation error is being considered. The lower order terms may be ignored. Next, we have an $L^{3/2} \to L^\infty$ intermittency loss of $(\lambda_{m'}\lambda_{q+\bar{n}/2}^{-1})^{4/3}$, which accounts for $4/3$ of the power in the intermittency losses and is used to pointwise dominate the high-frequency portion (at frequency $\lambda_{m'}$ due to the frequency projector) of the pressure increment using the $L^{3/2}$ norm. By simply pointwise dominating the high-frequency portion of the pressure increment, using this to compute the L^1 norm of the resulting current error, and showing that the result is dominated by existing pressure, we prevent a loop of new current error and new pressure creation. Finally, we have an inverse divergence gain depending on which synthetic Littlewood-Paley shell of the pressure increment we are considering. The net effect is that the Λ_q from "frequency gain" and the $\lambda_{m'}^{-1}$ from "inv. div. gain" upgrade the $\pi_\ell^{3/2}$ to $(\pi_q^{m'})^{3/2}$, and the remaining $\lambda_q\lambda_m^{-1}$ from the $D_{t,q}$ cost and the frequency gain is strong enough to absorb the intermittency loss since $m' \leq m$, with a perfect balance in the case

$$m = m' = q + \bar{n} \qquad \Longrightarrow \qquad \left(\frac{\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}/2}}\right)^2 \lambda_q\lambda_{q+\bar{n}}^{-1} \approx 1\,.$$

In order to fill in the details, we now recall the choices of $\bar{m}$ and $\mu_{m'}$ from (10.39). For the sake of brevity we ignore the slight variation in the case of the first projector for $m = q + \bar{n}/2 + 2$ and focus on the second projector for $m = q + \bar{n}/2 + 2$ and the other cases $q + \bar{n}/2 + 2 < m \leq q + \bar{n} + 1$. We have from (A.182) that

$$\sigma_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} = \sigma^+_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} - \sigma^-_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} = \sigma^*_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} + \sum_{\iota=0}^{m-q-\bar{n}/2} \sigma^\iota_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,.$$

We then define the frequency-projected pressure increments by

$$
\begin{aligned}
\sigma^{*}_{S^m_O} &= \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{*}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,, \qquad \sigma^{q+\bar{n}/2+1}_{S^m_O} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{0}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,,\\
\sigma^{q+\bar{n}/2+2}_{S^m_O} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond\\ \iota=1,2}} \sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,,\\
\sigma^{q+\bar{n}/2+m'}_{S^m_O} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond\\ \iota=m'}} \sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} \quad \text{if } q+\bar{n}/2+m' = q+\bar{n}/2+\iota \le m \le q+\bar{n}-1\,, \qquad (10.51)\\
\sigma^{q+\bar{n}}_{S^m_O} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond\\ \iota=}} \quad \text{if } \iota\ m = q+\bar{n}, q+\bar{n}+1\,.
\end{aligned}
$$

Using (A.188), we may define the current errors

$$
\begin{aligned}
\phi^{*}_{S^m_O} &= \sum_{i,j,k,\xi,\vec{l},I,R} (\mathcal{H}+\mathcal{R}^*)\left(D_{t,q}\sigma^{*}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\,, \quad \phi^{q+\bar{n}/2+1}_{S^m_O} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} (\mathcal{H}+\mathcal{R}^*)\left(D_{t,q}\sigma^{0}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\,,\\
\phi^{q+\bar{n}/2+2}_{S^m_O} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond\\ \iota=1,2}} (\mathcal{H}+\mathcal{R}^*)\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\,,\\
\phi^{q+\bar{n}/2+m'}_{S^m_O} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond\\ \iota=m'}} (\mathcal{H}+\mathcal{R}^*)\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right) \quad \text{if } q+\bar{n}/2+m' = q+\bar{n}/2+\iota < m\,,\\
\phi^{q+\bar{n}}_{S^m_O} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond\\ \iota=m-q-\bar{n}/2,m-q-\bar{n}/2+1}} (\mathcal{H}+\mathcal{R}^*)\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\,.
\end{aligned}
$$

As in the previous case, we may append superscripts of l and $*$ for $q+\bar{n}/2+1 \le m \le q+\bar{n}$ corresponding to the $\mathcal{H}$ and $\mathcal{R}^*$ portions, respectively. We have thus verified item (i) immediately from these definitions and from (A.188) and item (ii). In order to check (10.46a), we define the temporary notation $m'(\iota)$ to make a correspondence between the value of ι above and the superscript on the left-hand side, which determines which bin the current errors go into. Specifically, we set $m'(0)=1$, $m'(1)=m'(2)=2$, $m'(\iota)=\iota$ if $q+\bar{n}/2+\iota<m$, and $m'(m-q-\bar{n}/2)=m'(m-q-\bar{n}/2+1)=m-q-\bar{n}/2$. Then from Part 4 of Proposition A.4.4,

specifically (A.189c), and (4.24a), we have that for each $i, j, k, \xi, \vec{l}, I, \diamond, \iota$ and $M, N \leq 2\mathsf{N}_{\text{ind}}$,

$$\begin{aligned}
&\left| D^N D^M_{t,q}(\mathcal{H} + \mathcal{R}^*) \left(D_{t,q} \sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right) \right| \\
&\leq \tau_q^{-1} \Gamma_q^{i+70} \pi_\ell \Lambda_q \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \lambda_{m-1}^{-2} \lambda_m \left(\frac{\min(\lambda_{q+\bar{n}/2+m'(\iota)}, \lambda_{q+\bar{n}}) \Gamma_q}{\lambda_{q+\bar{n}/2}} \right)^{4/3} \\
&\quad \times \lambda^{-2}_{q+\bar{n}/2+m'(\iota)-1} \lambda_{q+\bar{n}/2+m'(\iota)} \left(\min(\lambda_{q+\bar{n}/2+m'(\iota)}, \lambda_m) \Gamma_q \right)^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}} - 1, \tau_q^{-1} \Gamma_q^{i+14}, \mathrm{T}_q^{-1} \Gamma_q^9 \right) .
\end{aligned}$$

Next, from (A.192) and the fact that $q + \bar{n}/2 + m'(\iota) \leq m$, we have that

$$\begin{aligned}
\text{supp} \left(\mathcal{H} \left(D_{t,q} \sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right) \right) &\subseteq B \left(\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}, 2\lambda_{q+\bar{n}/2+m'(\iota)-1} \Gamma_q^{-2} \right) \\
&\subseteq B \left(\text{supp} \left(a_{(\xi),\diamond} (\varrho^{\diamond}_{(\xi)} \boldsymbol{\zeta}^I_\xi) \circ \Phi_{(i,k)} \rho^I_{(\xi),\diamond} \right), \lambda_{m-1}^{-1} + 2\lambda_{q+\bar{n}/2+m'(\iota)-1} \Gamma_q^{-2} \right) \\
&\subseteq B \left(\text{supp} \left(a_{(\xi),\diamond} (\varrho^{\diamond}_{(\xi)} \boldsymbol{\zeta}^I_\xi) \circ \Phi_{(i,k)} \rho^I_{(\xi),\diamond} \right), 2\lambda_{q+\bar{n}/2+m'(\iota)-1} \right) .
\end{aligned}$$

Then applying (9.22), we have that (10.47) is verified for $m' = q + \bar{n}/2 + m'(\iota)$. Returning to the proof of (10.46a), we can now apply Corollary 8.6.4 with

$$\begin{aligned}
H &= \mathcal{H} \left(D_{t,q} \sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right) , \\
\varpi &= \Gamma_q^{70} \pi_\ell \Lambda_q \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \lambda_{m-1}^{-2} \lambda_m \left(\frac{\min(\lambda_{q+\bar{n}/2+m'(\iota)}, \lambda_{q+\bar{n}}) \Gamma_q}{\lambda_{q+\bar{n}/2}} \right)^{4/3} \lambda^{-2}_{q+\bar{n}/2+m'(\iota)-1} \lambda_{q+\bar{n}/2+m'(\iota)} .
\end{aligned}$$

From (4.10h), (8.56b), (6.6), (5.20), and (4.26), we have that

$$
\begin{aligned}
&\left|\psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} \mathcal{H}\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}T^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\right| \\
&\quad \underset{(8.56b)}{\lesssim} \Gamma_q^{76} r_q^{-1}\lambda_q \left(\pi_q^q\right)^{1/2} \pi_\ell \Lambda_q \left(\frac{\min(\lambda_m,\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}}\right)^{2/3} \lambda_{m-1}^{-2}\lambda_m \left(\frac{\min(\lambda_{q+\bar{n}/2+m'(\iota)},\lambda_{q+\bar{n}})\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3} \\
&\qquad \times \lambda_{q+\bar{n}/2+m'(\iota)-1}^{-2}\lambda_{q+\bar{n}/2+m'(\iota)} \left(\min(\lambda_{q+\bar{n}/2+m'(\iota)},\lambda_m)\Gamma_q\right)^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\quad \underset{(6.6),(5.20)}{\lesssim} \Gamma_q^{76} r_q^{-1}\lambda_q \left(\pi_q^{q+\bar{n}/2+m'(\iota)} \frac{\delta_{q+\bar{n}}}{\delta_{q+\bar{n}/2+m'(\iota)+\bar{n}}}\right)^{3/2} \Lambda_q \left(\frac{\min(\lambda_m,\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}}\right)^{2/3} \\
&\qquad \times \lambda_{m-1}^{-2}\lambda_m \left(\frac{\min(\lambda_{q+\bar{n}/2+m'(\iota)},\lambda_{q+\bar{n}})\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3} \lambda_{q+\bar{n}/2+m'(\iota)-1}^{-2}\lambda_{q+\bar{n}/2+m'(\iota)} \\
&\qquad\quad \times \left(\min(\lambda_{q+\bar{n}/2+m'(\iota)},\lambda_{m'})\Gamma_q\right)^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\quad \underset{(4.26),(4.10h)}{\leq} \Gamma_{m'}^{-150} r_{m'}^{-1} \left(\pi_q^{q+\bar{n}/2+m'(\iota)}\right)^{3/2} \left(\min(\lambda_{q+\bar{n}/2+m'(\iota)},\lambda_m)\Gamma_q\right)^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right),
\end{aligned}
$$

for $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$ from (4.24a), which verifies (10.46a) at level m'. In order to achieve (10.46b), we appeal to (A.190)–(A.191), the choice of $K_\circ$ in item xvi, and (4.24a). □

10.2.2 Transport and Nash stress errors S_{TN}

Lemma 10.2.6 (**Applying inverse divergence**). *There exist symmetric stresses $S_{TN} = S_{TN}^l + S_{TN}^*$ defined by*

$$
S_{TN} = \mathcal{H}\left(D_{t,q}w_{q+1} + w_{q+1}\cdot\nabla\widehat{u}_q\right) + \mathcal{R}^*\left(D_{t,q}w_{q+1} + w_{q+1}\cdot\nabla\widehat{u}_q\right) =: S_{TN}^l + S_{TN}^*
$$

which satisfy the following.

(i) For all $N, M \leq \mathsf{N}_{\mathrm{fin}}/10$, the local part S_{TN}^l satisfies

$$
\left\|\psi_{i,q} D^N D_{t,q}^M S_{TN}^l\right\|_{3/2} \lesssim \Gamma_{q+\bar{n}}^{-100}\delta_{q+2\bar{n}}\lambda_{q+\bar{n}}^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.52a}
$$

$$
\left\|\psi_{i,q} D^N D_{t,q}^M S_{TN}^l\right\|_{\infty} \lesssim \Gamma_{q+\bar{n}}^{\mathsf{C}_\infty-100}\lambda_{q+\bar{n}}^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right). \tag{10.52b}
$$

Furthermore, we have that

$$B\left(\operatorname{supp}\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} S_{TN}^{l} = \emptyset \tag{10.53a}$$

for all $q+1 \leq q' \leq q+\bar{n}-1$.

(ii) For $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$ *the nonlocal part satisfies*

$$\left\|D^N D_{t,q}^M S_{TN}^*\right\|_\infty \leq \mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^2 \lambda_{q+\bar{n}}^N \tau_q^{-M} . \tag{10.54}$$

Remark 10.2.7 (Abstract formulation of the transport and Nash stress errors). For the purposes of analyzing the transport and Nash current errors in subsubsection 11.2.2 and streamlining the creation of pressure increments, it will again be useful to abstract the properties of these error terms. We will prove every one of the following claims in the course of of proving Lemma 10.2.6. First, there exists a q-independent constant $\mathcal{C}_{\mathcal{H}}$ such that

$$S_{TN}^{l} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sum_{j'=0}^{\mathcal{C}_{\mathcal{H}}} H_{i,j,k,\xi,\vec{l},I,\diamond}^{\alpha(j')} \rho_{i,j,k,\xi,\vec{l},I,\diamond}^{\beta(j')} \circ \Phi_{(i,k)} . \tag{10.55}$$

Next, the functions H and ρ (with subscripts and superscripts suppressed for convenience) defined above satisfy the following.

(i) H satisfies

$$\left|D^N D_{t,q}^M H\right| \lesssim \pi_\ell \Lambda_q \lambda_{q+\bar{n}/2}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \tag{10.56}$$

for all $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_{10}$.

(ii) We have that

$$\operatorname{supp} H \subseteq \operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}_\xi^{I,\diamond}\right) . \tag{10.57}$$

(iii) For d as in (xvii), there exist a tensor potential ϑ (we suppress the indices at the moment for convenience) such that $\rho = \partial_{i_1 \dots i_{\mathsf{d}}} \vartheta^{(i_1,\dots,i_{\mathsf{d}})}$. Furthermore, ϑ is $(\mathbb{T}/\lambda_{q+\bar{n}/2}\Gamma_q)^3$-periodic and satisfies the estimates

$$\left\| D^N \partial_{i_1} \dots \partial_{i_k} \vartheta^{(i_1,\dots,i_{\mathsf{d}})} \right\|_{L^p} \lesssim r_q^{2/p-2} \lambda_{q+\bar{n}}^{-1+N+k-\mathsf{d}} . \tag{10.58}$$

for $p = 3/2, \infty$, all $N \leq \mathsf{N}_{\text{fin}}/5$, and $0 \leq k \leq \mathsf{d}$.

(iv) We have that

$$\text{supp } (H\rho \circ \Phi) \cap B\left(\text{supp } \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) = \emptyset \tag{10.59}$$

for all $q+1 \leq q' \leq q+\bar{n}-1$. We will prove this claim in the course of proving Lemma 10.2.6.

Proof of Lemma 10.2.6. We start by considering either a Reynolds or current corrector defined in subsection 9.1 and expanding

$$\begin{aligned}
D_{t,q} w_{q+1,\diamond} &= D_{t,q}\Bigg(\sum_{i,j,k,\xi,\vec{l},I} \text{curl } \left(a_{(\xi),\diamond} (\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} \nabla \Phi^T_{(i,k)} \mathbb{U}^I_{(\xi),\diamond} \circ \Phi_{(i,k)} \right) \Bigg) \\
&= \sum_{i,j,k,\xi,\vec{l},I} D_{t,q} \left(a_{(\xi),\diamond} \nabla \Phi^{-1}_{(i,k)} \right) (\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} \mathbb{W}^I_{(\xi),\diamond} \circ \Phi_{(i,k)} \\
&\quad + \sum_{i,j,k,\xi,\vec{l},I} D_{t,q} \nabla \left((\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} a_{(\xi),\diamond} \right) \times \left(\nabla \Phi_{(i,k)} \mathbb{U}^I_{(\xi),\diamond} \circ \Phi_{(i,k)} \right) \\
&\quad + \sum_{i,j,k,\xi,\vec{l},I} \nabla \left((\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} a_{(\xi),\diamond} \right) \times \left(D_{t,q} \nabla \Phi_{(i,k)} \mathbb{U}^I_{(\xi),\diamond} \circ \Phi_{(i,k)} \right)
\end{aligned} \tag{10.60}$$

and

$$
\begin{aligned}
w_{q+1,\diamond}\cdot\nabla\widehat{u}_q &= \sum_{i,j,k,\xi,\vec{l},I} \operatorname{curl}\left(a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi})\circ\Phi_{(i,k)}\nabla\Phi^{T}_{(i,k)}\mathbb{U}^{I}_{(\xi),\diamond}\circ\Phi_{(i,k)}\right)\cdot\nabla\widehat{u}_q \\
&= \sum_{i,j,k,\xi,\vec{l},I}\left(a_{(\xi),\diamond}\nabla\Phi^{-1}_{(i,k)}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi})\circ\Phi_{(i,k)}\mathbb{W}^{I}_{(\xi),\diamond}\circ\Phi_{(i,k)}\right)\cdot\nabla\widehat{u}_q \\
&\quad+\sum_{i,j,k,\xi,\vec{l},I}\left(\nabla\left((\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi})\circ\Phi_{(i,k)}a_{(\xi),\diamond}\right)\times\left(\nabla\Phi_{(i,k)}\mathbb{U}^{I}_{(\xi),\diamond}\circ\Phi_{(i,k)}\right)\right)\cdot\nabla\widehat{u}_q\,.
\end{aligned}
\tag{10.61}
$$

We shall only consider the worst terms, which are the ones containing $\mathbb{W}^{I}_{(\xi),\diamond}$. Since $D_{t,q}w_{q+1,\diamond}$ and $w_{q+1,\diamond}\cdot\nabla\widehat{u}_q$ are mean-zero (see the argument below the display in (10.1)), we can apply $\mathcal{H}$ and $\mathcal{R}^*$ from Proposition A.3.3 to each term in (10.60) while ignoring the last term in (A.56).

We now fix values of i, j, k, ξ, $\vec{l}$, I, and $\diamond$ so that we are simply considering

$$
\begin{aligned}
T_{i,j,k,\xi,\vec{l},I,\diamond} := D_{t,q}&\left(a_{(\xi),\diamond}\nabla\Phi^{-1}_{(i,k)}\right)(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi})\circ\Phi_{(i,k)}\mathbb{W}^{I}_{(\xi),\diamond}\circ\Phi_{(i,k)} \\
&+\nabla\widehat{u}_q\cdot\left(a_{(\xi),\diamond}\nabla\Phi^{-1}_{(i,k)}\right)(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi})\circ\Phi_{(i,k)}\mathbb{W}^{I}_{(\xi),\diamond}\circ\Phi_{(i,k)}\,.
\end{aligned}
\tag{10.62}
$$

We apply Proposition A.3.3 along with Remark A.3.9 with the following choices. Let $p\in\{3/2,\infty\}$. We set $v=\widehat{u}_q$, and $D_t=D_{t,q}=\partial_t+\widehat{u}_q\cdot\nabla$. In order to verify the low-frequency assumptions from Part 1 of Proposition A.3.3 and Remark A.3.9, we set

$$
\begin{aligned}
&G_{i,j,k,\xi,\vec{l},I,R} = r_q\left[D_{t,q}\left(a_{(\xi),R}\nabla\Phi^{-1}_{(i,k)}\right)(\boldsymbol{\rho}^{R}_{(\xi)}\boldsymbol{\zeta}^{I,R}_{\xi})\circ\Phi_{(i,k)}\xi+\nabla\widehat{u}_q\cdot\left(a_{(\xi),R}\nabla\Phi^{-1}_{(i,k)}\right)(\boldsymbol{\rho}^{R}_{(\xi)}\boldsymbol{\zeta}^{I,R}_{\xi})\circ\Phi_{(i,k)}\xi\right]\,,\\
&G_{i,j,k,\xi,\vec{l},I,\varphi} = r_q^{4/3}\left[D_{t,q}\left(a_{(\xi),\varphi}\nabla\Phi^{-1}_{(i,k)}\right)(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi})\circ\Phi_{(i,k)}\xi+\nabla\widehat{u}_q\cdot\left(a_{(\xi),\varphi}\nabla\Phi^{-1}_{(i,k)}\right)(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi})\circ\Phi_{(i,k)}\xi\right]\,,\\
&N_* = \mathsf{N}_{\text{fin}}/4\,,\quad M_* = \mathsf{N}_{\text{fin}}/5\,,\quad \mathcal{C}_{G,3/2} = r_q\left|\operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{2/3}\delta^{1/2}_{q+\bar{n}}\Gamma_q^{i+j+20}\tau_q^{-1}+r_q\lambda^{-10}_{q+\bar{n}}\,,\\
&\mathcal{C}_{G,\infty} = \Lambda_q\Gamma_q^{2+\mathsf{C}_\infty}\,,\quad \lambda=\lambda_{q+\lfloor\bar{n}/2\rfloor}\,,\quad \nu=\tau_q^{-1}\Gamma_q^{i+13}\,,\quad M_t=\mathsf{N}_{\text{ind,t}}\,,\quad \nu'=\mathrm{T}_q^{-1}\Gamma_q^{8}\,,\\
&v=\widehat{u}_q\,,\quad \Phi=\Phi_{(i,k)}\,,\quad D_t=D_{t,q}\,,\quad \lambda'=\Lambda_q\,,\quad \pi=\pi_\ell\Lambda_q\,.
\end{aligned}
\tag{10.63}
$$

Then we have that (A.39) is satisfied by definition, and (A.41)–(A.42b) are satisfied as in the proof of Lemma 10.2.1. In order to check (A.40), we appeal to Lemma 9.3.1, estimate (8.13b) for $(\nabla\Phi_{(i,k)})^{-1}$, estimate (8.40) from Lemma 8.4.3 to estimate $\boldsymbol{\zeta}_{\xi}^{I,\diamond}\circ\Phi_{(i,k)}$, Proposition 7.2.1, and (5.34). Specifically, we have that for all $N, M \leq 9\mathsf{N}_{\text{ind}}$,

$$\begin{aligned}
\left\| D^N D^M_{t,q} G_{i,j,k,\xi,\vec{l},I,\diamond} \right\|_{3/2} &\lesssim \mathcal{C}_{G,3/2}\lambda^N_{q+\lfloor \bar{n}/2\rfloor}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}-1, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
&\lesssim \mathcal{C}_{G,3/2}\lambda^N_{q+\lfloor \bar{n}/2\rfloor}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \qquad (10.64)\\
\left| D^N D^M_{t,q} G_{i,j,k,\xi,\vec{l},I,\diamond} \right| &\lesssim r_q\Gamma_q^{50}\pi_\ell^{1/2}\tau_q^{-1}\Gamma_q^i\lambda^N_{q+\lfloor \bar{n}/2\rfloor}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}-1, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
&\lesssim r_q r_{q-\bar{n}}^{-1}\Gamma_q^{100}\pi_\ell\Lambda_q\lambda^N_{q+\lfloor \bar{n}/2\rfloor}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}-1, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) \\
&\lesssim \pi_\ell\Lambda_q\lambda^N_{q+\lfloor \bar{n}/2\rfloor}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) , \qquad (10.65)
\end{aligned}$$

where we have used (4.18) to upgrade the sharp derivatives to $\mathsf{N}_{\text{ind,t}}$ in both inequalities, (5.23), (4.10b), and (6.6) to convert $\tau_q^{-1}\Gamma_q^i$ into $\pi_\ell^{1/2}\Gamma_q^{50}\Lambda_q r_{q-\bar{n}}^{-1}$ in the pointwise bounds, and (4.10h) to absorb the Γ_q^{100}. In order to obtain an L^∞ bound, we can appeal to (10.65) and (6.14b). Thus we have that (A.40) and (A.59) are satisfied in all cases.

In order to verify the high-frequency assumptions from Part 2 of Proposition A.3.3, we set

$$\begin{aligned}
&r_q\varrho_R = \varrho^I_{(\xi),R}\,, \quad r_q\vartheta_R \text{ as defined in item (1) from Proposition 7.1.5}\\
&r_q^{4/3}\varrho_\varphi = \varrho^I_{(\xi),\varphi}\,, \quad r_q^{4/3}\vartheta_\varphi \text{ defined similarly but adjusted to fit Proposition 7.1.6}\\
&\mathsf{N}_{\text{dec}} \text{ as in (xv)}\,, \quad \mathsf{d} \text{ as in (xvii)}\,, \quad \mathcal{C}_{*,3/2} = r_q^{-2/3}\,, \quad \mathcal{C}_{*,\infty} = r_q^{-2}\,,\\
&\mu = \lambda_{q+\bar{n}}r_q = \lambda_{q+\bar{n}/2}\Gamma_q\,, \quad \Upsilon = \Upsilon' = \Lambda = \lambda_{q+\bar{n}}\,. \qquad (10.66)
\end{aligned}$$

Then we have that (i) is satisfied from (7.9), (ii) is satisfied by the construction of w_{q+1} in subsection 9.1, and (A.43) is satisfied from Proposition 7.1.5 or the corresponding estimates in Proposition 7.1.6. Finally, we have that (A.44) follows by definition and from (4.24a),

while (A.45) is satisfied from (4.21).

We therefore may appeal to the local conclusions (i)–(vi) and (A.56)–(A.57), from which we have the following. First, we note that from (iii), we have that (10.55) is satisfied. Next, we have from (A.46), (A.50), and (A.60) that for $N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$,

$$\left\| D^N D^M_{t,q} \left(\mathcal{H} \left(T_{i,j,k,\xi,\vec{l},I,\diamond} \right) \right) \right\|_{3/2} \lesssim \left(\left| \text{supp} \left(\eta_{i,j,k,\xi,\vec{l},I,\diamond} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \right) \right|^{2/3} \delta^{1/2}_{q+\bar{n}} r_q^{1/3} \Gamma_q^{i+j+25} \tau_q^{-1} + \lambda^{-10}_{q+\bar{n}} \right) \times \lambda^{-1+N}_{q+\bar{n}} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+14}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) , \tag{10.67}$$

$$\left| D^N D^M_{t,q} \left(\mathcal{H} \left(T_{i,j,k,\xi,\vec{l},I,\diamond} \right) \right) \right| \lesssim \pi_\ell \Lambda_q r_q^{-2} \lambda^{-1+N}_{q+\bar{n}} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+14}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) . \tag{10.68}$$

Notice that from (ii), the support of $\text{div} \mathcal{H} T_{i,j,k,\xi,\vec{l},I,R}$ is contained in the support of $T_{i,j,k,\xi,\vec{l},I,R}$, which itself is contained in the support of $\eta_{i,j,k,\xi,\vec{l},R} \boldsymbol{\zeta}^{I,R}_{\xi}$. From this observation, we have that (10.57) is satisfied. Furthermore, we have that (10.58) is satisfied from (A.49a) and the estimates from Proposition 7.1.5 and 7.1.6. Next, we have that (10.56) is satisfied from (A.60). Finally, we have that (10.59) holds due to item (ii) and item (7) from Proposition 7.1.5. We note also that (10.53a) follows from (10.57), (10.59), and (9.24).

In order to aggregate $L^{3/2}$ estimates, we appeal to Corollary 8.6.1 with $\theta_1 = \theta_2 = 1$, $H = \mathcal{H} \left(T_{i,j,k,\xi,\vec{l},I,\diamond} \right)$, (5.8) at level q, and (4.10i) to write that

$$\begin{aligned} &\left\| \psi_{i,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q} \left(\mathcal{H} \left(T_{i',j,k,\xi,\vec{l},I,\diamond} \right) \right) \right\|_{3/2} \\ &\lesssim \Gamma_q^{50+\mathsf{C}_b} \delta^{1/2}_{q+\bar{n}} r_q^{1/3} \lambda^{-1+N}_{q+\bar{n}} \tau_q^{-1} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) \\ &\lesssim \Gamma^{-25}_{q+\bar{n}} \delta_{q+2\bar{n}} \lambda^N_{q+\bar{n}} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^8 \right) . \end{aligned} \tag{10.69}$$

In order to aggregate pointwise estimates, we appeal to Corollary 8.6.3 with the same choice of H and $\varphi = \pi_\ell \Lambda_q r_q^{-2} \mathbf{1}_{\text{supp} \left(\eta_{i,j,k,\xi,\vec{l},R} \boldsymbol{\zeta}^{I,R}_{\xi} \right)}$. Then from (8.53b), (8.45), (6.3b), and (4.13a), we

have that

$$\left|\psi_{i,q}\sum_{i',j,k,\xi,\vec{l},I,\diamond} D^N D^M_{t,q}\left(\mathcal{H}\left(T_{i',j,k,\xi,\vec{l},I,\diamond}\right)\right)\right| \lesssim \pi_\ell \Lambda_q r_q^{-2}\lambda_{q+\bar{n}}^{-1+N}\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^8\right)$$
$$\leq \Gamma_{q+\bar{n}}^{\mathsf{C}_\infty-200}\lambda_{q+\bar{n}}^N\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^8\right) .$$

To conclude the proof for the leading order term from $D_{t,q}w_{q+1}$, we must still estimate the nonlocal $\mathcal{R}^*$ portion of the inverse divergence. In order to check the nonlocal assummptions, we again set

$$M_\circ = N_\circ = 2\mathsf{N}_{\mathrm{ind}}\,, \quad K_\circ \text{ as in (xvi)}\,.$$

Then from (4.23b) and Remark A.3.4, we have that (A.52)–(A.55) are satisfied. We note that $D_{t,q}w_{q+1} + w_{q+1}\cdot\nabla\widehat{u}_q$ has zero mean, and so we may ignore the means of individual terms that get plugged into the inverse divergence since their sum will vanish. Then from (A.56), (A.57), and Remark A.3.4, we have that for $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\left\|D^N D^M_{t,q}\sum_{i,j,k,\xi,\vec{l}}\mathcal{R}^* T_{i,j,k,\xi,\vec{l},\diamond}\right\|_\infty \leq \delta^2_{q+3\bar{n}}\mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}}\lambda_{q+\bar{n}}^N\tau_q^{-M}\,,$$

matching the desired estimate in (10.54). □

At this point, we can construct the pressure increment and associated current error coming from the Nash and transport errors. Since the proofs of both lemmas are completely analogous to the proofs of the corresponding lemmas for the *highest frequency shell* from (10.18e) of the oscillation error, we omit the majority of the details and merely note the minor differences required in a combined proof.

Lemma 10.2.8 (Pressure increment). *There exists a function $\sigma_{S_{TN}} = \sigma^+_{S_{TN}} - \sigma^-_{S_{TN}}$ such that the following hold.*

(i) We have that

$$\left|\psi_{i,q} D^N D^M_{t,q} S_{TN}\right| < \left(\sigma^+_{S_{TN}} + \delta_{q+3\bar{n}}\right) \left(\lambda_{q+\bar{n}}\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.70a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S_{TN}}\right| < \left(\sigma^+_{S_{TN}} + \delta_{q+3\bar{n}}\right) \left(\lambda_{q+\bar{n}}\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.70b}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S_{TN}}\right\|_{3/2} \leq \Gamma^{-9}_{q+\bar{n}} \delta_{q+2\bar{n}} \left(\lambda_{q+\bar{n}}\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.70c}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S_{TN}}\right\|_{\infty} \leq \Gamma^{\mathsf{C}_\infty - 9}_{q+\bar{n}} \left(\lambda_{q+\bar{n}}\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.70d}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^-_{S_{TN}}\right| \leq \Gamma^{-100}_{q+\bar{n}/2} \pi_q^{q+\bar{n}/2} \left(\lambda_{q+\bar{n}/2}\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.70e}$$

for all $N, M < \mathsf{N}_{\text{fin}}/100$.

(ii) For all $q+1 \leq q' \leq q + \bar{n}/2$ *and* $q+1 \leq q'' \leq q + \bar{n} - 1$, *we have that*

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} \sigma^-_{S_{TN}} = B\left(\operatorname{supp} \widehat{w}_{q''}, \lambda_{q''}^{-1}\Gamma_{q''+1}\right) \cap \operatorname{supp} \sigma^+_{S_{TN}} = \emptyset\,. \tag{10.71}$$

(iii) Define

$$\mathfrak{m}_{\sigma_{S_{TN}}}(t) = \int_0^t \left\langle D_{t,q}\sigma_{S_{TN}}\right\rangle (s)\, ds\,. \tag{10.72}$$

Then we have that for $0 \leq M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left|\frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{\sigma_{S_{TN}}}\right| \leq (\max(1,T))^{-1} \delta_{q+3\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right)\,. \tag{10.73}$$

Lemma 10.2.9 (Pressure current). *There exists a current error* $\phi_{S_{TN}}$ *associated to the pressure increment* $\sigma_{S_{TN}}$ *defined by Lemma 10.2.8 which satisfies the following properties.*

(i) We have the decomposition and equalities

$$\phi_{S_{TN}} = \phi^{*}_{S_{TN}} + \sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}} \phi^{m'}_{S_{TN}}\,, \qquad \phi^{m'}_{S_{TN}} = \phi^{m',l}_{S_{TN}} + \phi^{m',*}_{S_{TN}} \tag{10.74a}$$

$$\mathrm{div}\phi_{S_{TN}} = D_{t,q}\sigma_{S_{TN}} - \langle D_{t,q}\sigma_{S_{TN}}\rangle\,. \tag{10.74b}$$

(ii) For all $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q}\phi^{k',l}_{S_{TN}}\right| < \Gamma_{k'}^{-100} r_{k'}^{-1}\left(\pi^{k'}_q\right)^{3/2}\left(\lambda_{k'}\Gamma^2_{m'}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+18}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,, \tag{10.75}$$

$$\left\|D^N D^M_{t,q}\phi^{k',*}_{S_{TN}}\right\|_{L^\infty} < \mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\delta^{3/2}_{q+3\bar{n}}(\lambda_{q+\bar{n}}\Gamma_q^2)^N\tau_q^{-M}\,. \tag{10.76}$$

(iii) For all m', q' with $q+1 \leq q' \leq m'-1$ and $q+\bar{n}/2+1 \leq m' \leq q+\bar{n}$, we have that

$$B\left(\mathrm{supp}\,\widehat{w}_{q'}, 1/2\lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \mathrm{supp}\,\phi^{k',l}_{S_{TN}} = \emptyset\,. \tag{10.77}$$

Proofs of Lemmas 10.2.8 and 10.2.9. As in Lemmas 10.2.4 and 10.2.5 in the case $m = q+\bar{n}$, the proofs of Lemmas 10.2.8 and 10.2.9 use Proposition A.4.4 to estimate a single error term indexed by $i, j, k, \xi, \vec{l}, I, \diamond$, and then aggregate estimates according to Corollaries 8.6.1–8.6.4. We now identify the minor differences between the applications of these various tools to the transport/Nash error and the oscillation error.

We first check the preliminary assumptions from Part 1 of Proposition A.4.4. Let us first compare the low-frequency parameter choices for the transport error in (10.63) to the low-frequency parameter choices for the error terms in (10.18e), which was analyzed in `Case 3b` from Lemma 10.2.1. First, we have that the vector field G in (10.63) is different than the vector field in (10.24), but it retains the exact same support properties due to the presence of $\boldsymbol{\rho}^{\diamond}_{\xi}\boldsymbol{\zeta}^{\diamond}_{\xi}$ in both. Next, we claim that $\mathcal{C}_{G,p}$ is effectively *smaller* in (10.63) than in (10.24). In the case $p = \infty$, this is immediate, so we focus on the case $3/2$. We use (4.10b), (4.10h),

and (4.10g) to write that

$$\tau_q^{-1} r_q \leq \Gamma_q^{50} \lambda_q \delta_q^{1/2} r_{q-\bar{n}}^{-1/3} r_q \leq \delta_{q+\bar{n}}^{1/2} \Lambda_q \Gamma_q^{-50} .$$

The difference between Γ_q^{i+j} in (10.63) and Γ_q^{2j} in (10.24) only matters in the application of Corollaries 8.6.1–8.6.4. Indeed, trading a j for an i simply necessitates a difference choice of θ_1 and θ_2, and the only difference in the output is the factor of $\Gamma_q^{\theta_1 \mathsf{C}_b}$ which must be absorbed in the latter case. The reader is invited to check inequalities (4.26), (4.27b), (4.13a), (4.27d), and (4.27e), each of which has a $\Gamma_q^{5\mathsf{C}_b}$ on the left-hand side that can therefore absorb this extra insignificant factor. Next, we have that the choices of $M_t, M_*, N_*, \lambda, \nu, \nu'$ are the same, and the choice of $\varpi = \pi_\ell \Gamma_q^{50} \Lambda_q$ from the beginning of Lemma 10.2.4 is larger than the choice of ϖ from (10.63) for the transport error. Finally, the vector field v and associated material derivative D_t from item (ii) are identical in both cases.

Next, we compare the high-frequency parameter choices from item (iii) in the case of the oscillation error in (10.25) to the choices for the transport error in (10.66). The potential ϑ in (10.66) is supported in a $\lambda_{q+\bar{n}}^{-1}$ neighborhood of $\varrho^I_{(\xi),\diamond}$, while for the oscillation error, the support is larger due to the presence of the synthetic Littlewood-Paley projector $\widetilde{\mathbb{P}}_{(\lambda_{q+\bar{n}-1}, q+\bar{n}]}$ applied to $(\varrho^I_{(\xi),\diamond})^2$. Thus the potential for transport error has more advantageous support properties than that of the oscillation error. Next, the choices of μ and Λ are identical, while the choices of Υ and Υ' are *more* advantageous for the transport error than they are for the oscillation error in the case $m = q + \bar{n}$. Indeed, this is because the inverse divergence gain in the transport error is a full $\lambda_{q+\bar{n}}$ from (7.9), while the highest shell of the oscillation error only gains $\lambda_{q+\bar{n}-1}$ due to the presence of the synthetic Littlewood-Paley projector. Next, the choices of $\mathcal{C}_{*,p}$ are identical due to our choice of rescaling in the transport error, and the choices of N_{dec} and d are identical as well. Therefore, all assumptions from item (iii) are *stronger* for the transport error than the oscillation error. Finally, we note that the nonlocal assumptions in item (v) are not changed in any significant way, and so we may treat the

nonlocal transport error terms in the same way as the nonlocal oscillation error terms.

Moving to the additional assumptions from Part 2 of Proposition A.4.4, we have that all inequalities in (A.179), (A.180a), (A.180c), (A.180d) are identical. The inequality in (A.180b) follows in the same was as in the oscillation error; indeed, all nonlocal error bounds can be treated in the same way via a large choice of d or N_{**}. The inequalities in item (iii) are the same for the transport error as for the highest shell of the oscillation error, since these inequalities relate to the synthetic Littlewood-Paley projection of a function which oscillates at frequency $\approx \Lambda = \lambda_{q+\bar{n}}$.

Now that we have highlighted the unimportant differences in the set-up, we merely note that the sharp material derivative cost in Lemmas 10.2.6–10.2.9 is worse by a factor of Γ_q than the corresponding estimates in Lemmas 10.2.1–10.2.5. This is due to the fact that the transport error loses a material derivative. This concludes the proofs of Lemmas 10.2.8 and 10.2.9. □

10.2.3 Divergence corrector error S_C

We will write the divergence corrector error as

$$S_C = S_{C1} + S_{C2}\,, \qquad \text{for} \qquad \mathrm{div} S_{C1} = \mathrm{div}\left(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)}\right)\,, \qquad S_{C2} = w_{q+1}^{(c)} \otimes w_{q+1}^{(c)}\,, \tag{10.78}$$

and estimate them in the following lemma.

Lemma 10.2.10 (**Basic estimates and applying inverse divergence).** *There exist symmetric stresses S_C^m for $m \in \{q + \lfloor \bar{n}/2 \rfloor + 1, \ldots, q + \bar{n}\}$ such that the following hold.*

(i) $\mathrm{div}\left(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)} + w_{q+1}^{(c)} \otimes w_{q+1}^{(c)}\right) = \sum_{m=q+\lfloor \bar{n}/2 \rfloor+1}^{q+\bar{n}} \mathrm{div} S_C^m$*, where S_C^m can be split into local and non-local errors as $S_C^m = S_C^{m,l} + S_C^{m,*}$.*

(ii) For the same range of m and for all $N, M \leq \mathsf{N}_{\text{fin}}/10$, the local parts $S_C^{m,l}$ satisfy

$$\left\| \psi_{i,q} D^N D_{t,q}^M S_C^{m,l} \right\|_{3/2} \lesssim \Gamma_m^{-9} \delta_{m+\bar{n}} \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^8\right) \tag{10.79a}$$

$$\left\| \psi_{i,q} D^N D_{t,q}^M S_C^{m,l} \right\|_{\infty} \lesssim \Gamma_m^{-9} \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^8\right) . \tag{10.79b}$$

(iii) For $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$ and $q + 1 \leq q' \leq m - 1$, the local parts satisfy

$$\operatorname{supp} S_C^{m,l} \cap B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1} \Gamma_{q'+1}\right) = \emptyset . \tag{10.80}$$

(iv) For the same range of m and $N, M \leq 2\mathsf{N}_{\text{ind}}$, the nonlocal parts $S_C^{m,*}$ satisfy

$$\left\| D^N D_{t,q}^M S_C^{m,*} \right\|_{\infty} \leq \mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\text{ind,t}}} \delta_{q+3\bar{n}} \lambda_m^N \tau_q^{-M} . \tag{10.81}$$

Remark 10.2.11 (Abstract formulation of the divergence corrector errors). For the purposes of analyzing the transport and Nash current errors in subsubsection 11.2.2 and streamlining the creation of pressure increments, it will again be useful to abstract the properties of these error terms. As we shall see in the course of the proofs of Lemma 10.2.10 and 10.2.12, these error terms may be decomposed and analyzed in *exactly* the same way as the oscillation errors with $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$ in Remark 10.2.2. This is not surprising, since both error terms are quadratic in w_{q+1}, and morally speaking, one expects the estimates for terms involving divergence correctors to be slightly better.

Proof of Lemma 10.2.10. The analysis in the proof generally follows that of the divergence corrector errors in [35], and we shall occasionally refer to algebraic identities from those arguments. The main difference is that we have to incorporate the synthetic Littlewood-Paley projector in certain terms before applying the inverse divergence operator in order to upgrade the material derivatives later. However, synthetic Littlewood-Paley projectors have already been applied to terms which are quadratic in high frequency objects in Lemma 10.2.1,

and so we may pirate a significant portion of the analysis from there as well.

Step 1: Analyze $\operatorname{div}\big(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)}\big)$.

We first write

$$\operatorname{div}\big(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)}\big)^{\bullet} = \sum_{\diamond,i,j,k,\xi,\vec{l},I} \partial_m \Bigg(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \varrho_{(\xi),\diamond}^{I} \right) \circ \Phi_{(i,k)} \xi^{\ell} \big(A_{\ell}^{m} \epsilon_{\bullet pr} + A_{\ell}^{\bullet} \epsilon_{mpr} \big) \\ \times \partial_p \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \right) \partial_r \Phi_{(i,k)}^{s} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \circ \Phi_{(i,k)} \Bigg), \tag{10.82}$$

where we have used Lemma 9.2.2, the definition of $\mathbb{W}_{(\xi),\diamond}^{I}$ in (7.9) (and the corresponding version for L^3 normalized pipes), $\epsilon_{i_1 i_2 i_3}$ is the Levi-Civita alternating tensor, we implicitly contract the repeated indices ℓ, m, p, r, s, and the $\bullet$ refers to the indices of the vectors on either side of the above display. Using that $\{\xi, \xi', \xi''\}$ is an orthonormal basis associated with the direction vector ξ with $\xi \times \xi' = \xi''$ and decomposing as in [35, (7.50)], we have that

$$\partial_p \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \right) = \underbrace{\partial_p \Phi_{(i,k)}^{n} \xi^{n} \xi^{\ell} A_{\ell}^{j} \partial_j \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \right)}_{=: a_{(\xi),\diamond}^{p,\mathrm{good}}} \tag{10.83}$$

$$+ \underbrace{\partial_p \Phi_{(i,k)}^{n} (\xi')^{n} (\xi')^{\ell} A_{\ell}^{j} \partial_j \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \right) + \partial_p \Phi_{(i,k)}^{n} (\xi'')^{n} (\xi'')^{\ell} A_{\ell}^{j} \partial_j \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \right) \circ \Phi_{(i,k)} \right)}_{=: a_{(\xi),\diamond}^{p,\mathrm{bad}}},$$

where we have also set $A = A_{(i,k)} = (\nabla \Phi_{(i,k)})^{-1}$. Indeed, the good differential operator appearing in $a_{(\xi),\diamond}^{p,\mathrm{good}}$ only costs $\Lambda_q \Gamma_q^{13}$ (see Lemma 9.3.1), so that we will leave $a_{(\xi),\diamond}^{p,\mathrm{good}}$ inside the divergence and dump the symmetric stress inside of the divergence into $S_C^{q+\bar{n}}$. On the other hand, $a_{(\xi),\diamond}^{p,\mathrm{bad}}$ contains an expensive derivative at $\lambda_{q+\lfloor \bar{n}/2 \rfloor}$, but $\xi^{\ell} A_{\ell}^{m} \partial_m$ only costs $\Lambda_q \Gamma_q^{13}$, which will be crucially used below.

Splitting the terms involved with $a^{p,\mathrm{bad}}_{(\xi),\diamond}$ from (10.82) as in [35, (7.52)], we further analyze

$$\sum_{\diamond,i,j,k,\xi,\vec{l},I} \partial_m \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \varrho^{I,\diamond}_{(\xi)} \right) \circ \Phi_{(i,k)} \xi^{\ell} \big(A^m_{\ell} \epsilon_{\bullet pr} + A^{\bullet}_{\ell} \epsilon_{mpr} \big) a^{p,\mathrm{bad}}_{(\xi),\diamond} \partial_r \Phi^s_{(i,k)} (\mathbb{U}^I_{(\xi),\diamond})^s \circ \Phi_{(i,k)} \right) = \mathbf{V}^{\bullet}_1 + \mathbf{V}^{\bullet}_2 \tag{10.84}$$

where $\mathbf{V}_1$ contains $A^m_{\ell} \epsilon_{\bullet pr}$, and $\mathbf{V}_2$ contains $A^{\bullet}_{\ell} \epsilon_{mpr}$. To analyze $\mathbf{V}_1$, we use that ∂_m and $\xi^{\ell} A^m_{\ell}$ commute, so that

$$\xi^{\ell} A^m_{\ell} \partial_m \left((\varrho^I_{(\xi),\diamond} (\mathbb{U}^I_{(\xi),\diamond})^s) \circ \Phi_{(i,k)} \right) = 0 \,.$$

Furthermore, the differential operator $\xi^{\ell} A^m_{\ell} \partial_m$ landing anywhere else costs only $\Lambda_q \Gamma^{13}_q$ from (9.36). Then we have in total that

$$\begin{aligned} \mathbf{V}^{\bullet}_1 &= \sum_{\diamond,i,j,k,\xi,\vec{l},I} \partial_m \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \right) \circ \Phi_{(i,k)} \xi^{\ell} A^m_{\ell} \epsilon_{\bullet pr} a^{p,\mathrm{bad}}_{(\xi),\diamond} \partial_r \Phi^s_{(i,k)} \right) \left(\varrho^I_{(\xi),\diamond} (\mathbb{U}^I_{(\xi),\diamond})^s \right) \circ \Phi_{(i,k)} \\ &=: \sum_{\diamond,i,j,k,\xi,\vec{l},I} (C^{1,I}_{(\xi),\diamond})^{\bullet s} \left(\varrho^I_{(\xi),\diamond} (\mathbb{U}^I_{(\xi),\diamond})^s \right) \circ \Phi_{(i,k)} \end{aligned} \tag{10.85}$$

is a product of a high-frequency, mean-zero potential which has gained one factor of $\lambda_{q+\bar{n}}$, and a low-frequency object which has lost one costly derivative at frequency $\lambda_{q+\lfloor \bar{n}/2 \rfloor}$, and

one cheap derivative at frequency $\Lambda_q \Gamma_q^{13}$. To analyze $\mathbf{V}_2$, we follow [35, 7.56] to get

$$
\begin{aligned}
\mathbf{V}_2^{\bullet} &= \sum_{\diamond,i,j,k,\xi,\vec{l},I} \partial_m \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \varrho_{(\xi)}^{I,\diamond} \right) \circ \Phi_{(i,k)} \xi^{\ell} A_{\ell}^{\bullet} \epsilon_{mpr} a_{(\xi),\diamond}^{p,\mathrm{bad}} \partial_r \Phi_{(i,k)}^{s} (\mathbb{U}_{(\xi)}^{I,\diamond})^{s} \circ \Phi_{(i,k)} \right) \\
&= \sum_{\diamond,i,j,k,\xi,\vec{l},I} \Big(\partial_m \big(\xi^{\ell} A_{\ell}^{\bullet} \epsilon_{mpr} \partial_r \Phi_{(i,k)}^{s} \big) a_{(\xi),\diamond} (\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond}) \circ \Phi_{(i,k)} a_{(\xi),\diamond}^{p,\mathrm{bad}} + a_{(\xi),\diamond}^{m,\mathrm{good}} \xi^{\ell} A_{\ell}^{\bullet} \epsilon_{mpr} a_{(\xi),\diamond}^{p,\mathrm{bad}} \partial_r \Phi_{(i,k)}^{s} \\
&\qquad\qquad - a_{(\xi),\diamond} (\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond}) \circ \Phi_{(i,k)} \xi^{\ell} A_{\ell}^{\bullet} \epsilon_{mpr} \partial_m (a_{(\xi),\diamond}^{p,\mathrm{good}}) \partial_r \Phi_{(i,k)}^{s} \Big) \left(\varrho_{(\xi)}^{I,\diamond} (\mathbb{U}_{(\xi)}^{I,\diamond})^{s} \right) \circ \Phi_{(i,k)} \\
&\quad + \sum_{\diamond,i,j,k,\xi,\vec{l},I} a_{(\xi),\diamond} \xi^{\ell} A_{\ell}^{\bullet} \epsilon_{mpr} a_{(\xi),\diamond}^{p,\mathrm{bad}} \partial_r \Phi_{(i,k)}^{s} \partial_m \left(\varrho_{(\xi)}^{I,\diamond} (\mathbb{U}_{(\xi)}^{I,\diamond})^{s} \right) \circ \Phi_{(i,k)} \,. \qquad (10.86) \\
&=: \sum_{\diamond,i,j,k,\xi,\vec{l},I} (C_{(\xi),\diamond}^{2,I})^{\bullet s} \left(\varrho_{(\xi),\diamond}^{I} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \right) \circ \Phi_{(i,k)}
\end{aligned}
$$

In the second equality above we have used the identities $\epsilon_{mpr} \partial_m (a_{(\xi),\diamond}^{p,\mathrm{bad}}) = -\epsilon_{mpr} \partial_m (a_{(\xi),\diamond}^{p,\mathrm{good}})$, which follows from (10.83), and $\epsilon_{mpr} a_{(\xi),\diamond}^{m,\mathrm{bad}} a_{(\xi),\diamond}^{p,\mathrm{bad}} = 0$. Furthermore, we recall from [35, pgs. 42-43] that the last term on the right-hand side of the second equality vanishes. As before, the slow function $C_{(\xi),\diamond}^{2,I}$ contains two spatial derivatives, one cheap and one expensive.

Step 2: Definition of S_C^m and their properties

Now, we define the stress error S_C^m from the divergence corrector. From (5) of Proposition 7.1.5 and (5) of Proposition 7.1.6, we know that $\varrho_{(\xi),\diamond}^{I} (\mathbb{U}_{(\xi),\diamond}^{I})^{s}$ has zero mean. As in the oscillation stress error, we decompose $\varrho_{(\xi),\diamond}^{I} (\mathbb{U}_{(\xi),\diamond}^{I})^{s}$, applying the synthetic Littlewood-Paley decomposition suggested in (7.34), and set for $q + \bar{n}/2 + 1 < m < q + \bar{n}$,

$$
S_C^{q+\bar{n}/2+1} := (\mathcal{H} + \mathcal{R}^*) \left[\sum_{\diamond,i,j,k,\xi,\vec{l},I} (C_{(\xi),\diamond}^{1,I} + C_{(\xi),\diamond}^{2,I})^{\bullet s} \widetilde{\mathbb{P}}_{q+\bar{n}/2+1} \left(\varrho_{(\xi),\diamond}^{I} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \right) \circ \Phi_{(i,k)} \right] \qquad (10.87a)
$$

$$
S_C^{m} := (\mathcal{H} + \mathcal{R}^*) \left[\sum_{\diamond,i,j,k,\xi,\vec{l},I} (C_{(\xi),\diamond}^{1,I} + C_{(\xi),\diamond}^{2,I})^{\bullet s} \widetilde{\mathbb{P}}_{(m-1,m]} \left(\varrho_{(\xi),\diamond}^{I} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \right) \circ \Phi_{(i,k)} \right] \qquad (10.87b)
$$

$$S_C^{q+\bar{n}} := w_{q+1}^{(c)} \otimes w_{q+1}^{(c)} \tag{10.87c}$$

$$+ \sum_{\diamond,i,j,k,\xi,\vec{l},I} a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \varrho_{(\xi),\diamond}^{I} \right) \circ \Phi_{(i,k)} \xi^{\ell} \left(A_{\ell}^{m} \epsilon_{\bullet pr} + A_{\ell}^{\bullet} \epsilon_{mpr} \right) a_{(\xi),\diamond}^{p,\text{good}} \partial_r \Phi_{(i,k)}^{s} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \circ \Phi_{(i,k)} \tag{10.87d}$$

$$+ \sum_{m=q+\bar{n}}^{q+\bar{n}+1} (\mathcal{H} + \mathcal{R}^*) \left[\sum_{\diamond,i,j,k,\xi,\vec{l},I} (C_{(\xi),\diamond}^{1,I} + C_{(\xi),\diamond}^{2,I})^{\bullet s} (\widetilde{\mathbb{P}}_{(m-1,m]} + \mathrm{Id} - \widetilde{\mathbb{P}}_{q+\bar{n}+1}) \left(\varrho_{(\xi),\diamond}^{I} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \right) \circ \Phi_{(i,k)} \right]. \tag{10.87e}$$

Here, the terms involved with the operators $\mathcal{R}^*$ or $\mathrm{Id} - \widetilde{\mathbb{P}}_{q+\bar{n}+1}$ will go into the nonlocal part and all the remaining terms will be included in the local parts.

The conclusions of Lemma 10.2.10 for the terms (10.87a), (10.87b), and the terms involving $\widetilde{\mathbb{P}}_{(m-1,m]}$ in (10.87e) follow similarly to `Case 3` from the proof of Lemma 10.2.1. Indeed, we fix indices $i, j, k, \xi, \vec{l}, I, s$, $\diamond = R$, and apply Proposition A.3.3 to

$$G_R^{\bullet} = \lambda_{q+\bar{n}}^{-1} (C_{(\xi),R}^{1,I} + C_{(\xi),R}^{2,I})^{\bullet s}, \quad \varrho_R = \begin{cases} \lambda_{q+\bar{n}} \widetilde{\mathbb{P}}_{q+\bar{n}/2+1} \left(\varrho_{(\xi),R}^{I} (\mathbb{U}_{(\xi),R}^{I})^{s} \right) & \text{for (10.87a)} \\ \lambda_{q+\bar{n}} \widetilde{\mathbb{P}}_{(m-1,m]} \left(\varrho_{(\xi),R}^{I} (\mathbb{U}_{(\xi),R}^{I})^{s} \right) & \text{for (10.87b), (10.87e),} \end{cases}$$

with the same choice of the rest of parameters as in `Case 3`. In the case of $\diamond = \varphi$, as in `Case 3`, G_φ and ϱ_φ will have extra $r_q^{2/3}$ and $r_q^{-2/3}$, respectively, with the replacement of R with φ in $C_{(\xi),R}^{1,I}$, $C_{(\xi),R}^{1,I}$, and $\varrho_{(\xi),R}^{I} (\mathbb{U}_{(\xi),R}^{I})^{s}$. The assumptions in (A.40) and (A.43) of Proposition A.3.3 can be verified using Lemma 9.3.1, Lemma 7.3.3, Lemma 7.3.4, item (6) from Proposition 7.1.5 and item (6) from Proposition 7.1.6.[2] The rest of the assumptions follow exactly as in `Case 3` from the proof of Lemma 10.2.1. We note now that the support of the low-frequency function G is the same as in the oscillation error due to the presence of $\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{\diamond}$ and their derivatives. In addition, the support of the high-frequency potentials is the same

[2]Note that we have traded $\lambda_{q+\bar{n}}$ between $G_R^{\bullet}$ and ρ_R so that the parameter choices are the same as the oscillation error. We also note that thanks to the extra gain $\lambda_{q+\bar{n}/2}/\lambda_{q+\bar{n}}$ in the estimate of G_R and G_φ compared with `Case 3`, all the error terms are actually small enough in amplitude to absorbed into the highest shell. The only reason to use the synthetic Littlewood-Paley decomposition here is to ensure that we can upgrade material derivatives via dodging later.

as in the oscillation error since $\mathbb{U}^{I}_{(\xi),\diamond}$ and $\varrho^{I}_{(\xi),\diamond}$ are both supported in a $2\lambda^{-1}_{q+\bar{n}}$ neighborhood of the pipe potential from (7.9) and item (7). Finally, to deal with the remaining term in (10.87e), we may use the same type of arguments as in `Case 4` in the proof of Lemma 10.2.1. For the sake of both the readers and authors, we omit these details.

Lastly, we consider (10.87c) and (10.87d), which are absorbed into $S^{q+\bar{n},l}_{C}$. From Lemma 9.2.2, we have that

$$w^{(c)}_{q+1} \otimes w^{(c)}_{q+1} = \sum_{\diamond,i,j,k,\xi,\vec{l},I} \left(\nabla \left(a_{(\xi),\diamond} (\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} \right) \times \left(\nabla \Phi^{T}_{(i,k)} \mathbb{U}^{I}_{(\xi),\diamond} \circ \Phi_{(i,k)} \right) \right) \otimes \left(\nabla \left(a_{(\xi),\diamond} (\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}) \circ \Phi_{(i,k)} \right) \times \left(\nabla \Phi^{T}_{(i,k)} \mathbb{U}^{I}_{(\xi),\diamond} \circ \Phi_{(i,k)} \right) \right) . \tag{10.88}$$

It follows immediately from estimate (9.44) with $r = 3, \infty$, (5.8) at level q, and Lemma 8.5.1 with $r_1 = \infty, r_2 = 1$ that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_{10}$,

$$\begin{aligned}
\left\| \psi_{i,q} D^{N} D^{M}_{t,q} \left(w^{(c)}_{q+1} \otimes w^{(c)}_{q+1} \right) \right\|_{\infty} &\lesssim \Gamma_{q}^{\mathsf{C}_{\infty}+9} \lambda^{N}_{q+\bar{n}} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q}^{-1} \Gamma_{q}^{i+15}, \mathrm{T}_{q}^{-1} \Gamma_{q}^{8} \right) \\
\left\| \psi_{i,q} D^{N} D^{M}_{t,q} \left(w^{(c)}_{q+1} \otimes w^{(c)}_{q+1} \right) \right\|^{3/2}_{3/2} &\lesssim r_{q}^{2} \sum_{\diamond,i,j,k,\xi,\vec{l},I} \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \right) \right| \delta^{3/2}_{q+\bar{n}} \Gamma_{q}^{3j+21} \lambda^{3N/2}_{q+\bar{n}} \\
&\qquad \times \left(\mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q}^{-1} \Gamma_{q}^{i+13}, \mathrm{T}_{q}^{-1} \Gamma_{q}^{8} \right) \right)^{3/2} \\
&\lesssim r_{q}^{2} \delta^{3/2}_{q+\bar{n}} \Gamma_{q}^{30} \lambda^{3N/2}_{q+\bar{n}} \left(\mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q}^{-1} \Gamma_{q}^{i+15}, \mathrm{T}_{q}^{-1} \Gamma_{q}^{8} \right) \right)^{3/2} .
\end{aligned}$$

The estimate for the L^{∞} norm matches (10.79b) for $m = q + \bar{n}$ after using (4.13a). For the $L^{3/2}$ estimate, taking cube roots and using the parameter inequality (4.10g) matches (10.79a) for $m = q + \bar{n}$. Finally, we have that the support of this error term is contained in w_{q+1}; then (10.80) is immediate from Lemma 9.2.2. On the other hand, one can observe that (10.87d) enjoys the exact same properties as $w^{(c)}_{q+1} \otimes w^{(c)}_{q+1}$, and hence we get the desired conclusion in a similar way.

□

Lemma 10.2.12 (Pressure increment). *For every $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$, there exists a function $\sigma_{S_C^m} = \sigma^+_{S_C^m} - \sigma^-_{S_C^m}$ such that the following hold.*

(i) We have that for all $N, M < \mathsf{N}_{\text{fin}}/100$ and $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n} - 1$,

$$\left|\psi_{i,q} D^N D^M_{t,q} S_C^{m,l}\right| < \left(\sigma^+_{S_C^m} + \delta_{q+3\bar{n}}\right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{10.89a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} S_C^{q+\bar{n},l}\right| < \left(\sigma^+_{S_C^{q+\bar{n}}} + \sigma^+_{\upsilon} + \delta_{q+3\bar{n}}\right) (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right), \tag{10.89b}$$

where σ^+_{υ} is defined as in (9.65). Furthermore, for any integer $q + \bar{n}/2 < m \leq q + \bar{n}$ and for all $N, M < \mathsf{N}_{\text{fin}}/100$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S_C^m}\right| < \left(\sigma^+_{S_C^m} + \delta_{q+3\bar{n}}\right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.90a}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{S_C^m}\right\|_{3/2} \leq \Gamma_m^{-9} \delta_{m+\bar{n}} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.90b}$$

$$\left\|D^N D^M_{t,q} \sigma^+_{S_C^m}\right\|_{\infty} \leq \Gamma_m^{\mathsf{C}_\infty - 9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.90c}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^-_{S_C^m}\right| < \Gamma_{q+\bar{n}/2}^{-100} \pi_q^{q+\bar{n}/2} \left(\lambda_{q+\bar{n}/2} \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right). \tag{10.90d}$$

(ii) For $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$, we have that

$$\begin{aligned} B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp}\left(\sigma^+_{S_C^m}\right) &= \emptyset \qquad \forall q+1 \leq q' \leq m-1 \\ B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'}\right) \cap \operatorname{supp}\left(\sigma^-_{S_C^m}\right) &= \emptyset \qquad \forall q+1 \leq q' \leq q + \bar{n}/2 \,. \end{aligned} \tag{10.91}$$

(iii) Define

$$\mathfrak{m}_{\sigma_{S_C^m}}(t) = \int_0^t \left\langle D_{t,q} \sigma_{S_C^m} \right\rangle (s)\, ds \,. \tag{10.92}$$

Then we have that for $0 \leq M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left| \frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{\sigma_{S_C^m}} \right| \leq (\max(1,T))^{-1} \delta_{q+3\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right) . \tag{10.93}$$

Lemma 10.2.13 (Pressure current). *For every* $q+{}^{\bar{n}}/_2 < m \leq q+\bar{n}$, *there exists a current error* $\phi_{S_C^m}$ *associated to the pressure increment* $\sigma_{S_C^m}$ *defined by Lemma 10.2.12 which satisfies the following properties.*

(i) We have the decompositions and equalities

$$\phi_{S_C^m} = \phi^*_{S_C^m} + \sum_{k=q+{}^{\bar{n}}/_2+1}^{m} \phi^k_{S_C^m} , \quad \phi^k_{S_C^m} = \phi^{k,l}_{S_C^m} + \phi^{k,*}_{S_C^m} \tag{10.94a}$$

$$\mathrm{div}\phi_{S_C^m} = D_{t,q}\sigma_{S_C^m} - \langle D_{t,q}\sigma_{S_C^m} \rangle . \tag{10.94b}$$

(ii) For $q + {}^{\bar{n}}/_2 + 1 \leq k \leq m$ *and* $N, M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left| \psi_{i,q} D^N D^M_{t,q} \phi^{k,l}_{S_C^m} \right| < \Gamma_k^{-100} r_k^{-1} \left(\pi_q^k\right)^{3/2} (\lambda_k \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.95a}$$

$$\left\| D^N D^M_{t,q} \phi^{k,*}_{S_C^m} \right\|_{L^\infty} \leq \delta^{3/2}_{q+3\bar{n}} \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \lambda_m^N \tau_q^{-M} . \tag{10.95b}$$

(iii) For all $q + {}^{\bar{n}}/_2 + 1 \leq k \leq m$ *and all* $q + 1 \leq q' \leq k - 1$,

$$B\left(\mathrm{supp}\, \widehat{w}_{q'}, {}^1/_2 \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap \mathrm{supp}\, \left(\phi^{k,l}_{S_C^m}\right) = \emptyset . \tag{10.96}$$

Proofs of Lemmas 10.2.12-10.2.13. `Case 0:` pressure for (10.87a), (10.87b), and (10.87e). The pressure increment and the current error associated to each piece in the local part of (10.87a), (10.87b), and (10.87e) can be constructed in the same way as in Lemma 10.2.4-10.2.5. Indeed, the proof relies on Proposition A.4.4, and (G_R, ϱ_R), $(G_\varphi, \varrho_\varphi)$ given in the proof of Lemma 10.2.10 have the exact same properties required in the proposition as the one given in `Case 3` of the proof of Lemma 10.2.1. In particular, the preliminary assumptions

(iv) holds with $\bar{\pi}$ given as in (10.28) due to (9.38). Therefore, we get the same conclusions by repeating the same arguments. In particular, all conclusions from Lemma 10.2.12–10.2.13 are obtained in the cases $m < q + \bar{n}$. Furthermore, when $m = q + \bar{n}$, we denote the pressure increment and the current error associated to (10.87e) by $\sigma_{(10.87e)} = \sigma^{+}_{(10.87e)} - \sigma^{-}_{(10.87e)}$ and $\phi^{k}_{(10.87e)} = \phi^{k,l}_{(10.87e)} + \phi^{k,*}_{(10.87e)}$, respectively. Since these error terms are defined using the same parameter choices as the oscillation error, we obtain estimates consistent with (10.90a)–(10.96) for these error terms. We note also that we obtain a version of (10.89b) which does not require the introduction of σ^{+}_{υ} on the right-hand side; later error terms will require σ^{+}_{υ}.

`Case 1:` (10.87c) needs no new pressure increment. From (9.70b), we have that

$$\left|\psi_{i,q} D^{N} D^{M}_{t,q}(10.87c)\right| \lesssim \Gamma_{q}^{-2}(\sigma^{+}_{\upsilon} + \delta_{q+3\bar{n}}) \left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}\right)^{N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q}^{-1}\Gamma_{q}^{i+16}, \mathrm{T}_{q}^{-1}\Gamma_{q}^{9}\right)$$

for $N, M \leq \mathsf{N}_{\text{fin}}/100$. This estimate is consistent with (10.89b), and since no pressure increment is created here, we need not check any of the conclusion in (10.90a)–(10.91).

`Case 2:` pressure for (10.87d). The general idea for this error term is that since it is given as a product of two slightly altered velocity increments, we can apply Lemma A.4.3 (which was used to construct pressure increments for velocity increments already in subsection 9.4) to construct pressure increments $\sigma^{\pm}_{(10.87d)}$ and current errors $\phi^{k}_{(10.87d)}$. So we fix the indices $i, j, k, \xi, \vec{l}, I, \diamond$ and apply Lemma A.4.3 to the functions $\widehat{\upsilon}_{b,\diamond} = \widehat{\upsilon}_{b,i,j,k,\xi,\vec{l},I,\diamond}$ defined by $\widehat{\upsilon}_{b,\diamond} =$

$G_{b\diamond}\rho_{b\diamond} \circ \Phi_{(i,k)}$, $b = 1, 2$, where

$$\widehat{\upsilon}_{1,\diamond} := r_q^{1/3}\lambda_q^{1/3}\lambda_{q+\bar{n}}^{-1/3} a_{(\xi),\diamond}\left(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\varrho^{I}_{(\xi),\diamond}\right)\circ\Phi_{(i,k)}$$

$$\widehat{\upsilon}_{2,\diamond} := r_q^{-1/3}\lambda_q^{-1/3}\lambda_{q+\bar{n}}^{1/3}\xi^{\ell}\big(A^m_{\ell}\epsilon_{\bullet pr} + A^{\bullet}_{\ell}\epsilon_{mpr}\big)a^{p,\mathrm{good}}_{(\xi),\diamond}\partial_r\Phi^s_{(i,k)}(\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}$$

$$G_{1R} := \lambda_q^{1/3}\lambda_{q+\bar{n}}^{-1/3}a_{(\xi),R}\left(\boldsymbol{\rho}^{R}_{(\xi)}\boldsymbol{\zeta}^{I,R}_{\xi}\right)\circ\Phi_{(i,k)},\quad \rho_{1R} := r_q^{1/3}\varrho^{I}_{(\xi),R}$$

$$G_{1\varphi} := r_q^{1/3}\lambda_q^{1/3}\lambda_{q+\bar{n}}^{-1/3}a_{(\xi),\varphi}\left(\boldsymbol{\rho}^{\varphi}_{(\xi)}\boldsymbol{\zeta}^{I,\varphi}_{\xi}\right)\circ\Phi_{(i,k)},\quad \rho_{1\varphi} := \varrho^{I}_{(\xi),\varphi}$$

$$G_{2R} := r_q^{-2/3}\lambda_q^{-1/3}\lambda_{q+\bar{n}}^{1/3}\lambda_{q+\bar{n}}^{-1}\xi^{\ell}\big(A^m_{\ell}\epsilon_{\bullet pr} + A^{\bullet}_{\ell}\epsilon_{mpr}\big)a^{p,\mathrm{good}}_{(\xi),R}\partial_r\Phi^s_{(i,k)},\quad \rho_{2R} := r_q^{1/3}\lambda_{q+\bar{n}}(\mathbb{U}^I_{(\xi),R})^s$$

$$G_{2\varphi} := r_q^{-1/3}\lambda_q^{-1/3}\lambda_{q+\bar{n}}^{1/3}\lambda_{q+\bar{n}}^{-1}\xi^{\ell}\big(A^m_{\ell}\epsilon_{\bullet pr} + A^{\bullet}_{\ell}\epsilon_{mpr}\big)a^{p,\mathrm{good}}_{(\xi),\varphi}\partial_r\Phi^s_{(i,k)},\quad \rho_{2\varphi} := \lambda_{q+\bar{n}}(\mathbb{U}^I_{(\xi),\varphi})^s\,.$$

We then set the following choices for the application of Lemma A.4.3:

$$N_* = M_* = \mathsf{N}_{\mathrm{fin}}/10,\quad M_t = \mathsf{N}_{\mathrm{ind,t}},\quad M_\circ = N_\circ = 2\mathsf{N}_{\mathrm{ind}},\quad K_\circ \text{ as in (xvi)},$$

$$\Phi = \Phi_{(i,k)},\quad v = \widehat{u}_q,\quad D_t = D_{t,q},\quad \lambda' = \Lambda_q,\quad \nu' = \mathrm{T}_q^{-1}\Gamma_8,\qquad \mathcal{C}_v = \Lambda_q^{1/2},\quad \Gamma = \Gamma_q^{1/10},$$

$$\mathcal{C}_{G,3} = \left|\mathrm{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{1/3}(\delta_{q+2\bar{n}}\Gamma^{-20}_{q+\bar{n}})^{1/2}\Gamma_q^j + \lambda^{-10}_{q+2\bar{n}},\quad \mathcal{C}_{G,\infty} = \Gamma^{\frac{\mathsf{C}_\infty}{2}-20}_{q+\bar{n}}r_q^{2/3},\quad \pi = \pi_\ell\Gamma_q^{30}r_q^{-2/3}\Lambda_q^{2/3}\lambda_{q+\bar{n}}^{-2/3},$$

$$\mathcal{C}_{\rho,3} := 1,\quad \mathcal{C}_{\rho,\infty} = r_q^{-2/3},\quad \lambda = \lambda_{q+\bar{n}/2},\quad \Lambda = \lambda_{q+\bar{n}},\quad \nu = \tau_q^{-1}\Gamma_q^{i+13},\quad r_G = r_{\widehat{\upsilon}} = 1,\quad \mu = \lambda_{q+\bar{n}}r_q$$

$$\delta_{\mathrm{tiny}} = \delta_{q+3\bar{n}},\quad \bar{m} = m + 1 - (q + \bar{n}/2),\quad \mu_0 = \lambda_{q+\bar{n}/2+1},\quad \mu_1 = \lambda_{q+\bar{n}/2+3/2},\quad \mu_k = \lambda_{q+\bar{n}/2+k}\,,$$

$\mathsf{N}_{\mathrm{cut,x}}$,$\mathsf{N}_{\mathrm{cut,t}}$ as in (xi), $\mathsf{N}_{\mathrm{dec}}$ as in (xv), d, N_{**} as in (xvii).

First, the verification of the assumptions from part 1 of Lemma A.4.3 can be done in a similar manner as in the proofs of Lemmas 9.4.4 and 9.4.6. We omit further details, but note that in this case, the intermittency parameters are chosen as 1 and G has extra factor $\lambda_q^{1/3}\lambda_{q+\bar{n}}^{-1/3}$ instead. From the definitions, the support properties of the low frequency functions $G_{b\diamond}$ and the high frequency functions $\rho_{b\diamond}$ are essentially the same as those of the corresponding functions in Lemmas 9.4.4 and 9.4.6.

As a consequence of (A.165), we have pressure increments associated to $\widehat{\upsilon}_{b,\diamond}$, $b = 1, 2$,

which satisfies

$$\left|D^N D^M_{t,q}\widehat{\upsilon}_{b,\diamond}\right| \lesssim (\sigma^+_{\widehat{\upsilon}_{b,\diamond}} + \delta_{q+3\bar{n}})^{1/2}(\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right)$$

for any $N, M \leq \mathsf{N}_{\text{fin}}/10$. This implies that

$$\left|D^N D^M_{t,q}(\widehat{\upsilon}_{1,\diamond}\widehat{\upsilon}_{2,\diamond})\right| \lesssim (\sigma^+_{\widehat{\upsilon}_{1,\diamond}} + \sigma^+_{\widehat{\upsilon}_{2,\diamond}} + \delta_{q+3\bar{n}})(\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right)$$

for any $N, M \leq \mathsf{N}_{\text{fin}}/10$. Then appealing to the same conclusions used in (9.77a)–(9.77f), we have that

$$\begin{aligned}
\left|D^N D^M_{t,q}\sigma^+_{\widehat{\upsilon}_b}\right| &\lesssim (\sigma^+_{\widehat{\upsilon}_b} + \delta_{q+3\bar{n}})(\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \\
\left\|D^N D^M_{t,q}\sigma^+_{\widehat{\upsilon}_b}\right\|_{3/2} &\lesssim \left[\left|\text{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{2/3}\delta_{q+2\bar{n}}\Gamma^{-20}_{q+\bar{n}}\Gamma_q^{2j} + \delta_{q+3\bar{n}}\right] \\
&\qquad \times (\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \\
\left\|D^N D^M_{t,q}\sigma^+_{\widehat{\upsilon}_b}\right\|_{\infty} &\lesssim \Gamma^{\mathsf{C}_\infty - 40}_{q+\bar{n}}(\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \\
\left|D^N D^M_{t,q}\sigma^-_{\widehat{\upsilon}_b}\right| &\lesssim \pi_\ell \Gamma_q^{41}\lambda_q^{1/3}\lambda_{q+\bar{n}}^{-1/3}(\lambda_{q+\bar{n}/2}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)
\end{aligned}$$

for all $N, M \leq \mathsf{N}_{\text{fin}}/100$. We reintroduce the indices $i, j, k, \xi, \vec{l}, I$ and define the pressure increment associated to (10.87d) by

$$\sigma^{\pm}_{(10.87\text{d})} := \sum_{i,j,k,\xi,\vec{l},I,b,\diamond} \sigma^{\pm}_{\widehat{\upsilon}_{b,i,j,k,\xi,\vec{l},I,\diamond}}.$$

The estimates (10.89a) and (10.90a) associated to (10.87d) follow using an aggregation procedure identical to that used in the proofs of Lemmas 9.4.4 and 9.4.6, and so we omit further details.

Lastly, we define $\phi^{k,l}_{(10.87\text{d})}$ and $\phi^{k,*}_{(10.87\text{d})}$ as in the proofs of Lemmas 9.4.4 and 9.4.6 and

obtain (10.95a), (10.95b), and (10.96) as in the cited Lemmas. Setting

$$\sigma^{\pm}_{S_C^{q+\bar{n}}} := \sigma^{\pm}_{(10.87e)} + \sigma^{\pm}_{(10.87d)}, \quad \phi^k_{S_C^{q+\bar{n},l}} := \phi^k_{(10.87e)} + \phi^k_{(10.87d)}$$

and collecting the properties of these objects obtained above, we conclude (10.90a)–(10.96) and (10.89b). □

10.2.4 Mollification error S_M

Recalling from subsection 10.1 that $\mathrm{div} S_{M2}$ has mean-zero, we first define the mollification error $S_M = S_{M1} + S_{M2}$ by

$$S_{M1} := R_q^q - R_\ell + \left(\pi_\ell - \pi_q^q\right) \mathrm{Id} =: S_M^{q+1,*} \tag{10.97}$$

$$S_{M2} := \mathcal{R}^* \left[(\partial_t + \widehat{u}_q \cdot \nabla)(\widehat{w}_{q+\bar{n}} - w_{q+1}) + (\widehat{w}_{q+\bar{n}} - w_{q+1}) \otimes \widehat{u}_q\right] + \widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1} =: S_M^{q+\bar{n},*} .$$

For the undefined mollification stress errors $S_M^{k,l}$, $S_M^{k,*}$, we set them as zero.

Lemma 10.2.14 (**Basic estimates and applying inverse divergence**). *The mollification error $S_M^{q+1,*}$ and $S_M^{q+\bar{n},*}$ satisfy*

$$\left\|D^N D_{t,q}^M S_M^{q+1,*}\right\|_\infty \leq \Gamma_{q+1}^9 \delta_{q+3\bar{n}} \mathrm{T}_{q+1}^{2\mathsf{N}_{\mathrm{ind,t}}} \left(\lambda_{q+1}\Gamma_{q+1}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \mathrm{T}_q^{-1}\right) . \tag{10.98a}$$

$$\left\|D^N D_{t,q+\bar{n}-1}^M S_M^{q+\bar{n},*}\right\|_\infty \leq \Gamma_{q+\bar{n}}^9 \delta_{q+3\bar{n}} \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}} \left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\right) . \tag{10.98b}$$

for all $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$.

Proof of Lemma 10.2.14. From (6.9), we have

$$\left\|D^N D_{t,q}^M S_M\right\|_\infty \lesssim \Gamma_{q+1} \mathrm{T}_{q+1}^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^2 \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \Gamma_q^{-1}\mathrm{T}_q^{-1}\right)$$

for all $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$, which immediately leads to (10.98a).

To deal with S_{M2}, we recall from (9.84) that

$$\left\|D^N D^M_{t,q+\bar{n}-1}\left(w_{q+1}-\widehat{w}_{q+\bar{n}}\right)\right\|_\infty \lesssim \delta^3_{q+3\bar{n}}\mathrm{T}^{25\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}\right)^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau^{-1}_{q+\bar{n}-1},\mathrm{T}^{-1}_{q+\bar{n}-1}\right) .$$

for all $N+M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_4$. Using Lemma 9.2.2, we note that $D_{t,q-\bar{n}-1}w_{q+1} = D_{t,q}w_{q+1}$ and $D_{t,q-\bar{n}-1}\widehat{w}_{q+\bar{n}} = D_{t,q}\widehat{w}_{q+\bar{n}}$. Then, writing $\widehat{w}_{q+\bar{n}}\otimes\widehat{w}_{q+\bar{n}} - w_{q+1}\otimes w_{q+1} = (\widehat{w}_{q+\bar{n}}-w_{q+1})\otimes\widehat{w}_{q+\bar{n}} + w_{q+1}\otimes(\widehat{w}_{q+\bar{n}}-w_{q+1})$ and using (9.83) and (9.87), we have

$$\begin{aligned}&\left\|\psi_{i,q+\bar{n}-1}D^N D^M_{t,q+\bar{n}-1}[\widehat{w}_{q+\bar{n}}\otimes\widehat{w}_{q+\bar{n}} - w_{q+1}\otimes w_{q+1}]\right\|_\infty\\ &\qquad\leq \delta_{q+3\bar{n}}\mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}\right)^N \mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau^{-1}_{q+\bar{n}-1},\mathrm{T}^{-1}_{q+\bar{n}-1}\right) ,\end{aligned} \tag{10.99}$$

for all $N+M \leq 2\mathsf{N}_{\mathrm{ind}}$.

As for the remaining term, we first upgrade the material derivative in the estimate for $\widehat{u}_q$. Applying Lemma A.5.1 to $F^l = 0$, $F^* = \widehat{u}_q$, $k = q+\bar{n}$, $N_\star = {}^{3\mathsf{N}_{\mathrm{fin}}}/_4$ with (5.35a), we get

$$\left\|D^N D^M_{t,q+\bar{n}-1}\widehat{u}_q\right\|_\infty \lesssim \mathrm{T}^{-1}_q\lambda^N_{q+\bar{n}}\mathrm{T}^{-M}_{q+\bar{n}-1}$$

Here, we used (4.15). Then, we use Remark A.3.5 with (4.15), setting

$$\begin{gathered}G = D_{t,q+\bar{n}-1}(\widehat{w}_{q+\bar{n}}-w_{q+1}) \quad (\text{or } G = (\widehat{w}_{q+\bar{n}}-w_{q+1})\otimes\widehat{u}_q), \quad v = \widehat{u}_{q+\bar{n}-1}\\ \mathcal{C}_{G,\infty} = \delta^3_{q+\bar{n}}\mathrm{T}^{20\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}, \quad \lambda = \lambda' = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}, \quad M_t = \mathsf{N}_{\mathrm{ind,t}}, \quad \nu = \nu' = \mathrm{T}^{-1}_{q+\bar{n}}, \quad \mathcal{C}_v = \Lambda^{1/2}_{q+\bar{n}-1}\\ N_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_9, \quad M_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_{10}, \quad N_\circ = M_\circ = 2\mathsf{N}_{\mathrm{ind}} .\end{gathered}$$

As a result, with a suitable choice of positive integer $K_\circ$ to have

$$\delta^3_{q+\bar{n}}\mathrm{T}^{20\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\lambda^5_{q+\bar{n}}2^{2\mathsf{N}_{\mathrm{ind}}} \leq \lambda^{-K_\circ}_{q+\bar{n}} \leq \delta_{q+3\bar{n}}\mathrm{T}^{10\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} ,$$

we get

$$\left\|D^N D^M_{t,q+\bar{n}-1}\mathcal{R}^*(D_{t,q}(\widehat{w}_{q+\bar{n}}-w_{q+1}))\right\|_\infty = \left\|D^N D^M_{t,q+\bar{n}-1}\mathcal{R}^*(D_{t,q+\bar{n}-1}(\widehat{w}_{q+\bar{n}}-w_{q+1}))\right\|_\infty \tag{10.100}$$

$$\lesssim \delta_{q+3\bar{n}}\mathrm{T}^{10\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N\mathrm{T}^{-M}_{q+\bar{n}} \tag{10.101}$$

$$\leq \delta_{q+3\bar{n}}\mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau^{-1}_{q+\bar{n}},\mathrm{T}^{-1}_{q+\bar{n}}\right), \tag{10.102}$$

for all $N+M\leq 2\mathsf{N}_{\mathrm{ind}}$. This completes the proof of (10.98b). □

10.3 Upgrading material derivatives and Dodging Hypothesis 5.4.4

Definition 10.3.1 (Definition of $\overline{R}_{q+1}$ and S^m_{q+1}). *Recalling Lemma 10.2.1, Lemma 10.2.6, Lemma 10.2.10, and Lemma 10.2.14, we define $S^m_{q+1} := S^{m,l}_{q+1}+S^{m,*}_{q+1}$ for all $q+1\leq m\leq q+\bar{n}$ by*

$$S^{m,l}_{q+1} := S^{m,l}_O + S^{m,l}_{TN} + S^{m,l}_C + S^{m,l}_M\,, \tag{10.103a}$$

$$S^{m,*}_{q+1} := S^{m,*}_O + S^{m,*}_{TN} + S^{m,*}_C + S^{m,*}_M\,. \tag{10.103b}$$

Here, any undefined terms are taken to be 0. We then define the primitive stress error $\overline{R}_{q+1}$ at $q+1$ step by

$$\overline{R}_{q+1} := \sum_{m=q+1}^{q+\bar{n}}\overline{R}^m_{q+1}\,,\qquad \overline{R}^m_{q+1} = R^m_q + S^m_{q+1}\,. \tag{10.104}$$

The local part $R^{m,l}_{q+1}$ and the non-local part $\overline{R}^{m,*}_{q+1}$ are defined by

$$R^{m,l}_{q+1} := R^{m,l}_{q} + S^{m,l}_{q+1}\,, \qquad \overline{R}^{m,*}_{q+1} := R^{m,*}_{q} + S^{m,*}_{q+1}\,. \tag{10.105}$$

We note that by the above definition, we have that

$$\overline{R}^{m}_{q+1} = R^{m,l}_{q+1} + \overline{R}^{m,*}_{q+1}\,. \tag{10.106}$$

We sometimes also use the notation $\overline{R}^{m,l}_{q+1}$ to denote $R^{m,l}_{q+1}$, since it will be shown later that the local portion of $\overline{R}^{m,l}_{q+1}$ remains unchanged throughout the rest of the analysis.

Lemma 10.3.2 (Upgrading material derivatives and verifying Hypothesis 5.4.4). *The new stress errors $S^{m}_{q+1} = S^{m,l}_{q+1} + S^{m,*}_{q+1}$ satisfy the following.*

(i) $R^{m,l}_{q+1}$ satisfies Hypothesis 5.4.4 with q replaced by $q+1$.

(ii) For $q+2 \le m \le q + \bar{n}/2$, the symmetric stresses $S^{m,l}_{q+1}$ obey the estimates

$$\left|\psi_{i,m-1} D^N D^M_{t,m-1} S^{m,l}_{q+1}\right| \lesssim \Gamma_m^{-50} \pi_q^m \Lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{m-1}^{i-5}\tau_{m-1}^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{10.107}$$

for $N, M \le \mathsf{N}_{\mathrm{fin}}/10$. For the same range of N, M, the symmetric stress $S^{q+1,l}_{q+1}$ obeys the estimates

$$\left|\psi_{i,q} D^N D^M_{t,q} S^{q+1,l}_{q+1}\right| \lesssim \Gamma_{q+1}^{-50} \pi_q^{q+1} \Lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q}^{i+19}\tau_{q}^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,. \tag{10.108}$$

(iii) For $q + \bar{n}/2 + 1 \le m \le q + \bar{n}$ and $N, M \le \mathsf{N}_{\mathrm{fin}}/100$, the symmetric stresses $S^{m,l}_{q+1}$ obey the estimates

$$\begin{aligned}\left|\psi_{i,m-1} D^N D^M_{t,m-1} S^{m,l}_{q+1}\right| \lesssim{}& \left(\sigma^+_{S^m_O} + \sigma^+_{S^{m,l}_C} + \mathbf{1}_{\{m=q+\bar{n}\}}\left(\sigma^+_{S_{TN}} + \sigma^+_{\upsilon}\right) + \delta_{q+3\bar{n}}\right) \\ &\times (\lambda_m \Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{m-1}^{i-5}\tau_{m-1}^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,.\end{aligned} \tag{10.109a}$$

(iv) For all $q+1 \leq m \leq q+\bar{n}$ and $N+M \leq 2\mathsf{N}_{\mathrm{ind}}$, the symmetric stresses $S_{q+1}^{m,*}$

$$\left\| D^N D_{t,m-1}^M S_{q+1}^{m,*} \right\|_{L^\infty} \leq \Gamma_{q+1}^2 \mathrm{T}_{q+1}^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^2 \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{m-1}^{-1}, \mathrm{T}_{m-1}^{-1}\right) . \tag{10.110}$$

Proof of Lemma 10.3.2. In order to prove the claim in item (i), note that for the portion of $R_{q+1}^{m,l}$ coming from $R_q^{m,l}$ (c.f. (10.104)), the claim follows by the inductive hypothesis itself. For the portion coming from $S_{q+1}^{m,l}$, we may appeal to (10.103) and (10.10), (10.53a), and (10.80).

Next, we may prove (10.108) directly from (10.27a), since from Lemma 10.2.6 and Lemma 10.2.10, the transport, Nash, and divergence corrector errors do not contribute to $S_{q+1}^{q+1,l}$. In order to prove (10.107), we note that from Lemma 10.2.6 and Lemma 10.2.10, the transport, Nash, and divergence corrector errors do not contribute to $S_{q+1}^{m,l}$ for $q+2 \leq m \leq q+\bar{n}/2$. Then from Lemmas 10.2.1 and 10.2.3, we need only consider the case $m = q+\bar{n}/2$, for which we have that for $N, M \leq \mathsf{N}_{\mathrm{fin}}/10$,

$$\begin{aligned}
\left| \psi_{i,m-1} D^N D_{t,m-1}^M S_{q+1}^{m,l} \right| &\underset{(5.8)}{=} \left| \psi_{i,m-1} \sum_{i'} \psi_{i',q}^6 D^N D_{t,m-1}^M S_{q+1}^{m,l} \right| \\
&\underset{(10.10)}{\lesssim} \sum_{i' : \psi_{i',q} \psi_{i,m-1} \neq 0} \left| \psi_{i',q} D^N D_{t,q}^M S_{q+1}^{m,l} \right| \\
&\underset{(10.27\mathrm{b}),(5.14)}{\lesssim} \Gamma_m^{-100} \pi_q^m \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{m-1}^{-1} \Gamma_{m-1}^{i-5}, \mathrm{T}_q^{-1} \Gamma_q^9\right) .
\end{aligned} \tag{10.111}$$

In order to prove (10.109a), we utilize a very similar argument to the one used to produce (10.111). The only difference is that instead of appealing to (10.27b), we appeal to (10.29a), (10.70a), (10.89a), and (10.89b). We omit further details.

Finally, we must prove (10.110). The proof is however very simliar to (6.9), and so we omit further details.

□

Chapter 11

Error estimates for the relaxed local energy inequality

11.1 Defining new current error terms

We will define $\overline{\phi}_{q+1}$ by adding $\widehat{w}_{q+\bar{n}}$ to u_q on the left-hand side of (5.3) and collecting new errors generated by the addition. Recall that in (10.3) we added $\widehat{w}_{q+\bar{n}}$ to the Euler-Reynolds system and obtained the equation

$$\partial_t u_{q+1} + \operatorname{div}(u_{q+1} \otimes u_{q+1}) + \nabla p_q = \operatorname{div}\left(\overline{R}_{q+1} - (\pi_q - \pi_q^q)\mathrm{Id}\right) \tag{11.1}$$

for $u_{q+1} = u_q + \widehat{w}_{q+\bar{n}}$, where

$$\overline{R}_{q+1} = R_q - R_q^q + S_{q+1} \tag{11.2a}$$

$$\operatorname{div} S_{q+1} = \partial_t \widehat{w}_{q+\bar{n}} + u_q \cdot \nabla \widehat{w}_{q+\bar{n}} + \widehat{w}_{q+\bar{n}} \cdot \nabla u_q + \operatorname{div}\left(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} + R_q^q - \pi_q^q \mathrm{Id}\right). \tag{11.2b}$$

We recall that $\kappa_q = {}^1\!/\!_2 \mathrm{tr}\,(R_q - \pi_q \mathrm{Id})$, $\kappa_q^q = {}^1\!/\!_2 \mathrm{tr}\,(R_q^q - \pi_q^q \mathrm{Id})$ and set[1]

$$\overline{\kappa}_{q+1} := \frac{1}{2}\mathrm{tr}\,(\overline{R}_{q+1} - (\pi_q - \pi_q^q)\mathrm{Id}) = \kappa_q - \kappa_q^q + \frac{1}{2}\mathrm{tr}\,(S_{q+1}) = \kappa_q - \kappa_\ell + \frac{1}{2}\mathrm{tr}\,(S_{q+1} - S_{M1})\,, \tag{11.3a}$$

$$\overline{\varphi}_{q+1} = \varphi_q - \varphi_q^q + \overline{\phi}_{q+1}\,, \tag{11.3b}$$

where $\mathrm{div}\overline{\phi}_{q+1}$ will include the new errors. We now introduce a function of time $\mathfrak{m}_{\overline{\phi}_{q+1}}(t)$ to account for the fact that the new errors may not have zero mean for each time and expect to obtain

$$\begin{aligned}\partial_t \left(\frac{1}{2}|u_{q+1}|^2\right) &+ \mathrm{div}\left(\left(\frac{1}{2}|u_{q+1}|^2 + p_q\right) u_{q+1}\right) \\ &= (\partial_t + \widehat{u}_{q+1}\cdot\nabla)\overline{\kappa}_{q+1} + \mathrm{div}\left((\overline{R}_{q+1} - \pi_q\mathrm{Id} + \pi_q^q\mathrm{Id})\widehat{u}_{q+1}\right) + \mathrm{div}\overline{\varphi}_{q+1} + \mathfrak{m}_{\overline{\phi}_{q+1}} - E\,.\end{aligned} \tag{11.4}$$

Towards this end, we first note that since $\mathrm{div}\,\widehat{u}_q = 0$, we have

$$\begin{aligned}&\mathrm{div}\,((R_q - \pi_q\mathrm{Id})\widehat{u}_q) + \widehat{u}_q\cdot\mathrm{div}\left(\overline{R}_{q+1} - R_q + \pi_q^q\mathrm{Id}\right) \\ &\qquad = \mathrm{div}\left((\overline{R}_{q+1} - \pi_q\mathrm{Id} + \pi_q^q\mathrm{Id})\widehat{u}_q\right) + \nabla\widehat{u}_q : \left(R_q - \pi_q^q\mathrm{Id} - \overline{R}_{q+1}\right)\,.\end{aligned} \tag{11.5}$$

[1] We are using the definition of S_{M1} from (10.1) to achieve the third equality.

We now add and subtract $\operatorname{div}\left((\overline{R}_{q+1} - \pi_q \mathrm{Id} + \pi_q^q \mathrm{Id})\widehat{u}_{q+1}\right)$ in the second to last equality below to obtain

$$
\begin{aligned}
&\partial_t \left(\frac{1}{2}|u_q + \widehat{w}_{q+\bar{n}}|^2\right) + \operatorname{div}\left(\left(\frac{1}{2}|u_q + \widehat{w}_{q+\bar{n}}|^2 + p_q\right)(u_q + \widehat{w}_{q+\bar{n}})\right) \\
&\underset{(5.3)}{=} (\partial_t + \widehat{u}_q \cdot \nabla)\kappa_q + \operatorname{div}((R_q - \pi_q \mathrm{Id})\widehat{u}_q) + \operatorname{div}\varphi_q + (\partial_t + u_q \cdot \nabla)\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2\right) + \operatorname{div}\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}}\right) \\
&\qquad + \widehat{w}_{q+\bar{n}} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) + \nabla u_q : \widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} \\
&\qquad + u_q \cdot (\partial_t \widehat{w}_{q+\bar{n}} + (u_q \cdot \nabla)\widehat{w}_{q+\bar{n}} + (\widehat{w}_{q+\bar{n}} \cdot \nabla)u_q + \operatorname{div}(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}})) - E \\
&\underset{(9.24)}{=} (\partial_t + \widehat{u}_q \cdot \nabla)\kappa_q + \operatorname{div}((R_q - \pi_q \mathrm{Id})\widehat{u}_q) + \operatorname{div}\varphi_q + (\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2\right) + \operatorname{div}\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}}\right) \\
&\qquad + \widehat{w}_{q+\bar{n}} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) + \nabla \widehat{u}_q : \widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} \\
&\qquad + \widehat{u}_q \cdot (\partial_t \widehat{w}_{q+\bar{n}} + (u_q \cdot \nabla)\widehat{w}_{q+\bar{n}} + (\widehat{w}_{q+\bar{n}} \cdot \nabla)u_q + \operatorname{div}(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}})) - E \\
&\underset{\substack{(11.2a),\\(11.2b)}}{=} (\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 + \kappa_q\right) + \operatorname{div}((R_q - \pi_q \mathrm{Id})\widehat{u}_q) + \operatorname{div}\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}} + \varphi_q\right) \\
&\qquad + \widehat{w}_{q+\bar{n}} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) + \nabla \widehat{u}_q : \widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} + \widehat{u}_q \cdot \operatorname{div}(\overline{R}_{q+1} - R_q + \pi_q^q \mathrm{Id}) - E \\
&\underset{(11.5)}{=} (\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 + \kappa_q\right) + \operatorname{div}\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}} + \varphi_q\right) + \operatorname{div}((\overline{R}_{q+1} - \pi_q \mathrm{Id} + \pi_q^q \mathrm{Id})\widehat{u}_{q+1}) \\
&\qquad + \widehat{w}_{q+\bar{n}} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) - \operatorname{div}\left((\overline{R}_{q+1} - \pi_q \mathrm{Id} + \pi_q^q \mathrm{Id})(\widehat{u}_{q+1} - \widehat{u}_q)\right) \\
&\qquad + \nabla \widehat{u}_q : (\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} + R_q - \pi_q^q \mathrm{Id} - \overline{R}_{q+1}) - E \\
&= (\partial_t + \widehat{u}_{q+1} \cdot \nabla)\overline{\kappa}_{q+1} + \operatorname{div}\left((\overline{R}_{q+1} - \pi_q \mathrm{Id} + \pi_q^q \mathrm{Id})\widehat{u}_{q+1}\right) + \operatorname{div}\overline{\varphi}_{q+1} + \mathfrak{m}_{\overline{\phi}_{q+1}} - E\,. \qquad (11.6)
\end{aligned}
$$

We see that the final quantity on the right-hand side of (11.6) will hold provided that

$$
\begin{aligned}
&\operatorname{div}\overline{\phi}_{q+1} + \mathfrak{m}'_{\overline{\phi}_{q+1}} \\
&\quad \underset{(11.3b)}{=} (\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 + \kappa_q\right) - (\partial_t + \widehat{u}_{q+1} \cdot \nabla)\overline{\kappa}_{q+1} \\
&\qquad - \operatorname{div}\left((\overline{R}_{q+1} - \pi_q \mathrm{Id} + \pi_q^q \mathrm{Id})(\widehat{u}_{q+1} - \widehat{u}_q)\right) + \widehat{w}_{q+\bar{n}} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) \\
&\qquad + \nabla \widehat{u}_q : (\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} + R_q - \pi_q^q \mathrm{Id} - \overline{R}_{q+1}) + \operatorname{div}\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}} + \varphi_q^q\right) \\
&\quad \underset{(11.3a)}{=} (\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 + \kappa_q^q - \frac{1}{2}\operatorname{tr}(S_{q+1})\right) + (\partial_t + \widehat{u}_q \cdot \nabla)\overline{\kappa}_{q+1} - (\partial_t + \widehat{u}_{q+1} \cdot \nabla)\overline{\kappa}_{q+1} \\
&\qquad - \operatorname{div}\left((\overline{R}_{q+1} - \pi_q \mathrm{Id} + \pi_q^q \mathrm{Id})(\widehat{u}_{q+1} - \widehat{u}_q)\right) + \widehat{w}_{q+\bar{n}} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) \\
&\qquad + \nabla \widehat{u}_q : (\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} + R_q - \pi_q^q \mathrm{Id} - \overline{R}_{q+1}) + \operatorname{div}\left(\frac{1}{2}|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}} + \varphi_q^q\right) \\
&\quad \underset{(11.3a)}{=} \underbrace{(\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|w_{q+1}|^2 + \kappa_q^q - \frac{1}{2}\operatorname{tr}(S_{q+1})\right)}_{=:\operatorname{div}\overline{\phi}_T + \mathfrak{m}'_T} \\
&\qquad - \underbrace{\operatorname{div}((\widehat{u}_{q+1} - \widehat{u}_q)\overline{\kappa}_{q+1}) - \operatorname{div}\left(\left(\overline{R}_{q+1} - (\pi_q - \pi_q^q)\mathrm{Id}\right)(\widehat{u}_{q+1} - \widehat{u}_q)\right)}_{=:\operatorname{div}\overline{\phi}_R} \\
&\qquad + \underbrace{w_{q+1} \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q)}_{=:\operatorname{div}\overline{\phi}_L + \mathfrak{m}'_L} + \underbrace{\nabla \widehat{u}_q : (w_{q+1} \otimes w_{q+1} + R_q - \pi_q^q \mathrm{Id} - \overline{R}_{q+1})}_{=:\operatorname{div}\overline{\phi}_N + \mathfrak{m}'_N}, \\
&\qquad + \underbrace{\operatorname{div}\left(\frac{1}{2}|w_{q+1}^{(p)}|^2 w_{q+1}^{(p)} + \varphi_\ell\right)}_{=:\operatorname{div}\overline{\phi}_O} + \underbrace{\operatorname{div}\left(\frac{1}{2}|w_{q+1}^{(c)}|^2 w_{q+1}^{(p)} + (w_{q+1}^{(c)} \cdot w_{q+1}^{(p)})w_{q+1}^{(p)} + \frac{1}{2}|w_{q+1}|^2 w_{q+1}^{(c)}\right)}_{=:\operatorname{div}\overline{\phi}_C} \\
&\qquad + \underbrace{\operatorname{div}(\varphi_q^q - \varphi_\ell) + (\partial_t + \widehat{u}_q \cdot \nabla)\frac{1}{2}\left(|\widehat{w}_{q+\bar{n}}|^2 - |w_{q+1}|^2\right) + \frac{1}{2}\operatorname{div}\left(|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}} - |w_{q+1}|^2 w_{q+1}\right)}_{=:\operatorname{div}\overline{\phi}_{M1} + \mathfrak{m}'_{M1}} \\
&\qquad + \underbrace{(\widehat{w}_{q+\bar{n}} - w_{q+1}) \cdot (\partial_t u_q + (u_q \cdot \nabla)u_q + \nabla p_q) + \nabla \widehat{u}_q : (\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1})}_{=:\operatorname{div}\overline{\phi}_{M2} + \mathfrak{m}'_{M2}},
\end{aligned}
$$

where $\mathfrak{m}_T, \mathfrak{m}_N, \mathfrak{m}_L, \mathfrak{m}_{M1}, \mathfrak{m}_{M2}$ are functions of time only and are given by

$$\mathfrak{m}_T(t) := \int_0^t \left\langle (\partial_t + \widehat{u}_q \cdot \nabla) \left(\frac{1}{2}|w_{q+1}|^2 + \kappa_\ell - \frac{1}{2}\mathrm{tr}\,(S_{q+1}) \right) \right\rangle (s)\, ds \tag{11.7a}$$

$$\mathfrak{m}_N(t) := \int_0^t \langle \nabla \widehat{u}_q : (w_{q+1} \otimes w_{q+1} + R_q - R_{q+1}) \rangle\, (s)\, ds \tag{11.7b}$$

$$\mathfrak{m}_L(t) := \int_0^t \langle w_{q+1} \cdot (\partial_t \widehat{u}_q + (\widehat{u}_q \cdot \nabla)\widehat{u}_q + \nabla p_q) \rangle\, (s)\, ds \tag{11.7c}$$

$$\mathfrak{m}_{M1}(t) := \int_0^t \left\langle (\partial_t + \widehat{u}_q \cdot \nabla) \frac{1}{2} \left(|\widehat{w}_{q+\bar{n}}|^2 - |w_{q+1}|^2 \right) \right\rangle (s)\, ds \tag{11.7d}$$

$$\begin{aligned} \mathfrak{m}_{M2}(t) := &\int_0^t \langle (\widehat{w}_{q+\bar{n}} - w_{q+1}) \cdot (\partial_t u_q + (u_q \cdot \nabla) u_q + \nabla p_q) \rangle\, (s)\, ds \\ &+ \int_0^t \langle \nabla \widehat{u}_q : (\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1}) \rangle\, (s)\, ds \end{aligned} \tag{11.7e}$$

and $\mathfrak{m}_{\overline{\phi}_{q+1}} := \mathfrak{m}_T + \mathfrak{m}_N + \mathfrak{m}_L + \mathfrak{m}_{M1} + \mathfrak{m}_{M2}$.

With these definitions in hand, we can rewrite (11.6) for $(u_{q+1}, p_q, \overline{R}_{q+1}, \overline{\varphi}_{q+1}, -(\pi_q - \pi_q^q))$ as

$$\begin{aligned} &\partial_t \left(\frac{1}{2}|u_{q+1}|^2 \right) + \mathrm{div} \left(\left(\frac{1}{2}|u_{q+1}|^2 + p_q \right) u_{q+1} \right) \\ &\quad = (\partial_t + \widehat{u}_{q+1} \cdot \nabla)(\overline{\kappa}_{q+1} + \mathfrak{m}_{\overline{\phi}_{q+1}}) + \mathrm{div}\left((\overline{R}_{q+1} - (\pi_q - \pi_q^q)\mathrm{Id})\widehat{u}_{q+1} \right) + \mathrm{div}\overline{\varphi}_{q+1} - E\,. \end{aligned} \tag{11.8}$$

The primitive current error $\overline{\phi}_{q+1}$ will consist of $\overline{\phi}_{q+1}^k$, so that $\overline{\phi}_{q+1} = \sum_{q+1}^{q+\bar{n}} \overline{\phi}_{q+1}^k$.

11.2 Error estimates

11.2.1 Oscillation current error

Recalling the definition of $\mathbb{B}_{(\xi),\varphi}$ and $\mathbb{B}_{(\xi),R}$ from Definition 7.2.6, we have

$$(\mathbb{B}_a\mathbb{B}_b\mathbb{B}_b)_{(\xi),\varphi} = \boldsymbol{\rho}^3_{(\xi),\varphi}\sum_I(\boldsymbol{\zeta}^I_\xi)^6\mathbb{P}_{\neq 0}\left[(\mathbb{W}^I_{(\xi),\varphi})_a(\mathbb{W}^I_{(\xi),\varphi})_b(\mathbb{W}^I_{(\xi),\varphi})_b\right] + \mathbb{P}_{\neq 0}\boldsymbol{\rho}^3_{(\xi),\varphi}\xi_a\xi_b\xi_b + \xi_a\xi_b\xi_b$$

$$(\mathbb{B}_a\mathbb{B}_b\mathbb{B}_b)_{(\xi),R} = \boldsymbol{\rho}^3_{(\xi),R}\sum_I(\boldsymbol{\zeta}^I_\xi)^9\mathbb{P}_{\neq 0}\left[(\mathbb{W}^I_{(\xi),R})_a(\mathbb{W}^I_{(\xi),R})_b(\mathbb{W}^I_{(\xi),R})_b\right]$$

where we used $\langle|\mathbb{W}^I_{(\xi),\varphi}|^2\mathbb{W}^I_{(\xi),\varphi}\rangle = |\xi|^2\xi$ from Proposition 7.1.6 item (5), $\langle\boldsymbol{\rho}^3_{(\xi),\varphi}\rangle = 1$ from Proposition 7.2.1 item (iii), $\sum_I(\boldsymbol{\zeta}^I_\xi)^6 = 1$ from (7.27), and $\langle|\mathbb{W}^I_{(\xi),R}|^2\mathbb{W}^I_{(\xi),R}\rangle = 0$ from Proposition 7.1.5 item (5). Using that all cross-terms from $|w^{(p)}_{q+1}|^2w^{(p)}_{q+1}$ (as defined in (9.5) and (9.13)) vanish due to Lemma 9.2.2 item (iii), and the fact that $(\nabla\Phi^{-1}_{(i,k)}\xi)\cdot\nabla$ gives zero when applied to $(\mathbb{W}^I_{(\xi),\diamond}\boldsymbol{\rho}_{(\xi),\diamond}\boldsymbol{\zeta}^I_\xi)\circ\Phi_{(i,k)}$, it then follows that

$$\begin{aligned}
&\frac{1}{2}\mathrm{div}\left(|w^{(p)}_{q+1}|^2w^{(p)}_{q+1}\right)\\
&\quad= \frac{1}{2}\mathrm{div}\left[\sum_{i,j,k,\xi,\vec{l},I,\diamond} a^3_{(\xi),\diamond}\left(\boldsymbol{\rho}^3_{(\xi),\diamond}(\boldsymbol{\zeta}^I_\xi)^{3\diamond}\right)\circ\Phi_{(i,k)}\left|\nabla\Phi^{-1}_{(i,k)}\mathbb{W}^I_{(\xi),\diamond}\circ\Phi_{(i,k)}\right|^2\nabla\Phi^{-1}_{(i,k)}\mathbb{W}^I_{(\xi),\diamond}\circ\Phi_{(i,k)}\right]\\
&\quad= \frac{1}{2}\sum_{\xi,i,j,k,\vec{l}}\mathrm{div}\left(a^3_{(\xi),\varphi}|\nabla\Phi^{-1}_{(i,k)}\xi|^2\nabla\Phi^{-1}_{(i,k)}\xi\right) + \sum_{\xi,i,j,k,\vec{l}} b_{(\xi),\varphi}\mathbb{P}_{\neq 0}\left(\boldsymbol{\rho}^3_{(\xi),\varphi}\right)(\Phi_{(i,k)})\\
&\qquad\quad + \sum_{\xi,i,j,k,\vec{l},I,\diamond} b_{(\xi),\diamond}\left(\boldsymbol{\rho}^3_{(\xi),\diamond}(\boldsymbol{\zeta}^I_\xi)^{3\diamond}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\diamond})^3\right)(\Phi_{(i,k)}),
\end{aligned}\tag{11.9}$$

where the γ component of $b_{(\xi),\diamond}$ is given by $b^\gamma_{(\xi),\diamond} = {}^1\!/\!_2\xi_a\partial_\gamma\left(a^3_{(\xi),\diamond}|\nabla\Phi^{-1}_{(i,k)}\xi|^2(\nabla\Phi^{-1}_{(i,k)})^\gamma_a\right)$, $\varrho_{(\xi),\diamond}$ is the pipe density defined in (10.7), and we are using the notation $3\diamond$ in the power of $\boldsymbol{\zeta}^I_\xi$ as a stand-in for 6 or 9 in the current and Reynolds cases, respectively.

By the choice of $a_{(\xi),\varphi}$, the first term cancels out φ_ℓ up to a higher-frequency error term. Indeed, using (9.2), Proposition 7.1.2, (8.36c) to yield a formula for the summation of

$\zeta^3_{q,\varphi,i,k,\xi,\vec{l}}$, (8.32), (5.8) at level q, (8.1) at level q, and (8.21), we have that

$$
\begin{aligned}
&\frac{1}{2}\sum_{\xi,i,j,k,\vec{l}} a^3_{(\xi),\varphi}|\nabla\Phi^{-1}_{(i,k)}\xi|^2(\nabla\Phi^{-1}_{(i,k)})^\gamma_a\xi_a \\
&\quad= \frac{1}{2}\delta^{3/2}_{q+\bar{n}}r_q^{-1}\sum_{\xi,i,j,k,\vec{l}}\Gamma_q^{3(j-1)}\psi^6_{i,q}\omega^6_{j,q}\chi^6_{i,k,q}\zeta^3_{q,\varphi,i,k,\xi,\vec{l}}\widetilde{\gamma}^3_\xi\left(\frac{\varphi_{q,i,k}}{\delta^{3/2}_{q+\bar{n}}r_q^{-1}\Gamma_q^{3(j-1)}}\right)(\nabla\Phi^{-1}_{(i,k)})^\gamma_a\xi_a \\
&\quad= \frac{1}{2}\delta^{3/2}_{q+\bar{n}}r_q^{-1}\sum_{\xi,i,j,k,l^\perp}\Gamma_q^{3(j-1)}\psi^6_{i,q}\omega^6_{j,q}\chi^6_{i,k,q}\mathcal{X}^3_{q,\xi,l^\perp}\circ\Phi_{(i,k)}\widetilde{\gamma}^3_\xi\left(\frac{\varphi_{q,i,k}}{\delta^{3/2}_{q+\bar{n}}r_q^{-1}\Gamma_q^{3(j-1)}}\right)(\nabla\Phi^{-1}_{(i,k)})^\gamma_a\xi_a \\
&\quad= \sum_{\xi,i,j,k}\psi^6_{i,q}\omega^6_{j,q}\chi^6_{i,k,q}c_3\left(-\frac{1}{c_3}\varphi^\gamma_\ell\right) \\
&\qquad+\frac{1}{2}\sum_{\xi,i,j,k,l^\perp}\underbrace{\delta^{3/2}_{q+\bar{n}}r_q^{-1}\Gamma_q^{3(j-1)}\psi^6_{i,q}\omega^6_{j,q}\chi^6_{i,k,q}\widetilde{\gamma}^3_\xi\left(\frac{\varphi_{q,i,k}}{\delta^{3/2}_{q+\bar{n}}r_q^{-1}\Gamma_q^{3(j-1)}}\right)(\nabla\Phi^{-1}_{(i,k)})^\gamma_a\xi_a}_{=:2\widetilde{b}^\gamma_{(\xi)}}\left(\mathbb{P}_{\neq 0}\mathcal{X}^3_{q,\xi,l^\perp}\right)\circ\Phi_{(i,k)} \\
&\quad= -(\varphi_\ell)^\gamma+\sum_{\xi,i,j,k,l^\perp}\widetilde{b}^\gamma_{(\xi)}\left(\mathbb{P}_{\neq 0}\mathcal{X}^3_{q,\xi,l^\perp}\right)\left(\Phi_{(i,k)}\right). \qquad (11.10)
\end{aligned}
$$

The inverse divergence of the remaining terms will form new current errors. We first recall the synthetic Littlewood-Paley decomposition (cf. Section 7.3). Since $\varrho^I_{(\xi),\diamond}$ is defined on the plane $\xi^\perp$ and is periodized to scale $(\lambda_{q+\bar{n}}r_q)^{-1}=(\lambda_{q+\lfloor\bar{n}/2\rfloor}\Gamma_q)^{-1}$ from (9.4), (9.12), and Propositions 7.1.5, 7.1.6, we can decompose $\mathbb{P}_{\neq 0}$ in front of $(\varrho^I_{(\xi),\diamond})^3$ into

$$
\mathbb{P}_{\neq 0}=\widetilde{\mathbb{P}}^\xi_{\lambda_{q+\lfloor n/2\rfloor+1}}\mathbb{P}_{\neq 0}+\sum_{m=q+\lfloor n/2\rfloor+2}^{q+\bar{n}+1}\widetilde{\mathbb{P}}^\xi_{(m-1,m]}+(1-\widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1}).
$$

Assuming we can apply the inverse divergence operator from Proposition A.3.3 (with the adjustments set out in Remark A.3.8 for scalar fields and the additions in Remark A.3.9 for

pointwise bounds), we define

$$\overline{\phi}_O^{q+1} := (\mathcal{H}+\mathcal{R}^*) \sum_{\xi,i,j,k,\vec{l}} \underbrace{b_{(\xi),\varphi}\left(\mathbb{P}_{\neq 0}\boldsymbol{\rho}^3_{(\xi),\varphi}\right)(\Phi_{(i,k)})}_{=:\, t^{q+1}_{i,j,k,\xi,\vec{l},\varphi}} \tag{11.11a}$$

$$+ (\mathcal{H}+\mathcal{R}^*) \sum_{\xi,i,j,k,l^\perp} \underbrace{\partial_\gamma \widetilde{b}^\gamma_{(\xi)}\left(\mathbb{P}_{\neq 0}\mathcal{X}^3_{q,\xi,l^\perp}\right)(\Phi_{(i,k)})}_{=:\, \widetilde{t}^{q+1}_{i,j,k,\xi,l^\perp,\varphi}} \tag{11.11b}$$

$$\overline{\phi}_O^{q+\lfloor n/2\rfloor+1} := (\mathcal{H}+\mathcal{R}^*) \sum_{\xi,i,j,k,\vec{l},I,\diamond} \underbrace{b_{(\xi),\diamond}\left(\boldsymbol{\rho}^3_{(\xi),\diamond}(\boldsymbol{\zeta}^I_\xi)^{3\diamond}\widetilde{\mathbb{P}}^\xi_{q+\lfloor n/2\rfloor+1}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\diamond})^3\right)(\Phi_{(i,k)})}_{=:\, t^{q+\lfloor n/2\rfloor+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \tag{11.11c}$$

$$\overline{\phi}_O^{m} := (\mathcal{H}+\mathcal{R}^*) \sum_{\xi,i,j,k,\vec{l},I,\diamond} \underbrace{b_{(\xi),\diamond}\left(\boldsymbol{\rho}^3_{(\xi),\diamond}(\boldsymbol{\zeta}^I_\xi)^{3\diamond}\widetilde{\mathbb{P}}^\xi_{(m-1,m]}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\diamond})^3\right)(\Phi_{(i,k)})}_{=:\, t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}} \tag{11.11d}$$

$$\overline{\phi}_O^{q+\bar{n}} := \sum_{m=q+\bar{n}}^{q+\bar{n}+1} (\mathcal{H}+\mathcal{R}^*) \sum_{\xi,i,j,k,\vec{l},I,\diamond} \underbrace{b_{(\xi),\diamond}\left(\boldsymbol{\rho}^3_{(\xi),\diamond}(\boldsymbol{\zeta}^I_\xi)^{3\diamond}\widetilde{\mathbb{P}}^\xi_{(m-1,m]}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\diamond})^3\right)(\Phi_{(i,k)})}_{=\, t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}} \tag{11.11e}$$

$$+ (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} b_{(\xi),\diamond}\left(\boldsymbol{\rho}^3_{(\xi),\diamond}(\boldsymbol{\zeta}^I_\xi)^{3\diamond}(1-\widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1})\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\diamond})^3\right)(\Phi_{(i,k)})\right], \tag{11.11f}$$

where (11.11d) is defined for $q + {}^{\bar{n}}\!/_2 + 1 < m < q + \bar{n}$. We justify these applications and record estimates on the outputs in the following Lemma.

Lemma 11.2.1 (Current error and pressure increment). *There exist current errors $\overline{\phi}_O^m$ for $m = q+1, \ldots, q+\bar{n}$ and pressure increments $\sigma^+_{\overline{\phi}_O^m} = \sigma_{\overline{\phi}_O^m} - \sigma^-_{\overline{\phi}_O^m}$ for $m = q+{}^{\bar{n}}\!/_2+1, \ldots, q+\bar{n}$ such that the following hold.*

(i) We have the equality

$$\frac{1}{2}\mathrm{div}\left(|w^{(p)}_{q+1}|^2 w^{(p)}_{q+1} + \varphi_\ell\right) = \sum_{m=q+1}^{q+\bar{n}} \mathrm{div}\overline{\phi}_O^m = \sum_{m=q+1}^{q+\bar{n}} \mathrm{div}\left(\overline{\phi}_O^{m,l} + \overline{\phi}_O^{m,*}\right).$$

(ii) The lowest shell has no pressure increment; more precisely, $\sigma^{+}_{\overline{\phi}_O^{q+1}} \equiv 0$*, and for* $N, M \leq \mathsf{N}_{\text{fin}}/100$*,*

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}_O^{q+1}\right| < \Gamma_q^{65} \Lambda_q \lambda_{q+1}^{-1} (\pi_q^q)^{3/2} r_q^{-1} \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q \Gamma_q^{i+14}, \Gamma_q^8 \mathrm{T}_q^{-1}\right) . \tag{11.12}$$

(iii) For all $m = q + \bar{n}/2 + 1, \ldots, q + \bar{n}$*, we have that*

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}_O^{m,l}\right| \lesssim \left((\sigma^{+}_{\overline{\phi}_O^m})^{3/2} r_m^{-1} + \delta^2_{q+3\bar{n}}\right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{11.13a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^{+}_{\overline{\phi}_O^m}\right| \lesssim \left(\sigma^{+}_{\overline{\phi}_O^m} + \delta^2_{q+3\bar{n}}\right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+16}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{11.13b}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^{+}_{\overline{\phi}_O^m}\right\|_{3/2} < \delta_{m+\bar{n}} \Gamma_m^{-9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+16}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{11.13c}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^{+}_{\overline{\phi}_O^m}\right\|_{\infty} < \Gamma_m^{\mathsf{C}_\infty - 9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+16}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{11.13d}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^{-}_{\overline{\phi}_O^m}\right| < \left(\frac{\lambda_q}{\lambda_{q+\lfloor \bar{n}/2 \rfloor}}\right)^{\frac{2}{3}} \pi_q^q \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1} \Gamma_q^{i+16}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{11.13e}$$

for all $N, M \leq \mathsf{N}_{\text{fin}}/100$*. Furthermore, we have that for all* $m' \geq q + \bar{n}/2 + 1$*,*

$$\begin{aligned} B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap \operatorname{supp} \overline{\phi}_O^{m,l} &= \emptyset \qquad \forall q+1 \leq q' \leq m-1 \\ B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap \operatorname{supp} (\sigma^{+}_{\overline{\phi}_O^{m'}}) &= \emptyset \qquad \forall q+1 \leq q' \leq m'-1 \\ B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q+1}^{-1} \Gamma_q^2\right) \cap \operatorname{supp} (\sigma^{-}_{\overline{\phi}_O^{m'}}) &= \emptyset \qquad \forall q+1 \leq q' \leq q + \bar{n}/2 . \end{aligned} \tag{11.14}$$

(iv) For all $m = q+1, \ldots, q+\bar{n}$ *and* $N, M \leq 2\mathsf{N}_{\text{ind}}$*, the non-local part* $\overline{\phi}_O^{m,*}$ *satisfies*

$$\left\|D^N D^M_{t,q} \overline{\phi}_O^{m,*}\right\|_{L^\infty} \leq \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}} \delta_{q+3\bar{n}}^{3/2} \lambda_m^N \tau_q^{-M} . \tag{11.15}$$

Proof. The equality in (i) follows from (11.9)–(11.11), assuming for the moment that all quantities in (11.11) are well-defined. We now split the proof up into cases. We first treat

$\overline{\phi}_O^{q+1}$ as defined in (11.11a)–(11.11b) and prove (11.12) and (11.15) for $m = q+1$. Next, we treat (11.11c) and prove (11.13a)–(11.15) for $m = q + \bar{n}/2 + 1$ using Proposition A.3.3 in conjunction with Remark A.3.9. Afterwards we treat the intermediate shells from (11.11d) and a portion of the last shell (11.11e), and prove (11.13a)–(11.15) for $q+\bar{n}/2+2 \leq m \leq q+\bar{n}$ using Proposition A.4.5. Finally, we treat (11.11f), which will be absorbed to the nonlocal part of the current error. We will therefore prove (11.15) using Proposition A.3.3. We fix the following choices throughout the proof,

$$v = \widehat{u}_q\,, \quad \Phi = \Phi_{(i,k)}\,, \quad \mathcal{C}_v = \Lambda_q^{1/2}\,, \quad \nu' = \mathrm{T}_q^{-1}\Gamma_q^8\,, \quad \lambda' = \Gamma_q^{13}\Lambda_q\,,$$
$$2M_* = N_* = \frac{\mathsf{N}_{\mathrm{fin}}}{3}\,, \quad M_t = \mathsf{N}_{\mathrm{ind,t}}\,, \quad M_\circ = N_\circ = 2\mathsf{N}_{\mathrm{ind}}\,, \qquad K_\circ \text{ as in (xvi)}$$

while the remaining parameters will vary depending on the case.

`Case 1a:` Analysis for (11.11a). Fix ξ, i, j, k, and $\vec{l}$. In order to check the low-frequency assumptions in Part 1, high-frequency assumptions in Part 2, and nonlocal assumptions in Part 4 of Proposition A.3.3, we set

$$p = \infty\,, \quad G = b_{(\xi),\varphi}\,, \quad \mathcal{C}_{G,\infty} = r_q^{-1}\delta_{q+\bar{n}}^{3/2}\Gamma_q^{3j+34}\Lambda_q\,, \quad \lambda = \Gamma_q^{13}\Lambda_q\,, \quad \nu = \tau_q^{-1}\Gamma_q^{i+13}\,, \quad \pi = \Gamma_q^{36}\pi_\ell^{3/2}r_q^{-1}\Lambda_q\Gamma_q^{10}\,,$$
$$\varrho = \mathbb{P}_{\neq 0}\boldsymbol{\rho}^3_{(\xi),\varphi}\,, \quad \vartheta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}} = \delta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}}\Delta^{-\mathsf{d}/2}\varrho\,, \quad \mathcal{C}_{*,1} = \mathcal{C}_{*,\infty} = \Gamma_q^6\lambda_{q+1}^{\alpha}\,, \quad \mu = \lambda_{q+1}\Gamma_q^{-4}\,,$$
$$\Upsilon = \Upsilon' = \lambda_{q+1}\Gamma_q^{-4}\,, \quad \Lambda = \lambda_{q+1}\Gamma_q^{-1}\,, \quad \mathsf{N}_{\mathrm{dec}} \text{ as in (xv)}\,, \quad \mathsf{d} \text{ as in (xvii)}\,,$$

where α is chosen as in (4.14). Then we have that (A.39)–(A.40) are satisfied by definition and by (9.36a), (A.41)–(A.42b) hold from Corollary 8.2.4 and (5.34) at level q, (A.59) holds from (9.38b), (i)–(ii) hold by definition and item i from Proposition 7.2.1, (A.43) holds due to standard Littlewood-Paley theory, (A.44) holds by definition and by (4.24a), (A.45) holds due to (4.21), (A.52) holds by (4.24a), (A.53)–(A.54) hold from Remark A.3.4, and (A.55) holds from (4.23b).

From (A.46) and (A.56), we have that (11.11a) is well-defined. From (A.47), (A.49),

(A.60), (6.6), and (4.24a), we have that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left| D^N D^M_{t,q} \mathcal{H}\, t^{q+1}_{i,j,k,\xi,\vec{l},\varphi} \right| \lesssim \Gamma_q^{60} (\pi_q^q)^{3/2} \Lambda_q r_q^{-1} \lambda_{q+1}^{-1} \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+13}, \mathrm{T}_q^{-1}\Gamma_q^8\right) .$$

Notice that from (ii), the support of $\mathcal{H} t^{q+1}_{i,j,k,\xi,\vec{l},\varphi}$ is contained in the support of $t_{i,j,k,\xi,\vec{l}}$, which is contained inside the support of $\eta_{i,j,k,\xi,\vec{l},\varphi}$ from the definition of $b_{(\xi),\varphi}$. Thus we may apply Corollary 8.6.3 with $H = \mathcal{H} t^{q+1}_{i,j,k,\xi,\vec{l},\varphi}$, $\varpi = \Gamma_q^{60} (\pi_q^q)^{3/2} r_q^{-1} \lambda_{q+1}^{-1} \Lambda_q \mathbf{1}_{\operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\varphi}}$, and $p = 1$ to deduce that

$$\left| \psi_{i,q} D^N D^M_{t,q} \sum_{i',j,k,\xi,\vec{l},\varphi} \mathcal{H}\, t^{q+1}_{i',j,k,\xi,\vec{l},\varphi} \right| \lesssim \Gamma_q^{60} \pi_\ell^{3/2} \Lambda_q r_q^{-1} \lambda_{q+1}^{-1} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) . \tag{11.17}$$

From (A.57) and summing over the values of $i, j, k, \xi, \vec{l}, \diamond$ which may be non-zero at a fixed point in time using (5.8) and (5.9) to control i, (8.5) to control k, (8.25) to control j, (8.30) and (8.33) to control $\vec{l}$, and using that ξ takes only finitely many values, we have from (A.57) and Remark A.3.4 that for all $M_0, N_0 \leq 2\mathsf{N}_{\text{ind}}$,

$$\left| D^N D^M_{t,q} \sum_{i,j,k,\xi,\vec{l},\varphi} \mathcal{R}^* t^{q+1}_{i,j,k,\xi,\vec{l},\varphi} \right| \leq \lambda_{q+\bar{n}}^{-2} \delta_{q+3\bar{n}}^{3/2} \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}} \lambda_{q+1}^N \tau_q^{-M} . \tag{11.18}$$

Combining (11.17)–(11.18) and using (6.6), we have an estimate consistent with (11.12), and an estimate consistent with (11.15) for $m = q + 1$.

`Case 1b:` Analysis for (11.11b). Fix ξ, i, j, and k. In order to check the low-frequency assumptions in Part 1, high-frequency assumptions in Part 2, and nonlocal assumptions in

Part 4 of Proposition A.3.3, we set

$$p=\infty\,,\quad G=\partial_\gamma\widetilde{b}^{\gamma}_{(\xi)}\,,\quad \mathcal{C}_{G,\infty}=r_q^{-1}\delta_{q+\bar{n}}^{3/2}\Gamma_q^{3j+34}\Lambda_q\,,\quad \lambda=\Gamma_q^{13}\Lambda_q\,,\quad \nu=\tau_q^{-1}\Gamma_q^{i+13}\,,\quad \pi=\Gamma_q^{36}\pi_\ell^{3/2}r_q^{-1}\Lambda_q\Gamma_q^{10}\,,$$
$$\varrho=\mathbb{P}_{\neq 0}\sum_{l^\perp}\mathcal{X}^{3}_{q,\xi,l^\perp}\,,\quad \vartheta^{i_1i_2\ldots i_{\mathsf{d}-1}i_{\mathsf{d}}}=\delta^{i_1i_2\ldots i_{\mathsf{d}-1}i_{\mathsf{d}}}\Delta^{-\mathsf{d}/2}\varrho\,,\quad \mathcal{C}_{*,1}=\mathcal{C}_{*,\infty}=\lambda_{q+1}^{\alpha}\,,\quad \mu=\mathcal{C}_\Gamma\lambda_{q+1}\Gamma_q^{-5}\,,$$
$$\Upsilon=\Upsilon'=\Lambda=\lambda_{q+1}\Gamma_q^{-5}\,,\quad \mathsf{N}_{\mathrm{dec}}\text{ as in (xv)}\,,\quad \mathsf{d}\text{ as in (xvii)}\,.$$

Then we have that (A.39)–(A.40) are satisfied by definition and by (5.37) at level q, (8.28), (8.3), Corollary 8.2.4 at level q, and (9.40), (A.41)–(A.42b) hold from Corollary 8.2.4 and (5.34) at level q, (A.59) holds from (8.22c) and the same estimates which justified (A.39)–(A.40), (i)–(ii) hold by definition and from (8.30), (A.43) holds due to standard Littlewood-Paley theory, (A.44) holds by definition and by (4.24a), (A.45) holds due to (4.21), (A.52) holds by (4.24a), (A.53)–(A.54) hold from Remark A.3.4, and (A.55) holds from (4.23b).

At this point, the remainder of the argument is essentially identical to that of `Case 1a`. Indeed, the only differences are that the support of the localized output is contained inside the support of $\psi_{i,q}\omega_{j,q}\chi_{i,k,q}$, and so instead of appealing to the abstract aggregation lemma, we may appeal directly to (5.8) and (8.21). Eschewing further details, we have concluded the analysis of ϕ_O^{q+1} and proven (11.12) and (11.15) at level $m=q+1$.

`Case 2:` Analysis for (11.11c). Fix ξ, i, j, k, $\vec{l}$, I, and $\diamond$. In order to check the low-frequency, preliminary assumptions in Part 1 of Proposition A.4.5, we set

$$\begin{aligned}
&p=1,\infty\,,\quad G_\varphi=b_{(\xi),\varphi}\left(\boldsymbol{\rho}^3_{(\xi),\varphi}(\boldsymbol{\zeta}^I_\xi)^{3\varphi}\right)\circ\Phi_{(i,k)}\,,\quad G_R=r_q^{-1}b_{(\xi),R}\left(\boldsymbol{\rho}^3_{(\xi),R}(\boldsymbol{\zeta}^I_\xi)^{3R}\right)\circ\Phi_{(i,k)}\,,\\
&\mathcal{C}_{G_\diamond,1}=\delta_{q+\bar{n}}^{3/2}r_q^{-1}\Gamma_q^{3j+40}\Lambda_q\left|\mathrm{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_\xi\right)\right|+\lambda_{q+\bar{n}}^{-15}\,,\quad \mathcal{C}_{G_\diamond,\infty}=\delta_{q+\bar{n}}^{3/2}r_q^{-1}\Gamma_q^{3j+40}\Lambda_q\,,\\
&\lambda=\lambda_{q+\bar{n}/2}\,,\quad \nu=\tau_q^{-1}\Gamma_q^{i+13}\,,\quad \pi=\Gamma_q^{35}\pi_\ell\Lambda_q^{2/3}\,,\quad r_G=r_q\,.
\end{aligned}\tag{11.20}$$

Then we have that (A.39) is satisfied by definition, (A.40) is satisfied by (9.36b), (9.36d), Corollary 8.2.4, (7.23), and Definition 7.2.4, (A.41)–(A.42b) hold from Corollary 8.2.4 and (5.34) at level q, and (A.132b) holds from (9.38). In order to check the high-frequency,

preliminary assumptions in Part 1 of Proposition A.4.5, we set

$$\varrho_R = \widetilde{\mathbb{P}}^{\xi}_{q+\bar{n}/2+1}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),R})^3 r_q\,, \quad \varrho_\varphi = \widetilde{\mathbb{P}}^{\xi}_{q+\bar{n}/2+1}\mathbb{P}_{\neq 0}(\varrho^I_{(\xi),\varphi})^3\,, \quad \vartheta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}}_{\diamond} = \delta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}}\Delta^{-\mathsf{d}/2}\varrho_\diamond\,,$$
$$\mathcal{C}_{*,1} = \Gamma_q \lambda^{\alpha}_{q+\bar{n}/2+1}\,, \quad \mathcal{C}_{*,\infty} = \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^2 \lambda^{\alpha}_{q+\bar{n}/2+1}\,, \quad \mu = \Upsilon = \Upsilon' = \lambda_{q+\bar{n}/2}\Gamma_q\,, \quad \Lambda = \lambda_{q+\bar{n}/2+1}\,,$$
$$\mathsf{N}_{\mathrm{dec}} \text{ as in (xv)}\,, \quad \mathsf{d} \text{ as in (xvii)}$$

Now (i)–(ii) hold by definition and from (9.4) and (9.12) (which specify the minimum frequency of $\mu = \lambda_{q+\bar{n}/2}\Gamma_q$), (A.43) holds due to Propositions 7.1.5 and 7.1.6 and estimate (7.37a) from Lemma 7.3.3 applied with $\lambda r = \mu$, $\lambda = \lambda_{q+\bar{n}}$, $\lambda_0 = \lambda_{q+\bar{n}/2+1}$, $\rho = \varrho_\diamond$, and $q = 1$, (A.44) holds by definition and by (4.24a), (A.45) holds due to (4.21), (A.52) holds by (4.24a), (A.53)–(A.54) hold from Remark A.3.4, and (A.55) holds from (4.23b). In order to check the additional assumptions in Part 2 of Proposition A.4.5, we set

$$N_{**} \text{ as in (xvii)}\,, \quad \mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}} \text{ as in (xi)}\,, \quad \Gamma = \Gamma_q^{\frac{1}{10}}\,, \quad \delta_{\mathrm{tiny}} = \delta^2_{q+3\bar{n}}\,, \quad r_\phi = r_{q+\bar{n}/2+1}\,,$$
$$\delta^{3/2}_{\phi,p} = \mathcal{C}_{G_\diamond,p}\mathcal{C}_{*,p}(\lambda_{q+\bar{n}/2}\Gamma_q)^{-1} r_{q+\bar{n}/2+1}\,, \quad \bar{m} = 1\,, \quad \mu_0 = \lambda_{q+\bar{n}/2+1}\Gamma_q^{-1}\,, \quad \mu_{\bar{m}} = \mu_1 = \lambda_{q+\bar{n}/2+1}\Gamma_q^2\,. \tag{11.22}$$

Then (A.197a)–(A.197b) hold from (4.24a), (A.197c) holds from (4.23a), (A.198a) holds by definition, (A.198b) holds from (4.17a), (A.198c) holds from (4.17b), (A.198d) holds from (4.24a), (A.198e) holds from (4.21), (A.199a) holds by definition, (A.199b) holds by definition and immediate computation, (A.199c) holds due to (4.23b), and (A.199d) holds due to (4.23c).

From (A.57) and summing over the values of $i, j, k, \xi, \vec{l}, \diamond, I$ which may be non-zero at a fixed point in time in a manner similar to that from `Case 1a`, we have from (A.57) and Remark A.3.4 that for all $M_\circ, N_\circ \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\left| D^N D^M_{t,q} \sum_{i,j,k,\xi,\vec{l},\diamond,I} \mathcal{R}^* t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond} \right| \leq \lambda^{-2}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} \Gamma^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \lambda^N_{q+1} \tau_q^{-M}\,. \tag{11.23}$$

This verifies (11.15) at level $m = q + \bar{n}/2 + 1$. From (A.200)–(A.201) and (4.24a), we have that there exists a pressure increment $\sigma_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} = \sigma^{+}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} - \sigma^{-}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}$ such that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left| D^N D^M_{t,q} \mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond} \right| \lesssim \left(\left(\sigma^{+}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right)^{3/2} r^{-1}_{q+1} + \delta^2_{q+3\bar{n}} \right) (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{11.24}$$

From (A.48) and (A.205), we have that

$$\operatorname{supp}\left(\sigma^{+}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}\right) \subseteq \operatorname{supp}\left(\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}\right) \subseteq \operatorname{supp}\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi}\right) \circ \Phi_{(i,k)}\right) . \tag{11.25}$$

Now define

$$\sigma^{\pm}_{\overline{\phi}_O^{q+\bar{n}/2+1}} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{\pm}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} . \tag{11.26}$$

Then (9.22) gives that (11.14) is satisfied for $m' = q + \bar{n}/2 + 1$. From (11.24), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}, \qquad \varpi = \left[\left(\sigma^{+}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right)^{3/2} r^{-1}_{q+\bar{n}/2+1} + \delta^2_{q+3\bar{n}} \right] \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}, \qquad p = 1,$$

we have that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left| \psi_{i,q} D^N D^M_{t,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} \mathcal{H}t^{q+\bar{n}/2+1}_{i',j,k,\xi,\vec{l},I,\diamond} \right| \lesssim \left(\left(\sigma^{+}_{\overline{\phi}_O^{q+\bar{n}/2+1}} \right)^{3/2} r^{-1}_{q+\bar{n}/2+1} + \delta^2_{q+3\bar{n}} \right) \times (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{11.27}$$

In combination with the bound in (11.23), we have that (11.13a) is satisfied for $m = q+\bar{n}/2+1$.

From (A.202), (4.24a), and (4.18), we have that for $N, M \leq \mathsf{N}_{\text{fin}}/7$,

$$\left| D^N D^M_{t,q} \sigma^+_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right| \lesssim \left(\sigma^+_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} + \delta^2_{q+3\bar{n}} \right) (\lambda_{q+\bar{n}/2+1}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{11.28}$$

From (11.28), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \sigma^+_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} , \qquad \varpi = \left[H + \delta^2_{q+3\bar{n}}\right] \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^I_\xi)\circ\Phi_{(i,k)}} , \qquad p = 1 ,$$

we have that (11.13b) is satisfied for $m = q + \bar{n}/2 + 1$.

Next, from (A.203), we have that

$$\left\| \sigma^\pm_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right\|_{3/2} \lesssim \left(\delta_{q+\bar{n}} r_q^{-2/3} \Gamma_q^{2j+30} \Lambda_q^{2/3} \left| \operatorname{supp} \left(\eta_{i,j,k,\xi,\vec{l},\diamond} \boldsymbol{\zeta}^{I,\diamond}_\xi \right) \right|^{2/3} + \lambda_{q+\bar{n}}^{-10} \right) (\lambda_{q+\bar{n}/2}\Gamma_q)^{-2/3} r^{2/3}_{q+\bar{n}/2+1} .$$

Now from (11.26), (4.27c), and Corollary 8.6.1 with $\theta = 2$, $\theta_1 = 0$, $\theta_2 = 2$, $H = \sigma^\pm_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}$, and $p = 3/2$, we have that

$$\begin{aligned} \left\| \psi_{i,q} \sigma^\pm_{\overline{\phi}_O^{q+\bar{n}/2+1}} \right\|_{3/2} &\lesssim \delta_{q+\bar{n}} r_q^{-2/3} \Gamma_q^{33} \Lambda_q^{2/3} (\lambda_{q+\bar{n}/2}\Gamma_q)^{-2/3} r^{2/3}_{q+\bar{n}/2+1} \\ &\leq \delta_{q+\bar{n}+\bar{n}/2+1} \Gamma^{-10}_{q+\bar{n}/2+1} . \end{aligned}$$

Combined with (11.13b), this verifies (11.13c) at level $q + \bar{n}/2 + 1$. Arguing now for $p = \infty$ from (A.203) and using (8.27), we have that

$$\begin{aligned} \left\| \sigma^\pm_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right\|_\infty &\lesssim \delta_{q+\bar{n}} r_q^{-2/3} \Gamma_q^{2j+30} \Lambda_q^{2/3} \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}\Gamma_q} \right)^{4/3} (\lambda_{q+\bar{n}/2}\Gamma_q)^{-2/3} r^{2/3}_{q+\bar{n}/2+1} \\ &\lesssim \Gamma_q^{36+\mathsf{C}_\infty} \left(\frac{\lambda_{q+\bar{n}/2+1}}{\lambda_{q+\bar{n}/2}\Gamma_q} \right)^{4/3} \Lambda_q^{2/3} (\lambda_{q+\bar{n}/2}\Gamma_q)^{-2/3} \leq \Gamma^{\mathsf{C}_\infty - 11}_{q+\bar{n}/2+1} . \end{aligned} \tag{11.29}$$

Now from (11.26) and Corollary 8.6.3 with $H = \sigma^\pm_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}$, $\varpi = \Gamma^{\mathsf{C}_\infty - 11}_{q+\bar{n}/2+1} \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^I_\xi)\circ\Phi_{(i,k)}}$

and $p=1$, we have that

$$\left\| \psi_{i,q} \sigma^{\pm}_{\overline{\phi}_O^{q+\bar{n}/2+1}} \right\|_\infty \leq \Gamma_{q+\bar{n}/2+1}^{\mathsf{C}_\infty - 10} \, .$$

Combined again with (11.13b), this verifies (11.13d) at level $q+\bar{n}/2+1$.

Finally, from (A.204), (6.6), (4.18), (4.10h), and (4.24a), we have that for $N, M \leq \mathsf{N}_{\mathrm{fin}}/7$,

$$\begin{aligned} \left| D^N D^M_{t,q} \sigma^-_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right| &\lesssim \left(\frac{r_{q+\bar{n}/2+1}}{r_q} \right)^{2/3} \Gamma_q^{28} \pi_q^q \Lambda_q^{2/3} \lambda_{q+\bar{n}/2}^{-2/3} (\lambda_{q+\bar{n}/2} \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \\ &\leq \Gamma_q^{-10} \left(\frac{\lambda_q}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \pi_q^q (\lambda_{q+\bar{n}/2} \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \, . \end{aligned}$$

Applying (11.26) and Corollary 8.6.3 with $H = \sigma^-_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}$, $\varpi = \left(\frac{\lambda_q}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \pi_q^q \Gamma_q^{-1} \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond} (\boldsymbol{\rho}^\diamond_{(\xi)} \boldsymbol{\zeta}^I_\xi) \circ \Phi_{(i,k)}}$ and $p=1$, we have that (11.13e) is verified at level $m = q+\bar{n}/2+1$.

`Case 3:` Analysis for (11.11d) and (11.11e). Fix ξ, i, j, k, $\vec{l}$, I, and $\diamond$. In order to check the low-frequency, preliminary assumptions in Part 1 of Proposition A.4.5, we may use the exact same choices as in (11.20). In order to check the high-frequency, preliminary assumptions in Part 1 of Proposition A.4.5, we set

$$\begin{aligned} &\varrho_R = \left(\widetilde{\mathbb{P}}^\xi_{(q+\bar{n}/2+1, q+\bar{n}/2+3/2]} + \widetilde{\mathbb{P}}^\xi_{(q+\bar{n}/2+3/2, q+\bar{n}/2+2]} \right) (\varrho^I_{(\xi),R})^3 r_q \quad \text{if} \quad m = q+\bar{n}/2+2 \, , \\ &\varrho_\varphi = \left(\widetilde{\mathbb{P}}^\xi_{(q+\bar{n}/2+1, q+\bar{n}/2+3/2]} + \widetilde{\mathbb{P}}^\xi_{(q+\bar{n}/2+3/2, q+\bar{n}/2+2]} \right) (\varrho^I_{(\xi),\varphi})^3 \quad \text{if} \quad m = q+\bar{n}/2+2 \\ &\varrho_R = \widetilde{\mathbb{P}}^\xi_{(m-1,m]} (\varrho^I_{(\xi),R})^3 r_q \, , \quad \varrho_\varphi = \widetilde{\mathbb{P}}^\xi_{(m-1,m]} (\varrho^I_{(\xi),\varphi})^3 \quad \text{if} \quad m \neq q+\bar{n}/2+2 \\ &\vartheta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}}_\diamond \text{ given by Lemma 7.3.4,} \, , \quad \mathcal{C}_{*,1} = 1 \, , \quad \mathcal{C}_{*,\infty} = \left(\frac{\min(\lambda_m, \lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2} \Gamma_q} \right)^2 \, , \quad \mu = \lambda_{q+\bar{n}/2} \Gamma_q \, , \\ &\Upsilon = \lambda_{q+\bar{n}/2+1} \, , \quad \Upsilon' = \Lambda = \lambda_{q+\bar{n}/2+3/2} \quad \text{for the first projector if} \quad m = q+\bar{n}/2+2 \, , \\ &\Upsilon = \lambda_{q+\bar{n}/2+3/2} \, , \quad \Upsilon' = \Lambda = \lambda_{q+\bar{n}/2+2} \quad \text{for the second projector if} \quad m = q+\bar{n}/2+2 \, , \\ &\Upsilon = \lambda_{m-1} \, , \quad \Upsilon' = \Lambda = \min(\lambda_m, \lambda_{q+\bar{n}}) \quad \text{if} \quad m \neq q+\bar{n}/2+2 \, , \\ &\mathsf{N}_{\mathrm{dec}} \text{ as in (xv)} \, , \quad \mathsf{d} \text{ as in (xvii)}, . \end{aligned}$$

Now (i)–(ii) hold by definition and from (9.4) and (9.12) as before, (A.43) holds due to Propositions 7.1.5 and 7.1.6 and estimate (7.40b) from Lemma 7.3.4 applied with the obvious choices, (A.44) holds by definition, by (4.24a), and by our extra splitting in the case $m = q + {}^{\bar{n}}\!/_2 + 2$, and (A.45) and (A.52)–(A.55) hold after appealing to the same parameter inequalities as the previous case. In order to check the additional assumptions in Part 2 of Proposition A.4.5, we set

$$
\begin{aligned}
&N_{**} \text{ as in (xvii)}\,, \quad \mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}} \text{ as in (xi)}\,, \quad \Gamma = \Gamma_q^{\frac{1}{10}}\,, \quad \delta_{\mathrm{tiny}} = \delta_{q+3\bar{n}}^2\,, \quad r_\phi = r_{\min(m,q+\bar{n})}\,,\\
&\delta_{\phi,p}^{3/2} = \mathcal{C}_{G_\diamond,p}\mathcal{C}_{*,p}\Upsilon'\Upsilon^{-2} r_{\min(m,q+\bar{n})}\,, \quad \mu_0 = \lambda_{q+{}^{\bar{n}}/2+1}\,, \quad \mu_1 = \lambda_{q+{}^{\bar{n}}/2+3/2}\Gamma_q^2\,,\\
&\mu_{m'} = \lambda_{q+{}^{\bar{n}}/2+m'}\Gamma_q^2 \quad \text{if} \quad 2 \le m' \le {}^{\bar{n}}\!/_2 + 1\,,\\
&\bar{m} = 1 \quad \text{for the first projector if} \quad m = q + {}^{\bar{n}}\!/_2 + 2\,,\\
&\bar{m} = 2 \quad \text{for the second projector if} \quad m = q + {}^{\bar{n}}\!/_2 + 2\,,\\
&\bar{m} = m - q - {}^{\bar{n}}\!/_2 \quad \text{if} \quad m > q + {}^{\bar{n}}\!/_2 + 2\,.
\end{aligned}
\tag{11.31}
$$

Then (A.197a)–(A.198e) hold after appealing to the same inequalities as in the previous case, (A.199a) holds by definition, (A.199b) holds by definition and immediate computation, and (A.199c)–(A.199d) hold as in the previous case.

First, we have that (11.15) at level m' for $q + {}^{\bar{n}}\!/_2 + 2 \le m' \le q + \bar{n}$ is satisfied by an argument essentially identical to that of the previous case. Next, from (A.200)–(A.201) and (4.24a), we have that for $q + {}^{\bar{n}}\!/_2 + 2 \le m \le q + \bar{n} + 1$, there exists a pressure increment $\sigma^+_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$ such that for $N, M \le {}^{\mathsf{N}_{\mathrm{fin}}}\!/_7$,

$$
\begin{aligned}
\left| D^N D_{t,q}^M \mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond} \right| \lesssim \Bigg(\left(\sigma^+_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right)^{3/2} r^{-1}_{\min(m,q+\bar{n})} + \delta^2_{q+3\bar{n}} \Bigg)\\
\times \left(\min(\lambda_m, \lambda_{q+\bar{n}})\Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\,.
\end{aligned}
\tag{11.32}
$$

From (A.48), (A.205), and (7.40c), we have that

$$\operatorname{supp}\left(\sigma^{+}_{\mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}}\right) \subseteq \operatorname{supp}\left(\mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}\right) \subseteq \operatorname{supp}\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi}\right)\circ\Phi_{(i,k)}\right)\cap B\left(\operatorname{supp}\varrho^{I}_{(\xi),\diamond},\lambda^{-1}_{m-1}\right) . \tag{11.33}$$

Now define

$$\sigma^{\pm}_{\overline{\phi}^{m}_{O}} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{\pm}_{\mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}} \quad \text{if} \quad m \neq q+\bar{n}\,, \tag{11.34}$$

$$\sigma^{\pm}_{\overline{\phi}^{q+\bar{n}}_{O}} = \sum_{m'=q+\bar{n}}^{q+\bar{n}+1} \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{\pm}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}} \quad \text{if} \quad m = q+\bar{n}\,, \tag{11.35}$$

Then (9.22)–(9.24) and (11.33) give that (11.14) is satisfied for $q+{}^{\bar{n}}/_{2}+2 \leq m' \leq q+\bar{n}$. From (11.32), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}\,, \qquad \varpi = \left[\left(\sigma^{+}_{\mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}}\right)^{3/2} r^{-1}_{\min(m,q+\bar{n})} + \delta^{2}_{q+3\bar{n}}\right]\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi}))\circ\Phi_{(i,k)}}\,, \qquad p=1\,,$$

we have that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_{7}$ and $q+{}^{\bar{n}}/_{2}+2 \leq m < q+\bar{n}$,

$$\left|\psi_{i,q} D^{N} D^{M}_{t,q} \sum_{i',j,k,\xi,\vec{l},I,\diamond} \mathcal{H}t^{m}_{i',j,k,\xi,\vec{l},I,\diamond}\right| \lesssim \left(\left(\sigma^{+}_{\overline{\phi}^{m}_{O}}\right)^{3/2} r^{-1}_{\min(m,q+\bar{n})} + \delta^{2}_{q+3\bar{n}}\right) \times (\lambda_{m}\Gamma_{q})^{N}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q}^{-1}\Gamma_{q}^{i+15}, \mathrm{T}_{q}^{-1}\Gamma_{q}^{9}\right) . \tag{11.36}$$

An analogous statement holds if $m = q+\bar{n}$, with the only change being the extra summation needed on the left-hand side, which leads to (11.13a) for $q+{}^{\bar{n}}/_{2}+2 \leq m \leq q+\bar{n}$. From (A.202), (4.24a), and (4.18), we have that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_{7}$,

$$\left|D^{N} D^{M}_{t,q} \sigma^{+}_{\mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}}\right| \lesssim \left(\sigma^{+}_{\mathcal{H}t^{m}_{i,j,k,\xi,\vec{l},I,\diamond}} + \delta^{2}_{q+3\bar{n}}\right)(\min(\lambda_{m},\lambda_{q+\bar{n}})\Gamma_{q})^{N}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q}^{-1}\Gamma_{q}^{i+15}, \mathrm{T}_{q}^{-1}\Gamma_{q}^{9}\right) . \tag{11.37}$$

From (11.37), (8.45), (5.8), and Corollary 8.6.3 with

$$H = \sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,, \qquad \varpi = \left[H + \delta^2_{q+3\bar{n}}\right] \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}\,, \qquad p = 1\,,$$

we have that (11.13b) is satisfied for $q + {}^{\bar{n}}\!/_2 + 2 \leq m \leq q + \bar{n}$.

Next, from (A.203), we have that

$$\left\|\sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right\|_{3/2} \lesssim \left(\delta_{q+\bar{n}} r_q^{-2/3}\Gamma_q^{2j+30}\Lambda_q^{2/3}\left|\operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{2/3} + \lambda^{-10}_{q+\bar{n}}\right)\left(\lambda^2_{m-1}\lambda^{-1}_m\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})}\,.$$

Now from (11.34)–(11.35), (4.27c), and Corollary 8.6.1 with $\theta = 2$, $\theta_1 = 0$, $\theta_2 = 2$, $H = \sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$, and $p = {}^3\!/_2$, we have that

$$\begin{aligned}\left\|\psi_{i,q}\sigma^{\pm}_{\overline{\phi}^m_O}\right\|_{3/2} &\lesssim \delta_{q+\bar{n}} r_q^{-2/3}\Gamma_q^{33}\Lambda_q^{2/3}\left(\lambda^2_{m-1}\lambda^{-1}_m\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})} \\ &\leq \delta_{m+\bar{n}}\Gamma_m^{-10}\,.\end{aligned}$$

Combined with (11.13b), this verifies (11.13c) for $q + {}^{\bar{n}}\!/_2 + 2 \leq m' \leq q + \bar{n}$. Arguing now for $p = \infty$ from (A.203), we have that

$$\left\|\sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right\|_{\infty} \lesssim \delta_{q+\bar{n}} r_q^{-2/3}\Gamma_q^{2j+30}\Lambda_q^{2/3}\left(\frac{\min(\lambda_m,\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^{4/3}\left(\lambda^2_{m-1}\lambda^{-1}_m\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})}\,.$$

Now from (11.34)–(11.35), (8.27), (4.13a), and Corollary 8.6.3 with $H = \sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$, $\varpi = \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}$ and $p = 1$, we have that

$$\begin{aligned}\left\|\psi_{i,q}\sigma^{\pm}_{\overline{\phi}^m_O}\right\|_{\infty} &\lesssim \Gamma_q^{36+\mathsf{C}_\infty}\left(\frac{\min(\lambda_m,\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^{4/3}\Lambda_q^{2/3}\left(\lambda^2_{m-1}\lambda^{-1}_m\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})} r_q^{-2/3} \\ &\leq \Gamma^{\mathsf{C}_\infty-10}_{q+\bar{n}/2+1}\,.\end{aligned}$$

Combined again with (11.13b), this verifies (11.13d) at level $q + {}^{\bar{n}}\!/_2 + 2 \leq m' \leq q + \bar{n}$.

Finally, from (A.204), (6.6), (4.18), (4.10h), and (4.24a), we have that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}\!/_7$

and $q + {}^{\bar{n}}/_{2} + 3 \leq m \leq q + \bar{n} + 1$,

$$\left| D^N D^M_{t,q} \sigma^-_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right| \lesssim \left(\frac{r_{\min(m,q+\bar{n})}}{r_q} \right)^{2/3} \Gamma_q^{40} \pi_q^q \Lambda_q^{2/3} \left(\lambda_{m-1}^2 \lambda_m^{-1} \right)^{-2/3} (\lambda_{q+\bar{n}/2} \Gamma_q)^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9 \right)$$
$$\leq \Gamma_q^{-10} \left(\frac{\lambda_q}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \pi_q^q (\lambda_{q+\bar{n}/2} \Gamma_q)^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+15}, \mathrm{T}_q^{-1} \Gamma_q^9 \right) .$$

A similar inequality holds for $m = q + {}^{\bar{n}}/_{2} + 2$ after using the extra splitting of the Littlewood-Paley projector to mitigate the loss from $\lambda_{m-1}^{-1} \lambda_m$. Then applying (11.34)–(11.35) and Corollary 8.6.3 with $H = \sigma^-_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$, $\varpi = \Gamma_q^{-10} \left(\frac{\lambda_q}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \pi_q^q \mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond} (\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^I_\xi) \circ \Phi_{(i,k)}}$ and $p = 1$, we have that (11.13e) is verified.

`Case 4:` Analysis for (11.11f). We expect that the error term in (11.11f) is vanishingly small due to the Littlewood-Paley projector on the cubed pipe density. Therefore no pressure increment will be necessary, and we do not even need a local portion of the inverse divergence. We thus apply Proposition A.3.3 with $p = \infty$ and the following choices. The low-frequency assumptions in Part 1 are exactly the same as the L^∞ low-frequency assumptions in the previous two steps. For the high-frequency assumptions, we recall the choice of N_{**} from (xvii) and set

$$\varrho_\varphi = (\mathrm{Id} - \widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1}) \mathbb{P}_{\neq 0} \left(\varrho^I_{(\xi),\varphi} \right)^3 , \quad \varrho_R = (\mathrm{Id} - \widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1}) \mathbb{P}_{\neq 0} \left(\varrho^I_{(\xi),R} \right)^3 r_q , \quad \vartheta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}}_\diamond = \delta^{i_1 i_2 \ldots i_{\mathsf{d}-1} i_{\mathsf{d}}} \Delta^{-\mathsf{d}/2} \varrho_\diamond ,$$
$$\mu = \Upsilon = \Upsilon' = \lambda_{q+\bar{n}/2} \Gamma_q , \quad \Lambda = \lambda_{q+\bar{n}} , \quad \mathcal{C}_{*,\infty} = \left(\frac{\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}+1}} \right)^{N_{**}} \lambda^3_{q+\bar{n}} , \quad \mathsf{N}_{\mathrm{dec}} \text{ as in (xv)} , \quad \mathsf{d} = 0 .$$

Then we have that item i is satisfied by definition, item ii is satisfied as in the previous steps, (A.43) is satisfied using Propositions 7.1.5 and 7.1.6 and (7.37b) from Lemma 7.3.3, (A.44) is satisfied by definition and as in the previous steps, and (A.45) is satisfied by (4.21). For the nonlocal assumptions, we choose $M_\circ, N_\circ = 2\mathsf{N}_{\mathrm{ind}}$ so that (A.52)–(A.54) are satisfied as in Case 1, and (A.55) is satisfied from (4.23c). We have thus satisfied all the requisite assumptions, and we therefore obtain nonlocal bounds very similar to those from the previous steps, which are consistent with (11.15) at level $q + \bar{n}$. We omit further details. □

Lemma 11.2.2 (Pressure current). *For every $m' \in \{q + {}^{\bar{n}}/_2 + 1, \dots, q + \bar{n}\}$, there exist a current error $\phi_{\overline{\phi}_O^{m'}}$ associated to the pressure increment $\sigma_{\overline{\phi}_O^{m'}}$ defined by Lemma 11.2.1 and a function of time $\mathfrak{m}_{\phi_{\overline{\phi}_O^{m'}}}$ which satisfy the following properties.*

(i) We have the decompositions and equalities

$$\phi_{\overline{\phi}_O^{m'}} = \phi^*_{\overline{\phi}_O^{m'}} + \sum_{m=q+{}^{\bar{n}}/_2+1}^{m'} \phi^m_{\overline{\phi}_O^{m'}}, \quad \phi^m_{\overline{\phi}_O^{m'}} = \phi^{m,l}_{\overline{\phi}_O^{m'}} + \phi^{m,*}_{\overline{\phi}_O^{m'}} \tag{11.38a}$$

$$\mathrm{div}\phi_{\overline{\phi}_O^{m'}} + \mathfrak{m}'_{\sigma_{\phi_{\overline{\phi}_O^{m'}}}} = D_{t,q}\sigma_{\overline{\phi}_O^{m'}} . \tag{11.38b}$$

(ii) For $q + {}^{\bar{n}}/_2 + 1 \le m \le m'$ and $N, M \le 2\mathsf{N}_{\mathrm{ind}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \phi^{m,l}_{\overline{\phi}_O^{m'}}\right| < \Gamma_m^{-100} \left(\pi_q^m\right)^{3/2} r_m^{-1} (\lambda_m \Gamma_m^2)^M \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.39a}$$

$$\left\|D^N D^M_{t,q} \phi^{m,*}_{\overline{\phi}_O^{m'}}\right\|_\infty + \left\|D^N D^M_{t,q} \phi^*_{\overline{\phi}_O^{m'}}\right\|_\infty < \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^{3/2} (\lambda_{m'}\Gamma_{m'}^2)^N \tau_q^{-M} . \tag{11.39b}$$

(iii) For all $q + {}^{\bar{n}}/_2 + 1 \le m \le m'$ and all $q + 1 \le q' \le m - 1$,

$$B\left(\mathrm{supp}\, \widehat{w}_{q'}, {}^1/_2 \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap \mathrm{supp}\left(\phi^{m,l}_{\overline{\phi}_O^{m'}}\right) = \emptyset . \tag{11.40}$$

(iv) The function of time $\mathfrak{m}_{\sigma_{\phi_{\overline{\phi}_O^{m'}}}}$ satisfies that for $M \le 2\mathsf{N}_{\mathrm{ind}}$,

$$\mathfrak{m}_{\sigma_{\overline{\phi}_O^{m'}}}(t) = \int_0^T \left\langle D_{t,q}\sigma_{\overline{\phi}_O^{m'}} \right\rangle (s)\, ds , \quad \left|\frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{\sigma_{\phi_{\overline{\phi}_O^{m'}}}}\right| \le (\max(1,T))^{-1} \delta_{q+3\bar{n}} \tau_q^{-M} . \tag{11.41}$$

Proof. We follow the case numbering from Lemma 11.2.1. Note that the only cases which require a pressure increments are `Cases 2` and `3`, which correspond to the analysis of (11.11c)–(11.11e).

`Case 2:` In this case, we recall from (11.22) that we have chosen $\bar{m} = 1$ in item iii, $\mu_0 =$

$\lambda_{q+\bar{n}/2+1}\Gamma_q^{-1}$, and $\mu_{\bar{m}} = \mu_1 = \lambda_{q+\bar{n}/2+1}\Gamma_q^2$. We therefore have from (A.206a) that

$$\phi_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} = \phi^*_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} + \phi^0_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} + \phi^1_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} .$$

and define the current error $\phi_{\overline{\phi}_O^{q+\bar{n}/2+1}} := \sum_{i,j,k,\xi,\vec{l},I,\diamond} \phi_{\overline{\phi}^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}$ which has a decomposition into

$$\phi^*_{\overline{\phi}_O^{q+\bar{n}/2+1}} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \phi^*_{\overline{\phi}^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} , \quad \phi^{q+\bar{n}/2+1}_{\overline{\phi}_O^{q+\bar{n}/2+1}} = \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond \\ \iota=0,1}} \phi^{\iota}_{\overline{\phi}^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \tag{11.42a}$$

which satisfies (11.38b) from (A.206). We make a further decomposition into the local and nonlocal parts, $\phi^{q+\bar{n}/2+1}_{\overline{\phi}_O^{q+\bar{n}/2+1}} = \phi^{q+\bar{n}/2+1,l}_{\overline{\phi}_O^{q+\bar{n}/2+1}} + \phi^{q+\bar{n}/2+1,*}_{\overline{\phi}_O^{q+\bar{n}/2+1}}$ from item (ii).

In order to check (11.39a), we recall the parameter choices from `Case 2` of the previous lemma and apply Part 4 of Proposition A.4.5, specifically (A.207c). We then have from (4.24a) and (6.6) that for each $i, j, k, \xi, \vec{l}, I, \diamond, \iota$ and $M, N \leq 2\mathsf{N}_{\text{ind}}$, (after appending a superscript l to refer to the local portion)

$$\begin{aligned}\left| D^N D^M_{t,q} \phi^{\iota,l}_{\overline{\phi}^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}} \right| \leq \tau_q^{-1}\Gamma_q^{i+60}\pi_q^q\Lambda_q^{2/3}\lambda_{q+\bar{n}/2}^{-2/3}\left(\frac{r_{q+\bar{n}/2+1}}{r_q}\right)^{2/3}\left(\frac{\lambda_{q+\bar{n}/2+1}\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3}\lambda_{q+\bar{n}/2}^{-1} \\ \times (\lambda_{q+\bar{n}/2+1}\Gamma_q^2)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}} - 1, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) .\end{aligned} \tag{11.43}$$

Next, from (A.210) and (11.25), we have that

$$\begin{aligned}\text{supp}\left(\phi^{\iota,l}_{\overline{\phi}^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}\right) &\subseteq B\left(\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}, 2\lambda_{q+\bar{n}/2+1}\Gamma_q^{-1}\right) \\ &\subseteq B\left(\text{supp}\left(a_{(\xi),\diamond}(\varrho^{\diamond}_{(\xi)}\boldsymbol{\zeta}^I_\xi) \circ \Phi_{(i,k)}\right), 2\lambda_{q+\bar{n}/2+1}\Gamma_q^{-1}\right) .\end{aligned}$$

Then applying (9.22), we have that (11.40) is verified for $m = q + \bar{n}/2 + 1$. Returning to the

proof of (11.39a), we can now apply Corollary 8.6.4 with

$$H = \phi^{\iota,l}_{\overline{\phi}^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}, \qquad \varpi = \Gamma_q^{50}\pi_\ell \Lambda_q^{2/3}\left(\frac{r_{q+\bar{n}/2+1}}{r_q}\right)^{2/3}\lambda_{q+\bar{n}/2}^{-2/3}\left(\frac{\lambda_{q+\bar{n}/2+1}\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3}\lambda_{q+\bar{n}/2}^{-1}\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}.$$

From (8.56b), (4.18), (6.6), (5.20), (4.24a), and (4.27a), we have that

$$\begin{aligned}
&\left|\psi_{i,q}\sum_{i',j,k,\xi,\vec{l},I,\diamond,\iota}\mathcal{H}\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}t^{q+\bar{n}/2+1}_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\right| \\
&\lesssim \underbrace{\Gamma_q r_q^{-1}\lambda_q\left(\pi_q^q\right)^{1/2}}_{\text{cost of } D_{t,q}}\ \underbrace{\pi_q^q}_{\substack{\text{dominates } 2/3 \text{ power}\\ \text{of low-freq. coeff's}}}\ \underbrace{\Lambda_q^{2/3}\lambda_{q+\bar{n}/2}^{-2/3}}_{2/3 \text{ power of freq. gain}}\ \underbrace{\Gamma_q^{51}\left(\frac{r_{q+\bar{n}/2+1}}{r_q}\right)^{2/3}}_{\text{lower order}}\underbrace{\left(\frac{\lambda_{q+\bar{n}/2+1}\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3}}_{\text{intermittency loss}}\ \underbrace{\lambda_{q+\bar{n}/2}^{-1}}_{\text{inv. div. gain}} \\
&\qquad\times(\lambda_{q+\bar{n}/2+1}\Gamma_q^2)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}}-\mathsf{N}_{\text{cut,t}}-1,\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\lesssim \Gamma_q r_q^{-1}\lambda_q\left(\pi_q^{q+\bar{n}/2+1}\frac{\delta_{q+\bar{n}}}{\delta_{q+\bar{n}/2+1+\bar{n}}}\right)^{3/2}\lambda_q^{2/3}\lambda_{q+\bar{n}/2}^{-2/3}\left(\frac{\lambda_{q+\bar{n}/2+1}\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3}\lambda_{q+\bar{n}/2}^{-1} \\
&\qquad\times(\lambda_{q+\bar{n}/2+1}\Gamma_q^2)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}}-\mathsf{N}_{\text{cut,t}}-1,\tau_q^{-1}\Gamma_q^{i+15},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\leq \Gamma_q^{-150}r_q^{-1}\left(\pi_q^{q+\bar{n}/2+1}\right)^{3/2}(\lambda_{q+\bar{n}/2+1}\Gamma_q^2)^N\mathcal{M}\left(M,\mathsf{N}_{\text{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right) \qquad (11.44)
\end{aligned}$$

for all $N, M \leq 2\mathsf{N}_{\text{ind}}$, which verifies (11.39a) at level $q+\bar{n}/2+1$. In order to achieve (11.39b), we appeal to (A.208)–(A.209), the choice of $K_\circ$ in item (xvi), (4.23b), and (4.24a). Finally, the proof of (11.41) follows from (A.211) in a very similar way, the only difference being that we need a large choice of a_* in item (xix) in order to have the advantageous prefactor of $\max(1,T)^{-1}$.

`Case 3:` In this case we consider the higher shells from the oscillation error. The general principle is that the estimate will only be sharp in the $m = m' = q+\bar{n}$ double endpoint case, for which the intermittency loss is most severe. We now explain why this is the case by parsing estimates (11.43) and (11.44). We incur a material derivative cost of $\tau_q^{-1}\Gamma_q^{i+60}$, which is converted into $r_q^{-1}\lambda_q(\pi_q^q)^{1/2}$ using (5.23) and the rough definition of $\tau_q^{-1} = \delta_q^{1/2}\lambda_q r_q^{-1/3}$, or equivalently Corollary 8.6.4. The rescaled size of the high-frequency coefficients from the oscillation error is always 1 (see the choices of $\mathcal{C}_{*,1}$ from the last lemma), and remains so upon

being raised to the $^2/_3$ power in the sample lemma. The low-frequency coefficient function from a trilinear oscillation error incurs a derivative cost of λ_q (which we have grouped with "frequency gain") and is dominated by $(\pi_\ell)^{3/2} r_q^{-1}$, at which point the r_q^{-1} is scaled out due to the $L^1 - L^{3/2}$ scaling balance between current and stress errors (see (A.136a)–(A.136b)). The negative power in the frequency gain is determined by which shell of the oscillation error is being considered. The lower order terms may essentially be ignored. Next, we have an $L^{3/2} \to L^\infty$ intermittency loss, which is used to pointwise dominate the high-frequency portion of the pressure increment using the $L^{3/2}$ norm and prevent a loop of new current error and new pressure creation. Finally, we have an inverse divergence gain depending on which synthetic Littlewood-Paley shell of the pressure increment we are considering. The net effect is that the "frequency gain" upgrades the π_ℓ to π_q^m since $m \leq m'$, the half power of π_q^q is upgraded using $\lambda_q^{1/3}$ from the cost of $D_{t,q}$ and $\lambda_m^{-1/3}$ from the inverse divergence gain, and the remaining $\lambda_q^{2/3}\lambda_m^{-2/3}$ is strong enough to absorb the intermittency loss, with a perfect balance in the case

$$m = m' = q + \bar{n} \qquad \Longrightarrow \qquad \left(\frac{\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}/2}}\right)^{4/3} \lambda_q^{2/3} \lambda_{q+\bar{n}}^{-2/3} \approx 1\,.$$

In order to fill in the details, we now recall the choices of $\bar{m}$ and μ_m from (11.31). For the sake of brevity we ignore the slight variation in the case of the first projector for $m' = q + {}^{\bar{n}}/_2 + 2$ and focus on the second projector for $m' = q + {}^{\bar{n}}/_2 + 2$ and the other cases $q + {}^{\bar{n}}/_2 + 2 < m' \leq q + \bar{n}$. We have from (A.206) that

$$\phi_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}} = \phi^*_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}} + \sum_{\iota=0}^{m'-q-\bar{n}/2} \phi^\iota_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}\,.$$

and define the current error $\phi_{\overline{\phi}_O^{m'}} := \sum_{i,j,k,\xi,\vec{l},I,\diamond} \phi_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}$ which has a decomposition into

$$\begin{aligned}
\phi^{*}_{\overline{\phi}_O^{m'}} &= \sum_{i,j,k,\xi,\vec{l},I,\diamond} \phi^{*}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}, \qquad \phi^{q+\bar{n}/2+1}_{\overline{\phi}_O^{m'}} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \phi^{0}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}, \\
\phi^{q+\bar{n}/2+2}_{\overline{\phi}_O^{m'}} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond \\ \iota=1,2}} \phi^{\iota}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}, \\
\sigma^{q+\bar{n}/2+m}_{\overline{\phi}_O^{m'}} &= \sum_{\substack{i,j,k,\xi,\vec{l},I,\diamond \\ \iota=m}} \phi^{\iota}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}} \qquad \text{if } q+\bar{n}/2+m = q+\bar{n}/2+\iota \leq m'.
\end{aligned}$$

As in the previous case, we make further decomposition into the local and nonlocal parts, $\phi^{q+\bar{n}/2+m}_{\overline{\phi}_O^{m'}} = \phi^{q+\bar{n}/2+m,l}_{\overline{\phi}_O^{m'}} + \phi^{q+\bar{n}/2+m,*}_{\overline{\phi}_O^{m'}}$ using (ii). We have thus verified (11.38a) and (11.38b) immediately from these definitions and from (A.206) and item ii. In order to check (11.39a), we define the temporary notation $m(\iota)$ to make a correspondence between the value of ι above and the superscript on the left-hand side, which determines which bin the current errors go into. Specifically, we set $m(0) = 1$, $m(1) = m(2) = 2$, $m(\iota) = \iota$ if $q + \bar{n}/2 + \iota \leq m'$.Then from Part 4 of Proposition A.4.5, specifically (A.207c), and (4.24a), we have that for each $i, j, k, \xi, \vec{l}, I, \diamond, \iota$ and $M, N \leq 2\mathsf{N}_{\text{ind}}$,

$$\begin{aligned}
&\left| D^N D^M_{t,q} \mathcal{H}\left(D_{t,q} \sigma^{\iota}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}} \right) \right| \\
&\quad \lesssim \tau_q^{-1} \Gamma_q^{i+60} \pi_q^q \Lambda_q^{2/3} \left(\frac{r_{q+\bar{n}/2+m(\iota)}}{r_q} \right)^{2/3} \left(\lambda^{-2}_{m'-1} \lambda_{m'} \right)^{2/3} \left(\frac{\min(\lambda_{q+\bar{n}/2+m(\iota)}, \lambda_{q+\bar{n}}) \Gamma_q}{\lambda_{q+\bar{n}/2}} \right)^{4/3} \\
&\qquad \times \lambda^{-2}_{q+\bar{n}/2+m(\iota)-1} \lambda_{q+\bar{n}/2+m(\iota)} \left(\min(\lambda_{q+\bar{n}/2+m(\iota)}, \lambda_{m'}) \Gamma_q^2 \right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}} - 1, \tau_q^{-1} \Gamma_q^{i+14}, \mathrm{T}_q^{-1} \Gamma_q^9 \right).
\end{aligned}$$

Next, from (A.210) and (11.33), we have that

$$\begin{aligned}
\operatorname{supp}\left(\mathcal{H}\left(D_{t,q} \sigma^{\iota}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}} \right) \right) &\subseteq B\left(\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}, 2\lambda_{q+\bar{n}/2+m(\iota)-1} \Gamma_q^{-2} \right) \\
&\subseteq B\left(\operatorname{supp}\left(a_{(\xi),\diamond} (\varrho^{\diamond}_{(\xi)} \boldsymbol{\zeta}^I_{\xi}) \circ \Phi_{(i,k)} \rho^I_{(\xi),\diamond} \right), \lambda^{-1}_{m-1} + 2\lambda_{q+\bar{n}/2+m(\iota)-1} \Gamma_q^{-2} \right).
\end{aligned}$$

Then applying (9.22), we have that (11.40) is verified for $m = q + \bar{n}/2 + m(\iota)$. Returning to

the proof of (11.39a), we can now apply Corollary 8.6.4 with

$$H = \mathcal{H}\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}\right),$$
$$\varpi = \Gamma_q^{60}\pi_q^q\left(\frac{\Lambda_q\lambda_{m'}}{\lambda_{m'-1}^2}\cdot\frac{r_{q+\bar{n}/2+m(\iota)}}{r_q}\right)^{2/3}\left(\frac{\min(\lambda_{q+\bar{n}/2+m(\iota)},\lambda_{q+\bar{n}})\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3}\frac{\lambda_{q+\bar{n}/2+m(\iota)}}{\lambda^2_{q+\bar{n}/2+m(\iota)-1}}\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^I_{\xi})\circ\Phi_{(i,k)}}.$$

From (4.10h), (8.56b), (4.18), (6.6), (5.20), (4.24a), and (4.25), we have that

$$\begin{aligned}
&\left|\psi_{i,q}\sum_{i',j,k,\xi,\vec{l},I,\diamond}\mathcal{H}\left(D_{t,q}\sigma^{\iota}_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}\right)\right| \\
&\quad\lesssim \Gamma_q r_q^{-1}\lambda_q\left(\pi_q^q\right)^{1/2}\Gamma_q^{60}\pi_q^q\Lambda_q^{2/3}\left(\frac{r_{q+\bar{n}/2+m(\iota)}}{r_q}\right)^{2/3}\left(\lambda^{-2}_{m'-1}\lambda_{m'}\right)^{2/3}\left(\frac{\min(\lambda_{q+\bar{n}/2+m(\iota)},\lambda_{q+\bar{n}})\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3} \\
&\qquad\times\lambda^{-2}_{q+\bar{n}/2+m(\iota)-1}\lambda_{q+\bar{n}/2+m(\iota)}\left(\min(\lambda_{q+\bar{n}/2+m(\iota)},\lambda_{m'})\Gamma_q^2\right)^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\quad\lesssim \Gamma_q r_q^{-1}\lambda_q\left(\pi_q^{q+\bar{n}/2+m(\iota)}\frac{\delta_{q+\bar{n}}}{\delta_{q+\bar{n}/2+m(\iota)+\bar{n}}}\right)^{3/2}\Lambda_q^{2/3}\left(\lambda^{-2}_{m'-1}\lambda_{m'}\right)^{2/3}\left(\frac{\min(\lambda_{q+\bar{n}/2+m(\iota)},\lambda_{q+\bar{n}})\Gamma_q}{\lambda_{q+\bar{n}/2}}\right)^{4/3} \\
&\qquad\times\lambda^{-2}_{q+\bar{n}/2+m(\iota)-1}\lambda_{q+\bar{n}/2+m(\iota)}\left(\min(\lambda_{q+\bar{n}/2+m(\iota)},\lambda_{m'})\Gamma_q^2\right)^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right) \\
&\quad\leq \Gamma_q^{-150}r_q^{-1}\left(\pi_q^{q+\bar{n}/2+m(\iota)}\right)^{3/2}\left(\min(\lambda_{q+\bar{n}/2+m(\iota)},\lambda_{m'})\Gamma_q^2\right)^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+16},\mathrm{T}_q^{-1}\Gamma_q^9\right)
\end{aligned} \tag{11.45}$$

for all $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$, which verifies (11.39a) at level $m > q + \bar{n}/2 + 1$. In order to achieve (11.39b) and (11.41), we appeal to (A.208)–(A.209), (A.211), the choice of $K_\circ$ in item (xvi), (4.23b), and (4.24a).

□

11.2.2 Transport and Nash current errors

In this section, we estimate the current error

$$\begin{aligned}
\operatorname{div}\left(\overline{\phi}_T+\overline{\phi}_N\right) &= (\partial_t+\widehat{u}_q\cdot\nabla)\left(\frac{1}{2}|w_{q+1}|^2+\kappa_q^q-\frac{\operatorname{tr}(S_{q+1})}{2}\right) \\
&\quad+(\nabla\widehat{u}_q):\left(w_{q+1}\otimes w_{q+1}+R_q-\pi_q^q\mathrm{Id}-\overline{R}_{q+1}\right)-\mathfrak{m}'_T-\mathfrak{m}'_N.
\end{aligned} \tag{11.46}$$

Recall that from (10.6) and (10.8), we have

$$\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}\right)^{\alpha,\bullet} - \pi_\ell \mathrm{Id} + R_\ell = \sum_{\xi,i,j,k,\vec{l}} A^{\alpha,\bullet}_{(\xi),R}(\mathbb{P}_{\neq 0}\boldsymbol{\rho}_\xi^6)(\Phi_{(i,k)}) \tag{11.47a}$$

$$+ \sum_{\xi,i,j,k,\vec{l}} A^{\alpha,\bullet}_{(\xi),\varphi}\mathbb{P}_{\neq 0}\boldsymbol{\rho}_\xi^4(\Phi_{(i,k)})c_0c_1r_q^{\frac{2}{3}} \tag{11.47b}$$

$$+ c_0 \sum_{\xi,i,j,k,\vec{l}} A^{\alpha,\bullet}_{(\xi),\varphi} r_q^{\frac{2}{3}} \left(\boldsymbol{\rho}_\xi^4 \mathbb{P}_{\neq 0} \sum_I (\boldsymbol{\zeta}_\xi^I)^4\right) \circ \Phi_{(i,k)} \tag{11.47c}$$

$$+ \sum_{\xi,i,j,k,\vec{l},\diamond} A^{\alpha,\bullet}_{(\xi),\diamond} \left(\boldsymbol{\rho}_\xi^{2\diamond} \sum_I (\boldsymbol{\zeta}_\xi^I)^{2\diamond} \mathbb{P}_{\neq 0}(\varrho^I_{\xi,\diamond})^2\right)(\Phi_{(i,k)}) \tag{11.47d}$$

where $A^{\alpha,\bullet}_{(\xi),\diamond} := \xi^\theta \xi^\gamma \left(a^2_{(\xi),\diamond}(\nabla\Phi^{-1}_{(i,k)})^\alpha_\theta (\nabla\Phi^{-1}_{(i,k)})^\bullet_\gamma\right)$. To shorten notation, we introduce the following notation to denote the operator

$$L_{TN} := (\partial_t + \widehat{u}_q \cdot \nabla)\frac{1}{2}\mathrm{tr} + (\nabla \widehat{u}_q) : . \tag{11.48}$$

Using (10.2), we then write

$$(\partial_t + \widehat{u}_q \cdot \nabla)\left(\frac{1}{2}|w_{q+1}|^2 + \kappa_q^q - \frac{\mathrm{tr}\,(S_{q+1})}{2}\right) + (\nabla \widehat{u}_q) : \left(w_{q+1} \otimes w_{q+1} + R_q - \pi_q^q \mathrm{Id} - \overline{R}_{q+1}\right)$$

$$= L_{TN}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + R_\ell - \pi_\ell \mathrm{Id}\right) \tag{11.49}$$

$$+ L_{TN}\left(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)}\right) \tag{11.50}$$

$$+ L_{TN}\left(R_q^q - \pi_q^q \mathrm{Id} - R_\ell + \pi_\ell \mathrm{Id} - S_{q+1} + w_{q+1}^{(c)} \otimes w_{q+1}^{(c)}\right). \tag{11.51}$$

From (11.47), we have that (11.49) is actually equal to

$$(11.49) = L_{TN}\big((11.47\mathrm{a}) + (11.47\mathrm{b}) + (11.47\mathrm{c}) + (11.47\mathrm{d})\big). \tag{11.52}$$

These terms will have a good form since $D_{t,q}$ can never land on the high-frequency object, and so we will estimate them directly using the inverse divergence. We will estimate (11.50)

directly, using the fact that the high-frequency part of a product of principal and corrector parts has zero mean from Proposition 7.1.5, item 5 and Proposition 7.1.6, item 5, and so is amenable to the inverse divergence. The last term, on the other hand, can be written as

$$(11.51) = -L_{TN}\left(S_O + S_{TN} + S_{C1} + S_{M2}\right) \tag{11.53}$$

using (10.97) and (10.78). We now split the analysis of these error terms into several lemmas.

Lemma 11.2.3 (Current error and pressure increment from (11.49)). *There exist vector fields $\overline{\phi}_{TNW}$ and a function $\mathfrak{m}_{TNW}$ of time such that*

$$(11.49) = L_{TN}\big((11.47\text{a}) + (11.47\text{b}) + (11.47\text{c}) + (11.47\text{d})\big) = \mathrm{div}\overline{\phi}_{TNW} + \mathfrak{m}'_{TNW}, \qquad \overline{\phi}_{TNW} = \sum_{m=q+1}^{q+\bar{n}} \overline{\phi}^{m}_{TNW}$$

where $\overline{\phi}^{m}_{TNW} = \overline{\phi}^{m,l}_{TNW} + \overline{\phi}^{m,*}_{TNW}$ for $m \in \{q+1, \ldots, q+\bar{n}\}$ satisfy the following.

(i) The errors $\overline{\phi}^{q+1}_{TNW}$ and $\overline{\phi}^{q+\lfloor \bar{n}/2 \rfloor}_{TNW}$ require no pressure increment. More precisely, we have that for $N, M \leq \mathsf{N}_{\text{fin}}/100$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}^{q+1,l}_{TNW}\right| < \Gamma_{q+1}^{-100} \left(\pi_q^{q+1}\right)^{3/2} r_{q+1}^{-1} \lambda_{q+1}^{N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^{8}\right), \tag{11.54a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}^{q+\lfloor \bar{n}/2 \rfloor,l}_{TNW}\right| < \Gamma_{q+\bar{n}/2}^{-100} \left(\pi_q^{q+\bar{n}/2}\right)^{3/2} r_{q+\bar{n}/2}^{-1} \lambda_{q+\lfloor \bar{n}/2 \rfloor}^{N} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^{8}\right). \tag{11.54b}$$

(ii) For $q + {}^{\bar{n}}/_2 + 1 \leq m \leq \bar{n}$, there exists functions $\sigma^{-}_{\overline{\phi}^{m}_{TNW}} = \sigma^{+}_{\overline{\phi}^{m}_{TNW}} - \sigma^{-}_{\overline{\phi}^{m}_{TNW}}$ such that

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}^{m,l}_{TNW}\right| \lesssim \left((\sigma^{+}_{\overline{\phi}^{m}_{TNW}})^{3/2} r_m^{-1} + \delta^2_{q+3\bar{n}} \right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.55a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^{+}_{\overline{\phi}^{m}_{TNW}}\right| \lesssim \left(\sigma^{+}_{\overline{\phi}^{m}_{TNW}} + \delta_{q+3\bar{n}} \right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.55b}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^{+}_{\overline{\phi}^{m}_{TNW}}\right\|_{3/2} \lesssim \delta_{m+\bar{n}} \Gamma_m^{-9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.55c}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^{+}_{\overline{\phi}^{m}_{TNW}}\right\|_{\infty} \lesssim \Gamma_{q+1}^{\mathsf{C}_\infty - 9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.55d}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^{-}_{\overline{\phi}^{m}_{TNW}}\right| \lesssim \left(\frac{\lambda_q}{\lambda_{q+\lfloor \bar{n}/2 \rfloor}} \right)^{2/3} \pi_q^q \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.55e}$$

for all $N, M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_{100}$. Furthermore, we have that for $q+1 \leq m' \leq m-1$ and $q+1 \leq q'' \leq q + {}^{\bar{n}}/_2$,

$$\operatorname{supp} \sigma^{-}_{\overline{\phi}^{m}_{TNW}} \cap B\left(\operatorname{supp} \widehat{w}_{q''}, \lambda_{q''}^{-1}\Gamma_{q''+1}\right) = \operatorname{supp} \sigma^{+}_{\overline{\phi}^{m}_{TNW}} \cap B\left(\operatorname{supp} \widehat{w}_{m'}, \lambda_{m'}^{-1}\Gamma_{m'+1}\right) = \emptyset\,. \tag{11.56}$$

(iii) When $m = q+2, \ldots, q+\bar{n}$ and $q+1 \leq q' \leq m-1$, the local parts satisfy

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} \overline{\phi}^{m,l}_{TNW} = \emptyset\,. \tag{11.57}$$

(iv) For $m = q+1, \ldots, q+\bar{n}$ and $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$, the non-local parts $\overline{\phi}^{m,*}_{O}$ satisfy

$$\left\|D^N D^M_{t,q} \overline{\phi}^{m,*}_{TNW}\right\|_{L^\infty} \leq \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^{3/2} \lambda_m^N \tau_q^{-M}\,. \tag{11.58}$$

(v) For $M \leq 2\mathsf{N}_{\mathrm{ind}}$, the time function $\mathfrak{m}_{TNW}$ satisfies

$$\mathfrak{m}_{TNW}(t) = \int_0^t \langle (11.49)(s) \rangle\, ds\,, \quad \left|\frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{TNW}\right| \leq (\max(1,T))^{-1} \delta^2_{q+3\bar{n}} \tau_q^{-M}\,. \tag{11.59}$$

Proof. The analysis of this error is similar to that of the oscillation stress error dealt with in subsection 10.2.1, Lemmas 10.2.1–10.2.5. We will invert the divergence on this error term using Proposition A.3.3 and apply Proposition A.4.5 to construct the pressure increment. Let us define

$$\overline{\phi}_{TNW}^{q+1} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l}} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),R}\right)(\mathbb{P}_{\neq 0}\boldsymbol{\rho}_\xi^6)(\Phi_{(i,k)})\right]$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l}} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\varphi}\right)\mathbb{P}_{\neq 0}\boldsymbol{\rho}_\xi^4(\Phi_{(i,k)})c_0 c_1 r_q^{\frac{2}{3}}\right] \tag{11.60a}$$

$$\overline{\phi}_{TNW}^{q+\lfloor n/2\rfloor} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l}} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\varphi}\right)c_0 r_q^{\frac{2}{3}}\left(\boldsymbol{\rho}_\xi^4\mathbb{P}_{\neq 0}\sum_I(\boldsymbol{\zeta}_\xi^I)^4\right)\circ\Phi_{(i,k)}\right] \tag{11.60b}$$

$$\overline{\phi}_{TNW}^{q+\lfloor n/2\rfloor+1} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\diamond}\right)\left(\boldsymbol{\rho}_\xi^{2\diamond}(\boldsymbol{\zeta}_\xi^I)^{2\diamond}\widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1}\mathbb{P}_{\neq 0}(\varrho^I_{\xi,\diamond})^2\right)(\Phi_{(i,k)})\right] \tag{11.60c}$$

$$\overline{\phi}_{TNW}^{m} := (\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\diamond}\right)\left(\boldsymbol{\rho}_\xi^{2\diamond}(\boldsymbol{\zeta}_\xi^I)^{2\diamond}\widetilde{\mathbb{P}}^\xi_{(m-1,m]}(\varrho^I_{\xi,\diamond})^2\right)(\Phi_{(i,k)})\right] \tag{11.60d}$$

$$\overline{\phi}_{TNW}^{q+\bar{n}} := \sum_{m=q+\bar{n}}^{q+\bar{n}+1}(\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\diamond}\right)\left(\boldsymbol{\rho}_\xi^{2\diamond}(\boldsymbol{\zeta}_\xi^I)^{2\diamond}\widetilde{\mathbb{P}}^\xi_{(m-1,m]}(\varrho^I_{\xi,\diamond})^2\right)(\Phi_{(i,k)})\right] \tag{11.60e}$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[\sum_{\xi,i,j,k,\vec{l},I,\diamond} L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\diamond}\right)\left(\boldsymbol{\rho}_\xi^{2\diamond}(\boldsymbol{\zeta}_\xi^I)^{2\diamond}\left(\mathrm{Id}-\widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1}\right)(\varrho^I_{\xi,\diamond})^2\right)(\Phi_{(i,k)})\right] \tag{11.60f}$$

for $m = q + \bar{n}/2 + 2, \cdots, q+\bar{n}-1$. We decompose $\overline{\phi}_{TNW}^{m}$ into the nonlocal part $\overline{\phi}_{TNW}^{m,*}$ which involves the operator $\mathcal{R}^*$ or $\mathrm{Id} - \widetilde{\mathbb{P}}^\xi_{q+\bar{n}+1}$ and the local part $\overline{\phi}_{TNW}^{m,l}$ containing the remaining terms. For the undefined $\overline{\phi}_{TNW}^{m}$ corresponding to $m = q+2, \cdots, q+\bar{n}/2-1$, we set them as identically zero.

The construction of the pressure increment and the desired estimates will follow from applying Propositions A.3.3 and A.4.5. While many of the parameter choices will vary depending on the case, we fix the following choices throughout the proof:

$$v = \widehat{u}_q\,, \quad D_t = D_{t,q}\,, \quad N_* = {}^{\mathsf{N}_{\text{fin}}}/_4\,, \quad M_* = {}^{\mathsf{N}_{\text{fin}}}/_5\,, \tag{11.61a}$$

$$\lambda' = \Lambda_q\,, \quad M_t = \mathsf{N}_{\text{ind,t}}\,, \quad \nu' = \mathrm{T}_q^{-1}\Gamma_q^8\,, \quad \mathsf{N}_{\text{dec}} \text{ as in (xv)}\,. \tag{11.61b}$$

`Case 1: Estimates for` (11.60a). Fix values of $i, j, k, \xi, \vec{l}$ and consider the term which includes $L_{TN}A_{(\xi),R}$. We apply Proposition A.3.3 with the low-frequency choices

$$G = L_{TN}A_{(\xi),R}\,, \quad \mathcal{C}_{G,3/2} = \left|\operatorname{supp}\left(\eta^2_{i,j,k,\xi,\vec{l},R}\right)\right| \tau_q^{-1}\Gamma_q^{i+13}\delta_{q+\bar{n}}\Gamma_q^{2j+8}\,, \quad \mathcal{C}_{G,\infty} = \Gamma_q^{\mathsf{C}_\infty+14}\tau_q^{-1}\Gamma_q^{i_{\max}+13}\,,$$
$$\pi = \Gamma_q^{50}\tau_q^{-1}\Gamma_q^i\psi_{i,q}\pi_\ell\,, \quad \lambda = \lambda_{q+1}\Gamma_q^{-5}\,, \quad \nu = \tau_q^{-1}\Gamma_q^{i+14}\,, \quad \Phi = \Phi_{(i,k)}\,,$$

and the choices from (11.61). By Corollary 8.2.4, $\Phi_{(i,k)}$ satisfies (A.41) and (A.42a), and by (5.34) at level q and (4.10b), we have that (A.42b) is satisfied. To check (A.40), we observe that L_{TN} involves a material derivative and a multiplication by $\nabla\widehat{u}_q$. Therefore, by (5.34), G satisfies (A.40) for $p = {}^3/_2$ from (9.36c) and for $p = \infty$ from the same inequality and (8.27). Also, (A.59) is satisfied by (9.38). To check the high-frequency assumptions, we set (exactly as in the analogous case for the oscillation stress error - see Lemmas 10.2.1–10.2.5)

$$\varrho = \left(\mathbb{P}_{\neq 0}\overline{\boldsymbol{\rho}}^6_\xi\right)\,, \quad \mathsf{d} \text{ as in (xvii)}\,, \quad \vartheta = \delta_{i_1 i_2}\delta_{i_3 i_4}\ldots\delta_{i_{\mathsf{d}-1} i_{\mathsf{d}}}\Delta^{-\mathsf{d}/2}\varrho\,,$$
$$\mu = \Upsilon = \Upsilon' = \lambda_{q+1}\Gamma_q^{-4}\,, \quad \overline{\Lambda} = \lambda_{q+1}\Gamma_q^{-1}\,, \quad \mathcal{C}_{*,1} = \Gamma_q^6\lambda_{q+1}^\alpha\,.$$

Since the choice of parameters is exactly the same as in the oscillation stress error, we see that the other high frequency assumptions are satisfied. In order to check the nonlocal assumptions, we set

$$M_\circ = N_\circ = 2\mathsf{N}_{\text{ind}}\,, \quad K_\circ \text{ as in (xvi)}\,, \quad \mathcal{C}_v = \Lambda_q^{1/2}\,. \tag{11.63}$$

Then from (4.23b) and Remark A.3.4, we have that (A.52)–(A.55) are satisfied.

We can therefore apply Remark A.3.9. Note that (A.59) follows from the definition of $L_{TN}A_{(\xi),R}$ in (10.7) and (9.38a). Then, abbreviating $G\varrho\circ\Phi$ as $t_{i,j,k,\xi,\vec{l},R}$, from (A.47), (A.49a), and (A.60), we have that for all $N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$

$$\left|D^N D^M_{t,q}\mathcal{H}t_{i,j,k,\xi,\vec{l},R}\right| \lesssim \tau_q^{-1}\Gamma_q^i\psi_{i,q}\pi_\ell\Gamma_q^{60}\lambda_{q+1}^{-1}\lambda_{q+1}^{N+\alpha}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) .$$

Notice that from (ii), we have

$$\operatorname{supp}\left(\operatorname{div}\mathcal{H}t_{i,j,k,\xi,\vec{l},R}\right) \subseteq \operatorname{supp} t_{i,j,k,\xi,\vec{l},R} \subseteq \operatorname{supp}\eta_{i,j,k,\xi,\vec{l},R} . \tag{11.64}$$

As for the terms which include $A^{\alpha,\bullet}_{(\xi),\varphi}$ from (11.60a), we note that from Lemma 9.3.1 $a^2_{(\xi),\varphi}$ differs in size relative to $a^2_{(\xi),R}$ by a factor of $r_q^{-2/3}$, which is exactly balanced out by the factor of $r_q^{2/3}$ in (11.60a). We therefore may argue exactly as above (in fact the estimates are slightly better since $\overline{\boldsymbol{\rho}}^4_\xi < \overline{\boldsymbol{\rho}}^6_\xi$), and we omit further details. In this case, we use the abbreviation $t_{i,j,k,\xi,\vec{l},\varphi}$ instead of $t_{i,j,k,\xi,\vec{l},R}$, which will satisfy an analogous support property to (11.64).

We now set

$$\overline{\phi}^{q+1,l}_{TNW} = \sum_{i,j,k,\xi,\vec{l},\diamond} \mathcal{H}t_{i,j,k,\xi,\vec{l},\diamond} .$$

Using (11.64) and applying the aggregation Corollary 8.6.4 with $H = \mathcal{H}t_{i,j,k,\xi,\vec{l},\diamond}$ and

$$\varpi = \pi_\ell\Gamma^{60}\lambda_{q+1}^{-1+\alpha}, \quad \lambda = \Lambda = \lambda_{q+1}, \quad \tau = \tau_q\Gamma_q^{-14}, \quad \mathrm{T} = \mathrm{T}_q\Gamma_q^{-8}$$

to get an estimate from (8.56a),

$$\left|\psi_{i,q}D^N D^M_{t,q}\overline{\phi}^{q+1,l}_{TNW}\right| \lesssim r_q^{-1}\lambda_q(\pi_q^q)^{1/2}\pi_\ell\Gamma_q^{61}\lambda_{q+1}^{-1}\lambda_{q+1}^{N+\alpha}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^8\right) .$$

for N, M in the same range as above. Then, (11.54a) follows from this term using (6.6), (5.20b) and (4.24a).

For the non-local term, from (A.57), and Remark A.3.4, we have that for $N, M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left\| D^N D_{t,q}^M \sum_{i,j,k,\xi,\vec{l}} \mathcal{R}^* t_{i,j,k,\xi,\vec{l},R} \right\|_\infty \leq \delta_{q+3\bar{n}}^{3/2} \mathrm{T}_q^{2\mathsf{N}_{\text{ind,t}}} \lambda_{q+1}^N \tau_q^{-M} ,$$

matching the desired estimate in (11.58). The estimate in (11.59) follows similarly using Remark A.3.7 and a large choice of a_* as in Lemma 11.2.2, and we omit further details. The version of these estimates in the later cases will again be similar, and so we do not address them again.

`Case 2: Estimates for` (11.60b). As before, we fix $i, j, k, \xi, \vec{l}$. We apply Proposition A.3.3 with Remark A.3.9 with the low-frequency choices

$$G = L_{TN} A_{(\xi),\varphi} c_0 r_q^{2/3} \overline{\boldsymbol{\rho}}_\xi^4 (\Phi_{(i,k)}) , \quad \mathcal{C}_{G,3/2} = \left| \operatorname{supp} \eta_{i,j,k,\xi,\vec{l},\varphi}^2 \right| \tau_q^{-1} \Gamma_q^i \delta_{q+\bar{n}} \Gamma_q^{20} , \quad \mathcal{C}_{G,\infty} = \Gamma_q^{\mathsf{C}_\infty + 20} \tau_q^{-1} \Gamma_q^{i_{\max}} , \tag{11.65a}$$

$$\pi = \Gamma_q^{50} \tau_q^{-1} \Gamma_q^i \psi_{i,q} \pi_\ell , \quad \lambda = \lambda_{q+1} \Gamma_q^{-1} , \quad \nu = \tau_q^{-1} \Gamma_q^{i+13} , \quad \Phi = \Phi_{(i,k)} , \tag{11.65b}$$

as well as the choices from (11.61). As in the previous substep, (A.41), (A.42a), and (A.42b) are satisfied. The estimates in (A.40) hold due to Proposition 7.2.1 and the estimates for $L_{TN} A_{(\xi),\varphi}$ from Case 1.

To check the high-frequency assumptions, we set the parameters and functions exactly as in `Case 2` in the proof of Lemma 10.2.1. Since we work with $p = 1$ instead of $p = \frac{3}{2}$, the only difference is that $\mathcal{C}_{*,1} := \mathcal{C}_{*,\infty} = \lambda_{q+\bar{n}/2}^\alpha$ instead of $\mathcal{C}_{*,3/2}$. Then, as before, high-frequency assumptions in (i)–(iv) can be verified. The nonlocal assumptions are identical to those of Case 1, and are satisfied trivially. The non-local parameters are set to be the same as in the previous case.

We therefore may appeal to the local conclusions (i)–(vi) and (A.56)–(A.57), from which

we have the following. First, abbreviating $G\varrho\circ\Phi$ as $t_{i,j,k,\xi,\vec{l},\varphi}$, we have from (A.46) and (A.50) that for $N \leq \frac{\mathsf{N}_{\text{fin}}}{4} - \mathsf{d}$ and $M \leq \frac{\mathsf{N}_{\text{fin}}}{5}$,

$$\left| D^N D^M_{t,q} \mathcal{H} t_{i,j,k,\xi,\vec{l},\varphi} \right| \lesssim \tau_q^{-1} \Gamma_q^i \psi_{i,q} \pi_\ell \Gamma_q^{50} \lambda_{q+\bar{n}/2}^{-1} \lambda_{q+\bar{n}/2}^{N+\alpha} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^8\right) ,$$

Notice that from (ii), the support of $\operatorname{div}\mathcal{H}t_{i,j,k,\xi,\vec{l},\varphi}$ is contained in $\operatorname{supp} t_{i,j,k,\xi,\vec{l},\varphi} \subset \operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\varphi}\right)$. Thus as before we may apply the aggregation Corollary 8.6.4 with $H = \mathcal{H}t_{i,j,k,\xi,\vec{l},R}$ and

$$\varpi = \pi_\ell \Gamma^{50} \lambda_{q+\bar{n}/2}^{-1}, \quad \lambda = \Lambda = \lambda_{q+\bar{n}/2}, \quad \tau = \tau_q \Gamma_q^{-14}, \quad \mathrm{T} = \mathrm{T}_q \Gamma_q^{-8}$$

to estimate

$$\overline{\phi}_{TNW}^{q+\bar{n}/2,l} = \sum_{i,j,k,\xi,\vec{l}} \mathcal{H}t_{i,j,k,\xi,\vec{l},\varphi} .$$

From (8.56a), we thus have that for N, M in the same range as above,

$$\left| \psi_{i,q} D^N D^M_{t,q} \overline{\phi}_{TNW}^{q+\bar{n}/2,l} \right| \lesssim r_q^{-1} \lambda_q (\pi_q^q)^{1/2} \pi_\ell \Gamma_q^{50} \lambda_{q+\bar{n}/2}^{-1} \lambda_{q+\bar{n}/2}^{N+\alpha} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q \Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^8\right) .$$

and so we can conclude (11.54b) as before. we must verify (11.57) for $\overline{\phi}_{TNW}^{q+\bar{n}/2,l}$. This however follows from (iii), which asserts that the support of $\overline{\phi}_{TNW}^{q+\bar{n}/2,l}$ is contained in $\cup_{(\xi)}\operatorname{supp}\left(a_{(\xi),\varphi}\boldsymbol{\rho}^{\varphi}_{(\xi)}\circ \Phi_{(i,k)}\right)$, and (i) of Lemma 9.2.2. The non-local conclusions also follow in much the same way as in Case 1, and we omit further details.

`Case 3: Estimates of the local portions of` (11.60c), (11.60d), `and` (11.60e). Fix ξ, i, j, k, $\vec{l}$, I, and $\diamond$. In order to check the low-frequency, preliminary assumptions in Part

1 of Proposition A.4.5, we set

$$
\begin{aligned}
&p = 1, \infty\,, \quad G_R = L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\diamond}\right)\left(\boldsymbol{\rho}^{2\diamond}_{\xi}(\boldsymbol{\zeta}^{I}_{\xi})^{2\diamond}\right)(\Phi_{(i,k)})\,, \quad G_\varphi = L_{TN}\left(A^{\alpha,\bullet}_{(\xi),\diamond}\right)\left(\boldsymbol{\rho}^{2\diamond}_{\xi}(\boldsymbol{\zeta}^{I}_{\xi})^{2\diamond}\right)(\Phi_{(i,k)})r_q^{2/3}\,,\\
&\mathcal{C}_{G_\diamond,1} = \delta_{q+\bar{n}}\tau_q^{-1}\Gamma_q^{i+2j+20}\left|\mathrm{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right| + \lambda_{q+\bar{n}}^{-10}\,, \quad \mathcal{C}_{G_\diamond,\infty} = \delta_{q+\bar{n}}\tau_q^{-1}\Gamma_q^{i+2j+20}\,,\\
&\lambda = \lambda_{q+\bar{n}/2}\,, \quad \nu = \tau_q^{-1}\Gamma_q^{i+14}\,, \quad \Phi = \Phi_{(i,k)}\,, \quad \pi = \Gamma_q^{50}\pi_\ell\lambda_q^{2/3}\,, \quad r_G = r_q\,.
\end{aligned}
\tag{11.66}
$$

Then we have that (A.39) is satisfied by definition, (A.40) is satisfied by (9.36b), (9.36d), Corollary 8.2.4, (7.23), and Definition 7.2.4, (A.41)–(A.42b) hold from Corollary 8.2.4 and (5.34) at level q, and (A.132b) holds from (9.38), Remark 5.3.2, and (6.6).

In order to check the high-frequency, preliminary assumptions in Part 1 of Proposition A.4.5, we choose parameters and functions exactly same as in `Case 3` and `Case 4` of Lemma 10.2.1. The only difference is that we use $C_{*,1}$ instead of $C_{*,3/2}$. Indeed, we choose $C_{*,1} = \lambda^{\alpha}_{q+\bar{n}/2+1}$ in both cases `Case 3a` and `Case 3b`. Then, it is enough to check (A.43), which holds true due to Propositions 7.1.5 and 7.1.6 and estimate (7.37a) from Lemma 7.3.3 or 7.3.4 with $q = 1$. In order to check the additional assumptions in Part 2 of Proposition A.4.5, we again choose the same parameters and functions as in as in `Case 3` and `Case 4` of Lemma 10.2.4, and set the extra parameters as $\delta_{\phi,p}$ and r_ϕ are

$$
\delta^{3/2}_{\phi,p} = \mathcal{C}_{G_\diamond,p}\mathcal{C}_{*,p}\Upsilon'\Upsilon^{-2}r_{\min(m,q+\bar{n})}\,, \quad r_\phi = r_{\min(m,q+\bar{n})}\,.
$$

Compared to Proposition A.4.4, we only need to check (A.198c), (A.199c), and (A.199d), which can be verified by (4.17b), (4.23b), and (4.23c).

Using the abbreviation $t^m_{i,j,k,\xi,\vec{l},I,\diamond}$ for $G\varrho\circ\Phi$ at the level of $q+\bar{n}/2+2 \le m \le q+\bar{n}+1$, as a consequence of (A.200)–(A.202), (4.24a), (4.18), (6.6), and (4.10h), there exists a pressure

increment $\sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$ such that for $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_{7}$,

$$\left| D^N D^M_{t,q} \mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond} \right| \lesssim \left(\left(\sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right)^{\frac{3}{2}} r^{-1}_{\min(m,q+\bar{n})} + \delta^2_{q+3\bar{n}} \right) \times (\min(\lambda_m, \lambda_{q+\bar{n}})\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{11.67}$$

$$\left| D^N D^M_{t,q} \sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right| \lesssim \left(\sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} + \delta^2_{q+3\bar{n}} \right) (\min(\lambda_m, \lambda_{q+\bar{n}})\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{11.68}$$

$$\begin{aligned} \left| D^N D^M_{t,q} \sigma^{-}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right| &\lesssim \left(\frac{r_{\min(m,q+\bar{n})}}{r_q} \right)^{2/3} \Gamma_q^{28} \pi_\ell \Lambda_q^{2/3} \left(\lambda^2_{m-1}\lambda^{-1}_m\right)^{-2/3} (\lambda_{q+\bar{n}/2}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \\ &\leq \Gamma_q^{-10} \left(\frac{\lambda_q}{\lambda_{q+\bar{n}/2}} \right)^{2/3} \pi_q^q (\lambda_{q+\bar{n}/2}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \end{aligned}$$

From (A.48), (A.205), and (7.40c), we have that

$$\begin{aligned} \operatorname{supp}\left(\sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right) &\subseteq \operatorname{supp}\left(\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond} \right) \\ &\subseteq \operatorname{supp}\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^I_\xi\right) \circ \Phi_{(i,k)} \right) \cap B\left(\operatorname{supp} \varrho^I_{(\xi),\diamond}, \lambda^{-1}_{m-1}\right) \circ \Phi_{(i,k)} , \end{aligned} \tag{11.69}$$

$$\operatorname{supp}\left(\sigma^{-}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \right) \subseteq \operatorname{supp}\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^I_\xi\right) \circ \Phi_{(i,k)} \right) \tag{11.70}$$

Then, we can obtain the desired estimates for

$$\overline{\phi}^{m.l}_{TNW} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}, \qquad \overline{\phi}^{q+\bar{n}.l}_{TNW} = \sum_{m=q+\bar{n}}^{q+\bar{n}+1} \sum_{i,j,k,\xi,\vec{l},I,\diamond} \mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}, ,$$

$$\sigma^{\pm}_{\overline{\phi}^m_{TNW}} = \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}} \qquad \sigma^{\pm}_{\overline{\phi}^{q+\bar{n}}_{TNW}} = \sum_{m=q+\bar{n}}^{q+\bar{n}+1} \sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$$

for $q + {}^{\bar{n}}/_2 + 1 \leq m < q + \bar{n}$ by applying Corollary 8.6.3 with $p = 1$ and

$$H = \mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}\,, \qquad \varpi = \mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}, \qquad \text{for (11.55a)}$$

$$H = \sigma^{+}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,, \qquad \varpi = \left[H + \delta^2_{q+3\bar{n}}\right]\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}, \qquad \text{for (11.55b)}$$

$$H = \sigma^{-}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\,, \qquad \varpi = \left(\frac{\lambda_q}{\lambda_{q+\bar{n}/2}}\right)^{2/3}\pi_\ell\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}(\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi})\circ\Phi_{(i,k)}}, \qquad \text{for (11.13e)}\,.$$

Also, (9.22)–(9.24), (11.69), and (11.70) give that (11.57) and (11.56) are satisfied for $q + {}^{\bar{n}}/_2 + 1 \leq m' \leq q + \bar{n}$.

Next, from (A.203), we have that

$$\begin{aligned}
\left\|\sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right\|_{3/2} &\lesssim \delta^{2/3}_{q+\bar{n}}\tau_q^{-2/3}\Gamma_q^{2/3(i+2j+24)}\left|\operatorname{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{2/3}\left(\lambda^2_{m-1}\lambda_m^{-1}\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})}\,,\\
\left\|\sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}\right\|_{\infty} &\lesssim \delta^{2/3}_{q+\bar{n}}\tau_q^{-2/3}\Gamma_q^{2/3(i+2j+24)}\left(\frac{\min(\lambda_m,\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^{4/3}\left(\lambda^2_{m-1}\lambda_m^{-1}\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})}\\
&\lesssim \Gamma_{q-\bar{n}}^{\frac{2}{3}\left(\frac{\mathsf{C}_\infty}{2}+18\right)} r^{-2/3}_{q-\bar{n}} r^{2/3}_{\min(m,q+\bar{n})}\Gamma_q^{\frac{2}{3}(40+\mathsf{C}_\infty)}\left(\frac{\min(\lambda_m,\lambda_{q+\bar{n}})}{\lambda_{q+\bar{n}/2}\Gamma_q}\right)^{4/3}\Lambda_q^{2/3}\left(\lambda^2_{m-1}\lambda_m^{-1}\right)^{-2/3}\\
&\leq \Gamma^{\mathsf{C}_\infty-11}_{q+\bar{n}/2+1}\,.
\end{aligned} \tag{11.71}$$

The last two inequalities follow from (5.10), (8.27) and (4.13a). Then, we apply Corollary 8.6.1 to $\theta = 2$, $\theta_1 = {}^2/_3$, $\theta_2 = {}^4/_3$, $H = \sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$, and $p = {}^3/_2$, which gives

$$\left\|\psi_{i,q}\sigma^{\pm}_{\overline{\phi}^m_{TNW}}\right\|_{3/2} \lesssim \delta^{2/3}_{q+\bar{n}}\tau_q^{-2/3}\Gamma_q^{20+2/3\mathsf{C}_b}\left(\lambda^2_{m-1}\lambda_m^{-1}\right)^{-2/3} r^{2/3}_{\min(m,q+\bar{n})} \leq \delta_{m+\bar{n}}\Gamma_m^{-10}\,.$$

from (4.27c). Combined with (11.55b), this verifies (11.55c) for $q + {}^{\bar{n}}/_2 + 2 \leq m' \leq q + \bar{n}$. On the other hand, from Corollary 8.6.3 with $H = \sigma^{\pm}_{\mathcal{H}t^m_{i,j,k,\xi,\vec{l},I,\diamond}}$, $\varpi = \Gamma^{\mathsf{C}_\infty-11}_{q+\bar{n}/2+1}\mathbf{1}_{\operatorname{supp} a_{(\xi),\diamond}\boldsymbol{\rho}^{\diamond}_{(\xi)}\boldsymbol{\zeta}^{I}_{\xi}}$ and $p = 1$, we have that

$$\left\|\psi_{i,q}\sigma^{\pm}_{\overline{\phi}^m_{O}}\right\|_{\infty} \leq \Gamma^{\mathsf{C}_\infty-10}_{q+\bar{n}/2+1}\,.$$

Combined again with (11.55b), this verifies (11.55d) at level $q+{}^{\bar{n}}/_{2}+1 \leq m' \leq q+\bar{n}$. Lastly, we have that (11.58) at level m' for $q+{}^{\bar{n}}/_{2}+1 \leq m' < q+\bar{n}$ and for the nonlocal part of (11.60e) are satisfied by an argument essentially identical to that of the previous case.

`Case 4: Estimate of` (11.60f). Here we apply Proposition A.3.3 with $p=\infty$ and the following choices. The low-frequency assumptions in Part 1 are exactly the same as the L^∞ low-frequency assumptions in the previous two steps. For the high-frequency assumptions, we recall the choice of N_{**} from (xvii) and set

$$\varrho_R = (\mathrm{Id} - \widetilde{\mathbb{P}}^{\xi}_{q+\bar{n}+1})\mathbb{P}_{\neq 0}\left(\varrho^I_{(\xi),R}\right)^2 , \quad \varrho_\varphi = (\mathrm{Id} - \widetilde{\mathbb{P}}^{\xi}_{q+\bar{n}+1})\mathbb{P}_{\neq 0}\left(\varrho^I_{(\xi),\varphi}\right)^2 r_q^{-2/3} , \quad \vartheta^{i_1 i_2 \dots i_{\mathsf{d}-1} i_{\mathsf{d}}}_{\diamond} = \delta^{i_1 i_2 \dots i_{\mathsf{d}-1} i_{\mathsf{d}}} \Delta^{-\mathsf{d}/2} \varrho_\diamond$$
$$\Lambda = \lambda_{q+\bar{n}} , \quad \mu = \Upsilon = \Upsilon' = \lambda_{q+{}^{\bar{n}}/_{2}} \Gamma_q , \quad \mathcal{C}_{*,\infty} = \left(\frac{\lambda_{q+\bar{n}}}{\lambda_{q+\bar{n}+1}}\right)^{N_{**}} \lambda^3_{q+\bar{n}} , \quad \mathsf{N}_{\mathrm{dec}} \text{ as in (xv)} , \quad \mathsf{d} = 0 .$$

Then we have that item (i) is satisfied by definition, item (ii) is satisfied as in the previous steps, (A.43) is satisfied using Propositions 7.1.5 and 7.1.6 and (7.37b) from Lemma 7.3.3, (A.44) is satisfied by definition and as in the previous steps, and (A.45) is satisfied by (4.21). For the non-local assumptions, we choose $M_\circ, N_\circ = 2\mathsf{N}_{\mathrm{ind}}$ so that (A.52)–(A.54) are satisfied as in Case 1, and (A.55) is satisfied from (4.23c). We have thus satisfied all the requisite assumptions, and we therefore obtain non-local bounds very similar to those from the previous steps, which are consistent with (11.58) at level $q+\bar{n}$. We omit further details. □

Lemma 11.2.4 (Current error and pressure increment from divergence correctors). *There exist vector fields $\overline{\phi}_{TNC}$ and a function $\mathfrak{m}_{TNC}$ of time such that*

$$(11.50) = \mathrm{div}\left(\overline{\phi}_{TNC}\right) + \mathfrak{m}'_{TNC} , \qquad \overline{\phi}_{TNC} = \sum_{m=q+{}^{\bar{n}}/_{2}+1}^{q+\bar{n}} \mathrm{div}\overline{\phi}^m_{TNC} , \tag{11.72}$$

where $\overline{\phi}^m_{TNC} = \overline{\phi}^{m,l}_{TNC} + \overline{\phi}^{m,*}_{TNC}$ for $q+{}^{\bar{n}}/_{2}+1 \leq m \leq q+\bar{n}$ satisfy the following.

(i) For $q + {}^{\bar{n}}/_2 + 1 \leq m \leq q + \bar{n}$, there exist functions $\sigma_{\overline{\phi}^m_{TNC}} = \sigma^+_{\overline{\phi}^m_{TNC}} - \sigma^-_{\overline{\phi}^m_{TNC}}$ such that

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}^m_{TNC}\right| \lesssim \left((\sigma^+_{\overline{\phi}^m_{TNC}})^{3/2} r_m^{-1} + \delta^2_{q+3\bar{n}} \right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.73a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{\overline{\phi}^m_{TNC}}\right| \lesssim \left(\sigma^+_{\overline{\phi}^m_{TNC}} + \delta_{q+3\bar{n}} \right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.73b}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{\overline{\phi}^m_{TNC}}\right\|_{3/2} \lesssim \delta_{m+\bar{n}} \Gamma_m^{-9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.73c}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{\overline{\phi}^m_{TNC}}\right\|_{\infty} \lesssim \Gamma_m^{\mathsf{C}_\infty - 9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.73d}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^-_{\overline{\phi}^m_{TNC}}\right| \lesssim \left(\frac{\lambda_q}{\lambda_{q+\lfloor \bar{n}/2 \rfloor}} \right)^{\frac{2}{3}} \pi_q^q \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.73e}$$

for all $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_{100}$. Furthermore, we have that for $q + 1 \leq m' \leq m - 1$ and $q + 1 \leq q'' \leq q + {}^{\bar{n}}/_2$,

$$\operatorname{supp} \sigma^-_{\overline{\phi}^m_{TNC}} \cap B\left(\operatorname{supp} \widehat{w}_{q''}, \lambda_{q''}^{-1}\Gamma_{q''+1}\right) = \operatorname{supp} \sigma^+_{\overline{\phi}^m_{TNC}} \cap B\left(\operatorname{supp} \widehat{w}_{m'}, \lambda_{m'}^{-1}\Gamma_{m'+1}\right) = \emptyset\,. \tag{11.74}$$

(ii) For all $q + {}^{\bar{n}}/_2 + 1 \leq m \leq q + \bar{n}$ and $q + 1 \leq q' \leq m - 1$, the local parts satisfy

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} \phi^{m,l}_{TNC} = \emptyset\,. \tag{11.75}$$

(iii) For all $q + {}^{\bar{n}}/_2 + 1 \leq m \leq q + \bar{n}$ and $N, M \leq 2\mathsf{N}_{\text{ind}}$, the non-local parts $\overline{\phi}^{m,*}_{TNC}$ satisfy

$$\left\|D^N D^M_{t,q} \overline{\phi}^{m,*}_{TNC}\right\|_{L^\infty} \leq \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}} \delta_{q+3\bar{n}}^{3/2} \lambda_m^N \tau_q^{-M}\,. \tag{11.76}$$

(iv) For $M \leq 2\mathsf{N}_{\text{ind}}$, the time function $\mathfrak{m}_{TNC}$ satisfies

$$\mathfrak{m}_{TNC}(t) = \int_0^t \langle (11.50)(s) \rangle \, ds \,, \quad \left| \frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{TNC} \right| \leq (\max(1,T))^{-1} \, \delta_{q+3\bar{n}}^2 \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right) \,. \tag{11.77}$$

Proof. The proof is similar to Step 2 of the proof of Lemma 10.2.10. In fact, it is much simpler since the $D_{t,q}$ in L_{TN} is always a "good" derivative. We provide a few details below.

First note that

$$\begin{aligned}
L_{TN}\left(w_{q+1}^{(p)} \otimes_s w_{q+1}^{(c)}\right) &= \sum_{\diamond,i,j,k,\xi,\vec{l},I} L_{TN}\Big[a_{(\xi),\diamond}\left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond} \varrho_{(\xi),\diamond}^{I}\right) \circ \Phi_{(i,k)} \xi^{\ell} \big(A_{\ell}^{m} \epsilon_{\bullet pr} + A_{\ell}^{\bullet} \epsilon_{mpr}\big) \\
&\qquad \times \partial_p \left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond}\right) \circ \Phi_{(i,k)} \right) \partial_r \Phi_{(i,k)}^{s} (\mathbb{U}_{(\xi),\diamond}^{I})^{s} \circ \Phi_{(i,k)} \Big] \\
&= \sum_{\diamond,i,j,k,\xi,\vec{l},I} L_{TN}\Big[a_{(\xi),\diamond}\left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond}\right) \circ \Phi_{(i,k)} \xi^{\ell} \big(A_{\ell}^{m} \epsilon_{\bullet pr} + A_{\ell}^{\bullet} \epsilon_{mpr}\big) \\
&\qquad \times \partial_p \left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}_{(\xi)}^{\diamond} \boldsymbol{\zeta}_{\xi}^{I,\diamond}\right) \circ \Phi_{(i,k)} \right) \partial_r \Phi_{(i,k)}^{s} \Big] (\varrho_{(\xi),\diamond}^{I} \mathbb{U}_{(\xi),\diamond}^{I})^{s} \circ \Phi_{(i,k)} \\
&=: \sum_{\diamond,i,j,k,\xi,\vec{l},I} G_{\diamond,i,j,k,\xi,\vec{l},I} (\varrho_{(\xi),\diamond}^{I} \mathbb{U}_{(\xi),\diamond}^{I})^{s} \circ \Phi_{(i,k)}
\end{aligned}$$

We note that $(\varrho_{(\xi),\diamond}^{I} \mathbb{U}_{(\xi),\diamond}^{I})^{s}$ has mean 0 (by property (5) of Proposition 7.1.5 and (5) of Proposition 7.1.6) and is $\frac{\mathbb{T}^d}{\lambda_{q+\bar{n}/2}\Gamma_q}$ -periodic. So just as in the Divergence corrector stress error, we apply the synthetic Littlewood-Paley decomposition suggested in (7.34) and define

the current errors as follows

$$\overline{\phi}_{TNC}^{q+\bar{n}/2+1} := \sum_{\diamond,i,j,k,\xi,\vec{l},I} (\mathcal{H}+\mathcal{R}^*)\left(G_{\diamond,i,j,k,\xi,\vec{l},I}\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}/2+1}}(\varrho^I_{(\xi),\diamond}\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}\right) \tag{11.78}$$

$$\overline{\phi}_{TNC}^{m} := \sum_{\diamond,i,j,k,\xi,\vec{l},I} (\mathcal{H}+\mathcal{R}^*)\left(G_{\diamond,i,j,k,\xi,\vec{l},I}\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}(\varrho^I_{(\xi),\diamond}\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}\right) \tag{11.79}$$

$$\overline{\phi}_{TNC}^{q+\bar{n}} := \sum_{m=q+\bar{n}}^{q+\bar{n}+1}\sum_{\diamond,i,j,k,\xi,\vec{l},I} (\mathcal{H}+\mathcal{R}^*)\left(G_{\diamond,i,j,k,\xi,\vec{l},I}\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}(\varrho^I_{(\xi),\diamond}\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}\right) \tag{11.80}$$

$$+\sum_{\diamond,i,j,k,\xi,\vec{l},I} (\mathcal{H}+\mathcal{R}^*)\left(G_{\diamond,i,j,k,\xi,\vec{l},I}\left(\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}}\right)(\varrho^I_{(\xi),\diamond}\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}\right) \tag{11.81}$$

We shall apply the inverse divergence operator to each term in the sum separately with the following choices. In all cases, we set

$$G_R = \lambda_{q+\bar{n}}^{-1}G_{R,i,j,k,\xi,\vec{l},I}\,,\quad G_\varphi = \lambda_{q+\bar{n}}^{-1}r_q^{2/3}G_{\varphi,i,j,k,\xi,\vec{l},I}\,.$$

We choose the high-frequency potentials as in **Step 2** of the proof of Lemma 10.2.10, and choose the rest of parameters and functions required in Proposition A.4.5 the same as in `Case 3` of the proof of Lemma 11.2.3. In fact, the size of $G_{\diamond,1}$ and $G_{\diamond,\infty}$ is smaller than the one in `Case 3`. By the same argument as in `Case 3`, we then get the same conclusion as in Lemma 11.2.3 for $\overline{\phi}_{TNC}^{m}$. We omit further details.

□

Lemma 11.2.5 (**Current error and pressure increment from** (11.51)**).** *There exist vector field $\overline{\phi}_{TNS}$ and a function $\mathfrak{m}_{TNS}$ of time such that*

$$(11.51) = -L_{TN}\left(S_O+S_{TN}+S_{C1}+S_{M2}\right) = \mathrm{div}\overline{\phi}_{TNS}+\mathfrak{m}'_{TNS}\,,\qquad \overline{\phi}_{TNS} = \sum_{m=q+1}^{q+\bar{n}}\overline{\phi}_{TNS}^{m}\,,$$

where $\overline{\phi}_{TNS}^{m} = \overline{\phi}_{TNS}^{m,l}+\overline{\phi}_{TNS}^{m,*}+\overline{\phi}_{TNS}^{*}$ satisfies the following properties.

(i) For $m = q+1,\ q+{}^{\bar{n}}/_{2}$, the local part $\overline{\phi}^{m}_{TNS}$ satisfies

$$\left|\psi_{i,q} D^{N} D^{M}_{t,q} \overline{\phi}^{m,l}_{TNS}\right| \lesssim \Gamma_q^{-12} (\pi_q^m)^{3/2} r_q^{-1} \lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^{8}\right) \tag{11.82}$$

for $M, N \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_{100}$.

(ii) For $m = q+{}^{\bar{n}}/_{2}+1, \ldots, q+\bar{n}$, there exists functions $\sigma_{\overline{\phi}^{m}_{TNS}} = \sigma^{+}_{\overline{\phi}^{m}_{TNS}} - \sigma^{-}_{\overline{\phi}^{m}_{TNS}}$ such that

$$\left|\psi_{i,q} D^{N} D^{M}_{t,q} \overline{\phi}^{m,l}_{TNS}\right| \lesssim \left((\sigma^{+}_{\overline{\phi}^{m}_{TNS}})^{3/2} r_m^{-1} + \delta^2_{q+3\bar{n}} \right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+17}, \mathrm{T}_q^{-1}\Gamma_q^{9}\right) \tag{11.83a}$$

$$\left|\psi_{i,q} D^{N} D^{M}_{t,q} \sigma^{+}_{\overline{\phi}^{m}_{TNS}}\right| < \left(\sigma^{+}_{\overline{\phi}^{m}_{TNS}} + \delta_{q+2\bar{n}} \right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+18}, \mathrm{T}_q^{-1}\Gamma_q^{9}\right) \tag{11.83b}$$

$$\left\|\psi_{i,q} D^{N} D^{M}_{t,q} \sigma^{+}_{\overline{\phi}^{m}_{TNS}}\right\|_{3/2} < \delta_{m+\bar{n}} \Gamma_m^{-9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+18}, \mathrm{T}_q^{-1}\Gamma_q^{9}\right) \tag{11.83c}$$

$$\left\|\psi_{i,q} D^{N} D^{M}_{t,q} \sigma^{+}_{\overline{\phi}^{m}_{TNS}}\right\|_{\infty} < \Gamma_{q+{}^{\bar{n}}/_{2}+1}^{\mathsf{C}_\infty - 9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+18}, \mathrm{T}_q^{-1}\Gamma_q^{9}\right) \tag{11.83d}$$

$$\left|\psi_{i,q} D^{N} D^{M}_{t,q} \sigma^{-}_{\overline{\phi}^{m}_{TNS}}\right| < \left(\frac{\lambda_q}{\lambda_{q+\lfloor \bar{n}/2 \rfloor}} \right)^{\frac{2}{3}} \pi_q^q \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+18}, \mathrm{T}_q^{-1}\Gamma_q^{9}\right) \tag{11.83e}$$

for $M, N \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_{200}$.

(iii) For $q+{}^{\bar{n}}/_{2}+1 \leq m \leq q+\bar{n}$, $q+1 \leq m' \leq m-1$, $q+1 \leq q'' \leq q+{}^{\bar{n}}/_{2}$, $q+1 \leq k \leq q+\bar{n}$, and $q+1 \leq k' \leq k-1$, we have that

$$\operatorname{supp} \sigma^{-}_{\overline{\phi}^{m}_{TNS}} \cap B\left(\operatorname{supp} \widehat{w}_{q''}, \lambda_{q''}^{-1}\Gamma_{q''+1}\right) = \operatorname{supp} \sigma^{+}_{\overline{\phi}^{m}_{TNS}} \cap B\left(\operatorname{supp} \widehat{w}_{m'}, \lambda_{m'}^{-1}\Gamma_{m'+1}\right) = \emptyset\,. \tag{11.84a}$$

$$B\left(\operatorname{supp} \widehat{w}_{k'}, \lambda_{k'}^{-1}\Gamma_{k'+1}\right) \cap \operatorname{supp} \overline{\phi}^{k,l}_{TNS} = \emptyset\,. \tag{11.84b}$$

(iv) For $m = q+1, \dots, q+\bar{n}$, the non-local parts $\overline{\phi}_{S^{m,*}_{\triangle}}$ and $\overline{\phi}^{*}_{S^{m,l}_{\triangle}}$ satisfy

$$\left\| D^N D^M_{t,q} \overline{\phi}^{m,*}_{TNS} \right\|_\infty \leq \mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} \lambda^N_m \tau_q^{-M} , \tag{11.85a}$$

$$\left\| D^N D^M_{t,q+\bar{n}-1} \overline{\phi}^{*}_{TNS} \right\|_\infty \leq \delta^{\frac{3}{2}}_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right) \tag{11.85b}$$

for all $N, M \leq \mathsf{N}_{\mathrm{ind}}/4$.

(v) For $M \leq 2\mathsf{N}_{\mathrm{ind}}$, the time function $\mathfrak{m}_{TNS}$ satisfies

$$\mathfrak{m}_{TNS}(t) = \int_0^t \langle (11.51)(s) \rangle \, ds \, , \quad \left| \frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{TNS} \right| \leq (\max(1,T))^{-1} \delta^2_{q+3\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \mathrm{T}^{-1}_{q+1}\right) . \tag{11.86}$$

Proof. Recall from (11.53) that (11.51) consists of $-L_{TN}(S_{\triangle})$ where $\triangle$ represents O, TN, $C1$, or $M2$. We first consider the terms involving the local part of $S_{\triangle}$, and then deal with the terms with the non-local parts.

`Case 1.` Current error from the terms $-L_{TN}(S^{m,l}_{\triangle})$ with $m = q+1$ or $m = q + \bar{n}/2$. In this case, we first note that $S^{m,l}_{\triangle}$ is non-trivial only when $\triangle = O$. Recall the expression of $S^{m,l}_{O}$ from (10.12a) of Remark 10.2.2, which gives

$$L_{TN} S^{m,l}_{O} = \sum_{i,j,k,\xi,\vec{l},\diamond} \sum_{j'=0}^{\mathcal{C}_{\mathcal{H}}} (L_{TN} H^{\alpha_{(j')}}_{i,j,k,\xi,\vec{l},\diamond}) \rho^{\beta_{(j')}}_{i,j,k,\xi,\vec{l},\diamond} \circ \Phi_{(i,k)} .$$

In order to get the associated current error, we fix indices $j', \diamond, i, j, k, \xi, \vec{l}$ and apply the inverse divergence Proposition A.3.3 and Remark A.3.9 with the following choice of parameters and functions. Set

$$G = -(\lambda_{q+1}\Gamma_q^{-4})^{-1} L_{TN} H^{\alpha_{(j')}}_{i,j,k,\xi,\vec{l},\diamond}, \quad \varrho = \lambda_{q+1}\Gamma_q^{-4} \rho^{\beta_{(j')}}_{i,j,k,\xi,\vec{l},\diamond}, \quad m = q+1$$

$$G = -\lambda^{-1}_{q+\bar{n}/2} L_{TN} H^{\alpha_{(j')}}_{i,j,k,\xi,\vec{l},\diamond}, \quad \varrho = \lambda_{q+\bar{n}/2} \rho^{\beta_{(j')}}_{i,j,k,\xi,\vec{l},\diamond}, \quad m = q + \bar{n}/2 .$$

We choose the rest of parameters and functions the same as in `Case 1` and `Case 2` in the proof of Lemma 11.2.3, except for $N_* = \mathsf{N}_{\text{fin}}/50$ and $M_* = \mathsf{N}_{\text{fin}}/100$. [2] With this change, (A.39) and (A.44) still hold from (4.24a). The rest of assumptions are satisfied as in `Case 1,2`. As a result, in the case of $m = q+1$ or $m = q + \bar{n}/2$, we obtain the associated current error $\overline{\phi}^m_{TNS} = \overline{\phi}^{m,l}_{TNS} + \overline{\phi}^{m,*}_{TNS}$ which satisfy

$$\mathrm{div}\overline{\phi}^m_{TNS} = -L_{TN}S^{m,l}_O + \langle L_{TN}S^{m,l}_O\rangle \tag{11.87}$$

and the same properties as $\overline{\phi}^m_{TNW}$ have, except that the range of N and M in the estimates are restricted to $N, M \leq \mathsf{N}_{\text{fin}}/100$. In particular, (11.82), (11.84b) for $k = q+1, q+\bar{n}/2$, and (11.85) with $m = q+1, q+\bar{n}/2$ hold. Finally, (11.86) holds due to similar arguments as in previous lemmas, and we omit further details throughout this proof.

`Case 2.` Current error and pressure increment from the terms $-L_{TN}(S^{m,l}_\triangle)$ with $q+\bar{n}/2+1 \leq m \leq q+\bar{n}$. Since S_{M2} only have the non-local parts, we consider only when $\triangle = O, TN, C1$. Recall from Remarks 10.2.2, 10.2.7 and 10.2.11 that for $\triangle = O, TN, C1$, we have

$$L_{TN}S^{m,l}_\triangle = \sum_{i,j,k,\xi,\vec{l},I,\diamond}\sum_{j'=0}^{\mathcal{C}_{\mathcal{H}}}(L_{TN}H^{\alpha_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond})\rho^{\beta_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}\circ\Phi_{(i,k)}\,. \tag{11.88}$$

With this representation (11.88), we fix indices $\triangle, j', \diamond, i, j, k, \xi, \vec{l}$ and apply Proposition A.4.5 to construct desired current errors and pressure increments.

`Case 2-1.` Consider $\triangle = O, C1$. Observe that $H^{\alpha_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}$ and $\rho^{\beta_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}$, $\triangle = O, C1$, have the same properties in Remark 10.2.2, 10.2.11. Set the parameters and functions in the proposition the same as in `Case 3` in the proof of Lemma 11.2.3, except for $N_* = \mathsf{N}_{\text{fin}}/50$,

[2] In fact, the actual size of G is smaller than the one in `Case 1` and `Case 2`.

$M_* = \mathtt{N}_{\mathrm{fin}}/100$,

$$G = -(\lambda_{q+\bar{n}/2}\Gamma_q)^{-1} L_{TN} H^{\alpha_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}, \quad \varrho = \lambda_{q+\bar{n}/2}\Gamma_q \rho^{\beta_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}, \qquad \text{when } m = q + \bar{n}/2 + 1$$
$$G = -(\lambda^2_{m-1}\lambda^{-1}_m)^{-1} L_{TN} H^{\alpha_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}, \quad \varrho = \lambda^2_{m-1}\lambda^{-1}_m \rho^{\beta_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}, \qquad \text{otherwise} \ .$$

Then, (A.39), (A.44), (A.197a), (A.197b), (A.198d) still hold from (4.24a) and (4.24a). The rest of assumptions are all satisfied as we see in `Case 3`. Therefore, as before, in each case of m, we obtain the associated current error $\overline{\phi}^m_{TN\triangle} = \overline{\phi}^{m,l}_{TN\triangle} + \overline{\phi}^{m,*}_{TN\triangle}$ and pressure increment $\sigma_{\overline{\phi}^m_{TN\triangle}} = \sigma^+_{\overline{\phi}^m_{TN\triangle}} - \sigma^-_{\overline{\phi}^m_{TN\triangle}}$, which satisfy

$$-L_{TN} S^m_\triangle + \left\langle L_{TN} S^m_\triangle \right\rangle = \mathrm{div}\overline{\phi}^m_{TN\triangle}, \tag{11.89}$$

and share the same properties as $\overline{\phi}^m_{TNW}$ and $\sigma_{\overline{\phi}^m_{TNW}}$ have in the restricted range of N, M. In particular, (11.83), (11.84), and (11.85) holds with the replacement of $\overline{\phi}^{m,l}_{TNS}$ and $\sigma^\pm_{\overline{\phi}^m_{TNS}}$ with $\overline{\phi}^{m,l}_{TN\triangle}$ and $\sigma^\pm_{\overline{\phi}^m_{TN\triangle}}$.

`Case 2-2`. Consider $\triangle = TN$. Comparing the properties of $H^{\alpha_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}$ and $\rho^{\beta_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}$ in Remark 10.2.2 when $m = q + \bar{n}$ with those in Remark 10.2.11, one can see that

$$G = -\lambda^{-1}_{q+\bar{n}} L_{TN} H^{\alpha_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}, \quad \varrho = \lambda_{q+\bar{n}} \rho^{\beta_{(j')}}_{\triangle,i,j,k,\xi,\vec{l},I,\diamond}$$

satisfies the same estimates as G and ϱ defined in `Case 2-1` when $m = q + \bar{n}$, except that G when $\triangle = TN$ has more expensive sharp material derivative cost by Γ_q. Thereefore, repeating the same argument, we can obtain the associated current error $\overline{\phi}^{q+\bar{n}}_{TN\triangle} = \overline{\phi}^{q+\bar{n},l}_{TN\triangle} + \overline{\phi}^{q+\bar{n},*}_{TN\triangle}$ and pressure increment $\sigma_{\overline{\phi}^{q+\bar{n}}_{TN\triangle}} = \sigma^+_{\overline{\phi}^{q+\bar{n}}_{TN\triangle}} - \sigma^-_{\overline{\phi}^{q+\bar{n}}_{TN\triangle}}$, which satisfy

$$-L_{TN} S^{q+\bar{n}}_\triangle + \left\langle L_{TN} S^{q+\bar{n}}_\triangle \right\rangle = \mathrm{div}\overline{\phi}^{q+\bar{n}}_{TN\triangle}, \tag{11.90}$$

and share the same properties as $\overline{\phi}^{q+\bar{n}}_{TNW}$ and $\sigma_{\overline{\phi}^{q+\bar{n}}_{TNW}}$ have in the restricted range of N, M

expect that the sharp material derivative have extra Γ_q cost. In particular, (11.83), (11.84), and (11.85) holds with the replacement of $\overline{\phi}^{q+\bar{n},l}_{TNS}$ and $\sigma^{\pm}_{\overline{\phi}^{q+\bar{n}}_{TNS}}$ with $\overline{\phi}^{q+\bar{n},l}_{TN\triangle}$ and $\sigma^{\pm}_{\overline{\phi}^{q+\bar{n}}_{TN\triangle}}$.

Lastly, we define

$$\overline{\phi}^{m}_{TNS} := \overline{\phi}^{m}_{TNO} + \overline{\phi}^{m}_{TNC1} + \overline{\phi}^{m}_{TNTN}, \quad \sigma_{\overline{\phi}^{m}_{TNS}} := \sigma_{\overline{\phi}^{m}_{TNO}} + \sigma_{\overline{\phi}^{m}_{TNC1}} + \sigma_{\overline{\phi}^{m}_{TNT}}$$

and the local and nonlocal parts of $\overline{\phi}^{m}_{TNS}$ and the superscript $\pm$ part of $\sigma_{\overline{\phi}^{m}_{TNS}}$ analogously. Here, we set undefined current errors $\overline{\phi}^{m}_{TN\triangle}$ and pressure increments $\sigma_{\overline{\phi}^{m}_{TN\triangle}} = 0$ as zero. Then, combining the analysis in `Case 2-1, 2-2`, (11.83), (11.84), and (11.85) for $\overline{\phi}^{m,*}_{TNS}$ can be verified.

`Case 3.` Current error from the terms $-L_{TN}(S^{m,*}_{\triangle})$ with $q+1 \le m \le q+\bar{n}$. Lastly, we construct $\overline{\phi}^{*}_{TNS}$ satisfying

$$\mathrm{div}\overline{\phi}^{*}_{TNS} = -\sum_{m=q+1}^{q+\bar{n}} \mathbb{P}_{\neq 0} L_{TN}\left(S^{m,*}_{O} + S^{m,*}_{TN} + S^{m,*}_{C1} + S^{m,*}_{M2}\right)$$

and (11.85). The terms on the right-hand side are not be intermittent, so there is no pressure increment generated from them. We fix $\triangle$ and m, and apply Remark A.3.5 of Proposition A.3.3. We first consider when $\triangle \neq M2$. Set $N_* = M_* = \mathsf{N}_{\mathrm{ind}} - 1$, $M_\circ = N_\circ = \mathsf{N}_{\mathrm{ind}}/4$,

$$G = -L_{TN}S^{m,*}_{\triangle}, \quad \mathcal{C}_{G,\infty} = \tau_q^{-1}\mathrm{T}^{4\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\delta_{q+3\bar{n}}, \quad \lambda = \lambda_{q+\bar{n}}, \quad \nu = \nu' = \mathrm{T}_q^{-1},$$
$$v = \widehat{u}_q, \quad D_t = D_{t,q}, \quad \lambda' = \lambda_q\Gamma_q, \quad \mathcal{C}_v = \Lambda_q^{1/2},$$

and choose a natural number $K_\circ$ such that

$$\mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}}\delta^{3/2}_{q+\bar{n}} \le \lambda^{-K_\circ}_{q+\bar{n}} \le \mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}+1}_{q+\bar{n}}\delta^{3/2}_{q+\bar{n}}$$

Then, all the assumptions are satisfied by (10.11), (10.54), (10.81), (5.34), Corollary 8.2.4.

In particular, (A.55) can be verified by the choice of sufficiently large a. As a result of Remark A.3.5, summing over m, we have $\overline{\phi}^*_{TN\triangle}$ which satisfies

$$\mathrm{div}\overline{\phi}^*_{TN\triangle} = -\sum_{m=q+1}^{q+\bar{n}} \mathbb{P}_{\neq 0} L_{TN} S^{m,*}_{\triangle}, \qquad \left\| D^N D^M_{t,q} \overline{\phi}^*_{TN\triangle} \right\|_\infty \leq \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}+1} \delta^{3/2}_{q+3\bar{n}} \lambda^N_{q+\bar{n}} \mathrm{T}_q^{-M}$$

for $N, M \leq \mathsf{N}_{\mathrm{ind}}/4$. Lastly, we apply Lemma A.5.1 to $\overline{\phi}^*_{TN\triangle}$, we have

$$\begin{aligned} \left\| D^N D^M_{t,q+\bar{n}-1} \overline{\phi}^*_{TN\triangle} \right\|_\infty &\leq \mathrm{T}_{q+\bar{n}}^{\mathsf{N}_{\mathrm{ind,t}}+1} \delta^{3/2}_{q+3\bar{n}} \lambda^N_{q+\bar{n}} (\mathrm{T}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^{-M} \\ &\leq \mathrm{T}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} \lambda^N_{q+\bar{n}} \mathcal{M}\left(M, \tau^{-1}_{q-\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\right) \end{aligned}$$

for $N, M \leq \mathsf{N}_{\mathrm{ind}}/4$.

Next, we consider $\triangle = M2$. As we see from (10.97), $S^{m,*}_{M2}$ is non-trivial only when $m = q + \bar{n}$. We first note that when $q + 1 \leq k < q + \bar{n}$,

$$\left\| D^N D^M_{t,q+\bar{n}-1} \widehat{w}_k \right\|_\infty = \left\| D^N D^M_{t,k} \widehat{w}_k \right\|_\infty \lesssim \Gamma_q^{\mathsf{C}_\infty/2+18} r_q^{-1} (\lambda_k \Gamma_k)^N (\mathrm{T}^{-1}_{k-1}\Gamma_{k-1})^M$$

for $N + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2 + 1$, from Hypothesis 5.4.1, (5.36), (5.10), and (4.2b). Also, applying Lemma A.5.1 to (5.34), we have

$$\left\| D^N D^M_{t,q+\bar{n}-1} \nabla \widehat{u}_q \right\|_\infty \lesssim \mathrm{T}^{-1}_{q+\bar{n}} \lambda_q \Gamma_q^{\mathsf{C}_\infty/2+18} r_q^{-1} (\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^N (\mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^M$$

Combining these with (10.98b), we have from (9.83b) that

$$\begin{aligned} \left\| D^N D^M_{t,q+\bar{n}-1} L_{TN} S^{q+\bar{n},*}_{M2} \right\|_\infty &\leq \left\| D^N D^{M+1}_{t,q+\bar{n}-1} S^{q+\bar{n},*}_{M2} \right\|_\infty \\ &\qquad + \left\| D^N D^M_{t,q+\bar{n}-1} [((\widehat{w}_{q+\bar{n}-1} - \widehat{w}_q) \cdot \nabla)\mathrm{tr} + \nabla \widehat{u}_q :] S^{q+\bar{n},*}_{M2} \right\|_\infty \\ &\leq \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}-2} \delta_{q+3\bar{n}} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N (\mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^M \end{aligned}$$

for $N + M \leq 2\mathsf{N}_{\mathrm{ind}} - 1$. Therefore, we apply Remark A.3.5 of Proposition A.3.3 by setting

$N_* = M_* = \mathsf{N}_{\text{ind}} - 1$, $M_\circ = N_\circ = {}^{\mathsf{N}_{\text{ind}}}/_4$,

$$G = -L_{TN} S^{m,*}_{\triangle}\,, \quad \mathcal{C}_{G,\infty} = \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}-2}\delta_{q+3\bar{n}}, \quad \lambda = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}, \quad \nu = \nu' = \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}\,,$$
$$v = \widehat{u}_{q+\bar{n}-1}\,, \quad D_t = D_{t,q+\bar{n}-1}\,, \quad \lambda' = \lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}\,, \quad \mathcal{C}_v = \Lambda^{1/2}_{q+\bar{n}-1}\,,$$

and choosing a natural number $K_\circ$ so that

$$\delta^{3/2}_{q+3\bar{n}}\mathrm{T}^{\mathsf{N}_{\text{ind}}}_{q+\bar{n}} \le (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-K_\circ} \le \delta^{3/2}_{q+3\bar{n}}\mathrm{T}^{\mathsf{N}_{\text{ind}}+1}_{q+\bar{n}}.$$

Then all required assumptions are satisfied as before. As a result of the remark, we obtain $\overline{\phi}^*_{TNM2}$ such that $\mathrm{div}\overline{\phi}^*_{TNM2} = -\sum_{m=q+1}^{q+\bar{n}} \mathbb{P}_{\neq 0} L_{TN} S^{m,*}_{M2}$, and for $N, M \le {}^{\mathsf{N}_{\text{ind}}}/_4$,

$$\begin{aligned}\left\| D^N D^M_{t,q+\bar{n}-1}\overline{\phi}^*_{TNM2}\right\|_\infty &\le \mathrm{T}^{\mathsf{N}_{\text{ind,t}}+1}_{q+\bar{n}}\delta^{3/2}_{q+3\bar{n}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N(\mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^M \\ &\le \mathrm{T}_{q+\bar{n}}\delta^{3/2}_{q+3\bar{n}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}\right)\,.\end{aligned}$$

Lastly, we set $\overline{\phi}^*_{TNS} := \overline{\phi}^*_{TNO} + \overline{\phi}^*_{TNC1} + \overline{\phi}^*_{TNTN} + \overline{\phi}^*_{TNM2}$ and collect the properties of $\overline{\phi}^*_{TN\triangle}$ to conclude (11.85). □

Remark 11.2.6 (Collecting pressure and current errors from transport-Nash). We now collect all current errors and pressure increments generated by (11.49)–(11.51) and set

$$\overline{\phi}^m_{TN} := \overline{\phi}^m_{TNW} + \overline{\phi}^m_{TNC} + \overline{\phi}^m_{TNS}\,, \qquad \sigma_{\overline{\phi}^m_{TN}} := \sigma_{\overline{\phi}^m_{TNW}} + \sigma_{\overline{\phi}^m_{TNC}} + \sigma_{\overline{\phi}^m_{TNS}}\,, \tag{11.91}$$

where the quantities on the right-hand side are constructed in Lemmas 11.2.3, 11.2.4, and 11.2.5. We use a similar notation for the various functions of time $\mathfrak{m}$, so that recalling (11.7), we have that $\mathfrak{m}_T + \mathfrak{m}_N = \mathfrak{m}_{TNW} + \mathfrak{m}_{TNC} + \mathfrak{m}_{TNS}$. Then summing over m, we have the transport and Nash current error $\overline{\phi}_{TN}$. We similarly collect the local and nonlocal parts of $\overline{\phi}^m_{TN}$ and the $\pm$ part of the pressure increments $\sigma_{\overline{\phi}^m_{TN}}$.

Lastly, we define and analyze the current error associated to the pressure increments $\overline{\phi}^m_{TN}$.

Lemma 11.2.7 (Pressure current). *For every $m' \in \{q + {}^{\bar{n}}/_2 + 1, \dots, q + \bar{n}\}$, there exist a current error $\phi_{\overline{\phi}^{m'}_{TN}}$ associated to the pressure increments $\sigma_{\overline{\phi}^{m'}_{TN}}$ and a function $\mathfrak{m}_{\sigma_{\overline{\phi}^{m'}_{TN}}}$ of time that satisfy the following properties.*

(i) We have the decompositions and equalities

$$\operatorname{div}\phi_{\overline{\phi}^{m'}_{TN}} + \mathfrak{m}'_{\sigma_{\overline{\phi}^{m'}_{TN}}} = D_{t,q}\sigma_{\overline{\phi}^{m'}_{TN}}\,, \tag{11.92a}$$

$$\phi_{\overline{\phi}^{m'}_{TN}} = \phi^*_{\overline{\phi}^{m'}_{TN}} + \sum_{m=q+{}^{\bar{n}}/_2+1}^{m'} \phi^m_{\overline{\phi}^{m'}_{TN}}\,, \quad \phi^m_{\overline{\phi}^{m'}_{TN}} = \phi^{m,l}_{\overline{\phi}^{m'}_{TN}} + \phi^{m,*}_{\overline{\phi}^{m'}_{TN}}\,. \tag{11.92b}$$

(ii) For $q + {}^{\bar{n}}/_2 + 1 \le m \le m'$ and $N, M \le 2\mathsf{N}_{\mathrm{ind}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}^{m,l}_{\overline{\phi}^{m'}_{TN}}\right| < \Gamma_m^{-100} \left(\pi^m_q\right)^{3/2} r_m^{-1} (\lambda_m \Gamma_m^2)^M \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+18}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.93a}$$

$$\left\|D^N D^M_{t,q} \phi^{m,*}_{\overline{\phi}^{m'}_{TN}}\right\|_\infty < \mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} (\lambda_{m'}\Gamma^2_{m'})^N \tau_q^{-M}, \tag{11.93b}$$

$$\left\|D^N D^M_{t,q} \phi^{*}_{\overline{\phi}^{m'}_{TN}}\right\|_\infty < \mathrm{T}^{2\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} (\lambda_{q+\bar{n}}\Gamma^2_{q+\bar{n}})^N \tau_q^{-M}\,. \tag{11.93c}$$

(iii) For all $q + {}^{\bar{n}}/_2 + 1 \le m \le m'$ and all $q + 1 \le q' \le m - 1$,

$$B\left(\operatorname{supp}\widehat{w}_{q'}, {}^1/_2\lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp}\left(\phi^{m,l}_{\overline{\phi}^{m'}_{TN}}\right) = \emptyset\,. \tag{11.94}$$

(iv) For $M \le 2\mathsf{N}_{\mathrm{ind}}$, the mean part $\mathfrak{m}_{\sigma_{\overline{\phi}^{m'}_{TN}}}$ satisfies

$$\left|\frac{d^{M+1}}{dt^{M+1}}\mathfrak{m}_{\sigma_{\overline{\phi}^{m'}_{TN}}}\right| \le (\max(1,T))^{-1}\delta_{q+3\bar{n}}\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right)\,. \tag{11.95}$$

Proof. Recall from (11.91) that the pressure increment $\sigma_{\overline{\phi}^{m'}_{TN}}$ consists of $\sigma_{\overline{\phi}^{m'}_{TNW}}$, $\sigma_{\overline{\phi}^{m'}_{TNC}}$, $\sigma_{\overline{\phi}^{m'}_{TNS}}$.

We first construct the pressure current for $\sigma_{\overline{\phi}^{m'}_{TNW}}$. Recall its construction in `Case 3` of the proof of Lemma 11.2.3. As a result of the application of Proposition A.4.5 to $t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}$, from Part 4 of the proposition, we obtain a pressure current $\phi_{i,j,k,\xi,\vec{l},I,\diamond}$ which has a decomposition

$$\overline{\phi}_{i,j,k,\xi,\vec{l},I,\diamond} = \overline{\phi}^{*}_{i,j,k,\xi,\vec{l},I,\diamond} + \sum_{m=0}^{\bar{m}} \overline{\phi}^{m}_{i,j,k,\xi,\vec{l},I,\diamond} = (\mathcal{H} + \mathcal{R}^{*})\, D_{t,q}\sigma_{\mathcal{H}t^{m'}_{i,j,k,\xi,\vec{l},I,\diamond}}.$$

Repeating a similar argument in the proof of Lemma 11.2.2 including aggregation over all indices $i, j, k, \xi, \vec{l}, I, \diamond$, we get the same point-wise estimate for $\overline{\phi}^{m,l}_{\overline{\phi}^{m'}_{TNW}}$ as (11.44) or (11.45) except for an extra factor Γ_q in the sharp material derivative cost. Applying the same analysis, the same estimate holds for the pressure current of the pressure increments $\sigma_{\overline{\phi}^{m'}_{TNC}}$ and $\sigma_{\overline{\phi}^{m'}_{TNS}}$ except for another extra factor Γ_q in the sharp material derivative cost. Combining all these estimates, we obtain (11.93a). The proof of (11.92a), (11.93b)–(11.95) also follows from arguments similar to those used to prove the corresponding properties in Lemma 11.2.2.

□

11.2.3 Linear current error

Lemma 11.2.8 (**Definition and basic estimates**). *There exists a current error $\overline{\phi}_L = \overline{\phi}^{q+\bar{n}}_L$ and a function of time $\mathfrak{m}_L$ such that the following hold.*

(i) We have the equality and decomposition

$$\mathrm{div}\overline{\phi}^{q+\bar{n}}_L + \mathfrak{m}'_L = w_{q+1} \cdot (\partial_t u_q + u_q \cdot \nabla u_q + \nabla p_q)\ , \qquad \overline{\phi}^{q+\bar{n}}_L = \overline{\phi}^{q+\bar{n},l}_L + \overline{\phi}^{q+\bar{n},*}_L. \quad (11.96)$$

(ii) For all $N+M \leq \mathsf{N}_{\mathrm{ind}}/4$, we have that

$$\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} \overline{\phi}_L^{q+\bar{n},l}\right| \leq \Gamma_{q+\bar{n}}^{-100} \left(\left(\pi_q^{q+\bar{n}}\right)^{3/2} + \left(\sigma_v^+\right)^{3/2}\right) r_{q+\bar{n}}^{-1} \times (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}^{i+4}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^2\right) \tag{11.97a}$$

$$\left\|D^N D^M_{t,q+\bar{n}-1} \overline{\phi}_L^{q+\bar{n},*}\right\|_\infty \leq \delta_{q+2\bar{n}}^{3/2} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^2\right). \tag{11.97b}$$

(iii) For all $q+1 \leq q' \leq q+q+\bar{n}-1$, we have that

$$\operatorname{supp}\left(\phi_L^{q+\bar{n},l}\right) \cap B\left(\widehat{w}_{q'}, \Gamma_{q'-1}\lambda_{q'}^{-1}\right) = \emptyset. \tag{11.98}$$

(iv) The time function $\mathfrak{m}_L$ satisfies $\mathfrak{m}_L' = \langle w_{q+1} \cdot (\partial_t u_q + u_q \cdot \nabla u_q + \nabla p_q)\rangle$ and

$$\left|\frac{d^{M+1}}{dt^{M+1}}\mathfrak{m}_L\right| \leq (\max(1,T))^{-1}\delta_{q+3\bar{n}}\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right) \quad \textit{for } M \leq \mathsf{N}_{\mathrm{ind}}/4. \tag{11.99}$$

Proof. `Step 0: Splitting the error term and upgrading material derivatives.` We use the Euler-Reynolds system (5.2), the mollified stresses and pressures from (6.8) and (6.4), respectively, and the formula for the velocity increment potential from Remark 9.4.2 to split the error

$$\begin{aligned} w_{q+1} \cdot (\partial_t u_q + u_q \cdot \nabla u_q + \nabla p_q) &= w_{q+1} \cdot \operatorname{div}(R_q - \pi_q \mathrm{Id}) \\ &= \underbrace{(w_{q+1} - e_{q+1}) \cdot \operatorname{div}\left(\sum_{m=q}^{q+\bar{n}-1} R_\ell^m - \sum_{m=q}^{q+\mathsf{N}_{\mathrm{pr}}-1} \pi_\ell^m \mathrm{Id}\right)}_{=\operatorname{div}\overline{\phi}_{L1}^{q+\bar{n}} + \mathfrak{m}_{L1}'} + \underbrace{e_{q+1} \cdot \operatorname{div}\left(\sum_{m=q}^{q+\bar{n}-1} R_\ell^m - \sum_{m=q}^{q+\mathsf{N}_{\mathrm{pr}}-1} \pi_\ell^m \mathrm{Id}\right)}_{=\operatorname{div}\overline{\phi}_{L2}^{q+\bar{n}} + \mathfrak{m}_{L2}'} \\ &\quad + \underbrace{w_{q+1} \cdot \operatorname{div}\left(\sum_{m=q}^{q+\bar{n}-1} (R_q^m - R_\ell^m) - \sum_{m=q}^{q+\mathsf{N}_{\mathrm{pr}}-1} (\pi_q^m - \pi_\ell^m)\mathrm{Id}\right)}_{=\operatorname{div}\overline{\phi}_{L3}^{q+\bar{n}} + \mathfrak{m}_{L3}'}. \end{aligned}$$

Notice that the tail of the sum $\sum_{q+\mathsf{N}_{\rm pr}}^{\infty} \pi_q^k$ of π_q does not appear because of (5.19). The term $\overline{\phi}_{L1}^{q+\bar{n}}$ is the main term and requires sharp estimates from the inverse divergence operator from Lemma A.3.12, while $\overline{\phi}_{L2}^{q+\bar{n}}$ and $\overline{\phi}_{L3}^{q+\bar{n}}$ may be estimated much more brutally.

`Step 1: Estimating the main local term.` In order to estimate $\overline{\phi}_{L1}^{q+\bar{n}}$, we need to upgrade the material derivative estimates on $w_{q+1} - e_{q+1} = \mathrm{div}^{\mathsf{d}} \upsilon_{q+1}$. Towards this end, we claim that on $\operatorname{supp} \psi_{i,m}$ and for all $N \leq {}^{\mathsf{N}_{\rm fin}}/_4 - 2\mathsf{d}^2$, $M \leq {}^{\mathsf{N}_{\rm fin}}/_5$, and $q+1 \leq m \leq q+\bar{n}-1$,

$$\lambda_{q+\bar{n}}^{\mathsf{d}-k} \left| D^N D_{t,m}^M \partial_{i_1} \cdots \partial_{i_k} \upsilon_{q+1} \right| \lesssim (\sigma_\upsilon^+ + \delta_{q+3\bar{n}})^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\rm ind,t}, \tau_m^{-1}\Gamma_m^{i-5}, \mathrm{T}_q^{-1}\Gamma_q^9\right) . \tag{11.100}$$

This estimate follows from dodging; more precisely, we appeal to (iii) and (9.49) from Lemma 9.4.1, and (ii) from Lemma 9.2.2 to assert that for all $m = q+1, \ldots, q+\bar{n}-1$, $\widehat{w}_m \cdot \nabla \upsilon_{q+1} \equiv 0$. Then using (9.64) from Lemma 9.4.4 and applying (5.8) and (5.14) concludes the proof.

We now fix $i \leq i_{\max}$ and $m = q+1, q+2, \ldots, q+\bar{n}-1$ (the cases $m = q$ and $m \geq q+\bar{n}$ will require minor modifications) and apply Lemma A.3.12 with the following choices:

$$\begin{aligned}
&G = \mathrm{div}\,(R_\ell^m - \pi_\ell^m \mathrm{Id})^\bullet\,, \quad \varrho = (w_{q+1} - e_{q+1})^\bullet\,, \quad \vartheta = \upsilon_{q+1}^\bullet\,, \quad v = \widehat{u}_{m-1}\,, \quad \lambda' = \Lambda_m \Gamma_m\,,\\
&\nu' = \mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^{12}\,, \quad \nu = \tau_{m-1}^{-1}\Gamma_{m-1}^{i+23}\,, \quad N_* = {}^{\mathsf{N}_{\rm fin}}/_4 - 2\mathsf{d}^2\,, \quad M_* = {}^{\mathsf{N}_{\rm fin}}/_5\,, \quad \mathsf{d} \text{ as in (xvii)}\,,\\
&\pi' = \left(\sigma_\upsilon^+ + \delta_{q+3\bar{n}}\right)^{1/2} r_q^{-1}\,, \quad \Omega = \operatorname{supp}\psi_{i,m-1}\,, \quad \pi = 2\Gamma_m^3 \pi_\ell^m \Lambda_m \Gamma_m\,, \quad M_t = \mathsf{N}_{\rm ind,t}\,,\\
&\lambda = \Lambda_m \Gamma_m\,, \quad \Upsilon = \Lambda = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}\,, \quad M_\circ = N_\circ = 3\mathsf{N}_{\rm ind}\,, \quad K_\circ \text{ as in (xvi)}\,.
\end{aligned}$$

Then we have that (A.42b) is satisfied (5.34), (A.97a) is satisfied from (6.4c) and (6.8), (A.97b) is satisfied from (9.64) and (9.94), and (A.98) is satisfied by definition and by (4.24a).

We then conclude from (A.100) that for all $N \leq {}^{\mathsf{N}_{\text{fin}}}/_{4} - 2\mathsf{d}^2 - \mathsf{d}$ and $M \leq {}^{\mathsf{N}_{\text{fin}}}/_{5}$,

$$\begin{aligned}
&\left|\mathbf{1}_{\operatorname{supp}\psi_{i,m-1}} D^N D^M_{t,m-1} \mathcal{H}\left((w_{q+1} - e_{q+1})^{\bullet} \operatorname{div}\left(R^m_\ell - \pi^m_\ell \mathrm{Id}\right)^{\bullet}\right)\right| \\
&\quad \lesssim \Gamma_m^3 \pi_\ell^m \Lambda_m \Gamma_m \left(\sigma_\upsilon^+ + \delta_{q+2\bar{n}}\right)^{1/2} r_q^{-1} \lambda_{q+\bar{n}}^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{m-1}^{-1} \Gamma_{m-1}^{i+4}, \mathrm{T}_m^{-1} \Gamma_{m-1}^2\right) .
\end{aligned} \tag{11.101}$$

From (A.99) and (9.49), we have that

$$\operatorname{supp}\left(\mathcal{H}\left((w_{q+1} - e_{q+1})^{\bullet} \operatorname{div}\left(R^m_\ell - \pi^m_\ell \mathrm{Id}\right)^{\bullet}\right)\right) \subseteq \operatorname{supp} \upsilon_{q+1} , \tag{11.102}$$

which leads to (11.98) from (9.49) and (9.23). Indeed, from (9.49) and Lemma 7.1.7, we have

$$\operatorname{supp}(\upsilon_{q+1}) \subseteq \bigcup_{\xi,i,j,k,\vec{l},I,\diamond} \operatorname{supp}\left(\chi_{i,k,q} \zeta_{q,\diamond,i,k,\xi,\vec{l}} \left(\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi}\right) \circ \Phi_{(i,k)}\right) \cap B\left(\operatorname{supp} \varrho^I_{(\xi),\diamond} \circ \Phi_{(i,k)}, 3\lambda_{q+\bar{n}}^{-1}\right) .$$

In order to have effective dodging with $\widehat{w}_{q'}$, $q+1 \leq q' \leq q + {}^{\bar{n}}/_{2}$, we appeal to (9.23). Also, using (11.101)–(11.102) and appealing to a similar dodging and upgrading argument which produced the bound (9.94), we have that for all $N \leq {}^{\mathsf{N}_{\text{fin}}}/_{4} - 2\mathsf{d}^2 - \mathsf{d}$ and $M \leq {}^{\mathsf{N}_{\text{fin}}}/_{5}$,

$$\begin{aligned}
&\left|\mathbf{1}_{\operatorname{supp}\psi_{i,q+\bar{n}-1}} D^N D^M_{t,q+\bar{n}-1} \mathcal{H}\left((w_{q+1} - e_{q+1})^{\bullet} \operatorname{div}\left(R^m_\ell - \pi^m_\ell \mathrm{Id}\right)^{\bullet}\right)\right| \\
&\quad \lesssim \Gamma_m^3 \pi_\ell^m \Lambda_m \Gamma_m \left(\sigma_\upsilon^+ + \delta_{q+2\bar{n}}\right)^{1/2} r_q^{-1} \lambda_{q+\bar{n}}^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{m-1}^{-1} \Gamma_{m-1}^{i+4}, \mathrm{T}_{m-1}^{-1} \Gamma_{m-1}^2\right) \\
&\quad \leq \Gamma_{q+\bar{n}}^{-101} \pi_q^{q+\bar{n}} \left(\sigma_\upsilon^+ + \delta_{q+2\bar{n}}\right)^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}^{i-5}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}^2\right) ,
\end{aligned} \tag{11.103}$$

where we have used (5.20), (6.6), and (4.10f) to conclude the last line. Finally, the estimates in (11.99) follow as usual from Remark A.3.7, and we omit the details now and in the rest of this proof.

In the case $m = q$, we make slight changes in the choices of υ, π, Ω, ν, and ν' based on (6.3c) and (5.34). Then, applying the same reasoning as for the case $m = q+1$ for example,

we find that in fact (11.103) holds for $m = q$. In the cases $q + \bar{n} \leq m < q + \mathsf{N}_{\mathrm{pr}}$, we set $G = \operatorname{div}(\pi_\ell^m \mathrm{Id})^\bullet$, $v = \widehat{u}_{q+\bar{n}-1}$, and make suitable changes to the parameters and functions based on (6.5c) and (5.34) for $q' = q + \bar{n} - 1$. Concluding as before, we find that

$$\begin{aligned}
&\left|\mathbf{1}_{\operatorname{supp}\psi_{i,q+\bar{n}-1}} D^N D^M_{t,q+\bar{n}-1} \mathcal{H}\left((w_{q+1} - e_{q+1}) \cdot \operatorname{div}(\pi_\ell^m \mathrm{Id})\right)\right| \\
&\quad \lesssim \Gamma_m^3 \pi_\ell^m \Lambda_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}^2 \left(\sigma_v^+ + \delta_{q+2\bar{n}}\right)^{1/2} r_q^{-1} \lambda_{q+\bar{n}}^{-1} \\
&\qquad \times (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}^{i+4}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}^2\right) \\
&\quad \leq \Gamma_{q+\bar{n}}^{-101} \pi_q^{q+\bar{n}} \left(\sigma_v^+ + \delta_{q+2\bar{n}}\right)^{1/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}^{i+4}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}^2\right)
\end{aligned} \tag{11.104}$$

for all $N \leq \mathsf{N}_{\mathrm{fin}}/4 - 2\mathsf{d}^2 - \mathsf{d}$ and $M \leq \mathsf{N}_{\mathrm{fin}}/5$. In the last inequality, we have used (6.6), (5.18), and (5.20) to write that $\pi_\ell^m \Lambda_{q+\bar{n}-1} \lambda_{q+\bar{n}}^{-1} \leq 2\pi_q^{q+\bar{n}-1} \Lambda_{q+\bar{n}-1} \lambda_{q+\bar{n}}^{-1} \leq \pi_q^{q+\bar{n}} \Gamma_{q+\bar{n}}^{-150}$.

We can now set

$$\overline{\phi}_L^{q+\bar{n},l} = \mathcal{H}\left((w_{q+1} - e_{q+1}) \cdot \operatorname{div}\left(\sum_{m=q}^{q+\bar{n}-1} R_\ell^m - \sum_{m=q}^{q+\bar{n}+\mathsf{N}_{\mathrm{pr}}-1} \pi_\ell^m \mathrm{Id}\right)^\bullet\right),$$

which is well-defined over various values of i since the algorithm used to define $\mathcal{H}$ is independent of the value of i. Summing the estimate in (11.103)–(11.104) over the various values of m and using Cauchy-Schwarz, (5.17), and (4.24a) gives (11.97a).

`Step 2: Estimating the main nonlocal term and remainder terms.` We first finish the application of Lemma A.3.12 to the main terms by setting up the nonlocal assumptions and output in Part 3. In the case of $q + 1 \leq m \leq q + \bar{n} - 1$, we first have that (A.52) is satisfied by (4.24a), (A.53) is satisfied with $\mathcal{C}_v = \Lambda_{m-1}^{1/2}$ by (5.35b), and (A.54) is satisfied by (4.15). Next, we set $\mathcal{C}_{G,\infty} = \mathcal{C}_{*,\infty} = \lambda_{q+\bar{n}}^2$. Then (A.101a) is satisfied from the bound for G from Step 1, the bounds in Lemma 6.0.1, equations (6.3)–(6.4) for π_ℓ and π_ℓ^m, respectively. Furthermore, (A.101b) is satisfied from (9.64) and (9.66b). Choosing d and $K_\circ$ according to items (xvi)–(xvii) so that they satisfy (4.22) and (4.23b), we have that (A.102) is satisfied.

Then from (A.103)–(A.104), we have that for $N, M \leq 3\mathsf{N}_{\text{ind}}$,

$$\left\| D^N D^M_{t,m-1} \mathcal{R}^* \left((w_{q+1} - e_{q+1})^\bullet \operatorname{div}(R^m_\ell - \pi^m_\ell \mathrm{Id})^\bullet \right) \right\|_\infty \leq \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathrm{T}^{-M}_{m-1} \,. \tag{11.105a}$$

In the remaining cases, we repeat the same argument and obtain that for $N, M \leq 3\mathsf{N}_{\text{ind}}$,

$$\left\| D^N D^M_{t,q} \mathcal{R}^* \left((w_{q+1} - e_{q+1})^\bullet \operatorname{div}(R_\ell - \pi_\ell \mathrm{Id})^\bullet \right) \right\|_\infty \leq \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathrm{T}^{-M}_{q} \tag{11.105b}$$

$$\left\| D^N D^M_{t,q+\bar{n}-1} \mathcal{R}^* \left((w_{q+1} - e_{q+1})^\bullet \operatorname{div}(\pi^m_\ell \mathrm{Id})^\bullet \right) \right\|_\infty \leq \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \delta^{3/2}_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathrm{T}^{-M}_{q+\bar{n}-1} \tag{11.105c}$$

for $q + \bar{n} \leq m < q + \mathsf{N}_{\text{pr}}$. We will upgrade the material derivatives at the end of `Step 2`.

Next, we treat the second error identified in `Step 0`, namely the remainder term which includes e_{q+1}. In the case of $q + 1 \leq m \leq q + \bar{n} - 1$, we apply Lemma A.3.12 with the following choices:

$$G = \operatorname{div}\left(R^m_\ell - \pi^m_\ell \mathrm{Id}\right)^\bullet \,, \quad \varrho = \vartheta = e^\bullet_{q+1} \,, \quad v = \widehat{u}_{m-1} \,, \quad \lambda' = \Lambda_m \Gamma_m \,,$$
$$\nu = \nu' = \mathrm{T}^{-1}_{m-1} \Gamma^2_{m-1} \,, \quad N_* = {}^{\mathsf{N}_{\text{fin}}}/_4 - 2\mathsf{d}^2 \,, \quad M_* = {}^{\mathsf{N}_{\text{fin}}}/_5 \,, \quad \mathsf{d} = 0 \,, \quad \lambda = \Lambda_m \Gamma_m \,,$$
$$\pi' = \mathcal{C}_{*,\infty} = \delta^3_{q+3\bar{n}} \mathrm{T}^{20\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \lambda^{-10}_{q+\bar{n}} \,, \quad \Omega = \mathbb{T}^3 \times \mathbb{R} \,, \quad \pi = 2\Gamma^3_m \pi^m_\ell \Lambda_m \Gamma_m \,, \,,$$
$$\Upsilon = \Lambda = \lambda_{q+\bar{n}} \Gamma_{q+\bar{n}} \,, \quad M_\circ = N_\circ = 3\mathsf{N}_{\text{ind}} \,, \quad K_\circ \text{ such that } \mathrm{T}^{-10\mathsf{N}_{\text{ind,t}}}_{q+\bar{n}} \leq \Lambda^{K_\circ} \leq \mathrm{T}^{-10\mathsf{N}_{\text{ind,t}}-1}_{q+\bar{n}} \,.$$

Then we have that (A.97a) is satisfied as in `Step 1`, (A.97b) is satisfied by (9.51), and (A.98) is satisfied as in `Step 1`. Since $\mathsf{d} = 0$, we move straight to the nonlocal assumptions and output, for which all assumptions from item (i) in Proposition A.3.3 and (A.101a) are satisfied as in the beginning of `Step 2`, (A.101b) is satisfied by (9.51), and (A.102) is satisfied

from direct computation. We therefore have from (A.104) that

$$\left\| D^N D^M_{t,m-1} \mathcal{R}^* \left(\operatorname{div} \left(R^m_\ell - \pi^m_\ell \mathrm{Id} \right)^\bullet e^\bullet_{q+1} \right) \right\|_\infty \lesssim \mathrm{T}^{5\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \delta^3_{q+3\bar{n}} (\mathrm{T}^{-1}_{m-1} \Gamma^2_{m-1})^M (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \tag{11.106a}$$

for $N, M \leq 3\mathsf{N}_{\mathrm{ind}}$. Similarly, we have that for $N, M \leq 3\mathsf{N}_{\mathrm{ind}}$ and $q + \bar{n} \leq m < q + \mathsf{N}_{\mathrm{pr}}$,

$$\left\| D^N D^M_{t,q} \mathcal{R}^* \left(\operatorname{div} \left(R_\ell - \pi_\ell \mathrm{Id} \right)^\bullet e^\bullet_{q+1} \right) \right\|_\infty \lesssim \mathrm{T}^{5\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \delta^3_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_q, \mathrm{T}^{-1}_q \Gamma^2_q \right) \tag{11.106b}$$

$$\begin{aligned} &\left\| D^N D^M_{t,q+\bar{n}-1} \mathcal{R}^* \left(\operatorname{div} \left(\pi^m_\ell \mathrm{Id} \right)^\bullet e^\bullet_{q+1} \right) \right\|_\infty \\ &\quad \lesssim \mathrm{T}^{4\mathsf{N}_{\mathrm{ind,t}}}_{q+\bar{n}} \delta^3_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma^2_{q+\bar{n}-1} \right) \end{aligned} \tag{11.106c}$$

Finally, we treat the third error identified in `Step 0`, namely the remainder term which includes the differences between mollified and inductive stresses and pressures. In the case of $q \leq m \leq q + \bar{n} - 1$, we apply Lemma A.3.12 with the following choices:

$$\begin{aligned} &G = \operatorname{div} \left(R^m_q - R^m_\ell - (\pi^m_q - \pi^m_\ell) \mathrm{Id} \right)^\bullet , \quad \varrho = \vartheta = w^\bullet_{q+1} , \quad v = \widehat{u}_{m-1} , \quad \lambda' = \Lambda_m \Gamma_m , \\ &\nu = \nu' = \mathrm{T}^{-1}_{m-1} \Gamma^2_{m-1} , \quad N_* = M_* = \mathsf{N}_{\mathrm{ind}} - 1 , \quad \mathsf{d} = 0 , \quad \lambda = \Lambda_m \Gamma_m , \\ &\pi' = \mathcal{C}_{*,\infty} = \lambda^2_{q+\bar{n}} , \quad \Omega = \mathbb{T}^3 \times \mathbb{R} , \quad \pi = \Gamma_{m+1} \mathrm{T}^{4\mathsf{N}_{\mathrm{ind,t}}}_{m+1} \delta^2_{m+3\bar{n}} \Lambda_m \Gamma_m , \quad M_t = \mathsf{N}_{\mathrm{ind,t}} , \\ &\Upsilon = \Lambda = \lambda_{q+\bar{n}} \Gamma_{q+\bar{n}} , \quad M_\circ = N_\circ = \mathsf{N}_{\mathrm{ind}}/4 , \quad K_\circ \text{ such that } \delta^3_{q+3\bar{n}} \mathrm{T}^{3\mathsf{N}_{\mathrm{ind,t}}+3}_{q+\bar{n}} \leq \Lambda^{-K_\circ} \leq \delta^3_{q+3\bar{n}} \mathrm{T}^{3\mathsf{N}_{\mathrm{ind,t}}+2}_{q+\bar{n}} . \end{aligned}$$

Then we have that (A.97a) is satisfied from (6.11), (A.97b) is satisfied by (9.50a) and Sobolev embedding, and (A.98) is satisfied as in `Step 1`. Since $\mathsf{d} = 0$, we move straight to the nonlocal assumptions and output, for which all assumptions from item (i) in Proposition A.3.3 are satisfied as in the beginning of `Step 2`. Next, we have that (A.101a) is equivalent to (A.97a), (A.101b) is equivalent to (A.97b), and (A.102) is satisfied from direct computation. We

therefore have from (A.104) that for $N, M \leq 3\mathsf{N}_{\text{ind}}$,

$$\begin{aligned}
&\left\| D^N D^M_{t,m-1} \mathcal{R}^* \left(\operatorname{div} \left((R^m_q - R^m_\ell) - (\pi^m_q - \pi^m_\ell) \mathrm{Id} \right)^\bullet w^\bullet_{q+1} \right) \right\|_\infty \\
&\quad \leq \mathrm{T}^{3\mathsf{N}_{\text{ind,t}}+1}_{q+\bar{n}} \delta^3_{q+3\bar{n}} (\mathrm{T}^{-1}_{m-1} \Gamma^2_{m-1})^M (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N
\end{aligned} \tag{11.107a}$$

In the remaining cases, we have

$$\begin{aligned}
&\left\| D^N D^M_{t,q} \mathcal{R}^* \left(\operatorname{div} \left((R^q_q - R_\ell) - (\pi^q_q - \pi_\ell) \mathrm{Id} \right) \cdot w_{q+1} \right) \right\|_\infty \\
&\quad \leq \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}+1}_{q+\bar{n}} \delta^3_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_q^{-1} \Gamma_q^2 \right)
\end{aligned} \tag{11.107b}$$

$$\begin{aligned}
&\left\| D^N D^M_{t,q+\bar{n}-1} \mathcal{R}^* \left(\operatorname{div} \left((\pi^m_q - \pi^m_\ell) \mathrm{Id} \right) w_{q+1} \right) \right\|_\infty \\
&\quad \leq \mathrm{T}^{2\mathsf{N}_{\text{ind,t}}+1}_{q+\bar{n}} \delta^3_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma^2_{q+\bar{n}-1} \right)
\end{aligned} \tag{11.107c}$$

for $N, M \leq 3\mathsf{N}_{\text{ind}}$ and $q + \bar{n} \leq m < q + \mathsf{N}_{\text{pr}}$.

We can now set

$$\begin{aligned}
\overline{\phi}_L^{q+\bar{n},*} = \mathcal{R}^* &\left[(w_{q+1} - e_{q+1}) \cdot \operatorname{div} \left[\sum_{m=q}^{q+\bar{n}-1} R^m_\ell - \sum_{m=q}^{q+\bar{n}+\mathsf{N}_{\text{pr}}-1} \pi^m_\ell \mathrm{Id} \right] \right] \\
&+ \mathcal{R}^* \left[\left[\sum_{m=q}^{q+\bar{n}-1} R^m_\ell - \sum_{m=q}^{q+\bar{n}+\mathsf{N}_{\text{pr}}-1} \pi^m_\ell \mathrm{Id} \right] \cdot e_{q+1} \right] \\
&+ \mathcal{R}^* \left[\operatorname{div} \left[\sum_{m=q}^{q+\bar{n}-1} (R^m_q - R^m_\ell) - \sum_{m=q}^{q+\bar{n}+\mathsf{N}_{\text{pr}}-1} (\pi^m_q - \pi^m_\ell) \mathrm{Id} \right] \cdot w_{q+1} \right] .
\end{aligned}$$

We must now upgrade the material derivatives in the estimates (11.105a), (11.106a), (11.107a) in order to match the bound in (11.97b). Specifically, we apply Remark A.2.6 with $p = \infty$, $N_x = N_t = \infty$, $N_* = \mathsf{N}_{\text{ind}}/4$, $\Omega = \mathbb{T}^3 \times \mathbb{R}$, $v = \widehat{u}_{m-1}$, $w = \widehat{u}_{q+\bar{n}-1} - \widehat{u}_{m-1}$, and parameter choices according to (5.32), which verifies (A.34), parameter choices according to (5.34), which verifies (A.27), and parameter choices according to (11.105)–(11.107), which verify (A.28). We however emphasize the choice of $\mathcal{C}_f = \mathrm{T}^{\mathsf{N}_{\text{ind,t}}+1}_{q+\bar{n}} \delta^3_{q+3\bar{n}}$, which can be used to absorb lossy material derivative estimates. We then have from (A.35) that (11.97b) holds,

concluding the proof. □

11.2.4 Stress current error

From subsection 11.1, the stress current error is given by

$$-\overline{\phi}_R = (\widehat{u}_{q+1} - \widehat{u}_q)\overline{\kappa}_{q+1} + \left(\overline{R}_{q+1} - (\pi_q - \pi_q^q)\mathrm{Id}\right)(\widehat{u}_{q+1} - \widehat{u}_q)\,.$$

Next, we recall from (5.4) that $\widehat{u}_{q+1} - \widehat{u}_q = \widehat{w}_{q+1}$, and that from (11.3a),

$$\overline{\kappa}_{q+1} = \frac{1}{2}\mathrm{tr}\left(-\pi_q\mathrm{Id} + \pi_q^q\mathrm{Id} + \overline{R}_{q+1}\right) = -\frac{3}{2}(\pi_q - \pi_q^q) + \frac{1}{2}\mathrm{tr}\,\overline{R}_{q+1}\,.$$

Lemma 11.2.9 (Definition and basic estimates). *The current error $\overline{\phi}_R$ satisfies the following.*

(i) We have the decomposition

$$\overline{\phi}_R = \overline{\phi}_R^{q+1,l} + \sum_{m=q+1}^{q+\bar{n}} \overline{\phi}_R^{m,*}\,. \tag{11.108}$$

(ii) For all N, M such that $N + M \leq \mathsf{N}_{\mathrm{ind}}/4$ and $q+1 \leq m \leq q+\bar{n}$, we have that

$$\left|\psi_{i,q} D^N D_{t,q}^M \overline{\phi}_R^{q+1,l}\right| \leq \Gamma_{q+1}^{-99}\left(\pi_q^{q+1}\right)^{3/2} r_{q+1}^{-1}\Lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+20}\tau_q^{-1}, \Gamma_q^{10}\mathrm{T}_q^{-1}\right) \tag{11.109a}$$

$$\left\|D^N D_{t,m-1}^M \overline{\phi}_R^{m,*}\right\|_{L^\infty} \leq \delta_{q+3\bar{n}}^2 \Lambda_m^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{m-1}^{-1}, \mathrm{T}_{m-1}^{-1}\right)\,. \tag{11.109b}$$

(iii) The local part $\overline{\phi}_R^{q+1,l}$ has the support property,

$$B\left(\mathrm{supp}\,\widehat{w}_q, \lambda_q^{-1}\Gamma_{q+1}\right) \cap \mathrm{supp}\,\overline{\phi}_O^{q+1,l} = \emptyset\,. \tag{11.110}$$

Proof. Recalling (5.5), (5.22), (10.103), and (10.104), we define

$$
\begin{aligned}
-\overline{\phi}_R^{q+1,l} &= \widehat{w}_{q+1}\left[-\frac{3}{2}(\pi_q-\pi_q^q)+\sum_{m=q+1}^{q+\bar{n}}\frac{1}{2}\mathrm{tr}\,\overline{R}_{q+1}^{m,l}\right]+\left[-(\pi_q-\pi_q^q)\mathrm{Id}+\sum_{m=q+1}^{q+\bar{n}}\overline{R}_{q+1}^{m,l}\right]\widehat{w}_{q+1}\\
&= \widehat{w}_{q+1}\left[-\frac{3}{2}(\pi_q-\pi_q^q)+\sum_{m=q+1}^{q+\bar{n}}\frac{1}{2}\mathrm{tr}\,\left(R_q^{m,l}+S_{q+1}^{m,l}\right)\right]+\left[-(\pi_q-\pi_q^q)\mathrm{Id}+\sum_{m=q+1}^{q+\bar{n}}R_q^{m,l}+S_{q+1}^{m,l}\right]\widehat{w}_{q+1}
\end{aligned}
\tag{11.111}
$$

$$
-\overline{\phi}_R^{m,*} = \widehat{w}_{q+1}\frac{1}{2}\mathrm{tr}\,\overline{R}_{q+1}^{m,*}+\overline{R}_{q+1}^{m,*}\widehat{w}_{q+1} \qquad \text{for } q+1\le m\le q+\bar{n}\,. \tag{11.112}
$$

In order to prove (11.109a) for $\overline{\phi}_R^{q+1,l}$, it suffices to prove the estimate for the second term from the second line of (11.111), as it is clear that the first term will obey identical estimates. We first consider the term with the stresses, before handling the term with the pressures next. The crucial first step is to employ *dodging* to eliminate most of the terms from (11.111). Specifically, we have from (5.5) (which gives that $R_q^{q+\bar{n},l}\equiv 0$) and (5.30) that

$$
\left(\sum_{m=q+1}^{q+\bar{n}}R_q^{m,l}\right)\widehat{w}_{q+1} = R_q^{q+1,l}\widehat{w}_{q+1}\,.
$$

Therefore, we have from (5.21a), (5.22) and (5.21c) that for $N+M\le 2\mathsf{N}_{\mathrm{ind}}$,

$$
\begin{aligned}
\left|\psi_{i,q}D^N D_{t,q}^M R_q^{q+1,l}\widehat{w}_{q+1}\right| &\le \sum_{N_1=0}^{N}\sum_{M_1=0}^{M}\sum_{i'=i-1}^{i+1}\left|\psi_{i,q}D^{N_1}D_{t,q}^{M_1}R_{q+1}^{q+1,l}\right|\left|\psi_{i',q}D^{N-N_1}D_{t,q}^{M-M_1}\widehat{w}_{q+1}\right|\\
&\le \pi_q^{q+1}(\pi_q^{q+1})^{1/2}r_{q-\bar{n}+1}^{-1}\Lambda_{q+1}^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_q^{i+20}\tau_q^{-1},\mathrm{T}_q^{-1}\Gamma_q^{10}\right)\\
&\le \Gamma_{q+1}^{-101}\left(\pi_q^{q+1}\right)^{3/2}r_{q+1}^{-1}\Lambda_{q+1}^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_q^{i+20}\tau_q^{-1},\mathrm{T}_q^{-1}\Gamma_q^{10}\right)\,,
\end{aligned}
$$

where we have used (4.10h) to achieve the final inequality.

In order to prove a similar estimate for the term with pressures, we appeal to (5.6) and

(5.31a) to write that for $N + M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q}\left[(\pi - \pi_q^q)\widehat{w}_{q+1}\right]\right| = \left|\psi_{i,q} D^N D^M_{t,q}\left[\left(\sum_{k=q+1}^{\infty} \pi_q^k\right)\widehat{w}_{q+1}\right]\right|$$
$$\lesssim \Gamma_q \Gamma_{q+1}^{-100} (\pi_q^{q+1})^{3/2} r_{q+1}^{-1} \Lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_q^{i+20}\tau_q^{-1}, \Gamma_q^{10}\mathrm{T}_q^{-1}\right) .$$

Combined with the previous estimate, this concludes the proof of (11.109a) for terms from (11.111) which involve stresses $R_q^{m,l}$ and pressure.

In order to prove (11.109a) for terms from (11.111) which involve stresses $S_{q+1}^{m,l}$ defined in (10.103), we again employ the dodging results from (10.10), (10.53a), and (10.80) to write that

$$\sum_{m=q+1}^{q+\bar{n}} S_{q+1}^{m,l}\widehat{w}_{q+1} = S_{q+1}^{q+1,l}\widehat{w}_{q+1} .$$

Then from (10.108) and (5.21c), we have that for $N + M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left|\psi_{i,q} D^N D^M_{t,q} S_{q+1}^{q+1,l}\widehat{w}_{q+1}\right| \leq \sum_{N_1=0}^{N}\sum_{M_1=0}^{M}\sum_{i'=i-1}^{i+1}\left|\psi_{i,q} D^{N_1} D^{M_1}_{t,q} S_{q+1}^{q+1,l}\right|\left|\psi_{i',q} D^{N-N_1} D^{M-M_1}_{t,q}\widehat{w}_{q+1}\right|$$
$$\lesssim \Gamma_{q+1}^{-50}\pi_q^{q+1}(\pi_q^{q+1})^{1/2} r_{q-\bar{n}+1}^{-1}\Lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_q^{i+20}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^{10}\right)$$
$$\leq \Gamma_{q+1}^{-101}\left(\pi_q^{q+1}\right)^{3/2} r_{q+1}^{-1}\Lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_q^{i+20}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^{10}\right) ,$$

concluding the proof of (11.109a).

Lastly, the nonlocal estimate (11.109b) follows immediately from (5.21c), (5.22), (10.110), and immediate computation. We omit further details. Also, the support property (11.110) can be easily obtained from the definition (11.111) of $\overline{\phi}_R^{q+1,l}$ and Hypothesis 5.4.1. □

11.2.5 Divergence correctors

Lemma 11.2.10 (**Divergence corrector current error and the associated pressure increment**). *There exist current errors* $\overline{\phi}_C^m = \overline{\phi}_C^{m,l} + \overline{\phi}_C^{m,*}$ *and pressure increments* $\sigma_{\overline{\phi}_C^m} = \sigma^+_{\overline{\phi}_C^m} - \sigma^-_{\overline{\phi}_C^m}$ *for* $m = q + \bar{n}/2 + 1, \ldots, q + \bar{n}$ *such that the following hold.*

(i) We have the equality

$$\operatorname{div}\left(\frac{1}{2}|w_{q+1}^{(c)}|^2 w_{q+1}^{(p)} + (w_{q+1}^{(c)} \cdot w_{q+1}^{(p)}) w_{q+1}^{(p)} + \frac{1}{2}|w_{q+1}|^2 w_{q+1}^{(c)}\right) = \sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} \operatorname{div}\phi_C^m \,.$$

(ii) For all $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$, we have that

$$\left|\psi_{i,q} D^N D_{t,q}^M \overline{\phi}_C^{m,l}\right| \lesssim (\sigma^+_{\overline{\phi}_C^m} + \delta_{q+3\bar{n}})^{3/2} r_m^{-1} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.113a}$$

$$\left|\psi_{i,q} D^N D_{t,q}^M \overline{\phi}_C^{q+\bar{n}}\right| \lesssim (\sigma^+_{\overline{\phi}_C^{q+\bar{n},l}} + \sigma_{\upsilon}^+ + \delta_{q+3\bar{n}})^{3/2} r_m^{-1} (\lambda_{q+\bar{n}} \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{11.113b}$$

where the first estimate holds for $m = q + \bar{n}/2 + 1, \ldots, q + \bar{n} - 1$, and σ_{υ}^+ is defined as in Lemma 9.4.4 in the second estimate. In addition, for all $m = q + \bar{n}/2 + 1, \ldots, q + \bar{n}$ and $N, M \leq \mathsf{N}_{\mathrm{fin}}/200$, we have that

$$\left|\psi_{i,q} D^N D_{t,q}^M \sigma^+_{\overline{\phi}_C^m}\right| \lesssim \left(\sigma^+_{\overline{\phi}_C^m} + \delta_{q+3\bar{n}}^2\right) (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.113c}$$

$$\left\|\psi_{i,q} D^N D_{t,q}^M \sigma^+_{\overline{\phi}_C^m}\right\|_{3/2} < \delta_{m+\bar{n}} \Gamma_m^{-9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.113d}$$

$$\left\|\psi_{i,q} D^N D_{t,q}^M \sigma^+_{\overline{\phi}_C^m}\right\|_{\infty} < \Gamma_m^{\mathsf{C}_\infty - 9} (\lambda_m \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.113e}$$

$$\left|\psi_{i,q} D^N D_{t,q}^M \sigma^-_{\overline{\phi}_C^m}\right| < \pi_q^{q+\bar{n}/2} \left(\lambda_{q+\lfloor \bar{n}/2 \rfloor} \Gamma_q\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.113f}$$

Finally, we have that for all $m = q + \bar{n}/2 + 1, \ldots, q + \bar{n}$,

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} \overline{\phi}_C^{m,l} = \emptyset \qquad \forall q+1 \leq q' \leq m-1 \tag{11.114a}$$

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}\right) \cap \operatorname{supp} (\sigma^+_{\overline{\phi}_C^m}) = \emptyset \qquad \forall q+1 \leq q' \leq m-1 \tag{11.114b}$$

$$B\left(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q+1}^{-1}\Gamma_q^2\right) \cap \operatorname{supp} (\sigma^-_{\overline{\phi}_C^m}) = \emptyset \qquad \forall q+1 \leq q' \leq q + \bar{n}/2 \,. \tag{11.114c}$$

(iii) For all $m = q + \bar{n}/2 + 1, \dots, q + \bar{n}$ and $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$, the non-local part $\bar{\phi}_C^{m,*}$ satisfies

$$\left\| D^N D_{t,q}^M \bar{\phi}_C^{m,*} \right\|_{L^\infty} \leq \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^{3/2} \lambda_m^N \tau_q^{-M} , \tag{11.115}$$

Lemma 11.2.11 (Pressure current). *For every $m' \in \{q + \bar{n}/2 + 1, \dots, q + \bar{n}\}$, there exists a current error $\phi_{\bar{\phi}_C^{m'}}$ associated to the pressure increment $\sigma_{\bar{\phi}_C^{m'}}$ in the sense of*

$$\mathrm{div} \phi_{\bar{\phi}_C^{m'}} = D_{t,q} \sigma_{\bar{\phi}_C^{m'}} - \int_{\mathbb{T}^3} D_{t,q} \sigma_{\bar{\phi}_C^{m'}}(t, x')\, dx' . \tag{11.116}$$

The current error $\phi_{\bar{\phi}_C^{m'}}$ has a decomposition

$$\phi_{\bar{\phi}_C^{m'}} = \phi^*_{\bar{\phi}_C^{m'}} + \sum_{m=q+\bar{n}/2+1}^{m'} \phi^m_{\bar{\phi}_C^{m'}} = \phi^*_{\bar{\phi}_C^{m'}} + \sum_{m=q+\bar{n}/2+1}^{m'} \phi^{m,l}_{\bar{\phi}_C^{m'}} + \phi^{m,*}_{\bar{\phi}_C^{m'}} ,$$

where the local parts $\phi^{m,l}_{\bar{\phi}_C^{m'}}$ and the nonlocal parts $\phi^{m,*}_{\bar{\phi}_C^{m'}}$ and $\phi^*_{\bar{\phi}_C^{m'}}$ satisfy the following properties.

(i) For $q + \bar{n}/2 + 1 \leq m \leq m'$ and $N, M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\left| \psi_{i,q} D^N D_{t,q}^M \phi^{m,l}_{\bar{\phi}_C^{m'}} \right| < \Gamma_m^{-100} \left(\pi_q^m\right)^{3/2} r_m^{-1} (\lambda_m \Gamma_m^2)^M \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1} \Gamma_q^{i+17}, \mathrm{T}_q^{-1} \Gamma_q^9\right) \tag{11.117a}$$

$$\left\| D^N D_{t,q}^M \phi^{m,*}_{\bar{\phi}_C^{m'}} \right\|_\infty \left\| D^N D_{t,q}^M \phi^*_{\bar{\phi}_C^{m'}} + \right\|_\infty < \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}} \delta_{q+3\bar{n}}^{3/2} (\lambda_m \Gamma_m^2)^N \tau_q^{-M} . \tag{11.117b}$$

(ii) For all $q + \bar{n}/2 + 1 \leq m \leq m'$ and all $q + 1 \leq q' \leq m - 1$,

$$B\left(\mathrm{supp}\, \widehat{w}_{q'}, {}^1/_2 \lambda_{q'}^{-1} \Gamma_{q'+1}\right) \cap \mathrm{supp}\, \phi^{m,l}_{\bar{\phi}_C^{m'}} = \emptyset . \tag{11.118}$$

(iii) For $M \leq 2\mathsf{N}_{\mathrm{ind}}$, the mean part $\langle D_{t,q} \sigma_{\bar{\phi}_C^{m'}} \rangle$ satisfies

$$\left| \frac{d^M}{dt^M} \langle D_{t,q} \sigma_{\bar{\phi}_C^{m'}} \rangle \right| \leq (\max(1, T))^{-1} \delta_{q+3\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_q^{-1}, \mathrm{T}_q^{-1} \Gamma_q^9\right) . \tag{11.119}$$

Proof of Lemma 11.2.10–11.2.11. `Step 1: Analyze the error`. We first decompose

$$\frac{1}{2}|w_{q+1}^{(c)}|^2 w_{q+1}^{(p)} + (w_{q+1}^{(c)} \cdot w_{q+1}^{(p)}) w_{q+1}^{(p)} + \frac{1}{2}|w_{q+1}|^2 w_{q+1}^{(c)}$$

$$= \frac{1}{2}|w_{q+1}^{(c)}|^2 w_{q+1}^{(p)} + \frac{1}{2} w_{q+1}^{(c)} |w_{q+1}^{(c)}|^2 + \left(w_{q+1}^{(c)} \cdot w_{q+1}^{(p)} \right) w_{q+1}^{(c)} \tag{11.120}$$

$$+ (w_{q+1}^{(c)} \cdot w_{q+1}^{(p)}) w_{q+1}^{(p)} + \frac{1}{2}|w_{q+1}^{(p)}|^2 w_{q+1}^{(c)} \,. \tag{11.121}$$

The first set (11.120) of terms is simpler because each term has two divergence correctors and thus will be absorbed directly into $\overline{\phi}_C^{q+\bar{n},l}$. The second set (11.121) is more delicate, so we now rewrite this term using a few algebraic identities similar to the divergence corrector error terms in the Euler-Reynolds system (analyzed in Lemma 10.2.10).

Taking the divergence operator to the first term in (11.121) and using $\mathbb{U}_{(\xi),\diamond}^{I,s}$ to denote the s component of the vector field $\mathbb{U}_{(\xi),\diamond}^{I}$ (the potential for $\mathbb{W}_{(\xi),\diamond}^{I,s}$), we have that

$$\begin{aligned}
&\operatorname{div}\left(w_{q+1,\diamond}^{(p)} (w_{q+1,\diamond}^{(c)} \cdot w_{q+1,\diamond}^{(p)}) \right) \\
&= \xi^\ell A_\ell^m \partial_m \left(\xi^\theta A_\theta^n a_{(\xi),\diamond}^2 \left(\boldsymbol{\rho}_{(\xi)}^\diamond \boldsymbol{\zeta}_\xi^{I,\diamond} \varrho_{(\xi),\diamond}^I \right)^2 \circ \Phi_{(i,k)} \epsilon_{npr} \partial_p \left(a_{(\xi),\diamond} \left(\boldsymbol{\rho}_{(\xi)}^\diamond \boldsymbol{\zeta}_\xi^{I,\diamond} \right) \circ \Phi_{(i,k)} \right) \partial_r \Phi_{(i,k)}^s \mathbb{U}_{(\xi),\diamond}^{I,s} \circ \Phi_{(i,k)} \right) \\
&=: (\mathbf{C}_0)_{(\xi),\diamond}^I \left((\varrho_{(\xi),\diamond}^I)^2 \mathbb{U}_{(\xi),\diamond}^{I,s} \right) \circ \Phi_{(i,k)} \,.
\end{aligned} \tag{11.122}$$

As we see in the second and the third line, for the time being we omit the summation over the indices $\diamond, i, j, k, \xi, \vec{l}, I$ for convenience, until we reintroduce them. In the first equality, observe that we have commuted $\xi^\ell A_\ell^m$ with ∂_m so that we see the good differential operator $\xi^\ell A_\ell^m \partial_m$, which can only cost $\Lambda_q \Gamma_q^{13}$ from Lemma 9.3.1. This is because it can never land on a high-frequency object (any of $\boldsymbol{\rho}_{(\xi)}^\diamond$, $\boldsymbol{\zeta}_\xi^{I,\diamond}$, $\varrho_{(\xi),\diamond}^I$, $\mathbb{U}_{(\xi),\diamond}^I$). In particular, this term can be written in the form appearing in the second equality. We will treat this term similar to the oscillation current error.

Next, we write the divergence of the second term in (11.121) as

$$
\begin{aligned}
&\operatorname{div}\left(w^{(c)}_{q+1,\diamond}|w^{(p)}_{q+1,\diamond}|^2\right)\\
&\quad=\partial_m\left(\epsilon_{mpr}\left(a^{p,\mathrm{bad}}_{(\xi),\diamond}+a^{p,\mathrm{good}}_{(\xi),\diamond}\right)\partial_r\Phi^s(\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}\,a^2_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\varrho^I_{(\xi),\diamond})^2\circ\Phi_{(i,k)}\xi^\ell A^j_\ell\xi^n A^j_n\right)\\
&\quad=:\mathbf{V}_3+\mathbf{V}_4\,,
\end{aligned}
$$

recalling the notation from (10.83). The term inside of the divergence in $\mathbf{V}_4$ enjoys properties identical to the terms in (11.120); indeed, the good differential operator in $a^{p,\mathrm{good}}_{(\xi),\diamond}$ only costs $\Lambda_q\Gamma_q^{13}$, and so we absorb these terms into $\phi_C^{q+\bar{n},l}$. On the other hand, we deal with the term $\mathbf{V}_3$ using (10.83) and (7.8) to expand

$$
\begin{aligned}
\mathbf{V}_3=\partial_m\Bigg[\epsilon_{mpr}\Bigg(&\partial_p\Phi^n_{(i,k)}(\xi')^n(\xi')^\ell A^j_\ell\partial_j\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\right)\circ\Phi_{(i,k)}\right)\\
&+\partial_p\Phi^n_{(i,k)}(\xi'')^n(\xi'')^\ell A^j_\ell\partial_j\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\right)\circ\Phi_{(i,k)}\right)\Bigg)\\
&\times\frac{1}{3}\partial_r\Phi^s_{(i,k)}\left(-(\xi')^s\varphi''_\xi+(\xi'')^s\varphi'_\xi\right)\circ\Phi_{(i,k)}\,a^2_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\varrho^I_{(\xi),\diamond})^2\circ\Phi_{(i,k)}\xi^\ell A^j_\ell\xi^n A^j_n\Bigg]\,.
\end{aligned}
$$

Note importantly that this term includes factors of either $\partial_p\Phi^n_{(i,k)}(\xi')^n$ or $\partial_p\Phi^n_{(i,k)}(\xi'')^n$ from $a^{p,\mathrm{bad}}_{(\xi),\diamond}$ and $\partial_r\Phi^s_{(i,k)}(\xi')^s$ or $\partial_r\Phi^s_{(i,k)}(\xi'')^s$ from $\mathbb{U}^{I,s}_{(\xi),\diamond}$. We immediately see from the alternating property of the Levi-Civita tensor that the terms including

$$
\epsilon_{mpr}\left(\partial_p\Phi^n_{(i,k)}(\xi')^n\partial_r\Phi^s_{(i,k)}(\xi')^s+\partial_p\Phi^n_{(i,k)}(\xi'')^n\partial_r\Phi^s_{(i,k)}(\xi'')^s\right)
$$

vanish. Thus we only have to consider the cross terms, for example the term

$$
\begin{aligned}
\partial_m\Bigg[\epsilon_{mpr}\partial_p\Phi^n_{(i,k)}(\xi')^n(\xi')^\ell A^j_\ell\partial_j\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\right)\circ\Phi_{(i,k)}\right)\\
\times\frac{1}{3}\partial_r\Phi^s_{(i,k)}(\xi'')^s\varphi'_\xi\circ\Phi_{(i,k)}\,a^2_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\varrho^I_{(\xi),\diamond})^2\circ\Phi_{(i,k)}\xi^\ell A^j_\ell\xi^n A^j_n\Bigg]\,.
\end{aligned}
\tag{11.123}
$$

Our first claim is that the vector indexed by m inside the parentheses is actually *parallel* to

$\xi^\ell A_\ell^m$, which means that we have the good differential operator! To see this, we write

$$\begin{aligned}
& \epsilon_{mpr}\partial_p\Phi^n(\xi')^n\partial_r\Phi^s(\xi'')^s \qquad \| \qquad \xi^\ell A_\ell^m \\
\iff\ & \epsilon_{mpr}\partial_p\Phi^n(\xi')^n\partial_r\Phi^s(\xi'')^s\partial_m\Phi^k \qquad \| \qquad \xi^\ell A_\ell^m\partial_m\Phi^k = \xi^k \\
\iff\ & \epsilon_{mpr}\partial_p\Phi^n(\xi')^n\partial_r\Phi^s(\xi'')^s\partial_m\Phi^k(\xi')^k = \epsilon_{mpr}\partial_p\Phi^n(\xi')^n\partial_r\Phi^s(\xi'')^s\partial_m\Phi^k(\xi'')^k = 0\,.
\end{aligned}$$

But the last two expressions are again equal to zero by the alternating property of the Levi-Civita tensor! Thus we have shown that the ∂_m on the outside of the expression in (11.123) will only cost $\Lambda_q\Gamma_q^{13}$, and furthermore that it *cannot* land on $\varrho^I_{(\xi),\diamond}\circ\Phi_{(i,k)}$ or $\varphi'_\xi\circ\Phi_{(i,k)}$ (which is a component of $\mathbb{U}^I_{(\xi),\diamond}$). Therefore, we write

$$\mathbf{V}_3 = (\mathbf{C}_{31})^I_{(\xi),\diamond}(\varphi'_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)} + (\mathbf{C}_{32})^I_{(\xi),\diamond}(\varphi''_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)} \tag{11.124}$$

where $(\mathbf{C}_{3r})^I_{(\xi),\diamond}$, $r=1,2$, are defined by

$$\begin{aligned}
(\mathbf{C}_{31})^I_{(\xi),\diamond} := \frac{1}{3}\xi_\ell A_\ell^m\partial_m\Bigg[\Bigg[\epsilon_{mpr}\partial_p\Phi^n_{(i,k)}(\xi')^n(\xi'')^s\Bigg]_{\|\xi}(\xi')^\ell A_\ell^j\partial_j\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\right)\circ\Phi_{(i,k)}\right) \\
\times\partial_r\Phi^s_{(i,k)}a^2_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi)^2\circ\Phi_{(i,k)}\xi^\ell A_\ell^j\xi^n A_n^j\Bigg] \\
(\mathbf{C}_{32})^I_{(\xi),\diamond} := \frac{1}{3}\xi_\ell A_\ell^m\partial_m\Bigg[\Bigg[\epsilon_{mpr}\partial_p\Phi^n_{(i,k)}(\xi'')^n(\xi')^s\Bigg]_{\|\xi}(\xi'')^\ell A_\ell^j\partial_j\left(a_{(\xi),\diamond}\left(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\right)\circ\Phi_{(i,k)}\right) \\
\times\partial_r\Phi^s_{(i,k)}a^2_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi)^2\circ\Phi_{(i,k)}\xi^\ell A_\ell^j\xi^n A_n^j\Bigg],
\end{aligned}$$

and $[f_m]_{\|\xi}$ denotes the mth component of the projection of f onto the vector $\xi^\ell A_\ell^\bullet$. The analysis of $\mathbf{V}_3$ will then mimic exactly the analysis of (11.122), since we have a good differential operator in ∂_m and one costly differential operator ∂_j landing on $\boldsymbol{\zeta}^{I,\diamond}_\xi$.

`Step 2: Define the current error, pressure increment, and current error, and verify their properties.` Based on the analysis above, we now define the current errors

as

$$\bar{\phi}_C^{q+\bar{n}/2+1} := (\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_0)^{I,s}_{(\xi),\diamond}\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}/2+1}}\left((\varrho^I_{(\xi),\diamond})^2\mathbb{U}^{I,s}_{(\xi),\diamond}\right)\circ\Phi_{(i,k)}\right] \tag{11.125a}$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_{31})^{I}_{(\xi),\diamond}\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}/2+1}}(\varphi'_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)}\right] \tag{11.125b}$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_{32})^{I}_{(\xi),\diamond}\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}/2+1}}(\varphi''_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)}\right] \tag{11.125c}$$

for the lowest shell,

$$\bar{\phi}_C^{m} := (\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_0)^{I,s}_{(\xi),\diamond}\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}\left((\varrho^I_{(\xi),\diamond})^2\mathbb{U}^{I,s}_{(\xi),\diamond}\right)\circ\Phi_{(i,k)}\right] \tag{11.126a}$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_{31})^{I}_{(\xi),\diamond}\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}(\varphi'_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)}\right] \tag{11.126b}$$

$$+(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_{32})^{I}_{(\xi),\diamond}\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}(\varphi''_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)}\right] \tag{11.126c}$$

for $q+{}^{\bar{n}}/_2+1<m<q+\bar{n}$, and

$$\bar{\phi}_C^{q+\bar{n}} := \sum_{m=q+\bar{n}}^{q+\bar{n}+1}(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_0)^{I,s}_{(\xi),\diamond}\left(\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}+\left(\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}}\right)\right)\left((\varrho^I_{(\xi),\diamond})^2\mathbb{U}^{I,s}_{(\xi),\diamond}\right)\circ\Phi_{(i,k)}\right] \tag{11.127a}$$

$$+\sum_{m=q+\bar{n}}^{q+\bar{n}+1}(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_{31})^{I}_{(\xi),\diamond}\left(\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}+\left(\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}}\right)\right)(\varphi'_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)}\right] \tag{11.127b}$$

$$+\sum_{m=q+\bar{n}}^{q+\bar{n}+1}(\mathcal{H}+\mathcal{R}^*)\left[(\mathbf{C}_{32})^{I}_{(\xi),\diamond}\left(\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}+\left(\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}}\right)\right)(\varphi''_\xi(\varrho^I_{(\xi),\diamond})^2)\circ\Phi_{(i,k)}\right] \tag{11.127c}$$

$$+(11.120)+\epsilon_{\bullet pr}a^{p,\mathrm{good}}_{(\xi),\diamond}\partial_r\Phi^s(\mathbb{U}^I_{(\xi),\diamond})^s\circ\Phi_{(i,k)}\,a^2_{(\xi),\diamond}(\boldsymbol{\rho}^\diamond_{(\xi)}\boldsymbol{\zeta}^{I,\diamond}_\xi\varrho^I_{(\xi),\diamond})^2\circ\Phi_{(i,k)}\xi^\ell A^j_\ell\xi^n A^j_n\,. \tag{11.127d}$$

The terms involved with $\mathcal{R}^*$ or $\mathrm{Id}-\widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}}$ go into the non-local parts while the rest goes into the local parts. Indeed, in the case of (11.125a), (11.126a), (11.127a) for example, fix

indices $\xi, i, j, k, \vec{l}, I$, and set when $\diamond = \varphi$

$$G_\varphi = \frac{(\mathbf{C}_0)^{I,s}_{(\xi),\varphi}}{\lambda_{q+\bar{n}}}, \quad \frac{\varrho_\varphi}{\lambda_{q+\bar{n}}} = \begin{cases} \widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}/2+1}}\left((\varrho^I_{(\xi),\varphi})^2 \mathbb{U}^{I,s}_{(\xi),\varphi}\right) & \text{for (11.125a)} \\ \widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}\left((\varrho^I_{(\xi),\varphi})^2 \mathbb{U}^{I,s}_{(\xi),\varphi}\right) & \text{for (11.126a), the first term of (11.127a)} \\ (\mathrm{Id} - \widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}})\left((\varrho^I_{(\xi),\varphi})^2 \mathbb{U}^{I,s}_{(\xi),\varphi}\right) & \text{for the second term of (11.127a),} \end{cases}$$

and when $\diamond = R$,

$$G_R = \frac{(\mathbf{C}_0)^{I,s}_{(\xi),R}}{\lambda_{q+\bar{n}} r_q}, \quad \frac{\varrho_R}{\lambda_{q+\bar{n}}} = \begin{cases} r_q \widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}/2+1}}\left((\varrho^I_{(\xi),R})^2 \mathbb{U}^{I,s}_{(\xi),R}\right) & \text{for (11.125a)} \\ r_q \widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}\left((\varrho^I_{(\xi),R})^2 \mathbb{U}^{I,s}_{(\xi),R}\right) & \text{for (11.126a), the first term of (11.127a)} \\ r_q(\mathrm{Id} - \widetilde{\mathbb{P}}_{\lambda_{q+\bar{n}+1}})\left((\varrho^I_{(\xi),R})^2 \mathbb{U}^{I,s}_{(\xi),R}\right) & \text{for the second term of (11.127a)}. \end{cases}$$

Notice that $\varrho^I_{(\xi),\diamond}(\mathbb{U}^I_{(\xi),\diamond})^s$ has zero mean from (5) of Proposition 7.1.5 and (5) of Proposition 7.1.6. The rest of parameters and the functions are chosen the same as in `Case 2`—`Case 4` of the proof of Lemma 11.2.1. The assumptions in (A.40) and (A.43) of Proposition A.3.3 can be verified using Lemma 9.3.1, Lemma 7.3.3, Lemma 7.3.4, item (6) from Proposition 7.1.5 and item (6) from Proposition 7.1.6, and we leave the details to the reader; the rest of the conditions of the inverse divergence are then satisfied exactly as the oscillation error, and we omit further details. We note only that the support properties for both $G_\diamond$ and $\rho_\diamond$ are also the same as in the oscillation error, and so we can expect the same support (and dodging) properties to hold for the output of the inverse divergence in this case.

Thus we can apply the inverse divergence from Proposition A.3.3 (with the adjustments set out in Remark A.3.8 for scalar fields). With these choices, we also apply Proposition A.4.5 to construct the associated pressure increments and pressure currents. Note that as in `Case 3` of the proof of Lemma 11.2.1, when $m = q + \bar{n}/2 + 2$, we split the synthetic Littlewood-Paley operator $\widetilde{\mathbb{P}}_{(\lambda_{m-1},\lambda_m]}$ further into $\widetilde{\mathbb{P}}_{(\lambda_{q+\bar{n}/2+1},\lambda_{q+\bar{n}/2+3/2}]} + \widetilde{\mathbb{P}}_{(\lambda_{q+\bar{n}/2+3/2},\lambda_{q+\bar{n}/2+2}]}$ and apply the propositions to each of them. The analysis of (11.125b), (11.125c), (11.126b), (11.126c),

(11.127b), (11.127c) is similar; we replace $\mathbb{U}^{I,s}_{(\xi),\diamond}$ by φ'_ξ or φ''_ξ and $(\mathbf{C}_0)^{I,s}_{(\xi),\diamond}$ by $(\mathbf{C}_{31})^{I}_{(\xi),\diamond}$ or $(\mathbf{C}_{32})^{I}_{(\xi),\diamond}$. As a result, we get the same conclusion as that for the oscillation current error. More precisely, we can verify (11.113a)–(11.115) for $q+{}^{\bar{n}}/_2+1 \leq m < q+\bar{n}$, (11.39a)–(11.40) for $q+{}^{\bar{n}}/_2+1 \leq m' < q+\bar{n}$, and these properties associated to (11.127a)–(11.127c).

Lastly, we consider (11.127d). From (9.70a) and (9.70b), the error terms in (11.120) satisfy

$$\left|\psi_{i.q} D^N D^M_{t,q}(11.120)\right| \lesssim (\sigma_v^+ + \delta_{q+3\bar{n}})^{3/2} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}}^{1/10})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+16}, \mathrm{T}_q^{-1}\Gamma_q^9\right) .$$

Therefore, (11.120) does not contribute to the pressure increment $\sigma_{\overline{\phi}_C^{m,l}}$. Recalling (9.24), one can also verify (11.114) for (11.120). On the other hand, the remaining term in (11.127d) generates a new pressure increment. Indeed, fix values of $i, j, k, \xi, \vec{l}, I, \diamond$ and apply Lemma A.4.3 to the functions $\widehat{\upsilon}_{b,\diamond} = \widehat{\upsilon}_{b,i,j,k,\xi,\vec{l},I,\diamond}$ defined by

$$\begin{aligned}
\widehat{\upsilon}_{1,\diamond} &:= r_q^{1/3} a_{(\xi),\diamond} \left(\boldsymbol{\rho}^{\diamond}_{(\xi)} \boldsymbol{\zeta}^{I,\diamond}_{\xi} \varrho^{I}_{(\xi),\diamond}\right) \circ \Phi_{(i,k)} \\
\widehat{\upsilon}_{3,\diamond} &:= r_q^{-1/3} \epsilon_{\bullet pr} a^{p,\text{good}}_{(\xi),\diamond} \partial_r \Phi^s \xi^\ell A^j_\ell \xi^n A^j_n \mathbb{U}^{I,s}_{(\xi),\diamond} \circ \Phi_{(i,k)} .
\end{aligned}$$

The pressure increment $\sigma_{\widehat{\upsilon}_{1,\diamond}}$ associated to $\widehat{\upsilon}_{1,\diamond}$ has already been constructed in the proof of Lemmas 10.2.12-10.2.13. The increment $\sigma_{\widehat{\upsilon}_{3,\diamond}}$ associated to $\widehat{\upsilon}_{3,\diamond}$ can be also constructed similar to $\sigma_{\widehat{\upsilon}_2}$ from the proof of Lemmas 10.2.12-10.2.13. In fact we can choose all the same parameters, and the support properties are also identical, as in the proofs of Lemmas 10.2.12 and 10.2.13. As a consequence, $\sigma_{\widehat{\upsilon}_{b,\diamond}}$, $b = 1, 3$ satisfy

$$\left|D^N D^M_{t,q} \widehat{\upsilon}_b\right| \lesssim (\sigma^+_{\widehat{\upsilon}_b} + \delta_{q+3\bar{n}})^{1/2} (\lambda_{q+\bar{n}} \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.128}$$

for any $N, M \leq {}^{\mathsf{N}_{\text{fin}}}/_{10}$, which implies that the second term in (11.127d) obeys

$$\left|D^N D^M_{t,q} (r_q^{-1/3} \widehat{\upsilon}_1^2 \widehat{\upsilon}_3)\right| \lesssim (\sigma^+_{\widehat{\upsilon}_1} + \sigma^+_{\widehat{\upsilon}_3} + \delta_{q+3\bar{n}})^{3/2} r_q^{-1/3} (\lambda_{q+\bar{n}} \Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)$$

for any $N, M \leq \mathsf{N}_{\text{fin}}/10$. Furthermore, we appeal to the same conclusions used in `Case 3` of the proof of Lemma 11.2.1 or (9.77a)–(9.77f) from Lemmas 9.4.4–9.4.6 to conclude that

$$\left|D^N D^M_{t,q}\sigma^+_{\widehat{v}_b}\right| \lesssim (\sigma^+_{\widehat{v}_b} + \delta_{q+3\bar{n}})(\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+15}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.129a}$$

$$\begin{aligned}\left\|D^N D^M_{t,q}\sigma^+_{\widehat{v}_b}\right\|_{3/2} \lesssim & \left[\left|\text{supp}\left(\eta_{i,j,k,\xi,\vec{l},\diamond}\boldsymbol{\zeta}^{I,\diamond}_{\xi}\right)\right|^{2/3}\delta_{q+\bar{n}} r_q^{4/3}\Gamma_q^{2j+14} + \delta_{q+3\bar{n}}\right] \\ & \times (\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right)\end{aligned} \tag{11.129b}$$

$$\left\|D^N D^M_{t,q}\sigma^+_{\widehat{v}_b}\right\|_{\infty} \lesssim \Gamma_q^{\mathsf{C}_\infty+30}(\lambda_{q+\bar{n}}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.129c}$$

$$\left|D^N D^M_{t,q}\sigma^-_{\widehat{v}_b}\right| \lesssim \pi_\ell \Gamma_q^{30}(\lambda_{q+\bar{n}/2}\Gamma_q)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}\Gamma_q^{i+14}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{11.129d}$$

for all $N, M \leq \mathsf{N}_{\text{fin}}/100$. We reintroduce the indices $i, j, k, \xi, \vec{l}, I$, and define the pressure increment associated to (11.127d) by

$$\sigma^{\pm}_{(11.127\text{d})} := \sum_{b=1,3}\sum_{i,j,k,\xi,\vec{l},I,\diamond} \sigma^{\pm}_{\widehat{v}_{b,i,j,k,\xi,\vec{l},I,\diamond}}.$$

The estimates (11.113b) and (11.113c) associated to (11.127d) follow using an aggregation procedure identical to that used in the proofs of Lemmas 9.4.4 and 9.4.6, and so we omit further details. Lastly, we define the pressure current $\phi^{k,l}_{(11.127\text{d})}$ and $\phi^{k,*}_{(11.127\text{d})}$ associated to $\sigma_{(11.127\text{d})}$, as in the proofs of Lemmas 9.4.4 and 9.4.6 and obtain conclusions consistent with Lemma 11.2.11. We summarize by setting

$$\begin{aligned}\sigma^{\pm}_{\overline{\phi}_C^{q+\bar{n}}} &:= \sigma^{\pm}_{(11.127\text{a})} + \sigma^{\pm}_{(11.127\text{b})} + \sigma^{\pm}_{(11.127\text{c})} + \sigma^{\pm}_{(11.127\text{d})} \\ \phi^{k}_{\overline{\phi}_C^{q+\bar{n}}} &:= \phi^{k}_{(11.127\text{a})} + \phi^{k}_{(11.127\text{b})} + \phi^{k}_{(11.127\text{c})} + \phi^{k}_{(11.127\text{d})},\end{aligned}$$

and collecting the properties of these objects obtained above, we conclude the proof. □

11.2.6 Mollification current error

Recall from subsection 11.1 the definition of the mollification errors $\overline{\phi}_{M1}$ and $\overline{\phi}_{M2}$. We recall the operators $\mathcal{R}^*$ from (A.56) and L_{TN} from (11.48) and regroup the terms by setting

$$\begin{aligned}
\overline{\phi}_M^{q+1} &:= \varphi_q^q - \varphi_\ell \\
\overline{\phi}_{M3}^{q+\bar{n}} &:= \frac{1}{2}\left(|\widehat{w}_{q+\bar{n}}|^2 \widehat{w}_{q+\bar{n}} - |w_{q+1}|^2 w_{q+1}\right) \\
\overline{\phi}_{M4}^{q+\bar{n}} &:= \mathcal{R}^*\left[L_{TN}\left(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1}\right) + \left(\widehat{w}_{q+\bar{n}} - w_{q+1}\right) \cdot \left(\partial_t u_q + (u_q \cdot \nabla) u_q + \nabla p_q\right)\right] .
\end{aligned}$$

Now notice that $\operatorname{div}(\overline{\phi}_M^{q+1} + \overline{\phi}_{M3}^{q+\bar{n}} + \overline{\phi}_{M4}^{q+\bar{n}}) = \operatorname{div}(\overline{\phi}_{M1} + \overline{\phi}_{M2})$. We also define

$$\overline{\phi}_M^{q+\bar{n}} := \overline{\phi}_{M3}^{q+\bar{n}} + \overline{\phi}_{M4}^{q+\bar{n}} , \tag{11.130}$$

and recalling (11.7), we set

$$\mathfrak{m}_{M4}(t) := \int_0^t \left\langle L_{TN}\left(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1}\right) + \left(\widehat{w}_{q+\bar{n}} - w_{q+1}\right) \cdot \left(\partial_t u_q + (u_q \cdot \nabla) u_q + \nabla p_q\right)\right\rangle (s)\, ds , \tag{11.131}$$

so that $\mathfrak{m}_{M4} = \mathfrak{m}_{M1} + \mathfrak{m}_{M2}$.

Lemma 11.2.12 (Basic estimates and applying inverse divergence). *For all $N+M \leq \mathsf{N}_{\text{ind}}/4$, the mollification errors $\overline{\phi}_M^{q+1}$ and $\overline{\phi}_M^{q+\bar{n}}$ satisfy*

$$\left\| D^N D_{t,q}^M \overline{\phi}_M^{q+1} \right\|_\infty \leq \delta_{q+3\bar{n}}^{3/2} \lambda_{q+1}^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \Gamma_q^{-1} \mathrm{T}_q^{-1}\right) , \tag{11.132a}$$

$$\left\| D^N D_{t,q+\bar{n}-1}^M \overline{\phi}_M^{q+\bar{n}} \right\|_\infty \leq \Gamma_{q+\bar{n}}^9 \delta_{q+3\bar{n}}^{3/2} \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\text{ind,t}}} \left(\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}\right) . \tag{11.132b}$$

In addition, the mean portion $\mathfrak{m}_{M4}$ satisfies

$$\left| \frac{d^{M+1}}{dt^{M+1}} \mathfrak{m}_{M4} \right| \leq (\max(1,T))^{-1} \delta_{q+3\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1}\right) \quad \textit{for } M \leq \mathsf{N}_{\text{ind}}/4 . \tag{11.133}$$

Proof of Lemma 11.2.12. We have that (11.132a) follows immediately from (6.10). Next, in order to handle $\overline{\phi}_{M3}^{q+\bar{n}}$, we recall from (9.84) that

$$\left\|D^N D_{t,q+\bar{n}-1}^M \left(w_{q+1} - \widehat{w}_{q+\bar{n}}\right)\right\|_\infty \lesssim \delta_{q+3\bar{n}}^3 \mathrm{T}_{q+\bar{n}}^{25\mathsf{N}_{\mathrm{ind,t}}} \left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\right) .$$

for all $N + M \leq {}^{\mathsf{N}_{\mathrm{fin}}}/_4$. Using Lemma 9.2.2, we note that $D_{t,q+\bar{n}-1} w_{q+1} = D_{t,q} w_{q+1}$ and $D_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}} = D_{t,q}\widehat{w}_{q+\bar{n}}$. Then writing

$$|\widehat{w}_{q+\bar{n}}|^2\widehat{w}_{q+\bar{n}} - |w_{q+1}|^2 w_{q+1} = (\widehat{w}_{q+\bar{n}} - w_{q+1})|\widehat{w}_{q+\bar{n}}|^2 + w_{q+1}(\widehat{w}_{q+\bar{n}} - w_{q+1})\cdot\widehat{w}_{q+\bar{n}} + w_{q+1}w_{q+1}\cdot(\widehat{w}_{q+\bar{n}} - w_{q+1})$$

and using (9.83), (9.84), and (9.87), we have that for all $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\begin{aligned}&\left\|D^N D_{t,q+\bar{n}-1}^M [|\widehat{w}_{q+\bar{n}}|^2\widehat{w}_{q+\bar{n}} - |w_{q+1}|^2 w_{q+1}]\right\|_\infty \\ &\quad \leq \delta_{q+3\bar{n}} \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}} \left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right) . \end{aligned} \tag{11.134}$$

As for the remaining term $\overline{\phi}_{M4}^{q+\bar{n}}$, we first upgrade the material derivative in the estimate for $\widehat{u}_q$. Applying Lemma A.5.1 to $F^l = 0$, $F^* = \widehat{u}_q$, $k = q + \bar{n}$, $N_\star = {}^{3\mathsf{N}_{\mathrm{fin}}}/_4$ with (5.35a) and using (4.15), we have that

$$\left\|D^N D_{t,q+\bar{n}-1}^M \widehat{u}_q\right\|_\infty \lesssim \mathrm{T}_q^{-1} \lambda_{q+\bar{n}}^N \mathrm{T}_{q+\bar{n}-1}^{-M} .$$

We can now tackle the part of the error term that involves L_{TN}. To estimate this, we use Remark A.3.5 with (4.15), setting

$$\begin{gathered} G = L_{TN}\left(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1}\right), \quad v = \widehat{u}_{q+\bar{n}-1} \\ \mathcal{C}_{G,\infty} = \delta_{q+3\bar{n}} \mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}}, \quad \lambda = \lambda' = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}, \quad M_t = \mathsf{N}_{\mathrm{ind,t}}, \quad \nu = \nu' = \mathrm{T}_{q+\bar{n}}^{-1}, \quad \mathcal{C}_v = \Lambda_{q+\bar{n}-1}^{1/2} \\ N_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_9, \quad M_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_{10}, \quad N_\circ = M_\circ = 2\mathsf{N}_{\mathrm{ind}} . \end{gathered}$$

As a result, with a suitable choice of positive integer $K_\circ$ so that

$$\delta_{q+3\bar{n}}\mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}}\lambda_{q+\bar{n}}^{5}2^{2\mathsf{N}_{\mathrm{ind}}} \leq \lambda_{q+\bar{n}}^{-K_\circ} \leq \delta_{q+3\bar{n}}\mathrm{T}_{q+\bar{n}}^{\mathsf{N}_{\mathrm{ind,t}}} \,,$$

we find that for all $N+M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\begin{aligned}\left\|D^N D_{t,q+\bar{n}-1}^M \mathcal{R}^* \left[L_{TN}\left(\widehat{w}_{q+\bar{n}} \otimes \widehat{w}_{q+\bar{n}} - w_{q+1} \otimes w_{q+1}\right)\right]\right\|_\infty &\lesssim \delta_{q+3\bar{n}}\mathrm{T}_{q+\bar{n}}^{\mathsf{N}_{\mathrm{ind,t}}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathrm{T}_{q+\bar{n}}^{-M} \\ &\leq \delta_{q+3\bar{n}}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}}^{-1}\right) .\end{aligned} \tag{11.135}$$

The estimate for the mean portion follows in the usual way from Remark A.3.7.

Now we deal with the other part of the error term. Recall from (5.2) that

$$\partial_t u_q + (u_q \cdot \nabla) u_q + \nabla p_q = \mathrm{div}(R_q - \pi_q \mathrm{Id}) \,.$$

We apply Lemma A.3.12 with the following choices:

$$\begin{aligned}
&G = \mathrm{div}\left(R_q - \pi_q \mathrm{Id}\right)^\bullet \,, \quad \varrho = \vartheta = \left(\widehat{w}_{q+\bar{n}} - w_{q+1}\right)^\bullet \,, \quad v = \widehat{u}_{q+\bar{n}-1} \,, \quad \lambda' = \lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1} \,, \\
&\nu = \nu' = \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^2 \,, \quad N_* = {}^{\mathsf{N}_{\mathrm{ind}}}/_2 \,, \quad M_* = {}^{\mathsf{N}_{\mathrm{ind}}}/_2 \,, \quad \mathsf{d} = 0 \,, \quad \lambda = \Lambda_{q+\bar{n}}\Gamma_{q+\bar{n}} \,, \\
&\pi' = \mathcal{C}_{*,\infty} = \delta_{q+3\bar{n}}^3 \mathrm{T}_{q+\bar{n}}^{25\mathsf{N}_{\mathrm{ind,t}}} \,, \quad \Omega = \mathbb{T}^3 \times \mathbb{R} \,, \quad \pi = \Gamma_{q+\bar{n}-1}\pi_q\Lambda_{q+\bar{n}-1} \,, \quad M_t = \mathsf{N}_{\mathrm{ind,t}} \,, \\
&\Upsilon = \Lambda = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1} \,, \quad M_\circ = N_\circ = {}^{\mathsf{N}_{\mathrm{ind}}}/_4 \,, \quad K_\circ \text{ such that } \mathrm{T}_{q+\bar{n}}^{-10\mathsf{N}_{\mathrm{ind,t}}} \leq \Lambda^{K_\circ} \leq \mathrm{T}_{q+\bar{n}}^{-10\mathsf{N}_{\mathrm{ind,t}}-1} \,.
\end{aligned}$$

The analysis here is similar to the analysis for the nonlocal transport-Nash current errors, and so we omit the details but note that one can easily check that (A.97a), (A.97b), and (A.98) are satisfied. Since $\mathsf{d} = 0$, we move straight to the non-local assumptions and output, which again can be easily checked by direct computation or using similar arguments as for

other nonlocal error terms. We therefore have from (A.104) that for $N + M \leq \mathsf{N}_{\mathrm{ind}}/4$,

$$\begin{aligned}
&\left\| D^N D^M_{t,q+\bar{n}-1} \mathcal{R}^* \left(\operatorname{div} \left(R_q - \pi_q \mathrm{Id} \right)^\bullet \left(\widehat{w}_{q+\bar{n}} - w_{q+1} \right)^\bullet \right) \right\|_\infty \\
&\quad \lesssim \mathrm{T}_{q+\bar{n}}^{3\mathsf{N}_{\mathrm{ind,t}}} \delta^3_{q+3\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M} \left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}} \right)
\end{aligned} \tag{11.136}$$

Promotion of the material derivatives again follows standard arguments and Lemma A.5.1, and we omit further details. □

11.3 Upgrading material derivatives

Definition 11.3.1 (Definition of $\overline{\phi}_{q+1}$ and $\overline{\varphi}_{q+1}$)**.** *Recalling Lemmas 11.2.1, 11.2.3, 11.2.4, 11.2.5, 11.2.8, 11.2.9, 11.2.10, and 11.2.12, we define* $\overline{\phi}_{q+1} = \sum_{m=q+1}^{q+\bar{n}} \overline{\phi}^m_{q+1}$ *and* $\overline{\phi}^m_{q+1} = \overline{\phi}^{m,l}_{q+1} + \overline{\phi}^{m,*}_{q+1}$ *for* $q+1 \leq m \leq q+\bar{n}$ *by*

$$\overline{\phi}^{m,l}_{q+1} = \overline{\phi}^{m,l}_O + \overline{\phi}^{m,l}_W + \overline{\phi}^{m,l}_{TNC} + \overline{\phi}^{l}_{S^{m,l}_O} + \overline{\phi}^{l}_{S^{m,l}_{TN}} + \overline{\phi}^{l}_{S^{m,l}_{C1}} + \overline{\phi}^{l}_{S^{m,l}_{M2}} + \mathbf{1}_{m=q+\bar{n}} \overline{\phi}^{q+\bar{n},l}_L + \mathbf{1}_{m=q+1} \overline{\phi}^{q+1,l}_R + \overline{\phi}^{m,l}_C \tag{11.137a}$$

$$\begin{aligned}
\overline{\phi}^{m,*}_{q+1} = {} & \overline{\phi}^{m,*}_O + \overline{\phi}^{m,*}_W + \overline{\phi}^{m,*}_{TNC} + \overline{\phi}^{*}_{S^{m,l}_O} + \overline{\phi}^{*}_{S^{m,l}_{TN}} + \overline{\phi}^{*}_{S^{m,l}_{C1}} + \overline{\phi}^{*}_{S^{m,l}_{M2}} + \overline{\phi}_{S^{m,*}_O} + \overline{\phi}_{S^{m,*}_{TN}} + \overline{\phi}_{S^{m,*}_{C1}} + \overline{\phi}_{S^{m,*}_{M2}} \\
& + \mathbf{1}_{m=q+\bar{n}} \overline{\phi}^{q+\bar{n},*}_L + \overline{\phi}^{m,*}_R + \overline{\phi}^{m,*}_C + \overline{\phi}^{m}_M
\end{aligned} \tag{11.137b}$$

Here, any undefined terms are taken to be 0*. We then define the primitive current error* $\overline{\varphi}_{q+1}$ *by*

$$\overline{\varphi}_{q+1} := \sum_{m=q+1}^{q+\bar{n}} \overline{\varphi}^m_{q+1}, \qquad \overline{\varphi}^m_{q+1} = \varphi^m_q + \overline{\phi}^m_{q+1}, \tag{11.138}$$

which we note is consistent with (11.3b)*.*

Lemma 11.3.2 (**Upgrading material derivatives**)**.** *The new current errors* $\overline{\phi}^m_{q+1} =$

$\overline{\phi}^{m,l}_{q+1} + \overline{\phi}^{m,*}_{q+1}$ *satisfy the following. For* $N + M \leq \mathsf{N}_{\mathrm{ind}}/4$*, we have that*

$$\left|\psi_{i,q+\bar{n}/2-1} D^N D^M_{t,q+\bar{n}/2-1} \overline{\phi}^{q+\bar{n}/2}_{q+1}\right| \lesssim \Gamma^{-50}_{q+\bar{n}/2} \pi^{q+\bar{n}/2}_q r^{-1}_{q+\bar{n}/2} \Lambda^N_{q+\bar{n}/2} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q+\bar{n}/2-1} \Gamma^{i-5}_{q+\bar{n}/2-1}, \mathrm{T}^{-1}_q \Gamma^{11}_q\right) . \tag{11.139}$$

For the same range of $N + M$*, the current error* $\overline{\phi}^{q+1}_{q+1}$ *obeys the estimate*

$$\left|\psi_{i,q} D^N D^M_{t,q} \overline{\phi}^{q+1}_{q+1}\right| \lesssim \Gamma^{-50}_{q+1} \pi^{q+1}_q r^{-1}_{q+1} \Lambda^N_m \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_q \Gamma^{i+20}_q, \mathrm{T}^{-1}_q \Gamma^{10}_q\right) . \tag{11.140}$$

Finally, we have that for $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$ *and the same range of* $N + M$*,*

$$\begin{aligned}\left|\psi_{i,m-1} D^N D^M_{t,m-1} \overline{\phi}^m_{q+1}\right| \lesssim & \left(\sigma^+_{m,q+1} + \mathbf{1}_{m=q+\bar{n}} \Gamma^{-50}_{q+\bar{n}} \pi^{q+\bar{n}}_q + \delta_{q+3\bar{n}}\right)^{3/2} r^{-1}_m \\ & \times (\lambda_m \Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+18}_q \tau^{-1}_q, \mathrm{T}^{-1}_q \Gamma^{11}_q\right) .\end{aligned} \tag{11.141}$$

Proof of Lemma 11.3.2. We have that (11.140) follows immediately from (11.12), (11.54a), (11.58), (11.82), (11.85), (11.109a), (11.109b), (11.132a), (5.17), (5.20), and (4.10f). In order to prove the remaining estimates, we appeal to Lemma A.5.1. The proof is very similar to the proofs of items (ii)–(iv), and so we omit most of the details. The basic idea is however that nonlocal error terms can be upgraded trivially using the minuscule amplitude, and the local error terms can be upgraded using the dodging conclusions that have been included in Lemmas 11.2.1, 11.2.3, 11.2.4, 11.2.5, 11.2.8, 11.2.9, 11.2.10, and 11.2.12. □

Chapter 12

Inductive cutoffs

12.1 New mollified velocity increment and definition of the velocity cutoff functions

We first recall the definition of $\widehat{w}_{q+\bar{n}}$ in (9.17). We have that for a mollifier $\widetilde{\mathcal{P}}_{q+\bar{n},x,t}$ at spatial scale $\lambda_{q+\bar{n}}^{-1}\Gamma_{q+\bar{n}-1}^{-1/2}$ and temporal scale T_{q+1}^{-1}, we have

$$\widehat{w}_{q+\bar{n}} = \widetilde{\mathcal{P}}_{q+\bar{n},x,t} w_{q+1} \,. \tag{12.1}$$

Before defining the velocity cutoff functions, we need the following translations between $\Gamma_{q'-1}$ and $\Gamma_{q'}$.

Definition 12.1.1 (Translating Γ's between $q'-1$ and q'). *Given $i, j, q' \geq 0$, we define*

$$\begin{aligned} i_* = i_*(j, q') = i_*(j) &= \min\{i \geq 0 \colon \Gamma_{q'}^{i} \geq \Gamma_{q'-1}^{j}\} \\ j_*(i, q') &= \max\{j : i_*(j) \leq i\} \,. \end{aligned}$$

A consequence of this definition is the inequality

$$\Gamma_{q'}^{i-1} < \Gamma_{q'-1}^{j_*(i,q)} \leq \Gamma_{q'}^{i} \,. \tag{12.3}$$

We also note that for $j = 0$, we have that $i_*(j) = 0$. Finally, a simple computation shows that $i_*(j)$ has an upper bound which depends on j but not q.

We may now define the velocity cutoff functions using the cutoff functions presented in Lemma 8.3.1, although Γ_q will be replaced with $\Gamma_{q+\bar{n}}$ throughout.

Definition 12.1.2 (Intermediate cutoff functions). *For stage $q+1$ of the iteration where $q + \bar{n} \geq 1$, $m \leq \mathsf{N}_{\mathrm{cut,t}}$, and $j_m \geq 0$, we define*

$$h^2_{m,j_m,q+\bar{n}}(x,t) = \Gamma_{q+\bar{n}}^{-2i_*(j_m)} \delta_{q+\bar{n}}^{-1} r_q^{2/3} \left(\tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}}^{i_*(j_m)+2}\right)^{-2m} \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}} \left(\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}}\right)^{-2N} \left| D^N D^m_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}} \right|^2 . \tag{12.4}$$

We then define $\psi_{m,i_m,j_m,q+\bar{n}}$ by

$$\psi_{m,i_m,j_m,q+\bar{n}}(x,t) = \gamma_{m,q+\bar{n}} \left(\Gamma_{q+\bar{n}}^{-2(i_m - i_*(j_m))(m+1)} h^2_{m,j_m,q+\bar{n}}(x,t)\right) \tag{12.5}$$

for $i_m > i_*(j_m)$, while for $i_m = i_*(j_m)$,

$$\psi_{m,i_*(j_m),j_m,q+\bar{n}}(x,t) = \widetilde{\gamma}_{m,q+\bar{n}} \left(h^2_{m,j_m,q+\bar{n}}(x,t)\right) . \tag{12.6}$$

The intermediate cutoff functions $\psi_{m,i_m,j_m,q+\bar{n}}$ are equal to zero for $i_m < i_*(j_m)$.

The indices i_m and j_m will be shown to take values no larger than $i_{\max}$. With these definitions and using (8.14) and (8.15), it follows that

$$\sum_{i_m \geq 0} \psi^6_{m,i_m,j_m,q+\bar{n}} = \sum_{i_m \geq i_*(j_m)} \psi^6_{m,i_m,j_m,q+\bar{n}} = \sum_{\{i_m \,:\, \Gamma^{i_m}_{q+\bar{n}} \geq \Gamma^{j_m}_{q+\bar{n}-1}\}} \psi^6_{m,i_m,j_m,q+\bar{n}} \equiv 1 \tag{12.7}$$

for any m, and for $|i_m - i'_m| \geq 2$,

$$\psi_{m,i_m,j_m,q+\bar{n}}\psi_{m,i'_m,j_m,q+\bar{n}} = 0\,. \tag{12.8}$$

Definition 12.1.3 (m^{th} **Velocity Cutoff Function).** *At stage $q+1$ and for $i_m \geq 0$, we inductively define the mth velocity cutoff function*

$$\psi^6_{m,i_m,q+\bar{n}} = \sum_{\{j_m\,:\,i_m \geq i_*(j_m)\}} \psi^6_{j_m,q+\bar{n}-1}\psi^6_{m,i_m,j_m,q+\bar{n}}\,. \tag{12.9}$$

We shall employ the notation

$$\vec{i} = \{i_m\}_{m=0}^{\mathsf{N}_{\text{cut,t}}} = \left(i_0, ..., i_{\mathsf{N}_{\text{cut,t}}}\right) \in \mathbb{N}_0^{\mathsf{N}_{\text{cut,t}}+1} \tag{12.10}$$

to signify a tuple of non-negative integers of length $\mathsf{N}_{\text{cut,t}} + 1$.

Definition 12.1.4 (**Velocity cutoff function).** *At stage $q+1$ and for $0 \leq i \leq i_{\max}$, we define*

$$\psi^6_{i,q+\bar{n}} = \sum_{\left\{\vec{i}:\ \max\limits_{0\leq m\leq \mathsf{N}_{\text{cut,t}}} i_m = i\right\}} \prod_{m=0}^{\mathsf{N}_{\text{cut,t}}} \psi^6_{m,i_m,q+\bar{n}}\,. \tag{12.11}$$

For $\vec{i}$ as in the sum of (12.11), we shall denote

$$\operatorname{supp}\left(\prod_{m=0}^{\mathsf{N}_{\text{cut,t}}} \psi_{m,i_m,q+\bar{n}}\right) = \bigcap_{m=0}^{\mathsf{N}_{\text{cut,t}}} \operatorname{supp}\left(\psi_{m,i_m,q+\bar{n}}\right) =: \operatorname{supp}\left(\psi_{\vec{i},q+\bar{n}}\right). \tag{12.12}$$

This implies that $(x,t) \in \operatorname{supp}(\psi_{i,q+\bar{n}})$ if and only if there exists $\vec{i} \in \mathbb{N}_0^{\mathsf{N}_{\text{cut,t}}+1}$ such that $\max_{0\leq m\leq \mathsf{N}_{\text{cut,t}}} i_m = i$, and $(x,t) \in \operatorname{supp}(\psi_{\vec{i},q+\bar{n}})$.

12.2 Partitions of unity, dodging, and simple bounds on velocity increments

Lemma 12.2.1 ($\psi_{m,i_m,q+\bar{n}}$ **- Partition of unity).** *For all m, we have that*

$$\sum_{i_m \geq 0} \psi^6_{m,i_m,q+\bar{n}} \equiv 1\,, \qquad \psi_{m,i_m,q+\bar{n}} \psi_{m,i'_m,q+\bar{n}} = 0 \quad \text{for} \quad |i_m - i'_m| \geq 2\,. \tag{12.13}$$

Proof of Lemma 12.2.1. The proof proceeds inductively in a manner very similar to the proof of [7, Lemma 6.7]. To show the first part of (12.13), we may use (12.7) and (12.9) and reorder the summation to obtain

$$\begin{aligned}\sum_{i_m \geq 0} \psi^6_{m,i_m,q} &= \sum_{i_m \geq 0} \sum_{\{j_m\,:\, i_*(j_m) \leq i_m\}} \psi^6_{j_m,q-1} \psi^6_{m,i_m,j_m,q}(x,t) \\ &= \sum_{j_m \geq 0} \psi^6_{j_m,q-1} \underbrace{\sum_{\{i_m\,:\, i_m \geq i_*(j_m)\}} \psi^6_{m,i_m,j_m,q}}_{\equiv 1 \text{ by } (12.7)} = \sum_{j_m \geq 0} \psi^6_{j_m,q-1} \equiv 1\end{aligned}$$

where the last ineqaulity follows from the inductive assumption (5.8).

The proof of the second claim is more involved and will be split into cases. Using the definition in (12.9), we have that

$$\psi_{m,i_m,q+\bar{n}} \psi_{m,i'_m,q+\bar{n}} = \sum_{\{j_m : i_m \geq i_*(j_m)\}} \sum_{\{j'_m : i'_m \geq i_*(j'_m)\}} \psi^6_{j_m,q+\bar{n}-1} \psi^6_{j'_m,q+\bar{n}-1} \psi^6_{m,i_m,j_m,q+\bar{n}} \psi^6_{m,i'_m,j'_m,q+\bar{n}}\,.$$

Recalling the inductive assumption (5.8), we have that the above sum only includes pairs of indices j_m and j'_m such that $|j_m - j'_m| \leq 1$. So we may assume that

$$(x,t) \in \operatorname{supp} \psi_{m,i_m,j_m,q} \cap \operatorname{supp} \psi_{m,i'_m,j'_m,q}, \tag{12.14}$$

where $|j_m - j'_m| \leq 1$. The first and simplest case is the case $j_m = j'_m$. We then appeal to (12.8) to deduce that it must be the case that $|i_m - i'_m| \leq 1$ in order for (12.14) to be true.

Before moving to the second and third cases, we recall from the proof of [7, Lemma 6.7] that by symmetry it will suffice to prove that $\psi_{m,i_m,q+\bar{n}}\psi_{m,i'_m,q+\bar{n}} \equiv 0$ when $i'_m \leq i_m - 2$. We then consider the second case in (12.14), in which $j'_m = j_m + 1$. When $i_m = i_*(j_m)$, we use that $i_*(j_m) \leq i_*(j_m + 1)$ to obtain

$$i'_m \leq i_m - 2 = i_*(j_m) - 2 < i_*(j_m + 1) = i_*(j'_m)\,,$$

and so by Definition 12.1.2, we have that $\psi_{m,i'_m,j'_m,q+\bar{n}} = 0$. Thus we need only now consider $i_m > i_*(j_m)$ in order to finish the proof of the second case from (12.14). From (12.14), items (1)–(2) from Lemma 8.3.1, and Definition 12.1.2, we have that

$$h_{m,j_m,q+\bar{n}}(x,t) \in \left[\frac{1}{2}\Gamma_{q+\bar{n}}^{(m+1)(i_m - i_*(j_m))}, \Gamma_{q+\bar{n}}^{(m+1)(i_m+1-i_*(j_m))}\right], \tag{12.15a}$$

$$h_{m,j_m+1,q+\bar{n}}(x,t) \leq \Gamma_{q+\bar{n}}^{(m+1)(i'_m+1-i_*(j_m+1))}. \tag{12.15b}$$

Note that from the definition of $h_{m,j_m,q+\bar{n}}$ in (12.4), we have that

$$\Gamma_{q+\bar{n}}^{(m+1)(i_*(j_m+1)-i_*(j_m))} h_{m,j_m+1,q+\bar{n}} = h_{m,j_m,q+\bar{n}}\,.$$

Then, since $i'_m \leq i_m - 2$, from (12.15b) we have that

$$\begin{aligned}
\Gamma_{q+\bar{n}}^{-(m+1)(i_m - i_*(j_m))} h_{m,j_m,q+\bar{n}} &= \Gamma_{q+\bar{n}}^{-(m+1)(i_m - i_*(j_m))} h_{m,j_m+1,q+\bar{n}} \Gamma_{q+\bar{n}}^{(m+1)(i_*(j_m+1)-i_*(j_m))} \\
&\leq \Gamma_{q+\bar{n}}^{-(m+1)(i_m - i_*(j_m))} \Gamma_{q+\bar{n}}^{(m+1)(i'_m+1-i_*(j_m+1))} \Gamma_{q+\bar{n}}^{(m+1)(i_*(j_m+1)-i_*(j_m))} \\
&= \Gamma_{q+\bar{n}}^{(m+1)(i'_m+1-i_m)} \\
&\leq \Gamma_{q+\bar{n}}^{-(m+1)}\,.
\end{aligned}$$

Since $m \geq 0$, the above estimate contradicts the lower bound on $h_{m,j_m,q+\bar{n}}$ in (12.15a) because $\Gamma_{q+\bar{n}}^{-1} \ll 1/2$ for a sufficiently large.

We move to the third and final case, $j'_m = j_m - 1$. As before, if $i_m = i_*(j_m)$, then since

$i_*(j_m) \leq i_*(j_m - 1) + 1$, we have that

$$i'_m \leq i_m - 2 = i_*(j_m) - 2 \leq i_*(j_m - 1) - 1 < i_*(j_m - 1) = i_*(j'_m)\,,$$

which by Definition 12.1.2 implies that $\psi_{m,i'_m,j'_m,q+\bar{n}} = 0$, and there is nothing to prove. Thus, we only must consider the case $i_m > i_*(j_m)$. Using the definition (12.4) we have that

$$h_{m,j_m,q+\bar{n}} = \Gamma_{q+\bar{n}}^{(m+1)(i_*(j_m-1)-i_*(j_m))} h_{m,j_m-1,q+\bar{n}}\,.$$

On the other hand, for $i'_m \leq i_m - 2$ we have from (12.15b) that

$$h_{m,j_m-1,q+\bar{n}} \leq \Gamma_{q+\bar{n}}^{(m+1)(i'_m+1-i_*(j_m-1))} \leq \Gamma_{q+\bar{n}}^{(m+1)(i_m-1-i_*(j_m-1))}\,.$$

Therefore, combining the above two displays and the inequality $-i_*(j_m) \geq -i_*(j_m - 1) - 1$, we obtain the bound

$$\begin{aligned}\Gamma_{q+\bar{n}}^{-(m+1)(i_m-i_*(j_m))} h_{m,j_m,q+\bar{n}} &\leq \Gamma_{q+\bar{n}}^{-(m+1)(i_m-i_*(j_m))} \Gamma_{q+\bar{n}}^{(m+1)(i_*(j_m-1)-i_*(j_m))} \Gamma_{q+\bar{n}}^{(m+1)(i_m-1-i_*(j_m-1))} \\ &= \Gamma_{q+\bar{n}}^{-(m+1)}\,,\end{aligned}$$

As before, since $m \geq 0$ this produces a contradiction with the lower bound on $h_{m,j_m,q+\bar{n}}$ given in (12.15a), since $\Gamma_{q+\bar{n}}^{-1} \ll 1/2$. □

Lemma 12.2.2 ($\psi_{i,q+\bar{n}}$ **- Partition of unity).** *We have that*

$$\sum_{i\geq 0} \psi_{i,q+\bar{n}}^6 \equiv 1\,, \qquad \psi_{i,q+\bar{n}}\psi_{i',q+\bar{n}} \equiv 0 \quad \text{for} \quad |i - i'| \geq 2\,. \tag{12.16}$$

Proof of Lemma 12.2.2. To prove the first claim for $q + \bar{n} \geq 1$, let us introduce the notation

$$\Lambda_i = \left\{ \vec{i} = (i_0, ..., i_{\mathsf{N}_{\mathrm{cut,t}}}) \colon \max_{0\leq m\leq \mathsf{N}_{\mathrm{cut,t}}} i_m = i. \right\} \tag{12.17}$$

Then

$$\psi^6_{i,q+\bar{n}} = \sum_{\vec{i}\in\Lambda_i} \prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \psi^6_{m,i_m,q+\bar{n}}\,,$$

and thus

$$\begin{aligned}\sum_{i\geq 0}\psi^6_{i,q} = \sum_{i\geq 0}\sum_{\vec{i}\in\Lambda_i}\prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\psi^6_{m,i_m,q} &= \sum_{\vec{i}\in\mathbb{N}_0^{\mathsf{N}_{\mathrm{cut,t}}+1}}\left(\prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\psi^6_{m,i_m,q}\right)\\ &= \prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\left(\sum_{i_m\geq 0}\psi^6_{m,i_m,q}\right) = \prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} 1 = 1\end{aligned}$$

after using (12.13).

To prove the second claim, assume towards a contradiction that there exists $|i-i'|\geq 2$ such that $\psi_{i,q}\psi_{i',q}\geq 0$. Then

$$0 \neq \psi^6_{i,q+\bar{n}}\psi^6_{i',q+\bar{n}} = \sum_{\vec{i}\in\Lambda_i}\sum_{\vec{i}'\in\Lambda_{i'}}\prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\psi^6_{m,i_m,q+\bar{n}}\psi^6_{m,i'_m,q+\bar{n}}\,. \tag{12.18}$$

In order for (12.18) to be non-vanishing, by (12.13), there must exist $\vec{i}=(i_0,...,i_{\mathsf{N}_{\mathrm{cut,t}}})\in\Lambda_i$ and $\vec{i}'=(i'_0,...,i'_{\mathsf{N}_{\mathrm{cut,t}}})\in\Lambda_{i'}$ such that $|i_m-i'_m|\leq 1$ for all $0\leq m\leq \mathsf{N}_{\mathrm{cut,t}}$. By the definition of i and i', there exist m_* and m'_* such that

$$i_{m_*} = \max_m i_m = i, \qquad i'_{m'_*} = \max_m i'_m = i'.$$

But then

$$i = i_{m_*} \leq i'_{m_*}+1 \leq i'_{m'_*}+1 = i'+1\,, \qquad\qquad i' = i'_{m'_*} \leq i_{m'_*}+1 \leq i_{m_*}+1 = i+1,$$

implying that $|i-i'|\leq 1$, a contradiction. □

Lemma 12.2.3 (Lower order derivative bounds on $\widehat{w}_{q+\bar{n}}$). *If* $(x,t)\in\operatorname{supp}(\psi_{m,i_m,j_m,q+\bar{n}})$

then

$$h_{m,j_m,q+\bar{n}} \leq \Gamma_{q+\bar{n}}^{(m+1)(i_m+1-i_*(j_m))}. \tag{12.19}$$

Moreover, if $i_m > i_*(j_m)$ *we have*

$$h_{m,j_m,q+\bar{n}} \geq (1/2)\Gamma_{q+\bar{n}}^{(m+1)(i_m-i_*(j_m))} \tag{12.20}$$

on the support of $\psi_{m,i_m,j_m,q+\bar{n}}$*. As a consequence, we have that for all* $0 \leq m, M \leq \mathsf{N}_{\mathrm{cut,t}}$ *and* $0 \leq N \leq \mathsf{N}_{\mathrm{cut,x}}$*,*

$$\left\|D^N D^m_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{m,i_m,q+\bar{n}})} \leq \delta_{q+\bar{n}}^{1/2} r_q^{-1/3}\Gamma_{q+\bar{n}}^{i_m+1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_m+3})^m \tag{12.21a}$$

$$\left\|D^N D^M_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \leq \delta_{q+\bar{n}}^{1/2} r_q^{-1/3}\Gamma_{q+\bar{n}}^{i+1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i+3})^M. \tag{12.21b}$$

Proof of Lemma 12.2.3. Estimates (12.19) and (12.20) follow directly from the definitions of $\widetilde{\gamma}_{m,q+\bar{n}}$ and $\gamma_{m,q+\bar{n}}$ in Lemma 8.3.1 and the definition of $h_{m,j_m,q+\bar{n}}$ in (12.4). In order to prove (12.21a), we note that for $(x,t) \in \operatorname{supp}(\psi_{m,i_m,q+\bar{n}})$, by (12.9) there must exist a j_m with $i_*(j_m) \leq i_m$ such that $(x,t) \in \operatorname{supp}(\psi_{m,i_m,j_m,q+\bar{n}})$. Using (12.19), we conclude that

$$\left\|D^N D^m_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{m,i_m,j_m,q+\bar{n}})} \leq \delta_{q+\bar{n}}^{1/2} r_q^{-1/3}\Gamma_{q+\bar{n}}^{i_m+1}\left(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}\right)^N\left(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_m+3}\right)^m \tag{12.22}$$

which completes the proof of (12.21a). The proof of (12.21b) follows from the fact that we have employed the *maximum* over m of i_m to define $\psi_{i,q+\bar{n}}$ in (12.1.4). □

Corollary 12.2.4 (Higher order derivative bounds on $\widehat{w}_{q+\bar{n}}$). *For* $N + M \leq 2\mathsf{N}_{\mathrm{fin}}$ *and* $i \geq 0$*, we have the bound*

$$\begin{aligned}&\left\|D^N D^M_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})}\\ &\qquad \leq \Gamma_{q+\bar{n}}^{i+1}\delta_{q+\bar{n}}^{1/2} r_q^{-1/3}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}\right).\end{aligned} \tag{12.23}$$

Proof of Corollary 12.2.4. When $0 \le N \le \mathsf{N}_{\text{cut,x}}$ and $0 \le M \le \mathsf{N}_{\text{cut,t}} \le \mathsf{N}_{\text{ind,t}}$, the desired bound was already established in (12.21b). For the remaining cases in which either $N > \mathsf{N}_{\text{cut,x}}$ or $M > \mathsf{N}_{\text{cut,t}}$, note that if $0 \le m \le \mathsf{N}_{\text{cut,t}}$ and $(x,t) \in \operatorname{supp} \psi_{m,i_m,q+\bar{n}}$, there exists $j_m \ge 0$ with $i_*(j_m) \le i_m$ such that $(x,t) \in \operatorname{supp} \psi_{j_m,q+\bar{n}-1}$. Thus, we may appeal to (9.83b), which gives that for $N + M \le 2\mathsf{N}_{\text{fin}}$,

$$\left|D^N D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}(x,t)\right| \lesssim \Gamma_q^{\mathsf{C}_\infty/2+16} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}-1}^{j_m-1} \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}\right) .$$

Since $i_*(j_m) \le i_m$ implies $\Gamma_{q+\bar{n}-1}^{j_m} \le \Gamma_{q+\bar{n}}^{i_m}$, we deduce that for $N + M \le 2\mathsf{N}_{\text{fin}}$,

$$\begin{aligned}
&\left\|D^N D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp} \psi_{m,i_m,q+\bar{n}})} \\
&\quad \lesssim \Gamma_q^{\mathsf{C}_\infty/2+16} r_q^{-1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}-1})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i_m} \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}\right) \\
&\quad \le \Gamma_{q+\bar{n}}^{i_m+1} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i_m+3} \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}-1}\right)
\end{aligned}$$

after using that either $N > \mathsf{N}_{\text{cut,x}}$ or $M > \mathsf{N}_{\text{cut,t}}$, the parameter inequality (4.17b), and a large choice of a to absorb the implicit constant in the spare factor of $\Gamma_{q+\bar{n}}$. The desired estimate in (12.23) then follows from taking the maximum over m from Definition 12.1.4. □

12.3 Pure spatial derivatives

In this section we prove that the cutoff functions $\psi_{i,q+\bar{n}}$ satisfy sharp spatial derivative estimates which are consistent with (5.11) for $q' = q + \bar{n}$.

Lemma 12.3.1 (Spatial derivatives for the cutoffs). *Fix $q + \bar{n} \ge 1$, $0 \le m \le \mathsf{N}_{\text{cut,t}}$, and $i_m \ge 0$. For all $j_m \ge 0$ such that $i_m \ge i_*(j_m)$, all $i \ge 0$, and all $N \le \mathsf{N}_{\text{fin}}$, we have*

$$\mathbf{1}_{\operatorname{supp}(\psi_{j_m,q+\bar{n}-1})} \frac{|D^N \psi_{m,i_m,j_m,q+\bar{n}}|}{\psi_{m,i_m,j_m,q+\bar{n}}^{1-N/\mathsf{N}_{\text{fin}}}} \lesssim (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N , \tag{12.24a}$$

$$\frac{|D^N \psi_{i,q+\bar{n}}|}{\psi_{i,q+\bar{n}}^{1-N/\mathsf{N}_{\text{fin}}}} \lesssim (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N . \tag{12.24b}$$

Proof of Lemma 12.3.1. **Step 1: proof of** (12.24a)**.** We distinguish two cases. The first case is when $\psi = \widetilde{\gamma}_{m,q}$, or $\psi = \gamma_{m,q}$ *and* we have the lower bound

$$h^2_{m,j_m,q+\bar{n}} \Gamma_{q+\bar{n}}^{-2(i_m - i_*(j_m))(m+1)} \geq \frac{1}{4} \Gamma_{q+\bar{n}}^{2(m+1)} , \tag{12.25}$$

so that (8.18) applies. The goal is then to apply [7, Lemma A.5] to the function $\psi = \widetilde{\gamma}_{m,q}$ or $\psi = \gamma_{m,q}$ with the choices $\Gamma_\psi = \Gamma_{q+\bar{n}}^{m+1}$, $\Gamma = \Gamma_{q+\bar{n}}^{(m+1)(i_m - i_*(j_m))}$, and $h = h^2_{m,j_m,q+\bar{n}}$. The assumption in [7, equation (A.24)] holds by (8.16) or (8.18) for all $N \leq \mathsf{N}_{\text{fin}}$, and so we need to obtain bounds on the derivatives of $h^2_{m,j_m,q+\bar{n}}$ which are consistent with assumption in [7, equation (A.25)] of [7, Lemma A.5]. For $B \leq \mathsf{N}_{\text{fin}}$, the Leibniz rule gives

$$\begin{aligned} &\left| D^B h^2_{m,j_m,q+\bar{n}} \right| \\ &\quad \lesssim (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^B \sum_{B'=0}^{B} \sum_{n=0}^{\mathsf{N}_{\text{cut,x}}} \Gamma_{q+\bar{n}}^{-i_*(j_m)} (\tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}}^{i_*(j_m)+2})^{-m} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{-n-B'} \delta_{q+\bar{n}}^{-1/2} r_q^{1/3} |D^{n+B'} D^m_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}| \\ &\qquad \times \Gamma_{q+\bar{n}}^{-i_*(j_m)} (\tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}}^{i_*(j_m)+2})^{-m} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{-n-B+B'} \delta_{q+\bar{n}}^{-1/2} r_q^{1/3} |D^{n+B-B'} D^m_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}| . \end{aligned} \tag{12.26}$$

For the terms with $L \in \{n+B', n+B-B'\} \leq \mathsf{N}_{\text{cut,x}}$, we may appeal to appeal to estimate (12.19), which gives

$$\begin{aligned} &\Gamma_{q+\bar{n}}^{-i_*(j_m)} (\tau_{q+\bar{n}-1}^{-1} \Gamma_{q+\bar{n}}^{i_*(j_m)+2})^{-m} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{-L} \delta_{q+\bar{n}}^{-1/2} r_q^{1/3} \left\| D^L D^m_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp} \psi_{m,i_m,j_m,q+\bar{n}})} \\ &\qquad \leq \Gamma_{q+\bar{n}}^{(m+1)(i_m+1-i_*(j_m))} . \end{aligned} \tag{12.27}$$

On the other hand, for $\mathsf{N}_{\text{cut,x}} < L \in \{n+B', n+B-B'\} \leq \mathsf{N}_{\text{cut,x}} + B \leq 2\mathsf{N}_{\text{fin}} - \mathsf{N}_{\text{ind,t}}$, we

may appeal to appeal to (9.83b), and since $m \leq \mathsf{N}_{\mathrm{cut,t}} < \mathsf{N}_{\mathrm{ind,t}}$, we deduce that

$$\begin{aligned}
&\Gamma_{q+\bar{n}}^{-i_*(j_m)}(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_*(j_m)+2})^{-m}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-L}\delta_{q+\bar{n}}^{-1/2}r_q^{1/3}\left\|D^L D_{t,q+\bar{n}-1}^m \widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{j_m,q+\bar{n}-1})} \\
&\lesssim \Gamma_{q+\bar{n}}^{-i_*(j_m)(m+1)-2m}\tau_{q+\bar{n}-1}^m(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-L}\delta_{q+\bar{n}}^{-1/2}r_q^{1/3}\Gamma_q^{\mathsf{C}_\infty/2+16}r_q^{-1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1})^L(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{j_m-1})^m \\
&\lesssim \Gamma_{q+\bar{n}}^{-i_*(j_m)(m+1)-2m}\delta_{q+\bar{n}}^{-1/2}r_q^{1/3}\Gamma_q^{\mathsf{C}_\infty/2+16}r_q^{-1}\left(\frac{\Gamma_{q+\bar{n}-1}}{\Gamma_{q+\bar{n}}}\right)^L\Gamma_{q+\bar{n}-1}^{m(j_m-1)} \\
&\leq \Gamma_{q+\bar{n}}^{(i_m+1-i_*(j_m))(m+1)}\,. \qquad (12.28)
\end{aligned}$$

In the last inequality we have used that $i_m \geq i_*(j_m)$ in order to convert $\Gamma_{q+\bar{n}-1}^{m(j_m-1)}$ into $\Gamma_{q+\bar{n}}^{mi_m}$ and (4.17c), which is applicable by the assumption that $L > \mathsf{N}_{\mathrm{cut,x}}$. Summarizing the bounds (12.26)–(12.28), since $n \leq \mathsf{N}_{\mathrm{cut,x}}$ and $\mathsf{N}_{\mathrm{ind,t}} \leq \mathsf{N}_{\mathrm{fin}}$, we arrive at

$$\mathbf{1}_{\operatorname{supp}(\psi_{j_m,q+\bar{n}-1}\psi_{m,i_m,j_m,q+\bar{n}})}\left|D^B h_{m,j_m,q+\bar{n}}^2\right| \lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^B\Gamma_{q+\bar{n}}^{2(m+1)(i_m+1-i_*(j_m))}$$

whenever $B \leq \mathsf{N}_{\mathrm{fin}}$. Thus, the assumption in [7, A.25] holds with $C_h = \Gamma_{q+\bar{n}}^{2(m+1)(i_m+1-i_*(j_m))}$, $\lambda = \widetilde{\lambda} = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}$, $N_* = \infty$, $N = \mathsf{N}_{\mathrm{fin}}$, $M = 0$. Note that with these choices of parameters, we have $C_h\Gamma_\psi^{-2}\Gamma^{-2} = 1$. We may thus apply [7, Lemma A.5] and conclude that

$$\mathbf{1}_{\operatorname{supp}(\psi_{j_m,q+\bar{n}-1})}\frac{\left|D^N\psi_{m,i_m,j_m,q+\bar{n}}\right|}{\psi_{m,i_m,j_m,q+\bar{n}}^{1-N/\mathsf{N}_{\mathrm{fin}}}} \lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N$$

for all $N \leq \mathsf{N}_{\mathrm{fin}}$, proving (12.24a) in the first case.

Recalling the inequality (12.25), the second case is when $\psi = \gamma_{m,q}$ and

$$h_{m,j_m,q+\bar{n}}^2\Gamma_{q+\bar{n}}^{-2(i_m-i_*(j_m))(m+1)} \leq \frac{1}{4}\Gamma_{q+\bar{n}}^{2(m+1)}\,. \qquad (12.29)$$

However, since $\gamma_{m,q}$ is uniformly equal to 1 when the left hand side of the above display takes values in $\left[1, \frac{1}{4}\Gamma_q^{2(m+1)}\right]$ from item (2) in Lemma 8.3.1, (12.24a) is trivially satisfied in this range of values of the left-hand side. Thus the analysis of the second case reduces to

analyzing the subcase when

$$h^2_{m,j_m,q+\bar{n}}\Gamma_{q+\bar{n}}^{-2(i_m-i_*(j_m))(m+1)} \leq 1\,. \tag{12.30}$$

As in the first case, we aim to apply [7, Lemma A.5] with $h = h^2_{m,j_m,q}$, but now with $\Gamma_\psi = 1$ and $\Gamma = \Gamma_{q+\bar{n}}^{(m+1)(i_m-i_*(j_m))}$. From (8.17), the assumption in [7, (A.24)] holds. Towards estimating derivatives of h, for the terms with $L \in \{n+B', n+B-B'\} \leq \mathsf{N}_{\text{cut,x}}$, (12.30) gives immediately that

$$\begin{aligned}\Gamma_{q+\bar{n}}^{-i_*(j_m)}(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_*(j_m)+2})^{-m}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-L}\delta_{q+\bar{n}}^{-1/2}r_q^{1/3}\left\|D^L D^m_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{m,i_m,j_m,q+\bar{n}})}\\ \leq \Gamma_{q+\bar{n}}^{(m+1)(i_m-i_*(j_m))}\,.\end{aligned} \tag{12.31}$$

Conversely, when $\mathsf{N}_{\text{cut,x}} > L$, we may argue as in the estimates which gave (12.28), except we achieve the slightly improved bound of $\Gamma_{q+\bar{n}}^{(m+1)(i_m-i_*(j_m))}$ as above. We then arrive at

$$\mathbf{1}_{\operatorname{supp}(\psi_{j_m,q+\bar{n}-1}\psi_{m,i_m,j_m,q+\bar{n}})}\left|D^B h^2_{m,j_m,q+\bar{n}}\right| \lesssim \Gamma_{q+\bar{n}}^{2(m+1)(i_m-i_*(j_m))}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^B$$

whenever $B \leq \mathsf{N}_{\text{fin}}$. Thus, the assumption in [7, (A.25)] now holds with the same choices as before, except now $\mathcal{C}_h = \Gamma_{q+\bar{n}}^{2(m+1)(i_m-i_*(j_m))}$, $\lambda = \widetilde{\lambda} = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}$. Note that with these new choices of parameters, we still have $\mathcal{C}_h\Gamma_\psi^{-2}\Gamma^{-2} = 1$. We may thus apply [7, Lemma A.5] and conclude that

$$\mathbf{1}_{\operatorname{supp}(\psi_{j_m,q+\bar{n}-1})}\frac{\left|D^N\psi_{m,i_m,j_m,q+\bar{n}}\right|}{\psi_{m,i_m,j_m,q+\bar{n}}^{1-N/\mathsf{N}_{\text{fin}}}} \lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N$$

for all $N \leq \mathsf{N}_{\text{fin}}$, proving (12.24a) in the second case.

Step 2: differentiating $\psi_{m,i_m,q}$. From the definition (12.9) and the bound (12.24a), we

next estimate derivatives of the mth velocity cutoff function $\psi_{m,i_m,q}$ and claim that

$$\frac{|D^N \psi_{m,i_m,q+\bar{n}}|}{\psi_{m,i_m,q+\bar{n}}^{1-N/\mathsf{N}_{\text{fin}}}} \lesssim (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \tag{12.32}$$

for all $i_m \geq 0$ and all $N \leq \mathsf{N}_{\text{fin}}$. We prove (12.32) by induction on N. When $N = 0$ the bound trivially holds, which gives the induction base. For the induction step, assume that (12.32) holds for all $N' \leq N - 1$. By the Leibniz rule from Lemma A.2.1 with $p = 6$, we obtain

$$D^N(\psi_{m,i_m,q+\bar{n}}^6) = 6\psi_{m,i_m,q+\bar{n}}^5 D^N \psi_{m,i_m,q+\bar{n}} + \sum_{\left\{\substack{\alpha : \sum_{i=1}^6 \alpha_i = N, \\ \alpha_i < N \, \forall i}\right\}} \binom{N}{\alpha_1, \ldots, \alpha_6} \prod_{i=1}^6 D^{\alpha_i} \psi_{m,i_m,q+\bar{n}} \tag{12.33}$$

and thus

$$\frac{D^N \psi_{m,i_m,q+\bar{n}}}{\psi_{m,i_m,q+\bar{n}}^{1-N/\mathsf{N}_{\text{fin}}}} = \frac{D^N(\psi_{m,i_m,q+\bar{n}}^6)}{6\psi_{m,i_m,q+\bar{n}}^{6-N/\mathsf{N}_{\text{fin}}}} - \frac{1}{6} \sum_{\left\{\substack{\alpha : \sum_{i=1}^p \alpha_i = N, \\ \alpha_i < N \, \forall i}\right\}} \binom{N}{\alpha_1, \ldots, \alpha_6} \prod_{i=1}^6 \frac{D^{\alpha_i} \psi_{m,i_m,q+\bar{n}}}{\psi_{m,i_m,q+\bar{n}}^{1-\alpha_i/\mathsf{N}_{\text{fin}}}} \, .$$

Since $\alpha_i \leq N - 1$, by the induction assumption (12.32) we obtain

$$\frac{\left|D^N \psi_{m,i_m,q+\bar{n}}\right|}{\psi_{m,i_m,q+\bar{n}}^{1-N/\mathsf{N}_{\text{fin}}}} \lesssim \frac{|D^N(\psi_{m,i_m,q+\bar{n}}^6)|}{\psi_{m,i_m,q+\bar{n}}^{6-N/\mathsf{N}_{\text{fin}}}} + (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N \, . \tag{12.34}$$

Thus establishing (12.32) for the Nth derivative reduces to bounding the first term on the

right side of the above. For this purpose we recall (12.9) and (A.21a) and compute

$$\begin{aligned}
\frac{\left|D^N(\psi^6_{m,i_m,q+\bar{n}})\right|}{\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,q+\bar{n}}} &= \frac{1}{\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,q+\bar{n}}} \sum_{\{j_m\,:\,i_*(j_m)\le i_m\}} \sum_{K=0}^{N} \binom{N}{K} D^K(\psi^6_{j_m,q+\bar{n}-1}) D^{N-K}(\psi^6_{m,i_m,j_m,q+\bar{n}}) \\
&= \frac{\psi^{6-K/\mathsf{N}_{\mathrm{fin}}}_{j_m,q+\bar{n}-1}\psi^{6-(N-K)/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,j_m,q+\bar{n}}}{\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,q+\bar{n}}} \sum_{\{j_m\,:\,i_*(j_m)\le i_m\}} \sum_{K=0}^{N} \binom{N}{K} \\
&\quad \times \sum_{\alpha:\sum_{i=1}^6 \alpha_i = K} \binom{K}{\alpha_1,\dots,\alpha_6} \prod_{i=1}^{6} \frac{D^{\alpha_i}\psi_{j_m,q+\bar{n}-1}}{\psi^{1-\alpha_i/\mathsf{N}_{\mathrm{fin}}}_{j_m,q+\bar{n}-1}} \\
&\quad \times \sum_{\beta:\sum_{i=1}^6 \beta_i = N-K} \binom{N-K}{\beta_1,\dots,\beta_6} \prod_{i=1}^{6} \frac{D^{\beta_i}\psi_{m,i_m,j_m,q+\bar{n}}}{\psi^{1-\beta_i/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,j_m,q+\bar{n}}} \,.
\end{aligned}$$

Since $K, N-K \le N$, and $\psi_{j_m,q+\bar{n}-1}, \psi_{m,i_m,j_m,q+\bar{n}} \le 1$, we have by (12.9) that

$$\frac{\psi^{6-K/\mathsf{N}_{\mathrm{fin}}}_{j_m,q+\bar{n}-1}\psi^{6-(N-K)/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,j_m,q+\bar{n}}}{\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,q+\bar{n}}} \le \frac{\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{j_m,q+\bar{n}-1}\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,j_m,q+\bar{n}}}{\psi^{6-N/\mathsf{N}_{\mathrm{fin}}}_{m,i_m,q+\bar{n}}} \le 1\,.$$

Then the estimate (12.24a) and the inductive assumption (5.11) conclude the proof of (12.32). In particular, note that this bound is independent of the value of i_m.

Step 3: proof of (12.24b) In order to conclude the proof of the Lemma, we must argue that (12.32) implies (12.24b). Recalling (12.11), we have that $\psi^6_{i,q+\bar{n}}$ is given as a sum of products of $\psi^6_{m,i_m,q+\bar{n}}$, for which suitable derivative bounds are available due to (12.32). Thus, the proof of (12.24b) is again done by induction on N, mutatis mutandi to the proof of (12.32). Indeed, we note that $\psi^6_{m,i_m,q+\bar{n}}$ was also given as a sum of squares of cutoff functions for which derivative bounds were available. The proof of the induction step is thus again based on the application of the Leibniz rule for $\psi^6_{i,q+\bar{n}}$; in order to avoid redundancy we omit these details. □

12.4 Maximal index appearing in the cutoff

Lemma 12.4.1 (Maximal i index in the definition of $\psi_{i,q+\bar{n}}$). *There exists $i_{\max} = i_{\max}(q+\bar{n}) \geq 0$, determined by (12.38) below, such that if λ_0 is sufficiently large, then*

$$\psi_{i,q+\bar{n}} \equiv 0 \quad \textit{for all} \quad i > i_{\max}\,, \tag{12.35a}$$

$$\Gamma_{q+\bar{n}}^{i_{\max}} \leq \Gamma_q^{\mathsf{C}_\infty/2+18}\delta_{q+\bar{n}}^{-1/2} r_q^{-2/3}\,, \tag{12.35b}$$

$$i_{\max}(q) \leq \frac{\mathsf{C}_\infty + 12}{(b-1)\varepsilon_\Gamma}\,. \tag{12.35c}$$

Proof of Lemma 12.4.1. Assume $i \geq 0$ is such that $\operatorname{supp}(\psi_{i,q+\bar{n}}) \neq \emptyset$. We will prove that

$$\Gamma_{q+\bar{n}}^{i} \leq \Gamma_q^{\mathsf{C}_\infty/2+18}\delta_{q+\bar{n}}^{-1/2} r_q^{-2/3}\,. \tag{12.36}$$

From (12.11) it follows that for any $(x,t) \in \operatorname{supp}(\psi_{i,q+\bar{n}})$, there must exist at least one $\vec{i} = (i_0, \ldots, i_{\mathsf{N}_{\mathrm{cut,t}}})$ such that $\max_{0\leq m\leq \mathsf{N}_{\mathrm{cut,t}}} i_m = i$ and $\psi_{m,i_m,q+\bar{n}}(x,t) \neq 0$ for all $0 \leq m \leq \mathsf{N}_{\mathrm{cut,t}}$. Therefore, in light of (12.9), for each such m there exists a maximal j_m such that $i_*(j_m) \leq i_m$, with $(x,t) \in \operatorname{supp}(\psi_{j_m,q+\bar{n}-1}) \cap \operatorname{supp}(\psi_{m,i_m,j_m,q+\bar{n}})$. In particular, this holds for any of the indices m such that $i_m = i$. For the remainder of the proof, we fix such an index $0 \leq m \leq \mathsf{N}_{\mathrm{cut,t}}$.

If we have $i = i_m = i_*(j_m) = i_*(j_m, q)$, then using that $(x,t) \in \operatorname{supp}(\psi_{j_m,q+\bar{n}-1})$ and the inductive assumption (5.10), we have that $j_m \leq i_{\max}(q+\bar{n}-1)$. Now using (5.10), (4.10j), and the inequalities $\Gamma_{q+\bar{n}}^{i-1} < \Gamma_{q+\bar{n}-1}^{j_m} \leq \Gamma_{q+\bar{n}-1}^{i_{\max}(q+\bar{n}-1)}$, we deduce that

$$\Gamma_{q+\bar{n}}^{i} \leq \Gamma_{q+\bar{n}}\Gamma_{q+\bar{n}-1}^{i_{\max}(q+\bar{n}-1)} \leq \Gamma_{q+\bar{n}}\Gamma_{q-1}^{\mathsf{C}_\infty/2+18}\delta_{q+\bar{n}-1}^{-1/2} r_{q-1}^{-2/3} \leq \Gamma_q^{\mathsf{C}_\infty/2+18}\delta_{q+\bar{n}}^{-1/2} r_q^{-2/3}\,,$$

Thus, in this case (12.36) holds.

On the other hand, if $i = i_m \geq i_*(j_m) + 1$, then from (12.20) we have that

$$|h_{m,j_m,q+\bar{n}}(x,t)| \geq (1/2)\Gamma_{q+\bar{n}}^{(m+1)(i_m - i_*(j_m))}\,.$$

Now from the pigeonhole principle, there exists $0 \le n \le \mathsf{N}_{\text{cut,x}}$ such that

$$\begin{aligned}|D^n D^m_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}(x,t)| &\ge \frac{1}{2\mathsf{N}_{\text{cut,x}}}\Gamma_{q+\bar{n}}^{(m+1)(i_m-i_*(j_m))}\Gamma_{q+\bar{n}}^{i_*(j_m)}\delta_{q+\bar{n}}^{1/2}r_q^{-1/3}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^n(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_*(j_m)+2})^m \\ &\ge \frac{1}{2\mathsf{N}_{\text{cut,x}}}\Gamma_{q+\bar{n}}^{i_m}\delta_{q+\bar{n}}^{1/2}r_q^{-1/3}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^n(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_m+2})^m\,,\end{aligned}$$

and we also know that $(x,t) \in \operatorname{supp}(\psi_{j_m,q+\bar{n}-1})$. By (9.83b) and the inequality $\mathsf{N}_{\text{cut,t}} \le \mathsf{N}_{\text{ind,t}}$ from (4.18), we know that

$$\begin{aligned}|D^n D^m_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}(x,t)| &\le \Gamma_q^{\mathsf{C}_\infty/2+17}r_q^{-1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}-1})^n(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{j_m-1})^m \\ &\le \Gamma_q^{\mathsf{C}_\infty/2+17}r_q^{-1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^n(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_m})^m\,,\end{aligned}$$

where in the last inequality we used the assumption that $i_m \ge i_*(j_m)$ and converted the $\Gamma_{q+\bar{n}-1}^{j_m-1}$ into $\Gamma_{q+\bar{n}}^{i_m}$. The proof is now completed, since the previous two inequalities and $i_m = i$ imply that

$$\Gamma_{q+\bar{n}}^{i} \le 2\mathsf{N}_{\text{cut,x}}\delta_{q+\bar{n}}^{-1/2}r_q^{-2/3}\Gamma_q^{\mathsf{C}_\infty/2+17} \le \delta_{q+\bar{n}}^{-1/2}r_q^{-2/3}\Gamma_q^{\mathsf{C}_\infty/2+18}\,, \tag{12.37}$$

where in the last inequality we used (4.12) and a large choice of a to ensure that $\Gamma_0 \ge 2\mathsf{N}_{\text{cut,x}}$.

In view of the above inequality, the value of $i_{\max}$ is chosen as

$$i_{\max}(q) = \sup\{i' \,:\, \Gamma_{q+\bar{n}}^{i'} \le \Gamma_q^{\mathsf{C}_\infty/2+18}r_q^{-2/3}\delta_{q+\bar{n}}^{-1/2}\}\,. \tag{12.38}$$

With this definition, if $i > i_{\max}(q+\bar{n})$, then $\operatorname{supp}(\psi_{i,q+\bar{n}}) = \emptyset$. To show that $i_{\max}(q+\bar{n})$ is bounded independently of q, simple (and brutal) computations give that

$$\frac{\log(\Gamma_q^{\mathsf{C}_\infty/2+18}\delta_{q+\bar{n}}^{-1/2}r_q^{-2/3})}{\log(\Gamma_{q+\bar{n}})} \le \frac{\mathsf{C}_\infty+12}{(b-1)\varepsilon_\Gamma}\,,$$

verifying that (12.35c) holds. □

12.5 Mixed derivative estimates

We will use the notation $D_{q+\bar{n}} = \widehat{w}_{q+\bar{n}} \cdot \nabla$ for the directional derivative in the direction of $\widehat{w}_{q+\bar{n}}$. With this notation we have $D_{t,q+\bar{n}} = D_{t,q+\bar{n}-1} + D_{q+\bar{n}}$. Next, we recall from [7, equations (6.54)-(6.55)] that

$$D^K_{q+\bar{n}} = \sum_{j=1}^{K} f_{j,K} D^j \,, \tag{12.39}$$

where

$$f_{j,K} = \sum_{\{\gamma\in\mathbb{N}^K\,:\,|\gamma|=K-j\}} c_{j,K,\gamma} \prod_{\ell=1}^{K} D^{\gamma_\ell} \widehat{w}_{q+\bar{n}} \,. \tag{12.40}$$

The $c_{j,K,\gamma}$'s are explicitly computable coefficients that depend only on K, j, and γ. With the notation in (12.40) we have the following bounds.

Lemma 12.5.1 (Bounds for $D^K_{q+\bar{n}}$). *For $q+\bar{n} \geq 1$ and $1 \leq K \leq 2\mathsf{N}_{\text{fin}}$, the functions $\{f_{j,K}\}_{j=1}^K$ defined in (12.40) obey the estimate*

$$\|D^a f_{j,K}\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \lesssim (\Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3})^K (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{a+K-j} \tag{12.41}$$

for any $a \leq 2\mathsf{N}_{\text{fin}} - K + j$, and any $0 \leq i \leq i_{\max}(q+\bar{n})$.

Proof of Lemma 12.5.1. Note that no material derivative appears in (12.40), and thus to establish (12.41) we appeal to Corollary 12.2.4 with $M = 0$ and (9.83b). From the product rule we obtain that

$$\begin{aligned}
\|D^a f_j\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} &\lesssim \sum_{\{\gamma\in\mathbb{N}^K\,:\,|\gamma|=K-j\}} \sum_{\{\alpha\in\mathbb{N}^k\,:\,|\alpha|=a\}} \prod_{\ell=1}^{K} \|D^{\alpha_\ell+\gamma_\ell} \widehat{w}_{q+\bar{n}}\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \\
&\lesssim \sum_{\{\gamma\in\mathbb{N}^K\,:\,|\gamma|=K-j\}} \sum_{\{\alpha\in\mathbb{N}^k\,:\,|\alpha|=a\}} \prod_{\ell=1}^{K} \Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{\alpha_\ell+\gamma_\ell} \\
&\lesssim (\Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3})^K (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{a+K-j}
\end{aligned}$$

since $|\gamma| = K - j$. $\square$

Lemma 12.5.2 (**Mixed derivatives for $\widehat{w}_{q+\bar{n}}$**). *For $q + \bar{n} \geq 1$ and $0 \leq i \leq i_{\max}$, we have that*

$$\begin{aligned}
&\left\| D^N D^K_{q+\bar{n}} D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \\
&\qquad \lesssim (\Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3})^{K+1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{N+K} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q+\bar{n}} \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right) \\
&\qquad \lesssim (\Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3}) (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{N} (\Gamma^{i-5}_{q+\bar{n}} \tau^{-1}_{q+\bar{n}})^{K} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q+\bar{n}} \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right)
\end{aligned}$$

holds for $0 \leq K + N + M \leq 2\mathsf{N}_{\mathrm{fin}}$.

Proof of Lemma 12.5.2. The second estimate in the Lemma follows from the parameter inequality (4.10b). In order to prove the first estimate, we let $0 \leq a \leq N$ and $1 \leq j \leq K$. From estimate (12.23), we obtain that

$$\begin{aligned}
\left\| D^{N-a+j} D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} &\lesssim \Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{N-a+j} \\
&\qquad \times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q+\bar{n}} \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right)
\end{aligned}$$

for $N - a + j + M \leq \mathsf{N}_{\mathrm{fin}}$, which may be combined with (12.39)–(12.41) to obtain that

$$\begin{aligned}
&\left\| D^N D^K_{q+\bar{n}} D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \\
&\quad \lesssim \sum_{a=0}^{N} \sum_{j=1}^{K} \| D^a f_{j,K} \|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \left\| D^{N-a+j} D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \\
&\quad \lesssim (\Gamma^{i+1}_{q+\bar{n}} \delta^{1/2}_{q+\bar{n}} r_q^{-1/3})^{K+1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{N+K} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q+\bar{n}} \tau^{-1}_{q+\bar{n}-1}, \mathrm{T}^{-1}_{q+\bar{n}-1} \Gamma_{q+\bar{n}-1}\right)
\end{aligned}$$

holds for $N + M + K \leq 2\mathsf{N}_{\mathrm{fin}}$, concluding the proof of the lemma. $\square$

Lemma 12.5.3 (**More mixed derivatives for $\widehat{w}_{q+\bar{n}}$ and derivatives for $\widehat{u}_{q+\bar{n}}$**). *For*

$q+\bar{n} \geq 1$, $k \geq 1$, $\alpha, \beta \in \mathbb{N}^k$ with $|\alpha| = K$, $|\beta| = M$, and $K + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2 + 1$, we have

$$\begin{aligned}
&\left\| \Big(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}-1}^{\beta_i} \Big) \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q+\bar{n}})} \\
&\qquad \lesssim \Gamma_{q+\bar{n}}^{i+1} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^K \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i+3} \tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1} \mathrm{T}_{q+\bar{n}-1}^{-1} \right) . \qquad (12.42)
\end{aligned}$$

Next, we have that

$$\begin{aligned}
&\left\| D^N \Big(\prod_{i=1}^{k} D_{q+\bar{n}}^{\alpha_i} D_{t,q+\bar{n}-1}^{\beta_i} \Big) \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q+\bar{n}})} \\
&\qquad \lesssim (\Gamma_{q+\bar{n}}^{i+1} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3})^{K+1} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{N+K} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i+3} \tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1} \mathrm{T}_{q+\bar{n}-1}^{-1} \right) \qquad (12.43a) \\
&\qquad \lesssim \Gamma_{q+\bar{n}}^{i+1} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^{N} (\Gamma_{q+\bar{n}}^{i-5} \tau_{q+\bar{n}}^{-1})^{K} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i+3} \tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1} \mathrm{T}_{q+\bar{n}-1}^{-1} \right) \qquad (12.43b)
\end{aligned}$$

holds for all $0 \leq K + M + N \leq {}^{3\mathsf{N}_{\text{fin}}}/_2 + 1$. Lastly, we have the estimate

$$\begin{aligned}
&\left\| \Big(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}}^{\beta_i} \Big) D \widehat{u}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q+\bar{n}})} \\
&\qquad \lesssim \tau_{q+\bar{n}}^{-1} \Gamma_{q+\bar{n}}^{i-5} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^K \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i-5} \tau_{q+\bar{n}}^{-1}, \Gamma_{q+\bar{n}-1} \mathrm{T}_{q+\bar{n}-1}^{-1} \right) \qquad (12.44)
\end{aligned}$$

for all $K + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2$, the estimate

$$\begin{aligned}
&\left\| \Big(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}}^{\beta_i} \Big) \widehat{u}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp} \psi_{i,q+\bar{n}})} \\
&\qquad \lesssim \Gamma_{q+\bar{n}}^{i+1} \delta_{q+\bar{n}}^{1/2} r_q^{-1/3} \lambda_{q+\bar{n}}^2 (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^K \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i-5} \tau_{q+\bar{n}}^{-1}, \Gamma_{q+\bar{n}-1} \mathrm{T}_{q+\bar{n}-1}^{-1} \right) \qquad (12.45)
\end{aligned}$$

for all $K + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2 + 1$, and the estimate

$$\left\| D^K \partial_t^M \widehat{u}_{q+\bar{n}} \right\|_\infty \leq \lambda_{q+\bar{n}}^{1/2} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^K \mathrm{T}_{q+\bar{n}}^{-M} \qquad (12.46)$$

for all $K + M \leq 2\mathsf{N}_{\text{fin}}$.

Proof of Lemma 12.5.3. We note that (12.43b) follows directly from (12.43a) by appealing to (4.10b). We first show that (12.42) holds, then establish (12.43a), and lastly, prove the bounds (12.44)–(12.46).

Proof of (12.42)**.** The statement is proven by induction on k. For $k = 1$ the estimate holds for $K + M \leq 2\mathsf{N}_{\text{fin}}$ from Corollary 12.2.4. For the induction step, assume that (12.42) holds for any $k' \leq k - 1$. We denote

$$P_{k'} = \Big(\prod_{i=1}^{k'} D^{\alpha_i} D^{\beta_i}_{t,q+\bar{n}-1} \Big) \widehat{w}_{q+\bar{n}} \tag{12.47}$$

and write

$$\begin{aligned} \Big(\prod_{i=1}^{k} D^{\alpha_i} D^{\beta_i}_{t,q+\bar{n}-1} \Big) \widehat{w}_{q+\bar{n}} &= (D^{\alpha_k} D^{\beta_k}_{t,q+\bar{n}-1})(D^{\alpha_{k-1}} D^{\beta_{k-1}}_{t,q+\bar{n}-1}) P_{k-2} \\ &= (D^{\alpha_k+\alpha_{k-1}} D^{\beta_k+\beta_{k-1}}_{t,q+\bar{n}-1}) P_{k-2} + D^{\alpha_k} \left[D^{\beta_k}_{t,q+\bar{n}-1}, D^{\alpha_{k-1}} \right] D^{\beta_{k-1}}_{t,q+\bar{n}-1} P_{k-2} \,. \end{aligned} \tag{12.48}$$

The first term in (12.48) already obeys the correct bound, since we know that (12.42) holds for $k' = k - 1$. In order to treat the second term on the right side of (12.48), we use [7, Lemma A.12] to write the commutator as

$$\begin{aligned} &D^{\alpha_k} \left[D^{\beta_k}_{t,q+\bar{n}-1}, D^{\alpha_{k-1}} \right] D^{\beta_{k-1}}_{t,q+\bar{n}-1} P_{k-2} \\ &= D^{\alpha_k} \sum_{1 \leq |\gamma| \leq \beta_k} \frac{\beta_k!}{\gamma!(\beta_k - |\gamma|)!} \left(\prod_{\ell=1}^{\alpha_{k-1}} (\operatorname{ad} D_{t,q+\bar{n}-1})^{\gamma_\ell}(D) \right) D^{\beta_k+\beta_{k-1}-|\gamma|}_{t,q+\bar{n}-1} P_{k-2} \,. \end{aligned} \tag{12.49}$$

From [7, Lemma A.13] and the Leibniz rule we claim that one may expand

$$\prod_{\ell=1}^{\alpha_{k-1}} (\operatorname{ad} D_{t,q+\bar{n}-1})^{\gamma_\ell}(D) = \sum_{j=1}^{\alpha_{k-1}} g_j D^j \tag{12.50}$$

for some explicit functions g_j which obey the estimate

$$\|D^a g_j\|_{L^\infty(\operatorname{supp}\psi_{i,q})} \lesssim (\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^{a+\alpha_{k-1}-j}\mathcal{M}\left(|\gamma|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i+1}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.51}$$

for all a such that $a + \alpha_{k-1} - j + |\gamma| \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. The claim (12.51) requires a proof, which we sketch next. Using the definition (12.9) and the inductive estimate (5.34) at level $q' = q + \bar{n} - 1$ and with $k = 1$, we have that

$$\begin{aligned}
&\left\|D^a D_{t,q+\bar{n}-1}^b D\widehat{u}_{q+\bar{n}-1}\right\|_{L^\infty(\operatorname{supp}\psi_{m,i_m,q+\bar{n}})} \\
&\lesssim \sum_{\{j_m\,:\,\Gamma_{q+\bar{n}-1}^{j_m}\leq\Gamma_{q+\bar{n}}^{i_m}\}} \left\|D^a D_{t,q+\bar{n}-1}^b D\widehat{u}_{q+\bar{n}-1}\right\|_{L^\infty(\operatorname{supp}\psi_{j_m,q+\bar{n}-1})} \\
&\lesssim \sum_{\{j_m\,:\,\Gamma_{q+\bar{n}-1}^{j_m}\leq\Gamma_{q+\bar{n}}^{i_m}\}} \tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{j_m+1}(\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^a\mathcal{M}\left(b, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}-1}^{j_m+1}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \\
&\lesssim (\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^a\mathcal{M}\left(b+1, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i_m+1}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)
\end{aligned}$$

for any $0 \leq m \leq \mathsf{N}_{\mathrm{cut,t}}$ and for all $a+b \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. Thus, from the definition (12.11) we deduce that

$$\left\|D^a D_{t,q+\bar{n}-1}^b D\widehat{u}_{q+\bar{n}-1}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \lesssim (\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^a\mathcal{M}\left(b+1, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i_m+1}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.52}$$

for all $a+b \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. When combined with the formula in [7, equation (A.49)], which allows us to write

$$(\operatorname{ad} D_{t,q+\bar{n}-1})^\gamma(D) = f_{\gamma,q+\bar{n}-1}\cdot\nabla \tag{12.53}$$

for an explicit function $f_{\gamma,q+\bar{n}-1}$ which is defined in terms of $\widehat{u}_{q+\bar{n}-1}$, estimate (12.52) and

the Leibniz rule gives the estimate

$$\|D^a f_{\gamma,q+\bar{n}-1}\|_{L^\infty(\operatorname{supp}\psi_{i,q})} \lesssim (\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^a \mathcal{M}\left(\gamma, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i+1}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.54}$$

for all $a+\gamma \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2$. In order to conclude the proof of (12.50)–(12.51), we use (12.53) to write

$$\prod_{\ell=1}^{\alpha_{k-1}} (\operatorname{ad} D_{t,q+\bar{n}-1})^{\gamma_\ell}(D) = \prod_{\ell=1}^{\alpha_{k-1}} (f_{\gamma_\ell,q+\bar{n}-1}\cdot\nabla) = \sum_{j=1}^{\alpha_{k-1}} g_j D^j\,,$$

and now the claimed estimate for g_j follows from the previously established bound (12.54) for the $f_{\gamma_\ell,q-1}$'s and their derivatives and the Leibniz rule.

With (12.50)–(12.51) and (12.42) with $k'=k-1$ in hand, we return to (12.49) and obtain

$$\begin{aligned}
&\left\|D^{\alpha_k}\left[D_{t,q+\bar{n}-1}^{\beta_k}, D^{\alpha_{k-1}}\right] D_{t,q+\bar{n}-1}^{\beta_{k-1}} P_{k-2}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})}\\
&\lesssim \sum_{j=1}^{\alpha_{k-1}} \sum_{1\leq|\gamma|\leq\beta_k} \left\|D^{\alpha_k}\left(g_j\ D^j D_{t,q+\bar{n}-1}^{\beta_k+\beta_{k-1}-|\gamma|} P_{k-2}\right)\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})}\\
&\lesssim \sum_{j=1}^{\alpha_{k-1}} \sum_{1\leq|\gamma|\leq\beta_k} \sum_{a'=0}^{\alpha_k} \left\|D^{\alpha_k-a'} g_j\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \left\|D^{a'+j} D_{t,q+\bar{n}-1}^{\beta_k+\beta_{k-1}-|\gamma|} P_{k-2}\right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})}\\
&\lesssim \sum_{j=1}^{\alpha_{k-1}} \sum_{|\gamma|=1}^{\beta_k} \sum_{a'=0}^{\alpha_k} \Gamma_{q+\bar{n}}^{i+1}\delta_{q+\bar{n}}^{1/2} r_q^{-1/3} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{\alpha_k-a'+\alpha_{k-1}-j} \mathcal{M}\left(|\gamma|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i+1}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)\\
&\qquad\qquad \times (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{a'+j+K-\alpha_{k-1}-\alpha_k} \mathcal{M}\left(M-|\gamma|, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)\\
&\lesssim \Gamma_{q+\bar{n}}^{i+1}\delta_{q+\bar{n}}^{1/2} r_q^{-1/3} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)
\end{aligned} \tag{12.55}$$

for $K+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2 + 1$. The $+1$ in the range of derivatives is simply a consequence of the fact that the summand in the third line of the above display starts with $j \geq 1$ and with $|\gamma| \geq 1$, so that only ${}^{3\mathsf{N}_{\mathrm{fin}}}/_2$ derivatives may fall on g_j, which is the extent of the bounds from (12.51). This concludes the proof of the inductive step for (12.42).

Proof of (12.43a)**.** This estimate follows from Lemma A.2.2. Indeed, letting $v = f = \widehat{w}_{q+\bar{n}}$,

$B = D_{t,q+\bar{n}-1}$, $\Omega = \operatorname{supp}\psi_{i,q+\bar{n}}$, $p = \infty$, the previously established bound (12.42) allows us to verify conditions (A.22)–(A.23) of Lemma A.2.2 with $N_* = {}^{3\mathsf{N}_{\text{fin}}}/_2 + 1$, $\mathcal{C}_v = \mathcal{C}_f = \Gamma_{q+\bar{n}}^{i+1}\widehat{\delta}_{q+\bar{n}}^{1/2} r_q^{-1/3}$, $\lambda_v = \lambda_f = \widetilde{\lambda}_v = \widetilde{\lambda}_f = \Gamma_{q+\bar{n}}\lambda_{q+\bar{n}}$, $N_x = \infty$, $\mu_v = \mu_f = \Gamma_{q+\bar{n}}^{i+3}\tau_{q+\bar{n}-1}^{-1}$, $\widetilde{\mu}_v = \widetilde{\mu}_f = \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}$, and $N_t = \mathsf{N}_{\text{ind,t}}$. The bound (12.43a) now is a direct consequence of (A.24).

Proof of (12.44)**.** First we consider the bound (12.44), inductively on k. For the case $k = 1$ we appeal to estimate (A.26) in Lemma A.2.2 with the operators $A = D_{q+\bar{n}}, B = D_{t,q+\bar{n}-1}$ and the functions $v = \widehat{w}_{q+\bar{n}}$ and $f = D\widehat{u}_{q+\bar{n}}$, so that $D^n(A+B)^m f = D^n D_{t,q+\bar{n}}^m D\widehat{u}_{q+\bar{n}}$. As before, the assumption (A.22) holds due to (12.42) with the same parameter choices. Verifying condition (A.23) is this time more involved, and follows by rewriting $f = D\widehat{u}_q = D\widehat{w}_q + D\widehat{u}_{q-1}$. By using (12.42), and the parameter inequality (4.10b), we conveniently obtain

$$\begin{aligned}\Big\|\Big(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}-1}^{\beta_i}\Big) D\widehat{w}_{q+\bar{n}}\Big\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \\ \lesssim \Gamma_{q+\bar{n}}^{i-5}\tau_{q+\bar{n}}^{-1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}}^{i-5}\tau_{q+\bar{n}}^{-1}, \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)\end{aligned} \tag{12.56}$$

for all $|\alpha| + |\beta| = K + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2$ (note that the maximal number of derivatives is not ${}^{3\mathsf{N}_{\text{fin}}}/_2 + 1$ anymore, but instead it is just ${}^{3\mathsf{N}_{\text{fin}}}/_2$; the reason is that we are estimating $D\widehat{w}_q$ and not $\widehat{w}_q$). On the other hand, from the inductive assumption (5.34) with $q' = q + \bar{n} - 1$ we obtain that

$$\begin{aligned}\Big\|\Big(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}-1}^{\beta_i}\Big) D\widehat{u}_{q+\bar{n}-1}\Big\|_{L^\infty(\operatorname{supp}\psi_{j,q+\bar{n}-1})} \\ \lesssim \tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{j-4}(\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^K \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{q+\bar{n}-1}^{j}\tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}\right)\end{aligned}$$

for $K + M \leq {}^{3\mathsf{N}_{\text{fin}}}/_2$. Recalling the definitions (12.9)–(12.11) and the notation (12.12), we have that $(x,t) \in \operatorname{supp}(\psi_{i,q+\bar{n}})$ if and only if $(x,t) \in \operatorname{supp}(\psi_{\vec{i},q+\bar{n}})$, and so for every $m \in \{0, \ldots, \mathsf{N}_{\text{cut,t}}\}$, there exists j_m with $\Gamma_{q+\bar{n}-1}^{j_m} \leq \Gamma_{q+\bar{n}}^{i_m} \leq \Gamma_{q+\bar{n}}^{i}$ and $(x,t) \in \operatorname{supp}(\psi_{j_m,q+\bar{n}-1})$.

Thus, the above stated estimate and (4.10b) imply that

$$\left\| \Big(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}-1}^{\beta_i} \Big) D\widehat{u}_{q+\bar{n}-1} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})}$$
$$\lesssim \tau_{q+\bar{n}}^{-1} \Gamma_{q+\bar{n}}^{i-10} (\lambda_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i-10}\tau_{q+\bar{n}}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}\right) \tag{12.57}$$

whenever $K + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. Combining (12.56) and (12.57), we may now verify condition (A.23) for $f = D\widehat{u}_{q+\bar{n}}$, with $p = \infty$, $\Omega = \operatorname{supp}(\psi_{i,q+\bar{n}})$, $\mathcal{C}_f = \Gamma_{q+\bar{n}}^{i-5}\tau_{q+\bar{n}}^{-1}$, $\lambda_f = \widetilde{\lambda}_f = \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}$, $N_x = \infty$, $\mu_f = \Gamma_{q+\bar{n}}^{i-5}\tau_{q+\bar{n}}^{-1}, \widetilde{\mu}_f = \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}, N_t = \mathsf{N}_{\mathrm{ind,t}}$, and $N_* = {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. We may thus appeal to (A.26) and obtain that

$$\left\| D^K D_{t,q+\bar{n}}^M D\widehat{u}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \lesssim \Gamma_{q+\bar{n}}^{i-5}\tau_{q+\bar{n}}^{-1}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}}^{i-5}\tau_{q+\bar{n}}^{-1}, \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)$$

whenever $K + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$, concluding the proof of (12.44) for $k = 1$.

In order to prove (12.44) for a general k, we proceed by induction. Assume the estimate holds for every $k' \leq k - 1$. Proving (12.44) at level k is done in the same way as we have established the induction step (in k) for (12.42). We let

$$\widetilde{P}_{k'} = \left(\prod_{i=1}^{k'} D^{\alpha_i} D_{t,q+\bar{n}}^{\beta_i} \right) D\widehat{u}_{q+\bar{n}}$$

and decompose

$$\left(\prod_{i=1}^{k} D^{\alpha_i} D_{t,q+\bar{n}}^{\beta_i} \right) D\widehat{u}_{q+\bar{n}} = (D^{\alpha_k+\alpha_{k-1}} D_{t,q+\bar{n}}^{\beta_k+\beta_{k-1}})\widetilde{P}_{k-2} + D^{\alpha_k}\left[D_{t,q+\bar{n}}^{\beta_k}, D^{\alpha_{k-1}}\right] D_{t,q+\bar{n}}^{\beta_{k-1}}\widetilde{P}_{k-2}\,.$$

Note that the first term is directly bounded using the induction assumption at level $k - 1$. To bound the commutator term, similarly to (12.49)–(12.51), we obtain that

$$D^{\alpha_k}\left[D_{t,q+\bar{n}}^{\beta_k}, D^{\alpha_{k-1}}\right] D_{t,q+\bar{n}}^{\beta_{k-1}}\widetilde{P}_{k-2} = D^{\alpha_k} \sum_{1\leq|\gamma|\leq\beta_k} \frac{\beta_k!}{\gamma!(\beta_k - |\gamma|)!} \left(\sum_{j=1}^{\alpha_{k-1}} \widetilde{g}_j D^j \right) D_{t,q+\bar{n}}^{\beta_k+\beta_{k-1}-|\gamma|}\widetilde{P}_{k-2}\,,$$

where one may use the previously established bound (12.44) with $k = 1$ (instead of (12.52)) to estimate $\|D^a \widetilde{g}_j\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})}$ The estimate

$$\begin{aligned}
&\left\| D^{\alpha_k} \left[D^{\beta_k}_{t,q+\bar{n}}, D^{\alpha_{k-1}} \right] D^{\beta_{k-1}}_{t,q+\bar{n}} \widetilde{P}_{k-2} \right\|_{L^\infty(\operatorname{supp}\psi_{i,q+\bar{n}})} \\
&\quad \lesssim \tau^{-1}_{q+\bar{n}} \Gamma^{i-5}_{q+\bar{n}} (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i-5}_{q+\bar{n}} \tau^{-1}_{q+\bar{n}}, \Gamma_{q+\bar{n}-1} \mathrm{T}^{-1}_{q+\bar{n}-1}\right)
\end{aligned} \tag{12.58}$$

follows similarly to (12.55), from the estimate for $\widetilde{g}_j$ and the bound (12.44) with $k-1$ terms in the product. This concludes the proof of estimate (12.44).

Proof of (12.45)**.** The proof of this bound is nearly identical to that of (12.44), as is readily seen for $k = 1$: we just need to replace $D\widehat{w}_{q+\bar{n}}$ estimates with $\widehat{w}_{q+\bar{n}}$ estimates, and $D\widehat{u}_{q+\bar{n}-1}$ bounds with $\widehat{u}_{q+\bar{n}-1}$ bounds. For instance, instead of (12.56), we appeal to (12.43b) and obtain a bound for $D^K D^M_{t,q+\bar{n}} \widehat{w}_{q+\bar{n}}$ which is better than (12.56) by a factor of $\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}}$, and which holds for $K + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2 + 1$. This estimate is sharper than required by (12.45). The estimate for $D^K D^M_{t,q+\bar{n}} \widehat{u}_{q+\bar{n}-1}$ is obtained similarly to (12.57), except that instead of appealing to the induction assumption (5.34) at level $q' = q + \bar{n} - 1$, we use (5.35a) with $q' = q + \bar{n} - 1$. The estimates hold for $K + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_2 + 1$. These arguments establish (12.45) with $k = 1$. The case of general $k \geq 2$ is treated inductively exactly as before, because the commutator term is bounded in the same way as (12.58), except that $K + 1$ is replaced by K. To avoid redundancy, we omit these details.

Proof of (12.46)**.** The proof of this bound is immediate from (9.83b), the definition of $\widehat{w}_{q+\bar{n}}$ in Lemma 9.5.1, the inductive assumption (5.35b), and the triangle inequality. □

12.6 Material derivatives

Remark 12.6.1 (**Rewriting** $\psi_{i,q+\bar{n}}$**).** In order to take material derivatives of $\psi_{i,q+\bar{n}}$, we need to take advantage of certain cancellations. For this purpose, we introduce the summed

cutoff function

$$\Psi^6_{m,i,q+\bar{n}} = \sum_{i_m=0}^{i} \psi^6_{m,i_m,q+\bar{n}} \tag{12.59}$$

for any given $0 \le m \le \mathsf{N}_{\mathrm{cut,t}}$ and note via Lemma 12.2.1 that

$$D(\Psi^6_{m,i,q+\bar{n}}) = D(\psi^6_{m,i,q+\bar{n}})\mathbf{1}_{\mathrm{supp}\,(\psi_{m,i+1,q+\bar{n}})}\,. \tag{12.60}$$

With the notation (12.59) we return to the definition (12.11) and note that

$$\begin{aligned}\psi^6_{i,q+\bar{n}} &= \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \psi^6_{m,i,q+\bar{n}} \prod_{m'=0}^{m-1} \Psi^6_{m',i,q+\bar{n}} \prod_{m''=m+1}^{\mathsf{N}_{\mathrm{cut,t}}} (\Psi^6_{m'',i,q+\bar{n}} - \psi^6_{m'',i,q+\bar{n}}) \\ &= \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \psi^6_{m,i,q+\bar{n}} \prod_{m'=0}^{m-1} \Psi^6_{m',i,q+\bar{n}} \prod_{m''=m+1}^{\mathsf{N}_{\mathrm{cut,t}}} \Psi^6_{m'',i-1,q+\bar{n}}\,. \end{aligned} \tag{12.61}$$

Inspecting (12.61) and using identity (12.60) and the definitions (12.12), (12.59), we see that

$$\begin{aligned}(x,t) \in \mathrm{supp}\,(D_{t,q+\bar{n}-1}\psi^6_{i,q+\bar{n}}) \quad \Longrightarrow \quad & \exists \vec{i} \in \mathbb{N}_0^{\mathsf{N}_{\mathrm{cut,t}}+1} \text{ and } \exists 0 \le m \le \mathsf{N}_{\mathrm{cut,t}} \\ & \text{with } i_m \in \{i-1, i\} \text{ and } \max_{0 \le m' \le \mathsf{N}_{\mathrm{cut,t}}} i_{m'} = i \\ & \text{such that } (x,t) \in \mathrm{supp}\,(\psi_{\vec{i},q+\bar{n}}) \cap \mathrm{supp}\,(D_{t,q+\bar{n}-1}\psi_{m,i_m,q+\bar{n}}) \\ & \text{and } i_{m'} \le i_m \text{ whenever } m < m' \le \mathsf{N}_{\mathrm{cut,t}}\,. \end{aligned} \tag{12.62}$$

The generalization of characterization (12.62) to higher order material derivatives $D^M_{t,q+\bar{n}-1}$ is direct: $(x,t) \in \mathrm{supp}\,(D^M_{t,q+\bar{n}-1}\psi^6_{i,q+\bar{n}})$ implies that there exists $\vec{i} \in \mathbb{N}_0^{\mathsf{N}_{\mathrm{cut,t}}+1}$ with maximal index equal to i, such that for every $0 \le m \le \mathsf{N}_{\mathrm{cut,t}}$ for which $(x,t) \in \mathrm{supp}\,(\psi_{\vec{i},q+\bar{n}}) \cap \mathrm{supp}\,(D_{t,q+\bar{n}-1}\psi_{m,i_m,q+\bar{n}})$, we have $i_{m'} \le i_m \in \{i-1, i\}$ whenever $m < m'$. Using this characterization, we may prove the following.

Lemma 12.6.2 (Mixed derivatives for intermediate velocity cutoff functions). *Let*

$q+\bar{n} \geq 1$, $0 \leq i \leq i_{\max}(q+\bar{n})$, and fix $\vec{i} \in \mathbb{N}_0^{\mathsf{N}_{\mathrm{cut,t}}+1}$ such that $\max_{0\leq m\leq \mathsf{N}_{\mathrm{cut,t}}} i_m = i$, as in the right side of (12.62)*. Fix $0 \leq m \leq \mathsf{N}_{\mathrm{cut,t}}$ such that $i_m \in \{i-1, i\}$ and such that $i_{m'} \leq i_m$ for all $m \leq m' \leq \mathsf{N}_{\mathrm{cut,t}}$, again as in the right hand side of* (12.62)*. Lastly, fix j_m such that $i_*(j_m) \leq i_m$. For $N, K, M, k \geq 0$, $\alpha, \beta \in \mathbb{N}^k$ such that $|\alpha| = K$ and $|\beta| = M$, we have*

$$\frac{\mathbf{1}_{\mathrm{supp}\,(\psi_{\vec{i},q+\bar{n}})}\mathbf{1}_{\mathrm{supp}\,(\psi_{j_m,q+\bar{n}-1})}}{\psi_{m,i_m,j_m,q+\bar{n}}^{1-(K+M)/\mathsf{N}_{\mathrm{fin}}}}\left|\left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q+\bar{n}-1}^{\beta_l}\right)\psi_{m,i_m,j_m,q+\bar{n}}\right|$$
$$\lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}} - \mathsf{N}_{\mathrm{cut,x}}, \Gamma_{q+\bar{n}}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.63}$$

for all K such that $0 \leq K+M \leq \mathsf{N}_{\mathrm{fin}}$. Moreover,

$$\frac{\mathbf{1}_{\mathrm{supp}\,(\psi_{\vec{i},q+\bar{n}})}\mathbf{1}_{\mathrm{supp}\,(\psi_{j_m,q+\bar{n}-1})}}{\psi_{m,i_m,j_m,q+\bar{n}}^{1-(N+K+M)/\mathsf{N}_{\mathrm{fin}}}}\left|D^N\left(\prod_{l=1}^{k} D_{q+\bar{n}}^{\alpha_l} D_{t,q+\bar{n}-1}^{\beta_l}\right)\psi_{m,i_m,j_m,q+\bar{n}}\right|$$
$$\lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N (\tau_{q+\bar{n}}^{-1}\Gamma_{q+\bar{n}}^{i-5})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}} - \mathsf{N}_{\mathrm{cut,x}}, \Gamma_{q+\bar{n}}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.64}$$

holds whenever $0 \leq N+K+M \leq \mathsf{N}_{\mathrm{fin}}$.

Proof of Lemma 12.6.2. Note that for $M = 0$ estimate (12.63) was already established in (12.24a). The bound (12.64) with $M = 0$, i.e., an estimate for the $D^N D_{q+\bar{n}}^K \psi_{m,i_m,j_m,q+\bar{n}}$, holds by appealing to the expansion (12.39)–(12.40), the bound (12.41) (which is applicable since in the context of estimate (12.64) we work on the support of $\psi_{\vec{i},q+\bar{n}}$), to the bound (12.63) with $M = 0$, and to (4.10b). The rest of the proof is dedicated to the case $M \geq 1$. The proofs are very similar to the proof of Lemma 12.3.1, but we additionally need to appeal to bounds and arguments from the proof of Lemma 12.5.3.

Proof of (12.63)**.** We start with the case $k = 1$ and estimate $D^K D_{t,q+\bar{n}-1}^M \psi_{m,i_m,j_m,q+\bar{n}}$ for $K + M \leq \mathsf{N}_{\mathrm{fin}}$ and $M \geq 1$. We note that the operator $D_{t,q+\bar{n}-1}$ is a scalar differential operator, and thus the Faa di Bruno argument which was used to bound (12.24a) may be repeated. As was done there, we recall the definitions (12.5)–(12.6) and split the analysis in two cases, according to whether (12.25) or (12.30) holds.

Let us first consider the case (12.25). Our goal is to apply [7, Lemma A.5] to the function $\psi = \gamma_{m,q+\bar{n}}$ or $\psi = \widetilde{\gamma}_{m,q+\bar{n}}$, with $\Gamma_\psi = \Gamma_{q+\bar{n}}^{m+1}$, $\Gamma = \Gamma_{q+\bar{n}}^{(m+1)(i_m - i_*(j_m))}$, $h(x,t) = h^2_{m,j_m,q+\bar{n}}(x,t)$, and $D_t = D_{t,q+\bar{n}-1}$. The estimate in [7, (A.24)] again holds by (8.16) and (8.18), and so it remains to obtain a bound on the derivatives of $(h_{m,j_m,q+\bar{n}}(x,t))^2$ on the set $\operatorname{supp}(\psi_{\vec{i},q}) \cap \operatorname{supp}(\psi_{j_m,q-1}\psi_{m,i_m,j_m,q})$ in order to satisfy [7, (A.25)]. Similarly to (12.26), for $K' + M' \leq \mathsf{N}_{\text{fin}}$ the Leibniz rule and definition (12.4) gives

$$\begin{aligned}
&\left| D^{K'} D^{M'}_{t,q+\bar{n}-1} h^2_{m,j_m,q+\bar{n}} \right| \\
&\quad \lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{K'} (\tau^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}})^{M'} \Gamma^{-2(m+1)i_*(j_m)}_{q+\bar{n}} \\
&\qquad \times \sum_{K''=0}^{K'} \sum_{M''=0}^{M'} \sum_{n=0}^{\mathsf{N}_{\text{cut,x}}} (\tau^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}})^{-m-M''} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-n-K''} \delta^{-1/2}_{q+\bar{n}} r_q^{1/3} |D^{n+K''} D^{m+M''}_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}| \\
&\qquad \times (\tau^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}})^{-m-M'+M''} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-n-K'+K''} \delta^{-1/2}_{q+\bar{n}} r_q^{1/3} |D^{n+K'-K''} D^{m+M'-M''}_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}| \,.
\end{aligned} \tag{12.65}$$

By the characterization (12.62), for every (x,t) in the support described on the left side of (12.63) we have that for every $m \leq R \leq \mathsf{N}_{\text{cut,t}}$, there exists $i_R \leq i_m$ and j_R with $i_*(j_R) \leq i_R$, such that $(x,t) \in \operatorname{supp} \psi_{j_R,q+\bar{n}-1}\psi_{R,i_R,j_R,q+\bar{n}}$. As a consequence, for the terms in the sum (12.65) with $L \in \{n+K'', n+K'-K''\} \leq \mathsf{N}_{\text{cut,x}}$ and $R \in \{m+M'', m+M'-M''\} \leq \mathsf{N}_{\text{cut,t}}$, we may appeal to estimate (12.19) which gives a bound on $h_{R,j_R,q+\bar{n}}$, and thus obtain

$$\begin{aligned}
&(\tau^{-1}_{q+\bar{n}-1}\Gamma^2_{q+\bar{n}})^{-R} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-L} \delta^{-1/2}_{q+\bar{n}} r_q^{1/3} \left\| D^L D^R_{t,q-1} \widehat{w}_{q+\bar{n}} \right\|_{L^\infty(\operatorname{supp}\psi_{R,i_R,j_R,q+\bar{n}})} \\
&\qquad \leq \Gamma^{(R+1)i_*(j_R)}_{q+\bar{n}} \Gamma^{(R+1)(i_R+1-i_*(j_R))}_{q+\bar{n}} \\
&\qquad \leq \Gamma^{(R+1)(i_m+1)}_{q+\bar{n}} \,.
\end{aligned}$$

On the other hand, if $L > \mathsf{N}_{\text{cut,x}}$, or if $R > \mathsf{N}_{\text{cut,t}}$, then by (9.83b), we have that

$$\begin{aligned}(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{2})^{-R}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-L}\delta_{q+\bar{n}}^{-1/2}r_q^{1/3}\left\|D^L D_{t,q+\bar{n}-1}^R \widehat{w}_{q+\bar{n}}\right\|_{L^\infty(\operatorname{supp}\psi_{j_m,q+\bar{n}-1})}& \\ \leq \Gamma_q^{\mathsf{c}_\infty/2+16} r_q^{-1}\Gamma_{q+\bar{n}}^{-L}\Gamma_{q+\bar{n}-1}^{L}\Gamma_{q+\bar{n}}^{-2R}\mathcal{M}\left(R,\mathsf{N}_{\text{ind,t}},\Gamma_{q+\bar{n}-1}^{j_m-1},\tau_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)& \\ \leq \mathcal{M}\left(R,\mathsf{N}_{\text{ind,t}},\Gamma_{q+\bar{n}}^{i_m-1},\tau_{q+\bar{n}-1}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)\ .& \qquad (12.66)\end{aligned}$$

since $\mathsf{N}_{\text{cut,x}}$ and $\mathsf{N}_{\text{cut,t}}$ were taken sufficiently large to obey (4.17) and $i_m \geq i_*(j_m)$. Combining (12.65)–(12.66), we have that

$$\begin{aligned}&\mathbf{1}_{\operatorname{supp}(\psi_{\vec{i},q+\bar{n}})}\mathbf{1}_{\operatorname{supp}(\psi_{j_m,q+\bar{n}-1})}\left|D^{K'}D_{t,q+\bar{n}-1}^{M'}h_{m,j_m,q+\bar{n}}^2\right| \\ &\lesssim \Gamma_{q+\bar{n}}^{2(m+1)(i_m-i_*(j_m)+1)}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{K'}\mathcal{M}\left(M',\mathsf{N}_{\text{ind,t}}-\mathsf{N}_{\text{cut,t}},\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i+3},\mathrm{T}_{q+\bar{n}-1}^{-1}\right) \qquad (12.67)\end{aligned}$$

for all $K' + M' \leq \mathsf{N}_{\text{fin}}$. The upshot of (12.67) is that the condition in [7, (A.25)] is now verified, with $\mathcal{C}_h = \Gamma_{q+\bar{n}}^{2(m+1)(i_m-i_*(j_m)+1)}$, and $\lambda = \widetilde{\lambda} = \Gamma_{q+\bar{n}}\lambda_{q+\bar{n}}$, $\mu = \tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i+3}$, $\widetilde{\mu} = \mathrm{T}_{q+\bar{n}-1}^{-1}$, and $N_t = \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}}$. We obtain from [7, (A.26)] and the fact that $(\Gamma_\psi\Gamma)^{-2}\mathcal{C}_h = 1$ that (12.63) holds when $k = 1$ for those (x,t) such that $h_{m,j_m,q+\bar{n}}(x,t)$ satisfies (12.25). The case when $h_{m,j_m,q+\bar{n}}(x,t)$ satisfies the bound (12.30) is nearly identical, as was the case in the proof of Lemma 12.3.1. The only changes are that now $\Gamma_\psi = 1$ (according to (8.17)), and that the constant $\mathcal{C}_h$ which we read from the right side of (12.67) is now improved to $\Gamma_{q+\bar{n}}^{2(m+1)(i_m-i_*(j_m))}$. These two changes offset each other, resulting in the same exact bound. Thus, we have shown that (12.63) holds when $k = 1$.

The general case $k \geq 1$ in (12.63) is obtained via induction on k, in precisely the same fashion as the proof of estimate (12.42) in Lemma 12.5.3. At the heart of the matter lies a commutator bound similar to (12.55), which is proven in precisely the same way by appealing to the fact that we work on $\operatorname{supp}(\psi_{\vec{i},q+\bar{n}}) \subset \operatorname{supp}(\psi_{i,q+\bar{n}})$, and thus bound (12.51) is available; in turn, this bound provides sharper space and material estimates than required in (12.63), completing the proof. In order to avoid redundancy we omit further details.

Proof of (12.64)**.** This estimate follows from Lemma A.2.2 in a manner identical to the proof of [7, (6.77)], and we omit the details. □

Lemma 12.6.3 (**Mixed spatial and material derivatives for velocity cutoffs**)**.** *Let* $q+\bar{n} \geq 1$, $0 \leq i \leq i_{\max}(q+\bar{n})$, $N, K, M, k \geq 0$, *and let* $\alpha, \beta \in \mathbb{N}^k$ *be such that* $|\alpha| = K$ *and* $|\beta| = M$. *Then we have*

$$\frac{1}{\psi_{i,q+\bar{n}}^{1-(K+M)/\mathsf{N}_{\text{fin}}}} \left| \left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q+\bar{n}-1}^{\beta_l} \right) \psi_{i,q+\bar{n}} \right| \lesssim (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}}, \Gamma_{q+\bar{n}-1}^{i+3} \tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}+1} \mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.68}$$

for $K + M \leq \mathsf{N}_{\text{fin}}$, *and*

$$\frac{1}{\psi_{i,q+\bar{n}}^{1-(N+K+M)/\mathsf{N}_{\text{fin}}}} \left| D^N \left(\prod_{l=1}^{k} D_{q+\bar{n}}^{\alpha_l} D_{t,q+\bar{n}-1}^{\beta_l} \right) \psi_{i,q+\bar{n}} \right| \lesssim (\lambda_{q+\bar{n}} \Gamma_{q+\bar{n}})^N (\Gamma_{q+\bar{n}}^{i-5} \tau_{q+\bar{n}}^{-1})^K \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}} - \mathsf{N}_{\text{cut,t}}, \Gamma_{q+\bar{n}-1}^{i+3} \tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}+1} \mathrm{T}_{q+\bar{n}-1}^{-1}\right) \tag{12.69}$$

holds for $N + K + M \leq \mathsf{N}_{\text{fin}}$.

Proof of Lemma 12.6.3. Note that for $M = 0$ estimate (12.68) holds by (12.24b). The bound (12.69) holds for $M = 0$, due to the expansion (12.39)–(12.40), the bound (12.41) on the support of $\psi_{i,q+\bar{n}}$, the bound (12.68) with $M = 0$, and to the parameter inequality (4.10b). The rest of the proof is dedicated to the cases $M \geq 1$ for both bounds.

The argument is very similar to the proof of Lemma 12.3.1 and so we only emphasize the main differences. We start with the proof of (12.68). We claim that in a the same way that (12.24a) was shown to imply (12.32), one may show that estimate (12.63) implies that for any $\vec{i}$ and $0 \leq m \leq \mathsf{N}_{\text{cut,t}}$ as on the right side of (12.62) (in particular, as in Lemma 12.5.3),

we have that

$$\frac{\mathbf{1}_{\mathrm{supp}\,(\psi_{\vec{i},q+\bar{n}})}}{\psi_{m,i_m,q+\bar{n}}^{1-(K+M)/\mathsf{N}_{\mathrm{fin}}}}\left|\left(\prod_{l=1}^{k} D^{\alpha_l} D_{t,q+\bar{n}-1}^{\beta_l}\right)\psi_{m,i_m,q+\bar{n}}\right|$$
$$\lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^K \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}} - \mathsf{N}_{\mathrm{cut,x}}, \Gamma_{q+\bar{n}-1}^{i+3}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}}\mathrm{T}_{q+\bar{n}-1}^{-1}\right) . \tag{12.70}$$

The proof of the above estimate is done by induction on k. For $k = 1$, the first step in establishing (12.70) is to use the Leibniz rule and induction on the number of material derivatives to reduce the problem to an estimate for $\psi_{m,i_m,q+\bar{n}}^{-6+(K+M)/\mathsf{N}_{\mathrm{fin}}} D^K D_{t,q+\bar{n}-1}^M(\psi_{m,i_m,q+\bar{n}}^6)$; this is achieved in precisely the same way that (12.34) was proven. The derivatives of $\psi_{m,i_m,q+\bar{n}}^6$ are now bounded via the Leibniz rule and the definition (12.9). Indeed, when $D^{K'} D_{t,q+\bar{n}-1}^{M'}$ derivatives fall on $\psi_{m,i_m,j_m,q+\bar{n}}^6$, the required bound is obtained from (12.63), which gives the same upper bound as the one required by (12.70). On the other hand, if $D^{K-K'} D_{t,q+\bar{n}-1}^{M-M'}$ derivatives fall on $\psi_{j_m,q+\bar{n}-1}^6$, the required estimate is provided by (5.37) with $q' = q + \bar{n} - 1$ and i replaced by j_m; the resulting estimates are strictly better than what is required by (12.70). This shows that estimate (12.70) holds for $k = 1$. We then proceed inductively in $k \geq 1$, in the same fashion as the proof of estimate (12.42) in Lemma 12.5.3; the corresponding commutator bound is applicable because we work on $\mathrm{supp}\,(\psi_{m,i_m,q+\bar{n}}) \cap \mathrm{supp}\,(\psi_{i,q+\bar{n}})$. In order to avoid redundancy we omit these details, and conclude the proof of (12.70).

As in the proof of Lemma 12.3.1, we are now able to show that (12.68) is a consequence of (12.70). As before, by induction on the number of material derivatives and the Leibniz rule we reduce the problem to an estimate for $\psi_{i,q+\bar{n}}^{-6+(K+M)/\mathsf{N}_{\mathrm{fin}}} \prod_{l=1}^k D^{\alpha_l} D_{t,q+\bar{n}-1}^{\beta_l}(\psi_{i,q+\bar{n}}^6)$; see the proof of (12.34) for details. In order to estimate derivatives of $\psi_{i,q+\bar{n}}^6$, we use identities (12.60)

and (12.61), which imply upon applying a differential operator, say $D_{t,q+\bar{n}-1}$, that

$$
\begin{aligned}
&D_{t,q+\bar{n}-1}(\psi^6_{i,q+\bar{n}}) \\
&\quad = D_{t,q+\bar{n}-1}\left(\sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\prod_{m'=0}^{m-1}\Psi^6_{m',i,q+\bar{n}}\cdot\psi^6_{m,i,q+\bar{n}}\cdot\prod_{m''=m+1}^{\mathsf{N}_{\mathrm{cut,t}}}\Psi^6_{m'',i-1,q+\bar{n}}\right) \\
&\quad = \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\sum_{\bar{m}'=0}^{m-1}D_{t,q+\bar{n}-1}(\psi^6_{\bar{m}',i,q+\bar{n}})\prod_{\substack{0\le m'\le m-1\\ m'\neq\bar{m}'}}\Psi^6_{m',i,q+\bar{n}}\cdot\psi^6_{m,i,q+\bar{n}}\cdot\prod_{m''=m+1}^{\mathsf{N}_{\mathrm{cut,t}}}\Psi^6_{m'',i-1,q+\bar{n}} \\
&\qquad + \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\sum_{\bar{m}''=m+1}^{\mathsf{N}_{\mathrm{cut,t}}}\prod_{m'=0}^{m-1}\Psi^6_{m',i,q+\bar{n}}\cdot\psi^6_{m,i,q+\bar{n}}\cdot D_{t,q+\bar{n}-1}(\Psi^6_{\bar{m}'',i-1,q+\bar{n}})\prod_{\substack{m+1\le m''\le\mathsf{N}_{\mathrm{cut,t}}\\ m''\neq\bar{m}''}}\Psi^6_{m'',i-1,q+\bar{n}} \\
&\qquad + \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\prod_{m'=0}^{m-1}\Psi^6_{m',i,q+\bar{n}}\cdot D_{t,q+\bar{n}-1}(\psi^6_{m,i,q+\bar{n}})\cdot\prod_{m''=m+1}^{\mathsf{N}_{\mathrm{cut,t}}}\Psi^6_{m'',i-1,q+\bar{n}}\,. \qquad (12.71)
\end{aligned}
$$

Higher order material derivatives of $\psi^6_{i,q+\bar{n}}$, and mixtures of space and material derivatives are obtained similarly, by an application of the Leibniz rule. Equality (12.71) in particular justifies why we have only proven (12.70) for $\vec{i}$ and $0 \le m \le \mathsf{N}_{\mathrm{cut,t}}$ as on the right side of (12.62)! With (12.70) and (12.71) in hand, we now repeat the argument from the proof of Lemma 12.3.1 (see the two displays below (12.34)) and conclude that (12.68) holds.

In order to conclude the proof of the Lemma, it remains to establish (12.69). This bound follows now directly from (12.68) and an application of Lemma A.2.2 (to be more precise, we need to use the proof of this Lemma), in precisely the same way that (12.63) was shown earlier to imply (12.64). As there are no changes to be made to this argument, we omit these details. □

12.7 L^r size of the velocity cutoffs

The purpose of this section is to show that the inductive estimate (5.13) holds with $q' = q + \bar{n}$.

Lemma 12.7.1 (Support estimate). *For all* $0 \le i \le i_{\max}(q+\bar{n})$ *and* $1 \le r \le \infty$, *we have*

that

$$\|\psi_{i,q+\bar{n}}\|_r \lesssim \Gamma_{q+\bar{n}}^{\frac{-3i+\mathsf{C}_b}{r}} \tag{12.72}$$

where C_b *is defined in* (5.13) *and thus depends only on* b.

Proof of Lemma 12.7.1. First, note that the cases $1 < r \leq \infty$ follow from the case $r = 1$ and interpolation. Next, observe that if $i \leq {}^1\!/\!_3\mathsf{C}_b$, then (12.72) trivially holds because $0 \leq \psi_{i,q+\bar{n}} \leq 1$ for all $q + \bar{n} \geq 1$ once a is chosen to be sufficiently large. Thus, we only consider i such that ${}^1\!/\!_3\mathsf{C}_b < i \leq i_{\max}(q + \bar{n})$.

First, we note that Lemma 12.2.1 implies that the functions $\Psi_{m,i',q+\bar{n}}$ defined in (12.59) satisfy $0 \leq \Psi^2_{m,i',q} \leq 1$, and thus (12.61) implies that

$$\|\psi_{i,q+\bar{n}}\|_1 \leq \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \|\psi_{m,i,q+\bar{n}}\|_1 \ . \tag{12.73}$$

Next, we let $j_*(i) = j_*(i, q + \bar{n})$ be the *maximal* index of j_m appearing in (12.9). In particular, recalling also (12.3), we have that

$$\Gamma_{q+\bar{n}}^{i-1} < \Gamma_{q+\bar{n}-1}^{j_*(i)} \leq \Gamma_{q+\bar{n}}^{i} < \Gamma_{q+\bar{n}-1}^{j_*(i)+1} \ . \tag{12.74}$$

Using (12.9), in which we simply write j instead of j_m, the fact that $0 \leq \psi^2_{j,q+\bar{n}-1}, \psi^2_{m,i,j,q+\bar{n}} \leq 1$, and the inductive assumption (5.13) at level $q + \bar{n} - 1$, we may deduce that

$$\begin{aligned}\|\psi_{m,i,q+\bar{n}}\|_1 &\leq \left\|\psi_{j_*(i),q+\bar{n}-1}\right\|_1 + \left\|\psi_{j_*(i)-1,q+\bar{n}-1}\right\|_1 + \sum_{j=0}^{j_*(i)-2} \|\psi_{j,q+\bar{n}-1}\psi_{m,i,j,q+\bar{n}}\|_1 \\ &\leq \Gamma_{q+\bar{n}-1}^{-3j_*(i)+\mathsf{C}_b} + \Gamma_{q+\bar{n}-1}^{-3j_*(i)+3+\mathsf{C}_b} + \sum_{j=0}^{j_*(i)-2} \left|\mathrm{supp}\left(\psi_{j,q+\bar{n}-1}\psi_{m,i,j,q+\bar{n}}\right)\right| \ . \end{aligned} \tag{12.75}$$

The second term on the right side of (12.75) is estimated using the last inequality in (12.74)

as

$$\Gamma_{q+\bar{n}-1}^{-3j_*(i)+3+\mathsf{C}_b} \leq \Gamma_{q+\bar{n}}^{-3i}\Gamma_{q+\bar{n}-1}^{6+\mathsf{C}_b} \leq \Gamma_{q+\bar{n}}^{-3i+\mathsf{C}_b-1}\Gamma_{q+\bar{n}-1}^{6+\mathsf{C}_b-b(\mathsf{C}_b-1)} = \Gamma_{q+\bar{n}}^{-3i+\mathsf{C}_b-1} \tag{12.76}$$

where in the last equality we have used the definition of C_b in (5.13). Clearly, the first term on the right side of (12.75) is also bounded by the right side of (12.76). We are left to estimate the terms appearing in the sum on the right side of (12.75). The key fact is that for any $j \leq j_*(i) - 2$ we have that $i \geq i_*(j) + 1$; this can be seen to hold because $b < 2$. Recalling (12.20), for $j \leq j_*(i) - 2$ we have that

$$\begin{aligned}\operatorname{supp}(\psi_{j,q+\bar{n}-1}\psi_{m,i,j,q+\bar{n}}) &\subseteq \left\{(x,t) \in \operatorname{supp}(\psi_{j,q+\bar{n}-1})\colon h_{m,j,q+\bar{n}}^3 \geq \frac{1}{8}\Gamma_{q+\bar{n}}^{3(m+1)(i-i_*(j))}\right\} \\ &\subseteq \left\{(x,t)\colon \psi_{j\pm,q+\bar{n}-1}^6 h_{m,j,q+\bar{n}}^3 \geq \frac{1}{8}\Gamma_{q+\bar{n}}^{3(m+1)(i-i_*(j))}\right\}. \end{aligned} \tag{12.77}$$

In the second inclusion of (12.77) we have appealed to (5.8) at level $q + \bar{n} - 1$. By Chebyshev's inequality and the definition of $h_{m,j,q+\bar{n}}$ in (12.4) we deduce that

$$\begin{aligned}|\operatorname{supp}(\psi_{j,q+\bar{n}-1}\psi_{m,i,j,q+\bar{n}})| \leq (2\mathsf{N}_{\mathrm{cut,x}})^3\Gamma_{q+\bar{n}}^{-3(m+1)(i-i_*(j))} &\sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}} \Gamma_{q+\bar{n}}^{-3i_*(j)}\delta_{q+\bar{n}}^{-3/2} r_q(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-3n} \\ &\times \left(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{i_*(j)+2}\right)^{-3m} \left\|\psi_{j\pm,q+\bar{n}-1}D^n D_{t,q+\bar{n}-1}^m \widehat{w}_{q+\bar{n}}\right\|_3^3 .\end{aligned}$$

Since in the above display we have that $m \leq \mathsf{N}_{\mathrm{cut,t}} \leq \mathsf{N}_{\mathrm{ind,t}}$ from (4.18), we may combine the above estimate with (9.83a) to deduce that

$$\begin{aligned}|\operatorname{supp}(\psi_{j,q+\bar{n}-1}\psi_{m,i,j,q+\bar{n}})| &\leq 8\mathsf{N}_{\mathrm{cut,x}}^4\Gamma_{q+\bar{n}}^{-3(m+1)(i-i_*(j))}\Gamma_{q+\bar{n}}^{-3i_*(j)}\Gamma_q^{60}\left(\Gamma_{q+\bar{n}-1}^{j-1}\Gamma_{q+\bar{n}}^{-i_*(j)-2}\right)^{3m} \\ &\leq 8\mathsf{N}_{\mathrm{cut,x}}^4\Gamma_q^{60}\Gamma_{q+\bar{n}}^{-3i} \\ &\leq \Gamma_{q+\bar{n}}^{-3i+\mathsf{C}_b-1}. \end{aligned} \tag{12.78}$$

We have used here that $\Gamma_{q+\bar{n}-1}^{j} \leq \Gamma_{q+\bar{n}}^{i_*(j)}$, that $m \geq 0$, and that $\mathsf{C}_b \geq 62$ since $b \leq {}^{25}\!/_{24}$ from

(ii).

Combining (12.73), (12.75), (12.76), and (12.78) we deduce that

$$\|\psi_{i,q+\bar{n}}\|_1 \leq \mathsf{N}_{\mathrm{cut,t}}\, j_*(i)\, \Gamma_{q+1}^{-3i+\mathsf{C}_b-1}\,.$$

In order to conclude the proof of the Lemma, we use that $\mathsf{N}_{\mathrm{cut,t}}$ is a constant independent of q, and that by (12.75) and (5.9) we have

$$j_*(i) \leq i\frac{\log \Gamma_{q+\bar{n}}}{\log \Gamma_{q+\bar{n}-1}} \leq i_{\max}(q+\bar{n}-1)b \leq \frac{\mathsf{C}_\infty + 12}{(b-1)\varepsilon_\Gamma} b\,.$$

Thus $j_*(i)$ is also bounded from above by a constant independent of q, and upon taking a sufficiently large we conclude the proof. □

12.8 Verifying Eqn. (5.14)

The following lemma verifies the inductive assumption (5.14) at level $q' = q + \bar{n}$.

Lemma 12.8.1 (Overlapping and timescales). *Let $q'' \in \{q+1, \ldots, q+\bar{n}\}$. Assume that $\psi_{i,q+\bar{n}}\psi_{i'',q''} \not\equiv 0$. Then it must be the case that $\tau_{q+\bar{n}}\Gamma_{q+\bar{n}}^{-i} \leq \tau_{q''}\Gamma_{q''}^{-i''-25}$.*

Proof of Lemma 12.8.1. We split the proof into two steps. In the first step, we prove the claim for $q'' = q+\bar{n}-1$, while in the second step we prove the claim for the remaining cases.

Step 1: We must prove that if $\psi_{i,q+\bar{n}}\psi_{i'',q+\bar{n}-1} \not\equiv 0$, then $\tau_{q+\bar{n}}\Gamma_{q+\bar{n}}^{-i} \leq \tau_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}^{-i''-25}$. By (12.11), if $\psi_{i,q+\bar{n}}(t,x) \neq 0$, then there exists $\vec{i} = (i_0, \ldots, i_{\mathsf{N}_{\mathrm{cut,t}}})$ such that $\max_m i_m = i$, and $\psi_{m,i_m,q+\bar{n}} \neq 0$ for all $0 \leq i \leq \mathsf{N}_{\mathrm{cut,t}}$. By (12.9) and Definition (12.1.1), for each i_m there exists a corresponding j_m such that $\psi_{j_m,q+\bar{n}-1}(t,x) \neq 0$ and $\Gamma_{q+\bar{n}}^{i_m} \geq \Gamma_{q+\bar{n}-1}^{j_m}$. From (5.8) and (4.10b), it then follows that if $\psi_{m,i_m,q+\bar{n}}\psi_{j',q+\bar{n}-1} \neq 0$, then

$$\tau_{q+\bar{n}}\Gamma_{q+\bar{n}}^{-i_m} \leq \tau_{q+\bar{n}-1}\Gamma_{q+\bar{n}-1}^{-j'-40}\,.$$

Then (12.11) gives that if $\psi_{i,q+\bar{n}}\psi_{i'',q+\bar{n}-1} \not\equiv 0$,

$$\tau_{q+\bar{n}}\Gamma^{-i}_{q+\bar{n}} \leq \tau_{q+\bar{n}-1}\Gamma^{-i''-30}_{q+\bar{n}-1} \, .$$

Step 2: Suppose that $q'' \leq q + \bar{n} - 2$ and that $\psi_{i,q+\bar{n}}(t,x)\psi_{i'',q''}(t,x) \neq 0$. Then from (5.8), there exists j such that $\psi_{i,q+\bar{n}}(t,x)\psi_{j,q+\bar{n}-1}(t,x)\psi_{i'',q''}(t,x) \neq 0$. Applying the result of Step 1 in combination with the inductive assumption (5.14) concludes the proof. $\square$

Chapter 13

Pressure increment

13.1 New pressure increment and new anticipated pressure

We collect the pressure increments generated by new errors and new velocity increment potentials. Recall that Lemma 9.4.4 defined a pressure increment (σ_υ) associated to velocity increment potentials, Lemmas 10.2.4, 10.2.8, and 10.2.12 defined pressure increments ($\sigma_{S_O^m}$, $\sigma_{S_{TN}}$, and $\sigma_{S_C^m}$, respectively) associated to various stress errors, and Lemmas 11.2.1, 11.2.3, 11.2.4, 11.2.5, and 11.2.10 and Remark 11.2.6 defined pressure increments ($\sigma_{\overline{\phi}_O^m}$, $\sigma_{\overline{\phi}_{TN}^m}$, and $\sigma_{\overline{\phi}_C^M}$) associated to various current errors. Then fixing m such that $q + \bar{n}/2 + 1 \leq m \leq q + \bar{n}$, we define

$$\sigma_{m,q+1} := \sigma_{S_O^m} + \sigma_{S_C^m} + \sigma_{\overline{\phi}_O^m} + \sigma_{\overline{\phi}_{TN}^m} + \sigma_{\overline{\phi}_C^m} + \mathbf{1}_{\{m=q+\bar{n}\}} \left(\sigma_{S_{TN}} + \sigma_\upsilon\right) . \tag{13.1}$$

Recalling that every pressure increment referenced above has a decomposition $\sigma_\bullet = \sigma_\bullet^+ - \sigma_\bullet^-$, we define $\sigma_{m,q+1}^+$ and $\sigma_{m,q+1}^-$ in the obvious way.

Next, associated to each pressure increment $\sigma_\bullet$ listed above is a function of time $\mathfrak{m}_{\sigma_\bullet}$ which satisfies $\mathfrak{m}'_{\sigma_\bullet} = \langle D_{t,q}\sigma_\bullet \rangle$ (see Lemmas 9.4.4, 10.2.4, 10.2.8, 10.2.12, 11.2.2, 11.2.7, and

11.2.11), and so we define

$$\mathfrak{m}_{m,q+1} := \mathfrak{m}_{\sigma_{S_O^m}} + \mathfrak{m}_{\sigma_{S_C^m}} + \mathfrak{m}_{\sigma_{\overline{\phi}_O^m}} + \mathfrak{m}_{\sigma_{\overline{\phi}_{TN}^m}} + \mathfrak{m}_{\overline{\phi}_C^m} + \mathbf{1}_{\{m=q+\bar{n}\}} \left(\mathfrak{m}_{\sigma_{S_{TN}}} + \mathfrak{m}_{\sigma_v} \right) . \tag{13.2}$$

Furthermore, recall that Lemma 9.4.6 defined a current error associated to velocity pressure increments, Lemmas 10.2.5, 10.2.9, and 10.2.13 defined current errors associated to various stress error pressure increments, and Lemmas 11.2.2, 11.2.7, and 11.2.11 defined current errors associated to various current error pressure increments. Then fixing m, m' such that $q + {}^{\bar{n}}/_2 + 1 \leq m' \leq m \leq q + \bar{n}$, we define

$$\phi_{m,q+1}^{m',l} := \phi_{S_O^m}^{m',l} + \phi_{S_C^m}^{m',l} + \phi_{\overline{\phi}_O^m}^{m',l} + \phi_{\overline{\phi}_{TN}^m}^{m',l} + \phi_{\overline{\phi}_C^m}^{m',l} + \mathbf{1}_{\{m=q+\bar{n}\}} \left(\phi_{S_{TN}}^{m',l} + \phi_{\sigma_v}^{m',l} \right) \tag{13.3a}$$

$$\begin{aligned} \phi_{m,q+1}^{m',*} := {} & \phi_{S_O^m}^{m',*} + \phi_{S_C^m}^{m',*} + \phi_{\overline{\phi}_O^m}^{m',*} + \phi_{\overline{\phi}_{TN}^m}^{m',*} + \phi_{\overline{\phi}_C^m}^{m',*} + \mathbf{1}_{\{m=q+\bar{n}\}} \left(\phi_{S_{TN}}^{m',*} + \phi_v^{m',*} \right) \\ & + \mathbf{1}_{\{m'=m\}} \Bigg(\phi_{S_O^m}^{*} + \phi_{S_C^m}^{*} + \phi_{\overline{\phi}_O^m}^{*} + \phi_{\overline{\phi}_{TN}^m}^{*} + \phi_{\overline{\phi}_C^m}^{*} + \mathbf{1}_{\{m=q+\bar{n}\}} \left(\phi_{S_{TN}}^{*} + \phi_v^{*} \right) \Bigg) . \end{aligned} \tag{13.3b}$$

Now we set

$$\phi_{m,q+1} := \sum_{m'=q+{}^{\bar{n}}/_2+1}^{m} \phi_{m,q+1}^{m',l} + \phi_{m,q+1}^{m',*} , \tag{13.4}$$

so that the aforementioned lemmas give the equality

$$\mathrm{div}\phi_{m,q+1} = D_{t,q}\sigma_{m,q+1} - \mathfrak{m}'_{m,q+1} = D_{t,q}\sigma_{m,q+1} - \langle D_{t,q}\sigma_{q+1,m} \rangle . \tag{13.5}$$

By appealing to the lemmas mentioned above, we have that the $\sigma_{m,q+1}$'s satisfy the properties listed in the following lemma; we refer the reader to Sections 9, 10, and 11 for more details.

Lemma 13.1.1 (Collected properties of error terms and pressure increments). *For each $q + {}^{\bar{n}}/_2 + 1 \leq m \leq q + \bar{n}$, $\sigma_{m,q+1}$ satisfies the following properties.*

(i) For any $0 \le k \le \mathsf{d}$, we have that

$$\left|\psi_{i,q} D^N D^M_{t,q} S^{m,l}_{q+1}\right| \lesssim \left(\sigma^+_{m,q+1} + \delta_{q+3\bar{n}}\right)(\lambda_m \Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+18}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{13.6a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \bar{\phi}^{m,l}_{q+1}\right| \lesssim \left(\sigma^+_{m,q+1} + \mathbf{1}_{m=q+\bar{n}}\Gamma^{-50}_{q+\bar{n}}\pi_q^{q+\bar{n}} + \delta_{q+3\bar{n}}\right)^{3/2} r_m^{-1} \times (\lambda_m \Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+18}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{13.6b}$$

where the first bound holds for $N+M \le 2\mathsf{N}_{\mathrm{ind}}$, and the second bound holds for $N+M \le \mathsf{N}_{\mathrm{ind}}/4$.

(ii) For $N, M \le \mathsf{N}_{\mathrm{fin}}/200$, we have that

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{m,q+1}\right\|_{3/2} \lesssim \Gamma_m^{-9}\delta_{m+\bar{n}}(\lambda_m\Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+18}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{13.7a}$$

$$\left\|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{m,q+1}\right\|_{\infty} \lesssim \Gamma_m^{\mathsf{C}_\infty - 9}(\lambda_m\Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+18}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{13.7b}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^+_{m,q+1}\right| \lesssim \left(\sigma^+_{m,q+1} + \delta_{q+3\bar{n}}\right)(\lambda_m \Gamma_m)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+18}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) \tag{13.7c}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \sigma^-_{m,q+1}\right| \lesssim \Gamma^{-100}_{q+\bar{n}/2}\pi_q^{q+\bar{n}/2}(\lambda_{q+\bar{n}/2}\Gamma_{q+\bar{n}/2})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+18}\tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right). \tag{13.7d}$$

(iii) $\sigma_{m,q+1}$ and $\sigma^+_{m,q+1}$ have the support properties

$$B(\operatorname{supp}\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}) \cap \sigma_{m,q+1} = \emptyset \qquad \forall q+1 \le q' \le q + \bar{n}/2\,, \tag{13.8a}$$

$$B(\operatorname{supp}\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}) \cap \sigma^+_{m,q+1} = \emptyset \qquad \forall q+1 \le q' \le m-1\,. \tag{13.8b}$$

Remark 13.1.2 (Upgrading material derivatives). As a consequence of (13.7a), (13.7b), (13.7c), and (13.8b), we may apply Lemma A.5.1 to $F = F^l = \sigma^{\pm}_{m,q+1}$ to upgrade the material

derivative estimates. In particular, we obtain that

$$\left\|\psi_{i,m-1}D^N D^M_{t,m-1}\sigma^+_{m,q+1}\right\|_{3/2} \lesssim \Gamma_m^{-9}\delta_{m+\bar{n}}(\lambda_m\Gamma_m)^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{m-1}^{i-5}\tau_{m-1}^{-1},\mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^{-1}\right) \tag{13.9a}$$

$$\left\|\psi_{i,m-1}D^N D^M_{t,m-1}\sigma^+_{m,q+1}\right\|_{\infty} \lesssim \Gamma_m^{\mathsf{C}_\infty-9}(\lambda_m\Gamma_m)^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{m-1}^{i-5}\tau_{m-1}^{-1},\mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^{-1}\right) \tag{13.9b}$$

$$\left|\psi_{i,m-1}D^N D^M_{t,m-1}\sigma^+_{m,q+1}\right| \lesssim \left(\sigma^+_{m,q+1}+\delta_{q+3\bar{n}}\right)(\lambda_m\Gamma_m)^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{m-1}^{i-5}\tau_{m-1}^{-1},\mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^{-1}\right) \tag{13.9c}$$

$$\begin{aligned}\left|\psi_{i,q+\bar{n}/2-1}D^N D^M_{t,q+\bar{n}/2-1}\sigma^-_{m,q+1}\right| &\lesssim \Gamma_{q+\bar{n}/2}^{-100}\pi_q^{q+\bar{n}/2}(\lambda_{q+\bar{n}/2}\Gamma_{q+\bar{n}/2})^N \\ &\quad\times\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{q+\bar{n}/2-1}^{i-5}\tau_{q+\bar{n}/2-1}^{-1},\mathrm{T}_{q+\bar{n}/2-1}^{-1}\Gamma_{q+\bar{n}/2-1}^{-1}\right).\end{aligned} \tag{13.9d}$$

for $N, M \leq \mathsf{N}_{\mathrm{fin}}/200$. Then, applying Lemma A.2.3 to $v = \widehat{u}_{m-1}$, $f = \sigma^{\pm}_{m,q+1}$, $p = \infty$, and $\Omega = \mathrm{supp}\,(\psi_{i,m-1})$ (or $\Omega = \mathcal{N}(x)$ where $\mathcal{N}(x)$ is a closed neighborhood of x contained in $\mathrm{supp}\,(\psi_{i,m-1})$), we have

$$\left\|D^N D^M_{t,m-1}\nabla\sigma^+_{m,q+1}\right\|_{L^\infty(\mathrm{supp}\,\psi_{i,m-1})} \lesssim \Gamma_m^{\mathsf{C}_\infty-9}(\lambda_m\Gamma_m)^{N+1}\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{m-1}^{i-5}\tau_{m-1}^{-1},\mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^{-1}\right) \tag{13.10a}$$

$$\begin{aligned}\left|\psi_{i,m-1}D^N D^M_{t,m-1}\nabla\sigma^+_{m,q+1}\right| &\lesssim \left(\sigma^+_{m,q+1}+\delta_{q+3\bar{n}}\right)(\lambda_m\Gamma_m)^{N+1} \\ &\quad\times\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{m-1}^{i-5}\tau_{m-1}^{-1},\mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^{-1}\right)\end{aligned} \tag{13.10b}$$

$$\begin{aligned}\left|\psi_{i,q+\bar{n}/2-1}D^N D^M_{t,q+\bar{n}/2-1}\nabla\sigma^-_{m,q+1}\right| &\lesssim \Gamma_{q+\bar{n}/2}^{-100}\pi_q^{q+\bar{n}/2}(\lambda_{q+\bar{n}/2}\Gamma_{q+\bar{n}/2})^{N+1} \\ &\quad\times\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\Gamma_{q+\bar{n}/2-1}^{i-5}\tau_{q+\bar{n}/2-1}^{-1},\mathrm{T}_{q+\bar{n}/2-1}^{-1}\Gamma_{q+\bar{n}/2-1}^{-1}\right),\end{aligned} \tag{13.10c}$$

for $N < \mathsf{N}_{\mathrm{fin}}/200$ and $M \leq \mathsf{N}_{\mathrm{fin}}/200$.

Definition 13.1.3 (Pressure increment σ_{q+1} and decomposition into σ^k_{q+1}). *Define constants*

$$a_{m,q,k} := 2^{k-q-1}\left(\frac{\delta_{k+\bar{n}}}{\delta_{m+\bar{n}}}\right)\Gamma_m^9, \quad A_{m,q} = \sum_{k=m}^{q+\mathsf{N}_{\mathrm{pr}}} a_{m,q,k}\,, \tag{13.11}$$

where N_{pr} *is chosen in item* (x)*. With these constants in hand, we define*

$$\sigma_{q+1} := \sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} \underbrace{A_{m,q}\sigma_{m,q+1}}_{=:\widetilde{\sigma}_{m,q+1}}, \qquad \sigma_{q+1}^{\pm} := \sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} \underbrace{A_{m,q}\sigma_{m,q+1}^{\pm}}_{=:\widetilde{\sigma}_{m,q+1}^{\pm}} . \tag{13.12}$$

Then reorganizing terms in σ_{q+1} *based on amplitude, have that*

$$\sigma_{q+1} = \sum_{k=q+\bar{n}/2}^{q+\mathsf{N}_{\mathrm{pr}}} \sigma_{q+1}^{k} , \tag{13.13}$$

where

$$\sigma_{q+1}^{q+\bar{n}/2} = -\sigma_{q+1}^{-} , \qquad \sigma_{q+1}^{k} = \sum_{m=q+\bar{n}/2+1}^{\min(k,q+\bar{n})} a_{m,q,k}\sigma_{m,q+1}^{+}, \quad \textit{for all } q+\bar{n}/2+1 \le k \le q+\mathsf{N}_{\mathrm{pr}} . \tag{13.14}$$

As a direct consequence of Lemma 13.1.1 and Definition 13.1.3, we have that σ_{q+1}^{k} satisfies the following properties.

Lemma 13.1.4 (Properties of σ_{q+1} and σ_{q+1}^{k}). *For all* $q+\bar{n}/2 \le k \le q+\mathsf{N}_{\mathrm{pr}}$*, the pressure increment* σ_{q+1}^{k} *has the following properties.*

(i) σ_{q+1}^{k} *has the support property*

$$B(\operatorname{supp} \widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'+1}) \cap \operatorname{supp}(\sigma_{q+1}^{k}) = \emptyset \qquad \forall q+1 \le q' \le q+\bar{n}/2 . \tag{13.15}$$

(ii) For all $q+\bar{n}/2+1 \le k \le q+\bar{n}$ *and* $N, M \le \mathsf{N}_{\mathrm{fin}}/200$*, we have that* σ_{q+1}^{k} *satisfies*

$$\left\|\psi_{i,k-1} D^{N} D_{t,k-1}^{M} \sigma_{q+1}^{k}\right\|_{3/2} \lesssim \delta_{k+\bar{n}} (\lambda_k \Gamma_k)^{N} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i-3}\tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}^{-1}\right) , \tag{13.16a}$$

$$\left\|\psi_{i,k-1} D^{N} D_{t,k-1}^{M} \sigma_{q+1}^{k}\right\|_{\infty} \lesssim \Gamma_{k}^{\mathsf{C}_{\infty}} (\lambda_k \Gamma_k)^{N} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i-3}\tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}^{-1}\right) . \tag{13.16b}$$

For all $q+\bar{n}+1 \le k \le q+\mathsf{N}_{\mathrm{pr}}$ *and* $N, M \le {}^{\mathsf{N}_{\mathrm{fin}}}/_{200}$*, we have that* σ^k_{q+1} *satisfies*

$$\left\|\psi_{i,q+\bar{n}} D^N D^M_{t,q+\bar{n}} \sigma^k_{q+1}\right\|_{3/2} \lesssim \delta_{k+\bar{n}} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i-3}_{q+\bar{n}}\tau^{-1}_{q+\bar{n}}, \mathrm{T}^{-1}_{q+\bar{n}}\Gamma^{-1}_{q+\bar{n}}\right) \tag{13.17a}$$

$$\left\|\psi_{i,q+\bar{n}} D^N D^M_{t,q+\bar{n}} \sigma^k_{q+1}\right\|_{\infty} \lesssim \Gamma^{\mathsf{C}_\infty}_{q+\bar{n}} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i-3}_{q+\bar{n}}\tau^{-1}_{q+\bar{n}}, \mathrm{T}^{-1}_{q+\bar{n}}\Gamma^{-1}_{q+\bar{n}}\right) . \tag{13.17b}$$

(iii) For $q+{}^{\bar{n}}/_2+1 \le k \le q+\bar{n}$ *and* $0 \le k' \le \mathsf{d}$*, we have that*

$$\left|\psi_{i,q} D^N D^M_{t,q} S^k_{q+1}\right| \le \Gamma_k^{-8}\left(\sigma^k_{q+1}+\delta_{q+2\bar{n}}\right)(\lambda_k\Gamma_k)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+18}_q \tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{13.18a}$$

$$\left|\psi_{i,q} D^N D^M_{t,q} \bar{\phi}^k_{q+1}\right| \le \Gamma_k^{-13}\left(\sigma^k_{q+1}+\mathbf{1}_{m=q+\bar{n}}\Gamma^{-50}_{q+\bar{n}}\pi^{q+\bar{n}}_q+\delta_{q+2\bar{n}}\right)^{3/2} r_k^{-1} \times (\lambda_k\Gamma_k)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+18}_q \tau_q^{-1}, \mathrm{T}_q^{-1}\Gamma_q^9\right) , \tag{13.18b}$$

where the first bound holds for $N+M \le 2\mathsf{N}_{\mathrm{ind}}$*, and the second bound holds for* $N+M \le {}^{\mathsf{N}_{\mathrm{ind}}}/_4$*.*

(iv) For $q+{}^{\bar{n}}/_2+1 \le k \le q+\mathsf{N}_{\mathrm{pr}}$ *and* $N, M \le {}^{\mathsf{N}_{\mathrm{fin}}}/_{200}$ *and* $q+1 \le k' \le \min(k-1, q+\bar{n})$*, we have that*

$$\left|\psi_{i,k'} D^N D^M_{t,k'} \sigma^k_{q+1}\right| \lesssim (\sigma^k_{q+1}+\Gamma_q^{-100}\delta_{k+\bar{n}})(\min(\lambda_k\Gamma_k, \lambda_{q+\bar{n}}\Gamma_{q+\bar{n}}))^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i-3}_{k'}\tau^{-1}_{k'}, \mathrm{T}^{-1}_{k'}\Gamma^{-1}_{k'}\right) . \tag{13.19}$$

For the same range of N *and* M*, we have that*

$$\left|\psi_{i,q+{}^{\bar{n}}/_2-1} D^N D^M_{t,q+{}^{\bar{n}}/_2-1} \sigma^{q+{}^{\bar{n}}/_2}_{q+1}\right| \le \Gamma^{-25}_{q+{}^{\bar{n}}/_2}\pi^{q+{}^{\bar{n}}/_2}_q (\lambda_{q+{}^{\bar{n}}/_2}\Gamma_{q+{}^{\bar{n}}/_2})^N \tag{13.20}$$

$$\times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i-3}_{q+{}^{\bar{n}}/_2-1}\tau^{-1}_{q+{}^{\bar{n}}/_2-1}, \mathrm{T}^{-1}_{q+{}^{\bar{n}}/_2-1}\Gamma^{-1}_{q+{}^{\bar{n}}/_2-1}\right) . \tag{13.21}$$

(v) For all $q + {}^{\bar{n}}/_{2} + 1 \leq k \leq k' \leq q + \mathsf{N}_{\mathrm{pr}}$, we have that

$$\frac{\delta_{k'+\bar{n}}}{\delta_{k+\bar{n}}} \sigma_{q+1}^{k} \leq 2^{k-k'} \sigma_{q+1}^{k'} \,. \tag{13.22}$$

(vi) For all $q + \bar{n} \leq k' \leq k \leq q + \mathsf{N}_{\mathrm{pr}}$, we have that

$$\sigma_{q+1}^{k} \leq \sigma_{q+1}^{k'}. \tag{13.23}$$

Proof of Lemma 13.1.4. `Proof of item` (i)`.` The proof of this item is immediate from Definition 13.1.3 and item (iii) from Lemma 13.1.1.

`Proof of item` (ii)`.` We first consider the estimates for $q + {}^{\bar{n}}/_{2} \leq k \leq q + \bar{n}$. From Remark 13.1.2, which ensures that every $\sigma_{m,q+1}^{+}$ has size $\Gamma_m^{-9}\delta_{m+\bar{n}}$ in $L^{3/2}$, and Definition 13.1.3, which ensures that the term in σ_{q+1}^{k} coming from $\sigma_{m,q+1}$ has been rescaled by a factor of $\delta_{k+\bar{n}}\delta_{m+\bar{n}}^{-1}\Gamma_m^{-9}$, we have that (13.16a) holds when $N = M = 0$. Similarly, when $N = M = 0$, we have that (13.16b) holds since $\Gamma_k^{\mathsf{C}_\infty}$ is increasing in k. In order to prove the versions of these estimates which involve derivatives, we must use Remark A.2.6 and eqn. (5.14) (at level q since we do not require $D_{t,q+\bar{n}-1}$) to upgrade the estimates in Remark 13.1.2, since σ_{q+1}^{k} is comprised of rescaled versions of $\sigma_{m,q+1}$ for $m \leq k$ which came with $D_{t,m-1}$ estimates. We omit further details and simply note that the material derivative cost and the assumptions required in (A.34) follow from (5.34) at level q (i.e. we apply (5.34) for $q' \leq q + \bar{n} - 1$), and that the pointwise bounds follow from the usual trick of choosing Ω to be a neighborhood centered at a point (x, t) and then shrinking the diameter of Ω to zero and using continuity. Finally, the proofs of the estimates for $q + \bar{n} + 1 \leq k \leq q + \mathsf{N}_{\mathrm{pr}}$ are quite similar, except that we have to use (5.14) at level $q + 1$ (which has been verified in Lemma 12.8.1), and (5.34) at level $q + 1$ (i.e. $q' = q + \bar{n}$), which has been verified in Lemma 12.5.3.

`Proof of item` (iii)`.` To obtain (iii), we use (13.6a), (13.6b), (13.11), and (13.12) to give the inequality $\sigma_{q+1}^{k} \geq \sigma_{k,q+1}^{+} 2^{k-q-1}\Gamma_k^{9}$, nonlocal estimates for $S_{q+1}^{k,*}$ from Lemma 10.3.2, nonlocal

estimates for current errors from Section 11, and spare factors of $\Gamma_k^{-1/2}$ to absorb implicit constants.

`Proof of item` (iv). In order to prove (13.19), we first prove the estimate when no derivatives have been applied. First note that $\delta_{q+3\bar{n}}\delta_{k+\bar{n}}\delta_{m+\bar{n}}^{-1}\Gamma_m^9 \leq \delta_{k+\bar{n}}\Gamma_q^{-100}$, since $m \leq q+\bar{n}$ so that the definition of $\delta_{q'}$ and (4.10k) can absorb Γ_q^{-500}. Then since both sides are linear in σ_{q+1}^k, the rescalings involved in the definition of σ_{q+1}^k, (13.9c), and the inequality just noted give the proper amplitude bound. At this point we must upgrade material derivative in a manner analogous to that which is required to prove the $L^{3/2}$ and L^∞ bounds from item (ii), and so omit further details. In order to prove (13.21), first note that $a_{m,q,k}$ is at most Γ_m^{10} if $k=m$ and we choose a_0 sufficiently large, and $a_{m,k,q} \ll 1$ if $k > m$. Then using (13.14), (13.12), and the fact that $\Gamma_q^2 > \Gamma_{q+\bar{n}}$ since $b^{\bar{n}} < 2$ from (4.2b), the estimate without derivatives follows from (13.9d). Upgrading material derivatives then follows in the usual way, and we omit further details.

`Proof of item` (v). Since $k' \geq k$, we have from (13.14) that

$$\begin{aligned}\frac{\delta_{k'+\bar{n}}}{\delta_{k+\bar{n}}}\sigma_{q+1}^k &= \sum_{m=q+\bar{n}/2+1}^{\min(k,q+\bar{n})} \frac{\delta_{k'+\bar{n}}}{\delta_{k+\bar{n}}} a_{m,q,k}\sigma_{m,q+1}^+ = \sum_{m=q+\bar{n}/2+1}^{\min(k,q+\bar{n})} 2^{k-q-1}\frac{\delta_{k'+\bar{n}}}{\delta_{m+\bar{n}}}\Gamma_m^9\sigma_{m,q+1}^+ \\ &= 2^{k-k'}\sum_{m=q+\bar{n}/2+1}^{\min(k,q+\bar{n})} a_{m,q,k'}\sigma_{m,q+1}^+ \leq 2^{k-k'}\sigma_{q+1}^{k'}\,.\end{aligned}$$

`Proof of item` (vi). This estimate follows from the observation in the proof of the previous item that $a_{m,k,q} \ll 1$ if $k > m$, (13.13), (13.14), and a large choice of a_0 which can be used to absorb implicit constants. □

Finally, we can define the new pressure π_{q+1}.

Definition 13.1.5 (New pressure π_{q+1} and decomposition into π_{q+1}^k). *We define π_{q+1}^k,*

$k \geq q+1$, *by*

$$\pi^k_{q+1} := \pi^k_q + \sigma^k_{q+1} + 2^{k-q-1}\delta_{k+\bar{n}} \qquad \text{for } q + \bar{n}/2 \leq k \leq q + \mathsf{N}_{\mathrm{pr}}\,, \tag{13.24}$$
$$\pi^k_{q+1} := \pi^k_q \qquad \text{for } q+1 \leq k \leq q + \bar{n}/2 - 1,\ q + \mathsf{N}_{\mathrm{pr}} + 1 \leq k < \infty\,.$$

Then $\pi_{q+1} = \sum_{k=q+1}^{\infty} \pi^k_{q+1}$ *satisfies*

$$\pi_{q+1} = \pi_q - \pi^q_q + \sigma_{q+1} + \sum_{k=q+\bar{n}/2}^{q+\mathsf{N}_{\mathrm{pr}}} 2^{k-q-1}\delta_{k+\bar{n}}\,. \tag{13.25}$$

13.2 Inductive assumptions on the new pressure

In this section, we verify the inductive assumptions on π^k_{q+1} which are required in subsections 5.3–5.5.

Lemma 13.2.1 ($L^{3/2}$, L^∞, **and pointwise bounds on** π^k_{q+1}**).** *The inductive assumptions* (5.15) *and* (5.16) *are verified at step* $q+1$.

Proof. We first consider (5.15a)–(5.15c). In the case that $q+1 \leq k \leq q+\bar{n}/2-1$, we have that $\pi^k_{q+1} = \pi^k_q$ from Definition 13.1.5, so that the desired estimates follow trivially from inductive assumptions (5.15a)–(5.15c) at step q. In the case that $q + \bar{n}/2 \leq k \leq q + \bar{n}$, we have from Definition 13.1.5 that $\pi^k_{q+1} = \pi^k_q + \sigma^k_{q+1} + 2^{k-q-1}\delta_{k+\bar{n}}$. Therefore, the desired estimates follow from the inductive assumptions and Lemma 13.1.4. In order to get (5.16a)–(5.16c), we have from Definition 13.1.5 that $\pi^k_{q+1} = \pi^k_q + \sigma^k_{q+1} + 2^{k-q-1}\delta_{k+\bar{n}}$. Then the desired estimates follow again from the inductive assumptions and Lemma 13.1.4. □

Lemma 13.2.2 (**Lower and upper bounds for** π^k_{q+1}**).** *Inductive assumptions* (5.17)–(5.20) *are verified at step* $q+1$.

Proof. In order to prove (5.17) at level $q+1$, we first consider the cases when $q+1 \leq k \leq$

$q + \bar{n}/2 - 1$. In these cases the inductive assumption (5.17) and Definition 13.1.5 imply that

$$\pi_{q+1}^k = \pi_q^k \geq \delta_{k+\bar{n}} \,.$$

For the case $k = q + \bar{n}/2$, we use (13.21) and (13.24) to write that

$$\pi_{q+1}^{q+\bar{n}/2} = \pi_q^{q+\bar{n}/2} + \sigma_{q+1}^{q+\bar{n}/2} + 2^{\bar{n}/2-1}\delta_{q+\bar{n}/2+\bar{n}} \geq \delta_{q+\bar{n}/2+\bar{n}} \,,$$

concluding the proof of (5.17) at level $q+1$. For the remaining cases, we use (5.20) at the level of $q+1$, so that we postpone the proof to the end.

Next, from Definition 13.1.5, the inductive assumption (5.18), and (13.23), we have that for $q + \bar{n} \leq k' < k < q + \mathsf{N}_{\mathrm{pr}}$,

$$\pi_{q+1}^k = \pi_q^k + \sigma_{q+1}^k + 2^{k-q-1}\delta_{k+\bar{n}} \leq \pi_q^{k'} + \sigma_{q+1}^{k'} + \delta_{k'+\bar{n}} \leq \pi_{q+1}^{k'} \,.$$

In the endpoint case when $k = q + \mathsf{N}_{\mathrm{pr}}$, we use that $\pi_q^{k+\mathsf{N}_{\mathrm{pr}}} \equiv \Gamma_{q+\mathsf{N}_{\mathrm{pr}}}\delta_{q+\mathsf{N}_{\mathrm{pr}}+\bar{n}}$ from (5.19), in which case a similar string of inequalities then concludes the proof that (5.18) is satisfied at level $q+1$.

From Definition 13.1.5 and (5.19) at level q, we have that

$$\pi_{q+1}^k = \Gamma_k \delta_{k+\bar{n}}$$

for $k \geq q + \mathsf{N}_{\mathrm{pr}} + 1$, so that the inductive assumption (5.19) for $q+1$ holds true.

Finally, we must prove (5.20) at level $q+1$. We split into cases depending on the value of q''. If $q'' \geq q + \mathsf{N}_{\mathrm{pr}} + 1$, then we have from Definition 13.1.5, the Sobolev inequality applied

to (5.15a), (5.16a), and (13.16a), and (4.16) that

$$\begin{aligned}
\frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\pi_{q+1}^{q'} &\leq \frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\left(\pi_q^{q'} + \mathbf{1}_{\{q+\bar{n}/2\leq q'\leq q+\mathsf{N}_{\mathrm{pr}}\}}\max(0,\sigma_{q+1}^{q'}) + 2^{\mathsf{N}_{\mathrm{pr}}}\delta_{q'+\bar{n}}\right)\\
&\leq \frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\left(\|\pi_q^{q'}\|_\infty + \mathbf{1}_{\{q+\bar{n}/2\leq q'\leq q+\mathsf{N}_{\mathrm{pr}}\}}\|\sigma_{q+1}^{q'}\|_\infty\right) + 2^{\mathsf{N}_{\mathrm{pr}}}\delta_{q''+\bar{n}}\\
&\leq \delta_{q''+\bar{n}}\left(\Gamma_q\Gamma_{q+\mathsf{N}_{\mathrm{pr}}}\Lambda_{q+\bar{n}}^3 + 2^{\mathsf{N}_{\mathrm{pr}}}\right) \leq \Gamma_q^{-1}\Gamma_{q+\mathsf{N}_{\mathrm{pr}}+1}\delta_{q''+\bar{n}}\\
&\leq 2^{q'-q''}\Gamma_{q''}\delta_{q''+\bar{n}} = 2^{q'-q''}\pi_{q+1}^{q''}\,.
\end{aligned}$$

Note that in the inequalities above, we have assumed a large choice of a_0 to absorb the implicit constant. Next, in the cases when $q+1+\bar{n}/2 \leq q'' \leq q+\mathsf{N}_{\mathrm{pr}}$, we first note that $\sigma_{q+1}^{q''} \geq 0$ and $\sigma_{q+1}^{q+\bar{n}/2} \leq 0$ since all the minus portions of the pressure have been absorbed into $\sigma_{q+1}^{q+\bar{n}/2}$ in (13.13) and (13.14). As a consequence of these facts, Definition 13.1.5, (5.20), and (13.22), we have that

$$\begin{aligned}
\frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\pi_{q+1}^{q'} &\leq \frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\left(\pi_q^{q'} + \mathbf{1}_{\{q+\bar{n}/2\leq q'\leq q+\mathsf{N}_{\mathrm{pr}}\}}\max\{0,\sigma_{q+1}^{q'}\} + 2^{q'-q-1}\delta_{q'+\bar{n}}\right)\\
&\leq 2^{q'-q''}\left(\pi_q^{q''} + \max\{0,\sigma_{q+1}^{q''}\} + 2^{q''-q-1}\delta_{q''+\bar{n}}\right)\\
&= 2^{q'-q''}\pi_{q+1}^{q''}\,.
\end{aligned}$$

Now for the case $q'' = q+\bar{n}/2$, we only must consider $q' \leq q+\bar{n}/2-1$, and so from Definitions 13.1.3 and 13.1.5 and (13.21),

$$\begin{aligned}
\frac{\delta_{q+\bar{n}/2+\bar{n}}}{\delta_{q'+\bar{n}}}\pi_{q+1}^{q'} &= \frac{\delta_{q+\bar{n}/2+\bar{n}}}{\delta_{q'+\bar{n}}}\pi_q^{q'} \leq 2^{q'-q+\bar{n}/2}\pi_q^{q+\bar{n}/2}\\
&= \left(\pi_q^{q+\bar{n}/2} + \sigma_{q+1}^{q+\bar{n}/2}\right) + \left(\left(2^{q'-q+\bar{n}/2}-1\right)\pi_q^{q+\bar{n}/2} - \sigma_{q+1}^{q+\bar{n}/2}\right)\\
&\leq \pi_{q+1}^{q+\bar{n}/2} + \left(-1/2 + \Gamma_{q+\bar{n}/2}^{-25}\right)\pi_q^{q+\bar{n}/2}\\
&\leq \pi_{q+1}^{q+\bar{n}/2}\,.
\end{aligned}$$

In the final cases $q'' < q+\bar{n}/2$, we have from Definition 13.1.5 and inductive assumption

(5.20) that

$$\frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\pi_{q+1}^{q'} = \frac{\delta_{q''+\bar{n}}}{\delta_{q'+\bar{n}}}\pi_{q}^{q'} \leq \pi_{q}^{q''} = \pi_{q+1}^{q''}\,,$$

concluding the proof of (5.20) at level $q+1$.

Lastly, we consider (5.17) for $k > q + {}^{\bar{n}}/_{2}$. We first note that from (5.17) for $q+1 \leq k \leq q + {}^{\bar{n}}/_{2}$ and (5.20) at level $q+1$, we have that for all $q + {}^{\bar{n}}/_{2} + 1 \leq k' < \infty$,

$$\pi_{q+1}^{k'} > 2^{k'-q-\bar{n}/2}\frac{\delta_{k'+\bar{n}}}{\delta_{q+\bar{n}/2+\bar{n}}}\pi_{q+1}^{q+\bar{n}/2} > \delta_{k'+\bar{n}}\,,$$

and so

$$\pi_{q+1}^{k} \geq \delta_{k+\bar{n}} \qquad \forall q+1 \leq k < \infty\,.$$

□

Lemma 13.2.3 (Pressure dominates velocity). *The inductive assumptions in* (5.21c), (5.23), *and* (5.40) *are verified at level* $q+1$.

Proof. `Step 1: Verification of` (5.40) `at level` $q+1$. From (9.92) and the definition of $\sigma_{q}^{q+\bar{n}}$ in Definition 13.1.3, which give an extra prefactor of $\Gamma_{q+\bar{n}}^{9}$, we have that

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1}D^{N}D_{t,q+\bar{n}-1}^{M}\widehat{\upsilon}_{q+\bar{n},k'}\right| \leq{}& \Gamma_{q+\bar{n}}^{-4}\left(\sigma_{q+1}^{q+\bar{n}}+\delta_{q+2\bar{n}}\right)^{1/2} r_{q}^{-1} \\ &\times (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{N}\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q+\bar{n}-1}^{i}\tau_{q+\bar{n}-1}^{-1}, \Gamma_{q+\bar{n}-1}^{2}\mathrm{T}_{q+\bar{n}-1}^{-1}\right)\end{aligned}$$

for all $N + M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$. Then using the definition of $\pi_{q+1}^{q+\bar{n}}$ from Definition 13.1.5 gives the proof of (5.40) for $q' = q+\bar{n}$; in fact we retain the extra smallness prefactor of $\Gamma_{q+\bar{n}}^{-4}$, which we shall use in the next step of this proof. In order to verify (5.40) for $q \leq q' \leq q+\bar{n}-1$, we appeal to Definition 13.1.5 for the definition of $\pi_{q+1}^{q'}$. Noticing that $\pi_{q+1}^{q'} \geq \pi_{q}^{q'}$ for all $q' \geq q+1$ except for $q' = q + {}^{\bar{n}}/_{2}$, we have that the verification of (5.40) for $q \leq q' \leq q+\bar{n}-1$,

$q' \neq q + {}^{\bar{n}}/_{2}$ is trivial. In the case $q' = q + {}^{\bar{n}}/_{2}$, we use (13.21) and (13.24) to write that

$$\pi_{q+1}^{q+\bar{n}/2} \geq {}^{1}/_{2}\pi_{q}^{q+\bar{n}/2},$$

from which (5.40) follows using the increase from Γ_q to Γ_{q+1} in (5.40) at level q versus level $q+1$, respectively.

`Step 2: Verification of` (5.21c) `at level` $q+1$. We first consider the cases $q+1 \leq k \leq q+\bar{n}-1$. From the same reasoning as above, which showed that $\pi_{q+1}^{k} \geq {}^{1}/_{2}\pi_{q}^{k}$ for $q+1 \leq k \leq q+\bar{n}-1$, we have that (5.21c) trivially holds. In the case $k = q+\bar{n}$, we use (5.38) at level $q' = q+\bar{n}$ (which has been verified in Proposition 9.5.2) and (5.40) for $q' = q+\bar{n}$, $k = \mathsf{d}$ (which we just verified with extra factor gain), (5.41) at level $q' = q+\bar{n}$ (which has been verified in (9.93)), and (13.24) to write that for $N+M \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_{2}$

$$\begin{aligned}
\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} \widehat{w}_{q+\bar{n}}\right| &= \left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1} \left(\widehat{v}_{q+\bar{n},\mathsf{d}} + \widehat{e}_{q+\bar{n}}\right)\right| \\
&\leq \Gamma_{q+\bar{n}}^{-3} \left((\pi_{q+1}^{q+\bar{n}})^{1/2} r_q^{-1} + \delta_{q+3\bar{n}}^{3}\right) (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \\
&\qquad \times \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i}_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \Gamma^{2}_{q+\bar{n}-1}\mathrm{T}^{-1}_{q+\bar{n}-1}\right) \\
&\leq (\pi_{q+1}^{q+\bar{n}})^{1/2} r_q^{-1} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i}_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \Gamma^{2}_{q+\bar{n}-1}\mathrm{T}^{-1}_{q+\bar{n}-1}\right),
\end{aligned}$$

which verifies (5.21c) at level $q+1$ with $q' = q+\bar{n}$.

`Step 3: Verification of` (5.23) `for` $q' = q+\bar{n}$. We will prove that

$$\sum_{i=0}^{i_{\max}} \psi_{i,q+\bar{n}}^2 \delta_{q+\bar{n}} r_q^{-2/3} \Gamma_{q+\bar{n}}^{2i} \lesssim r_q^{-2} \pi_{q+1}^{q+\bar{n}} \tag{13.26}$$

for a q-independent implicit constant, from which (5.23) for $q' = q+\bar{n}$ follows by using the extra factor of $\Gamma_{q+\bar{n}}$ to absorb the implicit constant and the powers of 2. From (12.11) and

the fact that all cutoff functions are bounded in between 0 and 1, we have that

$$\sum_{i=0}^{i_{\max}} \psi^2_{i,q+\bar{n}} \delta_{q+\bar{n}} r_q^{-2/3} \Gamma^{2i}_{q+\bar{n}} \lesssim \delta_{q+\bar{n}} r_q^{-2/3} \sum_{i=0}^{i_{\max}} \Gamma^{2i}_{q+\bar{n}} \sum_{\left\{\vec{i}:\ \max\limits_{0\le m\le \mathsf{N}_{\mathrm{cut,t}}} i_m = i\right\}} \prod_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \psi^2_{m,i_m,q+\bar{n}}$$
$$\le \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \delta_{q+\bar{n}} r_q^{-2/3} \sum_{i_m\ge 0} \psi^2_{m,i_m,q+\bar{n}} \Gamma^{2i_m}_{q+\bar{n}} \,. \tag{13.27}$$

Therefore it will suffice to show that the right-hand side of (13.26) dominates the double sum above. We will in fact fix m, take the sum over $i_m \ge 0$, multiply by $\Gamma_{q+\bar{n}}$, and show that this is dominated by the right-hand side of (13.26). Using that m is bounded by $\mathsf{N}_{\mathrm{cut,t}}$ and choosing a large enough will then conclude the proof.

From the definition of $\psi_{m,i_m,q+\bar{n}}$ in (12.9), we have that

$$\Gamma^{2i_m}_{q+\bar{n}} \psi^2_{m,i_m,q+\bar{n}} \lesssim \Gamma^{2i_m}_{q+\bar{n}} \sum_{\{j_m : i_*(j_m)\le i_m\}} \psi^2_{j_m,q+\bar{n}-1} \psi^2_{m,i_m,j_m,q+\bar{n}}$$
$$= \Gamma^{2i_*(j_m)}_{q+\bar{n}} \psi^2_{j_m,q+\bar{n}-1} \psi^2_{m,i_*(j_m),j_m,q+\bar{n}} + \Gamma^{2i_m}_{q+\bar{n}} \sum_{\{j_m : i_*(j_m) < i_m\}} \psi^2_{j_m,q+\bar{n}-1} \psi^2_{m,i_m,j_m,q+\bar{n}} \,. \tag{13.28}$$

From (12.3), we know that the first term above is dominated by

$$\Gamma^{2j_m+4}_{q+\bar{n}-1} \psi^2_{j_m,q+\bar{n}-1} \,.$$

Since m and i_m only take finitely many values, we may bound the contribution to the right-hand sides of (13.27) and (13.28) from the terms with j_m such that $i_*(j_m) = i_m$ by an implicit constant multiplied by

$$\sum_{j_m\ge 0} \Gamma^{2j_m+4}_{q+\bar{n}-1} \psi^2_{j_m,q+\bar{n}-1} \delta_{q+\bar{n}} r_q^{-2/3} \le r_{q-1}^{-2} \pi_q^{q+\bar{n}-1} \Gamma^5_{q+\bar{n}-1} \frac{\delta_{q+\bar{n}} r_q^{-2/3}}{\delta_{q+\bar{n}-1} r_{q-1}^{-2/3}}$$
$$\le \Gamma^{-2}_{q+\bar{n}} r_q^{-2} \pi_q^{q+\bar{n}} \,.$$

Here we have used the inductive assumption (5.23) to achieve the first inequality above and the inequalities (4.10c) and (5.20) to achieve the second inequality. We have thus concluded that the *lowest terms* with $i_m = i_*(j_m)$ from (13.28), summed over i_m and appropriately weighted, are indeed dominated by the right-hand side of (13.26).

We now must consider the rest of the terms in (13.28), for which $i_*(j_m) < i_m$. Assume that $(t,x) \in \operatorname{supp}(\psi^2_{j_m,q+\bar{n}-1}\psi^2_{m,i_m,j_m,q+\bar{n}})$. By (12.4) and Lemma 8.3.1, item (2), and there exists $n \leq \mathsf{N}_{\text{cut,x}}$ such that

$$\frac{1}{4\mathsf{N}_{\text{cut,x}}} \leq \Gamma_{q+\bar{n}}^{-2i_m(m+1)}\delta_{q+\bar{n}}^{-1}r_q^{2/3}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-2n}(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^2)^{-2m}|D^nD^m_{t,q+\bar{n}-1}\widehat{w}_{q+\bar{n}}|^2\,.$$

Note that due to Definition 12.1.1, the fact that we consider $(t,x) \in \operatorname{supp}(\psi^2_{j_m,q+\bar{n}-1}\psi^2_{m,i_m,j_m,q+\bar{n}})$, and (12.7), which gives $i_m \geq i_*(j_m)$, we have that $\Gamma_{q+\bar{n}}^{-i_m}\Gamma_{q+\bar{n}-1}^{j_m} \leq 1$. Now using (4.18) and that we are on the support of $\psi_{j,q+\bar{n}-1}$ by assumption so that we may appeal to (9.71), we have that

$$\begin{aligned}\Gamma_{q+\bar{n}}^{2i_m}\delta_{q+\bar{n}}r_q^{-2/3} &\lesssim (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{-2n}(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}}^{2+i_m})^{-2m}\left(\sigma_\upsilon^+ + \delta_{q+3\bar{n}}\right)r_q^{-2}(\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^{2n}(\tau_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{j_m-5})^{2m}\\ &\leq \left(\sigma_\upsilon^+ + \delta_{q+3\bar{n}}\right)r_q^{-2}\,. \qquad (13.29)\end{aligned}$$

Note now that from (13.1) and the subsequent sentence, which shows that $\sigma^+_{q+\bar{n},q+1} \geq \sigma^+_\upsilon$, Definition 13.1.3, which shows that $\sigma^{q+\bar{n}}_{q+1} \geq \sigma^+_{q+\bar{n},q+1}$, and Definition 13.1.5, which shows that $\pi^{q+\bar{n}}_{q+1} \geq \sigma^{q+\bar{n}}_{q+1} + \delta_{q+3\bar{n}}$, we have that $\sigma^+_\upsilon + \delta_{q+3\bar{n}} \leq \pi^{q+\bar{n}}_{q+1}$. Thus, (13.26) follows from summing (13.29) over $i_m \geq 0$, from which we find that

$$\sum_{i_m\geq 0}\sum_{\{j_m:i_*(j_m)<i_m\}}\psi^2_{j_m,q+\bar{n}-1}\psi^2_{m,i_m,j_m,q+\bar{n}}\Gamma_{q+\bar{n}}^{2i_m}\delta_{q+\bar{n}}r_q^{-2/3} \lesssim r_q^{-2}\pi^{q+\bar{n}}_{q+1}\,.$$

Now summing over $0 \leq m \leq \mathsf{N}_{\text{cut,t}}$ concludes the proof of (13.26) and thus (5.23) at level $q+1$ with $q' = q+\bar{n}$.

`Step 4: Verification of` (5.23) `for` $q+1 \leq q' \leq q+\bar{n}-1$. Recall that in `Step 1`, we

showed that

$$\pi_{q+1}^{q'} \geq {}^1\!/\!_2 \pi_q^{q'} \qquad q' \geq q+1\,. \tag{13.30}$$

Therefore we may use (5.23) at level q to write that

$$\begin{aligned}
\sum_{i=0}^{i_{\max}} \psi_{i,q'}^2 \delta_{q'} r_{q'-\bar{n}}^{-2/3} \Gamma_{q'}^{2i} &\leq 2^{q-q'} \Gamma_{q'} r_{q'-\bar{n}}^{-2} \pi_q^{q'} \\
&= 2^{q+1-q'} \Gamma_{q'} r_{q'-\bar{n}}^{-2} \left({}^1\!/\!_2 \pi_q^{q'} \right) \\
&\leq 2^{q+1-q'} \Gamma_{q'} r_{q'-\bar{n}}^{-2} \pi_{q+1}^{q'}\,,
\end{aligned}$$

concluding the proof of (5.23) at level $q+1$. □

Lemma 13.2.4 (**Pressure dodging at level** $q+1$**).** *Hypothesis 5.4.5 is verified at step* $q+1$.

Proof. We must show that for all $q+1 < k \leq q+\bar{n}$, $k \leq k'$, and $N+M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left| \psi_{i,k-1} D^N D_{t,k-1}^M \left(\widehat{w}_k \pi_{q+1}^{k'} \right) \right| < \Gamma_{q+1} \Gamma_k^{-100} \left(\pi_{q+1}^k \right)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1} \tau_{k-1}^{-1}, \Gamma_k^{-1} \mathrm{T}_k^{-1} \right)\,.$$

We divide up the proof into cases based on the value of k'.

`Case 1:` $q+1 < k \leq k' < q + {}^{\bar{n}}\!/\!_2$. From (13.24), we have that $\pi_{q+1}^{k'} = \pi_q^{k'}$, and so using Hypothesis 5.4.5 at level q and (13.30), we have that for $N+M \leq 2\mathsf{N}_{\text{ind}}$,

$$\begin{aligned}
\left| \psi_{i,k-1} D^N D_{t,k-1}^M \left(\widehat{w}_k \pi_{q+1}^{k'} \right) \right| &= \left| \psi_{i,k-1} D^N D_{t,k-1}^M \left(\widehat{w}_k \pi_q^{k'} \right) \right| \\
&< \Gamma_q \Gamma_k^{-100} \left(\pi_q^k \right)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1} \tau_{k-1}^{-1}, \Gamma_k^{-1} \mathrm{T}_k^{-1} \right) \\
&< \Gamma_{q+1} \Gamma_k^{-100} \left(\pi_{q+1}^k \right)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1} \tau_{k-1}^{-1}, \Gamma_k^{-1} \mathrm{T}_k^{-1} \right)\,.
\end{aligned}$$

`Case 2:` $q+1 < k \leq k' = q + {}^{\bar{n}}\!/\!_2$. In this case, we have from (13.24) that $\pi_{q+1}^{q+\bar{n}/2} = \pi_q^{q+\bar{n}/2} + \sigma_{q+1}^{q+\bar{n}/2} + 2^{\bar{n}/2-1} \delta_{q+\bar{n}/2+\bar{n}}$. Then considering just the contribution $\widehat{w}_k \pi_q^{q+\bar{n}/2}$ to $\widehat{w}_k \pi_{q+1}^{q+\bar{n}/2}$ from the

first term, we have from Hypothesis 5.4.5 at level q and (13.30) that for $N + M \leq 2\mathsf{N}_{\text{ind}}$,

$$\begin{aligned}\left|\psi_{i,k-1} D^N D^M_{t,k-1}\left(\widehat{w}_k \pi_q^{q+\bar{n}/2}\right)\right| &< \Gamma_q \Gamma_k^{-100}\left(\pi_q^k\right)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) \\ &< {}^1\!/_2 \Gamma_{q+1}\Gamma_k^{-100}\left(\pi_{q+1}^k\right)^{3/2} r_k^{-1}\Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) .\end{aligned}$$

Next, we have from (13.15) that $\widehat{w}_k \sigma_{q+1}^{q+\bar{n}/2} \equiv 0$ for $q+1 < k \leq q + \bar{n}/2$, and so we may ignore the contribution from $\sigma_{q+1}^{q+\bar{n}/2}$. Finally, in order to bound the contribution coming from the constant term $\delta_{q+\bar{n}/2+\bar{n}}$, we use (5.17) and (5.21c) at level $q+1$ and (4.10h) to write that

$$\begin{aligned}\left|\psi_{i,k-1} D^N D^M_{t,k-1}\left(\widehat{w}_k 2^{\bar{n}/2-1}\delta_{q+\bar{n}/2+\bar{n}}\right)\right| &\leq 2^{\bar{n}/2-1}\Gamma_{q+1} r_{k-\bar{n}}^{-1}(\pi_{q+1}^k)^{1/2}\delta_{k+\bar{n}} \\ &\qquad \times \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) \\ &< {}^1\!/_2 \Gamma_{q+1}\Gamma_k^{-100}\left(\pi_{q+1}^k\right)^{3/2} r_k^{-1}\Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) .\end{aligned}$$

`Case 3:` $q+1 < k \leq k'$, $q+1 < k \leq q+\bar{n}$, `and` $q + \bar{n}/2 + 1 \leq k' < \infty$. From Definition 13.1.5, we have that in these cases, either $\pi_{q+1}^k = \pi_q^k + \sigma_{q+1}^k + 2^{k-q-1}\delta_{k+\bar{n}}$ or $\pi_{q+1}^k = \pi_q^k$. We therefore first make a few preliminary calculations to help bound the contributions from π_q^k and $2^{k-q-1}\delta_{k+\bar{n}}$ before dividing up further into subcases. We first recall (5.17) at level $q+1$,

$$\pi_{q+1}^k \geq \delta_{k+\bar{n}} \qquad \forall q+1 \leq k < \infty . \tag{13.32}$$

Then, we have from (5.21c) at level $q+1$ that for all $q+1 < k \leq k' \leq q+\bar{n}-1$, $k \leq k'$,

$$\begin{aligned}\left|\psi_{i,k-1} D^N D^M_{t,k-1}\left(\widehat{w}_k \delta_{k'+\bar{n}}\right)\right| &\leq \Gamma_{q+1} r_{k-\bar{n}}^{-1}(\pi_{q+1}^k)^{1/2}\delta_{k+\bar{n}}\Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) \\ &< {}^1\!/_3 \Gamma_{q+1}\Gamma_k^{-100}\left(\pi_{q+1}^k\right)^{3/2} r_k^{-1}\Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+1}\tau_{k-1}^{-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) .\end{aligned} \tag{13.33}$$

Next, we have from Hypothesis 5.4.5 at level q and (13.30) that for $q+1 < k \leq q+\bar{n}-1$

and $k \leq k' < \infty$,

$$\begin{aligned}\left|\psi_{i,k-1} D^N D^M_{t,k-1}\left(\widehat{w}_k \pi^{k'}_q\right)\right| &< \Gamma_q \Gamma_k^{-100} \left(\pi^k_q\right)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+1}_{k-1}\tau^{-1}_{k-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) \\ &< {}^1\!/_3 \Gamma_{q+1}\Gamma_k^{-100}\left(\pi^k_{q+1}\right)^{3/2} r_k^{-1}\Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+1}_{k-1}\tau^{-1}_{k-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) .\end{aligned} \tag{13.34}$$

We claim that the above estimate holds in addition for $k = q + \bar{n}$ and $k \leq k' < \infty$. Indeed from (5.16c) and (5.18) at level q, (5.21c) at level $q+1$, (13.30), and (4.10h), we have that

$$\begin{aligned}\left|\psi_{i,q+\bar{n}-1} D^N D^M_{t,q+\bar{n}-1}\left(\widehat{w}_{q+\bar{n}} \pi^{k'}_q\right)\right| &\lesssim \Gamma_{q+1} r_q^{-1} (\pi^{q+\bar{n}}_{q+1})^{1/2} \Gamma_q \pi^{k'}_q \Lambda^N_{q+\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+1}_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \Gamma^{-1}_{q+\bar{n}}\mathrm{T}^{-1}_{q+\bar{n}}\right) \\ &< \Gamma_{q+1}\Gamma_k^{-101}\left(\pi^{q+\bar{n}}_{q+1}\right)^{3/2} r^{-1}_{q+\bar{n}}\Lambda^N_{q+\bar{n}}\mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+1}_{q+\bar{n}-1}\tau^{-1}_{q+\bar{n}-1}, \Gamma^{-1}_{q+\bar{n}}\mathrm{T}^{-1}_{q+\bar{n}}\right) .\end{aligned} \tag{13.35}$$

Finally, we use Definition 13.1.3, equations (13.11)–(13.14) and the dodging ensured by (13.8b) to write that for $q+1 < k \leq q+\bar{n}$, $k \leq k'$, and $q + {}^{\bar{n}}\!/_2 + 1 \leq k' < \infty$,

$$\begin{aligned}\left|\psi_{i,k-1} D^N D^M_{t,k-1}\left(\widehat{w}_k \sigma^{\min(k',q+\bar{n})}_{q+1}\right)\right| &= \left|\sum_{m=q+{}^{\bar{n}}\!/_2+1}^{k'} \psi_{i,k-1} D^N D^M_{t,k-1}\left(a_{m,q,k'}\widehat{w}_k \sigma^+_{m,q+1}\right)\right| \\ &= \left|\sum_{m=q+{}^{\bar{n}}\!/_2+1}^{k} \psi_{i,k-1} D^N D^M_{t,k-1}\left(a_{m,q,k'}\widehat{w}_k \sigma^+_{m,q+1}\right)\right| .\end{aligned}$$

Then using (5.21c) and eqn. (5.14) at level $q+1$, (13.7c), and (13.8b), we have that the quantity above is controlled by

$$\begin{aligned}&\sum_{N_1=0}^{N}\sum_{M_1=0}^{M} \left|\psi_{i,k-1} D^{N_1} D^{M_1}_{t,k-1}\widehat{w}_k\right| \left|\sum_{m=q+{}^{\bar{n}}\!/_2+1}^{k} a_{m,q,k'} \mathbf{1}_{\operatorname{supp}\psi_{i,k-1}} D^{N-N_1} D^{M-M_1}_{t,k-1}\sigma^+_{m,q+1}\right| \\ &\lesssim \sum_{N_1,M_1} \left|\psi_{i,k-1} D^{N_1} D^{M_1}_{t,k-1}\widehat{w}_k\right| \sum_{i':\psi_{i',q}\psi_{i,k-1}\neq 0} \sum_{m=q+{}^{\bar{n}}\!/_2+1}^{k} a_{m,q,k}\mathbf{1}_{\operatorname{supp}\psi_{i',q}} \left|D^{N-N_1} D^{M-M_1}_{t,q}\sigma^+_{m,q+1}\right| \\ &\lesssim \Gamma_{q+1} r^{-1}_{k-\bar{n}} (\pi^k_{q+1})^{1/2} (\sigma^k_{q+1} + \Gamma_q^{-100}\delta_{k+\bar{n}}) \Lambda^N_k \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^i_{k-1}\tau^{-1}_{k-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) \\ &\leq \Gamma_k^{-101}\left(\pi^k_{q+1}\right)^{3/2} r_k^{-1}\Lambda_k^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^i_{k-1}\tau^{-1}_{k-1}, \Gamma_k^{-1}\mathrm{T}_k^{-1}\right) .\end{aligned} \tag{13.36}$$

Here, the second inequality follows from $0 \le a_{m,q,k'} \le a_{m,q,k}$ because of $k \le k'$, and the third inequality follows from the proof of (13.19).

`Case 3a:` $q+1 < k \le q+\bar{n}$, $k \le k'$, `and` $q + \bar{n}/2 + 1 \le k' \le q + \mathsf{N}_{\rm pr}$. In these cases, we have from Definition 13.1.5 that $\pi^k_{q+1} = \pi^k_q + \sigma^k_{q+1} + 2^{k-q-1}\delta_{k+\bar{n}}$. Then combining (13.33), (13.34), (13.35), and (13.36) concludes the proof.

`Case 3b:` $q+1 < k \le q+\bar{n}$, $k \le k'$, `and` $q + \mathsf{N}_{\rm pr} + 1 \le k' < \infty$. In these cases, we have from Definition 13.1.5 that $\pi^k_{q+1} = \pi^k_q$. Then (13.34) gives the desired estimate. □

13.3 The Euler-Reynolds system and the relaxed LEI adapted to new pressure

In this section, we upgrade the Euler-Reynold system (10.3) and the relaxed local energy inequality (11.8) adapted to the new pressure defined in subsections 13.1 and 13.2.

Lemma 13.3.1 (Relaxed equations at level $q+1$)**.** *The inductive assumptions* (5.2)–(5.7) *are satisfied at level* $q+1$.

Proof. We first set a few notations and definitions. Referring to (13.2), (13.3), and Definition (13.1.5), we first define ϕ_{P1} and $\mathfrak{m}'_{P1}$ by

$$\mathfrak{m}'_{P1} := \frac{3}{2}\sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} A_{m,q}\mathfrak{m}'_{m,q+1}\,, \qquad \phi_{P1} := \frac{3}{2}\sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} A_{m,q}\phi_{m,q+1}\,. \tag{13.37}$$

From the definition (13.12) of σ_{q+1} and (13.5), we therefore have that

$$\frac{3}{2}D_{t,q}\sigma_{q+1} = \mathrm{div}\phi_{P1} + \mathfrak{m}'_{P1}\,. \tag{13.38}$$

In particular, $\mathfrak{m}'_{P1} = \frac{3}{2}\langle D_{t,q}\sigma_{q+1}\rangle$. Recalling (10.105) and (11.7), we now define

$$R^{q+\bar{n},*}_{q+1} = \overline{R}^{q+\bar{n},*}_{q+1} + \frac{2}{3}(\mathfrak{m}_T + \mathfrak{m}_N + \mathfrak{m}_L + \mathfrak{m}_{M1} + \mathfrak{m}_{M2} + \mathfrak{m}_{P1})\mathrm{Id}\,, \tag{13.39}$$

and we do not modify the local part of the Reynolds stress that was defined in (10.105), nor do we modify any of the nonlocal portions for $q+1 \leq m \leq q+\bar{n}-1$. We then define a new pressure and a new stress error at step $q+1$ by

$$
\begin{aligned}
&p_{q+1} = -\sigma_{q+1} + p_q \,, \\
&R_{q+1} = \overline{R}_{q+1} + \frac{2}{3}(\mathfrak{m}_T + \mathfrak{m}_N + \mathfrak{m}_L + \mathfrak{m}_{M1} + \mathfrak{m}_{M2} + \mathfrak{m}_{P1})\mathrm{Id} \underset{(11.7)}{=} \overline{R}_{q+1} + \frac{2}{3}\left(\mathfrak{m}_{\overline{\phi}_{q+1}} + \mathfrak{m}_{P1}\right)\mathrm{Id} \,,
\end{aligned}
\tag{13.40}
$$

which verifies (5.5).

Recalling Definition 13.1.5, (13.25), and (10.3), we now have that $(u_{q+1}, p_{q+1}, R_{q+1}, -\pi_{q+1})$ solves the Euler-Reynolds system

$$
\partial_t u_{q+1} + \mathrm{div}\,(u_{q+1} \otimes u_{q+1}) + \nabla p_{q+1} = \mathrm{div}(-\pi_{q+1}\mathrm{Id} + R_{q+1}) \,, \tag{13.41}
$$

where we have used that the constant term in (13.25) and the terms with functions of time $\mathfrak{m}_\bullet$ in R_{q+1} vanish inside of the divergence. Thus we have verified (5.2). Note that we have also verified (5.6) as well. Recalling (9.17), we have in addition that (5.4) is verified, and so it only remains to check (5.7) and (5.3) at level $q+1$.

Let us set

$$
\kappa_{q+1} = \frac{1}{2}\mathrm{tr}\,(R_{q+1} - \pi_{q+1}\mathrm{Id}))\,. \tag{13.42}
$$

Then we can now rewrite (11.8) as the relaxed local energy inequality for $(u_{q+1}, p_{q+1}, R_{q+1}, -\pi_{q+1}, \varphi_{q+1})$ adapted to the upgraded stress error R_{q+1} and the new pressure π_{q+1}. Specifically, we have

that

$$
\begin{aligned}
&\partial_t\left(\frac{1}{2}|u_{q+1}|^2\right)+\operatorname{div}\left(\left(\frac{1}{2}|u_{q+1}|^2+p_{q+1}\right)u_{q+1}\right)\\
&\underset{(11.8),(13.40)}{=}(\partial_t+\widehat{u}_{q+1}\cdot\nabla)(\overline{\kappa}_{q+1}+\mathfrak{m}_{\overline{\phi}_{q+1}})+\operatorname{div}\left((\overline{R}_{q+1}-(\pi_q-\pi_q^q)\mathrm{Id})\widehat{u}_{q+1}\right)\\
&\qquad+\operatorname{div}\overline{\varphi}_{q+1}-E-\operatorname{div}(\sigma_{q+1}u_{q+1})\\
&\underset{\substack{(11.3a),\\ \operatorname{div}\widehat{u}_{q+1}\equiv 0\\ \nabla\mathfrak{m}_\bullet\equiv 0}}{=}(\partial_t+\widehat{u}_{q+1}\cdot\nabla)\left(\frac{1}{2}\mathrm{tr}\left(\overline{R}_{q+1}-\left(\pi_q-\pi_q^q+\sigma_{q+1}+\sum_{k=q+\bar{n}/2}^{q+\mathsf{N}_{\mathrm{pr}}}2^{k-q-1}\delta_{k+\bar{n}}-\frac{2}{3}\left(\mathfrak{m}_{\overline{\phi}_{q+1}}+\mathfrak{m}_{P1}\right)\right)\mathrm{Id}\right)\right)\\
&\qquad+\operatorname{div}\left(\left((\overline{R}_{q+1}-\left(\pi_q-\pi_q^q+\sigma_{q+1}+\sum_{k=q+\bar{n}/2}^{q+\mathsf{N}_{\mathrm{pr}}}2^{k-q-1}\delta_{k+\bar{n}}-\frac{2}{3}\left(\mathfrak{m}_{\overline{\phi}_{q+1}}+\mathfrak{m}_{P1}\right)\right)\mathrm{Id}\right)\widehat{u}_{q+1}\right)\\
&\qquad\qquad+\operatorname{div}\overline{\varphi}_{q+1}-E-\operatorname{div}(\sigma_{q+1}(u_{q+1}-\widehat{u}_{q+1}))+\frac{3}{2}(\partial_t+\widehat{u}_{q+1}\cdot\nabla)\sigma_{q+1}-\mathfrak{m}'_{P1}\\
&\underset{\substack{(13.25),(13.40),\\ (13.42)}}{=}(\partial_t+\widehat{u}_{q+1}\cdot\nabla)\kappa_{q+1}+\operatorname{div}\left((R_{q+1}-\pi_{q+1}\mathrm{Id})\widehat{u}_{q+1}\right)\\
&\qquad\qquad+\operatorname{div}\overline{\varphi}_{q+1}-E-\operatorname{div}(\sigma_{q+1}(u_{q+1}-\widehat{u}_{q+1}))+\frac{3}{2}\underbrace{(\partial_t+\widehat{u}_{q+1}\cdot\nabla)}_{=\partial_t+(\widehat{u}_q+\widehat{w}_{q+1})\cdot\nabla}\sigma_{q+1}-\mathfrak{m}'_{P1}\\
&\underset{(13.15),(13.13)}{=}(\partial_t+\widehat{u}_{q+1}\cdot\nabla)\kappa_{q+1}+\operatorname{div}\left((R_{q+1}-\pi_{q+1}\mathrm{Id})\widehat{u}_{q+1}\right)\\
&\qquad\qquad+\operatorname{div}\overline{\varphi}_{q+1}-E-\underbrace{\operatorname{div}(\sigma_{q+1}(u_{q+1}-\widehat{u}_{q+1}))}_{=:\operatorname{div}\phi_{P2}}+\underbrace{\frac{3}{2}(\partial_t+\widehat{u}_q\cdot\nabla)\sigma_{q+1}-\mathfrak{m}'_{P1}}_{=\operatorname{div}\phi_{P1}}\\
&\quad=(\partial_t+\widehat{u}_{q+1}\cdot\nabla)\kappa_{q+1}+\operatorname{div}\left((R_{q+1}-\pi_{q+1}\mathrm{Id})\widehat{u}_{q+1}\right)+\underbrace{\operatorname{div}\varphi_{q+1}}_{=\operatorname{div}(\overline{\varphi}_{q+1}+\phi_{P1}+\phi_{P2})}-E\,. \qquad (13.43)
\end{aligned}
$$

Thus we have verified (5.3), and recalling (11.138), we have that (5.7) is verified as well. □

13.4 Pressure current error

In this subsection, we analyze the new pressure current errors ϕ_{P1} and ϕ_{P2} defined in (13.43).

13.4.1 Pressure current error I

Recalling the definition of $\phi_{m,q+1}$ from (13.3)–(13.5) and the definition of ϕ_{P1} from (13.37), we decompose ϕ_{P1} into

$$\begin{aligned}\phi_{P1} &= \frac{3}{2}\sum_{m=q+\bar{n}/2+1}^{q+\bar{n}}\sum_{m'=q+\bar{n}/2+1}^{m}\left(A_{m,q}\phi_{m,q+1}^{m',l}+A_{m,q}\phi_{m,q+1}^{m',*}\right)\\ &= \frac{3}{2}\sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}}\left(\sum_{m=m'}^{q+\bar{n}}A_{m,q}\phi_{m,q+1}^{m',l}+\sum_{m=m'}^{q+\bar{n}}A_{m,q}\phi_{m,q+1}^{m',*}\right)\\ &=: \sum_{m'=q+\bar{n}/2+1}^{q+\bar{n}}\phi_{P1}^{m',l}+\phi_{P1}^{m',*}\,. \end{aligned}\tag{13.44}$$

Lemma 13.4.1 (Properties of ϕ_{P1}). *For all $q+\bar{n}/2+1\le m\le q+\bar{n}$, the terms $\phi_{P1}^{m,l}$ and $\phi_{P1}^{m,*}$ satisfy the following properties.*

(i) The local part $\phi_{P1}^{m,l}$ satisfies

$$\left|\psi_{i,q}D^N D_{t,q}^M\phi_{P1}^{m,l}\right| < \Gamma_m^{-80}(\pi_q^m)^{3/2}r_m^{-1}\Lambda_m^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1}\Gamma_q^{i+18},\mathrm{T}_q^{-1}\Gamma_q^{11}\right)\tag{13.45a}$$

$$\operatorname{supp}\phi_{P1}^{m,l}\cap B(\operatorname{supp}\widehat{w}_{q'},\lambda_{q'}^{-1}\Gamma_{q'})=\emptyset,\qquad q+1\le q'\le m-1\tag{13.45b}$$

for all $N,M<\mathsf{N}_{\mathrm{fin}}/200$.

(ii) For all $N,M\le 2\mathsf{N}_{\mathrm{ind}}$, the non-local part $\phi_{P1}^{m,*}$ satisfies

$$\left\|\psi_{i,q}D^N D_{t,q}^M\phi_{P1}^{m,*}\right\|_\infty \lesssim \delta_{q+3\bar{n}}^{3/2}\mathrm{T}_{q+\bar{n}}^{2\mathsf{N}_{\mathrm{ind,t}}}\Lambda_m^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_q^{-1},\mathrm{T}_q^{-1}\Gamma_q^9\right)\tag{13.46}$$

Remark 13.4.2. Applying Lemma A.5.1 to $F^\ell=\phi_{P1}^{k,l}$ and $F^*=\phi_{P1}^{k,*}$, for $q+\bar{n}/2+1\le k\le q+\bar{n}$ we can upgrade the material derivatives in the estimates (13.45a) to obtain that for $N+M\le 2\mathsf{N}_{\mathrm{ind}}$,

$$\left|\psi_{i,k-1}D^N D_{t,k-1}^M\phi_{P1}^{k}\right| < \Gamma_k^{-60}(\pi_q^k)^{3/2}r_k^{-1}\Lambda_k^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_{k-1}^{-1}\Gamma_{k-1}^{i},\mathrm{T}_{k-1}^{-1}\right)\,.\tag{13.47}$$

Proof of Lemma 13.4.1. From the definition (13.11) of $A_{m,q}$, we have that $A_{m,q} \leq \Gamma_m^{10}$. Recalling the definition (13.3) of $\phi_{m,q+1}^{m'}$, it is immediate that $\phi_{m,q+1}^{m'}$ satisfy the same properties delineated in Lemma 9.4.6 (for the current errors associated to velocity pressure increments), Lemmas 10.2.5, 10.2.9, and 10.2.13 (for the current errors associated to stress error pressure increments), and Lemmas 11.2.2, 11.2.7, and 11.2.11 (for the current errors associated to current error pressure increments). Therefore, the lemma follows from the definition (13.44). □

13.4.2 Pressure current error II

Here we deal with the error

$$\mathrm{div}\phi_{P2} = \mathrm{div}(\sigma_{q+1}(u_{q+1} - \widehat{u}_{q+1}))\,.$$

Recall from (5.4) at level $q+1$ that

$$u_{q+1} - \widehat{u}_{q+1} = \sum_{q'=q+2}^{q+\bar{n}} \widehat{w}_{q'}\,.$$

By the definition of σ_{q+1} in (13.12) and the dodging properties (13.8a), (13.8b), and (13.15), we first write

$$\begin{aligned}
\sigma_{q+1} \sum_{q'=q+2}^{q+\bar{n}} \widehat{w}_{q'} &= \sum_{q'=q+\bar{n}/2+1}^{q+\bar{n}} \sigma_{q+1} \widehat{w}_{q'} \\
&= \sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} \sum_{q'=m+1}^{q+\bar{n}} \widetilde{\sigma}_{m,q+1}^{+} \widehat{w}_{q'} + \sum_{m=q+\bar{n}/2+1}^{q+\bar{n}} \widetilde{\sigma}_{m,q+1}^{+} \widehat{w}_{m} - \sum_{q'=q+\bar{n}/2+1}^{q+\bar{n}} \sigma_{q+1}^{-} \widehat{w}_{q'}\,.
\end{aligned}$$

Since $\operatorname{div}\left(\sigma_{q+1}\sum_{q'=q+2}^{q+\bar{n}}\widehat{w}_{q'}\right)$ has zero mean, we recall the identity (5.38) at level $q+1$ and set

$$\phi_{P2}^{q'} := \underbrace{\left[\sum_{m=q+\bar{n}/2+1}^{q'-1}(\mathcal{H}+\mathcal{R}^*)(\nabla\widetilde{\sigma}_{m,q+1}^{+}\cdot(\operatorname{div}^{\mathsf{d}}\widehat{\upsilon}_{q'}))\right]}_{=:\phi_{P21}^{q'}} + \underbrace{\widetilde{\sigma}_{q',q+1}^{+}\widehat{w}_{q'}}_{=:\phi_{P22}^{q'}} - \underbrace{(\mathcal{H}+\mathcal{R}^*)(\nabla\sigma_{q+1}^{-}\cdot(\operatorname{div}^{\mathsf{d}}\widehat{\upsilon}_{q'}))}_{=:\phi_{P23}^{q'}}$$
$$+ \underbrace{\left[\sum_{m=q+\bar{n}/2+1}^{q'-1}\mathcal{R}^*(\nabla\widetilde{\sigma}_{m,q+1}^{+}\cdot\widehat{e}_{q'})\right] - \mathcal{R}^*(\nabla\sigma_{q+1}^{-}\cdot\widehat{e}_{q'})}_{=:\phi_{P*}^{q'}} \tag{13.48}$$

$$\phi_{P2} := \sum_{q'=q+\bar{n}/2+1}^{q+\bar{n}}\phi_{P2}^{q'}\,.$$

As before, the terms including $\mathcal{R}^*$ are non-local and the rest are local.

Lemma 13.4.3 (Properties of σ_{P2}^m). *For $q+\bar{n}/2+1 \le m \le q+\bar{n}$, the error ϕ_{P2}^m has no pressure increment; for $N+M \le 2\mathsf{N}_{\mathrm{ind}}$, it satisfies*

$$\left|\psi_{i,m-1}D^N D_{t,m-1}^M\phi_{P2}^m\right| < \Gamma_m^{-80}\left(\pi_{q+1}^m\right)^{3/2} r_m^{-1}\Lambda_m^N\mathcal{M}\left(M,\mathsf{N}_{\mathrm{ind,t}},\tau_{m-1}^{-1}\Gamma_{m-1}^{i+3},\mathrm{T}_{m-1}^{-1}\Gamma_{m-1}^2\right)\,. \tag{13.49}$$

Proof. We divide the proof up into cases based on the decomposition in (13.48).

`Step 1: Estimate of` ϕ_{P22}^m. By (13.9c), the definition of $A_{m,q}$ and $\widetilde{\sigma}_{m,q+1}^{+}$ in Definition 13.1.3, Definition 13.1.5, which shows that $\pi_{q+1}^m \ge \sigma_{m,q+1}^{+} \ge \widetilde{\sigma}_{m,q+1}\Gamma_m^{-25}$, (5.21c) at level $q+1$ to bound $\widehat{w}_m$, and (4.10h) to absorb errant factors of Γ_m, we have that for

$N+M \leq 2\mathsf{N}_{\text{ind}}$ and $q+{}^{\bar{n}}/_{2}+1 \leq m \leq q+\bar{n}$,

$$\begin{aligned}
\left|\psi_{i,m-1} D^N D^M_{t,m-1}(\widetilde{\sigma}^+_{m,q+1}\widehat{w}_m)\right| &\lesssim \sum_{\substack{N_1+N_2=N\\ M_1+M_2=M}} \left|\psi_{i,m-1} D^{N_1} D^{M_1}_{t,m-1}\widetilde{\sigma}^+_{m,q+1}\right| \left|\psi_{i,m-1} D^{N_2} D^{M_2}_{t,m-1}\widehat{w}_m\right| \\
&\lesssim \left(\widetilde{\sigma}^+_{m,q+1}+\delta_{q+3\bar{n}}\right)\left(\pi^m_{q+1}\right)^{1/2}\Gamma_{q+1} r^{-1}_{m-\bar{n}}\Lambda^N_m \\
&\qquad \times \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{m-1}\Gamma^i_{m-1}, \mathrm{T}^{-1}_{m-1}\Gamma^2_{m-1}\right) \\
&\lesssim \Gamma^{-101}_m \left(\pi^m_{q+1}\right)^{3/2} r^{-1}_m \Lambda^N_m \mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{m-1}\Gamma^i_{m-1}, \mathrm{T}^{-1}_{m-1}\Gamma^2_{m-1}\right) .
\end{aligned}$$

Therefore, ϕ^m_{P22} satisfies the desired pointwise estimate.

`Step 2: Estimates of` $\phi^{q'}_{P21}$, $\phi^{q'}_{P23}$, `and` $\phi^{q'}_{P*}$. We first carry out the preliminary step of upgrading material derivatives on $\nabla\sigma^{\pm}_{m,q+1}$, which will be required for all three terms. We apply Remark A.2.6 inductively to $v=\widehat{u}_{m-1}$, $f=\nabla\sigma^{\pm}_{m,q+1}$, and $w=\widehat{w}_m,\ldots,\widehat{w}_{q'-1}$, and $\Omega=\operatorname{supp}(\psi_{i,q'-1})$. The assumptions in the remark are satisfied due to (5.34), Remark 13.1.2, and (5.32). As a result, we have that for $q+{}^{\bar{n}}/_{2}+1 \leq m \leq q'-1$,

$$\left|\psi_{i,q'-1} D^N D^M_{t,q'-1}\nabla\sigma^+_{m,q+1}\right| \lesssim (\sigma^+_{m,q+1}+\delta_{q+3\bar{n}})\Lambda^N_{q'-1}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{q'-1}\tau^{-1}_{q'-1}, \Gamma_{q'-1}\mathrm{T}^{-1}_{q'-1}\right) \tag{13.50a}$$

$$\left|\psi_{i,q'-1} D^N D^M_{t,q'-1}\nabla\sigma^+_{m,q+1}\right| \lesssim \Gamma^{-100}_{q+\bar{n}/2}\pi^{q+\bar{n}/2}_{q+1}\Lambda^N_{q'-1}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \Gamma^{i+3}_{q'-1}\tau^{-1}_{q'-1}, \Gamma_{q'-1}\mathrm{T}^{-1}_{q'-1}\right) \tag{13.50b}$$

for $N,M < {}^{\mathsf{N}_{\text{fin}}}/_{200}$. In a similar way, we have

$$\left\|D^N D^M_{t,q'-1}\nabla\widetilde{\sigma}^{m,+}_{q+1}\right\|_\infty \lesssim \lambda_m \Gamma^{\mathsf{C}_\infty+2}_m \Lambda^N_{q'-1}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q'-1}\Gamma^{i_{\max}+3}_{q'-1}, \mathrm{T}^{-1}_{q'-1}\Gamma^{-1}_{q'-1}\right) \tag{13.51a}$$

$$\left\|D^N D^M_{t,q'-1}\nabla\sigma^-_{q+1}\right\|_\infty \lesssim \lambda_{q+\bar{n}/2}\Gamma^{\mathsf{C}_\infty-50}_{q+\bar{n}/2}\Lambda^N_{q'-1}\mathcal{M}\left(M, \mathsf{N}_{\text{ind,t}}, \tau^{-1}_{q'-1}\Gamma^{i_{\max}+3}_{q'-1}, \mathrm{T}^{-1}_{q'-1}\Gamma^{-1}_{q'-1}\right) \tag{13.51b}$$

for $N,M < {}^{\mathsf{N}_{\text{fin}}}/_{200}$ and $q+{}^{\bar{n}}/_{2} \leq m \leq q+\bar{n}$. The first inequality holds for $m+1 \leq q' \leq q+\bar{n}$, while the second one holds for $q+{}^{\bar{n}}/_{2}+1 \leq q' \leq q+\bar{n}$.

`Step 2a:` We now estimate $\phi_{P21}^{m'}$ by applying Lemma A.3.12 with

$$G = \nabla\widetilde{\sigma}_{m,q+1}^{+}, \quad \vartheta = \widehat{\upsilon}_{q'}, \quad \pi = \lambda_m \Gamma_m^{25} \pi_{q+1}^{m}, \quad \pi' = \Gamma_{q+1}\Gamma_{q'}(\pi_{q+1}^{q'})^{1/2} r_{q'-\bar{n}}^{-1}, \quad M_t = \mathsf{N}_{\mathrm{ind,t}}, \quad v = \widehat{u}_{q'-1},$$

$$\Omega = \mathrm{supp}\left(\widehat{\upsilon}_{q'}\psi_{i,q'-1}\right), \quad \lambda = \lambda' = \lambda_m \Gamma_m, \quad \Upsilon = \lambda_{q'}, \quad \Lambda = \lambda_{q'}\Gamma_{q'}, \quad \nu = \Gamma_{q'-1}^{i+3}\tau_{q'-1}^{-1}, \quad \nu' = \mathrm{T}_{q'-1}^{-1}\Gamma_{q'-1}^{2},$$

$$\mathcal{C}_{G,\infty} = \lambda_m \Gamma_m^{\mathsf{C}_\infty+1}, \qquad \mathcal{C}_{*,\infty} = \Gamma_{q+1}\Gamma_{q'}(\Gamma_{q+1}\Gamma_{q'}^{\mathsf{C}_\infty+1})^{1/2} r_{q'-\bar{n}}^{-1}, \quad \mathsf{d} \text{ as in (xvii)/(9.48)},$$

$$M_\circ = N_\circ = 2\mathsf{N}_{\mathrm{ind}}, \quad K_\circ \text{ as in (4.22)}, \quad N_* = M_* = {}^{\mathsf{N}_{\mathrm{fin}}}/_{300}.$$

Then we have that (A.42b) is satisfied due to (5.34) at level $q+1$, (A.97a) is satisfied due to and Definitions 13.1.5 and 13.1.3 and (13.10b), (A.97b) is satisfied due to (5.40) at level $q+1$, all assumptions from item (i) in Part 4 of Proposition A.3.3 are satisfied due to Remark A.3.4, (A.101a) and (A.101b) are satisfied due to (5.15) at level $q+1$, and (A.102) is satisfied due to (4.23b). Then from (A.100), (4.10h), (xviii), and (5.20) at level $q+1$, we have that for $q + {}^{\bar{n}}/_{2} + 1 \leq m \leq q+\bar{n}$ and $m+1 \leq q' \leq q+\bar{n}$ and $N+M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$\begin{aligned}
&\left|\psi_{i,q'-1} D^N D_{t,q'-1}^M \mathcal{H}(\nabla\widetilde{\sigma}_{q+1}^{m,+} \cdot \mathrm{div}^{\mathsf{d}}\widehat{\upsilon}_{q'})\right| \\
&\quad \lesssim \lambda_m \Gamma_m^{25} \pi_{q+1}^{m} \cdot \lambda_{q'}^{-1}\Gamma_{q+1}\Gamma_{q'} \left(\pi_{q+1}^{q'}\right)^{1/2} r_{q'-\bar{n}}^{-1} (\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'-1}^{i+3}\tau_{q'-1}^{-1}, \mathrm{T}_{q'-1}^{-1}\Gamma_{q'-1}^{2}\right) \\
&\quad \lesssim \Gamma_{q'}^{-100} \left(\pi_{q+1}^{q'}\right)^{3/2} r_{q'}^{-1} (\lambda_{q'}\Gamma_{q'})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{q'-1}^{i+3}\tau_{q'-1}^{-1}, \mathrm{T}_{q'-1}^{-1}\Gamma_{q'-1}^{2}\right).
\end{aligned}$$

From (A.104) and for the same range of N and M, we also have that

$$\left\|D^N D_{t,q'-1}^M \mathcal{R}^*(\nabla\widetilde{\sigma}_{q+1}^{m,+} \cdot \mathrm{div}^{\mathsf{d}}\widehat{\upsilon}_{q'})\right\|_\infty \lesssim \Gamma_{q+\bar{n}}^{-100} \delta_{q+3\bar{n}}^{\frac{3}{2}} (\lambda_{q+\bar{n}}\Gamma_{q+\bar{n}})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{q+\bar{n}-1}^{-1}, \mathrm{T}_{q+\bar{n}-1}^{-1}\Gamma_{q+\bar{n}-1}^{2}\right),$$

concluding the proof of the desired estimates for $\phi_{P21}^{q'}$.

`Step 2b:` In the case of $\phi_{P23}^{q'}$, we instead set

$$G = \nabla\sigma_{q+1}^{-}, \quad \mathcal{C}_{G,\infty} = \Gamma_{q+1}\lambda_{q+{}^{\bar{n}}/_{2}}\Gamma_{q+{}^{\bar{n}}/_{2}}^{\mathsf{C}_\infty+2}, \quad \lambda = \lambda' = \lambda_{q+{}^{\bar{n}}/_{2}}\Gamma_{q+{}^{\bar{n}}/_{2}}, \quad \pi = \lambda_{q+{}^{\bar{n}}/_{2}}\Gamma_{q+{}^{\bar{n}}/_{2}}^{25}\pi_{q+1}^{q+{}^{\bar{n}}/_{2}},$$

while the remaining parameters stay the same. Concluding again as before, we have that

$$
\begin{aligned}
&\left|\psi_{i,q'-1} D^N D^M_{t,q'-1} \mathcal{H}(\nabla \sigma^-_{q+1} \cdot \mathrm{div}^{\mathsf{d}} \widehat{\upsilon}_{q'})\right| \\
&\quad \lesssim \lambda_{q+\bar{n}/2} \Gamma_{q+\bar{n}/2} \pi^{q+\bar{n}/2}_{q+1} \cdot \lambda^{-1}_{q'} \Gamma_{q+1} \Gamma_{q'} \left(\pi^{q'}_{q+1}\right)^{1/2} r^{-1}_{q'-\bar{n}} \lambda^N_{q'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q'-1} \tau^{-1}_{q'-1}, \mathrm{T}^{-1}_{q'-1} \Gamma^2_{q'-1}\right) \\
&\quad \leq \Gamma^{-100}_{q'} \left(\pi^{q'}_{q+1}\right)^{3/2} r^{-1}_{q'} \left(\lambda_{q'} \Gamma_{q'}\right)^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q'-1} \tau^{-1}_{q'-1}, \mathrm{T}^{-1}_{q'-1} \Gamma^2_{q'-1}\right) , \\
&\left\| D^N D^M_{t,q'-1} \mathcal{R}^*(\nabla \sigma^-_{q+1} \cdot \mathrm{div}^{\mathsf{d}} \widehat{\upsilon}_{q'})\right\|_\infty \leq \Gamma^{-100}_{q'} \delta^{\frac{3}{2}}_{q'+\bar{n}} r^{-1}_{q'} \lambda^N_{q'+\bar{n}} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma^{i+3}_{q'+\bar{n}-1} \tau^{-1}_{q'+\bar{n}-1}, \mathrm{T}^{-1}_{q'+\bar{n}-1} \Gamma^2_{q'+\bar{n}-1}\right) ,
\end{aligned}
$$

for $q + \bar{n}/2 + 1 \leq q' \leq q + \bar{n}$, where the range of N and M are the same as before.

`Step 2c:` Finally, we must estimate $\phi^{q'}_{P*}$. By Remark A.3.5 and using (13.51), (5.41), and (9.93), we have that for $N + M \leq 2\mathsf{N}_{\mathrm{ind}}$,

$$
\begin{aligned}
\sum_{m=q+\bar{n}/2}^{q+\bar{n}} \sum_{q'=m+1}^{q+\bar{n}} \left\| D^N D^M_{t,q'-1} \mathcal{R}^*(\nabla \widetilde{\sigma}^{m,+}_{q+1} \cdot \widehat{e}_{q'})\right\|_\infty + \sum_{q'=q+\lfloor \bar{n}/2 \rfloor+1}^{q+\bar{n}} \left\| D^N D^M_{t,q'-1} \mathcal{R}^*(\nabla \sigma^-_{q+1} \cdot \widehat{e}_{q'})\right\|_\infty \\
\leq \Gamma^{-100}_{q+\bar{n}} \delta^{\frac{3}{2}}_{q+2\bar{n}} r^{-1}_q \Lambda^N_{q'} \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau^{-1}_{q'+\bar{n}-1}, \mathrm{T}^{-1}_{q'-1} \Gamma^2_{q'-1}\right) .
\end{aligned}
$$

[1] □

13.5 Inductive estimates on the new errors

Lemma 13.5.1 (Inductive pointwise error estimates). *The inductive assumptions* (5.21a), (5.21b), *and* (5.22) *are satisfied at level* $q + 1$.

Proof. We recall from (10.104) and (13.40) the definition of the stress error

$$
R_{q+1} = \overline{R}_{q+1} + \frac{2}{3}\left(\mathfrak{m}_{\overline{\phi}_{q+1}} + \mathfrak{m}_{P1}\right) \mathrm{Id} = \sum_{m=q+1}^{q+\bar{n}} \left(R^m_q + S^m_{q+1}\right) + \frac{2}{3}\left(\mathfrak{m}_{\overline{\phi}_{q+1}} + \mathfrak{m}_{P1}\right) \mathrm{Id} .
$$

[1] need $\delta_{q+2\bar{n}} \mathrm{T}^{3\mathsf{N}_{\mathrm{ind,t}}}_q \lambda^{-10}_q \lambda_{q+\bar{n}} \Gamma^{\mathsf{C}_\infty+2}_{q+\bar{n}} \Lambda^5_{q+\bar{n}} 2^{\mathsf{N}_{\mathrm{ind,t}}} \leq \Gamma^{-100}_{q+\bar{n}} \delta^{\frac{3}{2}}_{q+2\bar{n}} r^{-1}_q$.

Recall also the definition of

$$\varphi_{q+1} = \overline{\varphi}_{q+1} + \phi_{P1} + \phi_{P2} = \sum_{m=q+1}^{q+\bar{n}} \left(\varphi_q^m + \overline{\phi}_{q+1}^m \right) + \phi_{P1} + \phi_{P2}$$

from (11.138), (13.43), (13.44), and (13.48). We therefore define the new current errors ϕ_{q+1}^k by

$$\varphi_{q+1}^k = \bar{\varphi}_{q+1}^k + \phi_{P1}^k + \phi_{P2}^k \,.$$

In order to prove (5.21a) and (5.21b) at level $q+1$, we first consider the cases $q+1 \leq k \leq q + {}^{\bar{n}}/_{2}$. Recall from Lemma 10.3.2 and Lemma 11.3.2 that

$$\begin{aligned}
\left| \psi_{i,k-1} D^N D_{t,k-1}^M S_{q+1}^k \right| &\lesssim \Gamma_k^{-10} \pi_q^k \Lambda_k^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+19} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^9 \right) \\
\left| \psi_{i,k-1} D^N D_{t,k-1}^M \overline{\phi}_{q+1}^k \right| &\lesssim \Gamma_k^{-10} (\pi_q^k)^{3/2} r_k^{-1} \Lambda_k^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+19} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^9 \right) \,,
\end{aligned}$$

where the first bound holds for $N + M \leq 2\mathsf{N}_{\text{ind}}$, the second holds for $N + M \leq {}^{\mathsf{N}_{\text{ind}}}/_{4}$, and we have used the lower bound on π_q^k given in (5.17). Then (5.21a) and (5.21b) at level $q+1$ follow from the definitions recalled at the beginning of the proof, (5.21a) and (5.21b) at level q, the estimates just recorded, and (13.30).

In the cases when $q + {}^{\bar{n}}/_{2} + 1 \leq k \leq q + \bar{n}$, we upgrade the material derivatives in (13.18a) and (13.19) applying Lemma A.5.1 to $F := S_{q+1}^k = S_{q+1}^{k,l} + S_{q+1}^{k,*} =: F^l + F^*$ and $F := \bar{\phi}_{q+1}^k = \bar{\phi}_{q+1}^{k,l} + \bar{\phi}_{q+1}^{k,*} =: F^l + F^*$, obtaining that

$$\left| \psi_{i,k-1} D^N D_{t,k-1}^M S_{q+1}^k \right| \leq \Gamma_k^{-8} \left(\sigma_{q+1}^k + \delta_{k+\bar{n}} \right) (\lambda_k \Gamma_k)^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^{i+18} \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{-1} \right) \tag{13.52a}$$

$$\begin{aligned}
\left| \psi_{i,k-1} D^N D_{t,k-1}^M \overline{\phi}_{q+1}^k \right| \leq \Gamma_k^{-13} &\left(\sigma_{q+1}^k + \mathbf{1}_{m=q+\bar{n}} \Gamma_{q+\bar{n}}^{-50} \pi_q^{q+\bar{n}} + \mathbf{1}_{m=q+1} \Gamma_{q+1}^{-50} \pi_q^{q+1} \delta_{q+2\bar{n}} \right)^{3/2} r_k^{-1} \\
&\times (\lambda_k \Gamma_k)^N \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \Gamma_{k-1}^i \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1} \Gamma_{k-1}^{-1} \right)
\end{aligned} \tag{13.52b}$$

for $N + M \leq 2\mathsf{N}_{\text{ind}}$. In addition, recalling the definitions in (13.2), (13.37), and (11.7), and the estimates given in Lemmas 9.4.4, 10.2.4, 10.2.8, 10.2.12, 11.2.2, 11.2.3, 11.2.4, 11.2.5, 11.2.7, 11.2.8, and 11.2.11, we have that for $M \leq 2\mathsf{N}_{\text{ind}}$,

$$\left| \frac{d^M}{dt^M} \left(\mathfrak{m}_{\overline{\phi}_{q+1}} + \mathfrak{m}_{P1} \right) \right| \leq \delta_{q+5\bar{n}/2} \mathcal{M} \left(M, \mathsf{N}_{\text{ind,t}}, \tau_q^{-1}, \mathrm{T}_{q+1}^{-1} \right) .$$

Then, using (13.47), (13.49), Definitions 13.1.3 and 13.1.5, and (13.30), we conclude the proofs of (5.21a) and (5.21b).

Finally, we note that the nonlocal bounds in (5.22) follow from (13.39), (10.110) from Lemma 10.3.2, and the estimate just above. □

Appendix A

Appendix and toolkit

A.1 Decoupling lemmas and consequences of the Faà di Bruno formula

We begin with an L^p decoupling lemma in the spirit of that from [7]. Some adjustments to the proof are required to treat the cases $p \neq 1, 2, \infty$ and $d \neq 3$, as well as the slight adjustment to the assumption (A.3) on the high-frequency function, which provides a slight increase in generality. Note that the first inequality in (A.1) is implied by the second and the assumption that $\lambda \geq 2$, and so in practice we shall only check the second inequality.

Lemma A.1.1 (L^p **decoupling).** *Let* $\mathsf{N}_{\mathrm{dec}}, \kappa, \lambda \geq 1$ *be such that*

$$\left(2 \cdot \frac{2\pi\sqrt{d}}{\kappa}\right) \cdot \lambda \leq \frac{2}{3}\,, \qquad \lambda^{\mathsf{N}_{\mathrm{dec}}+d+1} \cdot \left(2 \cdot \frac{2\pi\sqrt{d}}{\kappa}\right)^{\mathsf{N}_{\mathrm{dec}}} \leq 1\,. \tag{A.1}$$

Let $p \in [1, \infty)$*, and for* $d \geq 1$*, let* f *be a* $\mathbb{T}^d$*-periodic function such that there exists* $\mathcal{C}_f$ *such that for all* $0 \leq j \leq \mathsf{N}_{\mathrm{dec}} + d + 1$*,*

$$\left\| D^j f \right\|_{L^p} \leq \mathcal{C}_f \lambda^j\,. \tag{A.2}$$

Let g be a $\mathbb{T}^d$-periodic function and $\mathcal{C}_g > 0$ a constant such that for any cube T of side-length $2\pi/\kappa$,

$$\kappa^{d/p} \|g\|_{L^p(T)} \leq \mathcal{C}_g \,. \tag{A.3}$$

Then there exists a dimensional constant $C = C(p,d)$ which is independent of f and g such that

$$\|fg\|_{L^p(\mathbb{T}^d)} \leq C(p,d)\mathcal{C}_f\mathcal{C}_g \,. \tag{A.4}$$

Proof of Lemma A.1.1. Let $\{T_j\}_j$ be disjoint cubes of side-length $2\pi/\kappa$ such that

$$\bigcup_j T_j = \mathbb{T}^d \,.$$

For any Lebesgue integrable function h, let

$$\bar{h}_j := \fint_{T_j} h(x)\,dx \,.$$

Note that from Jensen's inequality, we have that

$$|\bar{h}_j|^p = \left|\fint_{T_j} h(x)\,dx\right|^p \leq \fint_{T_j} |h(x)|^p\,dx = \overline{|h|^p}_j \,. \tag{A.5}$$

For any $x \in T_j$, we have that

$$
\begin{aligned}
|f(x)|^p &= \left(|\bar{f}_j| + |f(x) - \bar{f}_j|\right)^p \\
&\leq 2^p \left(|\bar{f}_j|^p + |f(x) - \bar{f}_j|^p\right) \\
&\leq 2^p \left(|\bar{f}_j|^p + \left(\sup_{x \in T_j} |f(x) - \bar{f}_j|\right)^p\right) \\
&\leq 2^p \left(|\bar{f}_j|^p + \left(\frac{2\pi\sqrt{d}}{\kappa} \sup_{T_j} |Df|\right)^p\right) \\
&\leq 2^p \overline{|f|^p}_j + 2^p \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^p \sup_{T_j} |Df|^p \,, \qquad \text{(A.6)}
\end{aligned}
$$

where in the last line we have used (A.5). Iterating, we obtain

$$
\begin{aligned}
|f(x)|^p &\leq 2^p \overline{|f|^p}_j + 2^p \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^p \left(2^p \overline{|Df|^p}_j + 2^p \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^p \sup_{T_j} |D^2 f|^p\right) \\
&\leq \sum_{m=0}^{\mathsf{N}_{\mathrm{dec}}-1} 2^{(m+1)p} \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^{mp} \overline{|D^m f|^p}_j + \left(2 \cdot \frac{2\pi\sqrt{d}}{\kappa}\right)^{\mathsf{N}_{\mathrm{dec}} p} \left\|D^{\mathsf{N}_{\mathrm{dec}}} f\right\|_{L^\infty}^p \,.
\end{aligned}
$$

Multiplying by g, integrating over T_j, and using (A.3), we obtain[1]

$$
\begin{aligned}
\|fg\|_{L^p}^p &= \sum_j \int_{T^j} |fg|^p \\
&\leq \sum_j \int_{T_j} |g|^p \sum_{m=0}^{\mathsf{N}_{\mathrm{dec}}-1} 2^{(m+1)p} \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^{mp} \overline{|D^m f|^p}_j + \left(2 \cdot \frac{2\pi\sqrt{d}}{\kappa}\right)^{\mathsf{N}_{\mathrm{dec}} p} \left\|D^{\mathsf{N}_{\mathrm{dec}}} f\right\|_{L^\infty}^p \mathcal{C}_g^p \\
&= \sum_j \fint_{T_j} |g|^p \sum_{m=0}^{\mathsf{N}_{\mathrm{dec}}-1} 2^{(m+1)p} \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^{mp} \|D^m f\|_{L^p(T_j)}^p + \left(2 \cdot \frac{2\pi\sqrt{d}}{\kappa}\right)^{\mathsf{N}_{\mathrm{dec}} p} \left\|D^{\mathsf{N}_{\mathrm{dec}}} f\right\|_{L^\infty}^p \mathcal{C}_g^p \\
&\leq (C(d))^p \mathcal{C}_g^p \sum_{m=0}^{\mathsf{N}_{\mathrm{dec}}-1} 2^{(m+1)p} \left(\frac{2\pi\sqrt{d}}{\kappa}\right)^{mp} \mathcal{C}_f^p \lambda^{mp} + \left(2 \cdot \frac{2\pi\sqrt{d}}{\kappa}\right)^{\mathsf{N}_{\mathrm{dec}} p} \left(C'(d) \mathcal{C}_f \lambda^{\mathsf{N}_{\mathrm{dec}}+d+1} \mathcal{C}_g\right)^p \\
&\leq (C(d))^p \mathcal{C}_g^p 2^p \cdot 3 \cdot \mathcal{C}_f^p + (C'(d))^p \mathcal{C}_f^p \mathcal{C}_g^p \\
&=: (C(p,d))^p \mathcal{C}_f^p \mathcal{C}_g^p \,. \qquad \text{(A.7)}
\end{aligned}
$$

[1]Note that in the third line, we move the average from $|D^m f|^p$ to $|g|^p$. In the fourth line, we used the assumption (A.3) on g. In the second to last line, we used the assumption (A.1).

Taking p^{th} roots on both sides concludes the proof. $\square$

We now recall the multivariable Faà di Bruno formula (see for example the appendix in [7]). Let $g = g(x_1, \ldots, x_d) = f(h(x_1, \ldots, x_d))$, where $f\colon \mathbb{R}^m \to \mathbb{R}$, and $h\colon \mathbb{R}^d \to \mathbb{R}^m$ are C^n functions. Let $\alpha \in \mathbb{N}_0^d$ be such that $|\alpha| = n$, and let $\beta \in \mathbb{N}_0^m$ be such that $1 \leq |\beta| \leq n$. We then define

$$p(\alpha, \beta) = \Bigg\{ (k_1, \ldots, k_n; \ell_1, \ldots, \ell_n) \in (\mathbb{N}_0^m)^n \times (\mathbb{N}_0^d)^n \colon \exists s \text{ with } 1 \leq s \leq n \text{ s.t.} \\ |k_j|, |\ell_j| > 0 \Leftrightarrow 1 \leq j \leq s,\, 0 \prec \ell_1 \prec \ldots \prec \ell_s, \sum_{j=1}^{s} k_j = \beta, \sum_{j=1}^{s} |k_j|\ell_j = \alpha \Bigg\}. \tag{A.8}$$

The multivariable Faà di Bruno formula states that

$$\partial^\alpha g(x) = \alpha! \sum_{|\beta|=1}^{n} (\partial^\beta f)(h(x)) \sum_{p(\alpha,\beta)} \prod_{j=1}^{n} \frac{(\partial^{\ell_j} h(x))^{k_j}}{k_j!(\ell_j!)^{k_j}}\,. \tag{A.9}$$

Throughout this manuscript, we must estimate only finitely many derivatives. Therefore we ignore the factorials in (A.9) and absorb them into the implicit constant of the symbol "$\lesssim$." We now recall the following lemma from [7], which gives a useful consequence of the Faà di Bruno formula.

Lemma A.1.2 (Compositions with flow maps). *Given a smooth function $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, suppose that for $\lambda \geq 1$ the vector field $\Phi\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ satisfies the estimate*

$$\left\| D^{N+1}\Phi \right\|_{L^\infty(\operatorname{supp} f)} \lesssim \lambda^N \tag{A.10}$$

for $0 \leq N \leq N_*$. Then for any $1 \leq N \leq N_*$ we have

$$\left| D^N \left(f \circ \Phi \right)(x,t) \right| \lesssim \sum_{m=1}^{N} \lambda^{N-m} \left| (D^m f) \circ \Phi(x,t) \right| \tag{A.11}$$

and thus trivially we obtain

$$\left|D^N\left(f\circ\Phi\right)(x,t)\right| \lesssim \sum_{m=0}^{N} \lambda^{N-m}\left|(D^m f)\circ\Phi(x,t)\right|.$$

for any $0 \le N \le N_*$.

Many estimates will require estimates for derivatives of products of functions which decouple and which are composed with a diffeomorphism.

Lemma A.1.3 (Decoupling with flow maps). *Let* $p \in [1,\infty]$*, and fix integers* $N_* \ge M_* \ge \mathsf{N}_{\mathrm{dec}} \ge 1$*. Fix* $d \ge 2$ *and* $f\colon \mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$*, and let* $\Phi\colon \mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$ *be a vector field satisfying* $D_t\Phi = (\partial_t + v\cdot\nabla)\Phi = 0$*. Denote by* Φ^{-1} *the inverse of the flow* Φ*, which is the identity at a time slice which intersects the support of* f*. Assume that for some* $\lambda, \tau^{-1}, \mathrm{T}^{-1} \ge 1$ *and* $\mathcal{C}_f > 0$ *the function* f *satisfies the estimates*

$$\left\|D^N D_t^M f\right\|_{L^p} \lesssim \mathcal{C}_f \lambda^N \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \tag{A.12}$$

for all $N \le N_*$ *and* $M \le M_*$*, and that* Φ *and* Φ^{-1} *are bounded for all* $N \le N_*$ *by*

$$\left\|D^{N+1}\Phi\right\|_{L^\infty(\operatorname{supp} f)} \lesssim \lambda^N \tag{A.13}$$

$$\left\|D^{N+1}\Phi^{-1}\right\|_{L^\infty(\operatorname{supp} f)} \lesssim \lambda^N. \tag{A.14}$$

Lastly, suppose that there exist $\varrho : \mathbb{T}^d \to \mathbb{R}$ *and parameters* $\Lambda \ge \Upsilon \ge \mu$ *and* $\mathcal{C}_\varrho > 0$ *such that for any cube* T *of side length* μ^{-1}*,*

$$\frac{1}{\mu^{d/p}}\left\|D^N\varrho\right\|_{L^p(T)} + \left\|D^N\varrho\right\|_{L^p(\mathbb{T}^d)} \lesssim \mathcal{C}_\varrho \mathcal{M}\left(N, N_x, \Upsilon, \Lambda\right) \tag{A.15}$$

for all $0 \le N \le N_*$*. If the parameters*

$$\lambda \le \mu \le \Upsilon \le \Lambda$$

satisfy

$$\Lambda^{d+1} \leq \left(\frac{\mu}{4\pi\sqrt{3}\lambda}\right)^{\mathsf{N}_{\mathrm{dec}}}, \tag{A.16}$$

and we have

$$2\mathsf{N}_{\mathrm{dec}} + d + 1 \leq N_* \,, \tag{A.17}$$

then for $N \leq N_*$ and $M \leq M_*$ we have the bound

$$\left\| D^N D_t^M \left(f \ \varrho \circ \Phi\right) \right\|_{L^p} \lesssim \mathcal{C}_f \mathcal{C}_\varrho \mathcal{M}\left(N, N_x, \Upsilon, \Lambda\right) \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) . \tag{A.18}$$

Remark A.1.4. We note that if estimate (A.12) is known to hold for $N+M \leq N_\circ$ for some $N_\circ \geq 2\mathsf{N}_{\mathrm{dec}} + d + 1$ (instead of $N \leq N_*$ and $M \leq M_*$), and if the bounds (A.13)–(A.14) hold for all $N \leq N_\circ$, then it follows from the method of proof that the bound (A.18) holds for $N + M \leq N_\circ$ and $M \leq N_\circ - 2\mathsf{N}_{\mathrm{dec}} - d - 1$. The only modification required is that instead of considering the cases $N' \leq N_* - \mathsf{N}_{\mathrm{dec}} - d - 1$ and $N' > N_* - \mathsf{N}_{\mathrm{dec}} - d - 1$, we now split into $N' + M \leq N_\circ - \mathsf{N}_{\mathrm{dec}} - d - 1$ and $N' + M > N_\circ - \mathsf{N}_{\mathrm{dec}} - d - 1$. In the second case we use that $N - N'' \geq N_0 - M - \mathsf{N}_{\mathrm{dec}} - d - 1 \geq \mathsf{N}_{\mathrm{dec}}$, where the last inequality holds precisely because $M \leq N_\circ - 2\mathsf{N}_{\mathrm{dec}} - d - 1$.

Proof of Lemma A.1.3. Since $D_t\Phi = 0$ we have $D_t^M(\varrho\circ\Phi) = 0$. Furthermore, since $\operatorname{div} v \equiv 0$, we have that Φ and Φ^{-1} preserve volume. Then using Lemma A.1.2, which we may apply due to (A.13), we have

$$\begin{aligned}
\left\| D^N D_t^M \left(f \ \varrho \circ \Phi\right) \right\|_{L^p} &\lesssim \sum_{N'=0}^{N} \left\| D^{N'} D_t^M f \ D^{N-N'} \left(\varrho \circ \Phi\right) \right\|_{L^p} \\
&\lesssim \sum_{N'=0}^{N} \sum_{N''=0}^{N-N'} \lambda^{N-N'-N''} \left\| D^{N'} D_t^M f \ \left(D^{N''} \varrho\right) \circ \Phi \right\|_{L^p} \\
&\lesssim \sum_{N'=0}^{N} \sum_{N''=0}^{N-N'} \lambda^{N-N'-N''} \left\| \left(D^{N'} D_t^M f\right) \circ \Phi^{-1} D^{N''} \varrho \right\|_{L^p} .
\end{aligned} \tag{A.19}$$

In (A.19) let us first consider the case $N' \leq N_* - \mathsf{N}_{\text{dec}} - d - 1$. Due to assumption (A.14), we may apply Lemma A.1.2, and appealing to (A.12) we have that

$$\begin{aligned}\left\| D^n \left((D^{N'} D_t^M f) \circ (\Phi^{-1}, t) \right) \right\|_{L^p} &\lesssim \sum_{n'=0}^{n} \lambda^{n-n'} \left\| (D^{n'+N'} D_t^M f) \circ \Phi^{-1} \right\|_{L^p} \\ &\lesssim \mathcal{C}_f \sum_{n'=0}^{n} \lambda^{n-n'} \lambda^{n'+N'} \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \\ &\lesssim \left(\mathcal{C}_f \lambda^{N'} \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \right) \lambda^n \end{aligned} \tag{A.20}$$

for all $n \leq \mathsf{N}_{\text{dec}} + d + 1$. This bound matches (A.2), with $\mathcal{C}_f$ replaced by $\mathcal{C}_f \lambda^{N'} \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right)$. Since the function $D^{N''} \varrho$ satisfies (A.15), we may apply (A.20), the fact that $\lambda \leq \Upsilon \leq \Lambda$, assumption (A.16), and Lemma A.1.1 to conclude that

$$\left\| \left(D^{N'} D_t^M f \right) \circ \Phi^{-1} D^{N''} \varrho \right\|_{L^p} \lesssim \mathcal{C}_f \lambda^{N'} \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \mathcal{C}_\varrho \mathcal{M}\left(N'', N_x, \Upsilon, \Lambda\right) .$$

Inserting this bound back into (A.19) concludes the proof of (A.18) for $N' \leq N_* - \mathsf{N}_{\text{dec}} - d - 1$ as considered in this case.

Next, let us consider the case $N' > N_* - \mathsf{N}_{\text{dec}} - d - 1$. Since $0 \leq N' \leq N$, in particular this implies that $N > N_* - \mathsf{N}_{\text{dec}} - d - 1$. Using furthermore that $N'' \leq N - N'$ and (A.17), we also obtain that $N - N'' \geq N' > N_* - \mathsf{N}_{\text{dec}} - d - 1 \geq \mathsf{N}_{\text{dec}}$. Then Hölder's inequality, the fact that Φ^{-1} is volume preserving, the Sobolev embedding $W^{d+1,1} \subset L^\infty$, the ordering

$\Lambda \geq \Upsilon \geq \mu \geq 1$, and assumption (A.16) implies that

$$\begin{aligned}
\lambda^{N-N'-N''} \left\| \left(D^{N'} D_t^M f \right) \circ \Phi^{-1} D^{N''} \varrho \right\|_{L^p} &\lesssim \lambda^{N-N'-N''} \left\| D^{N'} D_t^M f \right\|_{L^p} \left\| D^{N''} \varrho \right\|_{L^\infty} \\
&\lesssim \lambda^{N-N'-N''} \mathcal{C}_f \lambda^{N'} \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \mathcal{C}_\varrho \mathcal{M}\left(N'' + d + 1, N_x, \Upsilon, \Lambda\right) \\
&\lesssim \mathcal{C}_f \mathcal{C}_\varrho \mathcal{M}\left(N, N_x, \Upsilon, \Lambda\right) \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \Lambda^{d+1} \left(\frac{\lambda}{\Upsilon}\right)^{N-N''} \\
&\lesssim \mathcal{C}_f \mathcal{C}_\varrho \mathcal{M}\left(N, N_x, \Upsilon, \Lambda\right) \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) \Lambda^{d+1} \left(\frac{\lambda}{\mu}\right)^{\mathsf{N}_{\mathrm{dec}}} \\
&\lesssim \mathcal{C}_f \mathcal{C}_\varrho \mathcal{M}\left(N, N_x, \Upsilon, \Lambda\right) \mathcal{M}\left(M, N_t, \tau^{-1}, \mathrm{T}^{-1}\right) .
\end{aligned}$$

Combining the above estimate with (A.19), we deduce that the bound (A.18) holds also for $N' > N_* - \mathsf{N}_{\mathrm{dec}} - d - 1$, concluding the proof of the lemma. □

A.2 Sums and iterates of operators and commutators with material derivatives

We first record the following identity for material and spatial derivatives applied to functions raised to a positive integer power.

Lemma A.2.1 (Leibniz rule with material and spatial derivatives). *Let $d \geq 2$ be given, $g : \mathbb{T}^d \to \mathbb{R}$ be a smooth function, $v : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}^d$ a divergence-free vector field, and set $D_t = \partial_t + v \cdot \nabla$, and $p \in \mathbb{N}$. Fix $M, N \in \mathbb{N}$, and use $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ to denote multi-indices with $|\alpha| = N, |\beta| = M$. Then we have the*

identities

$$D^N D_t^M g^p = \sum_{\left\{\substack{\alpha,\beta:\sum_{i=1}^p \alpha_i = N\,,\\ \sum_{i=1}^p \beta_i = M}\right\}} \binom{N}{\alpha_1,\ldots,\alpha_p}\binom{M}{\beta_1,\ldots,\beta_p}\prod_{i=1}^p D^{\alpha_i} D_t^{\beta_i} g \tag{A.21a}$$

$$pg^{p-1} D^N D_t^M g = D^N D_t^M g^p - \sum_{\left\{\substack{\alpha,\beta:\sum_{i=1}^p \alpha_i = N\,,\\ \sum_{i=1}^p \beta_i = M\,,\\ \alpha_i+\beta_i<N+M\,\forall\, i}\right\}} \binom{N}{\alpha_1,\ldots,\alpha_p}\binom{M}{\beta_1,\ldots,\beta_p}\prod_{i=1}^p D^{\alpha_i} D_t^{\beta_i} g\,. \tag{A.21b}$$

We recall [7, Lemma A.10]. We have generalized the statement slightly so that it applies in $\mathbb{T}^d$ rather than just $\mathbb{T}^3$; in fact the statement and proof should have nothing to do with the dimension.

Lemma A.2.2. *Fix $N_x, N_t, N_* \in \mathbb{N}$, $\Omega \in \mathbb{T}^d \times \mathbb{R}$ a space-time domain, and let v be a vector field and B a differential operator. For $k \geq 1$ and $\alpha, \beta \in \mathbb{N}^k$ such that $|\alpha| + |\beta| \leq N_*$, we assume that we have the bounds*

$$\left\| \left(\prod_{i=1}^k D^{\alpha_i} B^{\beta_i} \right) v \right\|_{L^\infty(\Omega)} \lesssim \mathcal{C}_v \mathcal{M}\left(|\alpha|, N_x, \lambda_v, \widetilde{\lambda}_v\right) \mathcal{M}\left(|\beta|, N_t, \mu_v, \widetilde{\mu}_v\right) \tag{A.22}$$

for some $\mathcal{C}_v \geq 0$, $1 \leq \lambda_v \leq \widetilde{\lambda}_v$, and $1 \leq \mu_v \leq \widetilde{\mu}_v$. With the same notation and restrictions on $|\alpha|, |\beta|$, let f be a function which for some $p \in [1, \infty]$ obeys

$$\left\| \left(\prod_{i=1}^k D^{\alpha_i} B^{\beta_i} \right) f \right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M}\left(|\alpha|, N_x, \lambda_f, \widetilde{\lambda}_f\right) \mathcal{M}\left(|\beta|, N_t, \mu_f, \widetilde{\mu}_f\right) \tag{A.23}$$

for some $\mathcal{C}_f \geq 0$, $1 \leq \lambda_f \leq \widetilde{\lambda}_f$, and $1 \leq \mu_f \leq \widetilde{\mu}_f$. Denote

$$\lambda = \max\{\lambda_f, \lambda_v\}, \quad \widetilde{\lambda} = \max\{\widetilde{\lambda}_f, \widetilde{\lambda}_v\}, \quad \mu = \max\{\mu_f, \mu_v\}, \quad \widetilde{\mu} = \max\{\widetilde{\mu}_f, \widetilde{\mu}_v\}.$$

Then, for

$$A = v \cdot \nabla$$

we have the bounds

$$\left\| D^n \left(\prod_{i=1}^{k} A^{\alpha_i} B^{\beta_i} \right) f \right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{C}_v^{|\alpha|} \mathcal{M}\left(n + |\alpha|, N_x, \lambda, \widetilde{\lambda}\right) \mathcal{M}\left(|\beta|, N_t, \mu, \widetilde{\mu}\right) \tag{A.24}$$

$$\lesssim \mathcal{C}_f \mathcal{M}\left(n, N_x, \lambda, \widetilde{\lambda}\right) (\mathcal{C}_v \widetilde{\lambda})^{|\alpha|} \mathcal{M}\left(|\beta|, N_t, \mu, \widetilde{\mu}\right)$$

$$\lesssim \mathcal{C}_f \mathcal{M}\left(n, N_x, \lambda, \widetilde{\lambda}\right) \mathcal{M}\left(|\alpha| + |\beta|, N_t, \max\{\mu, \mathcal{C}_v \widetilde{\lambda}\}, \max\{\widetilde{\mu}, \mathcal{C}_v \widetilde{\lambda}\}\right) \tag{A.25}$$

as long as $n + |\alpha| + |\beta| \leq N_*$. *As a consequence, if* $k = m$ *then* (A.25) *and an expansion of the operator* $(A + B)^M$ *imply that for all* $n + m \leq N_*$,

$$\|D^n (A + B)^m f\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M}\left(n, N_x, \lambda, \widetilde{\lambda}\right) \mathcal{M}\left(m, N_t, \max\{\mu, \mathcal{C}_v \widetilde{\lambda}\}, \max\{\widetilde{\mu}, \mathcal{C}_v \widetilde{\lambda}\}\right). \tag{A.26}$$

A corollary of the previous lemma is the commutator lemma [7, Lemma A.14], which we now record along with several useful remarks.

Lemma A.2.3. *Let* $p \in [1, \infty]$. *Fix* $N_x, N_t, N_*, M_* \in \mathbb{N}$, *let* v *be a vector field, let* $D_t = \partial_t + v \cdot \nabla$ *be the associated material derivative, and let* Ω *be a space-time domain. Assume that the vector field* v *obeys*

$$\left\| D^N D_t^M Dv \right\|_{L^\infty(\Omega)} \lesssim \mathcal{C}_v \mathcal{M}\left(N + 1, N_x, \lambda_v, \widetilde{\lambda}_v\right) \mathcal{M}\left(M, N_t, \mu_v, \widetilde{\mu}_v\right) \tag{A.27}$$

for $N \leq N_*$ *and* $M \leq M_*$. *Moreover, let* f *be a function which obeys*

$$\left\| D^N D_t^M f \right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M}\left(N, N_x, \lambda_f, \widetilde{\lambda}_f\right) \mathcal{M}\left(M, N_t, \mu_f, \widetilde{\mu}_f\right) \tag{A.28}$$

for all $N \leq N_*$ and $M \leq M_*$. Denote

$$\lambda = \max\{\lambda_f, \lambda_v\}, \quad \widetilde{\lambda} = \max\{\widetilde{\lambda}_f, \widetilde{\lambda}_v\}, \quad \mu = \max\{\mu_f, \mu_v\}, \quad \widetilde{\mu} = \max\{\widetilde{\mu}_f, \widetilde{\mu}_v\}.$$

Let $m, n, \ell \geq 0$ be such that $n + \ell \leq N_*$ and $m \leq M_*$. Then, we have that the commutator $[D_t^m, D^n]$ is bounded as

$$\left\|D^\ell \left[D_t^m, D^n\right] f\right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{C}_v \widetilde{\lambda}_v \mathcal{M}\left(\ell + n, N_x, \lambda, \widetilde{\lambda}\right) \mathcal{M}\left(m - 1, N_t, \max\{\mu, \mathcal{C}_v \widetilde{\lambda}_v\}, \max\{\widetilde{\mu}, \mathcal{C}_v \widetilde{\lambda}_v\}\right) \tag{A.29}$$

$$\lesssim \mathcal{C}_f \mathcal{M}\left(\ell + n, N_x, \lambda, \widetilde{\lambda}\right) \mathcal{M}\left(m, N_t, \max\{\mu, \mathcal{C}_v \widetilde{\lambda}_v\}, \max\{\widetilde{\mu}, \mathcal{C}_v \widetilde{\lambda}_v\}\right). \tag{A.30}$$

Moreover, we have that for $k \geq 2$, and any $\alpha, \beta \in \mathbb{N}^k$ with $|\alpha| \leq N_*$ and $|\beta| \leq M_*$, the estimate

$$\left\|\left(\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i}\right) f\right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M}\left(|\alpha|, N_x, \lambda, \widetilde{\lambda}\right) \mathcal{M}\left(|\beta|, N_t, \max\{\mu, \mathcal{C}_v \widetilde{\lambda}_v\}, \max\{\widetilde{\mu}, \mathcal{C}_v \widetilde{\lambda}_v\}\right) \tag{A.31}$$

holds.

Remark A.2.4. If instead of (A.27) and (A.28) holding for $N \leq N_*$ and $M \leq M_*$, we know that both of these inequalities hold for all $N + M \leq N_\circ$ for some $N_\circ \geq 1$, then the conclusions of the Lemma hold as follows: the bounds (A.29) and (A.30) hold for $\ell + n + m \leq N_\circ$, while (A.31) holds for $|\alpha| + |\beta| \leq N_\circ$. We refer to [7] for further discussion.

Remark A.2.5. If the assumption (A.28) is replaced by

$$\left\|D^N D_t^M f\right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M}\left(N - 1, N_x, \lambda_f, \widetilde{\lambda}_f\right) \mathcal{M}\left(M, N_t, \mu_f, \widetilde{\mu}_f\right), \tag{A.32}$$

whenever $1 \leq N \leq N_*$, then the conclusion (A.31) instead becomes

$$\left\| \left(\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} \right) f \right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M} \left(|\alpha| - 1, N_x, \lambda, \widetilde{\lambda} \right) \mathcal{M} \left(|\beta|, N_t, \max\{\mu, \mathcal{C}_v \widetilde{\lambda}_v\}, \max\{\widetilde{\mu}, \mathcal{C}_v \widetilde{\lambda}_v\} \right) \tag{A.33}$$

whenever $|\alpha| \geq 1$. We again refer to [7] for further discussion.

Remark A.2.6. Fix $p \in [1, \infty]$, $N_x, N_t, N_* \in \mathbb{N}$, and a space-time domain $\Omega \in \mathbb{T}^d \times \mathbb{R}$. Define $D_t = \partial_t + (v \cdot \nabla)$ as in Lemma A.2.3. Suppose that for $k \geq 1$ and $\alpha, \beta \in \mathbb{N}^k$ such that $|\alpha| + |\beta| \leq N_*$, we have the bounds

$$\left\| \left(\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} \right) w \right\|_{L^\infty(\Omega)} \lesssim \mathcal{C}_w \mathcal{M} \left(|\alpha|, N_x, \lambda_w, \widetilde{\lambda}_w \right) \mathcal{M} \left(|\beta|, N_t, \mu_w, \widetilde{\mu}_w \right) \tag{A.34}$$

for some $\mathcal{C}_w \geq 0$, $1 \leq \lambda_w \leq \widetilde{\lambda}_w$, and $1 \leq \mu_w \leq \widetilde{\mu}_w$. Then, under the assumption (A.27) and (A.28) in Lemma A.2.3 with $M_* = N_*$, we have that for all $N, M \leq N_*$,

$$\left\| D^N (D_t + (w \cdot \nabla))^M f \right\|_{L^p(\Omega)} \lesssim \mathcal{C}_f \mathcal{M} \left(n, N_x, \lambda, \widetilde{\lambda} \right) \mathcal{M} \left(m, N_t, \mu, \widetilde{\mu} \right) \tag{A.35}$$

where

$$\lambda = \max\{\lambda_f, \lambda_v, \lambda_w\}, \quad \widetilde{\lambda} = \max\{\widetilde{\lambda}_f, \widetilde{\lambda}_v, \widetilde{\lambda}_w\}, \quad \mu = \max\{\mu_f, \mu_v, \mu_w, \mathcal{C}_v \widetilde{\lambda}_v, \mathcal{C}_w \widetilde{\lambda}_w\},$$
$$\widetilde{\mu} = \max\{\widetilde{\mu}_f, \widetilde{\mu}_v, \widetilde{\mu}_w, \mathcal{C}_v \widetilde{\lambda}_v, \mathcal{C}_w \widetilde{\lambda}_w\}.$$

If (A.27) and (A.28) hold for $N + M \leq N_*$, as in Remark A.2.4, then (A.35) holds also for $N + M \leq N_*$.

A.3 Inversion of the divergence

Proposition A.3.1 (Inverse divergence iteration step). *Let $n \geq 2$ be given. Fix a zero-mean $\mathbb{T}^n$-periodic function ϱ and a zero-mean $\mathbb{T}^n$-periodic symmetric tensor field $\vartheta^{(i,j)}$ which are related by $\varrho = \partial_{ij}\vartheta^{(i,j)}$. Let Φ be a volume preserving diffeomorphism of $\mathbb{T}^n$. Define the matrix $A = (\nabla\Phi)^{-1}$. Given a vector field G^k, we have*

$$G^k(\varrho \circ \Phi) = \partial_\ell R^{k\ell} + E^k \tag{A.36}$$

where the symmetric stress $R^{k\ell}$ is given by

$$R^{k\ell} = G^k A_i^\ell(\partial_j \vartheta^{(i,j)} \circ \Phi) + G^\ell A_i^k(\partial_j \vartheta^{(i,j)} \circ \Phi) - G^n \partial_n \Phi^m A_i^k A_j^\ell(\partial_m \vartheta^{(i,j)} \circ \Phi)\,, \tag{A.37}$$

and the error term E^k is given by

$$E^k = -\partial_\ell(G^\ell A_i^k)(\partial_j \vartheta^{(i,j)} \circ \Phi) - (\partial_\ell G^k) A_i^\ell(\partial_j \vartheta^{(i,j)} \circ \Phi) + \partial_n(G^\ell A_i^k \partial_\ell \Phi^m) A_j^n(\partial_m \vartheta^{(i,j)} \circ \Phi)\,. \tag{A.38}$$

Remark A.3.2 (Linearity with respect to G). From (A.37) and (A.38), it is clear that the symmetric stress and error term are *linear* in G; more precisely, each term of the symmetric stress and error may be written as a product of flow maps, high frequency functions, and a single component of either G or ∇G. This will be a useful observation when determining the support properties of the symmetric stresses and error terms.

Proof of Proposition A.3.1. By the definition of A, we have $A_\ell^n \partial_k \Phi^\ell = \delta_{nk}$, and the volume-preserving property of Φ gives the Piola identity $\partial_n A_\ell^n = 0$. These then imply a useful

identity $(\partial_\ell \varphi) \circ \Phi = \partial_n (A^n_\ell (\varphi \circ \Phi))$. Using this, we first get

$$
\begin{aligned}
G^k(\varrho \circ \Phi) &= G^k(\partial_i \partial_j \vartheta^{(i,j)} \circ \Phi) = G^k \partial_\ell (A^\ell_i (\partial_j \vartheta^{(i,j)} \circ \Phi)) = \partial_\ell (G^k A^\ell_i (\partial_j \vartheta^{(i,j)} \circ \Phi)) - (\partial_\ell G^k) A^\ell_i (\partial_j \vartheta^{(i,j)} \circ \Phi) \\
&= \partial_\ell (G^k A^\ell_i (\partial_j \vartheta^{(i,j)} \circ \Phi) + G^\ell A^k_i (\partial_j \vartheta^{(i,j)} \circ \Phi)) - G^\ell A^k_i \partial_\ell \Phi^m (\partial_m \partial_j \vartheta^{(i,j)}) \circ \Phi \\
&\quad - \partial_\ell (G^\ell A^k_i)(\partial_j \vartheta^{(i,j)} \circ \Phi) - (\partial_\ell G^k) A^\ell_i (\partial_j \vartheta^{(i,j)} \circ \Phi) \,.
\end{aligned}
$$

In the last equality,the first two terms match the first two terms in $\partial_\ell R^{k\ell}$, while the last two terms will go into the error term E^k. To deal with the remaining term, we use

$$
\begin{aligned}
G^\ell A^k_i \partial_\ell \Phi^m (\partial_m \partial_j \vartheta^{(i,j)}) \circ \Phi &= G^\ell A^k_i \partial_\ell \Phi^m \partial_n (A^n_j (\partial_m \vartheta^{(i,j)} \circ \Phi)) \\
&= \partial_n (G^\ell \partial_\ell \Phi^m A^k_i A^n_j (\partial_m \vartheta^{(i,j)} \circ \Phi)) - \partial_n (G^\ell A^k_i \partial_\ell \Phi^m) A^n_j (\partial_m \vartheta^{(i,j)} \circ \Phi) \,.
\end{aligned}
$$

Indeed, plugging this identity into the second term, we obtain the symmetric stress $R^{k\ell}$ and error term E^k. Note that the first term above is symmetric due to the assumed symmetry of $\vartheta^{(i,j)}$. □

With the iterative step in hand, we can now state the proposition which contains our main inverse divergence algorithm. The spirit of the statement and proof is similar to the corresponding statements and proofs in [7, 35], modulo minor adjustments. After stating the main proposition, we record a number of useful remarks which follow from the proof.

Proposition A.3.3 (Main inverse divergence operator). *Let dimension $n \geq 2$ and Lebesgue exponent $p \in [1, \infty]$ be free parameters. The remainder of the proposition is composed first of* low and high-frequency assumptions*, which then produce a* localized output *satisfying a number of properties. Finally, the proposition concludes with* nonlocal assumptions and output*.*

Part 1: Low-frequency assumptions

(i) Let G be a vector field and assume there exist a constant $\mathcal{C}_{G,p} > 0$ and parameters

$$N_* \geq M_* \geq 1\,, \tag{A.39}$$

M_t, and $\lambda, \nu, \nu' \geq 1$ such that

$$\left\|D^N D_t^M G\right\|_{L^p} \lesssim \mathcal{C}_{G,p} \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.40}$$

for all $N \leq N_*$ and $M \leq M_*$.

(ii) Fix an incompressible vector field $v(t,x) : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}^n$ and denote its material derivative by $D_t = \partial_t + v \cdot \nabla$. Let Φ be a volume preserving diffeomorphism of $\mathbb{T}^n$ such that

$$D_t \Phi = 0 \qquad \text{and} \qquad \|\nabla \Phi - \mathrm{Id}\|_{L^\infty(\operatorname{supp} G)} \leq {}^1\!/_2\,. \tag{A.41}$$

Denote by Φ^{-1} the inverse of the flow Φ, which is the identity at a time slice which intersects the support of G. Assume that the velocity field v and the flow functions Φ and Φ^{-1} satisfy the bounds

$$\left\|D^{N+1}\Phi\right\|_{L^\infty(\operatorname{supp} G)} + \left\|D^{N+1}\Phi^{-1}\right\|_{L^\infty(\operatorname{supp} G)} \lesssim \lambda'^N \tag{A.42a}$$

$$\left\|D^N D_t^M D v\right\|_{L^\infty(\operatorname{supp} G)} \lesssim \nu \lambda'^N \mathcal{M}\left(M, M_t, \nu, \nu'\right)\,, \tag{A.42b}$$

for all $N \leq N_*$, $M \leq M_*$, and some $\lambda' > 0$.

Part 2: High-frequency assumptions

(i) Let $\varrho\colon \mathbb{T}^n \to \mathbb{R}$ be a zero mean scalar function such that there exists a large positive even integer $\mathsf{d} \gg 1$ and a smooth, mean-zero, adjacent-pairwise symmetric tensor potential[2]

[2]We use i_j for $1 \leq j \leq \mathsf{d}$ to denote any number in the set $\{1, \ldots, n\}$. We refer to Lemma 7.3.3 for the meaning of adjacent-pairwise symmetric.

$\vartheta^{(i_1,\dots,i_{\mathsf{d}})} : \mathbb{T}^n \to \mathbb{R}^{\binom{n}{\mathsf{d}}}$ *such that* $\varrho(x) = \partial_{i_1} \dots \partial_{i_{\mathsf{d}}} \vartheta^{(i_1 \dots i_{\mathsf{d}})}(x)$.

(ii) There exists a parameter $\mu \geq 1$ *such that* ϱ *and* ϑ *are* $(\mathbb{T}/\mu)^n$*-periodic.*

(iii) There exist parameters $1 \ll \Upsilon \leq \Upsilon' \leq \Lambda$, $\mathcal{C}_{*,p} > 0$ *such that for all* $0 \leq N \leq N_*$ *and all* $0 \leq k \leq \mathsf{d}$,

$$\left\| D^N \partial_{i_1} \dots \partial_{i_k} \vartheta^{(i_1,\dots,i_{\mathsf{d}})} \right\|_{L^p} \lesssim \mathcal{C}_{*,p} \Upsilon^{k-\mathsf{d}} \mathcal{M}\left(N, \mathsf{d}-k, \Upsilon', \Lambda\right) . \tag{A.43}$$

(iv) There exists $\mathsf{N}_{\mathrm{dec}}$ *such that the above parameters satisfy*

$$\lambda', \lambda \ll \mu \leq \Upsilon \leq \Upsilon' \leq \Lambda \,, \qquad \max(\lambda, \lambda') \Upsilon^{-2} \Upsilon' \leq 1 \,, \qquad N_* - \mathsf{d} \geq 2\mathsf{N}_{\mathrm{dec}} + n + 1 \,, \tag{A.44}$$

where by in the first inequality in (A.44) *we mean that*

$$\Lambda^{n+1} \left(\frac{\mu}{2\pi\sqrt{3}\max(\lambda, \lambda')} \right)^{-\mathsf{N}_{\mathrm{dec}}} \leq 1 \,. \tag{A.45}$$

Part 3: Localized output

(i) There exist tensors R *and* E *such that*

$$G\ \varrho \circ \Phi = \mathrm{div} R + E =: \mathrm{div}\left(\mathcal{H}\left(G\varrho \circ \Phi\right)\right) + E \,. \tag{A.46}$$

We use the notation $R = \mathcal{H}(G\varrho \circ \Phi)$ *for the symmetric stress.*

(ii) The support of R *is a subset of* $\operatorname{supp} G \cap \operatorname{supp} \vartheta$.

(iii) There exists an explicitly computable positive integer $\mathcal{C}_{\mathcal{H}}$, *an explicitly computable func-*

tion $r(j) : \{0, 1, \ldots, \mathcal{C}_{\mathcal{H}}\} \to \mathbb{N}$ *and explicitly computable tensors*

$$\rho^{\beta(j)}\,, \qquad \beta(j) = (\beta_1, \beta_2, \ldots, \beta_{r(j)}) \in \{1, \ldots, n\}^{r(j)}\,,$$
$$H^{\alpha(j)}\,, \qquad \alpha(j) = (\alpha_1, \alpha_2, \ldots, \alpha_{r(j)}, k, \ell) \in \{1, \ldots, n\}^{r(j)+2}$$

of rank $r(j)$ *and* $r(j) + 2$*, respectively, all of which depend only on* G*,* ϱ*,* Φ*,* n*,* d*, such that the following holds. The symmetric, localized stress* R *can be decomposed into a sum of symmetric, localized stresses as*[3]

$$\mathcal{H}^{k\ell}(G\varrho \circ \Phi) = R^{k\ell} = \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} H^{\alpha(j)} \rho^{\beta(j)} \circ \Phi\,. \tag{A.47}$$

Furthermore, we have that

$$\operatorname{supp} H^{\alpha(j)} \subseteq \operatorname{supp} G\,, \qquad \operatorname{supp} \rho^{\beta(j)} \subseteq \operatorname{supp} \vartheta\,. \tag{A.48}$$

(iv) For all $N \leq N_* - \mathsf{d}/2$*,* $M \leq M_*$*, and* $j \leq \mathcal{C}_{\mathcal{H}}$*, we have the subsidiary estimates*[4]

$$\left\| D^N \rho^{\beta(j)} \right\|_{L^p} \lesssim \mathcal{C}_{*,p} \Upsilon^{-2} \Upsilon' \mathcal{M}\left(N, 1, \Upsilon', \Lambda\right) \tag{A.49a}$$
$$\left\| D^N D_t^M H^{\alpha(j)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \left(\max(\lambda, \lambda')\right)^N \mathcal{M}\left(M, M_t, \nu, \nu'\right)\,. \tag{A.49b}$$

(v) For all $N \leq N_* - \mathsf{d}/2$ *and* $M \leq M_*$*, we have the main estimate*

$$\left\| D^N D_t^M R \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \Upsilon' \Upsilon^{-2} \mathcal{M}\left(N, 1, \Upsilon', \Lambda\right) \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.50}$$

[3]The contraction is on the first $r(j)$ indices, and the resulting rank two tensor is symmetric.

[4]In fact it is clear from the algorithm that as j increases, the estimates become much stronger. For simplicity's sake we simply record identical estimates for each term which are sufficient for our aims.

(vi) For $N \leq N_* - \mathsf{d}/2$ and $M \leq M_*$ the error term E in (A.46) *satisfies*

$$\left\|D^N D_t^M E\right\|_{L^p} \lesssim \mathcal{C}_{G,p}\mathcal{C}_{*,p} \max(\lambda,\lambda')^{\mathsf{d}/2} \left(\Upsilon'\Upsilon^{-2}\right)^{\mathsf{d}/2} \Lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) . \tag{A.51}$$

Part 4: Nonlocal assumptions and output

(i) Let $N_\circ, M_\circ$ be integers such that

$$1 \leq M_\circ \leq N_\circ \leq {}^{M_*}/_2 \,, \tag{A.52}$$

and let $K_\circ$ be a positive integer.[5] *Assume that in addition to the bound* (A.42b) *we have the following global lossy estimates*

$$\left\|D^N \partial_t^M v\right\|_{L^\infty} \lesssim \mathcal{C}_v \lambda'^N \nu'^M \tag{A.53}$$

for all $M \leq M_\circ$ and $N + M \leq N_\circ + M_\circ$, where

$$\mathcal{C}_v \lambda' \lesssim \nu' \,. \tag{A.54}$$

(ii) Assume that d is large enough so that

$$\mathcal{C}_{G,p}\mathcal{C}_{*,p} \max(\lambda,\lambda')^{\mathsf{d}/4} (\Upsilon'\Upsilon^{-2})^{\mathsf{d}/4} \Lambda^{n+2+K_\circ} \left(1 + \frac{\max\{\nu', \mathcal{C}_v\Lambda\}}{\nu}\right)^{M_\circ} \leq 1 \,. \tag{A.55}$$

Then we may write

$$E = \operatorname{div}\overline{R}_{\text{nonlocal}} + \fint_{\mathbb{T}^3} G\varrho \circ \Phi \, dx =: \operatorname{div}\left(\mathcal{R}^*(G\varrho \circ \Phi)\right) + \fint_{\mathbb{T}^3} G\varrho \circ \Phi \, dx \,, \tag{A.56}$$

[5] $K_\circ$ serves as an extra amplitude gain which will be used later to eat some material derivative losses.

where $\overline{R}_{\text{nonlocal}} = \mathcal{R}^*(G\varrho \circ \Phi)$ *is a traceless symmetric stress which satisfies*

$$\left\| D^N D_t^M \overline{R}_{\text{nonlocal}} \right\|_{L^\infty} \leq \frac{1}{\Lambda^{K_\circ}} \max(\lambda, \lambda')^{\mathsf{d}/4} (\Upsilon' \Upsilon^{-2})^{\mathsf{d}/4} \Lambda^N \nu^M \tag{A.57}$$

for $N \leq N_\circ$ *and* $M \leq M_\circ$.

Remark A.3.4 (**Lossy derivatives on** v **and estimates for** $\overline{R}_{\text{nonlocal}}$**).** Let us specify the estimates we expect to obtain from (A.57) for the nonlocal error term $\overline{R}_{\text{nonlocal}}$. For our applications, we need to choose parameters so that the estimate reads

$$\left\| D^N D_t^M \overline{R}_{\text{nonlocal}} \right\|_{L^\infty} \leq \lambda_{q+\bar{n}}^{-10} \delta_{q+3\bar{n}}^2 \mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\text{ind,t}}} \lambda_{q+\bar{n}}^N \tau_q^{-M} \tag{A.58}$$

for $N, M \leq 2\mathsf{N}_{\text{ind}}$. We therefore choose $N_\circ = M_\circ = 2\mathsf{N}_{\text{ind}}$, and since in applications M_* will be at least $\mathsf{N}_{\text{fin}}/10000$, we have from (4.24a) that $M_\circ \leq N_\circ \leq M_*/2$. Next, we choose $K_\circ$ large enough so that $\lambda_q^{-K_\circ} \leq \delta_{q+3\bar{n}}^2 \mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\text{ind,t}}} \lambda_{q+\bar{n}}^{-100}$, which follows from (4.22). The lossy estimates in (A.53) follow from the inductive assumption (5.35b) with $\mathcal{C}_v = \Lambda_q^{1/2}$; note that (A.54) is precisely (4.15). Finally, the inequality in (A.55) will be a consequence of our choices of $\lambda, \lambda', \Upsilon', \Upsilon$, which from (4.10d) give a gain of at least $\Gamma_q^{-\lfloor \mathsf{d}/40 \rfloor}$, and (4.23b).

Remark A.3.5 (**Special case for negligible error terms).** The inverse divergence operator defined in the proposition can be applied to an input without the structure of low and high frequency parts when $\varrho = 1$ and $\mathcal{C}_{G,p}$ are sufficiently small. More precisely, we keep the low-frequency assumption (Part 1), replace the high-frequency assumptions (Part 2) with $\varrho = 1$, and set $\Upsilon = \Upsilon' = \Lambda = \max(\lambda, \lambda')$, $\mathcal{C}_{*,p} = 1$, $\mathsf{d} = 0$. [6] Then, as long as $\mathcal{C}_{G,p}$ is small enough to satisfy (A.55), the conclusions in Part 4 hold. In particular, we have that

$$G = \operatorname{div}\mathcal{R}^* G + \fint_{T^3} G \, dx \, .$$

Note that $\mathcal{R}^* G = \mathcal{R} G$ in the special case, where $\mathcal{R}$ is the usual inverse divergence operator

[6] Since we do not need decoupling, μ does not need to be specified.

defined in (A.80).

Remark A.3.6 (High frequency part of the output as a potential). In order to obtain the conclusions in Remarks 10.2.2, 10.2.7, and 10.2.11, we need to write $\rho^{\beta(j)}$ as a potential. This can be done if the potentials $\vartheta^{(i_1,\cdots,i_{\mathsf{d}})}$ used in the application of the inverse divergence in Section 10 can be written as $\vartheta^{(i_1,\cdots,i_{\mathsf{d}})} = \partial_{i_{\mathsf{d}+1}\cdots i_{2\mathsf{d}}}\theta^{(i_{\mathsf{d}+1},\cdots,i_{2\mathsf{d}})}$, where θ satisfies

$$\left\|D^N \mathrm{div}^k \theta^{(i_1,\cdots,i_{2\mathsf{d}})}\right\|_{L^p} \lesssim \mathcal{C}_{*,p}\Upsilon^{k-2\mathsf{d}}\mathcal{M}\left(N, 2\mathsf{d}-k, \Upsilon', \Lambda\right)$$

for $0 \leq k \leq 2\mathsf{d}$ and $N \leq N_*$. This is easily ensured by *initially* choosing ϱ as $\varrho = \partial_{i_1\cdots i_{2\mathsf{d}}}\theta^{(i_1,\cdots,i_{2\mathsf{d}})}$, where we save half of the divergences for later to enable the application of the inverse divergence algorithm a second time in Section 11. Since the inverse divergence algorithm shows that $\rho^{\beta(j)}$ consists of spatial derivatives and divergences of ϑ, it is clear that $\rho^{\beta(j)}$ can be written in potential form as $\rho^{\beta(j)} = \partial_{i_{\mathsf{d}+1}\cdots i_{\mathsf{d}+k}}\overline{\theta}^{(i_{\mathsf{d}+1},\cdots,i_k,\beta(j))}$ for some potential $\overline{\theta}^{(i_{\mathsf{d}+1},\cdots,i_k,\beta(j))}$. Furthermore, we have

$$\left\|D^N \partial_{i_{\mathsf{d}+1}\cdots i_{\mathsf{d}+k}}\overline{\theta}^{(i_{\mathsf{d}+1},\cdots,i_{\mathsf{d}+k},\beta(j))}\right\|_{L^p} \lesssim \mathcal{C}_{*,p}(\Upsilon^{-2}\Upsilon')\Upsilon^{k-\mathsf{d}}\mathcal{M}\left(N, \mathsf{d}-k+1, \Upsilon', \Lambda\right)$$

for $0 \leq k \leq \mathsf{d}$ and $N \leq N_* - \mathsf{d}/2$.

Remark A.3.7 (Mean of the error term). We claim that the mean $\langle G(\varrho \circ \Phi)\rangle$ satisfies

$$\left|\frac{d^M}{dt^M}\langle G(\varrho \circ \Phi)\rangle\right| \leq \Lambda^{-K_\circ}(\max(\lambda,\lambda')\Upsilon^{-1})^{\frac{3}{4}\mathsf{d}}\mathcal{M}\left(M, M_t, \nu, \nu'\right)$$

for $M \leq M_\circ$. To see this, first note that since v is incompressible, $\frac{d^M}{dt^M}\langle G(\varrho\circ\Phi)\rangle = \langle (D_t^M G)(\varrho\circ\Phi)\rangle$. Then using Lemma A.1.1 with (A.45), (A.40), (A.42a), (A.43), and (A.55), we have

the desired estimate

$$\begin{aligned}\left|\int_{\mathbb{T}^3}(D_t^M G)(\varrho\circ\Phi)dx\right| &= \left|\int_{\mathbb{T}^3}(D_t^M G)\circ\Phi^{-1}\mathrm{div}^{\mathsf{d}}\vartheta dx\right| = \left|\int_{\mathbb{T}^3}\partial_{(i_1,\cdots,i_{\mathsf{d}})}((D_t^M G)\circ\Phi^{-1})\vartheta^{(i_1,\cdots,i_{\mathsf{d}})}dx\right| \\ &\lesssim \left\|\partial_{(i_1,\cdots,i_{\mathsf{d}})}((D_t^M G)\circ\Phi^{-1})\right\|_1 \left\|\vartheta^{(i_1,\cdots,i_{\mathsf{d}})}\right\|_1 \\ &\lesssim \mathcal{C}_{G,p}\mathcal{C}_{*,p}(\max(\lambda,\lambda'))^{\mathsf{d}}\Upsilon^{-\mathsf{d}}\mathcal{M}\left(M,M_t,\nu,\nu'\right) \\ &\leq \Lambda^{-K_\circ}(\max(\lambda,\lambda')\Upsilon^{-1})^{\frac{3}{4}\mathsf{d}}\mathcal{M}\left(M,M_t,\nu,\nu'\right)\ .\end{aligned}$$

Inn particular, under the same choice of parameters suggested in Remark A.3.4, we have

$$\left|\frac{d^M}{dt^M}\langle G(\varrho\circ\Phi)\rangle\right| \leq \lambda_{q+\bar{n}}^{-10}\delta_{q+3\bar{n}}^2\mathrm{T}_{q+\bar{n}}^{4\mathsf{N}_{\mathrm{ind,t}}}\tau_q^{-M}$$

for $M \leq 2\mathsf{N}_{\mathrm{ind}}$.

Remark A.3.8 (Inverse divergence for scalar fields). Adjusting the above proposition so that G is a scalar field and the output is a vector field is simple; one can make the substitution $G \to \left(G, \underbrace{0,\ldots,0}_{n-1\ 0'\mathrm{s}}\right)$, apply the Proposition to the newly constructed vector field, and take the first row or column of the symmetric stress as the output.

Remark A.3.9 (Inverse divergence with pointwise bounds). Let us consider the setting in which all the inductive assumptions from the proposition hold, or are adjusted according to Remark A.3.8, but we know in addition that there exists a smooth, non-negative function π such that

$$\left|D^N D_t^M G\right| \lesssim \pi\lambda^N\mathcal{M}\left(M,M_t,\nu,\nu'\right)\ . \tag{A.59}$$

for $N \leq N_*$ and $M \leq M_*$. Then it is clear from the algorithm utilized in the proof that we may additionally conclude that

$$\left|D^N D_t^M H^{\alpha_{(j)}}\right| \lesssim \pi\left(\max(\lambda,\lambda')\right)^N\mathcal{M}\left(M,M_t,\nu,\nu'\right) \tag{A.60}$$

for $N \leq N_* - \lfloor \mathsf{d}/2 \rfloor$ and $M \leq M_*$.

Remark A.3.10 (Avoiding abuses of notations). Proposition A.3.3, and indeed many of the other "abstract nonsense" lemmas and propositions in the manuscript, are written using generic notations such as $\lambda, \mathcal{C}_{G,3/2}$, etc. Application of the lemma or proposition then requires specification of values for these various inputs. Occasionally several such lemmas or propositions will be applied in succession; for example, repeated applications of the inverse divergence as in Corollary A.3.11. In such situations, we shall add bars above all symbols in the statements of the "abstract nonsense" lemmas, and then specify an input for the "bar variable." For example, applying Proposition A.3.3 to a term from the sum in (A.47) (which has the same form as the input of the inverse divergence, just with different parameters!) would be done using the parameter choices $\overline{\mathcal{C}}_{G,p} = \mathcal{C}_{G,p}$, $\overline{\lambda} = \max(\lambda, \lambda')$, $\overline{\mathcal{C}}_{*,p} = \mathcal{C}_{*,p}\Upsilon^{-2}\Upsilon'$, and $\overline{N}_* = N_* - \lfloor \mathsf{d}/2 \rfloor$, which are all valid choices due to (A.49).

Proof of Proposition A.3.3. We divide the proof into four steps. First, we collect some simple preliminary bounds. Next, we apply Proposition A.3.1 the first time and show that an error term is produced which obeys the estimates required in (A.50). Afterwards we indicate how to apply the algorithm $\lfloor \mathsf{d}/2 \rfloor - 1$ more times to produce R and E obeying (A.50) and (A.51), respectively. By construction, both R and E will be supported in $\operatorname{supp} G \cap \operatorname{supp} \vartheta \circ \Phi$. The support property for R and the conclusions in (A.47), (A.49), (A.50), and (A.51) will be proven along the way. Finally, we outline how to obtain the bounds in (A.57) for the nonlocal portion of the inverse divergence. The entire proof follows closely the method of proof of [7, Proposition A.18], the main differences being the slight adjustment to the iteration step due to the difference between Proposition A.3.1 and [7, Proposition A.17], and the slightly more general assumption in (A.43) compared to [7, A.69]. The only significant difference to the conclusion is that the amplitude gain is $\Upsilon'\Upsilon^{-2}$, cf. (A.50) compared to [7, A.73].

`Step 1:` An application of Lemma A.2.3, or more precisely Remark A.2.5, yields

$$\left\| D^{N''} D_t^M D^{N'} D\Phi \right\|_{L^\infty(\operatorname{supp} G)} \lesssim \lambda'^{N'+N''} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.61}$$

whenever $N' + N'' \leq N_*$ and $M \leq M_*$. We similarly obtain

$$\left\| D^{N''} D_t^M D^{N'} (D\Phi)^{-1} \right\|_{L^\infty(\operatorname{supp} G)} \lesssim \lambda'^{N'+N''} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.62}$$

from the Fa'a di Bruno formula (A.9), the formula for matrix inversion in $B_{1/2}(\mathrm{Id})$, the Liebniz rule, and (A.61). Another application of Lemma A.2.3 yields

$$\left\| D^{N''} D_t^M D^{N'} G \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \lambda^{N'+N''} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.63}$$

whenever $N' + N'' \leq N_*$ and $M \leq M_*$. These preliminary bounds are similar to those from the beginning of the proof of [7, Proposition A.18], and we refer there for further details.

`Step 2:` For notational purposes, let $\varrho_{(0)} = \varrho$ and $\varrho_{(\mathsf{d})}^{(i_1,\dots,i_\mathsf{d})} = \vartheta^{(i_1,\dots,i_\mathsf{d})}$, and for $1 \leq k < \mathsf{d}$ let $\varrho_{(k)}^{i_{\mathsf{d}-k+1},\dots,i_\mathsf{d}} = \partial_{i_1} \dots \partial_{i_{\mathsf{d}-k}} \vartheta^{(i_1,\dots,i_\mathsf{d})}$. Then $\varrho_{(k-1)} = \operatorname{div} \varrho_{(k)}$ (assuming contraction along the proper index, which we omit in a slight abuse of notation), and for any "pairwise permutation"[7] $\sigma : \{\mathsf{d}-k+1, \dots, \mathsf{d}\} \to \{\mathsf{d}-k+1, \dots, \mathsf{d}\}$, $\varrho_{(k)}^{i_{\mathsf{d}-k+1},\dots,i_\mathsf{d}} = \varrho_{(k)}^{i_{\sigma(\mathsf{d}-k+1)},\dots,i_{\sigma(\mathsf{d})}}$, so that $\varrho_{(k)}$ is pairwise symmetric. We also define $G_{(0)} = G$. Since $\rho_{(0)} = \operatorname{div}\operatorname{div}\rho_{(2)}$ where $\rho_{(2)}$ is pairwise symmetric, we deduce from Proposition A.3.1, identities (A.36)–(A.38) that

$$G_{(0)}^k \varrho_{(0)} \circ \Phi = \partial_\ell R_{(0)}^{k\ell} + G_{(1)}^{ijkm} \partial_m \varrho_{(2)}^{(i,j)} \circ \Phi \,. \tag{A.64}$$

The symmetric stress $R_{(0)}$ is given by

$$R_{(0)}^{k\ell} = \underbrace{\left(G_{(0)}^k A_i^\ell \delta_{mj} + G_{(0)}^\ell A_i^k \delta_{mj} - G^n \partial_n \Phi^m A_i^k A_j^\ell\right)}_{=: S_{(0)}^{ijk\ell m}} (\partial_m \varrho_{(2)}^{(i,j)}) \circ \Phi \,, \tag{A.65}$$

[7] We refer again to Lemma 7.3.3 for the meaning of this.

and the error terms are given by

$$G_{(1)}^{ijkm} = -\partial_\ell (G_{(0)}^{\ell} A_i^k)\delta_{jm} - \partial_\ell G_{(0)}^{k} A_i^\ell \delta_{jm} + \partial_n (G_{(0)}^{\ell} A_i^k \partial_\ell \Phi^m) A_j^n \,, \tag{A.66}$$

where as before we denote $(\nabla\Phi)^{-1} = A$. We first show that the symmetric stress $R_{(0)}^{k\ell}$ defined in (A.65) satisfies the estimate (A.50). First, we note that from (i) and (ii), the function $\partial_m \varrho_{(2)}^{(i,j)}$ has zero mean, is $(\mathbb{T}/\mu)^3$ periodic, and satisfies

$$\left\| D^N \partial_m \varrho_{(2)}^{(i,j)} \right\|_{L^p} \lesssim \mathcal{C}_{*,p} \Upsilon^{-2} \Upsilon' \mathcal{M}\left(N, 1, \Upsilon', \Lambda\right) \tag{A.67}$$

for $N \leq N_* - 1$, in view of (A.43). Second, we note that since $D_t \Phi = 0$, material derivatives may only land on the components of the 5-tensor $S_{(0)}$. Third, the components of the 5-tensor $S_{(0)}$ are sums of terms which are linear in $G_{(0)}$ and multilinear in A and $D\Phi$. In particular, due to our assumption (A.40) and the previously established bounds in (A.61) and (A.62), upon applying the Leibniz rule, we obtain that

$$\left\| D^N D_t^M S_{(0)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \max(\lambda, \lambda')^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.68}$$

for $N \leq N_*$ and $M \leq M_*$. Having collected these estimates, the L^p norm of the space-material derivatives of $R_{(0)}$ is obtained from Lemma A.1.3. As dictated by (A.65) we apply this lemma with $f = S_{(0)}$ and $\varphi = \partial_m \varrho_{(2)}^{(i,j)}$. Due to (A.68), the bound (A.12) holds with $\mathcal{C}_f = \mathcal{C}_G$ and a spatial derivative cost of $\max(\lambda, \lambda')$. Due to (A.42a), the assumptions (A.13) and (A.14) are verified. Next, due to (A.67), the assumption (A.15) is verified, with $N_x = 1$ and $\mathcal{C}_\varphi = \mathcal{C}_{*,p} \Upsilon^{-2} \Upsilon' \Lambda^\alpha$. Lastly, assumption (A.45) verifies the condition (A.16) of Lemma A.1.3. Thus, applying estimate (A.18) we deduce that

$$\left\| D^N D_t^M R_{(0)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \Upsilon^{-2} \Upsilon' \mathcal{M}\left(N, 1, \Upsilon', \Lambda\right) \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.69}$$

for all $N \leq N_* - 1$ and $M \leq M_*$, which is precisely the bound stated in (A.50). Here we have used that $N_* \geq 2\mathsf{N}_{\mathrm{dec}} + n + 1$, which gives that (A.17) is satisfied.

`Step 3:` To continue the iteration, we first analyze the second term in (A.64). The point is that this term has the *same structure* as what we started with; for every fixed i, j, m, we may replace $G^k_{(0)}$ by $G^{ijkm}_{(1)}$, and we replace $\varrho_{(0)}$ with $\partial_m \varrho^{(i,j)}_{(2)}$; the only difference is that the bounds for this term are *better*. Indeed, from (A.66) we see that the 4-tensor $G_{(1)}$ is the sum of various entries from the tensors $DG_{(0)} \otimes A$ and $DG_{(0)} \otimes A \otimes A \otimes D\Phi$. Recalling (A.61), (A.62), and (A.63) and using the Leibniz rule, we deduce that

$$\left\| D^{N''} D^M_t D^{N'} G^{ijkm}_{(1)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \max(\lambda, \lambda')^{N'+N''+1} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.70}$$

for $N' + N'' \leq N_* - 1$ and $M \leq M_*$. The only caveat is that the bounds hold for one fewer spatial derivative. In order to iterate Proposition A.3.1, for simplicity we ignore the i, j, k, m indices, since the argument works in exactly the same way in each case. Specifically, we write $G^{ijkm}_{(1)}$ simply as $G^k_{(1)}$, and for the sake of convenience we suppress indices on the tensors $D\varrho_{(k)}$ and use D as a stand-in for ∂_m. We first note that $D\varrho_{(2)} = \mathrm{divdiv}\left(D\varrho_{(4)}\right)$, where $D\varrho_{(4)}$ is a symmetric 2-tensor once both indices have been specified on the left-hand side of the equality for $D\varrho_{(2)}$. Thus, using identities (A.36)–(A.38) and (in a slight abuse of notation) reusing the indices we previously tossed away, we obtain that the second term in (A.64) may be written as

$$G^k_{(1)}(D\varrho_{(2)}) \circ \Phi = \partial_\ell R^{k\ell}_{(1)} + G^{ijkm}_{(2)}(\partial_m D\varrho^{(i,j)}_{(4)}) \circ \Phi \tag{A.71}$$

where the symmetric stress $R_{(1)}$ is given by

$$R^{k\ell}_{(1)} = \underbrace{\left(G^k_{(1)} A^\ell_i \delta_{mj} + G^\ell_{(1)} A^k_i \delta_{mj} - G^n_{(1)} \partial_n \Phi^m A^k_i A^\ell_j\right)}_{=: S^{ijk\ell m}_{(1)}} (\partial_m D\varrho^{(i,j)}_{(4)}) \circ \Phi\,, \tag{A.72}$$

the error terms are computed as

$$G_{(2)}^{ijkm} = -\partial_\ell (G_{(1)}^\ell A_i^k)\delta_{jm} - \partial_\ell G_{(1)}^k A_i^\ell \delta_{jm} + \partial_n (G_{(1)}^\ell A_i^k \partial_\ell \Phi^m) A_j^n \,. \tag{A.73}$$

We emphasize that by combining (A.65) and (A.66) with (A.72) and (A.73), we may compute the tensors $S_{(1)}$ and $G_{(2)}$ *explicitly in terms of just space derivatives* of G, $D\Phi$, and A. Using a similar argument to the one which was used to prove (A.68), but by appealing to (A.70) instead of (A.63), we deduce that for $N \leq N_* - 1$ and $M \leq M_*$,

$$\left\| D^N D_t^M S_{(1)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \max(\lambda, \lambda')^{N+1} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \,. \tag{A.74}$$

Using the bound (A.74) and the estimate

$$\left\| D^N (\partial_m D \varrho_{(4)}) \right\|_{L^p} \lesssim \mathcal{C}_{*,p} \Upsilon^{-4} \Upsilon'^2 \mathcal{M}\left(N, 2, \Upsilon', \Lambda\right) \,,$$

which is a consequence of (A.43), we may deduce from Lemma A.1.3 that

$$\left\| D^N D_t^M R_{(1)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \max(\lambda, \lambda')(\Upsilon^{-2}\Upsilon')^2 \mathcal{M}\left(N, 2, \Upsilon', \Lambda\right) \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.75}$$

for $N \leq N_* - 2$ and $M \leq M_*$, which is an estimate that is even better than (A.69), aside from the fact that we have lost a spatial derivative. This shows that the first term in (A.71) satisfies the expected bound. The low-frequency portion of the second term in (A.71) may in turn be shown to satisfy

$$\left\| D^{N''} D_t^M D^{N'} G_{(2)}^{ijkm} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \max(\lambda, \lambda')^{2+N'+N''} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.76}$$

for $N' + N'' \leq N_* - 2$ and $M \leq M_*$.

At this point there is a clear roadmap for iterating this procedure $\lfloor \mathsf{d}/2 \rfloor$ times, where the limit on the number of steps comes from that fact that $\varrho_{(k)}$ is only defined for $0 \leq k \leq \mathsf{d}$, and

each step in the iteration increases the value of k by 2. Without spelling out these details, the iteration procedure described above produces

$$G_{(0)}\varrho_{(0)} \circ \Phi = \sum_{k=0}^{\lfloor \mathsf{d}/2 \rfloor - 1} \operatorname{div} R_{(k)} + \underbrace{G_{(\lfloor \mathsf{d}/2 \rfloor)} \,:\, \left(D^{\lfloor \mathsf{d}/2 \rfloor} \varrho_{(2\lfloor \mathsf{d}/2 \rfloor)}\right) \circ \Phi}_{=:E} \tag{A.77}$$

where each of the $\lfloor \mathsf{d}/2 \rfloor$ symmetric stresses satisfies

$$\left\| D^N D_t^M R_{(k)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \max(\lambda, \lambda')^k \left(\Upsilon^{-2}\Upsilon'\right)^{k+1} \Lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.78}$$

for $N \leq N_* - k - 1$ and $M \leq M_*$. Furthermore, the formulae in (A.47) and (A.48) can be computed explicitly from the algorithm already detailed above by keeping track of the high-low product structure of each term in each $R_{(k)}$ and Remark A.3.2, although we forego the details. The subsidiary estimates are precisely those from (A.67) and (A.68), which are immediate for the terms from the first step of the parametrix expansion, and which follow for the higher order terms by transferring the amplitude gains from the high-frequency function onto the low-frequency function, and using (A.44). Each component of the the error tensor $G_{(\lfloor \mathsf{d}/2 \rfloor)}$ in (A.77) is recursively computable solely in terms of G, $D\Phi$, and A and their spatial derivatives and satisfies

$$\left\| D^{N''} D_t^M D^{N'} G_{(\lfloor \mathsf{d}/2 \rfloor)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \max(\lambda, \lambda')^{\lfloor \mathsf{d}/2 \rfloor + N' + N''} \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.79}$$

for $N' + N'' \leq N_* - \lfloor \mathsf{d}/2 \rfloor$ and $M \leq M_*$. Lastly, a final application of Lemma A.1.3, which is valid due to with (A.79) and the assumption $N_* - \mathsf{d} \geq 2\mathsf{N}_{\text{dec}} + n + 1$, shows that estimate (A.51) holds.

`Step 4:` Finally, we turn to the proof of (A.56) and (A.57). Recall that E is defined by the second term in (A.77), and thus $\fint_{\mathbb{T}^n} G\varrho \circ \Phi dx = \fint_{\mathbb{T}^n} E dx$. Using the standard nonlocal

inverse-divergence operator

$$(\mathcal{R}f)^{ij} = -\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k f^k - \frac{1}{2}\Delta^{-1}\partial_k\delta_{ij}f^k + \Delta^{-1}\partial_i\delta_{jk}f^k + \Delta^{-1}\partial_j\delta_{ik}f^k \tag{A.80}$$

we may define

$$\overline{R}_{\text{nonlocal}} = \mathcal{R}E\,.$$

By the definition of $\mathcal{R}$ we have that $\overline{R}_{\text{nonlocal}}$ is traceless, symmetric, and satisfies $\operatorname{div}\overline{R}_{\text{nonlocal}} = E - \fint_{\mathbb{T}^n} E dx$, i.e. (A.56) holds.

Using the formulas in (A.224a), (A.224b), the assumption (A.53), and the fact that D and ∂_t commute with $\mathcal{R}$, we deduce that for every $N \leq N_\circ$ and $M \leq M_\circ$ we have

$$\begin{aligned}\left\|D^N D_t^M \overline{R}_{\text{nonlocal}}\right\|_{L^\infty} &\lesssim \sum_{\substack{M'\leq M\\ N'+M'\leq N+M}} \sum_{K=0}^{M-M'} \mathcal{C}_v^K (\lambda')^{N-N'+K}\nu'^{-(M-M'-K)} \left\|D^{N'}\partial_t^{M'}\mathcal{R}E\right\|_{L^\infty}\\ &\lesssim \sum_{\substack{M'\leq M\\ N'+M'\leq N+M}} (\lambda')^{N-N'}\nu'^{-(M-M')}\left\|D^{N'}\partial_t^{M'}E\right\|_{L^\infty}\end{aligned} \tag{A.81}$$

where in the last inequality we have used that by assumption $\mathcal{C}_v\lambda' \lesssim \nu'^{-1}$, and that $\mathcal{R}\colon L^p(\mathbb{T}^n) \to L^p(\mathbb{T}^n)$ is a bounded operator.

Our goal is to appeal to estimate (A.26) in Lemma A.2.2, with $A = -v\cdot\nabla$, $B = D_t$ and $f = E$ in order to estimate the L^∞ norm of $D^{N'}\partial_t^{M'}E = D^{N'}(A+B)^{M'}E$. First, we claim that v satisfies the lossy estimate

$$\left\|D^N D_t^M v\right\|_{L^\infty} \lesssim \mathcal{C}_v\lambda'^N\nu'^{-M} \tag{A.82}$$

for $M \leq M_\circ$ and $N + M \leq N_\circ + M_\circ$. This estimate does not follow immediately from either (A.42b) or (A.53). For this purpose, we apply Lemma A.2.2 with $f = v$, $B = \partial_t$, $A = v\cdot\nabla$, and $p = \infty$. Using (A.53), and the fact that $B = \partial_t$ and D commute, we

obtain that bounds (A.22) and (A.23) hold with $\mathcal{C}_f = \mathcal{C}_v$, $\lambda_v = \widetilde{\lambda}_v = \lambda_f = \widetilde{\lambda}_f = \lambda'$, and $\mu_v = \widetilde{\mu}_v = \mu_f = \widetilde{\mu}_f = \nu'^{-1}$. Since $A + B = D_t$, we obtain from the bound (A.26) and the assumption $\mathcal{C}_v \lambda' \lesssim \nu'^{-1}$ that (A.82) holds.

Second, we claim that for any $k \geq 1$ we have

$$\left\| \left(\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} \right) v \right\|_{L^\infty(\operatorname{supp} G)} \lesssim \mathcal{C}_v \lambda'^{|\alpha|} \nu'^{|\beta|} \tag{A.83}$$

whenever $|\beta| \leq M_\circ$ and $|\alpha| + |\beta| \leq N_\circ + M_\circ$. To see this, we use Lemma A.2.3 with $f = v$, $p = \infty$, and $\Omega = \operatorname{supp} G$. From (A.42b) we have that (A.27) holds with $\mathcal{C}_v = \nu/\lambda'$, $\lambda_v = \widetilde{\lambda}_v = \lambda'$, $\mu_v = \nu$, and $\widetilde{\mu}_v = \nu'$. On the other hand, from (A.82) we have that (A.28) holds with $\mathcal{C}_f = \mathcal{C}_v$, $\lambda_f = \widetilde{\lambda}_f = \lambda'$, and $\mu_f = \widetilde{\mu}_f = \nu'^{-1}$. We then deduce from (A.31) that (A.83) holds.

Third, we claim that

$$\left\| \left(\prod_{i=1}^{k} D^{\gamma_i} D_t^{\beta_i} \right) E \right\|_{L^\infty(\operatorname{supp} G)} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \max(\lambda, \lambda')^{\lfloor \mathsf{d}/2 \rfloor} (\Upsilon' \Upsilon^{-2})^{\lfloor \mathsf{d}/2 \rfloor} \Lambda^{|\gamma| + n + 1} \mathcal{M}\left(|\beta|, M_t, \nu, \nu'\right) \tag{A.84}$$

holds whenever $|\gamma| \leq N_* - \lfloor \mathsf{d}/2 \rfloor - n - 1$ and $|\beta| \leq M_*$. This estimate again follows from Lemma A.2.3, this time with $f = E$, by appealing to the previously established bound (A.51) and the Sobolev embedding $W^{n+1,1}(\mathbb{T}^n) \hookrightarrow L^\infty(\mathbb{T}^n)$.

At last, we are in the position to apply Lemma A.2.2. The bound (A.83) implies that assumption (A.22) holds with $B = D_t$, $\lambda_v = \widetilde{\lambda}_v = \lambda'$, and $\mu_v = \widetilde{\mu}_v = \nu'$. The bound (A.84) implies that assumption (A.23) of Lemma A.2.2 holds with $\mathcal{C}_f = \mathcal{C}_{G,p} \mathcal{C}_{*,p} \max(\lambda, \lambda')^{\lfloor \mathsf{d}/2 \rfloor} (\Upsilon' \Upsilon^{-2})^{\lfloor \mathsf{d}/2 \rfloor} \Lambda^{n+1}$, $\lambda_f = \widetilde{\lambda}_f = \Lambda$, $\mu_f = \nu$, and $\widetilde{\mu}_f = \nu'$. We may now use estimate (A.26), and the assumption that $\Lambda \geq \lambda, \lambda'$ to deduce that

$$\left\| D^{N'} \partial_t^{M'} E \right\|_{L^\infty} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \max(\lambda, \lambda')^{\lfloor \mathsf{d}/2 \rfloor} (\Upsilon' \Upsilon^{-2})^{\lfloor \mathsf{d}/2 \rfloor} \Lambda^{N' + n + 1} (\max\{\mathcal{C}_v \Lambda, \nu'\})^{M'} \tag{A.85}$$

holds whenever $M' \leq M_\circ$ and $N' + M' \leq N_\circ + M_\circ$. Combining (A.81) and (A.85) we deduce that

$$\begin{aligned}\left\|D^N D_t^M \overline{R}_{\text{nonlocal}}\right\|_{L^\infty} &\lesssim \mathcal{C}_{G,p}\mathcal{C}_{*,p} \max(\lambda, \lambda')^{\lfloor \mathsf{d}/2 \rfloor} (\Upsilon'\Upsilon^{-2})^{\lfloor \mathsf{d}/2 \rfloor} \Lambda^{n+1} \\ &\qquad \times \sum_{\substack{M' \leq M \\ N'+M' \leq N+M}} \lambda'^{N-N'} \nu'^{-(M-M')} \Lambda^{N'} (\max\{\mathcal{C}_v \Lambda, \nu'\})^{M'} \\ &\lesssim \mathcal{C}_{G,p}\mathcal{C}_{*,p} \max(\lambda, \lambda')^{\lfloor \mathsf{d}/2 \rfloor} (\Upsilon'\Upsilon^{-2})^{\lfloor \mathsf{d}/2 \rfloor} \Lambda^{N+n+1} (\max\{\mathcal{C}_v \Lambda, \nu'\})^{M} \quad \text{(A.86)}\end{aligned}$$

whenever $N \leq N_\circ$ and $M \leq M_\circ$. Estimate (A.57) follows by appealing to the assumption (A.55). □

Observe that in the proof of Proposition A.3.3, $\rho^{\beta(j)}$ consists of $\nabla \varrho_{(2)}, \nabla^2 \varrho_{(4)}, \cdots, \nabla^{\lfloor \mathsf{d}/2 \rfloor} \varrho_{2\lfloor \mathsf{d}/2 \rfloor}$; recall that $\varrho_{(0)} = \varrho = \operatorname{div}^{\mathsf{d}} \vartheta$ and $\varrho_{(k-1)} = \operatorname{div} \varrho_{(k)} = \operatorname{div}^{\mathsf{d}-(k-1)} \vartheta$. Keeping this in mind, when ϱ is given as $\operatorname{div}^{(2\mathsf{d})^2} \vartheta$, we can apply the proposition iteratively to get

$$G(\rho \circ \Phi) = \operatorname{div}^{\mathsf{d}} R + E.$$

The details are described in the following corollary. Since this operator will be applied to velocity increments, some of the adjustments are specified for this particular application.

Corollary A.3.11 (Iterated inverse divergence for scalar fields). *We suppose that the same assumptions hold as in Proposition A.3.3 together with Remark A.3.8 except for the following substitutions.*

(i) Fix $\mathsf{N}_{\text{dec}}, N_*, M_* \mathsf{d} \geq 1$ *such that* d *is even and* $N_* - \mathsf{d}^2 \geq 2\mathsf{N}_{\text{dec}} + n + 1 + M_*$ *(replacing* (A.39) *and the last inequality in* (A.44)*).*

(ii) ϱ *is given as an iterated divergence* $\varrho = \operatorname{div}^{(\mathsf{d}^2)} \widetilde{\vartheta}$ *(replacing* (i)*).*

(iii) There exist parameters $1 \ll \Upsilon \leq \Upsilon' = \Lambda$ *and* $\mathcal{C}_{*,p} > 0$ *such that for all* $0 \leq N \leq N_*$

and all $0 \leq k \leq \mathsf{d}^2$, (A.43) *is replaced with*

$$\left\| D^N \partial_{i_1} \dots \partial_{i_k} \widetilde{\vartheta}^{(i_1,\dots,i_{\mathsf{d}^2})} \right\|_{L^p} \lesssim \mathcal{C}_{*,p} \Upsilon^{k-\mathsf{d}^2} \Upsilon'^N \,. \tag{A.87}$$

Additionally, we assume that there exists a smooth, non-negative function π *such that*

$$\left| D^N D_t^M G \right| \lesssim \pi^{\frac{1}{2}} r_G^{-\frac{1}{3}} \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.88}$$

for $N \leq N_*$ *and* $M \leq M_*$. *Then, we have that*

$$G(\varrho \circ \Phi) = \operatorname{div}^{\mathsf{d}} R + E \tag{A.89}$$

for a rank dpot tensor R *and error* E *satisfying the following properties.*

(i) The support of R *is a subset of* $\operatorname{supp} G \cap \operatorname{supp}(\widetilde{\vartheta} \circ \Phi)$, *and hence so is the support of* E.

(ii) There exists an explicitly computable positive integer $\overline{\mathcal{C}}_H$, *an explicitly computable function* $r(j) : \{0, 1, \dots, \overline{\mathcal{C}}_H\}$ *and explicitly computable tensors*

$$\begin{aligned} &\rho^{\beta(j)}\,, \qquad \beta(j) = (\beta_1, \beta_2, \dots, \beta_{r(j)}) \in \{1, \dots, n\}^{r(j)}\,, \\ &H^{\alpha(j)}\,, \qquad \alpha(j) = (\alpha_1, \alpha_2, \dots, \alpha_{r(j)}) \in \{1, \dots, n\}^{r(j)+\mathsf{d}}\,, \end{aligned}$$

of rank $r(j)$ *and* $r(j)+\mathsf{d}$, *respectively, all of which depend only on* $G, \varrho, \Phi, n, \mathsf{d}$ *such that the following holds. The localized stress* R *can be decomposed into a sum of localized stresses as*

$$R = \sum_{j=0}^{\overline{\mathcal{C}}_{\mathcal{H}}} H^{\alpha(j)} (\rho^{\beta(j)} \circ \Phi)\,.$$

Furthermore, we have that

$$\operatorname{supp} H^{\alpha(j)} \subseteq \operatorname{supp} G\,, \qquad \operatorname{supp} \rho^{\beta(j)} \subseteq \operatorname{supp} \widetilde{\vartheta}\,. \tag{A.90}$$

(iii) We have the subsidiary estimates

$$\left\| D^N \rho^{\beta(j)} \right\|_{L^p} \lesssim \mathcal{C}_{*,p} (\Upsilon^{-2}\Upsilon')^{\mathsf{d}} \Lambda^N \tag{A.91a}$$

for all $N \leq N_* - \mathsf{d}^2$ and $j \leq \overline{\mathcal{C}}_{\mathcal{H}}$, and

$$\left\| \prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} H^{\alpha(j)} \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \left(\max(\lambda, \lambda')\right)^{|\alpha|} \mathcal{M}\left(|\beta|, M_t, \nu, \nu'\right) \tag{A.91b}$$

$$\left| \prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} H^{\alpha(j)} \right| \lesssim \pi^{\frac{1}{2}} r^{-\frac{1}{3}} (\max(\lambda, \lambda'))^{|\alpha|} \mathcal{M}\left(|\beta|, M_t, \nu, \widetilde{\nu}\right). \tag{A.91c}$$

for all integer $k \geq 1$, multi-indices $\alpha, \beta \in \mathbb{N}^k$ with $|\alpha| \leq N_* - \mathsf{d}^2$ and $|\beta| \leq M_*$, and $j \leq \overline{\mathcal{C}}_{\mathcal{H}}$.

(iv) We have the main estimate

$$\left\| \prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} R \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} (\Upsilon'\Upsilon^{-2})^{\mathsf{d}} \Upsilon'^{|\alpha|} \mathcal{M}\left(|\beta|, M_t, \nu, \nu'\right) \tag{A.92}$$

for all integer $k \geq 1$, multi-indices $\alpha, \beta \in \mathbb{N}^k$ with $|\alpha| \leq N_* - \mathsf{d}^2$ and $|\beta| \leq M_*$, and $j \leq \overline{\mathcal{C}}_{\mathcal{H}}$.

(v) For $N \leq N_* - \mathsf{d}^2$ and $M \leq M_*$ the error term E in (A.89) *satisfies*[8]

$$\left\| D^N D_t^M E \right\|_{L^p} \lesssim \mathcal{C}_{G,p} \mathcal{C}_{*,p} \max(\lambda, \lambda')^{\mathsf{d}/2} \left(\Upsilon'\Upsilon^{-2}\right)^{\mathsf{d}/2} \Lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \sum_{k=0}^{\mathsf{d}-1} \left(\frac{\Upsilon'}{\Upsilon}\right)^{2k}. \tag{A.93}$$

[8] In our applications, $\Upsilon = \Upsilon'$, so the sum of loss factors is irrelevant. If one wanted to be more precise, this loss could be eliminated using a more careful algorithm and a few more conditions on the relative sizes of all the frequencies.

Proof. The proof is based on applying Proposition A.3.3 d times. In the first iteration, we get

$$G(\varrho \circ \Phi) = \sum_{j_1=0}^{\mathcal{C}_{\mathcal{H}}} \operatorname{div}\left(H^{\alpha(j_1)}(\rho^{\beta(j_1)} \circ \Phi)\right) + E_{(1)}$$

where $H^{\alpha(j_1)}$ satisfies (A.49b) and (A.53). From (A.47) and Remark A.3.8, we have that the rank of $H^{\alpha(j_1)}$ is one larger than the rank of $\rho^{\beta(j_1)}$. Also, replacing π by $\pi^{1/2}r^{-1/3}$ in Remark A.3.9, we get

$$|D^N D_t^M H^{\alpha(j_1)}| \lesssim \pi^{\frac{1}{2}} r^{-\frac{1}{3}} \lambda^N \mathcal{M}\left(M, M_t, \nu, \widetilde{\nu}\right)$$

for $N \leq N_* - \mathsf{d}/2$ and $M \leq M_*$. In addition, $E_{(1)}$ satisfies (A.93). Since we use the same Φ, all assumptions on G and Φ in the proposition holds for N_* replaced with $N_* - \mathsf{d}/2$. From the proof of Proposition A.3.3 we note that $\rho^{\beta(j)}$ consists of $\nabla^k \varrho_{(2k)}$, $1 \leq k \leq \mathsf{d}/2$, which can be written as $\nabla^k \operatorname{div}^{\mathsf{d}^2-2k}\widetilde{\vartheta} = \operatorname{div}^{\mathsf{d}}(\nabla^k \operatorname{div}^{\mathsf{d}^2-2k-\mathsf{d}}\widetilde{\vartheta})$. Then, $\nabla^k \varrho_{(2k)}$ and its potential $\nabla^k \operatorname{div}^{\mathsf{d}^2-2k-\mathsf{d}}\widetilde{\vartheta}$ satisfy (i), (ii) in the assumption of Proposition A.3.3 and

$$\left\| D^N \partial_{i_1} \cdots \partial_{i_{k'}} (\nabla^k \operatorname{div}^{\mathsf{d}^2-2k-\mathsf{d}}\widetilde{\vartheta}) \right\| \lesssim \mathcal{C}_{*,p} \Upsilon^{-2k-\mathsf{d}+k'} \Upsilon'^{N+k}$$

for any $N \leq N_* - k$ and $0 \leq k' \leq \mathsf{d}$. In particular, we have

$$\left\| D^N \rho^{\beta(j_1)} \right\|_{L^p} \lesssim \mathcal{C}_{*,p} \Upsilon^{-2} \Upsilon' \Upsilon'^N \tag{A.94}$$

for $N \leq N_* - \mathsf{d}/2$ and $j_1 \leq \mathcal{C}_{\mathcal{H}}$. This implies that (A.43) holds for $C_{*,p}$ replaced with $C_{*,p}\Upsilon'\Upsilon^{-2}$ and N_* with $N_* - \mathsf{d}/2$ and ϑ with the potential of $\rho^{\beta(j)}$, respectively. Furthermore, from the construction it is easy to see that

$$\operatorname{supp}\left(\rho^{\beta(j)}\right) \subset \operatorname{supp}\left(\widetilde{\vartheta}\right).$$

Iterating this process d times, we get

$$G(\varrho\circ\Phi)=\sum_{j_1=0}^{\mathcal{C}_{\mathcal{H}}}\operatorname{div}\left(H^{\alpha(j_1)}(\rho^{\beta(j_1)}\circ\Phi)\right)+E_{(1)}=\sum_{j_1,j_2=0}^{\mathcal{C}_{\mathcal{H}}}\operatorname{div}^2\left(H^{\alpha(j_1,j_2)}(\rho^{\beta(j_1,j_2)}\circ\Phi)\right)+\operatorname{div}E_{(2)}+E_{(1)}$$
$$=:\sum_{j=0}^{\overline{\mathcal{C}}_{\mathcal{H}}}\operatorname{div}^{\mathsf{d}}\left(H^{\alpha(j)}(\rho^{\beta(j)}\circ\Phi)\right)+\sum_{k=1}^{\mathsf{d}}\operatorname{div}^{k-1}E_{(k)}\,.$$

As a result, we get (A.89), where E is defined by

$$E:=\sum_{k=1}^{\mathsf{d}}\operatorname{div}^{k-1}E_{(k)}\,.$$

Since we have

$$\operatorname{supp}H^{\alpha(j)}\subset\cdots\subset\operatorname{supp}\left(H^{\alpha(j_1)}\right)\subset\operatorname{supp}(G),\quad \operatorname{supp}\rho^{\beta(j_1)}\subset\operatorname{supp}\left(\widetilde{\vartheta}\right),$$

(A.90) holds. Therefore, (i) and (ii) have been verified, as has (A.94) and (A.91a). Furthermore, we have

$$\left\|D^N D_t^M H^{\alpha(j)}\right\|_{L^p}\lesssim\mathcal{C}_{G,p}\left(\max(\lambda,\lambda')\right)^N\mathcal{M}\left(M,M_t,\nu,\nu'\right)$$
$$\left|D^N D_t^M H^{\alpha(j)}\right|\lesssim\pi^{\frac{1}{2}}r^{-\frac{1}{3}}(\max(\lambda,\lambda'))^N\mathcal{M}\left(M,M_t,\nu,\widetilde{\nu}\right).$$
$$\left\|D^N D_t^M R\right\|_{L^p}\lesssim\mathcal{C}_{G,p}\mathcal{C}_{*,p}(\Upsilon'\Upsilon^{-2})^{\mathsf{d}}\Lambda^N\mathcal{M}\left(M,M_t,\nu,\nu'\right)$$

for all integers $N\le N_*-\mathsf{d}^2$ and $M\le M_*$. Also, $E^{(k)}$ satisfies

$$\left\|D^N D_t^M E^{(k)}\right\|_{L^p}\lesssim\mathcal{C}_{G,p}\mathcal{C}_{*,p}(\Upsilon'\Upsilon^{-2})^{k-1}\max(\lambda,\lambda')^{\mathsf{d}/2}\left(\Upsilon'\Upsilon^{-2}\right)^{\mathsf{d}/2}\Upsilon'^N\mathcal{M}\left(M,M_t,\nu,\nu'\right)$$

for $1\le k\le\mathsf{d}$, $N\le N_*-k\cdot\mathsf{d}/2$, and $M\le M_*$.

Finally, we apply Lemma A.2.3 to upgrade these estimates to the one with commutations of the operators, (A.91b), (A.91c), (A.92), and (A.93). We will work only for (A.91b), then

the last will follow by a similar argument. To avoid confusion in the notations, we rewrite some repeated symbols from Lemma A.2.3 with bars above on the left-hand side of the equalities below, while the right-hand side are parameters given in the assumptions of the Corollary. Set $\overline{p} = p$, $\overline{N}_t = M_t$, $\overline{N}_* = N_* - \mathsf{d}^{\mathsf{d}}/2$, $\overline{M}_* = M_*$, $\overline{v} = v$, $\overline{\Omega} = \operatorname{supp} G$, $\mathcal{C}_v = \nu(\lambda')^{-1}$, $\lambda_v = \widetilde{\lambda}_v = \lambda'$, $\mu_v = \mu_f = \nu$, $\widetilde{\mu}_v = \widetilde{\mu}_f = \widetilde{\nu}$, $f = H^{\alpha(j)}$, and $\lambda_f = \widetilde{\lambda}_f = \max(\lambda, \lambda')$. Then, as a consequence of the lemma, we have (A.91b). For (A.91c), we work at each point x in a similar way, but set $\overline{\Omega} = \Omega(x)$ as a small closed neighborhood of x contained in $\operatorname{supp}(G)$ and use the continuity of π so that $\sup_{\Omega(x)} \pi \leq 2\pi(x)$.

□

Finally, we shall need a simpler case of the inverse divergence, when the density is not flowed and the input is a scalar field.

Lemma A.3.12 (Inverse divergence without flow map). *Fix dimension $n \geq 2$. Let G be a smooth scalar field and let d be a non-negative integer such that the smooth scalar field ϱ and tensor field ϑ defined on $\mathbb{R} \times \mathbb{T}^n$ satisfy $\varrho = \partial_{i_1} \dots \partial_{i_{\mathsf{d}}} \vartheta^{(i_1 \dots i_{\mathsf{d}})}(x)$ (note that no symmetry assumptions needed).*

Part 1: Algorithm for inverse divergence

We have a decomposition

$$G\varrho =: \operatorname{div}(\mathcal{H}(G\varrho)) + E \tag{A.95}$$

where the vector field $\mathcal{H}(G\varrho)$ and scalar field E are defined by

$$\mathcal{H}(G\varrho)^{\bullet} := \sum_{k=0}^{\mathsf{d}-1} (-1)^{\mathsf{d}-k+1} \partial_{i_{k+2}} \dots \partial_{i_{\mathsf{d}}} G \underbrace{\operatorname{div}^{(k)}}_{\partial_{i_1}, \dots, \partial_{i_k}} \vartheta^{(i_1, \dots, i_k, \bullet, i_{k+2}, \dots, i_{\mathsf{d}})}, \quad E = (-1)^{\mathsf{d}} \nabla^{\mathsf{d}} G : \vartheta \,, \tag{A.96}$$

where we use the convention $\partial_{i_{k+2}} \cdots \partial_{i_{\mathsf{d}}} G = G$ and $\vartheta^{(i_1, \dots, i_k, \bullet, i_{k+2}, \dots, i_{\mathsf{d}})} = \vartheta^{(i_1, \dots, i_{\mathsf{d}-1}, \bullet)}$ when $k = \mathsf{d} - 1$.

Part 2: Localized assumptions and output

Fix a set $\Omega \subset \mathbb{R} \times \mathbb{T}^n$*. Let parameters* $N_* \geq M_* \geq 1$ *be given. Define* v *and* D_t *as in Part 1 of Proposition A.3.3, where* v *satisfies* (A.42b) *with* $\lambda', \nu, \nu', N_*, M_*$ *and* $L^\infty(\operatorname{supp} G)$ *replaced with* $L^\infty(\Omega)$*. Let smooth, non-negative functions* π *and* π' *be given such that*

$$\left|D^N D_t^M G\right| \lesssim \pi \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \qquad \text{on } \Omega \tag{A.97a}$$

$$\Upsilon^{\mathsf{d}-k} \left|D^N D_t^M \partial_{i_1} \ldots \partial_{i_k} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})}\right| \lesssim \pi' \Lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \qquad \text{on } \Omega \tag{A.97b}$$

for $N \leq N_*$ *and* $M \leq M_*$*, where the parameters satisfy*

$$\lambda', \lambda \leq \Upsilon \leq \Lambda, \quad \max(\lambda, \lambda')\Upsilon^{-1} \leq 1, \quad N_* \geq \mathsf{d}, \quad \lambda, \nu, \nu' \geq 1\,. \tag{A.98}$$

Then $\mathcal{H}(G\varrho)$ *satisfies*

$$\operatorname{supp}\left(\mathcal{H}(G\varrho)\right) \subseteq \operatorname{supp}\left(G\vartheta\right), \tag{A.99}$$

and for $N \leq N_* - \mathsf{d}$ *and* $M \leq M_*$*,*

$$\left|D^N D_t^M \mathcal{H}(G\varrho)\right| \lesssim \pi\pi' \Upsilon^{-1} \Lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \quad \text{on } \Omega\,. \tag{A.100}$$

Part 3: Nonlocal assumptions and output

Finally, we assume that all assumptions from (i) *in Part 4 in Proposition A.3.3 hold. Next, we assume that for* $N \leq N_*$ *and* $M \leq M_*$*,*

$$\left\|D^N D_t^M G\right\|_{L^\infty} \lesssim \mathcal{C}_{G,\infty} \lambda^N (\nu')^M\,, \tag{A.101a}$$

$$\left\|D^N D_t^M \partial_{i_1} \ldots \partial_{i_k} \vartheta^{(i_1,\ldots,i_{\mathsf{d}})}\right\|_{L^\infty} \lesssim \mathcal{C}_{*,\infty} \Upsilon^{k-\mathsf{d}} \Lambda^N (\nu')^M\,. \tag{A.101b}$$

Also, we choose d *large enough to satisfy*

$$\mathcal{C}_{G,\infty}\mathcal{C}_{*,\infty}(\max(\lambda,\lambda')\Upsilon^{-1})^{\mathsf{d}/2}\Lambda^{K_\circ}\left(1+\frac{\max\{\nu',\mathcal{C}_v\Lambda\}}{\nu}\right)^{M_\circ}\leq 1\,. \tag{A.102}$$

Then we may write

$$E =: \operatorname{div}(\mathcal{R}^*(G\varrho)) + \fint_{\mathbb{T}^3} G\varrho\,dx\,, \tag{A.103}$$

where $\mathcal{R}^*(G\varrho)$ *is a vector field which satisfies*

$$\left\|D^N D_t^M \mathcal{R}^*(G\varrho)\right\|_{L^\infty} \lesssim \frac{1}{\Lambda^{K_\circ}}(\max(\lambda,\lambda')\Upsilon^{-1})^{\mathsf{d}/2}\Lambda^N\nu^M \tag{A.104}$$

for $N\leq N_\circ$ *and* $M\leq M_\circ$.

Proof of Lemma A.3.12. With the definition (A.96) in hand, we can easily check (A.95)–(A.100). To define $\mathcal{R}^*(G\varrho)$, we use the standard operator $(\mathcal{R}f)^i = \Delta^{-1}\partial_i$ and let $\mathcal{R}^*(G\varrho) = \mathcal{R}E$. The desired estimate for $\mathcal{R}^*(G\varrho)$ follows as in the Proof of Proposition A.3.3 with minor modifications, and we leave the details to the reader. □

A.4 Sample lemma

Lemma A.4.1 (Pressure increment for stress error). *Let* v *be an incompressible vector field on* $\mathbb{R}\times\mathbb{T}^3$. *Denote its material derivative by* $D_t = \partial_t + v\cdot\nabla$. *We use large positive integers* $N_\dagger \geq M_\dagger \gg M_t$ *for counting derivatives and specify additional constraints that they must satisfy in assumptions* (i)–(iv).

Suppose a stress error $S = H\,\rho\circ\Phi$ *and a non-negative, continuous function* π *are given such that the following hold.*

(i) There exist constants $\mathcal{C}_{G,p}$ *and* $\mathcal{C}_{\rho,p}$[9] *for* $p = 3/2$ *and* $p=\infty$ *and frequency paramaters*

[9] In practice, $\mathcal{C}_{\rho,p} = \mathcal{C}_{*,p}\zeta^{-2}\xi\Lambda^\alpha$ from (A.49a). We shall also assume that these constants are ordered in the obvious way, i.e. $\mathcal{C}_{\bullet,3/2}\leq\mathcal{C}_{\bullet,\infty}$.

$\lambda, \Lambda, \nu, \nu'$ *such that*

$$\left\|D^N D_t^M H\right\|_p \lesssim \mathcal{C}_{G,p}\lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.105a}$$

$$\left|D^N D_t^M H\right| \lesssim \pi\lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.105b}$$

$$\left\|D^N \rho\right\|_p \lesssim \mathcal{C}_{\rho,p}\Lambda^N \tag{A.105c}$$

$$\|S\|_p \lesssim \mathcal{C}_{G,p}\mathcal{C}_{\rho,p} =: \delta_{S,p}\,. \tag{A.105d}$$

for all $N \leq N_\dagger$, $M \leq M_\dagger$.

(ii) There exist a frequency parameter μ, *a parameter* Γ *for measuring small losses in derivative costs,*[10] *and a positive integer* $\mathsf{N}_{\mathrm{dec}}$ *such that* ρ *is* $(\mathbb{T}/\mu)^3$*-periodic and* $\lambda \ll \mu \leq \Lambda$, *whereby we mean that*

$$(\Lambda\Gamma)^4 \leq \left(\frac{\mu}{4\pi\sqrt{3}(\lambda\Gamma)}\right)^{\mathsf{N}_{\mathrm{dec}}}\,. \tag{A.106}$$

(iii) Let Φ *be a volume preserving flow of* $\mathbb{T}^3$ *such that* $D_t\Phi = 0$ *and* Φ *is the identity at a time slice which intersects the support of* H, *and*

$$\left\|D^{N+1}\Phi\right\|_{L^\infty(\operatorname{supp} H)} + \left\|D^{N+1}\Phi^{-1}\right\|_{L^\infty(\operatorname{supp} H)} \lesssim \lambda^N \tag{A.107a}$$

$$\left\|D^N D_t^M Dv\right\|_{L^\infty(\operatorname{supp} H)} \lesssim \nu\lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.107b}$$

for all $N \leq N_\dagger$, $M \leq M_\dagger$.

[10] In practice, $\Gamma = \Gamma_{q'}$ for some q', which then makes Γ a small power of λ or Λ.

(iv) There exist positive integers $\mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}}$ *and a small parameter* $\delta_{\mathrm{tiny}} \leq 1$ *such that*[11]

$$\mathsf{N}_{\mathrm{cut,t}} \leq \mathsf{N}_{\mathrm{cut,x}} \,, \tag{A.108a}$$

$$(\mathcal{C}_{G,\infty} + 1)\,(\mathcal{C}_{\rho,\infty} + 1)\,\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}} \leq \delta_{\mathrm{tiny}} \,, \mathcal{C}_{G,3/2} \,, \mathcal{C}_{\rho,3/2} \,, \tag{A.108b}$$

$$2\mathsf{N}_{\mathrm{dec}} + 4 \leq N_{\dagger} - \mathsf{N}_{\mathrm{cut,x}} \,. \tag{A.108c}$$

Then one can construct a pressure increment $\sigma_S = \sigma_S^+ - \sigma_S^-$ *associated to the stress error* S*, where*

$$\sigma_S := \mathsf{\Pi}(H)\,(\mathsf{\Pi}(\rho) \circ \Phi - \langle \mathsf{\Pi}(\rho) \rangle) \,, \tag{A.109a}$$

$$\sigma_S^+ := \mathsf{\Pi}(H)\mathsf{\Pi}(\rho) \circ \Phi) \,, \tag{A.109b}$$

and

$$\mathsf{\Pi}(H) := \left(\mathcal{C}_{G,3/2}^2 + \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}} \sum_{M=0}^{\mathsf{N}_{\mathrm{cut,t}}} (\lambda\Gamma)^{-2N} (\nu\Gamma)^{-2M} |D^N D_t^M H|^2 \right)^{\frac{1}{2}} - \mathcal{C}_{G,3/2} \,, \tag{A.110a}$$

$$\mathsf{\Pi}(\rho) := \left(\mathcal{C}_{\rho,3/2}^2 + \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}} (\Lambda\Gamma)^{-2N} |D^N \rho|^2 \right)^{\frac{1}{2}} - \mathcal{C}_{\rho,3/2} \,, \tag{A.110b}$$

and which has the properties listed below.

(i) σ_S^+ *dominates derivatives of* S *with suitable weights, so that for all* $N \leq N_{\dagger}$ *and* $M \leq M_{\dagger}$*,*

$$\left| D^N D_t^M S \right| \lesssim (\sigma_S^+ + \delta_{\mathrm{tiny}})(\Lambda\Gamma)^N \mathcal{M}\,(M, M_t, \nu\Gamma, \nu'\Gamma) \,. \tag{A.111}$$

(ii) σ_S^+ *dominates derivatives of itself with suitable weights, so that for all* $N \leq N_{\dagger} - \mathsf{N}_{\mathrm{cut,x}}$*,*

[11] The choice of $\mathsf{N}_{\mathrm{cut,t}}$ is such that $\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}$ can absorb a Sobolev loss from H or ρ, or help absorb small remainder terms into the miniscule constant δ_{tiny}.

$M \leq M_\dagger - \mathsf{N}_{\mathrm{cut,t}}$,

$$\left|D^N D_t^M \sigma_S^+\right| \lesssim (\sigma_S^+ + \delta_{\mathrm{tiny}})(\Lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) . \tag{A.112}$$

(iii) σ_S^+ *and* σ_S^- *have the same size as* S*, so that*

$$\left\|\sigma_S^+\right\|_p \lesssim \delta_{S,p}, \quad \left\|\sigma_S^-\right\|_p \lesssim \delta_{S,p} . \tag{A.113}$$

Furthermore $\mathsf{\Pi}(H)$ *and* $\mathsf{\Pi}(\rho)$ *have the same size as* H *and* ρ*, so that for* $N \leq N_\dagger - \mathsf{N}_{\mathrm{cut,x}}$*,* $M \leq M_\dagger - \mathsf{N}_{\mathrm{cut,t}}$*, and* $p = 3/2, \infty$

$$\left\|D^N D_t^M \mathsf{\Pi}(H)\right\|_p \lesssim \mathcal{C}_{G,p}(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) , \qquad \left\|D^N \mathsf{\Pi}(\rho)\right\|_p \lesssim \mathcal{C}_{\rho,p}(\Lambda\Gamma)^N . \tag{A.114}$$

We note also that $\mathsf{\Pi}(\rho)$ *is* $(\mathbb{T}/\mu)^3$*-periodic.*

(iv) π *dominates* σ_S^- *and* $\mathsf{\Pi}(H)$ *and their derivatives with suitable weights, so that for all* $N \leq N_\dagger - \mathsf{N}_{\mathrm{cut,x}}$ *and* $M \leq M_\dagger - \mathsf{N}_{\mathrm{cut,t}}$*,*

$$\left|D^N D_t^M \sigma_S^-\right| \lesssim \pi \left\|\mathsf{\Pi}(\rho)\right\|_1 (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) , \tag{A.115a}$$

$$\left|D^N D_t^M \mathsf{\Pi}(H)\right| \lesssim \pi(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) . \tag{A.115b}$$

(v) σ_S^+ *and* σ_S^- *are supported on* $\mathrm{supp}\,(S)$ *and* $\mathrm{supp}\,(H)$*, respectively.*

Proof of Lemma A.4.1. We break the proof into steps in which we prove each of the items (i)–(v).

Proof of (i)**:** We first use (A.107a) and $D_t\Phi = 0$ from (iii) and Lemma A.1.2 to deduce

that for $N \leq N_\dagger$ and $M \leq M_\dagger$,

$$|D^N D_t^M S| = |D^N((D_t^M H)(\rho) \circ \Phi)| \leq \sum_{N_1+N_2=N} |D^{N_1}(D_t^M H)||D^{N_2}(\rho \circ \Phi))|$$
$$\lesssim \sum_{N_1+N_2=N} |D^{N_1}(D_t^M H)| \sum_{n_2=1}^{N_2} (\lambda\Gamma)^{N_2-n_2} |(D^{n_2}\rho) \circ \Phi| \ . \tag{A.116}$$

Estimate (A.111) will then follow from (A.116) and the following claims;

$$\Pi(H) \lesssim \mathcal{C}_{G,\infty} \tag{A.117a}$$
$$\Pi(\rho) \lesssim \mathcal{C}_{\rho,\infty} \tag{A.117b}$$
$$|D^{N_1} D_t^M H| \lesssim (\Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})(\lambda\Gamma)^{N_1} \mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \tag{A.117c}$$
$$\lambda^{N_2-n_2}|D^{n_2}\rho| \lesssim (\Pi(\rho) + \mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})(\Lambda\Gamma)^{N_2} \tag{A.117d}$$

for any integers $0 \leq N_1, n_2 \leq N_\dagger$, $M \leq M_\dagger$. Indeed, the above claims, (A.108a)–(A.108b), and (A.116) give that for $N \leq N_\dagger$ and $M \leq M_\dagger$,

$$\begin{aligned}
|D^N D_t^M S| &\lesssim (\Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})(\Pi(\rho) \circ \Phi + \mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})(\Lambda\Gamma)^N \mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \\
&\lesssim \left(\Pi(H)\Pi(\rho) \circ \Phi + \Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\left(\mathcal{C}_{G,\infty}\Pi(\rho) \circ \Phi + \mathcal{C}_{\rho,\infty}\Pi(H) + \mathcal{C}_{G,\infty}\mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)\right) \\
&\qquad \times (\Lambda\Gamma)^N \mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \\
&\lesssim (\sigma_s^+ + \delta_{\mathrm{tiny}})(\Lambda\Gamma)^N \mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \ .
\end{aligned}$$

The proofs of the claims are then given as follows. The first is immediate from the definition of $\Pi(H)$ and the computation

$$\begin{aligned}
& \Pi(H) \lesssim \mathcal{C}_{G,\infty} \\
\Longleftarrow \qquad & \left(\Pi(H) + \mathcal{C}_{G,3/2}\right)^2 \lesssim \mathcal{C}_{G,\infty}^2 + \mathcal{C}_{G,3/2}^2 \\
\Longleftarrow \qquad & (\lambda\Gamma)^{-2N}(\nu\Gamma)^{-2M}|D^N D_t^M H|^2 \lesssim \mathcal{C}_{G,\infty}^2 \, ,
\end{aligned}$$

which holds for $N \leq \mathsf{N}_{\text{cut,x}}$ and $M \leq \mathsf{N}_{\text{cut,t}}$ from (A.105a). A similar computation holds for $\Pi(\rho)$. For the next two claims, if $M \leq \mathsf{N}_{\text{cut,t}}$ and $N_1, N_2 \leq \mathsf{N}_{\text{cut,x}}$, an argument quite similar to the above computation shows that

$$|D^{N_1}(D_t^M H)| \lesssim \Pi(H)(\lambda\Gamma)^{N_1}(\nu\Gamma)^M \,, \tag{A.118a}$$

$$\lambda^{N_2-n_2}\,|(D^{n_2}\rho)\circ\Phi| \lesssim (\Lambda\Gamma)^{N_2}\Pi(\rho)\circ\Phi \,. \tag{A.118b}$$

If however $M > \mathsf{N}_{\text{cut,t}}$, $N_1 > \mathsf{N}_{\text{cut,x}}$, or $N_2 > \mathsf{N}_{\text{cut,x}}$, we use (A.108a)–(A.108b) and (A.105a) in the first two cases and (A.105c) in the third case to obtain, respectively, that

$$\left\|D^{N_1}(D_t^M H)\right\|_{L^\infty} \lesssim \mathcal{C}_{G,\infty}\lambda^{N_1}\mathcal{M}\left(M, M_t, \nu, \nu'\right) \lesssim \Gamma^{-\mathsf{N}_{\text{cut,t}}}\mathcal{C}_{G,\infty}\lambda^{N_1}\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \tag{A.119a}$$

$$\left\|D^{N_1}(D_t^M H)\right\|_{L^\infty} \lesssim \Gamma^{-\mathsf{N}_{\text{cut,t}}}\mathcal{C}_{G,\infty}(\lambda\Gamma)^{N_1}\mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.119b}$$

$$\lambda^{N_2-n_2}\left\|D^{n_2}\rho\right\|_{L^\infty} \lesssim \Gamma^{-\mathsf{N}_{\text{cut,t}}}\mathcal{C}_{\rho,\infty}(\Lambda\Gamma)^{N_2} \,, \tag{A.119c}$$

concluding the proof of the claims and thus (A.111).

Proof of (ii)**:** We first show by induction that for integers $K \geq 0$ and N, M such that $N + M = K, N \leq N_\dagger - \mathsf{N}_{\text{cut,x}}$, and $M \leq M_\dagger - \mathsf{N}_{\text{cut,t}}$,

$$|D^N D_t^M \Pi(H)| \lesssim \left(\Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\text{cut,t}}}\right)(\lambda\Gamma)^N\mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \,. \tag{A.120}$$

When $K = 0$ the claim is immediate. Now, suppose by induction that (A.120) holds true for any $K \leq K_0$, $K_0 \in \mathbb{N} \cup \{0\}$. To obtain (A.120) for $K_0 + 1$, we first note that for N'', M'' such that $0 < N'' + M''$, $|D^{N''}D_t^{M''}\Pi(H)| = |D^{N''}D_t^{M''}(\Pi(H) + \mathcal{C}_{G,3/2})|$. We then obtain the

inequality

$$\begin{aligned}\left|D^N D_t^M \mathsf{\Pi}(H)\right| &= \left|D^N D_t^M \left(\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right)\right| \\ &\lesssim \frac{1}{\left|\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right|}\Bigg[\left|D^N D_t^M \left((\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2})^2\right)\right| \\ &\qquad + \sum_{\substack{0\le N'\le N\\ 0\le M'\le M\\ 0<N'+M'\le K_0}} \left|D^{N'} D_t^{M'} \mathsf{\Pi}(H)\right| \left|D^{N-N'} D_t^{M-M'} \mathsf{\Pi}(H)\right|\Bigg],\end{aligned} \tag{A.121}$$

which follows from Lemma A.2.1 with $p = 2$ and the positivity of $\left|\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right|$. Using the inductive assumption (A.120), which is valid since $0 < N' + M' \le K_0$, and (A.108b), the second term can be controlled by

$$\begin{aligned}&\frac{1}{\left|\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right|}\left(\mathsf{\Pi}(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)^2 (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \Gamma\nu, \Gamma\nu'\right) \\ &\quad \lesssim \left(\mathsf{\Pi}(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \Gamma\nu, \Gamma\nu'\right).\end{aligned} \tag{A.122}$$

As for the first term, we have that

$$\begin{aligned}&\frac{\left|D^N D_t^M \left((\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2})^2\right)\right|}{\left|\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right|} \\ &\quad \le \frac{1}{\left|\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right|}\sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}}\sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} (\lambda\Gamma)^{-2n}(\nu\Gamma)^{-2m}\left|D^N D_t^M \left|D^n D_t^m H\right|^2\right| \\ &\quad = \frac{1}{\left|\mathsf{\Pi}(H) + \mathcal{C}_{G,3/2}\right|}\sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}}\sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}}\sum_{\substack{0\le N'\le N\\ 0\le M'\le M}} (\lambda\Gamma)^{-2n}(\nu\Gamma)^{-2m}\left|D^{N'} D_t^{M'} D^n D_t^m H\right|\left|D^{N-N'} D_t^{M-M'} D^n D_t^m H\right|.\end{aligned} \tag{A.123}$$

To bound the quantity above, we first claim that for multi-indices $\alpha, \beta \in \mathbb{N}^k$ with $k \ge 2$, $|\alpha| \le N_\dagger$, and $|\beta| \le M_\dagger$,

$$\left|\prod_{i=1}^k D^{\alpha_i} D_t^{\beta_i} H\right|(x) \lesssim \left(\mathsf{\Pi}(H)(x) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\lambda\Gamma)^{|\alpha|}\mathcal{M}\left(|\beta|, M_t, \nu\Gamma, \nu'\Gamma\right). \tag{A.124}$$

To prove this claim, let $\Omega(x) \subseteq \operatorname{supp}(H)$ be a closed set containing x. Then applying Lemma A.2.3 with $p = \infty$, $N_t = M_t$, $N_* = N_\dagger$, $M_* = M_\dagger$, $\Omega = \Omega(x)$, $\mathcal{C}_v = \nu\lambda^{-1}$, $\lambda_v = \widetilde{\lambda}_v = \lambda$, $\mu_v = \nu$, $\widetilde{\mu}_v = \nu'$, $f = H$, $\mathcal{C}_f = \sup_{\Omega(x)}(\Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})$, $\lambda_f = \widetilde{\lambda}_f = \lambda\Gamma$, $\mu_f = \nu\Gamma$, and $\widetilde{\mu}_f = \nu'\Gamma$, we have that (A.27) is satisfied from (A.107b), and (A.28) is satisfied by (A.117c) and the assumption on $|\alpha|, |\beta|$. Then (A.31) gives that

$$\left|\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} H\right|(x) \lesssim \left(\sup_{\Omega(x)} \Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\lambda\Gamma)^{|\alpha|}\mathcal{M}\left(|\beta|, M_t, \nu\Gamma, \nu'\Gamma\right) . \tag{A.125}$$

Since $\Omega(x)$ is arbitrary and $\Pi(H)$ is continuous, we have proven (A.124). Plugging the bound in (A.124) into (A.123), we find that

$$\begin{aligned}\frac{\left|D^N D_t^M \left((\Pi(H) + \mathcal{C}_{G,3/2})^2\right)\right|}{\left|\Pi(H) + \mathcal{C}_{G,3/2}\right|} \lesssim \frac{1}{\left|\Pi(H) + \mathcal{C}_{G,3/2}\right|}&\left(\Pi(H)(x) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)^2 \\ &\times (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) ,\end{aligned}$$

which matches the desired bound in (A.120) after using (A.108b). This concludes the proof of (A.120).

Arguing in a similar way (in fact the proof is simpler since only spatial derivatives are required), we also have that for each integer $0 \le N \le N_\dagger - \mathsf{N}_{\mathrm{cut,x}}$,

$$\left|D^N \Pi(\rho)\right| \lesssim \left(\Pi(\rho) + \mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)\ (\Lambda\Gamma)^N , \tag{A.126a}$$

$$\left|D^N (\Pi(\rho) \circ \Phi)\right| \lesssim \left(\Pi(\rho) \circ \Phi + \mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)\ (\Lambda\Gamma)^N . \tag{A.126b}$$

Combining (A.120), (A.126b), and the choice of δ_{tiny} from (A.108b), we obtain the desired estimate (A.112).

Proof of (iii)**:** Observe that by the construction of $\Pi(H)$, (A.105a), and a computation similar to that used to produce (A.117a), we have $\left\|\Pi(H) + \mathcal{C}_{G,3/2}\right\|_p \lesssim \mathcal{C}_{G,p}$ for $p = 3/2, \infty$,

and so $\|\Pi(H)\|_p \lesssim \mathcal{C}_{G,p}$. It follows from (A.120) and (A.108b) that

$$\left\|D^N D_t^M \Pi(H)\right\|_p \lesssim \mathcal{C}_{G,p}(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.127}$$

for $N \leq N_\dagger - \mathsf{N}_{\mathrm{cut,x}}$ and $M \leq M_\dagger - \mathsf{N}_{\mathrm{cut,t}}$. Similarly, by the construction of $\Pi(\rho)$, (A.105c) and (A.126a), we have that $\|\Pi(\rho)\|_p \lesssim \mathcal{C}_{\rho,p}$, and so

$$\left\|D^N \Pi(\rho)\right\|_p \lesssim \mathcal{C}_{\rho,p}(\Lambda\Gamma)^N \tag{A.128}$$

for $N \leq N_\dagger - \mathsf{N}_{\mathrm{cut,x}}$. Thus (A.114) is verified. Also, by the construction of $\Pi(\rho)$, its periodicity easily follows from (ii). Next, we can immediately deduce from the definition of σ_S^- the easier bound

$$\left\|\sigma_S^-\right\|_p \lesssim \|\Pi(H)\|_p \|\Pi(\rho)\|_1 \lesssim \mathcal{C}_{G,p}\mathcal{C}_{\rho,p} = \delta_{S,p}\,.$$

In the case of σ_S^+ and $p = 3/2$, we additionally apply Lemma A.1.3 by setting

$$\begin{aligned}
&N_* = N_\dagger - \mathsf{N}_{\mathrm{cut,x}}, \quad M_* = M_\dagger - \mathsf{N}_{\mathrm{cut,t}}, \quad f = \Pi(H), \quad \Phi = \Phi\,,\\
&\lambda = \lambda\Gamma, \quad \tau^{-1} = \nu\Gamma, \quad \mathrm{T}^{-1} = \nu'\Gamma,\\
&\mathcal{C}_f = \mathcal{C}_{G,3/2}, \quad v = v, \quad \varrho = \Pi(\rho), \quad \mu = \mu,\\
&\Upsilon = \Lambda = \Lambda\Gamma, \quad \mathcal{C}_\varrho = \mathcal{C}_{\rho,3/2}, \quad N_t = M_t - \mathsf{N}_{\mathrm{cut,t}}\,.
\end{aligned}$$

Then (A.12) is verified from (A.127), (A.13)–(A.14) follow from (A.107a), (A.15) follows from (A.128) and the periodicity of $\Pi(\rho)$, (A.16) follows from (A.106), and (A.17) follows from (A.108c). We then obtain from (A.18) that

$$\left\|\sigma_S^+\right\|_{3/2} \lesssim \mathcal{C}_{G,3/2}\mathcal{C}_{\rho,3/2} = \delta_{S,3/2}\,.$$

Finally, the estimate for $\left\|\sigma_S^+\right\|_\infty$ is trivial, so that (A.113) holds and (iii) is totally verified.

Proof of (iv)**:** We first prove (A.115b) by induction; namely, for each integer $K = N+M \geq 0$, $N \leq N_\dagger - \mathsf{N}_{\text{cut,x}}$, $M \leq M_\dagger - \mathsf{N}_{\text{cut,t}}$,

$$|D^N D_t^M \Pi(H)| \lesssim \pi(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu\Gamma'\right) . \tag{A.129}$$

The proof uses an argument quite similar to the proof of (A.120). The base case follows from writing that

$$\begin{aligned} &\Pi(H) \lesssim \pi \\ \iff &\Pi(H) + \mathcal{C}_{G,3/2} \lesssim \pi + \mathcal{C}_{G,3/2} \\ \impliedby &\left(\Pi(H) + \mathcal{C}_{G,3/2}\right)^2 \lesssim \pi^2 + \mathcal{C}_{G,3/2}^2 , \end{aligned}$$

which can be seen to hold from the definition of $\Pi(H)$ and (A.105b). For the inductive step, we argue starting from (A.121), although with slightly different steps to follow. Using the inductive assumption from (A.129) to control one term and the bound (A.120) to control the other term, and (A.108b), we have that the second term from (A.121) may be bounded by

$$\begin{aligned} &\frac{1}{\left|\Pi(H) + \mathcal{C}_{G,3/2}\right|} \pi \left(\Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\text{cut,t}}}\right) (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \Gamma\nu, \Gamma\nu'\right) \\ &\quad \lesssim \pi(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \Gamma\nu, \Gamma\nu'\right) . \end{aligned} \tag{A.130}$$

Thus it remains to control the first term from (A.121). Towards this end, we claim that for multi-indices $\alpha, \beta \in \mathbb{N}^k$ with $k \geq 2$, $|\alpha| \leq N_\dagger$, and $|\beta| \leq M_\dagger$,

$$\left|\prod_{i=1}^k D^{\alpha_i} D_t^{\beta_i} H\right| (x) \lesssim \pi(x)(\lambda\Gamma)^{|\alpha|} \mathcal{M}\left(|\beta|, M_t, \nu\Gamma, \nu'\Gamma\right) . \tag{A.131}$$

We apply Lemma A.2.3 with precisely the same choices as in the proof of (A.124), save for

the choice of $\mathcal{C}_f = \sup_{\Omega(x)} \pi$. Then (A.27) is satisfied from (A.107b), and (A.28) is satisfied by (A.105b). Then applying (A.31), shrinking $\Omega(x)$ to a point, and using the continuity of π provides (A.131). Plugging this bound into (A.123) and using (A.124) and (A.108b), we find that for $N \leq N_\dagger - \mathsf{N}_{\text{cut,x}}$ and $M \leq M_\dagger - \mathsf{N}_{\text{cut,t}}$,

$$\begin{aligned}
&\frac{\left|D^N D_t^M \left((\Pi(H) + \mathcal{C}_{G,3/2}^{2/3})^2\right)\right|}{\left|\Pi(H) + \mathcal{C}_{G,3/2}\right|} \\
&\quad \lesssim \frac{1}{\left|\Pi(H) + \mathcal{C}_{G,3/2}\right|} \pi\left(\Pi(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\text{cut,t}}}\right)(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \\
&\quad \lesssim \pi(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) ,
\end{aligned}$$

which combined with (A.130) concludes the proof of (A.115b). To prove (A.115a), we use (A.115b) and the definition of σ_S^-.

Proof of (v)**:** By the definition of $\Pi(H)$ and $\Pi(\rho)$, it is easy to see that $\operatorname{supp}(\Pi(H)) \subseteq \operatorname{supp}(H)$ and $\operatorname{supp}(\Pi(\rho)) \subseteq \operatorname{supp}(\rho)$, and so (v) is verified. □

Lemma A.4.2 (**Pressure increment for current error).** *Let v be an incompressible vector field on $\mathbb{R} \times \mathbb{T}^3$. Denote its material derivative by $D_t = \partial_t + v \cdot \nabla$. We use large positive integers $N_* \geq M_* \gg M_t$ for counting derivatives and specify additional constraints that they must satisfy in assumptions* (i)–(iv)*.*

Suppose a current error $\phi = H\, \rho \circ \Phi$ and a non-negative, continuous function π are given such that the following hold.

(i) There exist constants $\mathcal{C}_{G,p}$ and $\mathcal{C}_{\rho,p}$ for $p = 1, \infty$, frequency parameters $\lambda, \Lambda, \nu, \nu'$, and intermittency parameters $0 < r_G, r_\phi \leq 1$ such that

$$\left\|D^N D_t^M H\right\|_p \lesssim \mathcal{C}_{G,p} \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.132a}$$

$$\left|D^N D_t^M H\right| \lesssim \pi^{3/2} r_G^{-1} \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.132b}$$

$$\left\|D^N \rho\right\|_p \lesssim \mathcal{C}_{\rho,p} \Lambda^N \tag{A.132c}$$

$$\|\phi\|_p \lesssim \mathcal{C}_{G,p} \mathcal{C}_{\rho,p} =: \delta_{\phi,p}^{3/2} r_\phi^{-1} \tag{A.132d}$$

for all $N \leq N_*$, $M \leq M_*$.

(ii) There exist a frequency parameter μ, *a parameter* Γ *for measuring small losses in derivative costs, and a positive integer* $\mathsf{N}_{\mathrm{dec}}$ *such that* ρ *is* $(\mathbb{T}/\mu)^3$*-periodic and* $\lambda \ll \mu \leq \Lambda$, *whereby we mean that*

$$(\Lambda\Gamma)^4 \leq \left(\frac{\mu}{4\pi\sqrt{3}(\lambda\Gamma)}\right)^{\mathsf{N}_{\mathrm{dec}}} . \tag{A.133}$$

(iii) Let Φ *be a volume preserving flow of* $\mathbb{T}^3$ *such that* $D_t\Phi = 0$ *and* Φ *is the identity at a time slice which intersects the support of* H, *and*

$$\left\|D^{N+1}\Phi\right\|_{L^\infty(\operatorname{supp} H)} + \left\|D^{N+1}\Phi^{-1}\right\|_{L^\infty(\operatorname{supp} H)} \lesssim \lambda^N \tag{A.134a}$$

$$\left\|D^N D_t^M Dv\right\|_{L^\infty(\operatorname{supp} H)} \lesssim \nu\lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.134b}$$

for all $N \leq N_*$, $M \leq M_*$.

(iv) There exist positive integers $\mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}}$ *and a small parameter* $\delta_{\mathrm{tiny}} \leq 1$ *such that*

$$\mathsf{N}_{\mathrm{cut,x}} \geq \mathsf{N}_{\mathrm{cut,t}} \tag{A.135a}$$

$$\left(\mathcal{C}_{G,\infty} + 1\right)\left(\mathcal{C}_{\rho,\infty} + 1\right)\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}} \leq \delta_{\mathrm{tiny}}^{3/2}, \mathcal{C}_{G,1}, \mathcal{C}_{\rho,1}, \tag{A.135b}$$

$$2\mathsf{N}_{\mathrm{dec}} + 4 \leq N_* - \mathsf{N}_{\mathrm{cut,x}} - 4 . \tag{A.135c}$$

Then one can construct a pressure increment σ_ϕ *associated to the current error* ϕ, *where*

$$\sigma_\phi = r_\phi^{2/3}\Pi(H)\left(\Pi(\rho)\circ\Phi - \langle\Pi(\rho)\rangle\right) , \tag{A.136a}$$

$$\sigma_\phi^+ := r_\phi^{2/3}\Pi(H)\Pi(\rho)\circ\Phi , \tag{A.136b}$$

and

$$\Pi(H) := \left(\mathcal{C}_{G,1}^2 + \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}} \sum_{M=0}^{\mathsf{N}_{\mathrm{cut,t}}} (\lambda\Gamma)^{-2N} (\nu\Gamma)^{-2M} |D^N D_t^M H|^2 \right)^{\frac{1}{3}} - \mathcal{C}_{G,1}^{2/3} \,, \tag{A.137a}$$

$$\Pi(\rho) := \left(\mathcal{C}_{\rho,1}^2 + \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}} (\Lambda\Gamma)^{-2N} |D^N \rho|^2 \right)^{\frac{1}{3}} - \mathcal{C}_{\rho,1}^{2/3} \,, \tag{A.137b}$$

and which has the properties listed below.

(i) σ_ϕ^+ *dominates derivatives of* ϕ *with suitable weights, so that for all* $N \leq N_*$ *and* $M \leq M_*$*,*

$$\left| D^N D_t^M \phi \right| \lesssim \left((\sigma_\phi^+)^{3/2} r_\phi^{-1} + \delta_{\mathrm{tiny}} \right) (\Lambda\Gamma)^N \mathcal{M} \left(M, M_t, \nu\Gamma, \nu'\Gamma \right) \,. \tag{A.138}$$

(ii) σ_ϕ^+ *dominates derivatives of itself with suitable weights, so that for all* $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$*,* $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$*,*

$$\left| D^N D_t^M \sigma_\phi^+ \right| \lesssim \left(\sigma_\phi^+ + \delta_{\mathrm{tiny}} \right) (\Lambda\Gamma)^N \mathcal{M} \left(M, M_t, \nu, \nu' \right) \,. \tag{A.139}$$

(iii) σ_ϕ^+ *and* σ_ϕ^- *have size comparable to* ϕ*, so that*

$$\left\| \sigma_\phi^+ \right\|_{3/2} \lesssim \delta_{\phi,1} \,, \qquad \left\| \sigma_\phi^- \right\|_{3/2} \lesssim \delta_{\phi,1} \,, \tag{A.140a}$$

$$\left\| \sigma_\phi^+ \right\|_{\infty} \lesssim \delta_{\phi,\infty} \,, \qquad \left\| \sigma_\phi^- \right\|_{\infty} \lesssim \delta_{\phi,\infty} \,. \tag{A.140b}$$

Furthermore, $\Pi(H)$ *and* $\Pi(\rho)$ *have size comparable to* H *and* ρ*, respectively, so that for*

all $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$ *and* $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$,

$$\left\|D^N D_t^M \Pi(H)\right\|_{3/2} \lesssim \mathcal{C}_{G,1}^{2/3}(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) , \qquad \left\|D^N \Pi(\rho)\right\|_{3/2} \lesssim \mathcal{C}_{\rho,1}^{2/3}(\Lambda\Gamma)^N , \tag{A.141a}$$

$$\left\|D^N D_t^M \Pi(H)\right\|_{\infty} \lesssim \mathcal{C}_{G,\infty}^{2/3}(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) , \qquad \left\|D^N \Pi(\rho)\right\|_{\infty} \lesssim \mathcal{C}_{\rho,\infty}^{2/3}(\Lambda\Gamma)^N , \tag{A.141b}$$

We note also that $\Pi(\rho)$ *is* $(\mathbb{T}/\mu)^3$*-periodic.*

(iv) π *dominates* σ_ϕ^- *and* $\Pi(H)$ *and their derivatives with suitable weights, so that for all* $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$ *and* $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$,

$$\left|D^N D_t^M \sigma_\phi^-\right| \lesssim \left(\frac{r_\phi}{r_G}\right)^{2/3} \pi \left\|\Pi(\rho)\right\|_1 (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) , \tag{A.142a}$$

$$\left|D^N D_t^M \Pi(H)\right| \lesssim r_G^{-2/3} \pi (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) . \tag{A.142b}$$

(v) σ_ϕ^+ *and* σ_ϕ^- *are supported on* $\mathrm{supp}\,(\phi)$ *and* $\mathrm{supp}\,(H)$*, respectivly.*

Proof of Lemma A.4.2. We break the proof into steps in which we prove each of the items (i)–(v). The proof follows quite closely the proof of Lemma A.4.1, save for various rescalings related to the different scalings for current errors versus stress errors.

Proof of (i)**:** We first use (A.134a) and $D_t\Phi = 0$ from (iii) and Lemma A.1.2 to deduce that for $N \leq N_*$ and $M \leq M_*$,

$$\begin{aligned} |D^N D_t^M \phi| = |D^N((D_t^M H)(\rho) \circ \Phi)| &\leq \sum_{N_1+N_2=N} |D^{N_1}(D_t^M H)||D^{N_2}(\rho \circ \Phi))| \\ &\lesssim \sum_{N_1+N_2=N} |D^{N_1}(D_t^M H)| \sum_{n_2=1}^{N_2} (\lambda\Gamma)^{N_2-n_2} \left|(D^{n_2}\rho) \circ \Phi\right| . \end{aligned} \tag{A.143}$$

Estimate (A.138) will then follow from (A.143) and the following claims;

$$\Pi(H) \lesssim \mathcal{C}_{G,\infty}^{2/3} \tag{A.144a}$$

$$\Pi(\rho) \lesssim \mathcal{C}_{\rho,\infty}^{2/3} \tag{A.144b}$$

$$|D^{N_1} D_t^M H| \lesssim \left(\Pi^{3/2}(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\lambda\Gamma)^{N_1}\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \tag{A.144c}$$

$$\lambda^{N_2-n_2}|D^{n_2}\rho| \lesssim \left(\Pi^{3/2}(\rho) + \mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\Lambda\Gamma)^{N_2} \tag{A.144d}$$

for any integers $0 \leq N_1, n_2 \leq N_*$, $M \leq M_*$. Indeed, the above claims, (A.135a)–(A.135b), and (A.143) give that for $N \leq N_*$ and $M \leq M_*$,

$$\begin{aligned}
\left|D^N D_t^M \phi\right| &\lesssim \left(\Pi^{3/2}(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)\left(\Pi^{3/2}(\rho)\circ\Phi + \mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\Lambda\Gamma)^N\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \\
&\lesssim \Big(\left(\Pi(H)\Pi(\rho)\circ\Phi\right)^{3/2} + \Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\left(\mathcal{C}_{G,\infty}\Pi^{3/2}(\rho)\circ\Phi + \mathcal{C}_{\rho,\infty}\Pi^{3/2}(H) + \mathcal{C}_{G,\infty}\mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)\Big) \\
&\qquad\qquad \times (\Lambda\Gamma)^N\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \\
&\lesssim \left((\sigma_s^+)^{3/2} r_\phi^{-1} + \delta_{\mathrm{tiny}}\right)(\Lambda\Gamma)^N\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) .
\end{aligned}$$

The proofs of the claims are then given as follows. The first is immediate from the definition of $\Pi(H)$ and the computation

$$\begin{aligned}
& \Pi(H) \lesssim \mathcal{C}_{G,\infty}^{2/3} \\
\Longleftarrow \qquad & \left(\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right)^3 \lesssim \mathcal{C}_{G,\infty}^2 + \mathcal{C}_{G,1}^2 \\
\Longleftarrow \qquad & (\lambda\Gamma)^{-2N}(\nu\Gamma)^{-2M}|D^N D_t^M H|^2 \lesssim \mathcal{C}_{G,\infty}^2 ,
\end{aligned}$$

which holds for $N \leq \mathsf{N}_{\mathrm{cut,x}}$ and $M \leq \mathsf{N}_{\mathrm{cut,t}}$ from (A.132a). A similar computation holds for $\Pi(\rho)$. Next, if $M \leq \mathsf{N}_{\mathrm{cut,t}}$ and $N_1, N_2 \leq \mathsf{N}_{\mathrm{cut,x}}$, a computation similar to the one above shows

that

$$|D^{N_1}(D_t^M H)| \lesssim \Pi^{3/2}(H)(\lambda\Gamma)^{N_1}(\nu\Gamma)^M \,, \tag{A.145a}$$

$$\lambda^{N_2-n_2}\,|(D^{n_2}\rho)\circ\Phi| \lesssim (\Lambda\Gamma)^{N_2}\Pi^{3/2}(\rho)\circ\Phi \,. \tag{A.145b}$$

If however $M > \mathsf{N}_{\mathrm{cut,t}}$, $N_1 > \mathsf{N}_{\mathrm{cut,x}}$, or $N_2 > \mathsf{N}_{\mathrm{cut,x}}$, we use (A.135a)–(A.135b) and (A.132a) in the first two cases and (A.132c) in the third case to obtain, respectively, that

$$\left\|D^{N_1}(D_t^M H)\right\|_{L^\infty} \lesssim \mathcal{C}_{G,\infty}\lambda^{N_1}\mathcal{M}\left(M, M_t, \nu, \nu'\right) \lesssim \Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\mathcal{C}_{G,\infty}\lambda^{N_1}\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \tag{A.146a}$$

$$\left\|D^{N_1}(D_t^M H)\right\|_{L^\infty} \lesssim \Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\mathcal{C}_{G,\infty}(\lambda\Gamma)^{N_1}\mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.146b}$$

$$\lambda^{N_2-n_2}\left\|D^{n_2}\rho\right\|_{L^\infty} \lesssim \Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\mathcal{C}_{\rho,\infty}(\Lambda\Gamma)^{N_2} \,, \tag{A.146c}$$

concluding the proof of the claims and thus of (A.138).

Proof of (ii)**:** We first show by induction that for integers $K \geq 0$ and N, M such that $N + M = K, N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$, and $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$,

$$|D^N D_t^M \Pi(H)| \lesssim \left(\Pi(H) + (\mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})^{2/3}\right)(\lambda\Gamma)^N\mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \,. \tag{A.147}$$

When $K = 0$ the claim is immediate. Now, suppose by induction that (A.147) holds true for any $K \leq K_0$, $K_0 \in \mathbb{N}\cup\{0\}$. To obtain (A.147) for $K_0 + 1$, we first note that for N'', M'' such that $0 < N'' + M''$, $|D^{N''}D_t^{M''}\Pi(H)| = |D^{N''}D_t^{M''}(\Pi(H) + \mathcal{C}_{G,1}^{2/3})|$. We then obtain the

inequality

$$
\begin{aligned}
\left|D^N D_t^M \Pi(H)\right| &= \left|D^N D_t^M \left(\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right)\right| \\
&\lesssim \frac{1}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2} \Bigg[\left|D^N D_t^M \left((\Pi(H) + \mathcal{C}_{G,1}^{2/3})^3\right)\right| \\
&\qquad + \sum_{\left\{\substack{\alpha,\beta : \sum_{i=1}^3 \alpha_i = N, \\ \sum_{i=1}^3 \beta_i = M, \\ \alpha_i + \beta_i < N+M \, \forall i}\right\}} \prod_{i=1}^3 \left|D^{\alpha_i} D_t^{\beta_i} \left(\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right)\right| \Bigg],
\end{aligned}
\tag{A.148}
$$

which follows from Lemma A.2.1 with $p = 3$ and the positivity of $\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|$. Using the inductive assumption (A.147), which is valid since $0 < N' + M' \leq K_0$, and (A.135b), the second term can be controlled by

$$
\begin{aligned}
&\frac{1}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2} \left(\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right) \left(\Pi(H) + (\mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})^{2/3}\right)^2 (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \Gamma\nu, \Gamma\nu'\right) \\
&\quad \lesssim \left(\Pi(H) + (\mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})^{2/3}\right) (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \Gamma\nu, \Gamma\nu'\right) .
\end{aligned}
\tag{A.149}
$$

As for the first term, we have that

$$
\begin{aligned}
&\frac{\left|D^N D_t^M \left((\Pi(H) + \mathcal{C}_{G,1}^{2/3})^3\right)\right|}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2} \\
&\quad \leq \frac{1}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2} \sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}} \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} (\lambda\Gamma)^{-2n} (\nu\Gamma)^{-2m} \left|D^N D_t^M \left|D^n D_t^m H\right|^2\right| \\
&\quad = \frac{1}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2} \sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}} \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \sum_{\substack{0 \leq N' \leq N \\ 0 \leq M' \leq M}} (\lambda\Gamma)^{-2n} (\nu\Gamma)^{-2m} \left|D^{N'} D_t^{M'} D^n D_t^m H\right| \left|D^{N-N'} D_t^{M-M'} D^n D_t^m H\right| .
\end{aligned}
\tag{A.150}
$$

To bound the quantity above, we first claim that for multi-indices $\alpha, \beta \in \mathbb{N}^k$ with $k \geq 2$,

$|\alpha| \leq N_*$, and $|\beta| \leq M_*$,

$$\left|\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} H\right|(x) \lesssim \left(\Pi(H)^{3/2}(x) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\lambda\Gamma)^{|\alpha|}\mathcal{M}\left(|\beta|, M_t, \nu\Gamma, \nu'\Gamma\right) . \tag{A.151}$$

To prove this claim, let $\Omega(x) \subseteq \operatorname{supp} H$ be a closed set containing x. Then applying Lemma A.2.3 with $p = \infty$, $N_t = M_t$, $N_* = \mathsf{N}_{\mathrm{cut,x}}$, $M_* = \mathsf{N}_{\mathrm{cut,t}}$, $\Omega = \Omega(x)$, $\mathcal{C}_v = \nu\lambda^{-1}$, $\lambda_v = \widetilde{\lambda}_v = \lambda$, $\mu_v = \nu$, $\widetilde{\mu}_v = \nu'$, $f = H$, $\mathcal{C}_f = \sup_{\Omega(x)}\left(\Pi^{3/2}(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)$, $\lambda_f = \widetilde{\lambda}_f = \lambda\Gamma$, $\mu_f = \nu\Gamma$, and $\widetilde{\mu}_f = \nu'\Gamma$, we have that (A.27) is satisfied from (A.134b), and (A.28) is satisfied by (A.144c). Then (A.31) gives that

$$\left|\prod_{i=1}^{k} D^{\alpha_i} D_t^{\beta_i} H\right|(x) \lesssim \left(\sup_{\Omega(x)} \Pi(H)^{3/2} + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)(\lambda\Gamma)^{|\alpha|}\mathcal{M}\left(|\beta|, M_t, \nu\Gamma, \nu'\Gamma\right) . \tag{A.152}$$

Since $\Omega(x)$ is arbitrary and $\Pi(H)$ is continuous, we have proven (A.151). Plugging this bound into (A.150), we find that

$$\frac{\left|D^N D_t^M\left((\Pi(H) + \mathcal{C}_{G,1}^{2/3})^3\right)\right|}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2} \lesssim \frac{1}{\left|\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right|^2}\left(\Pi^{3/2}(H) + \mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)^2 \times (\lambda\Gamma)^N\mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) ,$$

which matches the desired bound in (A.147) after using (A.135b). This concludes the proof of (A.147).

Arguing in a similar way (in fact the proof is simpler since only spatial derivatives are required), we also have that for each integer $0 \leq N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$,

$$\left|D^N(\Pi(\rho)\circ\Phi)\right| \lesssim \left(\Pi(\rho)\circ\Phi + (\mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})^{2/3}\right)(\Lambda\Gamma)^N , \tag{A.153a}$$

$$\left|D^N\Pi(\rho)\right| \lesssim \left(\Pi(\rho) + (\mathcal{C}_{\rho,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}})^{2/3}\right)(\Lambda\Gamma)^N . \tag{A.153b}$$

Combining (A.147), (A.153a), and the choice of δ_{tiny} from (A.135b), we obtain the desired

estimate (A.139).

Proof of (iii)**:** Observe that by the construction of $\Pi(H)$, (A.132a), and a computation similar to that used to produce (A.144a), we have $\left\|\Pi(H)+\mathcal{C}_{G,1}^{2/3}\right\|_{3/2} \lesssim \mathcal{C}_{G,1}^{2/3}$, and so $\|\Pi(H)\|_{3/2} \lesssim \mathcal{C}_{G,1}^{2/3}$, with analogous bounds holding for ρ. It follows from (A.147) and (A.135b) that

$$\left\|D^N D_t^M \Pi(H)\right\|_{3/2} \lesssim \mathcal{C}_{G,1}^{2/3}(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.154}$$

for $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$ and $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$. If the left-hand side is measured instead in L^∞, we may appeal to (A.144a) to deduce that (A.154) holds with $\mathcal{C}_{G,\infty}$ in place of $\mathcal{C}_{G,1}$. Arguing similarly for $\Pi(\rho)$ but appealing to (A.153a) and (A.144b), we have that (A.141a)–(A.141b) are verified. Also, by the construction of $\Pi(\rho)$, its periodicity easily follows from (ii). Next, we can immediately deduce from the definition of σ_S^- and for $p = 3/2, \infty$ the easier bound

$$\left\|\sigma_S^-\right\|_p \lesssim r_\phi^{2/3} \left\|\Pi(H)\right\|_p \left\|\Pi(\rho)\right\|_1 ,$$

which matches the desired bounds in (A.140a)–(A.140b) for σ_ϕ^- after using the aforementioned bounds for $\Pi(H), \Pi(\rho)$ and recalling the definition of $\delta_{\phi,\cdot}$ from (A.132d). In the case of σ_ϕ^+ and $p = 3/2$, we additionally apply Lemma A.1.3 by setting

$$\begin{aligned}
&N_* = N_* - \mathsf{N}_{\mathrm{cut,x}}, \quad M_* = M_* - \mathsf{N}_{\mathrm{cut,t}}, \quad f = \Pi(H), \quad \Phi = \Phi\,, \\
&\lambda = \lambda\Gamma, \quad \tau^{-1} = \nu\Gamma, \quad \mathrm{T}^{-1} = \nu'\Gamma, \\
&\mathcal{C}_f = \mathcal{C}_{G,1}^{2/3}, \quad v = v, \quad \varrho = \Pi(\rho), \quad \mu = \mu, \\
&\Upsilon = \Lambda = \Lambda\Gamma, \quad \mathcal{C}_\varrho = \mathcal{C}_{\rho,1}^{2/3}, \quad N_t = M_t - \mathsf{N}_{\mathrm{cut,t}}\,.
\end{aligned}$$

Then (A.12) is verified from (A.154), (A.13)–(A.14) follow from (A.134a), (A.15) follows from (A.153b) and the periodicity of $\Pi(\rho)$, (A.16) follows from (A.133), and (A.17) follows

from (A.135c). We then obtain from (A.18) that

$$\left\|\sigma_S^+\right\|_{3/2} \lesssim r_\phi^{2/3}\mathcal{C}_{G,1}^{2/3}\mathcal{C}_{\rho,1}^{2/3} = \delta_{\phi,1}\,.$$

Finally, the estimate for $\left\|\sigma_S^+\right\|_\infty$ is trivial, so that (A.140a)–(A.140b) holds for σ_ϕ^+, and (iii) is totally verified.

Proof of (iv)**:** We first prove (A.142b) by induction; namely, for each integer $K = N+M \geq 0$, $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$, $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$,

$$|D^N D_t^M \Pi(H)| \lesssim r_G^{-2/3}\pi(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu\Gamma'\right)\,. \tag{A.155}$$

The proof uses an argument quite similar to the proof of (A.147). The base case follows from writing that

$$\begin{aligned} &\Pi(H) \lesssim \pi\, r_G^{-2/3} \\ \iff\ &\Pi(H) + \mathcal{C}_{G,1}^{2/3} \lesssim \pi\, r_G^{-2/3} + \mathcal{C}_{G,1}^{2/3} \\ \Longleftarrow\ &\left(\Pi(H) + \mathcal{C}_{G,1}^{2/3}\right)^3 \lesssim \pi^3 r_G^{-2} + \mathcal{C}_{G,1}^2\,, \end{aligned}$$

which can be seen to hold from the definition of $\Pi(H)$ and (A.132b). For the inductive step, we argue starting from (A.148), although with slightly different steps to follow. Using the inductive assumption from (A.155) to control the term from the trilinear product in the second term with the *highest* number of derivatives,[12] the bound (A.147) to control the other two terms from the trilinear product, and (A.135b), we have that the second term

[12]In fact any term which has been differentiated at all will suffice, so that we may replace $\Pi(H) + \mathcal{C}_{G,1}^{2/3}$ with simply $\Pi(H)$.

from (A.148) may be bounded by

$$\frac{1}{\left|\Pi(H)+\mathcal{C}_{G,1}^{2/3}\right|^2} r_G^{-2/3}\pi\left(\Pi(H)+\left(\mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)^{2/3}\right)^2(\lambda\Gamma)^N\mathcal{M}\left(M,M_t-\mathsf{N}_{\mathrm{cut,t}},\Gamma\nu,\Gamma\nu'\right)$$
$$\lesssim r_G^{-2/3}\pi(\lambda\Gamma)^N\mathcal{M}\left(M,M_t-\mathsf{N}_{\mathrm{cut,t}},\Gamma\nu,\Gamma\nu'\right)\ . \tag{A.156}$$

Thus it remains to control the first term from (A.148). Towards this end, we claim that for multi-indices $\alpha,\beta\in\mathbb{N}^k$ with $k\geq 2$, $|\alpha|\leq N_*$, and $|\beta|\leq M_*$,

$$\left|\prod_{i=1}^k D^{\alpha_i}D_t^{\beta_i}H\right|(x)\lesssim \pi^{3/2}(x)r_G^{-1}(\lambda\Gamma)^{|\alpha|}\mathcal{M}\left(|\beta|,M_t,\nu\Gamma,\nu'\Gamma\right)\ . \tag{A.157}$$

As in the proof of (A.151), we apply Lemma A.2.3 with precisely the same choices as led to the bound in (A.152), save for the choice of $\mathcal{C}_f=\sup_{\Omega(x)}\pi^{3/2}r_G^{-1}$. Then (A.27) is satisfied from (A.134b), and (A.28) is satisfied by (A.132b). Then applying (A.31), shrinking $\Omega(x)$ to a point, and using the continuity of π provides (A.157). Then plugging this bound into (A.150) and using (A.151) and (A.135b), we find that for $N\leq N_*-\mathsf{N}_{\mathrm{cut,x}}$ and $M\leq M_*-\mathsf{N}_{\mathrm{cut,t}}$,

$$\begin{aligned}
&\frac{\left|D^ND_t^M\left((\Pi(H)+\mathcal{C}_{G,1}^{2/3})^3\right)\right|}{\left|\Pi(H)+\mathcal{C}_{G,1}^{2/3}\right|^2}\\
&\lesssim \frac{1}{\left|\Pi(H)+\mathcal{C}_{G,1}^{2/3}\right|^2}\pi r_G^{-2/3}\left(\Pi^{3/2}(H)+\mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)^{4/3}(\lambda\Gamma)^N\mathcal{M}\left(M,M_t-\mathsf{N}_{\mathrm{cut,t}},\nu\Gamma,\nu'\Gamma\right)\\
&\lesssim \pi r_G^{-2/3}\frac{\Pi^2(H)+\left(\mathcal{C}_{G,\infty}\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}}\right)^{4/3}}{\left|\Pi(H)+\mathcal{C}_{G,1}^{2/3}\right|^2}(\lambda\Gamma)^N\mathcal{M}\left(M,M_t-\mathsf{N}_{\mathrm{cut,t}},\nu\Gamma,\nu'\Gamma\right)\\
&\lesssim \pi r_G^{-2/3}(\lambda\Gamma)^N\mathcal{M}\left(M,M_t-\mathsf{N}_{\mathrm{cut,t}},\nu\Gamma,\nu'\Gamma\right)\ ,
\end{aligned}$$

which combined with (A.156) concludes the proof of (A.142b). To prove (A.142a), we use (A.142b) and the definition of σ_ϕ^-.

Proof of (v)**:** By the definition of $\Pi(H)$ and $\Pi(\rho)$, it is easy to see that $\mathrm{supp}\,(\Pi(H))\subseteq\mathrm{supp}\,(H)$ and $\mathrm{supp}\,(\Pi(\rho))\subseteq\mathrm{supp}\,(\rho)$, and so (v) is verified. □

Lemma A.4.3 (Pressure increment and upgrade error from velocity increment potential). *We begin with* assumptions *which allow for the construction of a pressure increment and an upgrade current error. Then we delineate a number of properties satisfied by the* pressure increment*, before applying the material derivative and inverse divergence to produce a* current error *satisfying additional properties.*

Part 1: Assumptions

Let v be an incompressible vector field on $\mathbb{R} \times \mathbb{T}^3$. Denote its material derivative by $D_t = \partial_t + v \cdot \nabla$. We use large positive integers N_{**}, d, $K_\circ$, $N_* \geq M_* \gg M_t$, and $1 \leq M_\circ \leq N_\circ \leq {}^1\!/_2(M_* - \mathsf{N}_{\mathrm{cut,t}} - 1 - N_{**})$ and specify additional constraints that they must satisfy below. Suppose a velocity increment potential $\widehat{v} = G(\rho \circ \Phi)$ and a non-negative continuous function π are given such that the following hold.

(i) There exist constants $\mathcal{C}_{G,p}$ and $\mathcal{C}_{\rho,p}$ for $p = 3, \infty$, frequency parameters $\lambda, \Lambda, \nu, \nu'$, and intermittency parameters $r_G, r_{\widehat{v}} \leq 1$ such that

$$\left\|D^N D_t^M G\right\|_p \lesssim \mathcal{C}_{G,p} \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.158a}$$

$$\left|D^N D_t^M G\right| \lesssim \pi^{\frac{1}{2}} r_G^{-\frac{1}{3}} \lambda^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.158b}$$

$$\left\|D^N \rho\right\|_p \lesssim \mathcal{C}_{\rho,p} \Lambda^N \tag{A.158c}$$

$$\|\widehat{v}\|_p \lesssim \mathcal{C}_{G,p} \mathcal{C}_{\rho,p} =: \delta_{\widehat{v},p}^{\frac{1}{2}} r_{\widehat{v}}^{-\frac{1}{3}} \tag{A.158d}$$

for all $N \leq N_*$, $M \leq M_*$.

(ii) There exist frequency parameters μ and λ', a parameter $\Gamma = \Lambda^\alpha$ for $0 < \alpha \ll 1$ for measuring small losses in derivative costs, and a positive integer $\mathsf{N}_{\mathrm{dec}}$ such that ρ is $({}^{\mathbb{T}}\!/_{\mu})^3$-periodic and $\lambda, \lambda' \ll \mu \leq \Lambda$, whereby we mean that

$$\max(\lambda, \lambda')\Gamma\mu^{-1} \leq 1\,, \qquad (\Lambda\Gamma)^4 \leq \left(\frac{\mu}{4\pi\sqrt{3}\max(\lambda', \lambda)\Gamma}\right)^{\mathsf{N}_{\mathrm{dec}}}. \tag{A.159a}$$

(iii) Let Φ be a volume preserving flow of $\mathbb{T}^3$ such that $D_t\Phi = 0$ and Φ is the identity at a

time slice which intersects the support of G, and

$$\left\|D^{N+1}\Phi\right\|_{L^\infty(\operatorname{supp} G)} + \left\|D^{N+1}\Phi^{-1}\right\|_{L^\infty(\operatorname{supp} G)} \lesssim \lambda'^N \tag{A.160a}$$

$$\left\|D^N D_t^M Dv\right\|_{L^\infty(\operatorname{supp} G)} \lesssim \nu\lambda'^N \mathcal{M}\left(M, M_t, \nu, \nu'\right) \tag{A.160b}$$

for all $N \leq N_*$, $M \leq M_*$. Furthermore, assume that we have the lossy estimate

$$\left\|D^N \partial_t^M v\right\|_{L^\infty} \lesssim \mathcal{C}_v \lambda'^N (\nu')^M\,, \qquad \mathcal{C}_v \lambda' \lesssim \nu' \tag{A.160c}$$

for all $M \leq M_\circ$ and $N + M \leq N_\circ + M_\circ$.

(iv) There exist positive integers $\mathsf{N}_{\mathrm{cut,x}}$, $\mathsf{N}_{\mathrm{cut,t}}$ and a small parameter $\delta_{\mathrm{tiny}} \leq 1$ such that

$$\mathsf{N}_{\mathrm{cut,t}} \leq \mathsf{N}_{\mathrm{cut,x}}\,, \tag{A.161a}$$

$$(\mathcal{C}_{G,\infty}^2 + 1)(\mathcal{C}_{\rho,\infty}^2 + 1)\Gamma^{-2\mathsf{N}_{\mathrm{cut,t}}} \leq \delta_{\mathrm{tiny}}\,,\ \mathcal{C}_{G,3}^2\,,\ \mathcal{C}_{\rho,3}^2\,, \tag{A.161b}$$

$$2\mathsf{N}_{\mathrm{dec}} + 4 \leq N_* - \mathsf{N}_{\mathrm{cut,x}} - N_{**}\,. \tag{A.161c}$$

(v) Let an increasing sequence of frequencies $\{\mu_0, \cdots, \mu_{\bar{m}}\}$, $\mu < \mu_0 < \cdots < \mu_{\bar{m}-1} < \Lambda\Gamma < \mu_{\bar{m}}$ be given satisfying

$$\max(\lambda, \lambda')\Gamma\mu_{m-1}^{-2}\mu_m \leq 1 \tag{A.162}$$

for all $1 \leq m < \bar{m}$.

(vi) Assume that d *and* N_{**} *are sufficiently large so that*

$$\nu\Gamma\mathcal{C}_{G,p}^2\mathcal{C}_{\rho,p}^2(\max(\lambda,\lambda')\Gamma)^{\lfloor \mathsf{d}/2\rfloor}\mu^{-\lfloor \mathsf{d}/2\rfloor}(\Lambda\Gamma)^{5+K_\circ}\left(1+\frac{\max\{\nu'\Gamma,\mathcal{C}_v\Lambda\Gamma\}}{\nu\Gamma}\right)^{M_\circ}\leq 1\,, \tag{A.163a}$$

$$\nu\Gamma\mathcal{C}_{G,p}^2\mathcal{C}_{\rho,p}^2(\max(\lambda,\lambda')\Gamma)^{\lfloor \mathsf{d}/2\rfloor}(\mu_m\mu_{m-1}^{-2})^{\lfloor \mathsf{d}/2\rfloor}(\Lambda\Gamma)^{5+K_\circ}\left(1+\frac{\max\{\nu'\Gamma,\mathcal{C}_v\Lambda\Gamma\}}{\nu\Gamma}\right)^{M_\circ}\leq 1\,, \tag{A.163b}$$

$$\nu\Gamma\mathcal{C}_{G,\infty}^2\mathcal{C}_{\rho,3}^2((\Lambda\Gamma)\mu_{\bar{m}}^{-1})^{N_{**}}(\Lambda\Gamma)^{5+K_\circ}\left(1+\frac{\max\{\nu'\Gamma,\mathcal{C}_v\Lambda\Gamma\}}{\nu\Gamma}\right)^{M_\circ}\leq 1\,, \tag{A.163c}$$

for $1\leq m\leq \bar{m}$.

Part 2: Pressure increment

There exists a pressure increment $\sigma_{\widehat{\upsilon}}=\sigma_{\widehat{\upsilon}}^+-\sigma_{\widehat{\upsilon}}^-$ *associated to the velocity increment potential* $\widehat{\upsilon}$ *which is defined by*

$$\sigma_{\widehat{\upsilon}} := r_{\widehat{\upsilon}}^2\Pi(G)\left(\Pi(\rho)\circ\Phi-\langle\Pi(\rho)\rangle\right) =: \sigma_{\widehat{\upsilon}}^+-\sigma_{\widehat{\upsilon}}^-\,, \tag{A.164a}$$

$$\Pi(G) := \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}}\sum_{M=0}^{\mathsf{N}_{\mathrm{cut,t}}}(\lambda\Gamma)^{-2N}(\nu\Gamma)^{-2M}|D^N D_t^M G|^2\,, \tag{A.164b}$$

$$\Pi(\rho) := \sum_{N=0}^{\mathsf{N}_{\mathrm{cut,x}}}(\Lambda\Gamma)^{-2N}|D^N\rho|^2\,, \tag{A.164c}$$

may be decomposed as

$$\sigma_{\widehat{\upsilon}} = \sigma_{\widehat{\upsilon}}^* + \sum_{m=0}^{\bar{m}}\sigma_{\widehat{\upsilon}}^m\,, \tag{A.164d}$$

and satisfies the properties listed below.

(i) $(\sigma_{\widehat{\upsilon}}^+)^{1/2}$ *dominates derivatives of* $\widehat{\upsilon}$ *with suitable weights, so that*

$$\left|D^N D_t^M\widehat{\upsilon}\right| \lesssim (\sigma_{\widehat{\upsilon}}^+ + \delta_{\mathrm{tiny}})^{1/2} r_{\widehat{\upsilon}}^{-1}(\Lambda\Gamma)^N\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right). \tag{A.165}$$

for all $N \leq N_*$, $M \leq M_*$.

(ii) $\sigma^+_{\widehat{\upsilon}}$ *dominates derivatives of itself with suitable weights, so that*

$$|D^N D_t^M \sigma^+_{\widehat{\upsilon}}| \lesssim (\sigma^+_{\widehat{\upsilon}} + \delta_{\text{tiny}})(\Lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.166}$$

for all $N \leq N_* - \mathsf{N}_{\text{cut,x}}$, $M \leq M_* - \mathsf{N}_{\text{cut,t}}$.

(iii) Let $(p, p') = (3, {}^3\!/_2)$ *or* (∞, ∞). *Then* $\sigma^+_{\widehat{\upsilon}}$ *and* $\sigma^-_{\widehat{\upsilon}}$ *satisfy*

$$\left\|\sigma^+_{\widehat{\upsilon}}\right\|_{p'} \lesssim \delta_{\widehat{\upsilon},p} r_{\widehat{\upsilon}}^{4/3}, \quad \left\|\sigma^-_{\widehat{\upsilon}}\right\|_{p'} \lesssim \delta_{\widehat{\upsilon},p} r_{\widehat{\upsilon}}^{4/3}.$$

We note also that $\Pi(\rho)$ *is* $({}^{\mathbb{T}}\!/_{\mu})^3$*-periodic. Furthermore,* $\Pi(G)$ *and* $\Pi(\rho)$ *have the same size as* G *and* ρ, *so that for* $N \leq N_* - \mathsf{N}_{\text{cut,x}}$ *and* $M \leq M_* - \mathsf{N}_{\text{cut,t}}$,

$$\left\|D^N D_t^M \Pi(G)\right\|_{p'} \lesssim \mathcal{C}^2_{G,p}(\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right), \qquad \left\|D^N \Pi(\rho)\right\|_{p'} \lesssim \mathcal{C}^2_{\rho,p}(\Lambda\Gamma)^N. \tag{A.167}$$

(iv) π *dominates* $\sigma^-_{\widehat{\upsilon}}$ *and* $\Pi(G)$ *and its derivatives with suitable weights, so that*

$$\left|D^N D_t^M \Pi(G)\right| \lesssim \pi r_G^{-2/3} (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right), \tag{A.168a}$$

$$|D^N D_t^M \sigma^-_{\widehat{\upsilon}}| \lesssim \pi r_G^{-2/3} \left\|\Pi(\rho)\right\|_1 r^2_{\widehat{\upsilon}} (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.168b}$$

for all $N \leq N_* - \mathsf{N}_{\text{cut,x}}$, $M \leq M_* - \mathsf{N}_{\text{cut,t}}$.

(v) We have the support properties

$$\operatorname{supp}(\sigma^+_{\widehat{\upsilon}}) \subset \operatorname{supp}(\widehat{\upsilon}), \quad \operatorname{supp}(\sigma^-_{\widehat{\upsilon}}) \subseteq \operatorname{supp}(G). \tag{A.169}$$

Part 3: Current error

There exists an upgrade current error $\phi_{\widehat{\upsilon}}$ *which satisfies the following properties.*

(i) We have the decomposition and equalities

$$\phi_{\widehat{\upsilon}} = \underbrace{\phi^*_{\widehat{\upsilon}}}_{\text{nonlocal}} + \underbrace{\sum_{m=0}^{\bar{m}} \phi^m_{\widehat{\upsilon}}}_{\text{local}} = \underbrace{(\mathcal{H}+\mathcal{R}^*)(D_t\sigma^*_{\widehat{\upsilon}}) + \sum_{m=0}^{\bar{m}} \mathcal{R}^*(D_t\sigma^m_{\widehat{\upsilon}})}_{\text{nonlocal}} + \underbrace{\sum_{m=0}^{\bar{m}} \mathcal{H}(D_t\sigma^m_{\widehat{\upsilon}})}_{\text{local}}, \tag{A.170a}$$

$$\operatorname{div}\left(\phi^m_{\widehat{\upsilon}}(t,x) + \mathcal{R}^*(D_t\sigma^m_{\widehat{\upsilon}})(t,x)\right) = D_t\sigma^m_{\widehat{\upsilon}}(t,x) - \int_{\mathbb{T}^3} D_t\sigma^m_{\widehat{\upsilon}}(t,x')\,dx', \tag{A.170b}$$

$$\operatorname{div}\left(\phi^*_{\widehat{\upsilon}}(t,x) - \sum_{m=0}^{\bar{m}} \mathcal{R}^*(D_t\sigma^m_{\widehat{\upsilon}})(t,x)\right) = D_t\sigma^*_{\widehat{\upsilon}}(t,x) - \int_{\mathbb{T}^3} D_t\sigma^*_{\widehat{\upsilon}}(t,x')\,dx'. \tag{A.170c}$$

(ii) Let $(p,p') = (3, {}^3\!/_2)$ or (∞,∞). The current error $\phi^m_{\widehat{\upsilon}}$ satisfies

$$\left\|D^N D_t^M \phi^0_{\widehat{\upsilon}}\right\|_{p'} \lesssim \nu\Gamma^2 \mathcal{C}^2_{G,p}\mathcal{C}^2_{\rho,3}\left(\frac{\mu_0}{\mu}\right)^{\frac{4}{3}-\frac{2}{p'}} r^2_{\widehat{\upsilon}}\mu^{-1}\mu_0^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right), \tag{A.171a}$$

$$\left|D^N D_t^M \phi^0_{\widehat{\upsilon}}\right| \lesssim \nu\Gamma^2 \pi r_G^{-2/3} \mathcal{C}^2_{\rho,3}\left(\frac{\mu_0}{\mu}\right)^{4/3} r^2_{\widehat{\upsilon}}\mu^{-1}\mu_0^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right), \tag{A.171b}$$

$$\begin{aligned}\left\|D^N D_t^M \phi^m_{\widehat{\upsilon}}\right\|_{p'} \lesssim{}& \nu\Gamma^2 \mathcal{C}^2_{G,p}\mathcal{C}^2_{\rho,3}\left(\frac{\min(\mu_m,\Lambda\Gamma)}{\mu}\right)^{\frac{4}{3}-\frac{2}{p'}} r^2_{\widehat{\upsilon}}(\mu_{m-1}^{-2}\mu_m) \\ &\times \min(\mu_m,\Lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right),\end{aligned} \tag{A.171c}$$

$$\begin{aligned}\left|D^N D_t^M \phi^m_{\widehat{\upsilon}}\right| \lesssim{}& \nu\Gamma^2 \pi r_G^{-2/3} \mathcal{C}^2_{\rho,3}\left(\frac{\min(\mu_m,\Lambda\Gamma)}{\mu}\right)^{4/3} r^2_{\widehat{\upsilon}}\mu_{m-1}^{-2}\mu_m \\ &\times (\min(\mu_m,\Lambda\Gamma))^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right),\end{aligned} \tag{A.171d}$$

for any $1 \le m \le \bar{m}$, $N \le N_* - {}^{\mathsf{d}}\!/_2 - \mathsf{N}_{\text{cut,x}} - N_{**}$, and $M \le M_* - \mathsf{N}_{\text{cut,t}} - 1 - N_{**}$. Furthermore, we have that $\phi^*_{\widehat{\upsilon}}$ satisfies

$$\left\|D^N D_t^M \phi^*_{\widehat{\upsilon}}\right\|_\infty \lesssim \mu_0^{-K_\circ}(\Lambda\Gamma)^N(\nu\Gamma)^M \tag{A.172}$$

for all $N \le N_\circ$ and $M \le M_\circ$.

(iii) We have the support properties[13]

$$\operatorname{supp}(\phi^0_{\widehat{v}}) \subseteq \operatorname{supp}(G), \quad \operatorname{supp}(\phi^m_{\widehat{v}}) \subseteq \operatorname{supp} G \cap B\left(\operatorname{supp}\rho, 2\mu_{m-1}^{-1}\right) \circ \Phi \tag{A.173}$$

for all $0 < m \leq \bar{m}$.

(iv) For all $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}} - 1$*, we have that the mean* $\langle D_t \sigma_{\widehat{v}} \rangle$ *satisfies*

$$\left| \frac{d^M}{dt^M} \langle D_t \sigma_{\widehat{v}} \rangle \right| \lesssim (\Lambda\Gamma)^{-K_\circ} \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, -1, \nu\Gamma, \nu'\Gamma\right) . \tag{A.174}$$

Proof. `Step 1:` Constructing $\sigma_{\widehat{v}}$ and verifying the properties in Part 2.

For the moment we ignore the decomposition in (A.164d) and handle the rest of the conclusions in Part 2. Towards a proof of (i), we first have that $\mathsf{\Pi}(G) \lesssim \mathcal{C}^2_{G,\infty}$ and $\mathsf{\Pi}(\rho) \lesssim \mathcal{C}^2_{\rho,\infty}$. The proof of these is similar to (A.117a) and (A.117b), and we omit the details. Also, using a method of proof similar to that used to obtain (A.117c) and (A.117d), we can show that

$$|D^{N_1} D_t^M G| \lesssim (\mathsf{\Pi}(G) + \mathcal{C}^2_{G,\infty}\Gamma^{-2\mathsf{N}_{\mathrm{cut,t}}})^{1/2} (\lambda\Gamma)^{N_1} \mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) \tag{A.175a}$$

$$\lambda^{N_2 - n_2} |D^{n_2}\rho| \lesssim (\mathsf{\Pi}(\rho) + \mathcal{C}^2_{\rho,\infty}\Gamma^{-2\mathsf{N}_{\mathrm{cut,t}}})^{1/2} (\Lambda\Gamma)^{N_2} \tag{A.175b}$$

for any integers $0 \leq N_1, N_2 \leq N_*$, $0 \leq n_2 \leq N_2$ and $M \leq M_*$. Then, (i) follows as in the proof of (A.111).

Next, to prove (ii), we again claim that for $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$ and $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$,

$$|D^N D_t^M \mathsf{\Pi}(G)| \lesssim \left(\mathsf{\Pi}(G) + \mathcal{C}^2_{G,\infty}\Gamma^{-2\mathsf{N}_{\mathrm{cut,t}}}\right) (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.176a}$$

$$\left|D^N \mathsf{\Pi}(\rho)\right| \lesssim \left(\mathsf{\Pi}(\rho) + \mathcal{C}^2_{\rho,\infty}\Gamma^{-2\mathsf{N}_{\mathrm{cut,t}}}\right) (\Lambda\Gamma)^N \tag{A.176b}$$

$$\left|D^N (\mathsf{\Pi}(\rho) \circ \Phi)\right| \lesssim \left(\mathsf{\Pi}(\rho) \circ \Phi + \mathcal{C}^2_{\rho,\infty}\Gamma^{-2\mathsf{N}_{\mathrm{cut,t}}}\right) (\Lambda\Gamma)^N . \tag{A.176c}$$

[13] For any $\Omega \in \mathbb{T}^3$, we use $\Omega \circ \Phi_{(i,k)}$ to refer to the space-time set $\Phi^{-1}_{(i,k)}(t,\cdot)\Omega$ whose characteristic function is annihilated by D_t.

The proof of the claims is similar to, and in fact easier, than the proofs of the analogous estimates in (A.120) and (A.126b). Indeed, instead of (A.121), we simply have from the Leibniz rule that

$$\begin{aligned}\left|D^N D_t^M \mathsf{\Pi}(G)\right| &\leq \sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}} \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} (\lambda\Gamma)^{-2n}(\nu\Gamma)^{-2m} \left|D^N D_t^M \left|D^n D_t^m G\right|^2\right| \\ &= \sum_{n=0}^{\mathsf{N}_{\mathrm{cut,x}}} \sum_{m=0}^{\mathsf{N}_{\mathrm{cut,t}}} \sum_{\substack{0\leq N'\leq N \\ 0\leq M'\leq M}} (\lambda\Gamma)^{-2n}(\nu\Gamma)^{-2m} \left|D^{N'} D_t^{M'} D^n D_t^m G\right| \left|D^{N-N'} D_t^{M-M'} D^n D_t^m G\right| ,\end{aligned}$$

at which point we apply (A.175a). A similar argument produces the other two bounds listed above. Then (A.176a)–(A.176c) imply (ii) as in the proof of Proposition A.4.1.

Regarding (iii), as before, the estimate for G in (A.167) follows from (A.158a), (A.176a), and (A.161b). The estimate for $\mathsf{\Pi}(\rho)$ follows similarly from (A.158c), (A.176b), and (A.161b). Therefore, (A.167) is verified, and as a consequence $\left\|\sigma_{\widehat{v}}^{-}\right\|_{p'} \lesssim \delta_{\widehat{v},p} r_{\widehat{v}}^{4/3}$ follows after using (A.158d). The periodicity of $\mathsf{\Pi}(\rho)$ is immediate from the definition and the periodicity assumption on ρ. To obtain $\left\|\sigma_{\widehat{v}}^{+}\right\|_3 \lesssim \delta_{\widehat{v},3/2} r_{\widehat{v}}^{4/3}$, we use Lemma A.1.3 as in the proof of (A.140a), for example. The assumptions in the lemma can be verified using (A.167), (A.160a), (A.159a), and (A.161c) and the recently observed periodicity. Therefore, the desired estimate for $\sigma_{\widehat{v}}^{+}$ in $L^{3/2}$ follows from (A.18). The L^∞ estimate follows trivially from (A.167).

Next, we consider (iv). Similar to the proof of (A.176a), one can obtain

$$\left|D^N D_t^M \mathsf{\Pi}(G)\right| \lesssim \pi r_G^{-2/3} (\lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu\Gamma'\right) \tag{A.177}$$

for any integer $N \leq N_* - \mathsf{N}_{\mathrm{cut,x}}$ and $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$. Then we have (A.168a), and hence (A.168b) holds. Finally, (A.169) is immediate from the definitions in (A.164), concluding the proof of all claims in Part 2 except (A.164d).

`Step 2:` Constructing the current errors $\phi_{\widehat{v}}^m$ and verifying the properties in Part 3.

We first define $\sigma^m_{\widehat{\upsilon}}$ in order to verify (A.164d). Using the synthetic Littlewood-Paley decomposition from (7.34) and Definition 7.3.1, we write

$$\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho) = \widetilde{\mathbb{P}}_{\mu_0}\mathbb{P}_{\neq 0}(\mathsf{\Pi}(\rho)) + \left(\sum_{m=1}^{\bar{m}} \widetilde{\mathbb{P}}_{(\mu_{m-1},\mu_m]}(\mathsf{\Pi}(\rho))\right) + \underbrace{\left(\mathrm{Id} - \widetilde{\mathbb{P}}_{\mu_{\bar{m}}}\right)}_{=:\mathbb{P}^*}(\mathsf{\Pi}(\rho))\,. \tag{A.178}$$

For convenience, we use the abbreviations $\mathbb{P}_0$ for $\widetilde{\mathbb{P}}_{\mu_0}\mathbb{P}_{\neq 0}$ and $\mathbb{P}_m$ for $\widetilde{\mathbb{P}}_{(\mu_{m-1},\mu_m]}$ for $1 \leq m \leq \bar{m}$. Define $\sigma^m_{\widehat{\upsilon}}$, $\sigma^*_{\widehat{\upsilon}}$, $\phi^m_{\widehat{\upsilon}}$, and $\phi^*_{\widehat{\upsilon}}$ by

$$\begin{aligned}
\sigma_{\widehat{\upsilon}} = \sigma^*_{\widehat{\upsilon}} + \sum_{m=0}^{\bar{m}} \sigma^m_{\widehat{\upsilon}} &:= r^2_{\widehat{\upsilon}}\mathsf{\Pi}(G)(\mathbb{P}^*\mathsf{\Pi}(\rho)) \circ \Phi) + r^2_{\widehat{\upsilon}} \sum_{m=0}^{\bar{m}} \mathsf{\Pi}(G)(\widetilde{\mathbb{P}}_m(\mathsf{\Pi}(\rho)) \circ \Phi)\,, \\
\phi^m_{\widehat{\upsilon}} := \mathcal{H}(D_t\sigma^m_{\widehat{\upsilon}}), \quad \phi^*_{\widehat{\upsilon}} &:= (\mathcal{H} + \mathcal{R}^*)\sigma^*_{\widehat{\upsilon}} + \sum_{m=0}^{\bar{m}} \mathcal{R}^*(D_t\sigma^m_{\widehat{\upsilon}})\,.
\end{aligned}$$

Assuming that everything above is well-defined, we have verified (i). We aim to apply Proposition A.3.3 with Remarks A.3.8 and A.3.9 in separate cases according to which projector is being applied above. In order to apply the inverse divergence, we may however first treat the low-frequency assumptions from Part 1, which are the same in all cases (irrespective of which projector is being applied). We therefore set

$$\begin{aligned}
&\overline{N}_* = N_* - \mathsf{N}_{\mathrm{cut,x}} - N_{**}\,, \quad \overline{M}_* = M_* - \mathsf{N}_{\mathrm{cut,t}} - 1 - N_{**}\,, \quad \overline{M}_t = M_t - \mathsf{N}_{\mathrm{cut,t}} - 1 \\
&\overline{G} = D_t\mathsf{\Pi}(G), \quad \overline{\mathcal{C}}_{G,3/2} = \nu\Gamma\mathcal{C}^2_{G,3}, \quad \overline{\mathcal{C}}_{G,\infty} = \nu\Gamma\mathcal{C}^2_{G,\infty}, \quad \overline{\mu} = \mu\,, \quad \overline{\lambda}' = \lambda'\,, \\
&\overline{\Phi} = \Phi, \quad \overline{\lambda} = \max(\lambda, \lambda')\Gamma, \quad \overline{\nu} = \nu\Gamma, \quad \overline{\nu}' = \nu'\Gamma\,, \quad \overline{\pi} = \nu\Gamma\pi r_G^{-2/3}\,, \quad \overline{\upsilon} = \upsilon\,,
\end{aligned}$$

where we have used the convention set out in Remark A.3.10 to rewrite the symbols from Lemma A.4.1 with bars above on the left-hand side of the equalities below, while the right-hand side are parameters given in the assumptions of this Lemma. Then we have that (A.39) is verified from the assumption $N_* \geq M_*$ and (A.161a), (A.40) follows from conclusion (A.167), and (A.59) follows from conclusion (A.168a). Next, we see that (A.41), (A.42a), (A.42b), and (A.53) hold from (A.160a)–(A.160c). At this point we split into cases based on

which projector is applied and address parts 2-4 of Proposition A.3.3 in order to conclude the proof of this Lemma.

`Step 2a: Lowest shell.` For the case $m = 0$, we appeal to Lemma 7.3.3 with $q = 3/2$, $\lambda = \Lambda\Gamma$, $\rho = \mathbb{P}_{\neq 0}\Pi(\rho)$, and α such that λ^α in (7.37a) is equal to Γ. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p' = 3/2, \infty$

$$\overline{\varrho} = \mathbb{P}_0\Pi(\rho)\,, \quad \overline{\vartheta} \text{ as defined in (7.37a)}\,, \quad \overline{\mathcal{C}}_{*,p'} = \mathcal{C}^2_{\rho,3}\left(\frac{\mu_0}{\mu}\right)^{\frac{4}{3}-\frac{2}{p'}}\,,$$
$$\overline{\mu} = \mu\,, \quad \overline{\Upsilon} = \overline{\Upsilon}' = \mu\,, \quad \overline{\Lambda} = \mu_0\,, \quad \overline{\mathsf{d}} = \mathsf{d}\,.$$

Then (7.35) is satisfied with $\mathcal{C}_{p,3/2} = \mathcal{C}^2_{\rho,3}$ and $\lambda = \Lambda\Gamma$ from standard Littlewood-Paley theory, (A.167), and the choices from Step 1 which led to that conclusion, and so from (7.37a) we have that (A.43) is satisfied. From (A.159a), (A.161c), and the choice of $\overline{N}_*$ above, we have that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied from (A.160c) and the assumptions from Part 1 on $M_\circ$ and $N_\circ$. We have that (A.55) is satisfied from (A.163a). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. From (A.50), we obtain (A.171a). The pointwise bound in (A.171b) holds due to (A.60), (A.49a), and (A.47). Next, we obtain (A.172) for the portion of $\phi^*_{\mathfrak{v}}$ coming from this case $m = 0$ from (A.57). Finally, we obtain (A.173) from (A.48), concluding the proof of the desired conclusions for $m = 0$.

`Step 2b: Intermediate shells.` For the cases $1 \leq m \leq \bar{m}$, we appeal to Lemma 7.3.4 with $q = 3/2$ and $\rho = \mathbb{P}_{\neq 0}\Pi(\rho)$. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p' = 3/2, \infty$

$$\overline{\varrho} = \mathbb{P}_m\Pi(\rho), \quad \overline{\mathcal{C}}_{*,3/2} = \mathcal{C}^2_{\rho,3}, \quad \overline{\mathcal{C}}_{*,\infty} = \min((\mu_m/\mu)^{4/3}\mathcal{C}^2_{\rho,3}, \mathcal{C}^2_{\rho,\infty})\,, \quad \overline{\Upsilon} = \mu_{m-1}\,,$$
$$\overline{\Upsilon}' = \overline{\Lambda} = \min(\mu_m, \Lambda\Gamma)\,, \quad \vartheta \text{ as defined in Lemma 7.3.4}\,, \qquad \alpha \text{ as in the previous substep}\,.$$

Then (7.39) is satisfied with $\mathcal{C}_{p,3/2} = \mathcal{C}^2_{\rho,3}$ as in the last substep, and so from (7.40b) we have that (A.43) is satisfied. From (A.159a), (A.161c), (A.162), and the choice of $\overline{N}_*$ above, we have that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied as in the last substep. We have that (A.55) is satisfied from (A.163b). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. From (A.50), we obtain (A.171c). The pointwise bound in (A.171d) holds due to (A.60), (A.49a), and (A.47). Next, we obtain (A.172) for the portion of $\phi^*_{\widehat{v}}$ coming from this case $1 \leq m \leq \bar{m}$ from (A.57). Finally, we obtain (A.173) from (A.48) and (7.40c), concluding the proof of the desired conclusions for $1 \leq m \leq \bar{m}$.

`Step 2c: Highest shell.` For the case $m = \bar{m}$, we appeal to Lemma 7.3.3 with $q = 3/2$, $\lambda = \Lambda\Gamma$, $\rho = \mathbb{P}_{\neq 0}\Pi(\rho)$, and α such that λ^α in (7.37a) is equal to Γ. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p' = \infty$

$$\overline{\varrho} = \mathbb{P}^*\mathbb{P}_0\Pi(\rho)\,, \quad \overline{\vartheta} \text{ as defined in (7.37b)}\,, \quad \overline{\mathcal{C}}_{*,p'} = \mathcal{C}^2_{\rho,3}(\Lambda\Gamma)^3\left(\frac{\Lambda\Gamma}{\mu_{\bar{m}}}\right)^{N_{**}}\,,$$
$$\overline{\mu} = \overline{\Upsilon} = \overline{\Upsilon}' = \mu\,, \quad \overline{\Lambda} = \Lambda\Gamma\,, \quad \overline{\mathsf{d}} = 0\,.$$

Then (7.35) is satisfied as in the previous substeps, and so from (7.37b) we have that (A.43) is satisfied. We have that (A.44)–(A.45) are satisfied as in the first substep. The nonlocal assumptions are satisfied as in the previous substeps, except that we now have (A.55) from (A.163c). The only conclusion we require at this point is to produce a bound matching (A.172), which follows from (A.57).

`Step 3: Verification of` (A.174). Since the vector field v is incompressible, $\frac{d^M}{dt^M}\langle D_t\sigma_{\widehat{v}}\rangle =$

$\langle D_t^{M+1}\sigma_{\widehat{v}}\rangle$. Since $\mathsf{\Pi}(\rho)$ is periodic in $(\mathbb{T}/\mu)^2$, we have that for $M+1 \leq M_* - \mathsf{N}_{\text{cut,t}} - 1$,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^3} D_t^{M+1}\mathsf{\Pi}\left(G\right)\left(\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho)\right)\circ\Phi\, dx \right| \\
&\quad = \left| \int_{\mathbb{T}^3} D_t^{M+1}\mathsf{\Pi}\left(G\right)\circ\Phi^{-1}\Delta^{\lfloor \frac{\mathsf{d}}{4}\rfloor}\Delta^{-\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho)\right)\, dx \right| \\
&\quad = \left| \int_{\mathbb{T}^3} \Delta^{\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(D_t^{M+1}\mathsf{\Pi}\left(G\right)\circ\Phi^{-1}\right)\Delta^{-\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho)\right) dx \right| \\
&\quad \lesssim \left\| \Delta^{\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(D_t^{M+1}\mathsf{\Pi}\left(G\right)\circ\Phi^{-1}\right)\right\|_{3/2} \left\| \Delta^{-\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho)\right)\right\|_{1} \\
&\quad \lesssim \mathcal{C}_{G,3/2}(\max(\lambda,\lambda')\Gamma)^{\mathsf{d}/2}\mu^{-\mathsf{d}/2}\mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'\mathcal{M}\left(M+1, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \\
&\quad \leq (\Lambda\Gamma)^{-K_\circ}\mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right) .
\end{aligned}$$

Here, we have used Lemma A.1.1, (A.167), (A.160a), (A.163b), and standard Littlewood-Paley theory. □

Proposition A.4.4 (Pressure increment and upgrade error for stress error). *We begin with* preliminary assumptions*, which include all of the assumptions and conclusions from the inverse divergence in Proposition A.3.3 and the pointwise bounds in Remark A.3.9. We then include* additional assumptions*, which allow for the application of Lemma A.4.1 to the stress error and Proposition A.3.3 to the material derivative of the output. We thus obtain a* pressure increment *which satisfies a number of properties. Finally, the material derivative of this pressure increment produces a* current error *which itself satisfies a number of properties.*

Part 1: Preliminary assumptions

(i) There exists a vector field G, constants $\mathcal{C}_{G,p}$ for $p = 3/2, \infty$, and parameters $M_t, \lambda, \nu, \nu', N_*, M_*$ such that (A.39) *and* (A.40) *are satisfied. There exists a smooth, non-negative scalar function π such that* (A.59) *holds.*

(ii) There exists an incompressible vector field v, associated material derivative $D_t = \partial_t + v\cdot\nabla$, a volume preserving diffeomorphism Φ, inverse flow Φ^{-1}, and parameter λ' such

that (A.41)–(A.42b) *are satisfied.*

(iii) There exists a zero mean scalar function ϱ*, a mean-zero tensor potential* ϑ*, constants* $\mathcal{C}_{*,p}$ *for* $p = 3/2, \infty$*, and parameters* $\mu, \Upsilon, \Upsilon', \Lambda, \mathsf{N}_{\mathrm{dec}}, \mathsf{d}$ *such that* (i)–(iii) *and* (A.43)–(A.45) *are satisfied.*

(iv) The symmetric stress $S = \mathcal{H}(G\varrho \circ \Phi)$ *and nonlocal error* E *satisfy the conclusions in* (A.46), (ii)–(vi)*, as well as the conclusion* (A.60) *from Remark A.3.9.*

(v) There exist integers $N_\circ, M_\circ, K_\circ$ *such that* (A.52)–(A.55) *are satisfied, and as a consequence conclusions* (A.56)–(A.57) *hold.*

Part 2: Additional assumptions

(i) There exists a large positive integer N_{**} *and integers positive* $\mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}}$ *such that we have the additional inequalities*

$$N_* - 2\mathsf{d} - \mathsf{N}_{\mathrm{cut,x}} - N_{**} - 3 \geq M_* \,, \tag{A.179a}$$

$$M_* - \mathsf{N}_{\mathrm{cut,t}} - 1 \geq 2N_\circ \,, \tag{A.179b}$$

$$N_{**} \geq 2\mathsf{d} + 3 \tag{A.179c}$$

(ii) There exist parameters $\Gamma = \Lambda^\alpha$ *for* $0 < \alpha \ll 1$ *and* δ_{tiny} *satisfying*

$$\mathsf{N}_{\mathrm{cut,t}} \leq \mathsf{N}_{\mathrm{cut,x}} \,, \tag{A.180a}$$

$$(\mathcal{C}_{G,\infty} + 1)\left(\mathcal{C}_{*,\infty}\Upsilon'\Upsilon^{-2} + 1\right)\Gamma^{-\mathsf{N}_{\mathrm{cut,t}}} \leq \delta_{\mathrm{tiny}} \,, \mathcal{C}_{G,3/2} \,, \mathcal{C}_{*,3/2}\Upsilon'\Upsilon^{-2} \,, \tag{A.180b}$$

$$2\mathsf{N}_{\mathrm{dec}} + 4 \leq N_* - N_{**} - \mathsf{N}_{\mathrm{cut,x}} - 3\mathsf{d} - 3 \,, \tag{A.180c}$$

$$(\Lambda\Gamma)^4 \leq \left(\frac{\mu}{2\pi\sqrt{3}\Gamma\max(\lambda, \lambda')}\right)^{\mathsf{N}_{\mathrm{dec}}} \,. \tag{A.180d}$$

(iii) There exists a parameter $\bar{m}$ *and an increasing sequence of frequencies* $\{\mu_0, \cdots, \mu_{\bar{m}}\}$

satisfying

$$\mu < \mu_0 < \cdots < \mu_{\bar{m}-1} \leq \Lambda < \Lambda\Gamma < \mu_{\bar{m}}\,, \tag{A.181a}$$

$$\max(\lambda,\lambda')\Gamma\left(\mu_{m-1}^{-2}\mu_m + \mu^{-1}\right) \leq 1\,, \tag{A.181b}$$

$$\begin{aligned}\mathcal{C}_{G,3/2}\mathcal{C}_{*,3/2}\nu\Gamma(\max(\lambda,\lambda')\Gamma)^{\lfloor \mathsf{d}/4\rfloor}\left(\max\left(\mu^{-1},\mu_m\mu_{m-1}^{-2}\right)\right)^{\lfloor \mathsf{d}/4\rfloor}\\ \times(\mu_{\bar{m}})^{5+K_\circ}\left(1+\frac{\max\{\nu',\mathcal{C}_v\mu_{\bar{m}}\}}{\nu}\right)^{M_\circ} \leq 1\,,\end{aligned} \tag{A.181c}$$

$$\mathcal{C}_{G,3/2}\nu\Gamma\mathcal{C}_{*,3/2}\left(\frac{\Lambda\Gamma}{\mu_{\bar{m}}}\right)^{N_{**}}(\mu_{\bar{m}})^{8+K_\circ}\left(1+\frac{\max\{\nu',\mathcal{C}_v\mu_{\bar{m}}\}}{\nu}\right)^{M_\circ} \leq 1\,, \tag{A.181d}$$

for all $1 \leq m \leq \bar{m}$.

Part 3: Pressure increment

(i) There exists a pressure increment σ_S*, where we have a decomposition*

$$\sigma_S = \sigma_S^+ - \sigma_S^- = \sigma_S^* + \sum_{m=0}^{\bar{m}} \sigma_S^m\,. \tag{A.182}$$

(ii) σ_S^+ *dominates derivatives of* S *with suitable weights, so that*

$$\left|D^N D_t^M S\right| \lesssim (\sigma_S^+ + \delta_{\text{tiny}})\,(\Lambda\Gamma)^N\,\mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right)\,. \tag{A.183}$$

for all $N \leq N_* - \lfloor \mathsf{d}/2\rfloor$, $M \leq M_*$.

(iii) σ_S^+ *dominates derivatives of itself with suitable weights, so that*

$$\left|D^N D_t^M \sigma_S^+\right| \lesssim (\sigma_S^+ + \delta_{\text{tiny}})\,(\Lambda\Gamma)^N\,\mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.184}$$

for all $N \leq N_* - \lfloor \mathsf{d}/2\rfloor - \mathsf{N}_{\text{cut,x}}$, $M \leq M_* - \mathsf{N}_{\text{cut,t}}$.

(iv) σ_S^+ *and* σ_S^- *have the same size as* S*, so that for* $p = 3/2, \infty$*,*

$$\left\|\sigma_S^+\right\|_p, \left\|\sigma_S^-\right\|_p \lesssim \mathcal{C}_{G,p}\mathcal{C}_{*,p}\Upsilon'\Upsilon^{-2}. \tag{A.185}$$

(v) π *dominates* σ_S^- *and its derivatives with suitable weights, so that*

$$\left|D^N D_t^M \sigma_S^-\right| \lesssim \mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'\pi(\max(\lambda,\lambda')\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.186}$$

for all $N \leq N_* - \lfloor \mathsf{d}/2 \rfloor - \mathsf{N}_{\mathrm{cut,x}}$*,* $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}}$*.*

(vi) We have the support properties

$$\mathrm{supp}\,(\sigma_S^+) \subseteq \mathrm{supp}\,(S), \quad \mathrm{supp}\,(\sigma_S^-) \subseteq \mathrm{supp}\,(G). \tag{A.187}$$

Part 4: Current error

(i) There exists a current error ϕ*, where we have the decomposition and equalities*

$$\phi = \phi_S^* + \sum_{m=0}^{\bar{m}} \phi_S^m = (\mathcal{H} + \mathcal{R}^*)(D_t\sigma_S^*) + \sum_{m=0}^{\bar{m}} (\mathcal{H} + \mathcal{R}^*)(D_t\sigma_S^m), \tag{A.188a}$$

$$\mathrm{div}\phi_S^m(t,x) = D_t\sigma_S^m(t,x) - \int_{\mathbb{T}^3} D_t\sigma_S^m(t,x')\,dx', \tag{A.188b}$$

$$\mathrm{div}\phi_S^*(t,x) = D_t\sigma_S^*(t,x) - \int_{\mathbb{T}^3} D_t\sigma_S^*(t,x')\,dx'. \tag{A.188c}$$

(ii) ϕ_S^m *can be written as* $\phi_S^m = \phi_S^{m,l} + \phi_S^{m,*}$*, and for* $1 \le m \le \bar{m}$*, these satisfy*

$$\left\|D^N D_t^M \phi_S^m\right\|_{3/2} \lesssim \nu\Gamma^2 \mathcal{C}_{G,3/2}\mathcal{C}_{*,3/2}\Upsilon'\Upsilon^{-2}\mu_{m-1}^{-2}\mu_m \left(\min(\mu_m, \Lambda\Gamma)\right)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right) , \tag{A.189a}$$

$$\begin{aligned}\left\|D^N D_t^M \phi_S^m\right\|_{\infty} \lesssim \nu\Gamma^2 \mathcal{C}_{G,\infty}\mathcal{C}_{*,3/2}\Upsilon'\Upsilon^{-2} &\left(\frac{\min(\mu_m, \Lambda\Gamma)}{\mu}\right)^{4/3} \mu_{m-1}^{-2}\mu_m \\ &\times \left(\min(\mu_m, \Lambda\Gamma)\right)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right) ,\end{aligned} \tag{A.189b}$$

$$\begin{aligned}\left|D^N D_t^M \phi_S^{m,l}\right| \lesssim \nu\Gamma^2 \pi \mathcal{C}_{*,3/2}\Upsilon'\Upsilon^{-2} &\left(\frac{\min(\mu_m, \Lambda\Gamma)}{\mu}\right)^{4/3} \mu_{m-1}^{-2}\mu_m \\ &\times \left(\min(\mu_m, \Lambda\Gamma)\right)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right) ,\end{aligned} \tag{A.189c}$$

for all $N \le N_* - 2\mathsf{d} - \mathsf{N}_{\mathrm{cut,x}}$*,* $M \le M_* - \mathsf{N}_{\mathrm{cut,t}} - 1$*. For* $m = 0$ *and the same range of* N *and* M*,* ϕ_S^m *and* $\phi_S^{m,l}$ *satisfy identical bounds but with* $\mu_{m-1}^2\mu_m$ *replaced with* $\Gamma\mu^{-1}$ *and* $\min(\mu_m, \Lambda\Gamma)$ *replaced with* μ_0 *in all three bounds. Furthermore, the nonlocal portions satisfy the improved estimate*

$$\left\|D^N D_t^M \phi_S^{m,*}\right\|_{\infty} \lesssim \left(\min(\mu_m, \Lambda\Gamma)\right)^{N-K_\circ} \left(\max(\lambda, \lambda')\Gamma\right)^{\lfloor \mathsf{d}/4 \rfloor} \left(\max\left(\mu^{-1}, \mu_m\mu_{m-1}^{-2}\right)\right)^{\lfloor \mathsf{d}/4 \rfloor} (\nu\Gamma)^M \tag{A.190}$$

for all $N \le N_\circ, M \le M_\circ$*, and the remainder term* ϕ_S^* *satisfies the improved estimate*

$$\left\|D^N D_t^M \phi_S^{*}\right\|_{\infty} \lesssim (\Lambda\Gamma)^{-K_\circ} \left(\max(\lambda, \lambda')\Gamma\right)^{\lfloor \mathsf{d}/4 \rfloor} \left(\max\left(\mu^{-1}, \mu_m\mu_{m-1}^{-2}\right)\right)^{\lfloor \mathsf{d}/4 \rfloor} (\Lambda\Gamma)^N (\nu\Gamma)^M \tag{A.191}$$

in the same range of N *and* M*.*

(iii) We have the support properties[14]

$$\operatorname{supp}(\phi_S^{m,l}) \subseteq \operatorname{supp} G \cap B\left(\operatorname{supp}\vartheta, 2\mu_{m-1}^{-1}\right) \circ \Phi \ \text{ for } \ 1 \le m \le \bar{m}\,, \qquad \operatorname{supp}\left(\phi_S^{0,l}\right) \subseteq \operatorname{supp} G\,. \tag{A.192}$$

(iv) For all $M \le M_* - \mathsf{N}_{\mathrm{cut,t}} - 1$, we have that the mean $\langle D_t \sigma_S \rangle$ satisfies

$$\left|\frac{d^M}{dt^M}\langle D_t\sigma_S\rangle\right| \lesssim (\Lambda\Gamma)^{-K_\circ}(\max(\lambda,\lambda')\Gamma)^{\lfloor \mathsf{d}/4\rfloor}\mu^{-\lfloor \mathsf{d}/4\rfloor}\mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}}, -1, \nu\Gamma, \nu'\Gamma\right)\,. \tag{A.193}$$

Proof. **Step 1: Defining and estimating σ_S to verify** (A.183)–(A.187)**.** From (A.47) of Proposition A.3.3, we have that S can be written as

$$S = \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} H^{\alpha(j)}\rho^{\beta(j)} \circ \Phi\,,$$

where $H^{\alpha(j)}$ and $\rho^{\beta(j)}$ satisfy the bounds in (A.49a), (A.49b). In addition, we have the pointwise bounds on $H^{\alpha(j)}$ in terms of π given by (A.60) in Remark A.3.9. For each $0 \le j \le \mathcal{C}_{\mathcal{H}}$, we shall apply Lemma A.4.1 with the following choices, where we have used the convention set out in Remark A.3.10 to rewrite the symbols from Lemma A.4.1 with bars above on the left-hand side of the equalities below, while the right-hand side are parameters given in the assumptions of this Proposition:

$$\begin{aligned}
&\overline{v} = v\,, \quad \overline{N}_\dagger = N_* - \lfloor \mathsf{d}/2\rfloor\,, \quad \overline{M}_\dagger = M_*\,, \quad \overline{M}_t = M_t\,,\\
&\overline{H} = H^{\alpha(j)}, \quad \overline{\mathcal{C}}_{G,3/2} = \mathcal{C}_{G,3/2}\,, \quad \overline{\mathcal{C}}_{G,\infty} = \mathcal{C}_{G,\infty}\,,\\
&\overline{\rho} = \rho^{\beta(j)}, \quad \overline{\mathcal{C}}_{\rho,3/2} = \mathcal{C}_{*,3/2}\Upsilon'\Upsilon^{-2}\,, \quad \overline{\mathcal{C}}_{\rho,\infty} = \mathcal{C}_{*\infty}\Upsilon'\Upsilon^{-2},\\
&\overline{\lambda} = \max(\lambda,\lambda')\,, \quad \overline{\Lambda} = \Lambda, \quad \overline{\Gamma} = \Gamma\,, \quad \overline{\Phi} = \Phi\,,\\
&\overline{\pi} = \pi\,, \quad \overline{\nu} = \nu\,, \quad \overline{\nu}' = \nu'\,, \quad \overline{\mu} = \mu\,, \quad \overline{\mathsf{N}_{\mathrm{dec}}} = \mathsf{N}_{\mathrm{dec}}\,,
\end{aligned}$$

[14]For any $\Omega \in \mathbb{T}^3$, we use $\Omega \circ \Phi_{(i,k)}$ to refer to the space-time set $\Phi_{(i,k)}^{-1}(t,\cdot)\Omega$ whose characteristic function is annihilated by D_t.

and $\mathsf{N}_{\text{cut,x}}$, $\mathsf{N}_{\text{cut,t}}$, and δ_{tiny} as in preliminary assumption (ii). From (A.49), (A.60), and (A.50), we have that (A.105a)-(A.105d) are satisfied. Assumption (A.106) is satisfied from (A.180d). All the assumptions in (iii) are satisfied from preliminary assumption (ii) from this proposition. Finally, all assumptions in (iv) are satisfied from the additional assumption (ii) from this Proposition.

We may then apply (A.109a)–(A.110b) from Lemma A.4.1 to obtain for $0 \leq j \leq \mathcal{C}_H$ the pressure increments $\sigma_S^j = \sigma_S^{+,j} - \sigma_S^{-,j}$, and we then collect terms to define

$$\sigma_S^+ := \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \sigma_S^{+,j}, \qquad \sigma_S^- := \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \sigma_S^{-,j}, \qquad \sigma_S := \sigma_S^+ - \sigma_S^- .$$

From conclusions (i)–(v) of Lemma A.4.1, we have that (A.183)–(A.187) are satisfied.

Step 2: Decomposing σ_S to verify (A.182)**, and defining and estimating ϕ_S^m to verify** (A.188)–(A.192)**.** From (A.109a)–(A.109b), we have that

$$\sigma_S = \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \Pi\left(H^{\alpha(j)}\right)\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right) \circ \Phi . \tag{A.194}$$

Note further that $\Pi(\rho^{\beta(j)})$ is $(\mathbb{T}/\mu)^3$-periodic and has derivative cost $\Lambda\Gamma$ from (A.114), conclusion (iii) from Lemma A.4.1. So we use the sequence of frequencies $\mu_0, \ldots, \mu_{\bar{m}}$ to apply the synthetic Littlewood-Paley decomposition (à la (7.34)) to $\Pi(\rho^{\beta(j)})$ and write

$$\Pi(\rho^{\beta(j)}) = \widetilde{\mathbb{P}}_{\mu_0}(\Pi(\rho^{\beta(j)})) + \left(\sum_{m=1}^{\bar{m}} \widetilde{\mathbb{P}}_{(\mu_{m-1},\mu_m]}(\Pi(\rho^{\beta(j)}))\right) + \left(\mathrm{Id} - \widetilde{\mathbb{P}}_{\mu_{\bar{m}}}\right)\Pi(\rho^{\beta(j)}) . \tag{A.195}$$

From now on, we shall abbreviate notation by writing $\mathbb{P}_0$ for $\widetilde{\mathbb{P}}_{\mu_0}$, $\mathbb{P}_m$ for $\widetilde{\mathbb{P}}_{(\mu_{m-1},\mu_m]}$ for $1 \leq m \leq \bar{m}$, and $\mathbb{P}^*$ for $\mathrm{Id} - \widetilde{\mathbb{P}}_{\mu_{\bar{m}}}$, so that we may use (A.195) to write

$$\sigma_S = \sigma_S^* + \sum_{m=0}^{\bar{m}} \sigma_S^m := \sum_{j=0}^{\mathcal{C}_H} \Pi\left(H^{\alpha(j)}\right)\mathbb{P}^*\left(\Pi\left(\rho^{\beta(j)}\right)\right) \circ \Phi + \sum_{m=0}^{\bar{m}}\sum_{j=0}^{\mathcal{C}_H} \Pi\left(H^{\alpha(j)}\right)\mathbb{P}_m\left(\Pi\left(\rho^{\beta(j)}\right)\right) \circ \Phi . \tag{A.196}$$

We aim to apply Proposition A.3.3 with Remarks A.3.8, A.3.9 to the material derivative of each of the terms in (A.196), which would produce

$$\begin{aligned}\phi := \phi_S^* + \sum_{m=0}^{\bar{m}} \phi_S^m =: \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} &\underbrace{(\mathcal{H}+\mathcal{R}^*)\left(D_t\Pi(H^{\alpha(j)})\left(\mathbb{P}^*\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\circ\Phi\right)}_{=:\phi^{*,j}} \\ &+ \sum_{m=0}^{\bar{m}}\sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \underbrace{(\mathcal{H}+\mathcal{R}^*)\left(D_t\Pi(H^{\alpha(j)})\left(\mathbb{P}_m\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\circ\Phi\right)}_{=:\phi^{m,j}} \\ &= (\mathcal{H}+\mathcal{R}^*)(D_t\sigma_S^*) + \sum_{m=0}^{\bar{m}}(\mathcal{H}+\mathcal{R}^*)(D_t\sigma_S^m)\,.\end{aligned}$$

Assuming that we succeed in doing so, we have at least verified (A.182) and (A.188). Now in order to apply the inverse divergence with the pointwise bounds from Remark A.3.9, we first treat the low-frequency assumptions from Part 1, which are the same in all cases (irrespective of the projector on $\Pi(\rho^{\beta(j)})$). Specifically, we shall use the convention from Remark A.3.10 and in all cases set

$$\begin{aligned}&\overline{p} = 3/2, \infty\,, \quad \overline{v} = v\,, \quad \overline{N}_* = N_* - \mathsf{d} - \lfloor \mathsf{d}/2 \rfloor - \mathsf{N}_{\mathrm{cut,x}}\,, \quad \overline{M}_* = M_* - \mathsf{N}_{\mathrm{cut,t}} - 1\,, \quad \overline{M}_t = M_t - \mathsf{N}_{\mathrm{cut,t}} - 1\,, \\ &\overline{G} = D_t\Pi(H^{\alpha(j)}), \quad \overline{\mathcal{C}}_{G,p} = \nu\Gamma\mathcal{C}_{G,p}\,, \quad \overline{\mu} = \mu\,, \quad \overline{\lambda} = \max(\lambda,\lambda')\Gamma\,, \quad \overline{\Phi} = \Phi\,, \quad \overline{\lambda}' = \lambda'\,, \\ &\overline{\nu} = \nu\Gamma\,, \quad \overline{\nu}' = \nu'\Gamma\,, \quad \overline{\Phi} = \Phi\,, \quad \overline{\pi} = \nu\Gamma\pi\,, \quad \overline{\mathsf{N}_{\mathrm{dec}}} = \mathsf{N}_{\mathrm{dec}}\,, \quad \overline{\mathsf{d}} = \mathsf{d}\,.\end{aligned}$$

Then (A.39) is satisfied from the additional assumption (A.179a), and (A.40) is satisfied from the conclusion (A.114) and the parameter choices from Step 1 which led to that conclusion. The estimates in (A.41), (A.42a) and (A.42b) hold from assumption (ii) from this Proposition. The pointwise bound in (A.59) holds with $\overline{M}_t = M_t - \mathsf{N}_{\mathrm{cut,t}} - 1$ and $\overline{\pi} = \nu\Gamma\pi$ due to (A.115b), which was verified in Step 1. At this point we split into cases based on which projector is applied to $\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$ in (A.196) and address parts 2-4 of Proposition A.3.3.

Step 2a: Lowest shell. For the case $m = 0$, we appeal to Lemma 7.3.3 with $q = 3/2$, $\lambda = \Lambda\Gamma$, $\rho = \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$, and α such that λ^α in (7.37a) is equal to Γ. Specifically, to verify

the assumptions in Part 2 of Proposition A.3.3, we set for $p = 3/2, \infty$

$$\overline{\varrho} = \mathbb{P}_0\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)}), \quad \overline{\vartheta} \text{ as defined in (7.37a)}, \quad \overline{\mathcal{C}}_{*,p} = \Gamma\mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'\left(\frac{\mu_0}{\mu}\right)^{\frac{4}{3}-\frac{2}{p}},$$
$$\overline{\mu} = \mu, \quad \overline{\Upsilon} = \overline{\Upsilon}' = \mu, \quad \overline{\Lambda} = \mu_0, \quad \overline{\mathsf{d}} = \mathsf{d}.$$

Then (7.35) is satisfied with $\mathcal{C}_{p,3/2} = \mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'$ and $\lambda = \Lambda\Gamma$ from standard Littlewood-Paley theory, (A.114), and the choices from Step 1 which led to that conclusion, and so from (7.37a) we have that (A.43) is satisfied. From (A.180d), (A.181a), (A.181b), the choice of $\overline{N}_*$ above, (A.114), and (A.180c), we have that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied from preliminary assumption (v) and (A.179b). We have that (A.55) is satisfied from (A.181c). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. First, we set

$$\phi_S^{0,l} = \mathcal{H}(D_t\sigma_S^0), \qquad \phi_S^{0,*} = \mathcal{R}^*(D_t\sigma_S^0).$$

From (A.50), we obtain both (A.189a) and (A.189b), but with the appropriate modifications for $m = 0$ as indicated. The pointwise bound in (A.189c) holds due to (A.60), (A.49a), and (A.47). Next, we obtain (A.190) for $m = 0$ from (A.57). Finally, we obtain (A.192) from (A.48), concluding the proof of the desired conclusions for $m = 0$.

Step 2b: Intermediate shells. For the cases $1 \leq m \leq \bar{m}$, we appeal to Lemma 7.3.4 with $q = 3/2$ and $\rho = \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p = 3/2, \infty$

$$\overline{\varrho} = \mathbb{P}_m\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)}), \quad \overline{\vartheta} = \mu_{m-1}^{-\mathsf{d}}\Theta_\rho^{\mu_{m-1},\mu_m} \text{ as defined in Lemma 7.3.4},$$
$$\overline{\mathcal{C}}_{*,p} = \mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'\left(\frac{\min(\mu_m, \Lambda\Gamma)}{\mu}\right)^{\frac{4}{3}-\frac{2}{p}}, \quad \overline{\Upsilon} = \mu_{m-1}, \quad \overline{\Upsilon}' = \overline{\Lambda} = \min(\mu_m, \Gamma\Lambda),$$
$$\overline{\mathsf{d}} = \mathsf{d}, \quad \overline{\mu} = \mu, \qquad \alpha \text{ as in the previous substep.}$$

Then (7.39) is satisfied exactly as in the previous substep, and so from (7.40a)–(7.40b) we

have that (A.43) is satisfied. As before, we use (A.180d), (A.181a), (A.181b), the choice of $\overline{N}_*$ above, (A.114), and (A.180c) to see that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied as in the previous substep, and (A.55) is satisfied from (A.181c). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. First, we set

$$\phi_S^{m,l} = \mathcal{H}(D_t\sigma_S^m)\,, \qquad \phi_S^{m,*} = \mathcal{R}^*(D_t\sigma_S^m)\,.$$

From (A.50), we obtain both (A.189a) and (A.189b). The pointwise bound in (A.189c) holds due to (A.60), (A.49a), and (A.47). Next, we obtain (A.190) from (A.57). Finally, we obtain (A.192) from (A.48) and (7.40c), concluding the proof for $1 \le m \le \bar{m}$.

Step 2c: Highest shell. For the case with the highest shell, corresponding to the projector $\mathbb{P}^*$ from (A.196), we appeal to Lemma 7.3.3 with $q = 3/2$, $\lambda = \Lambda\Gamma$, $\rho = \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p = 3/2, \infty$

$$\begin{aligned}
&\overline{\varrho} = \mathbb{P}^*\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\,, \quad \overline{\vartheta} = \vartheta \text{ as defined in (7.37b)}\,,\\
&\overline{\mathcal{C}}_{*,p} = \left(\frac{\Lambda\Gamma}{\mu_{\bar{m}}}\right)^{N_{**}} \mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'(\Lambda\Gamma)^3\,, \quad \overline{\Upsilon} = \overline{\Upsilon}' = \mu\,, \quad \overline{\Lambda} = \Gamma\Lambda\,,\\
&\overline{\mathsf{d}} = 0\,, \quad \overline{N}_* = N_* - \mathsf{N}_{\mathrm{cut,x}} - N_{**} - 3\,.
\end{aligned}$$

We note that we have altered the definition of N_* compared to the previous two substeps for convenience. But from (A.179c), we have in fact made it *smaller*, so that the low-frequency assumptions from the inverse divergence are still satisfied. Then (7.35) is satisfied exactly as in the first substep, and so from (7.37b) we have that (A.43) is satisfied. We use (A.180d), (A.181a), (A.181b), the altered choice of $\overline{N}_*$ above, (A.114), and (A.180c) to see that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied as in the previous substep, and (A.55) is satisfied from (A.181d). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude

as follows. First, we set

$$\phi_S^* = (\mathcal{H} + \mathcal{R}^*)(D_t \sigma_S^*)\,.$$

We may ignore (A.50) since $\mathsf{d} = 0$. Then the only conclusion we require is (A.191), which follows from (A.57).

Step 3: Verification of (A.193)**.** Since the vector field v is incompressible, $\frac{d^M}{dt^M}\langle D_t \sigma_S\rangle = \langle D_t^{M+1}\sigma_S\rangle$. From (A.194), we have

$$D_t^{M+1}\sigma_S = \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\circ\Phi\,.$$

Since $\Pi(\rho^{\beta(j)})$ is periodic in $(\mathbb{T}/\mu)^2$, we have that for $M+1 \leq M_* - \mathsf{N}_{\mathrm{cut,t}} - 1$

$$\begin{aligned}
&\left|\int_{\mathbb{T}^3} D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\circ\Phi dx\right| \\
&\quad = \left|\int_{\mathbb{T}^3} D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\circ\Phi^{-1}\Delta^{\lfloor\frac{\mathsf{d}}{4}\rfloor}\Delta^{-\lfloor\frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\,dx\right| \\
&\quad = \left|\int_{\mathbb{T}^3} \Delta^{\lfloor\frac{\mathsf{d}}{4}\rfloor}\left(D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\circ\Phi^{-1}\right)\Delta^{-\lfloor\frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)dx\right| \\
&\quad \lesssim \left\|\Delta^{\lfloor\frac{\mathsf{d}}{4}\rfloor}\left(D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\circ\Phi^{-1}\right)\right\|_{3/2}\left\|\Delta^{-\lfloor\frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\right\|_1 \\
&\quad \lesssim \mathcal{C}_{G,3/2}(\max(\lambda,\lambda')\Gamma)^{\mathsf{d}/2}\mu^{-\mathsf{d}/2}\mathcal{C}_{*,3/2}\Upsilon^{-2}\Upsilon'\mathcal{M}\left(M+1, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \\
&\quad \leq (\Lambda\Gamma)^{-K_\circ}(\max(\lambda,\lambda')\Gamma)^{\lfloor\mathsf{d}/4\rfloor}\mu^{-\lfloor\mathsf{d}/4\rfloor}\mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right)\,.
\end{aligned}$$

Here, we have used Lemma A.1.1, (A.114), (A.107a), (A.181c), and standard Littlewood-Paley theory. □

Proposition A.4.5 (**Pressure increment and upgrade error from current error).** *We begin with* preliminary assumptions*, which include all of the assumptions and conclusions from the inverse divergence in Proposition A.3.3 and the pointwise bounds in Remark A.3.9. We then include* additional assumptions*, which allow for the application of Lemma A.4.2 to the current error and Proposition A.3.3 to the material derivative of the output. We thus obtain a* pressure increment *which satisfies a number of properties. Finally, the material*

derivative of this pressure increment produces a current error *which itself satisfies a number of properties.*

Part 1: Preliminary assumptions

(i) There exists a scalar field G, constants $\mathcal{C}_{G,p}$ for $p = 1, \infty$, and parameters $M_t, \lambda, \nu, \nu', N_*, M_*$ such that (A.39) *and* (A.40) *are satisfied. There exists a smooth, non-negative scalar function π and a parameter r_G such that* (A.132b) *holds with H replaced by G.*

(ii) There exists an incompressible vector field v, associated material derivative $D_t = \partial_t + v \cdot \nabla$, a volume preserving diffeomorphism Φ, inverse flow Φ^{-1}, and parameter λ' such that (A.41)–(A.42b) *are satisfied.*

(iii) There exists a zero mean scalar function ϱ, a mean-zero tensor potential ϑ, constants $\mathcal{C}_{*,p}$ for $p = 1, \infty$, and parameters $\mu, \Upsilon, \Upsilon', \Lambda, \mathsf{N}_{\mathrm{dec}}, \mathsf{d}$ such that (i)–(iii) *and* (A.43)–(A.45) *are satisfied.*

(iv) The current error $\varphi = \mathcal{H}(G\varrho \circ \Phi)$ and nonlocal error E satisfy the conclusions in (A.46), (ii)–(vi), *as well as the conclusion* (A.60) *from Remark A.3.9 with π replaced by $\pi^{3/2} r_G^{-1}$.*

(v) There exist integers $N_\circ, M_\circ, K_\circ$ such that (A.52)–(A.55) *are satisfied, and as a consequence conclusions* (A.56)–(A.57) *hold.*

Part 2: Additional assumptions

(i) There exists a large positive integer N_{**} and positive integers $\mathsf{N}_{\mathrm{cut,x}}, \mathsf{N}_{\mathrm{cut,t}}$ such that we have the additional inequalities

$$N_* - 2\mathsf{d} - \mathsf{N}_{\mathrm{cut,x}} - N_{**} - 3 \geq M_* \,, \tag{A.197a}$$

$$M_* - \mathsf{N}_{\mathrm{cut,t}} - 1 \geq 2N_\circ \,, \tag{A.197b}$$

$$N_{**} \geq 2\mathsf{d} + 3 \tag{A.197c}$$

(ii) There exist parameters $\Gamma = \Lambda^\alpha$ for $0 < \alpha \ll 1$, $\delta_{\text{tiny}}, r_\phi$, and $\delta_{\phi,p}$ for $p = 1, \infty$ satisfying

$$0 < r_\phi \leq 1\,, \qquad \delta_{\phi,p}^{3/2} = \mathcal{C}_{G,p}\mathcal{C}_{*,p}\Upsilon'\Upsilon^{-2}r_\phi\,, \tag{A.198a}$$

$$\mathsf{N}_{\text{cut,t}} \leq \mathsf{N}_{\text{cut,x}}\,, \tag{A.198b}$$

$$(\mathcal{C}_{G,\infty}+1)\left(\mathcal{C}_{*,\infty}\Upsilon'\Upsilon^{-2}+1\right)\Gamma^{-\mathsf{N}_{\text{cut,t}}} \leq {\delta_{\text{tiny}}}^{3/2}\,, \mathcal{C}_{G,1}\,, \mathcal{C}_{*,1}\Upsilon'\Upsilon^{-2}\,, \tag{A.198c}$$

$$2\mathsf{N}_{\text{dec}} + 4 \leq N_* - N_{**} - \mathsf{N}_{\text{cut,x}} - 3\mathsf{d} - 3\,, \tag{A.198d}$$

$$(\Lambda\Gamma)^4 \leq \left(\frac{\mu}{2\pi\sqrt{3}\Gamma\max(\lambda,\lambda')}\right)^{\mathsf{N}_{\text{dec}}}\,. \tag{A.198e}$$

(iii) There exists a parameter $\bar{m}$ and an increasing sequence of frequencies $\{\mu_0, \cdots, \mu_{\bar{m}}\}$ satisfying

$$\mu < \mu_0 < \cdots < \mu_{\bar{m}-1} \leq \Lambda < \Lambda\Gamma < \mu_{\bar{m}}\,, \tag{A.199a}$$

$$\max(\lambda,\lambda')\Gamma\left(\mu_{m-1}^{-2}\mu_m + \mu^{-1}\right) \leq 1\,, \tag{A.199b}$$

$$\begin{aligned}(\mathcal{C}_{G,1}\mathcal{C}_{*,1}r_\phi)^{2/3}\nu\Gamma(\max(\lambda,\lambda')\Gamma)^{\lfloor \mathsf{d}/4\rfloor}\left(\max\left(\mu^{-1},\mu_m\mu_{m-1}^{-2}\right)\right)^{\lfloor \mathsf{d}/4\rfloor}\\ \times(\mu_{\bar{m}})^{5+K_\circ}\left(1+\frac{\max\{\nu',\mathcal{C}_v\mu_{\bar{m}}\}}{\nu}\right)^{M_\circ} \leq 1\,,\end{aligned} \tag{A.199c}$$

$$(\mathcal{C}_{G,1}\mathcal{C}_{*,1}r_\phi)^{2/3}\,\nu\Gamma\left(\frac{\Lambda\Gamma}{\mu_{\bar{m}}}\right)^{N_{**}}(\mu_{\bar{m}})^{8+K_\circ}\left(1+\frac{\max\{\nu',\mathcal{C}_v\mu_{\bar{m}}\}}{\nu}\right)^{M_\circ} \leq 1\,, \tag{A.199d}$$

for all $1 \leq m \leq \bar{m}$.

Part 3: Pressure increment

(i) There exists a pressure increment σ_φ, where we have a decomposition

$$\sigma_\varphi = \sigma_\varphi^+ - \sigma_\varphi^- = \sigma_\varphi^* + \sum_{m=0}^{\bar{m}}\sigma_\varphi^m\,. \tag{A.200}$$

(ii) σ_φ^+ *dominates derivatives of* φ *with suitable weights, so that*

$$\left|D^N D_t^M \varphi\right| \lesssim \left((\sigma_\varphi^+)^{3/2} r_\phi^{-1} + \delta_{\text{tiny}}\right) (\Lambda\Gamma)^N \mathcal{M}\left(M, M_t, \nu\Gamma, \nu'\Gamma\right) . \tag{A.201}$$

for all $N \leq N_* - \lfloor \mathsf{d}/2 \rfloor$, $M \leq M_*$.

(iii) σ_φ^+ *dominates derivatives of itself with suitable weights, so that*

$$\left|D^N D_t^M \sigma_\varphi^+\right| \lesssim \left(\sigma_\varphi^+ + \delta_{\text{tiny}}\right) (\Lambda\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.202}$$

for all $N \leq N_* - \lfloor \mathsf{d}/2 \rfloor - \mathsf{N}_{\text{cut,x}}$, $M \leq M_* - \mathsf{N}_{\text{cut,t}}$.

(iv) σ_φ^+ *and* σ_φ^- *have size comparable to* φ*, so that*

$$\left\|\sigma_\varphi^+\right\|_{3/2}, \left\|\sigma_\varphi^-\right\|_{3/2} \lesssim \delta_{\phi,1} , \qquad \left\|\sigma_\varphi^+\right\|_\infty, \left\|\sigma_\varphi^-\right\|_\infty \lesssim \delta_{\phi,\infty} . \tag{A.203}$$

(v) π *dominates* σ_φ^- *and its derivatives with suitable weights, so that*

$$\left|D^N D_t^M \sigma_\varphi^-\right| \lesssim \left(\frac{r_\phi}{r_G}\right)^{2/3} \left(\mathcal{C}_{*,1} \Upsilon^{-2} \Upsilon'\right)^{2/3} \pi(\max(\lambda, \lambda')\Gamma)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\text{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \tag{A.204}$$

for all $N \leq N_* - \lfloor \mathsf{d}/2 \rfloor - \mathsf{N}_{\text{cut,x}}$, $M \leq M_* - \mathsf{N}_{\text{cut,t}}$.

(vi) We have the support properties

$$\text{supp}\left(\sigma_\varphi^+\right) \subseteq \text{supp}\left(\varphi\right), \quad \text{supp}\left(\sigma_\varphi^-\right) \subseteq \text{supp}\left(G\right). \tag{A.205}$$

Part 4: Current error

(i) There exists a current error ϕ_φ, where we have the decomposition and equalities

$$\phi_\varphi = \phi_\varphi^* + \sum_{m=0}^{\bar{m}} \phi_\varphi^m = (\mathcal{H}+\mathcal{R}^*)(D_t\sigma_\varphi^*) + \sum_{m=0}^{\bar{m}} (\mathcal{H}+\mathcal{R}^*)(D_t\sigma_\varphi^m)\,, \tag{A.206a}$$

$$\mathrm{div}\phi_\varphi^m(t,x) = D_t\sigma_\varphi^m(t,x) - \int_{\mathbb{T}^3} D_t\sigma_\varphi^m(t,x')\,dx'\,, \tag{A.206b}$$

$$\mathrm{div}\phi_\varphi^*(t,x) = D_t\sigma_\varphi^*(t,x) - \int_{\mathbb{T}^3} D_t\sigma_\varphi^*(t,x')\,dx'\,, \tag{A.206c}$$

(ii) ϕ_φ^m can be written as $\phi_\varphi^m = \phi_\varphi^{m,l} + \phi_\varphi^{m,*}$ and for $1 \le m \le \bar{m}$ these satisfy

$$\left\|D^N D_t^M \phi_\varphi^m\right\|_{3/2} \lesssim \nu\Gamma^2 \left(\mathcal{C}_{G,1}\mathcal{C}_{*,1}\Upsilon'\Upsilon^{-2}r_\phi\right)^{2/3} \mu_{m-1}^{-2}\mu_m \left(\min(\mu_m,\Lambda\Gamma)\right)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right)\,, \tag{A.207a}$$

$$\begin{aligned}\left\|D^N D_t^M \phi_\varphi^m\right\|_{\infty} \lesssim{}& \nu\Gamma^2 \left(\mathcal{C}_{G,\infty}\mathcal{C}_{*,1}\Upsilon'\Upsilon^{-2}r_\phi\right)^{2/3} \left(\frac{\min(\mu_m,\Lambda\Gamma)}{\mu}\right)^{4/3} \mu_{m-1}^{-2}\mu_m \\ &\times \left(\min(\mu_m,\Lambda\Gamma)\right)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right)\,,\end{aligned} \tag{A.207b}$$

$$\begin{aligned}\left|D^N D_t^M \phi_\varphi^{m,l}\right| \lesssim{}& \nu\Gamma^2\pi \left(\frac{r_\phi}{r_G}\right)^{2/3} \left(\mathcal{C}_{*,1}\Upsilon'\Upsilon^{-2}\right)^{2/3} \left(\frac{\min(\mu_m,\Lambda\Gamma)}{\mu}\right)^{4/3} \mu_{m-1}^{-2}\mu_m \\ &\times \left(\min(\mu_m,\Lambda\Gamma)\right)^N \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right)\,,\end{aligned} \tag{A.207c}$$

for all $N \le N_* - 2\mathsf{d} - \mathsf{N}_{\mathrm{cut,x}}$, $M \le M_* - \mathsf{N}_{\mathrm{cut,t}} - 1$. For $m = 0$ and the same range of N and M, ϕ_φ^m and $\phi_\varphi^{m,l}$ satisfy identical bounds but with $\mu_{m-1}^2\mu_m$ replaced with $\Gamma\mu^{-1}$ and $\min(\mu_m,\Lambda\Gamma)$ replaced with μ_0 in all three bounds. Furthermore, the nonlocal portions satisfy the improved estimate

$$\left\|D^N D_t^M \phi_\varphi^{m,*}\right\|_{\infty} \lesssim \left(\min(\mu_m,\Lambda\Gamma)\right)^{N-K_\circ} \left(\max(\lambda,\lambda')\Gamma\right)^{\lfloor \mathsf{d}/4\rfloor} \left(\max\left(\mu^{-1}, \mu_m\mu_{m-1}^{-2}\right)\right)^{\lfloor \mathsf{d}/4\rfloor} (\nu\Gamma)^M\,, \tag{A.208}$$

for all $N \leq N_\circ, M \leq M_\circ$, and the remainder term ϕ^*_φ satisfies the improved estimate

$$\left\| D^N D_t^M \phi^*_\varphi \right\|_\infty \lesssim (\Lambda\Gamma)^{-K_\circ} (\max(\lambda, \lambda')\Gamma)^{\lfloor \mathsf{d}/4 \rfloor} \left(\max\left(\mu^{-1}, \mu_m \mu_{m-1}^{-2}\right)\right)^{\lfloor \mathsf{d}/4 \rfloor} (\Lambda\Gamma)^N (\nu\Gamma)^M \tag{A.209}$$

in the same range of N and M.

(iii) We have the support properties

$$\operatorname{supp}(\phi^{m,l}_\varphi) \subseteq \operatorname{supp} G \cap B\left(\operatorname{supp}\vartheta, 2\mu_{m-1}^{-1}\right) \circ \Phi \ \text{ for } \ 1 \leq m \leq \bar{m}\,, \qquad \operatorname{supp}\left(\phi^{0,l}_\varphi\right) \subseteq \operatorname{supp} G\,. \tag{A.210}$$

(iv) For all $M \leq M_* - \mathsf{N}_{\mathrm{cut,t}} - 1$, we have that the mean $\langle D_t \sigma_S \rangle$ satisfies

$$\left| \frac{d^M}{dt^M} \langle D_t \sigma_\varphi \rangle \right| \lesssim (\Lambda\Gamma)^{-K_\circ} (\max(\lambda, \lambda')\Gamma)^{\lfloor \mathsf{d}/4 \rfloor} \mu^{-\lfloor \mathsf{d}/4 \rfloor} \mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right) \tag{A.211}$$

Proof. `Step 1: Defining and estimating` σ_φ `to verify` (A.201)--(A.205). From (A.47) of Proposition A.3.3, we have that φ can be written as

$$\varphi = \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} H^{\alpha(j)} \rho^{\beta(j)} \circ \Phi\,,$$

where $H^{\alpha(j)}$ and $\rho^{\beta(j)}$ satisfy the bounds in (A.49a), (A.49b). In addition, we have the pointwise bounds on $H^{\alpha(j)}$ in terms of $\pi^{3/2} r_G^{-1}$ given by (A.60) in Remark A.3.9, but with the modifications listed in preliminary assumption (i). For each $0 \leq j \leq \mathcal{C}_{\mathcal{H}}$, we shall apply Lemma A.4.2 with the following choices, where we have used the convention set out in Remark A.3.10 to rewrite the symbols from Lemma A.4.2 with bars above on the left-hand side of the equalities below, while the right-hand side are parameters given in the

assumptions of this Proposition:

$$\overline{v} = v\,, \quad \overline{N}_* = N_* - \lfloor {}^{\mathsf{d}}/_2 \rfloor\,, \quad \overline{M}_* = M_*\,, \quad \overline{M}_t = M_t\,,$$
$$\overline{H} = H^{\alpha(j)}, \quad \overline{\mathcal{C}}_{G,1} = \mathcal{C}_{G,1}\,, \quad \overline{\mathcal{C}}_{G,\infty} = \mathcal{C}_{G,\infty}\,,$$
$$\overline{\rho} = \rho^{\beta(j)}, \quad \overline{\mathcal{C}}_{\rho,1} = \mathcal{C}_{*,1}\Upsilon^{-2}\Upsilon'\,, \quad \overline{\mathcal{C}}_{\rho,\infty} = \mathcal{C}_{*,\infty}\Upsilon^{-2}\Upsilon'\,, \quad \overline{r}_G = r_G\,, \quad \overline{r}_\phi = r_\phi$$
$$\overline{\lambda} = \max(\lambda, \lambda')\,, \quad \overline{\Lambda} = \Lambda, \quad \overline{\Gamma} = \Gamma\,, \quad \overline{\Phi} = \Phi\,,$$
$$\overline{\pi} = \pi\,, \quad \overline{\nu} = \nu\,, \quad \overline{\nu}' = \nu'\,, \quad \overline{\mu} = \mu\,, \quad \overline{\mathsf{N}_{\mathrm{dec}}} = \mathsf{N}_{\mathrm{dec}}\,,$$

and $\mathsf{N}_{\mathrm{cut,x}}$, $\mathsf{N}_{\mathrm{cut,t}}$, and δ_{tiny} as in preliminary assumption (ii). From (A.49), the modified version of (A.60), which is listed in preliminary assumption (i), (A.50), and (A.198a), we have that (A.132a)–(A.132d) are satisfied. Assumption (A.133) is satisfied from (A.198e). All the assumptions in (iii) are satisfied from preliminary assumption (ii) from this proposition. Finally, all assumptions in (iv) are satisfied from the additional assumption (ii) from this Proposition.

We may then apply (A.136a)–(A.137b) from Lemma A.4.2 to obtain for $0 \le j \le \mathcal{C}_H$ the pressure increments $\sigma_\varphi^j = \sigma_\varphi^{+,j} - \sigma_\varphi^{-,j}$, and we then collect terms to define

$$\sigma_\varphi^+ := \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \sigma_\varphi^{+,j}, \qquad \sigma_\varphi^- := \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \sigma_\varphi^{-,j}, \qquad \sigma_\varphi := \sigma_\varphi^+ - \sigma_\varphi^-\,.$$

From conclusions (i)–(v) of Lemma A.4.2, we have that (A.201)–(A.205) are satisfied.

`Step 2: Decomposing` σ_φ `to verify` (A.200)`, and defining and estimating` ϕ_φ^m `to verify` (A.206)`--`(A.210) From (A.136a)–(A.137b), we have that

$$\sigma_\varphi = r_\phi^{2/3} \sum_{j=0}^{\mathcal{C}_{\mathcal{H}}} \Pi\left(H^{\alpha(j)}\right)\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right) \circ \Phi\,. \tag{A.212}$$

Note further that $\Pi(\rho^{\beta(j)})$ is $({}^{\mathbb{T}}/_{\mu})^3$-periodic and has derivative cost $\Lambda\Gamma$ from (A.141a), con-

clusion (iii) from Lemma A.4.2. So we decompose as in (A.195) to write

$$\mathsf{\Pi}(\rho^{\beta(j)}) = \widetilde{\mathbb{P}}_{\mu_0}(\mathsf{\Pi}(\rho^{\beta(j)})) + \left(\sum_{m=1}^{\bar{m}} \widetilde{\mathbb{P}}_{(\mu_{m-1},\mu_m]}(\mathsf{\Pi}(\rho^{\beta(j)}))\right) + \left(\mathrm{Id} - \widetilde{\mathbb{P}}_{\mu_{\bar{m}}}\right)\mathsf{\Pi}(\rho^{\beta(j)})\,. \tag{A.213}$$

Using the same abbreviations used in (A.196), from (A.213) we may write

$$\sigma_\varphi = \sigma_\varphi^* + \sum_{m=0}^{\bar{m}} \sigma_\varphi^m := r_\phi^{2/3} \sum_{j=0}^{\mathcal{C}_H} \mathsf{\Pi}\left(H^{\alpha(j)}\right) \mathbb{P}^* \left(\mathsf{\Pi}\left(\rho^{\beta(j)}\right)\right) \circ \Phi + r_\phi^{2/3} \sum_{m=0}^{\bar{m}} \sum_{j=0}^{\mathcal{C}_H} \mathsf{\Pi}\left(H^{\alpha(j)}\right) \mathbb{P}_m \left(\mathsf{\Pi}\left(\rho^{\beta(j)}\right)\right) \circ \Phi\,. \tag{A.214}$$

We aim to apply Proposition A.3.3 with Remarks A.3.8, A.3.9 to the material derivative of each of the terms in (A.214), which would produce

$$\begin{aligned}
\phi_\varphi := \phi_\varphi^* + \sum_{m=0}^{\bar{m}} \phi_\varphi^m =: r_\phi^{2/3} \sum_{j=0}^{\mathcal{C}_\mathcal{H}} & \underbrace{(\mathcal{H}+\mathcal{R}^*)\left(D_t\mathsf{\Pi}(H^{\alpha(j)})\left(\mathbb{P}^*\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho^{\beta(j)})\right)\circ\Phi\right)}_{=:\phi^{*,j}} \\
& + r_\phi^{2/3} \sum_{m=0}^{\bar{m}} \sum_{j=0}^{\mathcal{C}_\mathcal{H}} \underbrace{(\mathcal{H}+\mathcal{R}^*)\left(D_t\mathsf{\Pi}(H^{\alpha(j)})\left(\mathbb{P}_m\mathbb{P}_{\neq 0}\mathsf{\Pi}(\rho^{\beta(j)})\right)\circ\Phi\right)}_{=:\phi^{m,j}} \\
= (\mathcal{H}+\mathcal{R}^*)(D_t\sigma_\varphi^*) & + \sum_{m=0}^{\bar{m}} (\mathcal{H}+\mathcal{R}^*)(D_t\sigma_\varphi^m)\,.
\end{aligned}$$

Assuming that we succeed in doing so, we have at least verified (A.200) and (A.206). Now in order to apply the inverse divergence with the pointwise bounds from Remark A.3.9, we again first treat the low-frequency assumptions from Part 1, which are the same in all cases (irrespective of the projector on $\mathsf{\Pi}(\rho^{\beta(j)})$). Specifically, we shall use the convention from Remark A.3.10 and in all cases set

$$\begin{aligned}
&\overline{p} = {}^3\!/_2, \infty\,, \quad \overline{v} = v\,, \quad \overline{N}_* = N_* - \mathsf{d} - \lfloor {}^{\mathsf{d}}\!/_2 \rfloor - \mathsf{N}_{\mathrm{cut,x}}\,, \quad \overline{M}_* = M_* - \mathsf{N}_{\mathrm{cut,t}} - 1\,, \quad \overline{M}_t = M_t - \mathsf{N}_{\mathrm{cut,t}} - 1\,, \\
&\overline{G} = r_\phi^{2/3} D_t\mathsf{\Pi}(H^{\alpha(j)}), \quad \overline{\mathcal{C}}_{G,3/2} = r_\phi^{2/3}\nu\Gamma\mathcal{C}_{G,1}^{2/3}\,, \quad \overline{\mu} = \mu\,, \quad \overline{\lambda} = \max(\lambda,\lambda')\Gamma\,, \quad \overline{\Phi} = \Phi\,, \quad \overline{\lambda}' = \lambda'\,, \\
&\overline{\nu} = \nu\Gamma\,, \quad \overline{\nu}' = \nu'\Gamma\,, \quad \overline{\Phi} = \Phi\,, \quad \overline{\pi} = \nu\Gamma\pi r_G^{-2/3}\,, \quad \overline{\mathsf{N}_{\mathrm{dec}}} = \mathsf{N}_{\mathrm{dec}}\,, \quad \overline{\mathsf{d}} = \mathsf{d}\,, \quad \overline{\mathcal{C}}_{G,\infty} = r_\phi^{2/3}\nu\Gamma\mathcal{C}_{G,\infty}^{2/3}\,.
\end{aligned}$$

Then (A.39) is satisfied from the additional assumption (A.197a), and (A.40) is satisfied from the conclusion (A.141a) and the parameter choices from Step 1 which led to that conclusion. The estimates in (A.41), (A.42a) and (A.42b) hold from assumption (ii) from this Proposition. The pointwise bound in (A.59) holds with $\overline{M}_t = M_t - \mathsf{N}_{\mathrm{cut,t}} - 1$ and $\overline{\pi} = \nu\Gamma\pi r_G^{-2/3}$ due to (A.142b), which was verified in Step 1. At this point we split into cases based on which projector is applied to $\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$ in (A.214) and address parts 2-4 of Proposition A.3.3.

`Step 2a: Lowest shell.` For the case $m = 0$, we appeal to Lemma 7.3.3 with $q = 3/2$, $\lambda = \Lambda\Gamma$, $\rho = \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$, and α such that λ^α in (7.37a) is equal to Γ. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p = 3/2, \infty$

$$\overline{\varrho} = \mathbb{P}_0\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\,, \quad \overline{\vartheta} \text{ as defined in (7.37a)}\,, \quad \overline{\mathcal{C}}_{*,p} = \Gamma\left(\mathcal{C}_{*,1}\Upsilon^{-2}\Upsilon'\right)^{2/3}\left(\frac{\mu_0}{\mu}\right)^{\frac{4}{3}-\frac{2}{p}}\,,$$
$$\overline{\mu} = \mu\,, \quad \overline{\Upsilon} = \overline{\Upsilon}' = \mu\,, \quad \overline{\Lambda} = \mu_0\,, \quad \overline{\mathsf{d}} = \mathsf{d}\,.$$

Then (7.35) is satisfied with $\mathcal{C}_{p,3/2} = \left(\mathcal{C}_{*,1}\Upsilon^{-2}\Upsilon'\right)^{2/3}$ and $\lambda = \Lambda\Gamma$ from standard Littlewood-Paley theory, (A.141a), and the choices from Step 1 which led to that conclusion, and so from (7.37a) we have that (A.43) is satisfied. From (A.198e), (A.199a), (A.199b), the choice of $\overline{N}_*$ above, (A.141a) and (A.141b), and (A.198d), we have that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied from preliminary assumption (v) and (A.197b). We have that (A.55) is satisfied from (A.199c). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. First, we set

$$\phi_\varphi^{0,l} = \mathcal{H}(D_t\sigma_\varphi^0)\,, \qquad \phi_\varphi^{0,*} = \mathcal{R}^*(D_t\sigma_\varphi^0)\,.$$

From (A.50), we obtain both (A.207a) and (A.207b), but with the appropriate modifications for $m = 0$ as indicated. The pointwise bound in (A.207c) holds due to (A.60), (A.49a), and (A.47). Next, we obtain (A.208) for $m = 0$ from (A.57). Finally, we obtain (A.210) from

(A.48), concluding the proof of the desired conclusions for $m = 0$.

`Step 2b: Intermediate shells.` For the cases $1 \le m \le \bar{m}$, we appeal to Lemma 7.3.4 with $q = 3/2$ and $\rho = \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$. Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p = 3/2, \infty$

$$\begin{aligned}
&\overline{\varrho} = \mathbb{P}_m \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\,, \quad \overline{\vartheta} = \mu_{m-1}^{-\mathsf{d}}\Theta_{\rho}^{\mu_{m-1},\mu_m} \text{ as defined in Lemma 7.3.4}\,,\\
&\overline{\mathcal{C}}_{*,p} = \left(\mathcal{C}_{*,1}\Upsilon^{-2}\Upsilon'\right)^{2/3}\left(\frac{\min(\mu_m,\Lambda\Gamma)}{\mu}\right)^{\frac{4}{3}-\frac{2}{p}}\,, \quad \overline{\Upsilon} = \mu_{m-1}\,, \quad \overline{\Upsilon}' = \overline{\Lambda} = \min(\mu_m,\Gamma\Lambda)\,,\\
&\overline{\mathsf{d}} = \mathsf{d}\,, \qquad \overline{\mu} = \mu\,, \qquad \alpha \text{ as in the previous substep}\,.
\end{aligned}$$

Then (7.39) is satisfied exactly as in the previous substep, and so from (7.40a)–(7.40b) we have that (A.43) is satisfied. As before, we use (A.198e), (A.199a), (A.199b), the choice of $\overline{N}_*$ above, (A.141a) and (A.141b), and (A.198d) to see that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied as in the previous substep, and (A.55) is satisfied from (A.199c). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. First, we set

$$\phi_{\varphi}^{m,l} = \mathcal{H}(D_t\sigma_{\varphi}^m)\,, \qquad \phi_{\varphi}^{m,*} = \mathcal{R}^*(D_t\sigma_{\varphi}^m)\,.$$

From (A.50), we obtain both (A.207a) and (A.207b). The pointwise bound in (A.207c) holds due to (A.60), (A.49a), and (A.47). Next, we obtain (A.208) from (A.57). Finally, we obtain (A.210) from (A.48) and (7.40c), concluding the proof for $1 \le m \le \bar{m}$.

`Step 2c: Highest shell.` For the case with the highest shell, corresponding to the projector $\mathbb{P}^*$ from (A.214), we appeal to Lemma 7.3.3 with $q = 3/2$, $\lambda = \Lambda\Gamma$, $\rho = \mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})$.

Specifically, to verify the assumptions in Part 2 of Proposition A.3.3, we set for $p = 3/2, \infty$

$$\begin{aligned}
&\overline{\varrho} = \mathbb{P}^*\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)}) \,, \quad \overline{\vartheta} = \vartheta \text{ as defined in (7.37b)} \,, \\
&\overline{\mathcal{C}}_{*,p} = \left(\frac{\Lambda\Gamma}{\mu_{\bar{m}}}\right)^{N_{**}} \left(\mathcal{C}_{*,1}\Upsilon^{-2}\Upsilon'\right)^{2/3} (\lambda\Gamma)^3 \,, \quad \overline{\Upsilon} = \overline{\Upsilon}' = \mu \,, \quad \overline{\Lambda} = \Gamma\Lambda \,, \\
&\overline{\mathsf{d}} = 0 \,, \quad \overline{N}_* = N_* - \mathsf{N}_{\mathrm{cut,x}} - N_{**} - 3 \,.
\end{aligned}$$

We note that we have altered the definition of N_* compared to the previous two substeps for convenience. But from (A.197c), we have in fact made it *smaller*, so that the low-frequency assumptions from the inverse divergence are still satisfied. Then (7.35) is satisfied exactly as in the first substep, and so from (7.37b) we have that (A.43) is satisfied. We use (A.198e), (A.199a), (A.199b), the altered choice of $\overline{N}_*$ above, (A.141a) and (A.141b), and (A.198d) to see that (A.44)–(A.45) are satisfied. Continuing onto the nonlocal assumptions from Proposition A.3.3, we have that (A.52)–(A.54) are satisfied as in the previous substep, and (A.55) is satisfied from (A.199d). We then appeal to the conclusions (A.46)–(A.51) and (A.56)–(A.57) to conclude as follows. First, we set

$$\phi^*_\varphi = (\mathcal{H} + \mathcal{R}^*)(D_t\sigma^*_\varphi) \,.$$

We may ignore (A.50) since $\mathsf{d} = 0$. Then the only conclusion we require is (A.209), which follows from (A.57).

`Step 3: Verification of` (A.211). The proof is similar to (A.193). Indeed, we have

$$\begin{aligned}
&r_\phi^{2/3}\left|\int_{\mathbb{T}^3} D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\circ\Phi dx\right| \\
&\qquad \lesssim r_\phi^{2/3}\left\|\Delta^{\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(D_t^{M+1}\Pi\left(H^{\alpha(j)}\right)\circ\Phi^{-1}\right)\right\|_{3/2}\left\|\Delta^{-\lfloor \frac{\mathsf{d}}{4}\rfloor}\left(\mathbb{P}_{\neq 0}\Pi(\rho^{\beta(j)})\right)\right\|_{3/2} \\
&\qquad \lesssim r_\phi^{2/3}\mathcal{C}_{G,1}^{2/3}(\max(\lambda,\lambda')\Gamma)^{\mathsf{d}/2}\mu^{-\mathsf{d}/2}(\mathcal{C}_{*,1}\Upsilon^{-2}\Upsilon')^{2/3}\mathcal{M}\left(M+1, M_t - \mathsf{N}_{\mathrm{cut,t}}, \nu\Gamma, \nu'\Gamma\right) \\
&\qquad \lesssim (\Lambda\Gamma)^{-K_\circ}(\Upsilon^{-2}\Upsilon')^{2/3}(\max(\lambda,\lambda')\Gamma)^{\lfloor \mathsf{d}/4\rfloor}\mu^{-\lfloor \mathsf{d}/4\rfloor}\mathcal{M}\left(M, M_t - \mathsf{N}_{\mathrm{cut,t}} - 1, \nu\Gamma, \nu'\Gamma\right) \,.
\end{aligned}$$

using Lemma A.1.1, (A.141a), (A.107a), (A.199c) with standard Littlewood-Paley theory. Then, recalling $\frac{d^M}{dt^M}\langle D_t\sigma_\varphi\rangle = \langle D_t^{M+1}\sigma_\varphi\rangle$ and using the representation (A.212) of $D_t\sigma_\varphi$, we obtain (A.211).

□

A.5 Upgrading material derivatives

Lemma A.5.1 (Upgrading material derivatives). *Fix $p \in [1,\infty]$ and a positive integer $N_\star \leq {}^{3\mathsf{N}_{\mathrm{fin}}}/_4$. Assume that a tensor F is given with a decomposition $F = F^l + F^*$ which satisfy*

$$\left\|\psi_{i,q} D^N D^M_{t,q} F^l\right\|_p \lesssim \mathcal{C}_{p,F}\lambda_F^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+c}\tau_q^{-1}, \Gamma_q^{-1}\mathrm{T}_q^{-1}\right) \tag{A.215a}$$

$$\left\|D^N D^M_{t,q} F^*\right\|_\infty \lesssim \mathcal{C}_{*,F}\mathrm{T}_{q+\bar{n}}^{\mathsf{N}_{\mathrm{ind,t}}}\lambda_F^N \tau_q^{-M} \tag{A.215b}$$

for all $M + N \leq N_\star$, an absolute constant $c \leq 20$, and constants $\mathcal{C}_{p,F}$ and $\mathcal{C}_{*,F}$. Assume furthermore that there exists k such that $q+1 < k \leq q+\bar{n}$ and

$$\mathrm{supp}\,(\widehat{w}_{q'}, \lambda_{q'}^{-1}\Gamma_{q'}) \cap \mathrm{supp}\,(F^l) = \emptyset \quad \forall q+1 \leq q' < k\,. \tag{A.216}$$

Finally, assume that

$$\lambda_F \Gamma_{q+\bar{n}}^{i_{\max}+2}\delta_{q+\bar{n}}^{\frac{1}{2}} r_q^{-\frac{1}{3}} \leq \mathrm{T}_{q+\bar{n}}^{-1}\,. \tag{A.217}$$

Then F obeys the following estimate with an upgraded material derivative for all $M+N \leq N_\star$;

$$\left\|\psi_{i,k-1} D^N D^M_{t,k-1} F\right\|_p \lesssim \left(\mathcal{C}_{p,F} + \mathcal{C}_{*,F}\right)\max(\lambda_F, \Lambda_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i}\tau_{k-1}^{-1}, \Gamma_{k-1}^{-1}\mathrm{T}_{k-1}^{-1}\right)\,. \tag{A.218}$$

In particular, the nonlocal part F^* obeys better estimate

$$\left\|D^N D^M_{t,k-1} F^*\right\|_\infty \lesssim \mathcal{C}_{*,F} \max(\lambda_F, \lambda_{k-1}\Gamma_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}^{-1}\right) \tag{A.219}$$

for $N + M \le N_\star$.

Similarly, if instead of (A.215a)*, F^l satisfies*

$$\left|\psi_{i,q} D^N D^M_{t,q} F^l\right| \lesssim \pi_F \lambda_F^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_q^{i+c}\tau_q^{-1}, \Gamma_q^{-1}\mathrm{T}_q^{-1}\right) \tag{A.220}$$

for all $M + N \le N_\star$, an absolute constant $c \le 24$, and a positive function π_F with $\pi_F \ge \mathcal{C}_{*,F}$, we have

$$\left|\psi_{i,k-1} D^N D^M_{t,k-1} F\right| \lesssim \ \pi_F \max(\lambda_F, \Lambda_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \Gamma_{k-1}^{i}\tau_{k-1}^{-1}, \Gamma_{k-1}^{-1}\mathrm{T}_{k-1}^{-1}\right) \tag{A.221}$$

for all $M + N \le N_\star$, provided that (A.217) *holds.*

Proof. We first handle the local portion F^l by upgrading $\psi_{i,q}$ in (A.215a) to the one with $\psi_{i,k-1}$, and then upgrading $D_{t,q}$ to $D_{t,k-1}$. Since $\psi_{i',q}^6$ forms a partition of unity from (5.8) and we have $\tau_q^{-1}\Gamma_q^{i'+24} \le \tau_{k-1}^{-1}\Gamma_{k-1}^{i}$ when $\psi_{i',q}\psi_{i,k-1} \ne 0$ by (5.14), we obtain that

$$\begin{aligned}
\left\|\psi_{i,k-1} D^N D^M_{t,q} F^l\right\|_p &= \left\|\psi_{i,k-1} \sum_{i'=0}^{i_{\max}} \psi_{i',q}^6 D^N D^M_{t,q} F^l\right\|_p \\
&\lesssim \sum_{i' : \psi_{i',q}\psi_{i,k-1} \ne 0} \left\|\psi_{i',q} D^N D^M_{t,q} F^l\right\|_p \\
&\lesssim \mathcal{C}_{p,F} \lambda_F^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{k-1}^{-1}\Gamma_{k-1}^{i}, \Gamma_{k-1}^{-1}\mathrm{T}_{k-1}^{-1}\right).
\end{aligned} \tag{A.222}$$

Here we used the maximal cardinality of i' is $i_{\max}$. Then, using (A.216), we have $D^M_{t,k-1}F^l = D^M_{t,q}F^l$ and the desired inequality (A.218) for F^l follows. In a similar way, we can also get (A.221) for F^l.

On the other hand, we handle the nonlocal portion F^* by claiming that for each $q \le k' \le$

$k-1$, we have

$$\left\|D^N D^M_{t,k'} F^*\right\|_\infty \lesssim \mathcal{C}_{*,F} \mathrm{T}_{q+\bar{n}}^{\mathsf{N}_{\mathrm{ind,t}}} \max(\lambda_F, \lambda_{k'}\Gamma_{k'})^N (\mathrm{T}_{k-1}^{-1}\Gamma_{k-1}^{-1})^M, \tag{A.223}$$

for all $N+M \leq N_\star$. In particular, this implies that

$$\left\|D^N D^M_{t,k-1} F^*\right\|_\infty \lesssim \mathcal{C}_{*,F} \max(\lambda_F, \lambda_{k-1}\Gamma_{k-1})^N \mathcal{M}\left(M, \mathsf{N}_{\mathrm{ind,t}}, \tau_{k-1}^{-1}, \mathrm{T}_{k-1}^{-1}\Gamma_{k-1}^{-1}\right)$$

for $N+M \leq N_\star$, which yields (A.218) and (A.221). The proof of the claim is then given by an inductive argument on k'. When $k' = q$, it easily follows from (A.215b). Next, suppose that (A.223) holds for some $k' < k-1$, and we apply Remark A.2.6 to $v = \widehat{u}_{k'}$, $w = \widehat{w}_{k'+1}$, $f = F$, $\Omega = \mathbb{T}^3$, $N_* = N_\star$, $N_t = \mathsf{N}_{\mathrm{ind,t}}$. Then (A.223) holds for $k'+1$, using (5.32), (5.34), the inductive assumption (A.223) for k', and (A.217). □

A.6 Mollification estimates

In this subsection, we require two algebraic identities originally stated in [7, (5.17a)–(5.17b)], which we now recall. Let v be a sufficiently smooth divergence-free vector field and let $D_t = \partial_t + v \cdot \nabla$ be the material derivative operator associated to v. For any sufficiently smooth function $F = F(x,t)$ and any $n, m \geq 0$, the Leibniz rule implies that

$$D^n D_t^m F = D^n(\partial_t + v \cdot \nabla_x)^m F = \sum_{\substack{m' \leq m \\ n'+m' \leq n+m}} d_{n,m,n',m'}(v)(x,t) D^{n'} \partial_t^{m'} F\,, \tag{A.224a}$$

$$d_{n,m,n',m'}(v) = \sum_{k=0}^{m-m'} \sum_{\substack{\{\gamma \in \mathbb{N}^k \colon |\gamma| = n-n'+k, \\ \beta \in \mathbb{N}^k \colon |\beta| = m-m'-k\}}} c(m,n,k,\gamma,\beta) \prod_{\ell=1}^{k} \left(D^{\gamma_\ell} \partial_t^{\beta_\ell} v(x,t)\right)\,, \tag{A.224b}$$

where $c(m,n,k,\gamma,\beta)$ denotes an explicitly computable combinatorial coefficient which depends only on the factors inside the parentheses. Identities (A.224a)–(A.224b) hold because D and ∂_t commute; the proof is based on induction on n and m and is left to the reader.

Proposition A.6.1 (Mollification with spatial and material derivatives). *Let $p \in [1,\infty]$, N_{g}, N_{c}, M_t, N_*, and N_γ be positive integers, v be a divergence-free vector field, and $D_t = \partial_t + v\cdot\nabla$. Fix parameters λ, Λ, τ, T, $\Gamma \geq 1$, i, $\mathcal{C}_{f,p} \leq \widetilde{\mathcal{C}}_f$, $\mathcal{C}_v$, and $c \in [0,30]$ such that*

$$N_{\mathrm{g}} \leq N_{\mathrm{c}} \leq N_*/4\,, \quad M_t \leq N_* \leq N_\gamma\,, \quad \lambda\Gamma \leq \Lambda\,, \quad \tau^{-1}\Gamma^{i+c} \leq \mathrm{T}^{-1}\,, \quad \mathcal{C}_v\lambda \leq \mathrm{T}^{-1}\,, \tag{A.225a}$$

$$(\mathrm{T}^{-1}\Gamma)^{M_t}\widetilde{\mathcal{C}}_f\Gamma^{-N_{\mathrm{c}}/2} \leq \Gamma^{-N_{\mathrm{g}}}\mathcal{C}_{f,p}\tau^{-M_t}\,. \tag{A.225b}$$

Let $(a,b)+\mathrm{T}$ be a time domain and $\Omega \subset (a,b)+\mathrm{T}\times\mathbb{T}^d$ be a subset in the space-time domain. Assume that v satisfies

$$\left\|D^N\partial_t^M v(x,t)\right\|_{L^\infty((a,b)+\mathrm{T}\times\mathbb{T}^3)} \lesssim \mathcal{C}_v\lambda^N\mathrm{T}^{-M} \tag{A.226}$$

for all $N+M \leq N_\gamma$. Assume that $f:(a,b)+\mathrm{T}\times\mathbb{T}^d \to \mathbb{R}$ satisfies the estimates[15]

$$\left\|D^N D_t^M f\right\|_{L^p(\Omega)} \lesssim \mathcal{C}_{f,p}\lambda^N\mathcal{M}\left(M, M_t, \tau^{-1}\Gamma^{i+c}, \mathrm{T}^{-1}\right) \tag{A.227a}$$

$$\left\|D^N\partial_t^M f\right\|_{L^\infty((a,b)+\mathrm{T}\times\mathbb{T}^d)} \lesssim \widetilde{\mathcal{C}}_f\lambda^N\mathrm{T}^{-M} \tag{A.227b}$$

for $N+M \leq N_*$. Let γ_x be a compactly supported mollifier in space at scale $(\lambda^{-1}\Lambda^{-1})^{1/2}$, γ_t be a compactly supported mollifier in time at scale $\mathrm{T}\Gamma^{-1/2}$, and assume that the kernels for both mollifiers have vanishing moments up to N_{c} and are C^{N_γ} differentiable.

Set $f_\gamma = \gamma_t * \gamma_x * f$. Then for $N+M \leq N_\gamma$, we have that

$$\left\|D^N D_t^M f_\gamma\right\|_{L^p(\Omega\cap(a,b)\times\mathbb{T}^d)} \lesssim \mathcal{C}_{f,p}\Lambda^N\mathcal{M}\left(M, M_t, \tau^{-1}\Gamma^{i+c+1}, \mathrm{T}^{-1}\Gamma\right)\,, \tag{A.228}$$

while for $N+M \leq N_*$, we have that

$$\left\|D^N D_t^M (f-f_\gamma)\right\|_{L^p(\Omega\cap(a,b)\times\mathbb{T}^d)} \lesssim \Gamma^{-N_{\mathrm{g}}}\mathcal{C}_{f,p}\Lambda^N\mathcal{M}\left(M, M_t, \tau^{-1}, \mathrm{T}^{-1}\Gamma\right)\,. \tag{A.229}$$

[15] By $L^p(\Omega)$, we mean L^p for each fixed timeslice $\Omega\cap\{t=t_0\}$, continuously in time which is non-empty.

Proof. We split the proof into steps. We first set up the Taylor expansion which allows us to take advantage of the vanishing moments. Next, we prove (A.228) and (A.229) for $N, M \leq {}^{N_*}/_4$. Finally, we prove (A.228) and (A.229) in the remaining cases where either $N > {}^{N_*}/_4$ or $M > {}^{N_*}/_4$. Note that since γ_t has a compact support in time at scale $\mathrm{T}\Gamma^{-1/2}$, f_γ is well-defined in the domain $(a, b) \times \mathbb{T}^d$.

`Step 1:` Let us denote by K_t the kernel for γ_t and K_x the kernel for γ_x so that $K := K_t K_x$ is the space-time kernel for $\gamma_t * \gamma_x$. We denote space-time points $(t, x) \in (a, b) \times \mathbb{T}^d$ and $(s, y) \in (a, b) + \mathrm{T} \times \mathbb{T}^d$ by

$$(t, x) = \theta, \qquad (s, y) = \kappa\,. \tag{A.230}$$

Using this notation we may write out f_γ explicitly as

$$f_\gamma(\theta) = \int_{\mathbb{T}^d \times \mathbb{R}} f(\theta - \kappa) K(\kappa)\, d\kappa\,. \tag{A.231}$$

Expanding f in a Taylor series in space and time around θ yields the formula

$$f(\theta - \kappa) = f(\theta) + \sum_{|\alpha| + m = 1}^{N_\mathrm{c} - 1} \frac{1}{\alpha! m!} D^\alpha \partial_t^m f(\theta) (-\kappa)^{(\alpha, m)} + R_{N_\mathrm{c}}(\theta, \kappa) \tag{A.232}$$

where

$$R_{N_\mathrm{c}}(\theta, \kappa) = \sum_{|\alpha| + m = N_\mathrm{c}} \frac{N_\mathrm{c}}{\alpha! m!} (-\kappa)^{(\alpha, m)} \int_0^1 (1 - \eta)^{N_\mathrm{c} - 1} D^\alpha \partial_t^m f(\theta - \eta\kappa)\, d\eta\,. \tag{A.233}$$

`Step 2:` Assume that $N, M \leq {}^{N_*}/_4$. Here we note that because of the vanishing moments of

K,

$$f_\gamma(\theta) - f(\theta) = \sum_{|\alpha|+m''=N_c} \frac{N_c}{\alpha! m''!} \int_{\mathbb{T}^d\times\mathbb{R}} K(\kappa)(-\kappa)^{(\alpha,m'')} \int_0^1 (1-\eta)^{N_c-1} D^\alpha \partial_t^{m''} f(\theta - \eta\kappa)\, d\eta\, d\kappa\,. \tag{A.234}$$

Now we appeal to the identity (A.224a) with $F = f_\gamma - f$ to obtain

$$\|D^n D_t^m (f_\gamma - f)\|_{L^\infty((a,b)\times\mathbb{T}^d)} \lesssim \sum_{\substack{m'\le m\\ n'+m'\le n+m}} \|d_{n,m,n',m'}(v)\|_{L^\infty} \left\|D^{n'}\partial_t^{m'}(f_\gamma - f)\right\|_{L^\infty((a,b)\times\mathbb{T}^d)}\,. \tag{A.235}$$

From assumptions (A.225) and (A.226) and the formula (A.224b), we have that

$$\|d_{n,m,n',m'}(v)\|_{L^\infty} \lesssim \sum_{k=0}^{m-m'} \mathcal{C}_v^k \lambda^{n-n'+k} (\mathrm{T}^{-1})^{m-m'-k} \lesssim \lambda^{n-n'} (\mathrm{T}^{-1})^{m-m'}\,. \tag{A.236}$$

Combining this estimate with the bound (A.227b), we deduce that

$$\begin{aligned}
&\left\|D^N D_t^M (f_\gamma - f)\right\|_{L^\infty((a,b)\times\mathbb{T}^d)}\\
&\quad\lesssim \sum_{\substack{m'\le M\\ n'+m'\le N+M}} \lambda^{N-n'}(\mathrm{T}^{-1})^{M-m'} \left\|D^{n'}\partial_t^{m'}(f_\gamma - f)\right\|_{L^\infty((a,b)\times\mathbb{T}^d)}\\
&\quad\lesssim \sum_{\substack{m'\le M\\ n'+m'\le N+M}} \sum_{|\alpha|+m''=N_c} \lambda^{N-n'}(\mathrm{T}^{-1})^{M-m'} \times \widetilde{\mathcal{C}}_f \lambda^{n'+|\alpha|}(\mathrm{T}^{-1})^{m'+m''} \int_{\mathbb{T}^3\times\mathbb{R}} \left|\kappa^{(\alpha,m'')}\right| |K(\kappa)| d\kappa\\
&\quad\lesssim \widetilde{\mathcal{C}}_f \sum_{|\alpha|+m''=N_c} \lambda^{N+|\alpha|}(\mathrm{T}^{-1})^{M+m''}(\Lambda\lambda)^{-|\alpha|/2}(\mathrm{T}\Gamma^{-1/2})^{m''}\\
&\quad\lesssim \widetilde{\mathcal{C}}_f \lambda^N \mathrm{T}^{-M}\Gamma^{-N_c/2} \lesssim \Gamma^{-N_g}\mathcal{C}_{f,p}\Lambda^N \mathcal{M}\left(M, M_t, \tau^{-1}, \mathrm{T}^{-1}\Gamma\right)\,,
\end{aligned} \tag{A.237}$$

where the last inequality follows from (A.225) and holds for $N, M \le {}^{N_*}/_4$. This establishes (A.229) in this range of N, M, and by the triangle inequality for $f_\gamma = f_\gamma - f + f$ establishes (A.228) in the same range of N, M.

`Step 3:` We now consider (A.228) in the case that either $M \geq {}^{N_*}/_4$ or $N \geq {}^{N_*}/_4$, and $N+M \leq N_\gamma$. We first note that when $N_* \leq N+M \leq N_\gamma$, applying the differential operator to the kernels for the mollifiers, we get

$$\left\|D^N \partial_t^M f_\gamma\right\|_{L^\infty((a,b)\times\mathbb{T}^d)} \lesssim \widetilde{\mathcal{C}}_f \min_{\substack{n+m=N_* \\ n\leq N,\, m\leq M}} \lambda^n \mathrm{T}^{-m} (\lambda\Lambda)^{\frac{1}{2}(N-n)} (\mathrm{T}^{-1}\Gamma^{1/2})^{M-m} \tag{A.238}$$

This implies that when either N or M exceeds ${}^{N_*}/_4$ but $N+M \leq N_\gamma$, we have

$$\begin{aligned}
\left\|D^N D_t^M f_\gamma\right\|_{L^\infty((a,b)\times\mathbb{T}^d)} &\lesssim \sum_{\substack{m\leq M \\ n+m\leq N+M}} \left\|d_{N,M,n,m}(v)\right\|_{L^\infty} \left\|D^n \partial_t^m f_\gamma\right\|_{L^\infty} \\
&\lesssim \widetilde{\mathcal{C}}_f \Gamma^{-\frac{N_*}{8}} \Lambda^N (\mathrm{T}^{-1}\Gamma)^M \lesssim \widetilde{\mathcal{C}}_f \Gamma^{-\frac{N_c}{2}} \Lambda^N (\mathrm{T}^{-1}\Gamma)^M \\
&\lesssim \Gamma^{-N_g} \mathcal{C}_{f,p} \Lambda^N \mathcal{M}\left(M, M_t, \tau^{-1}, \mathrm{T}^{-1}\Gamma\right)
\end{aligned} \tag{A.239}$$

where we have used (A.236), (A.227b), (A.238), (A.225), and (A.225b) . In the second inequality, the factor $\Gamma^{-\frac{N_*}{8}}$ gain has been obtained by paying lossy derivative costs. This completes the proof of (A.228) when either N or M exceeds ${}^{N_*}/_4$ and $N+M \leq N_\gamma$.

Finally, in order to prove (A.229) when either N or M exceeds ${}^{N_*}/_4$ and $N+M \leq N_*$, we use the triangle inequality as in the previous step, the estimate just shown, and the estimate

$$\begin{aligned}
\left\|D^N D_t^M f\right\|_{L^p(\Omega\cap(a,b)\times\mathbb{T}^d)} &\lesssim \mathcal{C}_{f,p} \Gamma^{-(M+N)} \Lambda^N \mathcal{M}\left(M, M_t, \tau^{-1}\Gamma^{i+c+1}, \mathrm{T}^{-1}\Gamma\right) \\
&\lesssim \Gamma^{-N_g} \mathcal{C}_{f,p} \Lambda^N \mathcal{M}\left(M, M_t, \tau^{-1}, \mathrm{T}^{-1}\Gamma\right) ,
\end{aligned}$$

which follows from (A.227a) and (A.225).

□

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