Quantitative rigidity theorems in differential geometry

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ABSTRACT

The aim of this thesis is to investigate certain rigidity theorems which relate equations between geometric tensors to the topology of a manifold, focusing in the case of hypersurfaces of $\mathbb{R}^n$. What inspired us are two works: The first one is [9], in which the authors prove that a 2-dimensional surface in $\mathbb{R}^3$ with small traceless second fundamental form $A - \frac{1}{2}\operatorname{tr} A \operatorname{Id}$ is close to a round sphere in an integral sense: namely they find a smooth parametrization $\psi \colon \mathbb{S}^2 \longrightarrow \Sigma$ and a constant $c_0 \in \mathbb{R}^3$ that satisfy the following estimate:

$$\|\psi - c_0 - \operatorname{Id}\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C \left\| A - \frac{\operatorname{tr} A}{2} \operatorname{Id} \right\|_{L^2(\Sigma)}.$$

The second one is [41], in which the author investigates conditions that allow the following oscillation inequality for a closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$.

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda \operatorname{Id}\|_{L^p(\Sigma)} \leqslant C \left\| A - \frac{\operatorname{tr} A}{n} \operatorname{Id} \right\|_{L^p(\Sigma)},$$

where p is a fixed exponent in $(1, \infty)$. Motivated by such results, we dug into [9] and [41] in order to combine them and obtain likewise-estimates.

The whole thesis is dedicated to analyse conditions that allow the following $W^{2,p}$-estimate:

$$\|\psi - \operatorname{Id}\|_{W^{2,p}(\Sigma)} \leqslant C \|T\|_{L^p(\Sigma)},$$

where Σ is a given closed hypersurface in $\mathbb{R}^{n+1}$ (typically a sphere), and T a tensor on Σ that satisfies a rigidity condition. In order to reach this type of inequality we have developed a successful linearisation scheme, that has revealed to be particularly robust.

In the first chapter we give a first n-dimensional version of the estimate in [9], which is valid for every $1 < p < \infty$, albeit under the additional hypothesis of convexity of Σ.

In the second chapter we consider how a small Ricci tensor affects a hypersurface, and prove a version of the aforementioned estimates for closed, convex, almost Einstein hypersurfaces.

In the third chapter we generalize the result exposed in the first one, considering the appropriate "anisotropic curvatures". We thereby study the case of the anisotropic second fundamental form, a tensor which has already been considered in the literature but which is not yet fully explored, and generalize the result in the first chapter for hypersurfaces with small L^p-norm of such tensor.

In the fourth chapter we attempt to remove the hypothesis of convexity that we have been always assumed. Such hypothesis is proven to be not entirely artificial by a counterexample, and we manage to give a version of the previous estimates, under alternative conditions.

In the fifth chapter we include some scattered partial results that we found throughout our investigation.

In the appendix we report some computational lemmas that are used often throughout the work, but might burden the reader and obscure the main ideas.

ZUSAMMENFASSUNG

Ziel dieser These ist es, gewisse Starrheitstheoreme zu analysieren, die Gleichungen zwischen geometrischen Tensoren mit Topologie einer Mannigfaltigkeit verbinden. Wir fokussieren uns auf den Fall von Hyperflächen in $\mathbb{R}^n$. Unsere Inspiration bestehet insbesondere aus zwei Arbeiten: Die Erste ist [9], wobei die Authoren beweisen, dass eine 2-dimensional Oberfläche in $\mathbb{R}^3$ mit kleiner spurfreien zweiten Fundamentalform $A - \frac{1}{2}\operatorname{tr} A \operatorname{Id}$ nah an eine runde Sphere ist. Tatsächlich finden sie eine glatte Parametrisierung $\psi \colon \mathbb{S}^2 \longrightarrow \Sigma$ und einen Vektor $c_0 \in \mathbb{R}^3$, so dass die folgende Abschätzung erfüllt wird:

$$\|\psi - c_0 - \operatorname{Id}\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C \left\| A - \frac{\operatorname{tr} A}{2} \operatorname{Id} \right\|_{L^2(\Sigma)}.$$

Die Zweite ist [41], wobei der Author Bedingungen analysiert, die die folgende *oscillation inequality* für abgeschlossene Hyperflächen erlauben:

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda \operatorname{Id}\|_{L^p(\Sigma)} \leqslant C \left\| A - \frac{\operatorname{tr} A}{n} \operatorname{Id} \right\|_{L^p(\Sigma)},$$

wobei p eine feste Zahl in $(1, \infty)$ ist. Diese Resultate haben uns motiviert, eine änliche Untersuchung von [9] und [41] durchzuführen, um die Beiden zu kombinieren und änliche Abschätzungen zu bekommen.

In der These untersuchen wir Bedingungen, die die folgende $W^{2,p}$-Abschätzung erlauben:

$$\|\psi - \operatorname{Id}\|_{W^{2,p}(\Sigma)} \leqslant C \|T\|_{L^p(\Sigma)},$$

wobei Σ eine abgeschlossene Hyperfläche in $\mathbb{R}^{n+1}$ (normalerweise eine Sphere) ist, und T ein Tensor auf Σ, der eine gewisse Starrheitsbedingung erfüllt. Um diese Klasse von Ungleichungen zu bekommen, haben wir eine Linearisierungsmethode entwickelt, die sich als speziell robust erwiesen hat.

Im ersten Kapitel zeigen wir eine erste n-dimensionale Version der Abschätzung in [9], die für jedes $p \in (1, \infty)$ gültig ist, obwohl wir benötigen, dass Σ convex ist.

Im zweiten Kapitel betrachten wir, wie ein kleiner Ricci Tensor eine Hyperfläche beeinflusst, und zeigen eine Version der vorigen Abschätzungen, die für abgeschlossene, konvexe, fast Einstein Hyperfläche gilt.

Im dritten Kapitel verallgemeinen wir die, im ersten Kapitel vorgelegten, Resultate, indem wir die geeignete anisotropische Fundamentalform betrachten. Dort untersuchen wir den Fall der anisotropischen zweiten Fundamentalform. Dieser Tensor wurde bereits in anderen Veröffentlichungen studiert, ist jedoch nicht komplett verstanden. Wir zeigen, dass die Resultate im ersten Kapitel auch für Mannigfaltigkeiten gelten, deren anisotropische zweite Fundamentalform L^p-klein ist.

Im vierten Kapitel versuchen wir, die Konvexitätshypothese, die wir immer betrachtet haben, abzuswächen. Diese Hypothese erweist sich als nicht völlig künstlich, da wir für

nicht konvexe Hyperflächen Gegenbeispiele konstruieren können. Wir beweisen eine Version unserer vorherigen Resultate, die unter alternativen Hypothesen gilt.

Kapitel 5 enthält partielle Resultate, die wir entdeckt haben, während wir die anderen Beweise geprüft haben.

Im Anhang finden sich einige Berechnungen, die wir in der These benutzt haben, aber die den Leser/die Leserin belasten könnten, und von der Hauptideen ablenken.

CONTENTS

PREFACE

INTRODUCTION

In the field of Differential Geometry there is an abundance of so called ”rigidity theorems”. These are remarkable propositions that establish a connection between the geometric properties of a manifold and its topology. To give a proper definition of what rigidity actually means is not the scope of this thesis. We shall focus rather on a particular class of propositions that connect tensorial relations to topological properties of a smooth manifold. By the very nature of tensors, such equations admit a natural interpretation in the language of analysis. In fact, they can be interpreted as zero sets of certain partial differential operators defined on vector bundles. Rephrasing the rigidity theorems in this language, they state that manifolds in which certain partial differential equations admit solutions are forced to satisfy certain topological constraints. In this work we aim to prove appropriate quantitative versions. We study the case in which the aforementioned operators are perturbed by a small error term, and prove stability results: the smaller is the error term, the closer is the manifold to the rigid configuration. We focus particularly on the quantitative aspects of our estimates, i.e. on the relation occurring between the smallness of the error term and the closeness of our manifold to satisfy the rigidity conditions.

Our research is mainly centered on the following problems.

The classical umbilical theorem

Given a smooth hypersurface $\Sigma \subset \mathbb{R}^3$, a point is called umbilical if its second fundamental form evaluated at it is diagonal. The umbilical theorem or *Nabelpunktsatz* says that a smooth, complete, connected surface of $\mathbb{R}^3$ whose points are all umbilical is either the plane or the sphere (see [14, Chap. 3.2, Prop. 4] or [14, 3.5(2)]). This result is one of the first rigidity results in Differential Geometry, with its first proofs dating back to 1776 or even before (we refer to the splendid historical note in [41, Sec. 1.6] for a clarification of the paternity of the theorem). While there have been innumerable and successful generalisations in the direction of rigidity, with extensions of the *Nabelpunktsatz* in the cases of higher dimension ([49, Lemma 1, p.8]), higher codimension ([49, Thm.26, p.75]), spaces of constant curvature ([49, Thm. 27, 29, p.75-77], lesser smoothness ([40]), there are still open questions about the stability of the problem. In this context, stability means understanding how much the closeness of the second fundamental form to be diagonal affects the closeness of the hypersurface to a sphere, where the concept of ”closeness” will be clarified later.

The first studies in this direction were made by the Russian school in the 60’s, where authors like A. V. Pogorelov (see [23] or [44]) considered the case of convex surfaces, and proved how the ratio between eigenvalues of the second fundamental form controls the ratio between the radius of the smallest outer sphere containing the surface and the radius of the largest inner sphere contained in the surface. We refer again to [41, 1.2] for a better insight in this direction. In the new millennium, works on foliations of asymptotically flat 3-manifolds like [32], [33] or [37] raised new questions about the stability of the umbilical theorem, more precisely:

How much do integral norms of the tensor $\mathring{A} := A - \frac{1}{n} \operatorname{tr} A \operatorname{Id}$ control the ratio between the two aforementioned radii?

Motivated by this question, posed by G. Huisken in 2003, C. De Lellis and S. Müller studied the problem in the case of closed surfaces of $\mathbb{R}^3$. Using the 2-dimensional structure they performed an extremely fine analysis of the considered quantities and found in a substantial way the following remarkable oscillation estimate, valid for all embedded surfaces in $\mathbb{R}^3$:

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda \operatorname{Id}\|_{L^2(\Sigma)} \leqslant C \|\mathring{A}\|_{L^2(\Sigma)}. \tag{DLM1}$$

They discovered moreover that, when the right hand side of (DLM1) is sufficiently small, then the surface is homeomorphic to a sphere, and found a conformal parametrization $\psi \colon \mathbb{S}^2 \longrightarrow \Sigma$ that satisfies the following estimate:

$$\|\psi - \operatorname{Id}\|_{W^{2,2}(\mathbb{S}^2)} \leqslant C \|\mathring{A}\|_{L^2(\Sigma)}. \tag{DLM2}$$

Such inequality was complemented with a C^0-one on the distance between the metric of Σ and the round metric of the sphere in [10]. Although such work represents a massive improvement in the understanding of quantitative stability properties of nearly umbilical surfaces, the methods do not seem to generalize to higher dimensions. In subsequent years this problem was approached by D. Perez, former student of C. De Lellis. In his PhD thesis [41] he focused on generalizing the inequality (DLM1) to closed hypersurfaces in $\mathbb{R}^n$ for $3 \leqslant n$ and general exponents $p \in (1, \infty)$. The approach followed by Perez is more direct and reduces the problem to a fine study of an elliptic differential equation on the unit ball in order to find a local version of (DLM1), and then via gluing charts globalizes the estimate. This approach allows Perez to find sufficient conditions to generalize (DLM1) and gives a comprehensive answer in the case of closed, convex hypersurfaces.

Einstein hypersurfaces

The study of rigidity and consequently stability properties for abstract manifolds presents more difficulties than the previous one, for a variety of reasons.

Firstly, the absence of an ambient space makes it more difficult to give a proper definition of "closeness". Ideas about the distance between manifolds already appeared in [7], but developments in the theory did not progress for two decades, until the groundbreaking work of M. Gromov in [22]. There the author introduced the concept of Gromov-Hausdorff distance, a way to measure the distance between abstract Riemannian manifolds. The Gromov-Hausdorff distance is however an abstract definition.

Secondly, the intrinsic quantities usually considered in Riemannian Geometry, i.e. scalar or sectional curvature, Ricci tensor... are invariant under diffeomorphism (see [2, Thm. 4.1] for instance). The differential equations underlying identities which involve them are naturally non-elliptic: indeed, for every single solution one can let the group of diffeomorphisms act and find an infinite dimensional space of solutions, contradicting the typical compactness properties of elliptic differential operators (see [2, Sec. 5.1, 5.2] or [25, Sec. 4] for a deeper insight in the subject).

A celebrated result in Differential Geometry characterizes all the complete manifolds with constant sectional curvature (see [19, Thm. 3.82]), and prove them to be, up to isometries, round spheres, Euclidean spaces or hyperbolic spaces. The collection [42] tries to solve the first problem we have exposed. The author gives in [42] and later in the book [43, Chap. 10] a more practical definition of distance between manifolds that represents the analogous of the Hölder distance for functions. This allows stronger compactness theorems to hold. Using these new propositions the author succeeds in proving stability results for the aforementioned rigidity theorem. These results require however a large number of restrictive assumptions.

We bypass such difficulties by considering the case of hypersurfaces in $\mathbb{R}^{n+1}$. Albeit being strongly restrictive, this gauge presents several advantages. Indeed, it fixes a clear ambient space in which we can measure distances, and at the same time it eliminates the problem of the diffeomorphism invariance we discussed before. In this case the classification theorem admits a strong relaxation. As observed by many authors (see [16], [46] or [52] for example) the rigidity assumption on the sectional curvature can be substituted by a rigidity one on the Ricci tensor: the only closed, connected hypersurfaces in $\mathbb{R}^{n+1}$ whose Ricci tensor is diagonal are round spheres. Manifolds satisfying such condition on the Ricci tensor are called Einstein manifolds. In this language, we can say that the only closed Einstein hypersurfaces in the Euclidean space are round spheres. The stability properties of such theorem, i.e. the properties of nearly Einstein hypersurfaces, are almost unknown, and only in recent years some authors have studied them. (See [45] or [53] for instance).

The anisotropic umbilicality

Spheres admit many characterizations. As shown in the seminal paper [4], round spheres $\mathbb{S}^2 \subset \mathbb{R}^3$ can be characterized as as the unique minimizers of the perimeter functional among the (smooth) boundaries of set with fixed volume. This characterization leads to a clear generalization: instead of considering the perimeter functional, one can also consider a positive function $F\colon \mathbb{S}^2 \longrightarrow (0, \infty)$ and study the associated variational problem, called often anisotropic surface energy:

$$\text{Minimize } \Sigma \longmapsto \int_\Sigma F(\nu_\Sigma)\, dV$$

among the surfaces which are boundaries of sets with fixed volume. Here ν_Σ denotes the outer normal of Σ. The generalization of the variational problem in higher dimension is straightforward. This problem is not an artificial generalization. It was formulated by J. E. Taylor and G. Wulff in [51] and [54], respectively, in the context of studying equilibrium configurations of solid crystals with sufficiently small grains, and it has been used later as a model for phase transitions in [24]. As conjectured in [54] under certain regularity assumptions it can be proven that the minimizer exists and is unique up to translation. Such minimizer is called Wulff shape and we will denote it by $\mathcal{W}$. Notice that when $F = 1$ the problem reduces to the isotropic one and the Wulff shape coincides with the round sphere.

There has been recently a lot of interest in anisotropic problems, and many properties enjoyed by the round spheres have been generalized to the Wulff shape, cf. [17], [18], [38] and the bibliographies therein. A rather new quantity arising in this field is the anisotropic second fundamental form. This tensor has proven to share the same rigidity properties of its anisotropic counterpart, with an anisotropic umbilical theorem which ensures that surfaces with a diagonal anisotropic second fundamental form are Wulff shapes.

PERSONAL CONTRIBUTIONS

The main scheme

In this work we focus mainly our attention on generalizing estimate (DLM2). Following a suggestion of C. De Lellis we have developed a simple but robust solution scheme to achieve the result. The scheme is rather flexible, and it is divided into 3 steps.

1) In the first step we establish a first preliminary qualitative C^1-estimate. This is normally achieved through a compactness argument. We consider sequences of hypersurfaces where some relevant quantities converge to 0, and prove the existence of a limit hypersurface. In this phase the rigidity statements play a crucial role, since they have to ensure the uniqueness of the limit.

2) Here we use the newly obtained C^1-estimate to give a proper parametrization of our hypersurfaces. We see then the main geometric quantities as differential operators in the derivatives of the chosen parametrization. Then we perform a first order approximation, linearising such operators and deriving estimates via classical PDE methods.

3) Normally the estimates that we have found insofar are not optimal. As a last step we optimize them. This step arises naturally, since the linearised operators we have derived typically have a kernel, which is defined by the invariance group of transformations that act on the problem. Such are for instance the translations, since our problems are translation-invariant. In this context the optimized estimate comes after as an appropriate centering of our hypersurface.

Summary of the thesis

The thesis is divided into six chapters and an appendix. Each of the chapters from 1 to 4 contains a different application of the method explained above. The results exhibited in Chapters 2 and 3 have already been published in [21], [12] respectively.

CHAPTER 0 We begin the work with an introductory chapter, which introduces some useful notation and collects important preliminary results, especially from reference [41].

CHAPTER 1 Here we establish the equivalent of (DLM2) in arbitrary dimension n and Sobolev exponent p, under the assumption of convexity of the considered hypersurface. This result should not be considered as a generalization of [41], but rather as a completion of it. Indeed, the techniques we use are natural consequences of the ones introduced in [41].

chapter 2 Here we apply our scheme to the case of closed, convex, almost Einstein hypersurfaces. This case presents more difficulties than the previous one. The moral reason behind them lies in the fact that the linearisation of the Ricci tensor is not an elliptic equation. This has forced us to find a new way to tackle the problem and reduce it to the previous case, under some additional auxiliary hypothesis.

chapter 3 Here we generalize the result of Chapter 1 in the case of anisotropic hypersurfaces. Many ideas in this chapter are not new and follow rather the previous ones. However, since this field is unexplored, we had to derive first analogous results to those in [41] and then conclude as in Chapter 1. The chapter includes an elegant characterization of the kernel of the stability operator associated to the Wulff shape, that uses some techniques related to the stability of the anisotropic isoperimetric inequality (cf. [18, Thm 1.1]).

chapter 4 All the results of previous chapter are derived under the hypothesis of convexity of the considered hypersurface. In this last chapter we attempt to remove this hypothesis. As an easy counterexample shows, the convexity is not artificial and it is necessary to avoid bubbling phenomena. Still, we can remove it in the supercritical case, i.e. when $n < p$ and substitute it with a L^p-integral bound of the second fundamental form. Under such hypothesis we are able to recover most results, although a rather surprising lack of linearity in the non convex nearly Einstein estimate appears.

chapter 5 The last chapter is a miscellanea of partial results obtained during the preparation of the other theorems.

appendix Here we collect some computational and technical lemmas used throughout the work.

0 NOTATIONS

0.1 CONVENTIONS

We write below a list of symbols we are going to use in the rest of the thesis. Some quantities that appear only sporadically or locally in a certain chapter will be defined when they will be useful.

Vol_n	n-dimensional Hausdorff measure.
δ	flat metric on $\mathbb{R}^k$ (or Kronecker delta, see below).
$\mathbb{S}^n$	n-dimensional sphere in $\mathbb{R}^{n+1}$.
σ	metric associated to the standard sphere.
g	restriction of the $\mathbb{R}^{n+1}$-flat metric to Σ.
$\mathcal{W}$	Wulff shape.
ω	metric associated to the Wulff shape.
tr_g	trace w.r.t. the metric g.
A	second fundamental form of Σ.
H	mean curvature of Σ, e.g. $\mathrm{tr}_g A$.
A_F	anisotropic second fundamental form of Σ.
H_F	anisotropic mean curvature of Σ, e.g. $\mathrm{tr}_g A_F$.
$\mathring{A}$	traceless second fundamental form of Σ.
$\mathring{A}_F$	traceless anisotropic second fundamental form of Σ.
$\mathbb{B}^\sigma_r(q)$	geodesic ball in $\mathbb{S}^n$ centered in q, of radius r.
$\mathbb{B}^k_r(x)$	k-dimensional ball in $\mathbb{R}^k$, centered in q, of radius r.
∇	Levi-Civita connection (see below).
∂	flat partial derivatives in $\mathbb{R}^k$.
Δ	Laplace-Beltrami operator.
div	divergence operator.
Γ^k_{ij}	Christoffel symbols.
$\mathrm{osc}(f, A)$	oscillation of f, i.e. $\sup_A f - \inf_A f$.
id	identity function from a set to itself.
Id	the identity $(1, 1)$-tensor from a bundle in itself.
Riem	Riemann tensor associated to a manifold.

Ric	Ricci tensor associated to a manifold.
Scal	scalar curvature associated to a manifold.
$\mathring{\mathrm{Ric}}$	traceless RIcci tensor.
$\owedge$	Nomizu operator between two $(2, 0)$-tensors.

We will always work keeping $\mathbb{R}^{n+1}$ as ambient space. In this framework, Σ will always denote a smooth, closed, connected n-dimensional submanifold in $\mathbb{R}^{n+1}$. For such a hypersurface, "convex" means that Σ is the boundary of a convex set. We will often need to parametrize Σ, or a portion of it. When doing this, we will identify geometric quantities of Σ with their respective pull-backs, whenever this does not lead to confusion.

We will also adopt the Einstein notation in order to omit the (possibly many) summation symbols. In this flavour it is crucial to point out an abuse of notation we will make throughout all the work. As said, we will make frequent use of parametrisations of Σ over spheres, balls or other manifolds and often we will need the explicit expression of geometric quantities in those parametrisations. In the formulae we will raise or lower the index in the left hand sides w.r.t. the metric of Σ, while the indices on the right hand side will be lowered or raised w.r.t. the natural metric associated to the parametrization (which will be the metric of the sphere, if we will parametrize the manifold over a sphere, and so on). When such convention cannot be followed, we will write the quantities with a subscript denoting the relevant metric (e.g. writing ${}_g\Gamma^k_{ij}$ instead of Γ^k_{ij} to denote the Christoffel symbols of (Σ, g)).

We will need to work with many types of derivative and will obey the following derivative conventions. The symbol ∇ shall be used for every possible Levi-Civita connection considered. When more than one such connection is involved, we will write it with a subscript that will express the relevant metric, e.g. we write ${}_g\nabla$ for the Levi-Civita connection w.r.t. the metric g. The same rule will apply to all differential operators considered, like the Laplace-Beltrami operator Δ. An exception is provided by the case of flat derivatives, for which we will always use the symbol ∂, and for 1-dimensional derivatives, for which we will use the typical d/dt or other classical notations.

Let now M be a smooth manifold. Given a function $f\colon M \longrightarrow \mathbb{R}$, we will use the notation ∇f to denote both the differential of f (which is a 1-form) and the gradient of f (which is a vector field), unless this abuse of notation leads to confusion. We also recall that at the first order all the notions of derivative coincide, so we will use sometimes the notation ∇ and sometimes ∂ depending on the context.

We shall use the letter δ to denote the flat metric of $\mathbb{R}^{n+1}$ and at the same time the Kronecker delta symbol. For every possible type of scalar product we shall use the symbol $\langle \cdot, \cdot \rangle$. We will also denote by $\{ e_i \}_{i=1}^k$ the standard basis of $\mathbb{R}^k$. With the definition outlined above, we have

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

Normally the metric that induces the scalar product is clear and we omit it. In all the other cases we will write the metric inducing it as subscript, e.g. $\langle \cdot, \cdot \rangle_g$. The same notation will be used for the norm $\|x\| := \sqrt{\langle x, x \rangle}$.

We will use the following sign conventions for curvature tensors. If X, Y and Z are vector fields on Σ, we write:

$$R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X - \nabla_X Y} Z,$$

and define the Riemann tensor of Σ as

$$\mathrm{Riem}(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle.$$

Given a (local) orthonormal frame $\{ E_i \}_{i=1}^n$, we define the Ricci tensor as the contraction of the Riemann tensor of the second and fourth index, i.e.

$$\mathrm{Ric}(X, Y) := \sum_{i=1}^n \mathrm{Riem}(X, E_i, Y, E_i),$$

and set the scalar curvature as its trace, i.e. $\mathrm{Scal} := \mathrm{tr}_g \mathrm{Ric}$. For the second fundamental form given two vector fields, we set

$$A(X, Y) := \langle X, \partial_Y \nu \rangle,$$

where ν denotes the outer normal of the hypersurface, and $\partial_Y \nu$ denotes the derivative of ν along the direction Y. Again, we will omit whenever possible the hypersurface whose ν is the normal, being that normally obvious. In the cases in which there may be confusion, e.g. when linearising quantities, we will specify the relevant surface in a subscript, writing for instance ν_Σ for the normal of Σ and so on. Such notations are chosen so that the round sphere $S^n \subset \mathbb{R}^{n+1}$ satisfies the following equalities:

$$H = n, \quad \mathrm{Scal} = n(n-1).$$

When having a $(2, 0)$-tensor T and a vector field X, we will use the notation $T(X)$ to denote the vector field

$$T(X)_i := g^{ik} T_{ij} X_k = T_i^j X_j.$$

Thanks to Einstein notation, the meaning will be the same also when we deal with $(1, 1)$-tensors, vector fields and differential forms respectively. In the same flavour, we will write

$$T(X, Y) := g^{ip} g^{jq} T_{ij} X_p Y_q = T^{pq} X_p Y_q.$$

Since we usually work with symmetric $(2, 0)$-tensors, the notation will not lead to abuse of notation.

For every Lebesgue-measurable set $A \subset \mathbb{R}^n$, we denote by $|A|$ its Lebesgue measure. The same notation shall be used to denote the volume measure $|A| := \mathrm{Vol}_g(A)$ for subsets $A \subset \Sigma$ when there is no confusion between the two. Given two sets $A, B \subset \mathbb{R}^{n+1}$, the set

$$A \Delta B := (A \setminus B) \cup (B \setminus A)$$

is called symmetric difference of A and B.

A measurable set $E \subset \mathbb{R}^{n+1}$ is said to be a set of finite perimeter if the distributional gradient $\partial\chi_E$ of the characteristic function of E is an $(n+1)$-valued Borel measure on $\mathbb{R}^{n+1}$ with total variation $|\partial\chi_E|(\mathbb{R}^{n+1}) < \infty$.

Let $F\colon \mathbb{S}^n \longrightarrow (0, \infty)$ be a smooth function defined on the n-sphere that we shall call *anisotropic integrand*. For every closed smooth hypersurface Σ in $\mathbb{R}^{n+1}$, we define its *anisotropic surface energy* as

$$\mathcal{F}(\Sigma) := \int_\Sigma F(\nu)\, dV,$$

where ν is the outer normal vector field associated to Σ. Notice that when $F = 1$, than the anisotropic surface energy $\mathcal{F}$ becomes the classical area of a hypersurface $\mathcal{F}(\Sigma) = \mathrm{Vol}_n(\Sigma)$.

For every anisotropic function F and every $m > 0$, it is natural to study the following problem

$$\inf\{\mathcal{F}(\Sigma) \,:\, \Sigma = \partial U,\ |U| = m\}, \tag{0.1}$$

which attains a minimum. Its solution is a dilation of a closed, convex hypersurface $\mathcal{W}$ called *Wulff shape*, see [51, Theorem 1.1]. In the context of differential geometry, the Wulff shape shares lots of similarities with the round sphere. For instance (see [51, Sec. 1]), it can be seen as the "sphere" for an anisotropic norm on $\mathbb{R}^{n+1}$, namely

$$\mathcal{W} = \{F^* = 1\}, \tag{0.2}$$

where F^* is the gauge function $F^* : \mathbb{R}^{n+1} \longrightarrow [0, +\infty)$ defined by

$$F^*(x) := \sup_{\nu \in \mathbb{R}^{n+1}} \left\{ \langle x, \nu\rangle : |\nu| F\left(\frac{\nu}{|\nu|}\right) \leqslant 1 \right\}.$$

A useful property of the differential of the gauge function, that we will use later, is the following:

$$dF^*|_z\,[c] = \langle \nu(z), c\rangle, \qquad \forall z \in \mathcal{W}, \tag{0.3}$$

where we denoted with ν the outer normal vector field associated to $\mathcal{W}$.

Denoting by ${}_\sigma\nabla^2 F\big|_x$ the intrinsic Hessian of F on $\mathbb{S}^n$ at the point x, we define the following map $S_F\colon x \in \mathbb{S}^n \mapsto S_F|_x$ taking values in the space of symmetric matrices:

$$S_F|_x\,[z] := {}_\sigma\nabla^2 F\big|_x\,[z] + F(x)z \quad \text{for every } x \in \mathbb{S}^n,\ z \in T_x\mathbb{S}^n. \tag{0.4}$$

We say that F is an *elliptic integrand* if $S_F|_x$ is positive definite at every $x \in \mathbb{S}^n$. For any smooth closed hypersurface Σ, we can define the anisotropic second fundamental form A_F as

$$A_F|_x : T_x\Sigma \longrightarrow T_x\Sigma,\ \ A_F|_x := S_F|_{\nu(x)} \circ d\nu|_x\,, \tag{0.5}$$

where ν denotes the outer normal vector field associated to Σ.

For integrable functions $f\colon \Sigma \longrightarrow \mathbb{R}$ we will set

$$\bar{f} := \fint_\Sigma f\, dV_g,$$

with dV_g being the measure induced by g. The same notation will be used for all the considered Riemannian manifolds, where the mean is always taken w.r.t. the associated measure.

For a function $u\colon A \longrightarrow \mathbb{R}$ where $A \subset \mathbb{R}^n$, we define its graph in the following way

$$\mathrm{Graph}(u, A) := \left\{ y \in \mathbb{R}^{n+1} \;\middle|\; y = \begin{pmatrix} x \\ u(x) \end{pmatrix}, \, x \in A \right\} \subset \mathbb{R}^{n+1}.$$

Let M be a smooth manifold. We define the Sobolev space $W_g^{k,p}(M)$ as the completion of the space

$$\left\{ f\colon C^\infty(M) \longrightarrow \mathbb{R} \;\middle|\; \int_M |f|^p \, dV_g + \sum_{j=1}^k \int_M |\nabla^j f|^p \, dV_g \right\}$$

w.r.t. the obvious norm

$$\|f - h\|^p_{W_g^{k,p}(M)} := \int_M |f - h|^p \, dV_g + \sum_{j=1}^k \int_M |\nabla^j f - \nabla^j h|^p \, dV_g.$$

The subscript in the definition is justified by the fact that this definition depends on the chosen metric g, which determines the Levi-Civita connection associated to the tensors $\nabla^j f$ and the volume measure. We will omit the index whenever it is clear which metric we are considering.

When making estimates, we will usually write inside brackets the main quantities upon which a constant C depends, i.e. we will write $C = C(x, y)$ to denote a constant depending on the quantities x and y. Plus, in the computational parts of the thesis we will adopt the convention, typical in the field of partial differential equations, of not relabelling the bounding constants at every computation line unless needed.

The core geometric quantities that we studied in the thesis can be better understood in the beautiful books [19], [35], [43] . For a better insight in the field of geometric analysis other books, as [3], [27], or even articles like [36] are recommended. Finally, for concepts as measures, perimeters and volumes we refer to the splendid books [1], [15], [31] or [48] for further studies.

0.2 PRELIMINARY KNOWLEDGE

The main inspiration of this work is the thesis [41] of D. Perez, and we refer mainly to the first two chapters. A portion of our thesis can be understood as a completion of this work, taking the main ideas from it and introducing new ones in order to reach the results in [9] and [10].

In [41, Ch. 1, 2], the author deals with the generalization of estimate (DLM1), namely he proves the following estimate for hypersurfaces in $\mathbb{R}^{n+1}$, albeit under some suitable assumptions:

$$\min_{\lambda \in \mathbb{R}} \|A - \lambda \, \mathrm{Id}\|_{L^p(\Sigma)} \leqslant C \|\mathring{A}\|_{L^p(\Sigma)}. \tag{0.6}$$

As stated in the introduction, in [9] the authors deal with the 2-dimensional case, where they take advantage of some special structural properties which are not available in higher dimensions. In [41] Perez found a way to tackle this problem, by applying the following scheme. Firstly he studied the inequality when the manifold Σ is a graph, and obtained local estimates. Then he made them global. This second step requires however some assumptions on the manifold, since one has to find an atlas of coordinate charts with some good controls. More precisely, the author obtains the following theorem:

Theorem 0.1 ([41], Thm 1.1 + Thm. 2.1)**.** *Let $2 \leqslant n$, $1 < p < +\infty$ be given, and let Σ be a smooth, closed hypersurface in $\mathbb{R}^{n+1}$. Assume Σ satisfies one of these two conditions:*

a) $n < p$, $\|A\|_{L^p(\Sigma)} \leqslant c_0$ *and* $\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n)$.

b) Σ *is convex.*

Then estimate (0.6) *holds, and the bounding constant* C *depends on* n, p, c_0 *in case* a), *just on* n *and* p *in case* b).

Observe that the assumption $\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n)$ can be omitted if the assumption on $\|A\|_{L^p}$ is replaced with a suitable scaling-invariant one.

In case b), the author proves that one can reduce himself to condition

b′) Σ is convex, $\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n)$ and $\|A\|_{L^p(\Sigma)} \leqslant c_0$.

A useful consequence of conditions a) and b′) is the following lemma, already proved in [41].

Lemma 0.2 ([41], Ch.1 + Ch.2)**.** *Let $2 \leqslant n$ be given, and let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying condition* a) *or* b′)*. Then there exist two numbers $0 < L$, R depending only on n, p and c_0 with the following property. For every $q \in \Sigma$ there exists a parametrisation*

$$\varphi_q \colon \mathbb{B}^n_R \longrightarrow \Sigma, \quad \varphi_q(x) = G_q(x, u_q(x)) \tag{0.7}$$

where u is a Lipschitz function satisfying

$$\mathrm{Lip}(f) \leqslant L, \quad u_q(0) = 0, \quad \partial u_q(0) = 0,$$

and $G_q \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$, $G_q := q + \Phi_q$ is an affine transformation obtained composing a translation $\tau_q(x) = x + q$ and a rotation Φ_q so that $\varphi_q(0) = q$ and $d\varphi_q|_0\,[\mathbb{R}^n] = T_q\Sigma$.

The parametrisation φ_q satisfies an additional property. Indeed, it parametrises Σ locally as a graph, and the radius R associated to the parametrisation satisfies the maximality property:

$$R = \sup\left\{ 0 < r \;\middle|\; \sup_{e \in \partial\mathbb{B}^n_1} \frac{|\partial u(re)|}{\sqrt{1 + |\partial u(re)|^2}} \leqslant \frac{1}{2} \right\}.$$

In [41, Section 2.1] it is proved that $0 < R < \infty$, and we can assume w.l.o.g. that R depends only on n and p. All our graph parametrisations will tacitly satisfy this property that uniquely characterizes them.

Lemma 0.2 gives the needed control over the hypersurface. Indeed, from it the author infers the following covering lemma.

Lemma 0.3 ([41], Lemma 1.7)**.** *Let $2 \leqslant n$ be given. Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$. Assume there exist $0 < L$, R with the property described in Lemma 0.2. Then, for every $0 < \rho \leqslant R$, the geodesic ball $\mathbb{B}^g_\rho(q)$ satisfies the inclusion*

$$\varphi_q \left(\mathbb{B}^n_{\frac{1}{1+L}\rho}\right) \subset \mathbb{B}^g_\rho(q) \subset \varphi_q \left(\mathbb{B}^n_\rho\right). \tag{0.8}$$

In particular, for every $q \in \Sigma$ the geodesic ball $\mathbb{B}^g_R(q)$ is contained in the image of φ_q, and Σ can be covered with N such geodesic balls, where N is a natural number depending on n, L, R.

The combination of these two triggers an elementary covering argument that allows local estimates to become global in a quite standardized procedure. Throughout the thesis we shall make frequent use of this it.

We conclude the section by reporting the following useful proposition, that is the cornerstone of Chapter 2 in [41] and will be used later in our work, too.

Proposition 0.4 ([41], Prop. 2.4 + Prop. 2.7)**.** *Let $2 \leqslant n$, $1 < p < \infty$ be given, and let $\Sigma = \partial U$ be a closed, convex hypersurface in $\mathbb{R}^{n+1}$. Assume that Σ satisfies the following condition:*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(S^n),$$
$$\|A\|_{L^p(\Sigma)} \leqslant c_0 \text{ or } \|\mathring{A}\|_{L^p(\Sigma)} \leqslant c_0.$$

Then there exist a vector $x \in \mathbb{R}^{n+1}$, two radii $0 < r < R$ depending on n, p, c_0 such that

$$\mathbb{B}_r(x) \subset U \subset \mathbb{B}_R(x).$$

1 THE CASE FOR ALMOST UMBILICAL HYPERSURFACES

The whole chapter is dedicated to the following theorem.

Theorem 1.1. *Let* $2 \leqslant n$ *and* $1 < p < \infty$ *be given, and let* Σ *be a smooth, closed and convex hypersurface in* $\mathbb{R}^{n+1}$*. There exists* $0 < \delta_0 = \delta_0(n, p)$ *with the following property. If* Σ *satisfies*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n), \tag{1.1}$$

$$\|\mathring{A}\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{1.2}$$

then there exist a vector $c = c(\Sigma) \in \mathbb{R}^{n+1}$ *and a smooth parametrization* $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ *such that the following estimate holds:*

$$\|\psi - \mathrm{id}\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p)\|\mathring{A}\|_{L^p(\Sigma)}. \tag{1.3}$$

It is immediate to see the correlation between our Theorem 1.1 and Theorem DLM2. There are differences in the formulation, though. The corresponding map $\psi\colon \mathbb{S}^2 \longrightarrow \Sigma$ found by the authors in [9] in the proof of Theorem DLM2 is indeed conformal. As stated in the introduction, our strategy will be completely different. We define here the parametrization with which we will work. Let U be the open, bounded set of which Σ is the boundary, and assume $0 \in U$. Then we can write Σ as graph over the sphere and define

$$\psi\colon \mathbb{S}^n \longrightarrow \Sigma, \quad \psi(x) = e^{f(x)}\, x. \tag{1.4}$$

The map ψ is clearly a smooth parametrization of Σ, and shall be called *radial parametrization* of Σ. We shall sometimes refer to such hypersurfaces as *radially parametrized hypersurfaces* and to the function f as *logarithmic radius associated to* ψ. What we will actually prove is the following theorem:

Theorem 1.2. *Let* $2 \leqslant n$ *and* $1 < p < \infty$ *be given, and let* $\Sigma = \partial U$ *be a smooth, closed and convex hypersurface in* $\mathbb{R}^{n+1}$*. There exists* $0 < \delta_0 = \delta_0(n, p)$ *with the following property.*

If Σ *satisfies conditions* (1.1) *and* (1.2)*, then there exists a vector* $c = c(\Sigma) \in \mathbb{R}^{n+1}$ *such that* $0 \in U - c$ *and the radial parametrization* $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ *of* (1.4) *satisfies:*

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p)\|\mathring{A}\|_{L^p(\Sigma)}. \tag{1.5}$$

It is clear that Theorem 1.2 implies 1.1. An interesting consequence we draw from 1.2 is the following corollary, which resembles the main result of [10]:

Corollary 1.3. *Under the assumptions of Theorem 1.1 the following estimate holds:*

$$\|\psi^* g - \sigma\|_{W^{1,p}(\mathbb{S}^n)} \leqslant C(n, p)\|\mathring{A}\|_{L^p(\Sigma)}. \tag{1.6}$$

This case is the simplest case in which we apply our scheme. Indeed, the proof of Theorem 1.2 can be subdivided into the three following propositions.

Proposition 1.4. *Let $2 \leqslant n$, $1 < p < \infty$ be given, and let Σ be a convex, closed hypersurface in $\mathbb{R}^{n+1}$. For every $0 < \varepsilon < 1$ there exists a $0 < \delta_0 = \delta_0(n, p, \varepsilon)$ with the following property.*

If Σ satisfies (1.1) and (1.2), then there exists a vector $c = c(\Sigma)$ such that its radial parametrization $\psi \colon \mathbb{S}^n \longrightarrow \Sigma - c$ satisfies

$$\|f\|_{C^1(\mathbb{S}^n)} \leqslant \varepsilon. \tag{1.7}$$

Proposition 1.5. *Let $2 \leqslant n$, $1 < p < \infty$ be given, and let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$. Let $0 < \varepsilon < 1$, $0 < \delta_0$ and $c = c(\Sigma)$ be chosen such that f satisfies estimate (1.7). Then the following estimate holds:*

$$\|f - \langle \nu_f, \cdot \rangle\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p)\Big(\|\mathring{A}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)}\Big), \tag{1.8}$$

where we have set

$$\nu_f := (n+1) \fint_{\mathbb{S}^n} z\, f(z)\, dV_\sigma(z).$$

Proposition 1.6. *Under the same hypothesis of Proposition 1.4 and the same notations of Proposition 1.5, we can find a vector $\tilde{c} \in \mathbb{R}^{n+1}$ such that $0 \in U - \tilde{c}$, and the associated radial parametrization $\psi \colon \mathbb{S}^n \longrightarrow \Sigma - \tilde{c}$ satisfies the conditions*

$$\|f\|_{C^1(\mathbb{S}^n)} \leqslant C(n, p)\varepsilon, \tag{1.9}$$

$$|\nu_f| \leqslant C(n, p)\varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)}. \tag{1.10}$$

Remark 1.7. The proof of Proposition 1.4 gives actually a more precise result. Namely, for every $0 < \varepsilon < 1$ there exists $0 < \delta_0 = \delta_0(n, p, \varepsilon)$ such that the following estimates hold:

$$\|f\|_{C^0(\mathbb{S}^n)} \leqslant \varepsilon, \quad \|\nabla f\|_{C^0(\mathbb{S}^n)} \leqslant 2\sqrt{\varepsilon}.$$

In any case we do not need such level of precision at this stage, because the compactness strategy used in Proposition 1.4 cannot be further improved. In order not to burden the notation, we shall omit the square root in this and in the future qualitative estimates: this can be easily avoided by considering for every $0 < \varepsilon < 1$ the threshold δ_0 associated to ε^2 and to use the simple inequality $\varepsilon^2 < \varepsilon$ in the C^0-estimate.

Remark 1.8. Theorem 1.2 follows immediately from these results. Indeed, up to choosing a smaller ε, we can consider the radial parametrization granted by (1.4). Condition (1.9) still triggers Proposition 1.5, granting us estimates (1.8) and (1.10), with a (possibly worsen) constant C which still depends only on n and p. Hence we obtain the conclusion:

$$\begin{aligned}
\|f\|_{W^{2,p}(\mathbb{S}^n)} &\leqslant C\Big(\|\mathring{A}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)} + \|\langle \nu_f, \cdot \rangle\|_{W^{2,p}(\mathbb{S}^n)}\Big) \\
&\leqslant C\Big(\|\mathring{A}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)} + |\nu_f|\Big) \\
&\leqslant C\Big(\|\mathring{A}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)}\Big).
\end{aligned}$$

Again, we notice that the bounding constant C depends only on n and p. If we choose $\varepsilon = \min\{1/2C, 1/2\}$, then the δ_0 that triggers the propositions depends only on n and p and estimate (1.8) becomes

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C\|\mathring{A}\|_{L^p(\Sigma)} + \frac{1}{2}\|f\|_{W^{2,p}(\mathbb{S}^n)}.$$

This clearly completes the proof.

We see now how to prove the three theorems. Before starting the proofs we need to report a computational lemma which gives suitable formulas for the main geometric quantities of Σ in the radial parametrization ψ. Again, we stress the fact that the indices in the left hand side are lowered or raised with respect to the metric $\psi^* g$, while the indexes in the left hand side are lowered or raised with respect to the metric σ. Moreover, the pull-back notation ψ^* will not be used, since there is no confusion.

Lemma 1.9. *Let ψ be as in (1.4). Then we have the following expressions:*

$$g_{ij} = e^{2f}\big(\sigma_{ij} + \nabla_i f \,\nabla_j f\big). \tag{1.11}$$

$$g^{ij} = e^{-2f}\left(\sigma_{ij} - \frac{\nabla_i f\,\nabla_j f}{1+|\nabla f|^2}\right). \tag{1.12}$$

$$\nu(x) = \frac{1}{\sqrt{1+|\nabla f|^2}}(x - \nabla f(x)). \tag{1.13}$$

$$A_{ij} = \frac{e^f}{\sqrt{1+|\nabla f|^2}}\big(\sigma_{ij} + \nabla_i f\,\nabla_j f - \nabla^2_{ij} f\big). \tag{1.14}$$

$$A^i_j = \frac{e^{-f}}{\sqrt{1+|\nabla f|^2}}\left(\delta^i_j - \nabla^i\nabla_j f + \frac{1}{1+|\nabla f|^2}\nabla^i f\,\nabla^2 f[\nabla f]_j\right). \tag{1.15}$$

$$dV_g = e^{nf}\sqrt{1+|\nabla f|^2}\, dV_\sigma. \tag{1.16}$$

$${}_g\Gamma^k_{ij} = \Gamma^k_{ij} + \frac{1}{1+|\nabla f|^2}\nabla^2_{ij} f\,\nabla^k f + \big(\nabla_i f\,\delta^k_i + \nabla_j f\,\delta^k_i - {}_g\,\nabla^k f\, g_{ij}\big). \tag{1.17}$$

The proof of the lemma is postponed in Appendix A.1.

1.1 PROOF OF PROPOSITION 1.4

In the scheme described above Proposition 1.4 represents the first step, that is, the qualitative C^1-result. This follows from a typical compactness argument, which we take from [41, Chap. 2]. We state here Corollary 2.5 from [41] for the reader's convenience.

Corollary 1.10. *Let $2 \leqslant n$, $1 < p < \infty$ be given, and let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth, closed, convex hypersurface in $\mathbb{R}^{n+1}$, satisfying condition (1.1).*

For every $0 < \varepsilon$ there exist $0 < \delta(n, p, \varepsilon)$ and $c(\Sigma) \in \mathbb{R}^{n+1}$, such that

$$\|\mathring{A}\|_{L^p(\Sigma)} \leqslant \delta \Rightarrow d_{HD}(\Sigma - c, S^n) \leqslant \varepsilon, \tag{1.18}$$

where d_{HD} denotes the Hausdorff distance between two sets, i.e.

$$d_{HD}(A, B) := \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}.$$

Our formulation of 1.10 is slightly more general than the ones given in [41]. The original version is stated with a different volume scaling and under the assumptions of $1 < p \leqslant n$. Anyway, since in a finite measure space the L^p-norm controls the L^q-norm for $q < p$ the corollary can be easily generalized to every exponent $1 < p$. At this point we just need to show how the convexity implies the C^1-estimate. This is granted by the following lemma.

Lemma 1.11. *Let Σ be a convex, closed radially parametrized hypersurface in $\mathbb{R}^{n+1}$. The following inequality holds:*

$$\|\nabla f\|^2_{C^0(\mathbb{S}^n)} \leqslant 2\,\mathrm{osc}(f,\, \mathbb{S}^n)\Big(1 + \|\nabla f\|^2_{C^0(\mathbb{S}^n)}\Big).$$

In particular if $\mathrm{osc}(f,\, \mathbb{S}^n) < \frac{1}{2}$ *we find the estimate:*

$$\|\nabla f\|_{C^0(\mathbb{S}^n)} \leqslant \sqrt{\frac{\mathrm{osc}(f,\, \mathbb{S}^n)}{1 - 2\,\mathrm{osc}(f,\, \mathbb{S}^n)}}. \tag{1.19}$$

Proof. It is known (e.g. [41, Prop. 3.2]) that a closed, connected hypersurface is the boundary of a convex, open set iff its second fundamental form satisfies the inequality $0 \leqslant A$. By Lemma 1.9 we obtain that Σ is convex iff f satisfies the inequality

$$\nabla^2 f \leqslant \sigma + \nabla f \otimes \nabla f, \tag{1.20}$$

in the sense of quadratic forms. Consider a point $x_0 \in \mathbb{S}^n$, and a unit vector ξ in $T_x\mathbb{S}^n$ which satisfies $\langle\, \nabla f(x_0),\, \xi\,\rangle = -\|\nabla f\|_{C^0}$. Setting $x_\tau = \exp_{x_0}(\tau\xi)$ the lemma follows by the simple equality

$$f(x_\tau) - f(x_0) = \langle\, \nabla f(x_0),\, \tau\xi\,\rangle + \int_0^1 t \int_0^1 \nabla^2 f(\gamma(st))[\dot\gamma(st),\, \dot\gamma(st)]\, ds\, dt,$$

where $\gamma\colon [0,\, 1] \longrightarrow \mathbb{S}^n$ is the geodesic which connects x_0 and x_τ. Applying (1.20) we find

$$\begin{aligned} f(x_\tau) - f(x_0) &\leqslant (\nabla f(x_0),\, \tau\xi) + \frac{\tau^2}{2}\Big(1 + \|\nabla f\|^2_{C^0}\Big) \\ &= -\tau\|\nabla f\|_{C^0} + \frac{\tau^2}{2}\Big(1 + \|\nabla f\|^2_{C^0}\Big). \end{aligned}$$

Finally we obtain the inequality

$$\|\nabla f\|_{C^0} \leqslant \frac{\mathrm{osc}(f,\, \mathbb{S}^n)}{\tau} + \frac{\tau}{2}\Big(1 + \|\nabla f\|^2_{C^0}\Big) \text{ for every } 0 < \tau.$$

Choosing $\tau = \sqrt{2\ \mathrm{osc}(f,\, \mathbb{S}^n)\,(1 + \|\nabla f\|_{C^0})}$ we obtain the result. □

From this result we easily infer Proposition 1.4.

1.2 PROOF OF PROPOSITION 1.5

In this section we prove Proposition 1.5. The result requires the following, preliminary proposition.

Proposition 1.12. *Let $2 \leqslant n$, and $1 < p < \infty$ be given, and let Σ be a convex, closed hypersurface in $\mathbb{R}^{n+1}$. Assume that Σ satisfies* (1.1) *and* (1.2), *with δ chosen such that estimate* (1.7) *holds with* $0 < \varepsilon < 1$. *There exists a constant* $C = C(n,\, p)$ *such that*

$$\|H - \overline{H}\|_{L^p(\Sigma)} \leqslant C\|\mathring{A}\|_{L^p(\Sigma)}. \tag{1.21}$$

Proposition 1.12 is a slight generalization of Theorem 2.3 in [41]. The latter one presents the same result, but assuming $1 < p \leqslant n$. However, as the author points out in [41, Remark 2.6], an avid reader of his thesis could recover exactly our proposition 1.12 by generalizing some of his arguments. We will prove it, following a similar approach. Proposition 1.12 follows trivially by the following two lemmas.

Lemma 1.13. *Let Σ be a radially parametrized hypersurface. Consider the second fundamental form as a $(1, 1)$-tensor $A^i_j = g^{ik} A_{jk}$, and let H be its mean curvature $H = \sum_i A^i_i$. Then we have the equality*

$$\nabla H = \frac{1}{n-1} \operatorname{div}_\sigma \mathring{A} + \frac{n}{n-1} \mathring{A}[\nabla f] \tag{1.22}$$

Lemma 1.14. *Let $2 \leqslant n$ and $p \in (1, \infty)$ be given. Let also $u \in C^\infty(\mathbb{S}^n)$, $\mathfrak{f} \in \Gamma(T^*\mathbb{S}^n \otimes T\mathbb{S}^n)$, $\mathfrak{h} \in \Gamma(T\mathbb{S}^n)$ be given so that the following equation holds:*

$$\nabla u = \operatorname{div} \mathfrak{f} + \mathfrak{f}[\mathfrak{h}] \tag{1.23}$$

There exists $\lambda_0 \in \mathbb{R}$ such that the following estimate holds:

$$\|u - \lambda_0\|_{L^p(\mathbb{S}^n)} \leqslant C(n, p)\Big(1 + \|\mathfrak{h}\|_{C^0(\mathbb{S}^n)}\Big)\|\mathfrak{f}\|_{L^p(\mathbb{S}^n)}. \tag{1.24}$$

Both lemmas have interest in their own, and will be used also in later passages of the thesis. Their proofs consist mainly in technicalities and computations, and will therefore be postponed in the appendix (see A.2).

From Lemmas 1.13 and 1.14 Proposition 1.12 follows immediately. Indeed, we find a constant $0 < C = C(n, p)$ and a number $\lambda_0 \in \mathbb{R}$ such that

$$\|H - \lambda_0\|_{L^p(\mathbb{S}^n)} \leqslant C(1 + \|\nabla f\|_{C^0})\|\mathring{A}\|_{L^p(\mathbb{S}^n)} \leqslant C\|\mathring{A}\|_{L^p(\mathbb{S}^n)},$$

where we have eliminated the dependence on $\|\nabla f\|_0$ simply by applying Proposition 1.4 and choosing $0 < \varepsilon < 1$. In this case we have to stress again how the considered quantities of H, and A are meant to be the pull-backs in the sphere $\mathbb{S}^n$. This justifies the use of the space $L^p(\mathbb{S}^n)$ rather than $L^p(\Sigma)$. Indeed, we are taking the integrals in the estimate above w.r.t. the measure dV_σ of the sphere, and not w.r.t. the measure dV_g associated to Σ. This is however not a problem, since condition (1.7) and formula (1.16) show us how to get a control: since

$$dV_g = e^{nf}\sqrt{1 + |\nabla f|^2}\, dV_\sigma$$

we obtain

$$e^{-n}\, dV_\sigma \leqslant dV_g \leqslant \sqrt{2} e^n\, dV_\sigma.$$

This shows that the measures are equivalent and with equivalence constants depending only on n. Thus we can substitute the measures without problem and obtain

$$\|H - \lambda_0\|_{L^p(\Sigma)} \leqslant C\|\mathring{A}\|_{L^p(\Sigma)},$$

Substituting λ_0 with $\overline{H}$ is straightforward. Indeed,

$$\begin{aligned}\|H-\overline{H}\|_{L^p(\Sigma)} &= \|H-\lambda_0+\lambda_0-\overline{H}\|_{L^p(\Sigma)} \leqslant \|H-\lambda_0\|_{L^p(\Sigma)} + \mathrm{Vol}(\Sigma)^{\frac{1}{p}}\left|\overline{H}-\lambda_0\right| \\ &\leqslant \|H-\lambda_0\|_{L^p(\Sigma)} + \mathrm{Vol}(\Sigma)^{\frac{1}{p}}\left|\fint_\Sigma H-\lambda_0\, dV_g\right| \\ &\leqslant \|H-\lambda_0\|_{L^p(\Sigma)} + \mathrm{Vol}(\Sigma)^{-1+\frac{1}{p}}\int_\Sigma |H-\lambda_0|\, dV_g \\ &\leqslant 2\|H-\lambda_0\|_{L^p(\Sigma)} \leqslant C\|\mathring{A}\|_{L^p(\Sigma)}.\end{aligned}$$

Now we have all the ingredients to perform our linearisation. Let us prove the following proposition.

Proposition 1.15. *Let Σ be a smooth, closed hypersurface in $\mathbb{R}^{n+1}$. Assume that Σ satisfies inequality* (1.21) *and admits a radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma$ as in* (1.4)*, with the logarithmic radius f satisfying estimate* (1.7) *for some $0 < \varepsilon < 1$. Then the following estimate is true:*

$$\|\Delta f + nf\|_{L^p(\mathbb{S}^n)} \leqslant C(n, p)\big(\|\mathring{A}\|_{L^p} + \varepsilon\|f\|_{W^{2,p}}\big). \tag{1.25}$$

Proof. We start by writing the explicit formula of the mean curvature. By formula (1.15) we obtain

$$H = -\operatorname{div}_\sigma\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)e^{-f} + \frac{ne^{-f}}{\sqrt{1+|\nabla f|^2}} = \frac{e^{-f}}{\sqrt{1+|\nabla f|^2}}\left(n-\Delta f + \frac{\nabla^2 f[\nabla f, \nabla f]}{1+|\nabla f|^2}\right). \tag{1.26}$$

Now we notice two simple estimates: first, we write

$$\frac{1}{\sqrt{1+|\nabla f|^2}} - 1 = \int_0^1 \frac{d}{dt}\frac{1}{\sqrt{1+t^2|\nabla f|^2}}\, dt = |\nabla f|^2\int_0^1 \frac{t}{\sqrt{(1+t^2|\nabla f|^2)^3}}$$

This gives us the pointwise inequality:

$$\left|\frac{1}{\sqrt{1+|\nabla f|^2}} - 1\right| \leqslant 2\varepsilon\, |\nabla f| \tag{1.27}$$

We use the same idea for simplifying the exponential: by standard calculus, we find

$$e^f = 1 + \int_0^1 \frac{d}{dt}e^{tf}\, dt = 1 + f\int_0^1 e^{tf}\, dt$$

and for $0 < \varepsilon < 1$ we obtain

$$\left|e^f - 1 - f\right| \leqslant 2\varepsilon\, |f|. \tag{1.28}$$

We use inequalities (1.27) and (1.28) to obtain the following estimate for the mean curvature:

$$\|H + \Delta f + nf\|_{L^p(\mathbb{S}^n)} \leqslant C(n, p)\varepsilon\, \|f\|_{W^{2,p}(\mathbb{S}^n)}. \tag{1.29}$$

We show now how to linearise the quantity $\overline{H}$. More precisely, we show

$$\left|\overline{H} - n\right| \leqslant C(n, p)\varepsilon \, \|f\|_{W^{2,p}(\mathbb{S}^n)}. \tag{1.30}$$

In order to achieve (1.30), we notice that the density of the measure dV_g w.r.t. the measure dV_σ satisfies the estimate:

$$\left| e^{nf}\sqrt{1+|\nabla f|^2} - 1 - nf \right| \leqslant C(n)\varepsilon \left(|f| + |\nabla f|\right).$$

We patch this latter estimate and (1.29) together and obtain

$$\left|\overline{H} - n + n(n-1) \fint_{\mathbb{S}^n} f \, dV_\sigma \right| \leqslant C(n, p)\varepsilon \, \|f\|_{W^{2,p}(\mathbb{S}^n)}. \tag{1.31}$$

We conclude by showing that the average of f is actually negligible, i.e. satisfies

$$\left|\overline{f}\right| \leqslant C(n, p)\varepsilon \, \|f\|_{W^{2,p}(\mathbb{S}^n)}.$$

Indeed, since Σ satisfies (1.1), then by the volume formula 1.16 we obtain the equality

$$\fint_{\mathbb{S}^n} e^{nf}\sqrt{1+|\nabla f|^2} \, dV_\sigma = 1.$$

With the previous approximations, we find

$$\left| \underbrace{\fint_{\mathbb{S}^n} e^{nf}\sqrt{1+|\nabla f|^2} \, dV_\sigma - 1}_{=0} - n \fint_{\mathbb{S}^n} f dV_\sigma \right| \leqslant C(n, p) \, \varepsilon \, \|f\|_{W^{2,p}(\mathbb{S}^n)}.$$

This means

$$\left| \fint_{\mathbb{S}^n} f dV_\sigma \right| \leqslant C(n, p)\varepsilon \, \|f\|_{W^{1,p}(\mathbb{S}^n)} \leqslant C(n, p)\varepsilon \, \|f\|_{W^{2,p}(\mathbb{S}^n)}.$$

All these estimates together give us the inequality

$$\|\Delta f + nf\|_{L^p(\mathbb{S}^n)} \leqslant C(n, p)\Big(\|H - \overline{H}\|_{L^p(\Sigma)} + \varepsilon \|f\|_{W^{2,p}(\mathbb{S}^n)} \Big),$$

and we conclude thanks to Proposition 1.12. □

In order to conclude the proof of Proposition 1.5, we just have to prove the estimate

$$\|f - \langle \nu_f, \cdot \rangle\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p)\|\Delta f + nf\|_{L^p(\Sigma)}.$$

This follows quite easily by the following characterization of the kernel of $\Delta + n$ (see [50, Chap.4] for a proof).

$$\ker \Delta + n := \{ \varphi \colon \mathbb{S}^n \longrightarrow \mathbb{R} \mid \varphi_\nu(x) := \langle \nu, x \rangle \}. \tag{1.32}$$

From this characterization and the concept of quotient norm (see [6, Prop. 11.8]), we obtain

$$\inf_{\nu\in\mathbb{R}^{n+1}} \|f - \langle \nu, \cdot \rangle\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p)\|\Delta f + nf\|_{L^p(\Sigma)}. \tag{1.33}$$

Now we can conclude. We write $\varphi_\nu(\cdot) := \langle \nu, \cdot \rangle$, and notice the integral equality:

$$\fint_{\mathbb{S}^n} x_i^2 = \frac{1}{n+1}, \text{ for every } i = 1, \dots n+1.$$

From this we deduce that the set

$$\left\{\varphi_i \colon \mathbb{S}^n \longrightarrow \mathbb{R} \,|\, \varphi_i := \langle\, \tilde{e}_i, x \,\rangle\right\}_{i=1}^{n+1}, \ \tilde{e}_i := \frac{e_i}{\sqrt{(n+1)\,\mathrm{Vol}_n(\mathbb{S}^n)}},$$

is a L^2-orthonormal frame for the vector space $\ker \Delta + n \subset L^2(\mathbb{S}^n)$ (here $\{\, e_i \,\}_{i=1}^{n+1}$ is the standard frame from $\mathbb{R}^{n+1}$). Now we write

$$\varphi_\nu = \sum_{i=1}^{n+1} \nu_i\, \varphi_i = \sum_{i=1}^{n+1} \langle\, \nu, \varphi_i \,\rangle_{L^2(\mathbb{S}^n)}\, \varphi_i, \text{ and } \varphi_{\nu_f} = \sum_{i=1}^{n+1} \langle\, f, \varphi_i \,\rangle_{L^2(\mathbb{S}^n)}\, \varphi_i.$$

Then we find, for every $c \in \mathbb{R}^{n+1}$:

$$\begin{aligned}
\|f - \varphi_{\nu_f}\|_{W^{2,p}(\mathbb{S}^n)} &\leqslant \|f - \varphi_c\|_{W^{2,p}(\mathbb{S}^n)} + \|\varphi_c - \varphi_{\nu_f}\|_{W^{2,p}(\mathbb{S}^n)} \\
&\leqslant \|f - \varphi_c\|_{W^{2,p}(\mathbb{S}^n)} + \sum_{i=1}^{n+1} \left\| \langle f - \varphi_c, \varphi_i \rangle_{L^2(\mathbb{S}^n)} \varphi_i \right\|_{W^{2,p}(\mathbb{S}^n)} \\
&\leqslant \|f - \varphi_c\|_{W^{2,p}(\mathbb{S}^n)} + \|f - \varphi_c\|_{L^2(\mathbb{S}^n)} \sum_{i=1}^{n+1} \|\varphi_i\|_{L^2(\mathbb{S}^n)} \|\varphi_i\|_{W^{2,p}(\mathbb{S}^n)} \\
&\leqslant \|f - \varphi_c\|_{W^{2,p}(\mathbb{S}^n)} + C(n, p)\|f - \varphi_c\|_{L^2(\mathbb{S}^n)} \\
&\leqslant C(n, p)\|f - \varphi_c\|_{W^{2,p}(\mathbb{S}^n)}.
\end{aligned}$$

By taking the inf over c and applying (1.33) we conclude the proof of Proposition 1.5.

1.3 PROOF OF PROPOSITION 1.6

Insofar we have proved that for every $0 < \varepsilon < 1$ there exist a $0 < \delta_0(n, p, \varepsilon)$ with the following property. If Σ is a closed, convex hypersurface in $\mathbb{R}^{n+1}$ satisfying (1.1) and (1.2) with $\delta \leqslant \delta_0$, then there exists a vector $c = c(\Sigma)$ such that the anisotropic radius associated to the radial parametrization $\psi \colon \mathbb{S}^n \longrightarrow \Sigma - c$ satisfies estimates (1.9) and (1.10), namely

$$\begin{aligned}
\|f\|_{C^1(\mathbb{S}^n)} &\leqslant \varepsilon, \\
\|f - \langle \nu_f, \cdot \rangle\|_{W^{2,p}(\mathbb{S}^n)} &\leqslant C(n, p)\Big(\|\mathring{A}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)}\Big).
\end{aligned}$$

We end the first chapter by finding the optimal vector. First of all, we assume $c = 0$, so the radial parametrization satisfying the estimates above is exactly $\psi\colon \mathbb{S}^n \longrightarrow \Sigma$. Now we define the barycenter of Σ: we set

$$b(\Sigma) := \fint_{\Sigma} z \, dV_g(z). \tag{1.34}$$

The convexity of Σ grants that $b(\Sigma)$ belongs to the bounded, convex, open set U of which Σ is boundary. We firstly consider the equality:

$$b(\Sigma) = \fint_{\Sigma} z \, e^{(n+1)f(z)} \sqrt{1 + |\nabla f(z)|^2} \, dV_\sigma(z).$$

The symmetries of the sphere easily imply $b(\mathbb{S}^n) = 0$. This fact, combined with the expression of $b(\Sigma)$ and estimates (1.27), (1.28) gives us the estimate:

$$\left| b(\Sigma) - b(\mathbb{S}^n) - (n+1) \fint_{\mathbb{S}^n} z \, f(z) \, dV_\sigma(z) \right| \leqslant C(n)\varepsilon \|f\|_{W^{1,1}(\mathbb{S}^n)} \leqslant C(n,p)\varepsilon \|f\|_{W^{2,p}(\mathbb{S}^n)}. \tag{1.35}$$

Estimate (1.35) clearly implies

$$|b(\Sigma)| \leqslant C(n,p)\varepsilon. \tag{1.36}$$

Therefore, the radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - b(\Sigma)$ satisfies all the hypothesis of Proposition 1.6. Indeed, inequality (1.36) still ensures that the logarithmic radius associated to ψ is C^0-close to 0 with a possibly worsen ε. Then we obtain Lemma 1.11, and via domino effect also Propositions 1.4, and 1.5. For the parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - b(\Sigma)$ we easily have from (1.35):

$$\left| \fint_{\mathbb{S}^n} z \, f(z) \, dV_\sigma(z) \right| \leqslant C(n,p)\varepsilon \|f\|_{W^{2,p}(\mathbb{S}^n)},$$

which is exactly (1.10), and therefore we complete the proof of Proposition 1.6.

Theorem 1.2 follows now as in Remark 1.8.

2 THE CASE FOR ALMOST EINSTEIN HYPERSURFACES

In this chapter we prove the following two theorems.

Theorem 2.1. *Let $3 \leqslant n$, $1 < p < \infty$ and $0 < \Lambda$ be given. There exists a $0 < \delta_0 = \delta_0(n, p, \Lambda)$ with the following property. If $\Sigma = \partial U$ is a closed, convex hypersurface in $\mathbb{R}^{n+1}$ satisfying*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n), \tag{2.1}$$

$$0 \leqslant A \leqslant \Lambda g, \tag{2.2}$$

$$\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{2.3}$$

then there exists a vector $c = c(\Sigma) \in \mathbb{R}^{n+1}$ such that $0 \in U - c$ and the radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ defined in (1.4) satisfies

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)}. \tag{2.4}$$

Theorem 2.2. *Let $3 \leqslant n$, $1 < p < \infty$ and $0 < \Lambda$ be given. There exists a $0 < \delta_1 = \delta_1(n, p, \Lambda)$ with the following property. If $\Sigma = \partial U$ is a closed, convex hypersurface in $\mathbb{R}^{n+1}$ satisfying condition* (2.1) *and*

$$\Lambda g \leqslant A, \tag{2.5}$$

$$\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} \leqslant \delta_1, \tag{2.6}$$

then there exists a vector $c = c(\Sigma) \in \mathbb{R}^{n+1}$ such that $0 \in U - c$ and the radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ defined in (1.4) satisfies

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)}. \tag{2.7}$$

Here we notice immediately differences with Theorem 1.1. First of all, we have a new constraint. The reason why conditions (2.2) and (2.5) appear is related to the intrinsic non-ellipticity of the Ricci tensor. As we shall see throughout the computations, the equation concerning the approximated Ricci operator is fully non-linear. This has forced us to find ways to bypass the problem and reduce ourselves to an application of Theorem 1.1. It is not clear how much conditions (2.2) and (2.5) are artificial. In order to prepare the road for the non-convex case in Chapter 4 and to let the reader understand that, we shall stress where and when these conditions are assumed.

Another important difference concerns the dimension in which the theorems hold. The results of this chapter are true only when the dimension of the hypersurface is *strictly* greater than 2. This condition does not appear just as result of our strategy: it is rather related to intrinsic geometric properties that are satisfied by the Ricci tensor. Indeed, in dimension 2 the tensor is a 1×1 matrix, i.e. a scalar quantity (which coincides with the scalar curvature), and therefore all surfaces satisfy $\mathring{\mathrm{Ric}} = \mathrm{Scal} - \mathrm{Scal} = 0$.

We also like to remark the following corollary.

Corollary 2.3. *Under the assumptions of Theorem 2.1 or 2.2 the following estimate holds:*

$$\|\psi^* g - \sigma\|_{W^{1,p}(S^n)} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)}. \tag{2.8}$$

Corollary 2.3 is particularly interesting. Indeed, in the theory of convergence for Riemannian manifolds there are many results about the $W^{k,p}$-closeness of a metric g to a constant curvature one (cf.[8], [42], [43, Ch. 10]) . However, these results are all of qualitative nature. Corollary 2.3 provides instead a quantitative estimate, and to our knowledge it is the first result of this type.

The theorems are the quantitative version of the following rigidity theorem.

Theorem 2.4 ([16], [46], [52]). *Let Σ be a closed, connected hypersurface in $\mathbb{R}^{n+1}$ such that*

$$\mathring{\mathrm{Ric}} = 0$$

at every point. Then Σ is a round sphere.

Again, we use our scheme to prove Theorem 2.1 and divide it into four main steps.

Proposition 2.5. *Let $2 \leqslant n$ be given, ad let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying* (2.1). *Assume Σ satisfies one of two following hypothesis.*

a) *Σ is convex, and $\|A\|_{L^p(\Sigma)} \leqslant c_0$ for some $1 < p < \infty$.*

b) *$\|A\|_{L^p(\Sigma)} \leqslant c_0$ for some $n < p < \infty$ and $0 < \overline{\mathrm{Scal}}$.*

Then the following inequality holds.

$$\left\| \mathrm{Riem} - \frac{\overline{\mathrm{Scal}}}{2n(n-1)} g \owedge g \right\|_{L^p(\Sigma)} \leqslant C(n, p, c_0)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)}. \tag{2.9}$$

Proposition 2.6. *Let $3 \leqslant n$, $1 < p < \infty$, $0 < \Lambda$ be given, and let $\Sigma = \partial U$ be a closed, convex hypersurface in $\mathbb{R}^{n+1}$. For every $0 < \varepsilon < 1$ there exists a $0 < \delta_0 = \delta_0(n, p, \Lambda, \varepsilon)$ with the following property.*

If Σ satisfies condition (2.1), (2.2) *and* (2.6) *then there exists a vector $c = c(\Sigma) \in U$ such that the radial parametrization $\psi \colon S^n \longrightarrow \Sigma - c$ satisfies*

$$\|f\|_{C^1(S^n)} \leqslant \varepsilon, \quad \|f\|_{C^2(S^n)} \leqslant c(n, \Lambda). \tag{2.10}$$

Proposition 2.7. *Let $3 \leqslant n$, $1 < p < \infty$ be given, and let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$. Let $0 < \varepsilon < 1$, $0 < \delta_0$ and $c = c(\Sigma)$ be chosen such that the logarithmic radius f satisfies estimate* (2.10) *given by Proposition 2.6. Then the following estimate holds:*

$$\|f - \langle \nu_f, \cdot \rangle\|_{W^{2,p}(S^n)} \leqslant C(n, p)\Big(\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(S^n)}\Big) \tag{2.11}$$

where we have set

$$\nu_f := (n+1) \fint_{S^n} z\, f(z)\, dV_\sigma(z)$$

Proposition 2.8. *Under the same hypothesis of Proposition 2.6 and the same notations of Proposition 2.7, we can find a vector* $\tilde{c} \in \mathbb{R}^{n+1}$ *such that the associated radial parametrization* $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - \tilde{c}$ *satisfies the conditions*

$$\|f\|_{C^1(\mathbb{S}^n)} \leqslant C(n, p, \Lambda)\varepsilon, \tag{2.12}$$

$$|\nu_f| \leqslant C(n, p, \Lambda)\varepsilon\|f\|_{W^{1,p}(\mathbb{S}^n)}. \tag{2.13}$$

Remark 2.9. Differently from Chapter 1 we have split the proof into four, rather than three sections. This happens because in the context of Chapter 1 the proof of the "Step 0" of our scheme had been proved by Perez in [41]. In the present case we have to prove a different version of it. Proposition 2.8 is identical to Proposition 1.6, with just the constant being worsened by Λ, and thus we do not report it. In the last section we give instead a proof of theorem 2.2, which shows how condition (2.5) allows us to reduce ourselves to Theorem 1.2.

Remark 2.10. Again, the combination of Propositions 2.6, 2.7 and 2.8 triggers Theorem 2.1. The proof is analogous as the one done in Remark 1.8 for Theorem 1.2, and we do not report it.

2.1 PROOF OF PROPOSITION 2.5

The proof of this Proposition 2.5 relies on a well known consequence of the second Bianchi identity (see [19, Cor. 3.135]).

Lemma 2.11. *Let* M *be a* n*-dimensional manifold, with* $3 \leqslant n$*. Then the following equation holds.*

$$\nabla R = \frac{1}{2}\operatorname{div}\operatorname{Ric}. \tag{2.14}$$

From this equation one can derive the following oscillation lemma, whose L^2-version has been proved under weaker assumptions in [11].

Lemma 2.12. *Let* Σ *be a closed, convex hypersurface in* $\mathbb{R}^{n+1}$*. Assume* Σ *satisfies condition* a) *or condition* b) *as in Proposition* (2.5)*. In the latter one, the positivity assumption of* $\overline{\mathrm{Scal}}$ *is not required. Then the following inequality holds.*

$$\left\|\mathrm{Scal} - \overline{\mathrm{Scal}}\right\|_{L^p(\Sigma)} \leqslant C(n, p, c_0)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)}. \tag{2.15}$$

We show the proof of Lemma 2.12 in Appendix A.2. From Lemma 2.12 we derive Proposition 2.5. In order to achieve such result, we recall the Gauss equation for hypersurfaces in a Euclidean space (see [19, Thm 5.5]): Let Σ be a hypersurface in $\mathbb{R}^{n+1}$. Then the following equation holds:

$$\mathrm{Riem}_{ijkl} = \frac{1}{2}(A \owedge A)_{ijkl} = A_{ik}A_{jl} - A_{il}A_{jk}. \tag{2.16}$$

Contracting the indices in (2.16) we obtain

$$\mathrm{Ric}_{ij} = HA_{ij} - A_i^k A_{kj}. \tag{2.17}$$

Since the second fundamental form is a symmetric tensor, we know by the spectral theorem that it is diagonalizable. Let $\lambda_1, \dots \lambda_n$ be its eigenvalues. Our idea to use equation (2.16) in order to interpret (2.9) as a polynomial inequality involving the eigenvalues of A. Let us define indeed the polynomials

$$p(\lambda) = \frac{1}{4}|A(\lambda) \owedge A(\lambda) - \kappa\delta \owedge \delta|^2 = \sum_{i\neq j} \big(\lambda_i\lambda_j - \kappa\big)^2, \tag{2.18}$$

$$q(\lambda) = \big|H(\lambda)A(\lambda) - A(\lambda)^2 - (n-1)\kappa\delta\big|^2 = \sum_i \left(\lambda_i \left(\sum_{i\neq j} \lambda_j\right) - (n-1)\kappa\right)^2. \tag{2.19}$$

Here $\kappa \in \mathbb{R}$, and in general we choose it so that $n(n-1)\kappa = \overline{\mathrm{Scal}}$. Using this notation we can let Proposition 2.5 easily follow from the following lemma.

Lemma 2.13. *Let $0 < \kappa$ be given. Then there exist constants c_1, c_2, depending on n such that*

$$c_1 \leqslant \frac{p(\lambda)}{q(\lambda)} \leqslant c_2, \textit{ for any } \lambda \in \mathbb{R}. \tag{2.20}$$

From the lemma we easily conclude by integrating the inequality for the eigenvalues of A. Indeed, if the mean of the scalar curvature $\overline{\mathrm{Scal}}$ is positive, then from Lemmas 2.12 and 2.13 we obtain:

$$\begin{aligned}\left\|\mathrm{Riem} - \frac{\overline{\mathrm{Scal}}}{2n(n-1)} g \owedge g\right\|_p = \left\|\mathrm{Riem} - \frac{\kappa}{2} g \owedge g\right\|_p &\leqslant C(n, p)\|\mathrm{Ric} - (n-1)\kappa g\|_p \\ &\leqslant C(n, p, c_0)\|\mathring{\mathrm{Ric}}\|_p.\end{aligned}$$

The positivity of the quantity $\overline{\mathrm{Scal}}$ is easily recovered: it is indeed, straightforward to prove that closed, convex and smooth manifolds have positive mean of the scalar curvature. This quantity is trivially non-negative since we have the formula

$$\mathrm{Scal} = \sum_{i\neq j} \lambda_i\lambda_j,$$

and all the λ_i are non-negative by convexity. Let us show that the quantity $\overline{\mathrm{Scal}}$ is actually positive. We consider the function

$$h \colon p \in \Sigma \longmapsto |p|^2.$$

Let p_0 be a maximum for h, and $\varphi_0 \colon \mathbb{B}^n_R \longrightarrow \Sigma$ be a graph parametrisation around p_0, i.e. $\varphi_0(0) = p_0$. Since p_0 is the maximum of h, we notice that φ_0 satisfies

$$|\varphi(0)|^2 = |p_0|^2 = \max_{z\in\mathbb{B}^n_{\rho_0}} |\varphi(z)|^2.$$

Deriving twice, we obtain the following equalities holding in 0:

$$\underbrace{\langle\, \partial_i\varphi_0,\ \varphi_0(0)\,\rangle = 0}_{\Rightarrow\langle\, p_0\,\rangle^\perp = T_{p_0}\Sigma}, \quad \partial^2\varphi(0) \leqslant 0 \Rightarrow \underbrace{\langle\, \partial^2_{ij}\varphi_0,\ \varphi_0(0)\,\rangle}_{=-|p_0|^{-1}A_{ij}} + \underbrace{\langle\, \partial_i\varphi_0,\ \partial_j\varphi_0\,\rangle}_{=g_{ij}} \leqslant 0,$$

from which we obtain the equality

$$A|_{p_0} \geqslant \frac{1}{|p_0|} g.$$

Thus the function $\mathrm{Scal} = \sum_{i \neq j} \lambda_i \lambda_j$ is non-negative and positive in a neighbourhood of p_0, hence $\overline{\mathrm{Scal}} > 0$.

Let us prove the lemma and conclude.

Proof of Lemma 2.13. We first show need to show that the polynomials p and q defined by (2.18) and (2.19) have the same zeros. Let $Z(p) := \{ p = 0 \}$ and $Z(q) := \{ q = 0 \}$ be the zero sets of p, q, respectively. We claim that:

$$Z(p) = Z(q) = \left\{ \sqrt{\kappa} e, -\sqrt{\kappa} e \right\}, \text{ where } e := \sum_{i=1}^{n} e_i. \tag{2.21}$$

We split the proof of Lemma 2.13 into four main parts. In the first two parts we prove Claim (2.21) for p and q respectively. In the third part we study the behaviour of the ratio p/q as $|\lambda|$ approaches ∞. In the fourth part we study the behaviour of p/r as $\lambda \to \pm\sqrt{\kappa} e$. From this analysis the lemma will easily follow.

Zeros of p Let $\lambda = (\lambda_1, \ldots \lambda_n)$ be given so that $p(\lambda) = 0$. Since p is a sum of squares, we get:

$$\lambda_i \lambda_j = \kappa, \text{ for every } i \neq j. \tag{2.22}$$

Since $0 < \kappa$ we also know that $\lambda_i \neq 0$ for every i. Then, for every $i \neq j \neq k$ we immediately find:

$$\lambda_i \lambda_j = \lambda_j \lambda_k \Rightarrow \lambda_j = \lambda_k =: t,$$

from which we deduce $\lambda = te$ for some $t \neq 0$. From (2.22) we immediately deduce $t^2 = \kappa$ and the thesis.

Zeros of q Let $\lambda = (\lambda_1, \ldots \lambda_n)$ be given so that $q(\lambda) = 0$. Since q is a sum of squares, we infer the following system:

$$H\lambda_i - \lambda_i^2 = (n-1)\kappa, \text{ for every } i, \tag{2.23}$$

where we have set

$$H := \sum_{i=1}^{n} \lambda_i = \langle \lambda, e \rangle.$$

Notice that from (2.23) we have that $\lambda_i \neq 0$ $\forall i$. Again, we claim that $\lambda_i = \lambda_j$ for every i, j. If the claim is true, system (2.23) for $\lambda = te$ is reduced to

$$(n-1)t^2 = (n-1)\kappa,$$

and this proves our claim. Let us assume by contradiction that there exist two indices i, j such that $\lambda_i \neq \lambda_j$. From (2.23) we infer

$$H\lambda_i - \lambda_i^2 = H\lambda_j - \lambda_j^2 \Rightarrow H(\lambda_i - \lambda_j) = \lambda_i^2 - \lambda_j^2 \Rightarrow H = \lambda_i + \lambda_j. \tag{2.24}$$

Substituting (2.24) in (2.23), we obtain

$$\lambda_i\lambda_j = (n-1)\kappa, \tag{2.25}$$

$$(\lambda_i + \lambda_j)\lambda_h - \lambda_h^2 = (n-1)\kappa, \text{ for every } h \neq i, j. \tag{2.26}$$

Assume there exists $\lambda_h \neq \lambda_i$. From equalities (2.25) and (2.26) we obtain:

$$\lambda_i\lambda_j = (\lambda_i + \lambda_j)\lambda_h - \lambda_h^2 \Rightarrow \lambda_j(\lambda_i - \lambda_h) = \lambda_h(\lambda_i - \lambda_h),$$

from which we easily infer $\lambda_h = \lambda_j$. Therefore the coefficients $\lambda_1, \ldots \lambda_n$ of the point λ can take at most two different values. Call them a and b, and assume a appears k times and b appears $n-k$ times in the coordinates of λ. From equality (2.24) we have

$$(k-1)a + (n-k-1)b = 0.$$

If both $k-1$ and $n-k-1$ are positive, then a and b must have different sign, and equation (2.25) is violated. If one of them is 0, say $k-1=0$, then we must have $b=0$, but again equation (2.25) would be violated. Hence all the values are equal, and we easily find the thesis. Notice how the estimate fails when $n=2$. In this case, equality (2.24) is not useful, and the polynomials p and q degenerate to

$$p(\lambda) = q(\lambda) = (\lambda_1\lambda_2 - \kappa)^2,$$

and therefore $Z(p) = Z(q) = \{ (x, y) \in \mathbb{R}^2 \mid xy = \kappa \}$.

Boundedness at infinity Now we show that the ratio $p(\lambda)/q(\lambda)$ is bounded from above and below when $|\lambda|$ attains large values. A simple computation shows:

$$\liminf_{|\lambda|\to\infty} \frac{p(\lambda)}{q(\lambda)} = \inf_{\lambda\in\mathbb{S}^n} \frac{\sum_{i\neq j}\lambda_i^2\lambda_j^2}{\sum_i \lambda_i^2\left(\sum_{i\neq j}\lambda_j\right)^2}, \qquad \limsup_{|\lambda|\to\infty} \frac{p(\lambda)}{q(\lambda)} = \sup_{\lambda\in\mathbb{S}^n} \frac{\sum_{i\neq j}\lambda_i^2\lambda_j^2}{\sum_i \lambda_i^2\left(\sum_{i\neq j}\lambda_j\right)^2}.$$

Note that this case represents the study of the ratio $p(\lambda)/q(\lambda)$ in the case $\kappa = 0$. Let us do the computation. Firstly, we claim that in this case the zero sets in the sphere of p and q are finite and satisfy

$$Z(p) = Z(q) = \{ \pm e_1, \cdots \pm e_n \}.$$

The claim is straightforward for p. For q, let us consider a point $\lambda \in \mathbb{S}^n$ so that $q(\lambda) = 0$. Keeping the notation used above, we have the equality:

$$\lambda_i^2(H - \lambda_i)^2 = 0, \text{ for every } i. \tag{2.27}$$

Since $\lambda \in \mathbb{S}^n$, then there must exist an index i such that $\lambda_i \neq 0$. Therefore $H = \lambda_i$ must hold. If $\lambda_j = 0$ for all indices $j \neq i$, then necessarily $\lambda = \lambda_i e_i$ and $\lambda_i = \pm 1$, as claimed. If $\lambda_j \neq 0$ for some j, then the equality $H = \lambda_j$ must hold and hence $\lambda_j = \lambda_i$. We immediately deduce that the set $\{\lambda_1, \dots \lambda_n\}$ can take only the values 0 and t for some $t \neq 0$, and not all λ_i can be 0 because $\lambda \in S^n$. Let us assume w.l.o.g. that $\lambda_1 = \cdots = \lambda_k = t$ and $\lambda_{k+1} = \cdots = \lambda_n = 0$. From this we can write the equation (2.27) as

$$k(k-1)t^4 = 0,$$

from which we infer $k = 1$, and thus the claim.

We show now how the ratio p/q is bounded near the zeros of p and q. By symmetry, it is enough to consider the limit for $\lambda \to e_1$. Now we write $\mu := \lambda - e_1$, so that we can study the limit as $\mu \to 0$. Denoting $\tilde{p}(\mu) := p(e_1 + \mu)$, $\tilde{q}(\mu) := q(e_1 + \mu)$, we easily obtain

$$\tilde{p}(\mu) = 2\sum_{j=2}^{n} \mu_j^2 + O(|\mu|^3), \quad \tilde{q}(\mu) = \sum_{j=2}^{n} \mu_j^2 + \left(\sum_{j=2}^{n} \mu_j\right)^2 + O(|\mu|^3),$$

where $O(|\mu|^k)$ is a quantity which satisfies $|O(|\mu|^k)| \leqslant C(n, k)|\mu|^k$. Therefore we can rewrite the ratio as

$$\frac{\tilde{p}(\mu)}{\tilde{q}(\mu)} = \frac{2 + O(|\mu|)}{1 + \mathcal{R}(\mu) + O(|\mu|)},$$

where $\mathcal{R}$ satisfies

$$0 \leqslant \mathcal{R}(\mu) = \frac{\left(\sum_{j=2}^{n} \mu_j\right)^2}{\sum_{j=2}^{n} \mu_j^2} \leqslant C(n),$$

from which we easily deduce the upper and lower bounds.

Boundedness near the zeros We study now the behaviour of the ratio $p(\lambda)/q(\lambda)$ when λ approaches the values $\pm\sqrt{\kappa}e$. Again, by symmetry it is enough to study the limit at $\sqrt{\kappa}e$. We write $\mu := \lambda - \sqrt{\kappa}e$, and define again $\tilde{p}(\mu) := p(\sqrt{\kappa}e_1 + \mu)$, $\tilde{q}(\mu) := q(\sqrt{\kappa}e_1 + \mu)$. A straight computation for $\tilde{p}$ shows:

$$\begin{aligned}
\tilde{p}(\mu) &= \sum_{i \neq j} \left((\mu_i + \sqrt{\kappa})(\mu_j + \sqrt{\kappa}) - \kappa \right)^2 = \sum_{i \neq j} \left(\sqrt{\kappa}(\mu_i + \mu_j) + \mu_i \mu_j \right)^2 \\
&= \kappa \sum_{i \neq j} (\mu_i + \mu_j)^2 + O(|\mu|^3) = \kappa \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_i^2 + 2\mu_i\mu_j + \mu_j^2 + O(|\mu|^3) \\
&= 2\kappa(n-2)|\mu|^2 + O(|\mu|^3).
\end{aligned}$$

For $\tilde{q}$ we have a similar expression:

$$\tilde{q}(\mu) = \sum_{i=1}^{n} \left((\mu_i + \sqrt{\kappa}) \sum_{j \neq i} (\mu_j + \sqrt{\kappa}) - (n-1)\kappa \right)^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(\sqrt{\kappa}\left((n-1)\mu_i + \sum_{j\neq i}\mu_j\right) + O(|\mu|^2)\right)^2 \\
&= \sum_{i=1}^{n}\left(\sqrt{\kappa}((n-2)\mu_i + H(\mu)) + O(|\mu|^2)\right)^2 = \kappa\sum_{i=1}^{n}((n-2)\mu_i + H)^2 + O(|\mu|^2) \\
&= (n-2)^2\kappa|\mu|^2 + (3n-4)\kappa H(\mu)^2 + O(|\mu|^3),
\end{aligned}
$$

where again $H(\mu) = \sum_i \mu$. From these computations we can easily deduce the lemma. Indeed,

$$
\frac{\tilde{p}(\mu)}{\tilde{q}(\mu)} = \frac{2 + O(|\mu|)}{n - 2 + \mathcal{R}(\mu) + O(|\mu|)},
$$

where

$$
0 \leqslant \mathcal{R}(\mu) = \frac{(3n-4)H(\mu)^2}{(n-2)|\mu|^2} \leqslant C(n),
$$

and the lemma is proved. □

Remark 2.14. Notice how from Lemma 2.13 we can infer our own proof of Theorem 2.4. Indeed, let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed hypersurface satisfying $\mathring{\mathrm{Ric}} = 0$. By the Bianchi identity (2.14) we immediately deduce

$$
\nabla\,\mathrm{Scal} = \frac{2n}{n-2}\,\mathrm{div}\,\mathring{\mathrm{Ric}} = 0. \tag{2.28}
$$

Therefore every hypersurface with traceless Ricci tensor being null has constant scalar curvature. Writing $\mathrm{Scal} = n(n-1)\kappa$ for some $\kappa \in \mathbb{R}$, we know that $\mathrm{Ric} = (n-1)\kappa g$. Again, we need to show how $\kappa > 0$. The strategy is the same we used in that we have used in the proof of Proposition 2.5 : we define the distance function

$$
q \in \Sigma \longmapsto |q|^2,
$$

and consider the point p_0 of maximum. As shown in the proof of Proposition 2.5, in this point p_0 the second fundamental form must be positive definite, hence $\kappa > 0$. This proves by the way that Σ is convex and satisfies $\mathrm{Ric} = (n-1)\kappa g$ for some positive κ. Then, from Lemma 2.13, the eigenvalues of A satisfy system 2.19, and therefore we obtain that either $\lambda_1 = \dots \lambda_n = \sqrt{\kappa}$ at every point or $\lambda_1 = \cdots = \lambda_n = -\sqrt{\kappa}$ at every point. Since $A|_{p_0}$ is positive definite, the first one must hold, and from the *Nabelpunktsatz*, the hypersurface must be a sphere.

2.2 PROOF OF PROPOSITION 2.6

Here we prove Theorem 2.6. We remark that the C^0-bound of the second fundamental form, namely (2.2), can be weakened, but this does not help us to improve the main results. This stronger version will be given in the appendix.

Proof of Proposition 2.6. Let us consider a sequence $\left(\Sigma^h\right)_{h\in\mathbb{N}}$ of closed hypersurfaces satisfying the following assumptions:

(i) $\Sigma^h = \partial U^h$, where U^h is an open, bounded, convex set.

(ii) $\mathrm{Vol}_n(\Sigma^h) = \mathrm{Vol}_n(\mathbb{S}^n)$.

(iii) $0 \leqslant A^h \leqslant \Lambda g^h$, where A^h is the second fundamental form associated to Σ^h.

(iv) $b(\Sigma^h) = 0$, where $b(\Sigma^h)$ denotes the barycenter of Σ^h, defined as in (1.34).

(v) $\lim_k \|\mathring{\mathrm{Ric}}^h\|_{L^p} = 0$, where $\mathring{\mathrm{Ric}}^h$ denotes the traceless Ricci tensor associated to Σ^h.

We are firstly going to show that necessarily we must have

$$\lim_h d_{HD}\left(\Sigma^h, \mathbb{S}^n\right) = 0.$$

Firstly, we notice how from assumptions $(i) - (iv)$ the hypersurfaces are all contained in a ball $\mathbb{B}^{n+1}_{R_0}$ for R_0 sufficiently large. Indeed Proposition 0.4 we obtain two radii $0 < r < R$ and a vector $x \in \mathbb{R}^{n+1}$ such that (3.20) holds, namely:

$$\mathbb{B}^{n+1}_r(x) \subset U^h \subset \mathbb{B}^{n+1}_R(x).$$

Since $\Sigma^h = \partial U^h$ we infer that $\operatorname{diam}\Sigma^h \leqslant D(r, R) < \infty$, and from condition (iv) we get our desired claim. Now we show that the Σ^h must converge to a sphere. We apply the Blaschke's selection theorem (see [47, Thm. 1.8.6]) and consider a (not relabeled) subsequence $\overline{U}^k \to V$ in the Hausdorff distance d_{HD}. From the inclusions in (3.20) we infer that the volumes $|U^h|$ do not converge to 0, hence V has positive measure and non-empty inner part. Necessarily it has the form $V = \overline{U}$ for some bounded, open and convex set U. We claim that $\Sigma = \partial V$ must be the round sphere $\mathbb{S}^n$

The proof of the claim follows from the following lemma.

Lemma 2.15. *Let $\kappa > 0$, and let $q = q(\lambda)$ defined as in (2.19). Define $r = r(\lambda)$ as*

$$r(\lambda) := |D(\lambda) - \sqrt{\kappa}\delta|^2\,|D(\lambda) + \sqrt{\kappa}\delta|^2 = \left(\sum_i \left(\lambda_i - \sqrt{\kappa}\right)^2\right)\left(\sum_i \left(\lambda_i + \sqrt{\kappa}\right)^2\right) \tag{2.29}$$

Then, there exists two constants $c_0(n, \Lambda)$ and $c_1(n, \Lambda)$ such that

$$c_0 \leqslant \frac{q}{r} \leqslant c_1 \text{ in the ball } \mathbb{B}^n_\Lambda.$$

Assuming Lemma 2.15, we show how it leads to the conclusion. We shall prove it at the end of the of the section. From Lemma 2.12 we easily find a sequence $\left(\kappa^h\right)_{h\in\mathbb{N}}$ of constants $\kappa^h \in \mathbb{R}$ such that

$$\|\mathrm{Ric}^h - (n-1)\kappa^h g^h\|_{L^p} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}^h\|_{L^p},$$

and thus

$$\lim_h \|\mathrm{Ric}^h - (n-1)\kappa^h g^h\|_{L^p} = 0.$$

Thanks to the analysis made in Section 2.1, we know that $0 < \kappa^h \leqslant \Lambda$ for every $h \in \mathbb{N}$, and we can assume $\kappa^h \to \kappa \in [0, \Lambda]$.

Firstly, we notice that the limit κ cannot be 0. Indeed, if $\kappa = 0$, then we would obtain a sequence $(\Sigma^h)_{h\in\mathbb{N}}$ of closed hypersurfaces satisfying assumptions $(i) - (iii)$, (v) and

$$\lim_h \|\mathrm{Ric}^h\|_{L^p(\Sigma^h)} = 0 \text{ for some } p \in (1, \infty). \tag{2.30}$$

Since assumption (iii) holds, we also know that $0 \leqslant \mathrm{Ric}^h \leqslant n(n-1)\Lambda g^h$, and therefore from (2.30) we infer

$$\lim_h \|\mathrm{Ric}^h\|_{L^p(\Sigma^h)} = 0 \text{ for every } p \in (1, \infty).$$

In particular, $\|\mathrm{Ric}^h\|_{L^{n/2}} \to 0$. This however is not possible, since in [53, Th.3] it is proved that for $3 \leqslant n$ any compact, connected manifold M^n admitting an isometric immersion into $\mathbb{R}^{n+1}$ satisfies the lower bound[1]

$$\int_M |\mathrm{Ric}|^{\frac{n}{2}} \geqslant a(n).$$

Therefore, we must have that any limit κ satisfies $\kappa \geqslant c(n) > 0$. We show now how to prove the proposition. Lemma 2.15 applied to the eigenvalues of the second fundamental forms A^h yields

$$\lim_h \| |A^h - \sqrt{\kappa^h} g^h| \, |A^h + \sqrt{\kappa^h} g^h| \|_{L^p(\Sigma^h)} = 0.$$

From condition (iii) we also know that

$$\sqrt{\kappa^h} g^h \leqslant A^h + \sqrt{\kappa^h} g^h \leqslant (\Lambda + \sqrt{\kappa^h}) g^h \Rightarrow |A^h + \sqrt{\kappa^h} g^h| \leqslant c(n, \Lambda),$$

from which we infer

$$\lim_h \|A^h - \kappa^h g^h\|_{L^p(\Sigma^h)} = 0. \tag{2.31}$$

Equation 2.31 is the key of the proof, and allows us to conclude by applying techniques used in the previous chapter. Indeed, from [41, Cor. 2.5] we are able to find a vector $x \in \mathbb{R}^{n+1}$ such that

$$\lim_{h\in\mathbb{N}} d_{HD}\big(\Sigma^h, \sqrt{\kappa}\mathbb{S}^n + x\big) = 0.$$

We show that $\kappa = 1$, $x = 0$ necessarily. Let us define $\tilde{\Sigma}^h = \big(\Sigma^h - x\big)/\sqrt{\kappa}$. Then, by construction we have

$$\lim_{h\in\mathbb{N}} d_{HD}\big(\tilde{\Sigma}^h, \mathbb{S}^n\big) = 0,$$

and can consider the radial parametrisations

$$\tilde{\psi}^h \colon \mathbb{S}^n \longrightarrow \Sigma^h, \ \tilde{\psi}(x) = e^{\tilde{f}^h(x)}\, x$$

1 The result is actually much finer than the one we expressed here. The precise statement involves however concepts taken from algebraic topology we do not need to introduce.

associated to every Σ^h as in (1.4). Since the hypersurfaces converge to the sphere in the Hausdorff distance, we get that necessarily

$$\lim_h \|\tilde{f}^h\|_{C^0(\mathbb{S}^n)} = 0.$$

Therefore, we can apply Lemma 1.11 and obtain that

$$\|\nabla \tilde{f}^h\|_{C^0(\mathbb{S}^n)} \leqslant \sqrt{\frac{\operatorname{osc}(\tilde{f}^h,\, \mathbb{S}^n)}{1 - 2\operatorname{osc}(\tilde{f}^h,\, \mathbb{S}^n)}} \to 0,$$

and get that $\tilde{f}^h \to 0$ in C^1. We can finally conclude: since the convergence is now C^1, we obtain

$$\begin{aligned} -x b(\tilde{\Sigma}^h) &= \fint_{\mathbb{S}^n} x e^{(n+1)\tilde{f}^h(x)} \sqrt{1 + |\tilde{f}^h(x)|^2}\, dV_\sigma \longrightarrow 0 = b(\mathbb{S}^n), \\ \kappa^{-\frac{n}{2}} \operatorname{Vol}_n(\mathbb{S}^n) &= \int_{\mathbb{S}^n} e^{n\tilde{f}^h(x)} \sqrt{1 + |\tilde{f}^h(x)|^2}\, dV_\sigma \longrightarrow \operatorname{Vol}_n(\mathbb{S}^n). \end{aligned}$$

Therefore $x = 0$, $\kappa = 1$, $\tilde{\Sigma}^h = \Sigma^h$ and our claim is proved.

We are now left to prove the C^2-bound. This will follow from assumption (iii) and expression (1.15) for the second fundamental form in the radial parametrisation. Let Σ be a closed, convex hypersurface such that $0 \leqslant A \leqslant \Lambda g$ and the logarithmic radius f given as in (1.4) satisfies $\|f\|_{C^1} \leqslant \varepsilon$ for $0 < \varepsilon < 1$. Let us recall (1.15):

$$A^i_j = \frac{e^{-f}}{\sqrt{1 + |\nabla f|^2}} \left(\delta^i_j - \nabla^i \nabla_j f + \frac{1}{1 + |\nabla f|^2} \nabla^i f\, \nabla^2 f[\nabla f]_j \right).$$

Since $0 \leqslant A \leqslant \Lambda g$, we get

$$-c(\lambda)\delta^i_j \leqslant \nabla^i \nabla_j f + \frac{1}{1 + |\nabla f|^2} \nabla^i f\, \nabla^2 f[\nabla f]_j \leqslant C(\Lambda)\delta^i_j,$$

hence

$$\|\nabla^2 f\|_{C^0(\mathbb{S}^n)} \leqslant C(\Lambda) + \varepsilon \|\nabla^2 f\|_{C^0(\mathbb{S}^n)}, \tag{2.32}$$

and we find our desired conclusion. Notice that the smallness of the gradient ∇f is actually not required: if we have $\|\nabla f\|_0 \leqslant c_0$ for some positive constant c_0, we can perform the estimate made in (2.32) as

$$\|\nabla^2 f\|_{C^0(\mathbb{S}^n)} \leqslant C(\Lambda) + \frac{c_0}{\sqrt{1 + c_0^2}} \|\nabla^2 f\|_{C^0(\mathbb{S}^n)},$$

and thus conclude, since $c_0/\sqrt{1 + c_0^2}$ is always smaller than 1. □

We prove Lemma 2.15 and conclude.

Proof of Lemma 2.15. From Lemma 2.13 we know that the zero set $Z(q)$ of the polynomial q contains only the vectors $\pm\kappa e$, with $e := \sum_i e_i$. We study first the behaviour of the ratio q/r near those points. Again, thanks to symmetry it is enough to consider just the case of κe. As done in the proof of Lemma 2.13, we consider $\tilde{q}(\mu) := q(\sqrt{\kappa}e + \mu)$, and define $\tilde{r}(\mu) := r(\sqrt{\kappa}e + \mu)$. Trivially,

$$\liminf_{\lambda\to e} \frac{q(\lambda)}{r(\lambda)} = \liminf_{\mu\to 0} \frac{\tilde{q}(\mu)}{\tilde{r}(\mu)}, \ \limsup_{\lambda\to e} \frac{q(\lambda)}{r(\lambda)} = \limsup_{\mu\to 0} \frac{\tilde{q}(\mu)}{\tilde{r}(\mu)}.$$

From the computations made in Lemma 2.13, we obtain:

$$\tilde{q}(\mu) = (n-2)^2\kappa|\mu|^2 + (3n-4)\kappa H^2 + O(|\mu|^3),$$

where $H = \sum_i e_i$ as usual. The computation of $\tilde{r}$ is straightforward:

$$\tilde{r}(\mu) = |\mu|^2\left(\sum_i \left(2\sqrt{\kappa} + \mu_i\right)^2\right) = 4n\kappa|\mu|^2 + O(|\mu|^3).$$

Therefore, we obtain:

$$\frac{\tilde{q}(\mu)}{\tilde{r}(\mu)} = \frac{(n-2)^2|\mu|^2 + (3n-4)H^2 + O(|\mu|^3)}{4n|\mu|^2 + O(|\mu|^3)},$$

and we find easily two constants $c(n)$, $C(n)$, such that

$$c(n) \leqslant \liminf_{\mu\to 0} \frac{\tilde{q}(\mu)}{\tilde{r}(\mu)} < \limsup_{\mu\to 0} \frac{\tilde{q}(\mu)}{\tilde{r}(\mu)} \leqslant C(n).$$

Thus, we are able to find a radius $0 < r$ such that

$$\frac{c(n)}{2} \leqslant \frac{q}{r} \leqslant 2C(n) \text{ in } \mathbb{B}^n_r(\sqrt{\kappa}e) \cup \mathbb{B}^n_r(-\sqrt{\kappa}e),$$

and since q/r is continuous in $\mathbb{B}^n_\Lambda \setminus \mathbb{B}^n_r(\sqrt{\kappa}e) \cup \mathbb{B}^n_r(-\sqrt{\kappa}e)$, we also obtain constants $c(n, \Lambda)$, $C(n, \Lambda)$ such that

$$c(n, \Lambda) \leqslant \frac{q}{r} \leqslant C(n, \Lambda) \text{ in } \mathbb{B}^n_\Lambda \setminus \mathbb{B}^n_r(\sqrt{\kappa}e) \cup \mathbb{B}^n_r(-\sqrt{\kappa}e).$$

The lemma is therefore proved. □

2.3 PROOF OF PROPOSITION 2.7

Since we have to linearise the same quantities of Chapter 1, a big part of the proof can be taken from Proposition 2.6. The main difference with it consists in the fact that we cannot proceed exactly as in the previous chapter, since the Ricci operator is non-linear. A rough computation can show

$$\text{Scal} = H^2 - |A|^2 \approx (\Delta f)^2 - \left|\nabla^2 f\right|^2 - 2(n-1)(\Delta f + nf)).$$

The problem here is that we are able to obtain $C^{1,\alpha}$-closeness of f to 0, but in order to obtain a proper linearisation, we would need a C^2-one. This is not possible, not even with our C^0-control on the second fundamental form. Thus we need an alternative to proceed, and here it is where condition (2.2) is used. In the same spirit of Proposition 2.5, we consider again the problem in terms of the eigenvalues of A, and search for a better expression of the quantities we are considering. In order to obtain it, however, the upper bound (2.2) will be used crucially.

Proof of Proposition 2.7 . The starting point of our analysis is estimate (2.9):

$$\left\|\mathrm{Riem} - \frac{\overline{\mathrm{Scal}}}{n(n-1)} g \owedge g\right\|_{L^p(\Sigma)} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}\|_{L^p_g(\Sigma)}.$$

We let Proposition 2.6 follow from two lemmas:

Lemma 2.16. *Under the hypothesis of Proposition 2.6, the mean of the scalar curvature can be approximated as follows:*

$$\mathrm{Scal} = n(n-1) + \mathcal{R}, \text{ where } |\mathcal{R}| \leqslant C(n, p, \Lambda)\varepsilon\|f\|_{W^{2,p}(S^n)}. \tag{2.33}$$

Lemma 2.17. *Let $p = p(\lambda)$ be defined as in (2.18) with $\kappa = 1$. We set $r = r(\lambda)$ as in (2.29). There exist $c_2 = c_2(n, \Lambda)$ and $c_3 = c_3(n, \Lambda)$ such that*

$$c_2 \leqslant \frac{p(\lambda)}{r(\lambda)} \leqslant c_3 \text{ in the ball } \mathbb{B}^n_\Lambda.$$

The proof of Lemma 2.17 follows easily by combining Lemma 2.13 and Lemma 2.15 with $\kappa = 1$. The proof of Lemma 2.16 is postponed at the end of the section.

Via (2.33) and we obtain the following estimate:

$$\left\|\mathrm{Riem} - \frac{1}{2} g \owedge g\right\|_{L^p(\Sigma)} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}\|_{L^p_g(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(S^n)}.$$

Now from (2.16) we find:

$$\|(A-g) \owedge (A+g)\|_{L^p(\Sigma)} \leqslant C\Big(\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(S^n)}\Big). \tag{2.34}$$

Applying Lemma 2.13 to the polynomials associate to the eigenvalues of A, we find:

$$|A-g||A+g| \leqslant c|(A-g) \owedge (A+g)|.$$

This allows us to improve (2.34) as follows:

$$\||A-g|\,|A+g|\|_{L^p(\Sigma)} \leqslant C\Big(\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(S^n)}\Big).$$

Now we get rid of the term $A+g$ as we did in the proof of Proposition 2.6 and obtain

$$\|A-g\|_{L^p(\Sigma)} \leqslant C\Big(\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(S^n)}\Big).$$

We end with the Cauchy-Schwartz inequality: indeed,

$$|H - n| = |\langle g, A - g \rangle| \leqslant n|A - g|,$$

and integrating

$$\|H - n\|_{L^p(\Sigma)} \leqslant C\Big(\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(S^n)}\Big).$$

Now we can proceed applying a proper version of Proposition 1.15 and we obtain the desired estimate. □

We finish the section by proving Lemma 2.16.

Proof of Lemma 2.16. Firstly, we have to find a suitable expression for the scalar curvature Scal. Again, we trace equation (2.16) twice and obtain

$$\mathrm{Scal} = H^2 - |A|^2.$$

Therefore we need to find an approximate expression for H^2 and $|A|^2$. Firstly, we need to deduce the following approximation:

$$H^2 = n^2 - 2n\Delta f + (\Delta f)^2 - 2n^2 f + \mathcal{R}_1, \tag{2.35}$$

$$|A|^2 = n - 2\Delta f + |\nabla^2 f|^2 + 2nf + \mathcal{R}_2, \tag{2.36}$$

where both $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy

$$|\mathcal{R}_{1,2}| \leqslant C(n, \Lambda)\varepsilon\,(|f| + |\nabla f| + |\nabla^2 f|) \tag{2.37}$$

The two expressions follows linearising expression (1.15) from Lemma 1.9 with the help of Proposition 2.6. From 2.6 we know

$$\|f\|_{C^0},\ \|\nabla f\|_{C^0} \leqslant \varepsilon,\ \|\nabla^2 f\|_{C^0} \leqslant \Lambda,$$

and we can linearise (1.15) as follows:

$$A^i_j = -(1-f)\nabla^i\nabla_j f + (1-f)\,\delta^i_j + \mathcal{R}^i_j,\ \text{ where } |\mathcal{R}| \leqslant C(n)\varepsilon\big(|f| + |\nabla f| + |\nabla^2 f|\big). \tag{2.38}$$

From (2.38), we obtain:

$$\begin{aligned} H^2 &= (-(1-f)\Delta f + n(1-f) + \mathcal{R}_1)^2 = n^2 - 2n\Delta f + (\Delta f)^2 - 2n^2 f + \mathcal{R}_1 \\ |A|^2 &= (1-f)^2|\nabla^2 f|^2 + n(1-f)^2 - 2(1-f)\Delta f + \mathcal{R}_2 \\ &= n - 2\Delta f + |\nabla^2 f|^2 - 2nf + \mathcal{R}_2, \end{aligned}$$

which are exactly expressions (2.35) and (2.36). Now we are able to find the following expression for the curvature.

$$\mathrm{Scal} = n(n-1) - 2(n-1)(\Delta_\sigma f + nf) + (\Delta f)^2 - |\nabla^2 f|^2 + \mathcal{R}, \tag{2.39}$$

where $\mathcal{R}$ satisfies inequality (2.37). Now we integrate Scal. Since f is C^1-close to the identity, we can perform the same volume estimate we did in the proof of Proposition 1.15, and easily notice

$$|e^{nf}\sqrt{1+|\nabla f|^2}-1-nf| \leqslant C(n)\varepsilon\,(|f|+|\nabla f|). \tag{2.40}$$

Estimate (2.40) and the C^0-bound on $\nabla^2 f$ allow us to perform the following computation:

$$\begin{aligned}\overline{\mathrm{Scal}} &= \fint_\Sigma \mathrm{Scal}\ dV_g = \fint_{S^n} \mathrm{Scal}\, e^{nf}\sqrt{1+|\nabla f|^2}\, dV_\sigma \\ &= \fint_{S^n} \mathrm{Scal}\ dV_\sigma + n^2(n-1)\fint_{S^n} f\, dV_\sigma + n\fint_{S^n} f(\mathrm{Scal}-n(n-1))\, dV_\sigma + \mathcal{R},\end{aligned}$$

where $\mathcal{R}$ satisfies (2.37). From (2.39) we infer:

$$\begin{aligned}\left|\fint_{S^n} f(\mathrm{Scal}-n(n-1))\, dV_\sigma\right| &\leqslant \fint_{S^n}|f|\Big|2(n-1)(\Delta_\sigma f+nf)-(\Delta f)^2+|\nabla^2 f|^2\Big| \\ &\leqslant C\varepsilon\fint_{S^n}|f|+|\nabla f|+|\nabla^2 f|\, dV_\sigma \leqslant C(n,\,p,\,\Lambda)\|f\|_{W^{2,p}(S^n)}.\end{aligned}$$

Therefore, we can write

$$\overline{\mathrm{Scal}} = \fint_{S^n} \mathrm{Scal}\ dV_\sigma + \mathcal{R},\ \text{ where } |\mathcal{R}| \leqslant C(n,\,p,\,\Lambda)\|f\|_{W^{2,p}(S^n)}. \tag{2.41}$$

From (2.41) we obtain

$$\fint_\Sigma \mathrm{Scal}\ dV_\sigma = n(n-1) + \fint_{S^n}(\Delta f)^2 - |\nabla^2 f|^2\, dV_\sigma + 2n(n-1)\fint_{S^n} f\, dV_\sigma + \overline{\mathcal{R}}. \tag{2.42}$$

We simplify the second-order terms with the following Bochner formula. Indeed, by definition of Riemann tensor we know the commutation formula

$$\nabla_j\nabla_i\alpha_k - \nabla_i\nabla_j\alpha_k = \mathrm{Riem}^l_{ijk}\,\alpha_l$$

Using this formula and integrating by parts we achieve our goal.

$$\begin{aligned}\int(\Delta f)^2\, dV &= \int \nabla^i\nabla_i f\cdot\nabla^j\nabla_j f\, dV = -\int \nabla_i f\cdot\nabla^i\nabla^j\nabla_j f\, dV \\ &= -\int \nabla_i f\cdot\nabla^j\nabla^i\nabla_j f\, dV + \int \mathrm{Ric}(\nabla f,\,\nabla f) dV \\ &= \int \nabla^j\nabla_i f\cdot\nabla^i\nabla_j f\, dV + \int \mathrm{Ric}(\nabla f,\,\nabla f) dV \\ &= \int |\nabla^2 f|^2\, dV + \int \mathrm{Ric}(\nabla f,\,\nabla f) dV\end{aligned}$$

In the case of the sphere, this computation gives us the equality

$$\leqslant \int_{S^n}(\Delta_\sigma f)^2\, dV_\sigma - \int_{S^n}|\nabla^2 f|^2\, dV_\sigma = (n-1)\int_{S^n}|\nabla f|^2 dV_\sigma \leqslant C\varepsilon\|\nabla f\|_{L^p},$$

and we can improve (2.42) and obtain

$$\overline{\text{Scal}} = n(n-1) + 2n(n-1) \fint_{S^n} f \, dV_\sigma + \overline{\mathcal{R}}.$$

With the same consideration made in the proof of Proposition 1.15 we notice that the mean of f is negligible, i.e.

$$\left| \fint_{S^n} f \, dV_\sigma \right| \leqslant C\varepsilon \, \|f\|_{W^{1,p}(S^n)}$$

and this proves the lemma. □

Again, the proof of Theorem 2.1 follows as outlined in Remarks 1.8 and 2.10.

Remark 2.18. Looking carefully at the proofs, one can notice how Theorem 2.1 can be simplified: it is indeed possible to obtain Proposition 2.7 without Proposition 2.5. One just has to use Lemmas 2.12, 2.16 and improve Proposition 2.6 to obtain 2.7. The polynomial study we made would be strongly simplified since we just need to study the zeros of the polynomial $q(\lambda)$ given by (2.19) and the behaviour of the ratio q/r given by Lemma 2.15. We have nevertheless chosen to proceed through this longer path because it highlights the importance of the Ricci tensor for closed hypersurfaces, by showing how under certain hypothesis it can control the Riemann tensor, and because Proposition 2.5 will be a necessary step in Chapter 4 for the proof of Theorem 4.2.

2.4 PROOF OF THEOREM 2.2

We conclude the chapter proving Theorem 2.2. This will follow directly by the proof of Theorem 1.1 in Chapter 1. Indeed, in this case the strictly convexity is translated into an inequality between the traceless Ricci and the traceless second fundamental form, that we would not normally have.

Proof of Theorem 2.2. Again, we consider the equation:

$$\text{Ric}^i_j = HA^i_j - A^i_k A^k_j.$$

Let then $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of A. Then the Ricci tensor has eigenvalues $\Lambda_1, \ldots \Lambda_n$ which satisfy the following equality:

$$\Lambda_j = \lambda_j \sum_{j \neq k} \lambda_k, \quad \forall j = 1, \ldots, n. \tag{2.43}$$

By assumption (2.5), we know that $\lambda_j \geqslant \Lambda$ for every $j = 1, \ldots, n$, and this allows us to perform the following estimate:

$$|\mathring{\text{Ric}}|^2 = \sum_{i \neq j} |\Lambda_i - \Lambda_j|^2 = \sum_{i \neq j} \left(\sum_{k \neq i, j} \lambda_k \right)^2 \left| \lambda_i - \lambda_j \right|^2$$

$$\geqslant (n-2)^2\Lambda^2 \sum_{i\neq j} \left|\lambda_i - \lambda_j\right|^2 = (n-2)^2\Lambda^2|\mathring{A}|^2,$$

from which we deduce

$$\|\mathring{A}\|_{L^p(\Sigma)} \leqslant C(n, p, \Lambda)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)}. \tag{2.44}$$

This shows how in the strictly convex case, having small L^p-norm of the traceless Ricci tensor implies having small L^p-norm of the traceless second fundamental form.

We choose δ_1 sufficiently small so that the hypothesis of 1.1 holds, and thus we find a vector $c = c(\Sigma)$ such that the associated radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ satisfies

$$\|\psi - \mathrm{Id}\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C\|\mathring{A}\|_{L^p} \leqslant C\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)},$$

as desired. □

3 THE ANISOTROPIC CASE

In this chapter we obtain the anisotropic counterpart of Chapter 1. Let us state the main theorem.

Theorem 3.1. *Let $2 \leqslant n$, $1 < p < \infty$ be given, and let F be an elliptic integrand. There exists $0 < \delta_0 = \delta_0(n, p, F)$ with the following property.*

If Σ is a closed, convex hypersurface in $\mathbb{R}^{n+1}$ satisfying the conditions

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathcal{W}), \tag{3.1}$$

$$\|\mathring{A}_F\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{3.2}$$

then there exist a vector $c = c(\Sigma) \in \mathbb{R}^{n+1}$ and a smooth parametrization $\psi\colon \mathcal{W} \longrightarrow \Sigma - c$ satisfying the following estimate:

$$\|\psi - \mathrm{id}\|_{W^{2,p}(\mathcal{W})} \leqslant C(n, p, F)\|\mathring{A}_F\|_{L^p(\Sigma)}, \tag{3.3}$$

where $\mathrm{id}\colon \mathcal{W} \longrightarrow \mathcal{W}$ denotes the identity map of the Wulff shape into itself, and $\mathring{A}_F$ is the tensor

$$\mathring{A}_F := A_F - \frac{1}{n} H_F \,\mathrm{Id}.$$

What allows Theorem 3.1 to be true is the following anisotropic version of the umbilical theorem, which states that $\mathcal{W}$ can be characterized as the only hypersurface with the anisotropic second fundamental form that is a constant multiple of the identity (see [26, Thm. 1.2]).

Theorem 3.2. *Let $2 \leqslant n$ be given, and let Σ be a closed, oriented hypersurface. If $A_F|_x$ is equal to a constant multiple of the identity at every point $x \in \Sigma$, then Σ is the Wulff shape.*

For the anisotropic case we introduce some change of notation, that will appear throughout this and the next chapter. The classic radial parametrization we used before is not suitable any more, and we shall need a version that reflects the fact that we are working in a non-symmetric environment. Let $B_\varepsilon(\mathcal{W})$ be the tubular neighbourhood associated to $\mathcal{W}$, that is the set

$$B_\varepsilon(\mathcal{W}) := \{ z \in \mathbb{R}^{n+1} \mid z = x + \rho\nu(x), \quad \forall x \in \mathcal{W},\ 0 \leqslant \rho < \varepsilon \}. \tag{3.4}$$

All the details needed on the tubular neighbourhood can be found in [28, Ch. 5]. We recall that for ε sufficiently small, $B_r(\mathcal{W})$ is an open, bounded set with smooth boundary diffeomorphic to $\mathcal{W}$, for every $r < \varepsilon$. Let Σ be a closed, convex hypersurface in $\mathbb{R}^{n+1}$ satisfying the closeness condition

$$\Sigma \subset B_\varepsilon(\mathcal{W}).$$

Then, up to translation, we can give the following parametrization for Σ:

$$\psi\colon \mathcal{W} \longrightarrow \Sigma,\ \psi(x) = x + u(x)\nu(x), \qquad \text{for some } u \in C^\infty(\mathcal{W}). \tag{3.5}$$

Clearly ψ is a smooth diffeomorphism. We call ψ *anisotropic radial parametrization of* Σ and u *anisotropic radius associated to* ψ. By the very definition, if Σ is ε-close to $\mathcal{W}$ in the Hausdorff distance, then it is contained in the tubular neighbourhood $B_\varepsilon(\mathcal{W})$. We are therefore going to prove the following theorem.

Theorem 3.3. *Let $2 \leqslant n$ and $1 < p < \infty$ be given, and let $\Sigma = \partial U$ be a smooth, closed and convex hypersurface in $\mathbb{R}^{n+1}$. There exists $0 < \delta_0 = \delta_0(n, p, F)$ with the following property.*

If Σ satisfies conditions (3.1) and (3.2), then there exists a vector $c = c(\Sigma) \in \mathbb{R}^{n+1}$ such that $0 \in U - c$ and the anisotropic radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ as in (3.5) satisfies:

$$\|u\|_{W^{2,p}(\mathcal{W})} \leqslant C(n, p)\|\mathring{A}_F\|_{L^p(\Sigma)}. \tag{3.6}$$

In order to avoid confusion when making the computations, we adopt the following notations throughout this chapter, and more in general when we are dealing with anisotropic quantities:

A_F	Anisotropic second fundamental form.
$\mathring{A}_F$	Anisotropic traceless second fundamental form.
H_F	Anisotropic mean curvature, i.e. $\operatorname{tr} A_F$.
D	Levi-Civita derivative on the sphere.
S_F	Anisotropy tensor, i.e. $S_F := D^2F + \mathrm{Id}$.
A	"Classical" second fundamental form, i.e. A_F with $F = 1$.

As before, we apply our scheme and divide the proof into the following steps.

Proposition 3.4. *Let $2 \leqslant n$, $1 < p < \infty$ and $0 < \delta_0$ be given. Let F be an elliptic integrand. There exists a constant $0 < C = C(n, p, \delta_0, F)$ such that the following holds.*

If Σ is a closed and convex hypersurface in $\mathbb{R}^{n+1}$ which satisfies (3.1) and (3.2) with a threshold $\delta \leqslant \delta_0$, then there exists $c \in \mathbb{R}^{n+1}$ such that

$$d_{HD}(\Sigma - c, \mathcal{W}) \leqslant \varepsilon. \tag{3.7}$$

In particular, this implies that the anisotropic radius u given by (3.5) is C^1-close to the identity, namely

$$\|u\|_{C^1(\mathcal{W})} \leqslant \varepsilon. \tag{3.8}$$

Proposition 3.5. *Let Σ be a closed, anisotropically radially parametrized hypersurface so that u satisfies (3.8). The following inequality holds:*

$$\|L[u]\|_{L^p(\mathcal{W})} \leqslant C(n, p, F)\|H_F - \overline{H_F}\|_{L^p(\Sigma)} + \varepsilon\|u\|_{W^{2,p}(\mathcal{W})}, \tag{3.9}$$

where L is defined as

$$L[u] := \operatorname{div}(S_F \nabla u) + Hu. \tag{3.10}$$

Proposition 3.6. *Let* L *be as in* (3.10). *Then:*

$$\ker L := \left\{ \varphi_c : y \in \mathcal{W} \longmapsto \langle c,\, \nu(y) \rangle \in \mathbb{R},\ \forall c \in \mathbb{R}^{n+1} \right\}. \tag{3.11}$$

Proposition 3.7. *Let* Σ *be a closed, anisotropically radially parametrized hypersurface so that the anisotropic radius* u *satisfies* (3.8), *and consequentially* (3.9). *Then there exists a vector* $c(\Sigma) \in \mathbb{R}^{n+1}$, *so that the anisotropic radius associated to the radial parametrization* $\psi\colon \Sigma \longrightarrow \mathcal{W} - c$ *still satisfies condition* (3.8) *with a possibly worsened bounding constant, and*

$$\langle\, u,\ \varphi\, \rangle_{L^2} = 0 \ \text{ for every } \varphi \in \ker L. \tag{3.12}$$

Remark 3.8. Via these propositions we are able to conclude as in Chapters 1 and 2. We notice that in this case an additional step appears, i.e. we have to characterize the kernel of the anisotropic stability operator L.

Again, before starting the proof, we give the anisotropic version of Lemma 1.9, and prove the following expressions for the main geometric quantities in our new definition of radial parametrization.

Lemma 3.9. *Let* ψ *be as in* (3.5), *and let us denote by* A^{Σ} *the second fundamental form of* Σ, *and by* A *the second fundamental form in* $\mathcal{W}$ *Then we have the following expressions.*

$$g_{ij} = \omega_{ij} + 2uA_{ij} + \nabla_i u\, \nabla_j u + u^2 A_i^k A_{kj}, \tag{3.13}$$

$$\nu^{\Sigma} = \frac{\nu - (\omega + uA)^{-1}[\nabla u]}{|\nu - (\omega + uA)^{-1}[\nabla u]|}, \tag{3.14}$$

$$A^{\Sigma}_{ij} = \frac{A_{ij} - \nabla^2_{ij} u + uA^2_{ij} - u\langle \nabla_i(\nabla u),\, \nabla_j \nu\rangle + A[\nabla u]_i\, \nabla_j u + \langle \nabla_i \mathcal{R},\, \nabla_j \psi\rangle}{\sqrt{1 + |(\mathrm{Id} + uA)^{-1}[\nabla u]|^2}}, \tag{3.15}$$

where $\mathcal{R}$ *is a combination of product of* u *and* ∇u. *In particular, if* Σ *is convex and* $\|u\|_{C^0(\Sigma)} \leqslant \varepsilon(\mathcal{W})$, *then we also have the following inequality:*

$$\nabla^2 u \leqslant C(\mathcal{W})(A + \nabla u \otimes \nabla u). \tag{3.16}$$

The proof of Lemma 3.9 is postponed in Appendix A.1

3.1 PROOF OF PROPOSITION 3.4

In this section we prove Proposition 3.4 and show a first qualitative, closeness result. As in its isotropic counterpart, we need an oscillation estimate in order to let the scheme work. This is given by the following proposition.

Proposition 3.10. *Let* $2 \leqslant n$, $1 < p < \infty$ *and* $0 < \delta_0$ *be given, and let* Σ *be a closed hypersurface in* $\mathbb{R}^{n+1}$ *with fixed volume* V. *Let* F *be an elliptic integrand. Assume* Σ *satisfies one of two following hypothesis.*

a) Σ *is convex, and* $\|\mathring{A}_F\|_{L^p(\Sigma)} \leqslant \delta_0$ *for some* $1 < p < \infty$.

b) $\|A\|_{L^p(\Sigma)} \leqslant \delta_0$ *for some* $n < p < \infty$.

Then the following estimate is satisfied:

$$\min_{\lambda\in\mathbb{R}}\|A_F - \lambda\,\mathrm{Id}\|_{L^p(\Sigma)} \leqslant C(n, p, \delta_0)\|\mathring{A}_F\|_{L^p(\Sigma)}. \tag{3.17}$$

In the proof of Proposition 3.4 we shall also need the following proposition.

Proposition 3.11. *Let* $2 \leqslant n$, $1 < p \leqslant n$ *be given, let* F *be an elliptic integrand and let* Σ *be a convex, closed hypersurface in* $\mathbb{R}^{n+1}$, *satisfying* $\mathrm{Vol}_n(\Sigma) = 1$. *Then, there exist two positive constants* c_1 *and* c_2, *depending only on* n, p *and* F, *such that*

$$\|A\|_{L^p(\Sigma)} \leqslant c_1\|A_F\|_{L^p(\Sigma)} \leqslant c_2\Big(1 + \|\mathring{A}_F\|_{L^p(\Sigma)}\Big). \tag{3.18}$$

The first inequality requires neither the upper bound $p < n$ *nor the assumption of convexity.*

Both the results are proved in the Appendix, see A.2 for 3.10 and A.3 for 3.11. Now we can prove Proposition 3.4. We show firstly the following proposition that, although being suboptimal, is the key point for our further study.

Proposition 3.12. *Let* $2 \leqslant n$, $1 < p < \infty$ *be given, and let* F *be an elliptic integrand. For every* $0 < \varepsilon < 1$ *there exists* $0 < \delta = \delta(n, p, F) < 1$ *with the following property.*

If Σ *is a closed, convex hypersurface satisfying* (3.1) *and* (3.2), *then there exists* $c \in \mathbb{R}^{n+1}$ *such that*

$$d_{HD}(\Sigma, \mathcal{W} + c) \leqslant \varepsilon \tag{3.19}$$

Proof. We argue by contradiction and assume there exist $\varepsilon_0 > 0$ and a sequence of closed, convex hypersurfaces $\{\Sigma^k\}_{k\in\mathbb{N}}$ satisfying

(i)′ $\mathrm{Vol}_n(\Sigma^k) = \mathrm{Vol}_n(\mathcal{W})$,

(ii)′ $\lim_k\|\mathring{A}^k_F\|_{L^p(\Sigma^k)} = 0$,

(iii)′ $d_{HD}\big(\Sigma^k, \mathcal{W} + c\big) \geqslant \varepsilon_0$ for every $c \in \mathbb{R}^{n+1}$ and $k \in \mathbb{N}$.

We notice that conditions (i)′, (ii)′, combined with Proposition 3.11 allow us to use Proposition 0.4. Thus, we are able to find two radii $0 < r < R$, depending only on n, p and F, such that, up to translating, the following inclusion holds:

$$\mathbb{B}_r \subset U^k \subset \mathbb{B}_R, \tag{3.20}$$

where U^k is the convex bounded set enclosed by Σ^k. As done in the proof of Proposition 2.6, we apply the Blaschke's selection theorem (see [47, Thm. 1.8.6])and consider a (not relabeled) subsequence $\overline{U}^k \to V$ in the Hausdorff distance d_{HD}. From the inclusions in (3.20) we infer that the volumes $|U^k|$ do not converge to 0, hence V has positive measure and non-empty inner part. Necessarily it has the form $V = \overline{U}$ for some bounded, open and convex set U.

Let Σ be the boundary of U. From the discussion above, we easily notice that Σ^k converges to Σ in the Hausdorff distance. Plugging this information in (iii)′, we deduce that $d_{HD}(\Sigma, \mathcal{W} + c) \geqslant \varepsilon_0$ for every $c \in \mathbb{R}^{n+1}$. If we show that Σ is a Wulff shape, we obtain the desired contradiction.

By Proposition 3.10, there exists a sequence $(\lambda^k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\|H_F^k - \lambda^k\|_{L^p(\Sigma^k)} \leqslant C\|\mathring{A}_F^k\|_{L^p(\Sigma^k)} \to 0. \tag{3.21}$$

Moreover, by Proposition 3.11 we know that for k large enough

$$\|H_F\|_{L^p(\Sigma^k)} \leqslant c_2\Big(1 + \|\mathring{A}_F\|_{L^p(\Sigma)}\Big) \leqslant 2c_2.$$

It follows that

$$|\lambda^k| = \frac{\|\lambda^k\|_{L^p(\Sigma^k)}}{\mathrm{Vol}_n(\Sigma^k)^{\frac{1}{p}}} \leqslant C\Big(\|H_F^k\|_{L^p(\Sigma^k)} + \|H_F^k - \lambda^k\|_{L^p(\Sigma^k)}\Big) \leqslant C.$$

We conclude that there exists $\lambda \in \mathbb{R}$ such that, up to subsequences, $\lambda^k \to \lambda$. Therefore, we can assume we are given a sequence $\{\Sigma^k\}$ satisfying the following properties:

(i)″ $\Sigma^k = \partial U^k$, with U^k being a convex, open, bounded set satisfying $\mathbb{B}_r^{n+1} \subset U^k \subset \mathbb{B}_R^{n+1}$,

(ii)″ there exists $\Sigma = \partial U$, with U convex, open, bounded such that $d_{HD}\big(\Sigma^k, \Sigma\big) \to 0$,

(iii)″ $\|A_F^k - \lambda\,\mathrm{Id}\|_{L^p(\Sigma^k)} \to 0$ for some $p \in (1, \infty)$.

We show how these three conditions imply that Σ is the Wulff shape. Firstly, condition (i)″ allows us to give the "classic" radial parametrization $\psi^k \colon \mathbb{S}^n \longrightarrow \Sigma^k$ for every k, i.e. as in (1.4). Clearly, ψ^k is a smooth parametrization for every k. By condition (i)″ and convexity, it is easy to see that every f^k satisfies

$$\log(r) \leqslant f^k \leqslant \log(R), \qquad \mathrm{Lip}(f^k) \leqslant L = L(r, R). \tag{3.22}$$

Moreover, by condition (ii)″, we find that f^k converges in C^0 to a function f satisfying (3.22) and such that the map

$$\psi \colon \mathbb{S}^n \longrightarrow \Sigma \text{ given by } \psi(x) = e^{f(x)}\,x,$$

is a Lipschitz parametrisation of Σ. We can improve the regularity of f. Indeed, by condition (iii)″, we can write

$$A_F^k = \lambda\,\mathrm{Id} + \mathcal{R}_k, \qquad \text{where } \|\mathcal{R}_k\|_{L^p(\Sigma^k)} \to 0. \tag{3.23}$$

We know that $A_F^k = S_F|_{\nu^k} \circ d\nu^k$, where S_F is the smooth 2-covariant tensor defined on the sphere as in (0.4) and recalled in the introduction in Chapter 3. Since $0 < S_F$, we multiply equality (3.23) by $(S_F)^{-1}$ and taking the L^p-norm, we obtain the estimate

$$\|A^k\|_{L^p(\Sigma^k)} \leqslant \|(S_F)^{-1}\|_{C^0}\Big(\lambda + \|\mathcal{R}_k\|_{L^p(\Sigma^k)}\Big). \tag{3.24}$$

We exploit the fact that we have ψ^k as a global parametrisation. Indeed, by Lemma 1.9 and the C^1-control as in (3.22) we get

$$\sup_k \left\| \mathrm{div}_\sigma\left(\frac{Df^k}{\sqrt{1+|Df^k|^2}}\right)\right\|_{L^p(\mathbb{S}^n)} < \infty. \tag{3.25}$$

Since f^k satisfies (3.22), standard elliptic regularity theory (see [20]) gives

$$\sup_k \|f^k\|_{W^{2,p}(\mathbb{S}^n)} < +\infty. \tag{3.26}$$

So the limit f must be in $W^{2,p}(\mathbb{S}^n)$ for some $p \in (1, \infty)$ and hence the limit Σ is a rough hypersurface with $W^{2,p}$-regularity. We have to prove that f is smooth. Using the expression (1.15) for the second fundamental form A of Σ and taking the trace, we obtain that every f^k satisfies (3.22) and the differential equation

$$\operatorname{div}_\sigma\left(\frac{Df^k}{\sqrt{1+|Df^k|^2}}\right) = \lambda \operatorname{tr}(S_F|_{\nu^k})^{-1} + \frac{ne^{-f^k}}{\sqrt{1+|Df^k|^2}} + \mathcal{R}_k =: h(f^k, Df^k) + \mathcal{R}_k, \tag{3.27}$$

where h is a smooth function. Equation (3.27) means that the limit f satisfies the equation

$$\operatorname{div}_\sigma\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = h(f, Df).$$

From elliptic regularity (see [20]), we obtain that f is smooth, and therefore the limit hypersurface Σ is smooth. Moreover, from the bounds (3.22) and (3.26) we know that the sequence $(f^k)_{k\in\mathbb{N}}$ converges to f weakly in $W^{2,p}(\mathbb{S}^n)$ and easily infer that the sequence $(\nabla f^k)_{k\in\mathbb{N}}$ converges to ∇f strongly in L^q for every $1 < q < \infty$. Since F is smooth, we obtain the following convergences:

$$\begin{aligned} \nu^k = \frac{x - \nabla f^k}{\sqrt{1+|\nabla f^k|^2}} &\to \nu = \frac{x - \nabla f}{\sqrt{1+|\nabla f|^2}} \text{ in } L^q,\ \forall q \in (1, \infty), \\ d\nu_k &\rightharpoonup d\nu_k \text{ in } L^p, \\ S_F|_{\nu^k} &\to S_F|_\nu \text{ in } L^q,\ \forall q \in (1, \infty). \end{aligned}$$

Summing the three convergences, it follows that $A_f|_{\nu^k} \rightharpoonup A_F|_\nu$. By condition (iii)$''$, we obtain:

$$A_F = \lambda \operatorname{Id}. \tag{3.28}$$

Hence, by Proposition 3.2 and the perimeter condition Σ must be the Wulff shape. □

Now the proof of Proposition 3.12 follows trivially

Proof of Proposition 3.12. Insofar we have obtained the C^0-closeness of u to the Wulff shape. Now we choose ε very small and consider inequality (3.16). Following the same argument of Lemma 1.11 we can improve the C^0-closeness to a C^1-one. □

3.2 PROOF OF PROPOSITION 3.5

As in Chapter 1, the proof of Proposition 3.5 follows by linearising the main geometric quantities that are given by Lemma 3.9.

Proof of Proposition 3.5 . The proof follows by linearising equalities (3.13), (3.14), (3.15). Again, here ν and A denote the outer normal and second fundamental form associated to the Wulff shape $\mathcal{W}$ respectively, while ν^Σ A^Σ denote the outer normal and the second fundamental form associated to Σ respectively. For the metric, we have:

$$|g_{ij} - \omega_{ij} - 2uA_{ij}| \leqslant C\varepsilon(|u| + |\nabla u|). \tag{3.29}$$

As an easy consequence of (3.29) we find the linearisation of the inverse:

$$|g^{ij} - \omega^{ij} + 2uA^{ij}| \leqslant C\varepsilon(|u| + |\nabla u|), \tag{3.30}$$

and its determinant

$$|\det g - \det \omega - 2Hu| \leqslant C\varepsilon(|u| + |\nabla u|). \tag{3.31}$$

The linearisation of ν^Σ follows easily:

$$\nu^\Sigma = \frac{\nu - (\omega + uA)^{-1}[\nabla u]}{|\nu - (\omega + uA)^{-1}[\nabla u]|} = \nu - \nabla u + \mathcal{R}$$

where $\mathcal{R}$ is given by linear combinations of products of u and components of ∇u. We obtain

$$|\nu^\Sigma - \nu + \nabla u| \leqslant C\varepsilon(|u| + |\nabla u|). \tag{3.32}$$

Now we linearise A^Σ. We write

$$\nu^\Sigma = \nu - \nabla u + \mathcal{R}$$

where $\mathcal{R}$ is again given linear combinations of products of u and components of ∇u. Thus, we deduce

$$A^\Sigma_{ij} = \langle \nabla_i \psi, \nabla_j \nu^\Sigma \rangle = \langle z_i + \nabla_i u\, \nu + u\nabla_i,\ \nabla_j(\nu + \nabla u + \mathcal{R})\rangle$$

and we obtain

$$|A^\Sigma - A + \nabla^2 u - A^2 u| \leqslant C\varepsilon\big(|u| + |\nabla u| + |\nabla^2 u|\big). \tag{3.33}$$

We linearise now the quantity A^Σ_F. We use again the shorthand notation $\mathcal{R}$ for a quantity that can be estimated by as

$$|\mathcal{R}| \leqslant C\big(|u| + |\nabla u| + |\nabla^2 u|\big).$$

We obtain

$$\begin{aligned} A^\Sigma_F &= S_F|_{\nu^\Sigma} A^\Sigma = (S_F|_\nu - DS_F|_\nu[\nabla u])\big(A - \nabla^2 u - A^2 u\big) + \mathcal{R} \\ &= \underbrace{S_F|_\nu A}_{=A^{\mathcal{W}}_F = \mathrm{Id}} - \underbrace{DS_F|_\nu[\nabla u] - S_F\nabla^2 u}_{\nabla(S_F\nabla u)} - S_F|_\nu - \underbrace{S_F|_\nu A^2 u}_{=Au} + \mathcal{R} \end{aligned}$$

and we obtain:

$$\big|A^\Sigma_F - \mathrm{Id} + \tilde{L}[u]\big| \leqslant C(n, F)\varepsilon\big(|u| + |\nabla u| + |\nabla^2 u|\big), \tag{3.34}$$

where $\tilde{L}$ is defined as

$$\tilde{L}[u] := \nabla(S_F \nabla u) + Au. \tag{3.35}$$

Now we take the trace in (3.34) and find the estimate for the mean curvature:

$$|H_F(\Sigma) - n + L[u]| \leqslant C\big(|u| + |\nabla u| + |\nabla^2 u|\big), \tag{3.36}$$

where L is defined as in (3.10). We complete the proof of Proposition 3.5 showing that we can substitute n with $\overline{H_F(\Sigma)}$ in (3.36). Namely, we prove:

$$|\overline{H_F(\Sigma)} - n| \leqslant C\varepsilon \|u\|_{W^{2,p}(S^n)}. \tag{3.37}$$

Let us prove (3.37). Integrating (3.36), we easily obtain

$$\left| \overline{H_F(\Sigma)} - n + \fint_{\mathcal{W}} Hu \, dV \right| \leqslant C\varepsilon \|u\|_{W^{2,p}(\mathcal{W})}. \tag{3.38}$$

We just have to prove that the integral quantity $\int Hu$ is negligible. This is again granted by condition (3.1). Indeed, using (3.31) we obtain

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathcal{W}) + \int_{\mathcal{W}} Hu \, dV + \mathcal{R}, \tag{3.39}$$

where $\mathcal{R}$ is again a quantity which can easily be approximated by $\varepsilon \|u\|_{W^{1,1}(\mathcal{W})}$. Since the volumes of Σ and $\mathcal{W}$ are equal, we obtain

$$\left| \fint_{\mathcal{W}} Hu \, dV \right| \leqslant C(n,p,F)\varepsilon \|u\|_{W^{1,1}(\mathcal{W})} \leqslant C(n,p,F)\varepsilon \|u\|_{W^{2,p}(\mathcal{W})},$$

and this concludes the proof. □

3.3 PROOF OF PROPOSITION 3.6

In this section we prove Proposition 3.6. Our proof uses the quantitative anisotropic perimeter inequality. Another proof of 3.6 has been done in [38, Prop. 1.9].

Proof of proposition 3.6. Firstly we show that every $u \in \ker L$ has mean $\overline{u}$ equal to 0. Assume by contradiction we are given a smooth function u satisfying

$$\begin{cases} L[u] = 0 \\ \overline{u} \neq 0 \end{cases} . \tag{3.40}$$

Let ν be the outer normal associated to $\mathcal{W}$, and $h \in C^\infty(\mathcal{W})$ be a positive function with $\int h = 1$. Using a construction shown in [4], we are able to find a smooth function $s\colon [0, \varepsilon) \longrightarrow \mathbb{R}$ such that the deformation

$$\psi_t\colon \mathcal{W} \longrightarrow \mathbb{R}^{n+1}, \ \psi_t(x) = x + (t(u(x) - \overline{u}) + s(t)h(x))\nu(x)$$

is volume preserving for $0 \leqslant t < \varepsilon$. By the computations made in [30, Prop. 2.1], for every deformation ψ_t with infinitesimal vector field

$$X := \frac{d}{dt}\psi_t\Big|_{t=0} = w\nu,$$

the following equalities hold:

$$\frac{d}{dt}\mathcal{F}(\mathcal{W}_t)\Big|_{t=0} = n\int_{\mathcal{W}} w\, dV, \tag{3.41}$$

$$\frac{d^2}{dt^2}\mathcal{F}(\mathcal{W}_t)\Big|_{t=0} = -\int_{\mathcal{W}} L[w]w\, dV, \tag{3.42}$$

where L is as in (3.10). We apply (3.41) and (3.42) to our deformation and obtain via Taylor approximation

$$\mathcal{F}(\psi_t(\mathcal{W})) = \mathcal{F}(\mathcal{W}) - \frac{t^2}{2}\int_{\mathcal{W}} L[u-\overline{u}](u-\overline{u})\, dV + O(t^3). \tag{3.43}$$

However it easy to notice that

$$\int_{\mathcal{W}} L[u-\overline{u}](u-\overline{u}) = \overline{u}^2\int_{\mathcal{W}} H\, dV =: c_0 > 0.$$

We plug this equality into (3.43) and obtain

$$\mathcal{F}(\psi_t(\mathcal{W})) = \mathcal{F}(\mathcal{W}) - \frac{c_0}{2}t^2 + O(t^3), \tag{3.44}$$

which is a contradiction, since $\mathcal{W}$ is the absolute minimizer among the closed hypersurfaces with constrained volume.

Therefore we can assume that every solution u has null mean. Again, we consider a positive function $h \in C^\infty(\mathcal{W})$ with $\int h = 1$ and use the construction shown in [4] to find a function $s\colon [0, \varepsilon) \longrightarrow \mathbb{R}$ such that the deformation

$$\psi_t\colon \mathcal{W} \longrightarrow \mathbb{R}^{n+1},\ \psi_t(x) = x + (tu(x) + s(t)h(x))\nu(x)$$

is volume preserving for every t. Moreover, the construction satisfies also $s(0) = \dot{s}(0) = 0$. For every $c \in \mathbb{R}^{n+1}$, we define the translation

$$\psi_t^c\colon \mathcal{W} \longrightarrow \mathbb{R}^{n+1},\ \psi_t^c(x) = x + tc := x + t\varphi_c(x)\,\nu(x) + t\xi_c(x),$$

where we have set $\xi_c := c - \varphi_c\,\nu$.

By the very definition of the L^1-norm, we find

$$\|\psi_t^c - \psi_t\|_{L^1} = |U_{\mathcal{W}_t}\,\Delta\, U_{\mathcal{W}+tc}|, \tag{3.45}$$

where $U_{\mathcal{W}_t}$ and $U_{\mathcal{W}+tc}$ denote the open sets enclosed respectively by $\mathcal{W}_t := \psi_t(\mathcal{W})$ and $\mathcal{W} + tc$. Moreover, we can also write

$$\|\psi_t^c - \psi_t\|_{L^1} = \|t(\varphi_c - u)\nu - s(t)h\nu + t\xi_c\|_{L^1} \tag{3.46}$$

From (3.45) and (3.46) we obtain the inequality

$$t\|(\varphi_c - u)\nu + \xi_c\|_{L^1} \leqslant s(t) + |U_{\mathcal{W}_t} \,\Delta\, U_{\mathcal{W}+tc}|. \tag{3.47}$$

However, since ν and ξ_c are pointwise orthogonal, we find

$$\|u - \varphi_c\|_{L^1} \leqslant \|(\varphi_c - u)\nu + \xi_c\|_{L^1}.$$

Hence we obtain

$$t\|u - \varphi_c\|_{L^1} \leqslant s(t) + |U_{\mathcal{W}_t} \,\Delta\, U_{\mathcal{W}+tc}|. \tag{3.48}$$

We take the infimum in c in (3.48) and obtain the expression

$$t \inf_{c \in \mathbb{R}^{n+1}} \|u - \varphi_c\|_{L^1} \leqslant s(t) + \inf_{c \in \mathbb{R}^{n+1}} |U_{\mathcal{W}_t} \,\Delta\, U_{\mathcal{W}+c}|. \tag{3.49}$$

We need to estimate the right hand side in (3.49). For this purpose we introduce a new concept.

Let E be a set of finite perimeter in $\mathbb{R}^{n+1}$. We define the *anisotropic asymmetry index* as

$$A(E) := \min_{x \in \mathbb{R}^{n+1}} \left\{ \frac{|U_{\mathcal{W}} \,\Delta\, (x + rE)|}{|E|} : \ |rU_{\mathcal{W}}| = |E| \right\},$$

where $U_{\mathcal{W}}$ is the open set enclosed by $\mathcal{W}$, and the *anisotropic isoperimetric deficit* as

$$\delta(E) := \frac{\mathcal{F}(\partial E)}{(n+1)|U_{\mathcal{W}}|^{\frac{1}{n+1}} |E|^{\frac{n}{n+1}}} - 1.$$

The relation among $A(E)$ and $\delta(E)$ is well studied in the framework of isoperimetric problems. In particular, the following anisotropic deficit estimate proved in [18, Thm 1.1] holds:

Theorem 3.13. *Every set E of finite perimeter in $\mathbb{R}^{n+1}$ satisfies the following inequality:*

$$A(E) \leqslant C(n)\sqrt{\delta(E)}. \tag{3.50}$$

If $|E| = |U_{\mathcal{W}}|$, then the inequality 3.50 can be written as

$$A(E) \leqslant C(n)\sqrt{\mathcal{F}(\partial E) - \mathcal{F}(\mathcal{W})}. \tag{3.51}$$

Since $U_{\mathcal{W}_t}$ and $U_{\mathcal{W}+c}$ share the same volume, we can apply Theorem 3.13 to deduce that

$$t \inf_{c \in \mathbb{R}^{n+1}} \|u - \varphi_c\|_{L^1} \leqslant s(t) + C(n)\sqrt{\mathcal{F}(\mathcal{W}_t) - \mathcal{F}(\mathcal{W})}. \tag{3.52}$$

We plug (3.41) and (3.42) in (3.52) to get

$$t \inf_{c \in \mathbb{R}^{n+1}} \|u - \varphi_c\|_{L^1} \leqslant s(t) + O\left(t^{\frac{3}{2}}\right). \tag{3.53}$$

Dividing by t and letting $t \to 0$, we obtain

$$\inf_{c \in \mathbb{R}^{n+1}} \|u - \varphi_c\|_{L^1} = 0$$

and since the infimum is attained, the thesis is proven. □

3.4 PROOF OF PROPOSITION 3.7

In this section we conclude the proof of Proposition 3.7. To this aim we give the following definition:

Definition 3.14. Let $u \in C^\infty(\mathcal{W})$ be given. We define

$$h(u) := \sum_{i=1}^{n+1} \langle u,\, \varphi_i \rangle_{L^2} w_i, \tag{3.54}$$

where $\{ w_i \}_{i=1}^{n+1} \subset \mathbb{R}^{n+1}$ are chosen such that the associated functions $\varphi_i := \varphi_{w_i}$ (defined as in (3.11)) are an orthonormal frame in L^2 for the vector space $\{ \varphi_c \}_{c \in \mathbb{R}^{n+1}} = \ker L$. We will denote $\varphi_u := \varphi_{h(u)}$.

We will need the following proposition, whose proof is postponed in Appendix, see A.3:

Proposition 3.15. *There exists* $C = C(n,\, p,\, F) > 0$ *such that, for every* $u \in C^\infty(\mathcal{W})$*, the following holds:*

$$\|u - \varphi_u\|_{W^{2,p}(\mathcal{W})} \leqslant C \inf_{c \in \mathbb{R}^{n+1}} \|u - \varphi_c\|_{W^{2,p}(\mathcal{W})}. \tag{3.55}$$

Proof of Proposition 3.7. Let $\varepsilon > 0$ to be fixed small enough at the end of the argument. Let δ_0 be so small, that Propositions 3.4, 3.5 apply, and assume, up to translations, that the parametrisation $\psi \colon \mathcal{W} \longrightarrow \Sigma$ defined as in (3.5) satisfies the estimates (3.8) and (3.9).

We notice that, for sufficiently small $c \in U$, we can define

$$\psi_c \colon \mathcal{W} \longrightarrow \Sigma - c, \quad \psi_c(x) := x + u_c(x)\nu(x). \tag{3.56}$$

For such c the mapping ψ_c is an alternative radial parametrization for Σ, and it is a well defined diffeomorphism. We also define:

$$\Phi \colon U \longrightarrow \mathbb{R}^{n+1}, \quad \Phi(c) := \sum_{i=1}^{n+1} \langle u_c,\, \varphi_i \rangle_{L^2} w_i,$$

where $\{ w_i \}_{i=1}^{n+1}$ are as in Definition 3.14. Our idea is to prove the existence of $c_0 \in U$ such that $\Phi(c_0) = 0$. This is enough to conclude the proof, because $\Phi(c_0) = 0$ implies that $\varphi_{u_{c_0}} = \langle \Phi(c_0), \nu \rangle = 0$, which, together with Proposition 3.15, implies

$$\|u_{c_0}\|_{W^{2,p}(\mathcal{W})} \leqslant C\Big(\|\mathring{A}_F\|_{L^p(\Sigma)} + \varepsilon\, \|u_{c_0}\|_{W^{2,p}(\mathcal{W})} \Big). \tag{3.57}$$

Therefore, if we set $\varepsilon_0 = \min\{ \frac{1}{2C},\, \frac{1}{2} \}$, then the second term in the right hand side of (3.57) can be absorbed in the left hand side, obtaining

$$\|\psi_{c_0} - \mathrm{id}\|_{W^{2,p}(\mathcal{W})} = \|u_{c_0}\|_{W^{2,p}(\mathcal{W})} \leqslant C \|\mathring{A}_F\|_{L^p(\Sigma)}. \tag{3.58}$$

In this case, with $c = c_0$ we would easily conclude.

We are just left to find $c_0 \in U$ such that $\Phi(c_0) = 0$. First of all, it is easy to notice that there exist $\tilde{\varepsilon}$ and $\tilde{r}$ depending only on n and $\mathcal{W}$, such that, for every $0 < \varepsilon < \tilde{\varepsilon}$, if $\Sigma = \partial U$ satisfies

$$d_{HD}(U, U_{\mathcal{W}}) < \varepsilon,$$

then the ball $\mathbb{B}_{\tilde{r}}$ is contained in U. Hence we consider $0 < \varepsilon < \tilde{\varepsilon}$ so small that the ball $\mathbb{B}_{\tilde{a}\,\varepsilon}$ is contained in U for some $\tilde{a}$, depending only on n and $\mathcal{W}$, that we will choose later. We study Φ inside $\mathbb{B}_{\tilde{a}\,\varepsilon}$. We will show that Φ admits the following linearisation:

$$|\Phi(c) - \Phi(0) + c| \leqslant C(n, \mathcal{W})\varepsilon^2 \quad \text{for every } c \in \mathbb{B}^{n+1}_{\tilde{a}\,\varepsilon}. \tag{3.59}$$

First of all, for every c such that $|c| < \tilde{a}\,\varepsilon$ we find

$$d_{HD}(\Sigma - c, \mathcal{W}) \leqslant d_{HD}(\Sigma - c, \Sigma) + d_{HD}(\Sigma, \mathcal{W}) \leqslant (\tilde{a} + 1)\varepsilon.$$

Therefore, it is easy to see that also the function u_c satisfies the estimates

$$\|u_c\|_{C^1(\mathcal{W})} \leqslant C(n, F)\varepsilon, \tag{3.60}$$

We start the linearisation with the following simple consideration: for every $z \in \mathcal{W}$ and $c \in \mathbb{R}^{n+1}$ such that $|c| \leqslant \tilde{a}\varepsilon$ there exists $x_c = x_c(z) \in \mathcal{W}$ so that

$$\psi_c(z) = \psi(x_c(z)) - c.$$

We expand this equality and find

$$z + u_c(z)\,\nu(z) = x_c(z) + u(x_c(z))\nu(x_c(z)) - c. \tag{3.61}$$

Using the C^0-smallness of u and u_c, we can easily see that $x_c(z)$ satisfies the relation

$$|x_c(z) - z| \leqslant C(n, \mathcal{W})\varepsilon. \tag{3.62}$$

This approximation, combined with (3.60), gives an estimate of u close to z:

$$|u(x_c(z)) - u(z)| \leqslant C(n, \mathcal{W})\varepsilon^2. \tag{3.63}$$

Now we recall that $\mathcal{W} = \{ F^* = 1 \}$, with F^* defined in (0.2) as

$$F^*(x) := \sup_{\nu \in \mathbb{R}^{n+1}} \left\{ \langle x, \nu \rangle : |\nu| F\left(\frac{\nu}{|\nu|}\right) \leqslant 1 \right\}.$$

We recall property (0.3) the differential dF^* enjoys, i.e.

$$dF^*|_z\,[c] = \langle \nu(z), c \rangle, \qquad \forall z \in \mathcal{W},$$

Using (0.2), we evaluate F^* in the point in (3.61) and find:

$$\underbrace{F^*(z + u_c(z)\,\nu(z))}_{=1 + u_c(z)\,dF^*|_z[\nu(z)] + \mathcal{R}} \quad = \quad \underbrace{F^*(x_c + u(x_c)\,\nu(x_c) - c)}_{=1 + u(x_c)\,dF^*|_{x_c}[\nu(x_c)] - dF^*|_{x_c}[c] + \mathcal{R}} \quad ,$$

where $\mathcal{R}$ satisfies

$$|\mathcal{R}| \leqslant C(n, \mathcal{W})\varepsilon^2.$$

Plugging (0.3) and (3.63) in the previous equality, we obtain

$$|u_c(z) - u(z) + \underbrace{\langle c, \nu(z)\rangle}_{=\varphi_c(z)}| \leqslant C(n, \mathcal{W})\varepsilon^2. \tag{3.64}$$

Integrating over $\mathcal{W}$ and using (3.64), we finally obtain (3.59). In order to obtain the thesis, we prove the following claim:

Claim Let G be a continuous map $G\colon B_1^{n+1} \longrightarrow \mathbb{R}^{n+1}$ which satisfies the estimate

$$|G(x) - a - x| \leqslant \varepsilon \text{ with } |a| < \frac{1}{10}. \tag{3.65}$$

Then G must have 0 in its image if ε is sufficiently small.

This claim gives us the thesis since we can always reduce to this case by choosing an $\tilde{a}$ big enough (depending only on n and $\mathcal{W}$) and via a proper rescaling. Indeed, we define

$$\varphi\colon \mathbb{B}_1^{n+1} \longrightarrow \mathbb{R}^{n+1}, \varphi(c) := -\frac{\Phi(\tilde{a}\varepsilon c)}{\tilde{a}\varepsilon}.$$

The rescaled map satisfies

$$\left|\varphi(c) + \frac{\Phi(0)}{\tilde{a}\varepsilon} - c\right| = \frac{1}{\tilde{a}\varepsilon}|\Phi(\tilde{a}\varepsilon c) - \Phi(0) + \tilde{a}\varepsilon c| \leqslant \frac{C(n, \mathcal{W})\varepsilon}{\tilde{a}}.$$

Moreover,

$$\frac{|\Phi(0)|}{\tilde{a}\varepsilon}\varepsilon \leqslant \frac{C(n, \mathcal{W})}{\tilde{a}} \leqslant \frac{1}{10}$$

if we choose the proper $\tilde{a}(n, \mathcal{W})$. Therefore, by the claim, we can find $\tilde{c} \in \mathbb{B}_1^{n+1}$ such that $\varphi(\tilde{c}) = 0$, i.e. $\Phi(\tilde{a}\varepsilon\tilde{c}) = 0$, and we have finished. Let us prove the claim.

We argue by contradiction, and assume that 0 is not in the image of G. Therefore, the rescaled map

$$g := \frac{G}{|G|}\colon \mathbb{B}_1^{n+1} \longrightarrow S^n$$

is well defined. Now, we know that G satisfies (3.65). Thus, we obtain:

$$\begin{aligned} |G(x)|^2 &= |a + x|^2 + |G(x) - a - x|^2 + 2\langle a + x, G(x) - a - x\rangle \\ &= 1 + |a|^2 + 2\langle a, x\rangle + \mathcal{R}, \end{aligned} \tag{3.66}$$

where $|\mathcal{R}| \leqslant C(n, \mathcal{W})\varepsilon$. From (3.66) we have:

$$\frac{79}{100} - C(n, \mathcal{W})\varepsilon \leqslant |G(x)|^2 \leqslant \frac{121}{100} + C(n, \mathcal{W})\varepsilon. \tag{3.67}$$

We use inequalities (3.65) (3.66) and (3.67) to infer the following estimate:

$$\begin{aligned}|g(x)-x| &= \left|\frac{G(x)}{|G(x)|}-x\right| = \frac{1}{|G(x)|}|G(x)-|G(x)|x| \\ &= \frac{1}{|G(x)|}|G(x)-a-x+a+x(1-|G(x)|)| \leqslant \frac{1}{|G(x)|}(|a|+C\varepsilon+|1-|G(x)||) \\ &\leqslant \frac{10}{\sqrt{79-C\varepsilon}}\left(\frac{1}{10}+\frac{\sqrt{21}}{10}+C\sqrt{\varepsilon}\right) \leqslant \frac{1+\sqrt{21}}{\sqrt{79-C\varepsilon}}(1+C\sqrt{\varepsilon}) \leqslant \frac{\sqrt{2}}{2}+C\sqrt{\varepsilon}\end{aligned}$$

where the constant C depends only on n and $\mathcal{W}$. Therefore, for every $0<\varepsilon<1$ sufficiently small, we obtain

$$|g(x)-x|<2 \text{ for every } x\in\mathbb{S}^n. \tag{3.68}$$

Therefore the map $\overline{g} := g|_{\mathbb{S}^n}$ defined as the restriction of g to the sphere is well defined. The thesis follows by a simple application of topological degree theory, which can be found in [28, Ch.5]: since $\overline{g}$ is the restriction of a map on the sphere, it must have degree equal to 0, but (3.68) easily implies that $\overline{g}$ is homotopic to the identity, and therefore it must have degree equal to 1, giving the desired contradiction. □

4 THE GENERALISATION IN THE NON CONVEX CASE

In this section we remove the convexity hypothesis used in the previous chapters. As stated in the introduction and as a counterexample in Chapter 5 shows, this hypothesis is not artificial, so we shall find conditions that substitute it. The theorems we are presenting are the following.

Theorem 4.1. *Let $2 \leqslant n < p$ be given, and let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$. We assume that Σ satisfies the conditions*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n), \tag{4.1}$$

$$\|A\|_{L^p(\Sigma)} \leqslant c_0. \tag{4.2}$$

There exists positive constants δ_0, C depending on n, p, c_0 such that, if

$$\|\mathring{A}\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{4.3}$$

then there exist a vector $c = c(\Sigma)$ such that the radial parametrization $\psi\colon \mathcal{W} \longrightarrow \Sigma - c$ as in (1.4) is well defined and satisfies

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C\|\mathring{A}\|_{L^p(\Sigma)}. \tag{4.4}$$

Theorem 4.2. *Let $3 \leqslant n < p$ be given, Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ We assume that Σ satisfies the conditions (4.1) and (4.2). Then for every $q \in (n, p)$ there exists δ_0, $C > 0$ depending only on n, p, q, c_0 with the following property: if*

$$\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{4.5}$$

then there exists a $c = c(\Sigma)$ such that the radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c$ as in (1.4) is well defined and satisfies

$$\|f\|_{W^{2,q}(\mathbb{S}^n)} \leqslant C_0\|\mathring{\mathrm{Ric}}\|^{\alpha}_{L^p(\Sigma)}, \tag{4.6}$$

where α is given by:

$$\alpha(p, q) := \begin{cases} 1, & \text{if } n < q \leqslant p/2, \\ p/q - 1, & \text{if } p/2 \leqslant q < p. \end{cases}$$

Theorem 4.3. *Let $2 \leqslant n < p$ be given, and let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$. Let also $F\colon \mathbb{S}^n \longrightarrow (0, \infty)$ be an elliptic integrand, and $\mathcal{W}$ be the associated Wulff shape as in Chapter 3. We assume that Σ satisfies the conditions:*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathcal{W}), \tag{4.7}$$

$$\|A_F\|_{L^p(\Sigma)} \leqslant c_0. \tag{4.8}$$

There exist positive constants δ_0, C *depending on* n, p, c_0 *and* $\mathcal{W}$ *such that, if*

$$\|\mathring{A}_F\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{4.9}$$

then there exists a vector $c = c(\Sigma)$ *such that the anisotropic parametrization* $\psi\colon \mathcal{W} \longrightarrow \Sigma - c$ *as in* (3.5) *is well defined and satisfies*

$$\|u\|_{W^{2,p}(\mathcal{W})} \leqslant C\|\mathring{A}_F\|_{L^p(\Sigma)}. \tag{4.10}$$

Remark 4.4. In all the three theorems the convexity hypothesis is substituted by a L^p-control on the second fundamental form. We remark how in Theorems 4.1 and 4.3 the final estimate is the same as in their convex counterparts 1.2 and 3.1, while Theorem 4.2 provides instead a weaker conclusion compared to Theorem 2.1

It has to be noted that Theorem 4.3 implies 4.1, which is just the version with trivial anisotropy $F = 1$; nevertheless, we have decided to keep them separate because the proof of the latter can be simplified using the natural symmetries of the problem. Theorem 4.2 is also heavily based on 4.1.

4.1 PROOF OF THEOREM 4.1

The main ingredient for Theorem 4.1 is the following proposition.

Proposition 4.5. *For every* $0 < \varepsilon$ *there exists* $0 < \delta_0 = \delta_0(n, p, c_0, \varepsilon)$ *with the following property. Let* Σ *be a closed hypersurface in* $\mathbb{R}^{n+1}$ *satisfying* (4.1) *and* (4.2). *If*

$$\|\mathring{A}\|_{L^p(\Sigma)} \leqslant \delta_0,$$

then up to translation the radial parametrization $\psi\colon S^n \longrightarrow \Sigma$ *as in* (1.4) *is well defined, and the logarithmic radius* f *satisfy*

$$\|f\|_{C^1(S^n)} \leqslant \varepsilon. \tag{4.11}$$

Proposition 4.5 is the cornerstone of the section, because it builds the radial parametrization and gives a qualitative estimate of it. The rest of the proof will indeed follow by an adaptation of Propositions 1.5 and 1.6 in the non-convex case.

Proof of Proposition 4.5

We split the proposition in two parts. In the first part we achieve a C^0-closeness in a certain sense, in the second part we show how to use this preliminary result to build the parametrization. Before proving the first results, we state the main tool of this chapter, i.e. the graph parametrisations.

Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$, and $q \in \Sigma$ a given point. We say that φ_q is a *graph parametrisation around* q *with width* R if φ_q has the following form:

$$\varphi_q\colon \mathbb{B}^n_R \longrightarrow \Sigma,\ \varphi_q(z) = q + \Phi_q \begin{pmatrix} z \\ u_q(z) \end{pmatrix}, \tag{4.12}$$

where $\Phi_q \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is a matrix in the orthogonal group $O(n+1)$ chosen so that $\Phi_q[\mathbb{R}^n \times \{0\}] = T_q\Sigma$, $\Phi_q[e_{n+1}] = \nu_\Sigma(q)$. Graph parametrisations have great importance in the non-convex case. Indeed, since Σ satisfies (4.1) and (4.2), by Lemma 0.2 we have the existence of two numbers $0 < R_0$ and $0 < L_0 \leqslant \frac{1}{\sqrt{3}}$ depending on n, p and c_0 with the following, useful property: for every q there exists a graph parametrisation φ_q has width $R \geqslant R_0$, and every function u_q is L_0-Lipschitz. Throughout all the chapter, we shall use only the parametrisations provided by Lemma 0.2 and will denote them by φ_q. We will use the φ_q to obtain local estimates and Lemma 0.3 to make them global. Since we will now work with graph parametrisations, we need an equivalent of Lemma 1.9. This exists and it is stated in [41, Lemma 1.3].

Lemma 4.6. *Let φ_q be a graph parametrisation. Then the following formulas hold:*

$$g_{ij} = \delta_{ij} + \partial_i u_q \partial_j u_q, \tag{4.13}$$

$$g^{ij} = \delta^{ij} - \frac{\partial^i u_q \partial^j u_q}{1 + |\partial u_q|^2}, \tag{4.14}$$

$$\nu = \frac{1}{\sqrt{1 + |\partial u_q|^2}} \Phi_q \begin{pmatrix} \partial u_q \\ -1 \end{pmatrix} \tag{4.15}$$

$$A^i_j = \partial_i \left(\frac{\partial^j u_q}{\sqrt{1 + |\partial u_q|^2}} \right) \tag{4.16}$$

The proof of Lemma 4.6 is actually made in [41] with the "standard" graph parametrisation $\varphi(x) = (x, u(x))$, i.e. with $q = 0$, $\Phi_q = \mathrm{Id}$. However, it can be noted that the action of the isometries does not change the obtained expressions, because the translations disappear with derivatives and the rotations satisfy $\langle \Phi[v], \Phi[w] \rangle = \langle v, w \rangle$.

Graph parametrisations are strongly used in [41], and the author explores much of their properties. Our proofs with them are basically slight improvements of the strategy developed there.

Lemma 4.7. *For every $0 < \varepsilon$ there exists $0 < \delta_0 = \delta_0(n, p, c_0, \varepsilon)$ with the following property.*

Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying (4.1) and (4.2). If $\|A - \lambda_0 g\|_{L^p} \leqslant \delta_0$ for some $\lambda_0 \neq 0$, then for every $q \in \Sigma$, for every graph parametrisation φ_q around q, we have the following estimate:

$$\left\| u_q - \lambda_0^{-1} \left(\sqrt{1 - \lambda_0^2 |x|^2} - 1 \right) \right\|_{C^1} \leqslant \varepsilon. \tag{4.17}$$

Remark 4.8. Notice how in Lemma 4.7 we do not claim that λ_0 has to be equal to 1. The problem of finding the "right" λ_0 will be solved in the second part, when we will build the parametrization. The requirement of being not 0 is instead necessary, but as shown in [41, Remark 1.9] a closed hypersurface Σ must satisfy the lower bound

$$\|A\|_{L^p(\Sigma)} \geqslant C(n, p, \mathrm{Vol}_n(\Sigma)). \tag{4.18}$$

Since in our case $\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n)$, we avoid such degenerate cases.

Proof. By contradiction, let $(\Sigma^k)_{k\in\mathbb{N}}$ be a sequence of closed hypersurfaces satisfying (4.1), (4.2), $\lim_k \|A-\lambda_0 g^k\|_{L^p_k} = 0$, and let $(q^k)_{k\in\mathbb{N}}$ be a sequence of points $q^k \in \Sigma^k$ such that the associated graph parametrisations satisfy

$$\left\| u^k - \lambda_0^{-1}\left(\sqrt{1-\lambda_0^2|\cdot|^2} - 1\right)\right\|_{C^1} \geqslant \varepsilon_0 > 0.$$

We show how this is not possible, using an idea of [41, Cor. 1.2]. Firstly, we can assume w.l.o.g. that every q^k is equal to $\lambda_0^{-1} e_{n+1}$ and $\Phi_{q^k} = \mathrm{Id}$. Thus, since every Σ^k satisfies (4.1) and (4.2), we consider the graph parametrisations φ^k associated to q^k. The properties φ^k satisfies combined with (4.2) grant us:

$$\sup_k \|u^k\|_{W^{2,p}(\mathbb{B}^n_R)} \leqslant c(n, p, c_0) < +\infty.$$

Let us set $\nu^k := \frac{\partial u^k}{\sqrt{1+|\partial u^k|^2}}$. Then, from (4.16) and the contradiction hypothesis, we obtain

$$\lim_k \|\partial \nu^k - \lambda_0 \,\mathrm{Id}\|_{L^p_k(\mathbb{B}_R)} = \lim_k \|\partial(\nu^k - \lambda_0 x)\|_{L^p_k(\mathbb{B}_R)} = 0.$$

Setting $c^k = \fint \nu^k$, we get from Sobolev inequalities

$$\lim_k \|\nu^k - c^k - \lambda_0 x\|_{W^{1,p}(\mathbb{B}^n_R)} = 0.$$

Now, since c^k is clearly bounded and $\nu^k(0) = 0$, $\forall\, k$ and $n < p$, we also obtain the convergence

$$\lim_k \|\nu^k - \lambda_0 x\|_{W^{1,p}(\mathbb{B}^n_R)} = 0.$$

Let us define the function

$$h\colon \mathbb{B}^n_1 \longrightarrow \mathbb{R}^n,\ h(x) := \frac{x}{\sqrt{1-|x|^2}}.$$

The function h is smooth and has bounded derivatives in the ball $\mathbb{B}^n_\rho$ with $\rho \leqslant \frac{1}{2}$. Moreover it satisfies the equality

$$h(\nu^k) = \frac{1}{\sqrt{1+|\partial u^k|^2}} \frac{\partial u^k}{\sqrt{1-|\partial u^k|^2/(1+|\partial u^k|^2)}} = \partial u^k.$$

We obtain:

$$\begin{aligned}\lim_k \|h(\nu^k) - h(\lambda_0 x)\|_{W^{1,p}(\mathbb{B}^n_R)} &= \left\| \partial u^k - \frac{\lambda_0 x}{\sqrt{1-\lambda_0^2|x|^2}} \right\|_{W^{1,p}(\mathbb{B}^n_R)} \\ &= \lim_k \left\| \partial\left(u^k - \lambda_0^{-1}\sqrt{1-\lambda_0^2|x|^2}\right)\right\|_{W^{1,p}(\mathbb{B}^n_R)} = 0.\end{aligned}$$

With the same argument as before, we observe that u^k is converging in $W^{2,p}$ to $\lambda_0^{-1}\sqrt{1-\lambda_0^2|x|^2}$, and this is the desired contradiction. □

Next we show how Lemma 4.7 leads to a C^0-closeness to the sphere.

Corollary 4.9. *Under the hypothesis of Lemma 4.7, for every* $0 < \varepsilon$ *there exists* $0 < \delta_0 = \delta_0(n, p, c_0, \varepsilon)$ *such that*

$$d_{HD}\left(\Sigma, \mathbb{S}^n_{|\lambda_0|^{-1}}\right).$$

Proof. Let Σ, $0 < \varepsilon$ and $0 < \delta_0$ be given as in Lemma 4.7. We choose a point $q_0 \in \Sigma$, then rotate and translate Σ so that $q_0 = -\lambda_0^{-1} e_{n+1}$, $T_{q_0}\Sigma = \mathbb{R}^n \times \{0\}$. Hence the parametrisation has the simpler form $\varphi_0(x) = -\lambda_0^{-1} e_{n+1} + (x, u_0(x))$ and parametrize a portion of the sphere $\mathbb{S}^n_{|\lambda_0|^{-1}}$. Now take $q_1 \in \varphi_0(\mathbb{B}^n_R)$. Writing $q_1 = \varphi(z_1)$, then the following inequalities easily hold:

$$\left|q_1 - \left(z_1, \lambda_0^{-1}\sqrt{1 - \lambda_0^2|z_1|^2}\right)\right| \leqslant \varepsilon, \quad \left|T_{q_1}\Sigma - \left\langle \left(z, \lambda_0^{-1}\sqrt{1 - \lambda_0^2|z|^2}\right)\right\rangle^{\perp}\right| \leqslant \varepsilon$$

Now we apply Lemma 0.3: For every parametrisation φ_q we can find a geodesic ball $\mathbb{B}^g_\rho(q)$ with $\rho = \rho(n, p, c_0)$ and satisfying condition (0.8), namely

$$\varphi_q\left(\mathbb{B}^n_{\frac{1}{1+L}\rho}\right) \subset \mathbb{B}^g_\rho(q) \subset \varphi_q\left(\mathbb{B}^n_\rho\right).$$

Via this lemma we can easily obtain a covering of N geodesic balls $\mathbb{B}^g(q_1), \ldots \mathbb{B}^g(q_N)$, where $N \leqslant N_0(n, p, c_0)$ such that Lemma 4.7 holds for $\varphi_{q_1}, \ldots \varphi_{q_N}$. Iterating the process, by a simple induction we easily find a constant $c(n, p, c_0)$ such that

$$\left|q - |\lambda_0|^{-1}\frac{q}{|q|}\right| \leqslant c\varepsilon, \quad \left|T_q\Sigma - \langle\, q\,\rangle^{\perp}\right| \leqslant c\varepsilon. \tag{4.19}$$

This proves the C^0-closeness. □

We finish the proof by proving that Σ can be parametrized as a sphere parametrisation given by (1.4) and that $\lambda_0 = 1$. Indeed, the proof of Corollary 4.9 does not only show a qualitative C^0-closeness, but also a C^1. Thus, we define the projection

$$p\colon \Sigma \longrightarrow \mathbb{S}^n_{|\lambda_0|^{-1}},\ p(q) := |\lambda_0|^{-1}\frac{q}{|q|}.$$

We start by proving that p is a local diffeomorphism. The map is clearly differentiable, and a straight computation proves that the differential of p at $q \in \Sigma$ is given by

$$dp|_q : T_x\Sigma \longrightarrow T_{p(q)}\mathbb{S}^n,\ dp|_q[v] = \frac{|\lambda_0|^{-1}}{|q|}\left(v - \left\langle v, \frac{q}{|q|}\right\rangle \frac{q}{|q|}\right) \tag{4.20}$$

It is easy to see that $\ker dp|_q = \{\, tq \mid t \in \mathbb{R}\,\}$. We want to prove that the differential $dp|_q$ has maximal rank at every q, and this will prove that p is a local diffeomorphism. In order to achieve this goal, we just need to show that for every $q \in \Sigma$, q does not belong to $T_q\Sigma$, and this is exactly what (4.19) implies. Hence p is a local diffeomorphism. Let us show that it is a global one. Indeed, we consider the multiplicity function

$$\eta\colon \mathbb{S}^n \longrightarrow \mathbb{N},\ \eta(x) := \sum_{p(q)=x} 1.$$

The function η is well-defined, and since p is a local diffeomorphism, it is continuous, thus necessarily constant, say $\eta \equiv Q$. Then it is a Q-covering, but since $\mathbb{S}^n$ is simply connected, we must have $Q = 1$, and hence p is a diffeomorphism. Let us define $\psi := p^{-1}$. By construction, we find that $\psi(x) = e^{f(x)}x$ as in (1.4), and (4.19) tells us that f has small C^1-norm. This concludes the construction.

Finally we can conclude the proof of the proposition. Let us argue by compactness and consider a sequence of closed hypersurfaces $\left(\Sigma^k\right)_{k\in\mathbb{N}}$ satisfying (4.1), (4.2), and $\lim_k \|\mathring{A}\|_{L^p_k} = 0$. By Theorem 0.1 we get the existence of a sequence $\left(\lambda^k\right)_{k\in\mathbb{N}}$ so that

$$\|A - \lambda^k g^k\|_{L^p_k} \leqslant C(n, p, c_0)\|\mathring{A}\|_{L^p_k} \downarrow 0.$$

The sequence $\left(\lambda^k\right)_{k\in\mathbb{N}}$ is clearly bounded. Up to extraction of a subsequence we can assume $\lambda^k \to \lambda_0$ which has to be non-zero because of (4.18). We show the equality $|\lambda_0| = 1$. This is then given by the area formula. Indeed, patching Lemma 4.7 and Corollary 4.9 we obtain that, up to translating, the hypersurfaces Σ^k are radially parametrized by a map

$$\psi^k \colon \mathbb{S}^n_{|\lambda_0|^{-1}} \longrightarrow \Sigma,\ \psi(x) = e^{f^k(x)}x,\ \|f\|_{C^1} \leqslant \varepsilon.$$

Then, we have:

$$1 = \frac{\mathrm{Vol}_n(\Sigma)}{\mathrm{Vol}_n(\mathbb{S}^n)} = |\lambda_0|^{-1} \fint_{\mathbb{S}^n_{|\lambda_0|^{-1}}} e^{nf^k}\sqrt{1+|\nabla f^k|^2}\,dV_\sigma = |\lambda_0|^{-1}\left(1 + O\left(\|f^k\|_{C^1}\right)\right).$$

For $k \to \infty$ we obtain that $|\lambda_0| = 1$. The conclusion of the proposition follows by showing that $\lambda_0 = 1$, thus implying that every subsequence of $\left(\lambda^k\right)_{k\in\mathbb{N}}$ converges to 1 and hence the whole sequence. Firstly, we notice that every $n\lambda^k$ must be very close to the average of the mean curvature $\overline{H^k}$. Indeed,

$$|\overline{H^k} - n\lambda^k| \leqslant \fint_{\Sigma^k}\left|H^k - n\lambda^k\right| = \fint_{\Sigma^k}\left|\langle A^k - \lambda^k g^k, g^k\rangle\right| \leqslant C(n, p, c_0)\|\mathring{A}\|_{L^p_k} \downarrow 0.$$

Now we show that $\overline{H^k}$ must be close to n and conclude. This follows by a simple estimate.

$$\begin{aligned}\overline{H^k} &= \fint_{\mathbb{S}^n} ne^{(n-1)f^k} - \fint_{\mathbb{S}^n}\mathrm{div}\left(\frac{\nabla f^k}{\sqrt{1+|\nabla f^k|^2}}\right)e^{(n-1)f^k}\sqrt{1+|\nabla f^k|^2}\\ &= n\fint_{\mathbb{S}^n} ne^{(n-1)f^k} + \fint_{\mathbb{S}^n} e^{(n-1)f^k}\left(\frac{(n-1)|\nabla f^k|^2}{\sqrt{1+|\nabla f^k|^2}} + \frac{\nabla^2 f^k[\nabla f^k, \nabla f^k]}{1+|\nabla f^k|^2}\right)\end{aligned}$$

Since every Σ^k satisfies (4.2), we easily obtain that the sequence $\left(f^k\right)_{k\in\mathbb{N}}$ is uniformly $W^{2,p}$-bounded, and thus

$$\left|\overline{H^k} - n\right| \leqslant C(n, p, c_0)\|f^k\|_{C^1} \downarrow 0.$$

This shows that λ_0 must be equal to 1, and all the computations we have made do not actually depend on the chosen subsequence.

Conclusion

Insofar we have found a qualitative convergence. We will next make it quantitative. This part is rather simple, because it follows from the arguments in Chapter 1. Indeed, although most propositions are stated under the convexity assumptions, one can easily notice that the computational proposition do not actually require convexity: what is needed are the C^1-closeness of f to 0 and the oscillation proposition stated in 0.1. From the latter one we can infer inequality (1.21) and apply Proposition 1.15 which was on purpose proved without convexity assumption, and obtain estimate (1.25). Then, the same arguments made below give us the linearised estimate

$$\|f - \varphi_{\nu_f}\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C\Big(\|\mathring{A}\|_{L^p(\Sigma)} + \varepsilon\|f\|_{W^{2,p}(\mathbb{S}^n)}\Big),$$

with the constant C depending this time on n, p and c_0. What is left is to prove that φ_{ν_f} is actually negligible, and this can be done by proving that we can center the hypersurface so that $b(\Sigma) = 0$.

Since Σ is not convex this time, it can be a priori impossible to translate it and keep a radial parametrization. However, this is not a problem, and it is done by looking carefully at the proof of Corollary 4.9, where we build the parametrization. In the proof of it, we chose a random point $q_k \in \Sigma_k$ and fix it to be $-\lambda_0 e_{n+1}$, then we perform our analysis. In order to center Σ better, we just improve to proof in 4.9 by choosing q_k more cleverly. Indeed, let again $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of hypersurfaces satisfying (4.1), (4.2) and $\lim_k\|\mathring{A}\|_{L^p_k} = 0$. We apply a translation so that $(b(\Sigma_k))_{k\in\mathbb{N}} = 0$ for every k, and choose q_k so that

$$|q_k|^2 = \max_{q\in\Sigma}|q|^2.$$

It is easy to see that for such choice we have the equality $T_{q_k}\Sigma_k = \langle\, q_k \,\rangle^{\perp}$. The study we made above also grants us the limit:

$$\lim_k\|A - \mathrm{Id}\|_{L^p_k} = 0.$$

We follow again the same argument of Lemma 4.7, choosing this time q_k as first point for the covering argument, and obtain that the sequence $(\Sigma_k)_{k\in\mathbb{N}}$ is converging to a sphere $\mathbb{S}^n(c)$ with center c. Since the barycenter condition $b(\Sigma_k) = 0$ passes to the limit, we also obtain that this sphere must satisfy $b(\mathbb{S}^n(c)) = 0$, therefore implying $c = 0$. Now we repeat the same argument, and obtain the following proposition:

Proposition 4.10. *For every $0 < \varepsilon$ there exists $0 < \delta_0 = \delta_0(n, p, c_0, \varepsilon)$ with the following property.*

If Σ is a closed hypersurface satisfying (4.1), (4.2) and $\|\mathring{A}\|_{L^p(\Sigma)} \leqslant \delta_0$, then there exists a vector $c \in \mathbb{R}^{n+1}$ such that $b(\Sigma - c) = 0$ and the radial parametrization

$$\psi\colon \mathbb{S}^n \longrightarrow \Sigma - c,\ \psi(x) = e^{f(x)}x$$

is well defined. Moreover, $\|f\|_{C^1(\Sigma)} \leqslant \varepsilon$.

Via this proposition and the previous discussion, we can obtain Theorem 4.1.

4.2 PROOF OF THEOREM 4.2

Theorem 4.2 requires a preliminary study. The strategy we would like to use is basically the same as the one used for the previous theorem, that is:

Let us consider a sequence of hypersurfaces $(\Sigma_k)_{k\in\mathbb{N}}$ satisfying (4.1), (4.2), and

$$\lim_k \|\mathring{\mathrm{Ric}}\|_{L^p_k} = 0.$$

Firstly, we estimate the diameter of Σ_k and consider a (not relabeled) subsequence Σ_k that converges in the Hausdorff distance to a subset $\Sigma_0 \subset \mathbb{R}^{n+1}$. *If* Σ_0 were a smooth manifold, and *if* the the decay of the traceless Ricci tensor passed to the limit, than Σ would be a smooth, closed Einstein manifold in $\mathbb{R}^{n+1}$, which is necessarily the round sphere. Then, performing a fine analysis of the φ_q, we would obtain than every graph parametrisation of Σ_k must converge to the graph parametrisation of the sphere, and thus we could build the same proof made for 4.1.

The problem here are the two ifs, which have to be motivated. First of all, the set Σ_0 we will find is a priori only a compact subset in $\mathbb{R}^{n+1}$; moreover, as we pointed out many times, the Ricci operator is not elliptic when viewed as a differential operator acting on the function which describes Σ as a graph parametrisation. Also if we consider the Gauss equation $\mathrm{Riem} = A \owedge A$ and consider the associated polynomial equation for the eigenvalues $\{x_1, \dots x_n\}$ of A, then the equality

$$\mathrm{Ric}(x) = (n-1)\lambda$$

implies $A = \lambda\,\mathrm{Id}$ only when $\lambda > 0$, as shown in Chapter 2. Thus, we also need to prove the positivity of λ in order to achieve our result. Lastly, even if we are able to fix these problems, the lack of a C^0-bound on the second fundamental form does not allow us to apply the strategies we have seen in Chapter 2. Thus, we must follow an alternative strategy. We split the proof of the qualitative closeness into two main propositions.

Proposition 4.11. *Let* $\varphi_k\colon \mathbb{B}^n_R \longrightarrow \mathbb{R}^{n+1}$ *be a sequence of graph parametrizations, and let then* $\mathrm{Graph}(u_k, \mathbb{B}^n_R)$ *be their image. Assume that every* u_k *satisfies the following:*

- $u_k(0) = 0$, $\partial u_k(0) = 0$.
- $\|u_k\|_{W^{2,p}} \leqslant c_0$.
- $u_k \rightharpoonup u_0$ *weakly in* $W^{2,p}$.
- *The sequence* $\big(\mathrm{Graph}(u_k, B^n_R)\big)_{k\in\mathbb{N}}$*, seen as sequence of hypersurfaces, satisfies*

$$\lim_k \|\mathrm{Ric} - (n-1)\lambda_0 g_k\|_{L^p_k} = 0.$$

Then there exists a radius $0 < \rho_0 = \rho_0(n, p, c_0)$ *such that the function* u_0 *is smooth (actually analytic) in* $\mathbb{B}^n_{\rho_0}$*, and the hypersurface* $\mathrm{Graph}(u_0, \mathbb{B}^n_{\rho_0})$ *is Einstein.*

Proposition 4.12. *For every $0 < \varepsilon$ there exists $0 < \delta = \delta(n, p, c_0, \varepsilon)$ with the following property.*
Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying (4.1) and (4.2). If $\|\mathrm{Ric} - (n-1)\lambda_0 g\|_{L^p} \leqslant \delta$, then $\lambda_0 > 0$, and for every $q \in \Sigma$, the graph parametrisation φ_q satisfies:

$$\left\| u_q - \mu_0^{-1}\left(\sqrt{1 - \mu_0^2 |\cdot|^2} - 1\right)\right\|_{C^1} \leqslant \varepsilon, \tag{4.21}$$

where $\mu_0 = \sqrt{\lambda_0}$.

Combining these two propositions, we obtain the C^1-closeness, and then we show how to conclude.

4.2.1 *Proof of the C^1-closeness*

We start by proving the first proposition.

Proof of Proposition 4.11. The proof uses the concept of *harmonic coordinates*. We recall the definition: given a manifold (M, g) and an open set $U \subset M$ a mapping $y\colon U \longrightarrow \mathbb{R}^{n+1}$ is said to be a *harmonic chart* if it is a diffeomorphism and if it satisfies the equation

$$\Delta_g y = 0.$$

The functions $y^1, \dots y^n$ are called *harmonic coordinates*. A detailed study on the topic can be found in [29, Sec. 8.10, p.523] or [43, Ch. 10, Sec. 2.3]. Harmonic coordinates have several properties which make them very suitable for our problem. Indeed, the following expression holds:

$$-\frac{1}{2}\Delta_g g_{ij} + Q_{ij}(g, \partial g) = \mathrm{Ric}^g_{ij} \text{ for every indices } i, j, \tag{4.22}$$

where $g_{ij} := g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$, Q_{ij} is a universal polynomial depending on g and its first derivatives ∂g. The computations can be found in [43, Ch. 10, Sec. 2.3].

In the aforementioned references however, the authors work under stronger regularity assumptions on the metric. In our case we ought to perform a finer study. We prove the following result.

Lemma 4.13. *Let $u\colon \mathbb{B}^n_R \longrightarrow \mathbb{R}$ be given so that $u(0) = 0$, $\partial u(0) = 0$, $\|u\|_{W^{2,p}(\mathbb{B}^n_R)} \leqslant c_0$. Set $G_\rho := \mathrm{Graph}(u, \mathbb{B}^n_\rho)$ for $0 < \rho \leqslant R$. Then there exist $0 < \rho_0 = \rho_0(n, p, c_0)$ and a diffeomorphism $\eta\colon G_{\rho_0} \longrightarrow \mathbb{R}^n$ such that*

$$\Delta_g \eta = 0, \ \|\eta\|_{W^{2,p}(G_{\rho_0})} \leqslant c_0,$$

with Δ_g being the Laplace-Beltrami operator associate to the manifold G_{ρ_0}.

Proof. By pull-back we work on the sequence $(\mathbb{B}^n_R, g_k)_{k\in\mathbb{N}}$, with $g_k = \delta + \partial u_k \otimes \partial u_k$. We are going to show the existence of a $0 < \rho_0 = (n, p, c_0) < R$ such that the map $\eta\colon \mathbb{B}^n_{\rho_0} \longrightarrow \mathbb{R}^n$ defined by

$$\begin{cases} \Delta_g \eta = 0 \text{ in } \mathbb{B}^n_{\rho_0}, \\ \eta|_{\partial \mathbb{B}^n_{\rho_0}} = x \end{cases}$$

is a diffeomorphism and satisfies $\|\eta\|_{2,p} \leqslant c_0$. In order to simplify the proof, we will consider a rescaled version the problem. Firstly, let us recall the expression in chart of the Laplace-Beltrami operator:

$$\Delta_g = \frac{1}{\sqrt{\det g}}\partial_i\Big(\sqrt{\det g}g^{ij}\partial_j\Big). \tag{4.23}$$

Let $\eta\colon \mathbb{B}^n_\rho \longrightarrow \mathbb{R}^n$ be a map satisfying $\Delta_g\eta = 0$. We say that the map

$$\eta_\rho\colon \mathbb{B}^n_1 \longrightarrow \mathbb{R}^n,\ \eta_\rho(z) := \frac{\eta(\rho z)}{\rho}$$

satisfies $\Delta_{g_p}\eta_p = 0$, where Δ_{g_ρ} is the Laplace-Beltrami operator associated to the metric $g_\rho(z) := g(\rho z)$, defined on the ball $\mathbb{B}^n_1$. Indeed, if we set $a^{ij} := \sqrt{\det g}g^{ij}$ and $a^{ij}_\rho := \sqrt{\det g_\rho}g^{ij}_\rho$, then

$$\begin{aligned}\partial_i\big(a^{ij}_\rho\partial_j\eta_\rho\big)(z) &= \partial_i a^{ij}_\rho(z)\partial_j\eta_\rho(z) + a^{ij}_\rho(z)\partial^2_{ij}\eta_\rho(z)\\ &= \rho\big(\partial_i a^{ij}(\rho z)\partial_j\eta(\rho z) + a^{ij}(\rho z)\partial^2_{ij}\eta(\rho z)\big) = \rho\partial_i\big(a^{ij}\partial_j\eta\big)(\rho z) = 0.\end{aligned}$$

Moreover, since $g = \delta + \partial u \otimes \partial u$ and u satisfies $u(0) = |\partial u(0)| = 0$ and $\|u\|_{2,p} \leqslant c_0$, then we also have

$$\lim_{\rho\to 0}\|g_\rho - \delta\|_{W^{1,p}(\mathbb{B}^n_1)} = 0.$$

We have reduced the problem to the following formulation:

> There exists $0 < \varepsilon_0 = \varepsilon_0(n, p)$ with the following property. If g is a metric on $\mathbb{B}^n_1$ such that $\|g - \delta\|_{W^{1,p}} \leqslant \varepsilon_0$, then there exists a diffeomorphism $\eta\colon \mathbb{B}^n_1 \longrightarrow \mathbb{R}^n$ such that
>
> $$\Delta_g\eta = 0,\ \|\eta - \mathrm{id}\|_{W^{2,p}} \leqslant \varepsilon_0.$$

As stated before, we prove that the only solution η of the problem

$$\begin{cases}\Delta_g\eta = 0,\\ \eta|_{\partial\mathbb{B}_1} = x\end{cases}$$

is a diffeomorphism, provided that ε_0 is sufficiently small. The solution η exists and it is smooth, since the coefficients are smooth. We prove that η satisfies the aforementioned a priori $W^{2,p}$-estimate and is a diffeomorphism in $\mathbb{B}_1$. From (4.23) we get that our equation is of the divergence form:

$$\partial_i\big(a^{ij}\partial_j\eta\big) = 0,\ \text{where } \|a^{ij} - \delta^{ij}\|_{W^{1,p}} \leqslant \varepsilon_0.$$

Since $n < p$, we have that the Sobolev closeness is also a $C^{0,\alpha}$-one, thus we obtain that for ε_0 sufficiently small the matrix $a = a^{ij}$ satisfies the bound

$$\frac{1}{2}\delta \leqslant a \leqslant 2\delta$$

in the sense of quadratic forms. This bound will be useful when we will deal with sequences of metrics converging weakly, because it passes to the limit and triggers the classical elliptic theory for weak solutions (see [20]). Now we conclude:

$$\|\eta - x\|_{W^{2,p}(\mathbb{B}_1^n)} \leqslant C(n, p)\|\Delta_g(\eta - x)\|_{L^p(\mathbb{B}_1^n)} = C\|\Delta_g x\|_{L^p(\mathbb{B}_1^n)},$$

and therefore

$$\begin{aligned}\Delta_g x^k &= \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ij}\partial_j x^k\right) = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ik}\right) \\ &= \partial_i g^{ik} + \mathrm{tr}(g^{pq}\partial_i g_{pq})g^{ik}.\end{aligned}$$

From this computation we obtain

$$\|\eta - x\|_{W^{2,p}(\mathbb{B}_1^n)} \leqslant C(n, p)\|g - \delta\|_{W^{1,p}(\mathbb{B}_1^n)} \leqslant C\varepsilon_0,$$

and for ε_0 sufficiently small we obtain the thesis. □

We use the lemma to prove 4.11. Indeed, let φ_k, u_k and $\mathrm{Graph}(u_k, \mathbb{B}_R^n)$ be as in the hypothesis of 4.11. Using the pull-back, we work in $\left(\mathbb{B}_R^n, g_k\right)$. Since $u_k \rightharpoonup u_0$ weakly in $W^{1,p}$ and $n < p$ by hypothesis, we know that $u_k \to u_0$ strongly in $C^{1,\alpha}$, then $\partial u_k \otimes \partial u_k \to \partial u_0 \otimes \partial u_0$ strongly in $C^{1,\alpha}$, and

$$\underbrace{\partial^2 u_k \otimes \partial u_k + \partial u_k \otimes \partial^2 u_k}_{\partial(\partial u_k \otimes \partial u_k)} \rightharpoonup \underbrace{\partial^2 u_0 \otimes \partial u_0 + \partial u_0 \otimes \partial^2 u_0}_{\partial(\partial u_0 \otimes \partial u_0)} \text{ in } L^p. \tag{4.24}$$

From (4.24) we deduce $g_k \rightharpoonup g_0 = \delta + \partial u_0 \otimes \partial u_0$ in $W^{1,p}$. We consider then harmonic coordinates $\eta_k \colon \mathbb{B}_{\rho_0}^n \longrightarrow \mathbb{R}^n$ built as in Lemma 4.13. From the discussion made above, one can easily infer that η_k converges weakly in $W^{2,p}$ to the vector valued function η_0 which is of class $W^{2,p}$ and weakly solves the system

$$\begin{cases}\Delta_{g_0}\eta_0 = 0, \\ \eta_0|_{\partial\mathbb{B}_{\rho_0}^n} = x,\end{cases}$$

Let us now call

$$g_{ij}^k := g_k\left(\frac{\partial}{\partial\eta_k^i}, \frac{\partial}{\partial\eta_k^j}\right), \text{ where } \frac{\partial}{\partial\eta^i} := d\eta\left[\frac{\partial}{\partial x^i}\right].$$

Then every g_{ij}^k solves system (4.22), namely

$$-\frac{1}{2}\Delta_{g^k} g_{ij}^k + Q_{ij}(g^k, \partial g^k) = \mathrm{Ric}_{ij}^k,$$

with $\mathrm{Ric}_{ij}^k := \mathrm{Ric}^k\left(\frac{\partial}{\partial\eta_k^i}, \frac{\partial}{\partial\eta_k^j}\right)$. Then, this equation passes to the limit in $g_{ij}^0 = g_0\left(\frac{\partial}{\partial\eta_0^i}, \frac{\partial}{\partial\eta_0^j}\right)$, which solves the distributional equation

$$-\frac{1}{2}\Delta_{g^0} g_{ij}^0 + Q_{ij}(g^0, \partial g^0) = \lambda_0 g_{ij}^0. \tag{4.25}$$

Following the computations leading to equation (4.22) as made in [43, Sec. 2.3], we can easily notice that the polynomial $Q_{ij}(g^0, \partial g^0)$ is of class $L^{p/2}$. Therefore we can apply the bootstrap technique to deduce regularity. Indeed, every g^0_{ij} is a $W^{1,p}$-weak solution of the equation

$$L[v] = f,$$

where f is a $L^{p/2}$-function. By the Morrey estimates, we know that every g^0_{ij} is actually in $W^{2,p/2}$, and in particular

$$\partial g^0 \in L^{(p/2)^*}, \text{ where } (p/2)^* = \frac{n(p/2)}{n-(p/2)} = \frac{np}{2n-p}.$$

A straight computation shows

$$(p/2)^* > p \Leftrightarrow \frac{np}{2n-p} > p \Leftrightarrow p > n,$$

and therefore every $Q_{ij}(g^0, \partial g^0) \in L^{p_1/2}$ for some $p_1 = (p/2) > p$. We proceed inductively until we find $Q_{ij}(g^0, \partial g^0) \in L^{p_N/2}$ for some $p_N > 2n$. In this case, we obtain that every $g^0_{ij} \in C^{1,\alpha}$. At this point, we notice that $Q_{ij}(g^0, \partial g^0) \in C^{0,\alpha}$. From the Schauder estimates we infer that every $g^0_{ij} \in C^{2,\alpha}$, thus rendering a^{ij}, $Q_{ij} \in C^{1,\alpha}$. Inductively we obtain that g^0 is in $C^{k,\alpha}$, therefore it is smooth. It can be also proved, that in this context, the metric is actually analytic, and hence we obtain our desired regularity. We refer to [20] for an overall synthesis on all the aforementioned estimates and elliptic regularity results. Since g^0 is regular and satisfies (4.25), then the hypersurface Graph$(u_0, \mathbb{B}^n_{\rho_0})$ is Einstein and $\mathrm{Ric}^0 = \lambda_0 g^0$. □

Now we deal with Proposition 4.12

Proof of Proposition 4.12. Again we need a useful lemma.

Lemma 4.14. *If* Σ *satisfies* (4.1) *and* (4.2)*, then there exists* $0 < D_0 = D_0(n, p, c_0)$ *such that*

$$\mathrm{diam}_g \Sigma \leqslant D_0.$$

Proof. As often argued throughout the work, a smooth, closed hypersurface satisfying (4.1) and (4.2) with $n < p$ allows us to apply Lemmas 0.2 and 0.3. Now we consider two points p_0, $q \in \Sigma$, such that $d_g(p_0, q) = \mathrm{diam}_g(\Sigma)$. Such points clearly exist by compactness. By virtue of Lemma 0.3 we are able to find Q geodesic balls $\mathbb{B}^g_1, \dots \mathbb{B}^g_Q$, with the following properties: $p_0 \in \mathbb{B}^g_1$, $q \in \mathbb{B}^g_Q$, $\mathbb{B}^g_i \cap \mathbb{B}^g_{i+1} \neq \varnothing$ and $Q \leqslant N$, where $N = N(n, c_0, p)$ is the natural number given by 0.3. Then, for every $i = 1, \dots, Q-1$ we choose a point $p_i \in \mathbb{B}^g_i \cap \mathbb{B}^g_{i+1}$, and set $p_Q := q$. Naturally, since $p_i, p_{i+1} \in B^g_i$, the following inequality holds:

$$d_g(p_i, q_i) \leqslant 2R.$$

Then by triangle inequality, we find our desired bound.

$$\mathrm{diam}_g(\Sigma) = d_g(p_0, q) = d_g(p_0, p_Q) \leqslant \sum_{i=0}^{Q-1} d_g(p_i, p_{i+1}) \leqslant 2QR = D(n, p, c_0).$$

□

We now come to the proof of Proposition 4.12. Let us argue by compactness, and let $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of closed hypersurfaces satisfying (4.1), (4.2) and satisfying

$$\lim_k \|\mathrm{Ric}^{g_k} - (n-1)\lambda_0 g_k\|_{L^p_k} = 0.$$

Up to translations, we can assume $b(\Sigma_k) = 0$ for every $k \in \mathbb{N}$, where

$$b(\Sigma) = \fint_\Sigma x \, dV_g(x)$$

denotes again the barycentre of Σ. Then the sequence $(\Sigma_k)_{k\in\mathbb{N}}$ is a sequence of compact sets, all enclosed in a ball, and thus we can use the classical compactness theorem of Hausdorff to extract a subsequence converging in the Hausdorff distance to a compact set $\Sigma_0 \subset \mathbb{R}^{n+1}$. Let $q_0 \in \Sigma_0$ be a point that attains the maximum distance from 0, i.e.

$$|q_0|^2 = \max_{q\in\Sigma_0} |q|^2.$$

Let then $(q_k)_{k\in\mathbb{N}}$ be a sequence of points $q_k \in \Sigma_k$ converging to q_0, and φ_k be the associated graph parametrisations with center q_k and width R. Then, up to subsequences, φ_k converges weakly in $W^{2,p}$ to a function $\varphi_0 \colon \mathbb{B}^n_R \longrightarrow \mathbb{R}^{n+1}$. Since

$$\varphi_k(z) = q_k + \Phi_k \begin{pmatrix} z \\ u^k(z) \end{pmatrix},$$

it is obvious that $\Phi_k \to \Phi_0$ and $u^k \rightharpoonup u_0$ weakly in $W^{2,p}$. Hence φ_0 is a graph parametrisation, and $\varphi_0(0) = q_0$, $\varphi_0(\mathbb{B}^n_R) \subset \Sigma_0$. Moreover, since the isometries Φ_k clearly alter neither the final result nor the proof, we are therefore in the hypothesis of Proposition 4.11, and obtain that u_0 is actually smooth and $\varphi_0(\mathbb{B}^n_{\rho_0}) = \mathrm{Graph}(u_0, \mathbb{B}^n_\rho) \subset \Sigma_0$ is a smooth, Einstein manifold. The map φ_0 has another remarkable property: it satisfies

$$|\varphi(0)|^2 = |q_0|^2 = \max_{z\in\mathbb{B}^n_{\rho_0}} |\varphi(z)|^2.$$

Deriving twice, we obtain the following equalities holding in 0:

$$\underbrace{\langle \partial_i\varphi_0, \varphi_0(0) \rangle = 0}_{\Rightarrow \langle q_0 \rangle^\perp = T_{q_0}\Sigma}, \quad \partial^2\varphi(0) \leqslant 0 \Rightarrow \underbrace{\langle \partial^2_{ij}\varphi_0, \varphi_0(0) \rangle}_{=-|q_0|^{-1}A_{ij}} + \underbrace{\langle \partial_i\varphi_0, \partial_j\varphi_0 \rangle}_{=g_{ij}} \leqslant 0,$$

from which we obtain the equality

$$A|_{q_0} \geqslant \frac{1}{|q_0|} \, g|_{q_0}. \tag{4.26}$$

Equality (4.26) holds just in one point, but it is enough: indeed, $\varphi_0(\mathbb{B}^n_{\rho_0})$ is smooth and Einstein, thus at q_0 we also have the estimate:

$$(n-1)\lambda_0 g = \mathrm{Ric} \geqslant \frac{(n-1)}{|q_0|^2} g \geqslant \frac{(n-1)}{D_0^2} g \Rightarrow \lambda \geqslant \frac{1}{D_0^2},$$

and hence $\lambda_0 > 0$. Since φ_0 parametrizes an Einstein hypersurface, the equality holds in the whole ball $\mathbb{B}^n_{\rho_0}$. Thanks to Theorem 2.4 we obtain that $A = \mu_0 g$, where $\mu_0 = \sqrt{\lambda_0}$. This tells us that φ_0 parametrizes a portion of a round sphere with radius μ_0^{-1}. Since φ_k converges weakly to φ_0 in $W^{2,p}$, we obtain that also the associated function u^k converge to $\mu_0^{-1}\left(\sqrt{1-\mu_0^2|x|^2}-1\right)$ weakly in $W^{2,p}$. Since $n < p$ the convergence is also strong in $C^{1,\alpha}$. The study we made insofar works not only for φ_0 but for every possible parametrization: let us go back to our sequence $(\Sigma_k)_{k\in\mathbb{N}}$ of closed hypersurface. Now we know that $\lambda_0 > 0$, and thus for every sequence $q_k \in \Sigma_k$, for every φ_k graph parametrisation with center q_k and width ρ_0, we obtain that every weak limit must parametrize a portion of a sphere with radius $\mathbb{S}^n_{\mu_0^{-1}}$ with $u_0(x) = \mu_0^{-1}\left(\sqrt{1-\mu_0^2|x|^2}-1\right)$ as parametrization, and the convergence is strong in C^1. This proves the proposition. □

Now we repeat the very same passages made in the proof of Theorem 4.1, and we easily obtain the corollary:

Corollary 4.15. *For every $0 < \varepsilon$ there exists $0 < \delta = \delta(n, p, c_0, \varepsilon)$ with the following property. Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying (4.1) and (4.2). If $\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma)} = 0$, then there exists a vector $c \in \mathbb{R}^{n+1}$ such that $b(\Sigma - c) = 0$, and the radial parametrization*

$$\psi\colon \mathbb{S}^n \longrightarrow \Sigma,\ \psi(x) = e^{f(x)}x$$

is well defined. Moreover $\|f\|_{C^1} \leqslant \varepsilon$.

This concludes the study of the qualitative C^1-closeness.

4.2.2 *Conclusion*

As in Chapter 2, Corollary 4.15 is not enough to conclude the estimate, because the Ricci operator seen as differential operator on f is not elliptic. We shall conclude the proof of Theorem 4.2 with an idea, that reduces it to an application of Theorem 4.1. First of all, let us show an easy corollary of 4.15.

Corollary 4.16. *Under the hypothesis of 4.15, we have the inequality:*

$$|\overline{\mathrm{Scal}} - n(n-1)| \leqslant C(n, p, c_0)\varepsilon.$$

Proof. The proof has basically already been given in Chapter 2, see Lemma 2.16. Indeed, from the C^1-closeness we are still able to obtain expression (2.39)

$$\mathrm{Scal} = n(n-1) - 2(n-1)(\Delta_\sigma f + f) + (\Delta f)^2 - |\nabla^2 f|^2 + \mathcal{R}, \tag{4.27}$$

where $\mathcal{R}$ satisfies

$$|\mathcal{R}| \leqslant C\varepsilon\left(|f| + |\nabla f| + |\nabla^2 f|\right).$$

Since

$$\left|\int_{\mathbb{S}^n} \mathcal{R}\right| \leqslant C(n, p, c_0)\varepsilon,$$

we integrate (4.27) and obtain the corollary. □

Now we can complete the proof of Theorem 4.2. Let us write $\mathrm{Scal} = n(n-1)\kappa$, and assume

$$|\kappa - 1| \leqslant \frac{1}{2}.$$

Again, we denote with $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ the eigenvalues of A and again we consider $\kappa := \frac{1}{n(n-1)}\overline{\mathrm{Scal}}$. As proved in Corollary 4.16, we can choose $\delta \leqslant \delta_0$ so that κ is between $1/2$ and 2. Then, given Proposition 2.5, we rewrite inequality (2.9) in terms of the eigenvalues of A and obtain

$$\left\|\lambda_i\lambda_j - \kappa\right\|_{L^p} \leqslant C\|\mathring{\mathrm{Ric}}\|_{L^p}, \ \forall\, i \neq j. \tag{4.28}$$

From (4.28), we easily infer for every $k = 1, \dots, n$

$$\left\|\lambda_k(\lambda_i - \lambda_j)\right\|_{L^p} \leqslant C\|\mathring{\mathrm{Ric}}\|_{L^p}. \tag{4.29}$$

Now, for every $0 < \Lambda^2 < \kappa$, we define

$$E_\Lambda := \{\, q \in \Sigma : |\, \lambda_n(q)| > \Lambda \,\}. \tag{4.30}$$

We use the set E_Λ and its complement in order to perform an estimate on the difference $\left|\lambda_i - \lambda_j\right|$. Indeed, since $\lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \Lambda$ for every $q \in E^c_\Lambda$, we get the bounds

$$|\kappa - \Lambda^2|\,|E^c_\Lambda|^{\frac{1}{p}} \leqslant \left\|\lambda_i\lambda_j - \kappa\right\|_{L^p(E^c_\Lambda)} \leqslant C\|\mathring{\mathrm{Ric}}\|_{L^p},$$

which hold for every $i \neq j$ and $0 < \Lambda^2 < \kappa$. Thus we have found

$$|E^c_\Lambda|^{\frac{1}{p}} \leqslant \frac{C}{|\kappa - \Lambda^2|}\|\mathring{\mathrm{Ric}}\|_{L^p}. \tag{4.31}$$

On the other hand, for any $i,\ j = 1, \dots n-1$, $i \neq j$ we find:

$$\left\|\lambda_i - \lambda_j\right\|_{L^p(E_\Lambda)} \leqslant \frac{1}{\Lambda}\left\|\lambda_n(\lambda_i - \lambda_j)\right\|_{L^p(E_\Lambda)} \overset{(4.29)}{\leqslant} \frac{C}{\Lambda}\|\mathring{\mathrm{Ric}}\|_{L^p},$$

which gives us

$$\left\|\lambda_i - \lambda_j\right\|_{L^p(E_\Lambda)} \leqslant \frac{C}{\Lambda}\|\mathring{\mathrm{Ric}}\|_{L^p}. \tag{4.32}$$

Combining (4.31) and (4.32) we obtain

$$\left\|\lambda_i - \lambda_j\right\|_{L^p} \leqslant C\left(\frac{1}{\Lambda} + \frac{1}{|\kappa - \Lambda^2|}\right)\|\mathring{\mathrm{Ric}}\|_{L^p}. \tag{4.33}$$

This estimate holds for every $i \neq j$, $i,\ j = 1, \dots n-1$ and for every $0 < \Lambda^2 < \kappa$. Equation (4.33) is not sufficient to conclude, because it does not give an estimate on the quantity $\left|\lambda_n - \lambda_j\right|$. This is the only quantity that prevents this proof to give a linear estimate in (4.6), forcing us to introduce the exponent α. Indeed, to deal with $\left|\lambda_n - \lambda_j\right|$, we define

$$\tilde{E}_\Lambda := \{\, q \in \Sigma \mid |\lambda_{n-1}(q)| > \Lambda \,\}.$$

With the very same considerations used to deduce (4.31), we obtain

$$\left|\tilde{E}^c_\Lambda\right|^{\frac{1}{p}} \leqslant \frac{C}{\kappa - \Lambda^2} \|\mathring{\mathrm{Ric}}\|_{L^p}. \tag{4.34}$$

Now we fix $q \in (n, p)$. Then, via Hölder inequality we get

$$\left\|\lambda_n - \lambda_j\right\|_{L^q(\tilde{E}^c_\Lambda)} \leqslant C(n, p, c_0) \|\mathring{\mathrm{Ric}}\|^\alpha_{L^p}, \tag{4.35}$$

where α is defined as in Theorem 4.2. Combining (4.34) with (4.35), we obtain

$$\left\|\lambda_n - \lambda_j\right\|_{L^q} \leqslant C \left(\frac{1}{|\kappa - \Lambda^2|} + 1\right) \|\mathring{\mathrm{Ric}}\|^\alpha_{L^p}. \tag{4.36}$$

Choosing $\Lambda = \sqrt{\frac{\kappa}{2}}$ and plugging together (4.33) and (4.36), we deduce

$$\|\mathring{A}\|_{L^q} \leqslant \frac{C}{\sqrt{\kappa}} \|\mathring{\mathrm{Ric}}\|^\alpha_{L^p} \leqslant \sqrt{2} C \|\mathring{\mathrm{Ric}}\|^\alpha_{L^p}.$$

We are thus under the assumptions of Theorem 1.2, which provide a radial parametrization $\psi\colon \mathbb{S}^n \longrightarrow \Sigma$, $\psi = e^f \, \mathrm{Id}$, and a vector $c = c(\Sigma)$ such that 4.6 holds.

4.3 PROOF OF THEOREM 4.3

We finish the chapter proving the anisotropic generalization of Theorem 4.1. As stated before, in the latter case we have to adopt a different strategy, since we lack the symmetry property of the sphere. The cornerstone of the proof is following proposition:

Proposition 4.17. *Let* $n \in \mathbb{N}$, $n < p$, $0 < \mathcal{A}$, $\mathcal{V}$, R *positive constants. Let* $\mathfrak{F}$ *be the set of all couples* (M, f) *with the following properties:*

- M *is an* n*-dimensional, compact manifold (without boundary).*
- $f \in W^{2,p}(M, \mathbb{R}^n)$ *is an immersion with*

$$\begin{aligned} \|A(f)\|_{L^p(M)} &\leqslant \mathcal{A}, \\ \mathrm{Vol}_n(M) &\leqslant \mathcal{V}, \\ f(M) &\subset \mathbb{B}^n_R. \end{aligned}$$

Then for every sequence $f_i\colon M_i \longrightarrow \mathbb{R}^n$ *in* $\mathfrak{F}$ *there exist a subsequence* f_j *, a mapping* $f\colon M \longrightarrow \mathbb{R}^n$ *in* $\mathfrak{F}$*, and a sequence of diffeomorphisms* $\varphi\colon M \longrightarrow M_j$*, such that* $f_j \circ \varphi_j$ *converges weakly in* $W^{2,p}(M, \mathbb{R}^n)$ *to* f.

The proposition is part of a series of compactness theorems on immersions, started in [34] where the author proves the result for immersed surfaces in $\mathbb{R}^3$ and then continued in [13] for immersed hypersurfaces, and in [5] for the general case. The proposition we want to prove is the following:

Proposition 4.18. *Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying* (4.7) *and* (4.8)*. For every $0 < \varepsilon$ sufficiently small there exists a $0 < \delta = \delta(\varepsilon, n, p, c_0, \mathcal{W})$ with the following property. If Σ satisfies* (4.9)*, then it admits an anisotropic radial parametrization as in* (3.5)*. Moreover the radius u satisfies the estimate*

$$\|u\|_{C^1} \leqslant \varepsilon. \tag{4.37}$$

As usual, we will see in the conclusion how the qualitative C^1-closeness will bring the desired quantitative one.

Proof of Proposition 4.18

The proof of Proposition 4.18 uses strongly the compactness result of Proposition 4.17. Firstly, we prove the following two lemmas.

Lemma 4.19. *Let $\varphi_k \colon M \longrightarrow \mathbb{R}^{n+1}$ be a sequence of immersions of a closed manifold. Assume φ_k satisfies* (4.7) *and* (4.8)*, and φ_k converges to an immersion φ_0, weakly in $W^{2,p}$. Then we have the inequality*

$$\|A_F(\varphi_0)\|_{L^p(M)} \leqslant \liminf_k \|A_F(\varphi_k)\|_{L^p(M)}. \tag{4.38}$$

Lemma 4.20. *Let Σ_k be a sequence of hypersurfaces satisfying* (4.7)*,* (4.8)*, and such that we have also $\lim_k \|\mathring{A}_F\|_{L^p(\Sigma_k)} = 0$. Then there exist a subsequence $(\Sigma_h)_{h\in M}$ and parametrizations $\eta_h \colon \mathcal{W} \longrightarrow \Sigma_h$ such that η_h converges weakly in $W^{2,p}$ to the identity map* $\mathrm{id} \colon \mathcal{W} \longrightarrow \mathcal{W}$

Let us prove the lemmas and then show how they bring the result.

Proof of Lemma 4.19. We introduce the map

$$\Psi \colon \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1},\ \Psi(x) := \mathrm{grad}_\sigma F(x) + F(x)x. \tag{4.39}$$

From [39] we know that the map Ψ parametrizes the Wulff shape. It is immediate to show the inequality

$$A_F := S_F \circ d\nu = d(\Psi \circ \nu). \tag{4.40}$$

Indeed, the differential of Ψ has the following form:

$$d\Psi\left[\frac{\partial}{\partial\vartheta^i}\right] = \underbrace{\frac{\partial}{\partial\vartheta^i}\big(\mathrm{grad}_\sigma F\big) + \partial_i F\ \mathrm{Id}}_{=D_i(DF)} + F\frac{\partial}{\partial\vartheta^i} = (S_F)_i^j \frac{\partial}{\partial\vartheta^j},$$

where we have denoted by D the Levi-Civita connection compatible with the canonical metric on the round sphere. Taking the composition we obtain (4.40). Let now $(\nu_k)_{k\in\mathbb{N}}$ be the sequence of outer normals associated to φ_k, i.e. the sequence of mappings $\nu_k \colon M \longrightarrow \mathbb{S}^n$ such that

$$\langle\, \nu_k(q),\ d\varphi_k|_q\,[v]\,\rangle = 0,\ \forall v \in T_qM,$$

and with orientation fixed so every ν_k is the outer normal for $\varphi_k(M) = \Sigma_k$. We claim that the sequence $(\nu_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(M, \mathbb{R}^{n+1})$. Firstly, since $A_F = S_F \circ A$, we obtain

$$\|A\|_{L^p(\Sigma)} = \left\|(S_F)^{-1} A_F\right\|_{L^p(\Sigma)} \leqslant c(F)\, c_0 = C(F, c_0), \tag{4.41}$$

and thus (4.8) implies inequality (4.2). Now we show how the L^p-boundedness of the second fundamental forms gives us the L^p-boundedness of the differential of the normals. The key is the following proposition, proved in [13, Thm. 6.3].

Proposition 4.21. *Let* $2 \leqslant p$, *and* $\psi\colon \mathbb{B}^n_R \longrightarrow \mathbb{R}^{n+1}$, $\psi(x) = (x, h(x))$ *be a graph parametrisation, with* h *smooth function. Then the following estimate holds:*

$$\|\partial^2 h\|_{L^p(\mathbb{B}^n_R)} \leqslant (1 + \|\partial h\|_0)^{\frac{3p-1}{p}} \|A\|_{L^p}. \tag{4.42}$$

Estimate (4.42) allows us to conclude. Since our hypersurfaces satisfy the volume condition (4.7) and the L^p-bound (4.8), then they also satisfy the assumptions of Lemma 0.2. Plugging (4.42) we can easily find a radius R depending on n, p, c_0 such that the estimate

$$\|d\nu\|_{L^p(\mathbb{B}^g_R(q))} \leqslant C(n, p, c_0)\|A\|_{L^p} \leqslant C(n, p, c_0, F)\|A_F\|_{L^p}$$

holds for every point q. Then we make this estimate global via Lemma 0.3, and obtain:

$$\|d\nu\|_{L^p} \leqslant C(n, p, c_0, \mathcal{W})\|A_F\|_{L^p}.$$

Therefore our sequence $(\nu_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,p}$. Since $n < p$, every weak $W^{1,p}$-limit point ν_0 is also a strong $C^{0,\alpha}$-limit point, and satisfies

$$|\nu_0(q)| = 1\ \forall\, q, \quad \langle\, \nu(q),\ d\varphi_0|_q\,[v]\,\rangle = 0.$$

This shows that ν_0 is the outer normal associated to the immersion φ_0, and moreover $d\nu_k \rightharpoonup d\nu_0$. In order to complete the proof, we simply consider equality (4.40): since the map Ψ is smooth, we obtain that $\Psi \circ \nu_k$ converges to $\Psi \circ \nu_0$ weakly in $W^{1,p}$, and the result follows from classical Sobolev theory. □

With the help of Lemma 4.19 we prove 4.20.

Proof of Lemma 4.20. Let us argue by contradiction, and assume there exists a sequence of closed hypersurfaces $(\Sigma_k)_{k\in\mathbb{N}}$ satisfying (4.7), (4.8), $\lim_k \|\mathring{A}_F\|_{L^p} = 0$, all enclosed in a ball $\mathbb{B}^{n+1}_R$, and such that the conclusion of the proposition does not hold.

We apply Proposition 4.17, and find a subsequence Σ_h, a closed manifold M, parametrizations $\varphi_h\colon M \longrightarrow \Sigma_h$ converging weakly in $W^{2,p}$ to an immersion φ_0. From Proposition 3.10 we find the existence of a bounded sequence $(\lambda_h)_{h\in\mathbb{N}}$ such that

$$\|A_F - \lambda_h\, \mathrm{Id}\|_{L^p_h} \leqslant C\|\mathring{A}_F\|_{L^p_h} \downarrow 0.$$

As usual, up to subsequences we assume $\lambda_h = \lambda_0$ for every h. As in the isotropic case, λ_0 must be different from 0 because of the estimate

$$\|A_F\|_{L^p} \geqslant C(n, p, F)\|A\|_{L^p} \geqslant C(n, p, F, \mathrm{Vol}_n(\Sigma)) = C(n, p, F).$$

Since $A_F = d(\Psi \circ \nu_h)$, we apply Lemma 4.19 to the sequence $\Psi \circ \nu_h - \lambda_0 x$, and obtain that the limit immersion φ_0 satisfies the equality

$$A_F = \lambda_0 \,\mathrm{Id}$$

weakly. From it we easily infer

$$A(\varphi_0) = \lambda_0 (S_F)^{-1}. \tag{4.43}$$

Now we take the trace in (4.43), and obtain that in every graph parametrisation around every point q, the function u_q that parametrizes the immersion is Lipschitz and satisfies an equality of the following type:

$$\operatorname{div}\left(\frac{\partial u_q}{\sqrt{1+|\partial u_q|^2}}\right) = f(u_q, \partial u_q),$$

for a certain smooth function f. This tells us that the function u_q is smooth. Since then (4.43) holds classically, u_q is also convex, and we obtain that φ_0 is a smooth immersion and $\Sigma_0 := \varphi_0(M)$ is a smooth, convex hypersurface of $\mathbb{R}^{n+1}$. Since Σ_0 is diffeomorphic to a round sphere, the same argument used to build the parametrization in the proof of 4.5 tells us that φ_0 is actually an embedding. From [39] and the volume condition (4.7) we conclude that $\lambda_0 = 1$ and $\varphi_0(M)$ must be a Wulff shape $\mathcal{W} + c$ for some vector $c \in \mathbb{R}^{n+1}$. Up to translation, we assume $c = 0$. Now we easily define $\eta_h \colon \mathcal{W} \longrightarrow \Sigma_h$, $\eta_h = \varphi_h \circ \varphi_0^{-1}$ and obtain that η_h converges to the identity map $\mathrm{id}\colon \mathcal{W} \longrightarrow \mathcal{W}$ weakly in $W^{2,p}$. □

The results obtained give us a priori only a qualitative C^0-closeness. We show how to build the radial parametrization and conclude.

Insofar we have proved the following result.

Corollary 4.22. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed hypersurface satisfying conditions* (4.7) *and* (4.8)*, namely*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathcal{W}), \quad \|A_F\|_{L^p(\Sigma)} \leqslant c_0.$$

Then for every $0 < \varepsilon$ there exist $0 < \delta_0(n, p, \mathcal{W}, c_0)$ with the following property. If Σ satisfies (4.9) *with $\delta \leqslant \delta_0$, then there exists a map $\eta\colon \mathcal{W} \longrightarrow \Sigma$ such that*

$$\|\eta - \mathrm{id}\|_{C^{1,\alpha}(\mathcal{W})} \leqslant \varepsilon. \tag{4.44}$$

We show how (4.44) yields the desired graph parametrisation. Let Σ be a closed hypersurface that satisfies the assumptions of Corollary 4.22. Let $B_\varepsilon(\mathcal{W})$ be the tubular neighbourhood associated to $\mathcal{W}$. We denote by P the natural projection over the Wulff shape, that is

$$P\colon B_\varepsilon(\mathcal{W}) \longrightarrow \mathcal{W}, \quad P\colon q = x + \rho\nu_{\mathcal{W}}(x) \longrightarrow x.$$

The map P is Lipschitz and smooth. Moreover, it can be proved that for every $q \in B_\varepsilon\mathcal{W}$, the differential $dP|_q : \mathbb{R}^{n+1} \longrightarrow T_{P(q)}\mathcal{W}$ is surjective and satisfies the property

$$dP|_q\,[z] = 0 \;\Leftrightarrow\; z = \lambda\nu_{\mathcal{W}}(P(q)),\; \lambda \in \mathbb{R}. \tag{4.45}$$

See [28, Ch. 5] for the details. Since Σ satisfies Corollary 4.22 and hence estimate (4.44), then $\Sigma \subset B_\varepsilon(\mathcal{W})$ and we can set $p := P|_\Sigma$. Therefore, p is a smooth, Lipschitz map from Σ to $\mathcal{W}$ and satisfies

$$\sup_{q\in\Sigma} |q - p(q)| \leqslant \varepsilon. \tag{4.46}$$

We claim that p also satisfies:

$$\sup_{q\in\Sigma} |\nu_\Sigma(q) - \nu_{\mathcal{W}}(p(q))| \leqslant C(n, \mathcal{W})\varepsilon. \tag{4.47}$$

If the claim is true, then p is a local diffeomorphism: indeed, since $\nu_{\mathcal{W}}(p(q)) \notin T_q\Sigma$ for every $q \in \Sigma$, by (4.45) $dp|_q$ has maximal rank at every point $q \in \Sigma$. Hence p is a local diffeomorphism, and since the Wulff shape is diffeomorphic to the sphere, the same argument made in the isotropic case proves it is a global diffeomorphism. Then the inverse $\psi(x) = x + u(x)\nu_{\mathcal{W}}(x)$ is the desired radial parametrization and from inequalities (4.46) and (4.47) we obtain that u is small in the C^1-norm.

Now we prove the claim. Let $q \in \Sigma$ be fixed, and let $z \in \mathcal{W}$ be given so that $q = \eta(z)$. From (4.46) we know that

$$|q - p(q)| \leqslant \varepsilon,$$

and from (4.44) we know that

$$|q - z|,\ |\nu_\Sigma(q) - \nu_{\mathcal{W}}(z)| \leqslant \varepsilon. \tag{4.48}$$

Patching the inequalities together, we get

$$|p(q) - z| \leqslant 2\varepsilon.$$

Since the Wulff shape is convex, necessarily z must belong to a graph parametrisation $\varphi_{p(q)} \colon \mathbb{B}^n_R \longrightarrow \mathcal{W}$ centered in $p(q)$, provided that $0 < \varepsilon$ is sufficiently small. By convexity, we easily notice that

$$|\nu_{\mathcal{W}}(p(q)) - \nu_{\mathcal{W}}(z)| \leqslant c(n, \mathcal{W})\varepsilon.$$

Patching this inequality with (4.48) we obtain the claim, and therefore the thesis.

Conclusion

Since also in this case the computations do not depend on the convexity property of the hypersurface Σ (see Section 3.2), we also reach:

Proposition 4.23. *Under the hypothesis of Proposition 4.18, we have the additional estimate:*

$$\|u - \varphi_u\|_{W^{2,p}(\mathcal{W})} \leqslant C\Big(\|\mathring{A}_F(\Sigma)\|_{L^p(\Sigma)} + \varepsilon\|u\|_{W^{2,p}(\mathcal{W})}\Big). \tag{4.49}$$

where $C = C(n, p, \mathcal{W})$ *and* φ_u *is defined as in Definition 3.14.*

We end the section by getting rid of the function φ_u in estimate (4.49), that is, proving the following:

Proposition 4.24. *Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying (4.1), (4.8) and (4.9), so that the estimates of Propositions 4.18 and 4.23 hold for a radial anisotropic parametrization ψ. There exist $\varepsilon_0 > 0$, $C_0 > 0$ depending only on $\mathcal{W}$ with the following property. If (4.37) holds with $\varepsilon \leqslant \varepsilon_0$, then there exists $c = c(\Sigma) \in \mathbb{R}^{n+1}$ such that $\Sigma - c$ still admits a radial parametrization*

$$\psi_c \colon \mathcal{W} \longrightarrow \Sigma - c, \quad \psi_c(x) := x + u_c(x)\nu_{\mathcal{W}}(x),$$

and u_c satisfies:

$$\begin{cases} \|u_c\|_{C^1} \leqslant C_0\varepsilon, \\ \langle u_c, \varphi_w\rangle_{L^2} = 0 \quad \text{for every } \varphi_w \text{ defined as in (3.11).} \end{cases}$$

Proof. The proof of 4.24 is similar to the one made for proposition 3.7 in the convex case, with some correction to remove the convexity assumption. We divide the proof into three main steps.

STEP 1 *For any positive constant C_1 there exist positive numbers ε, C_2 depending only on $\mathcal{W}$, C_1 with the following property. For every $c \in \mathbb{B}_{C_1\varepsilon}$, the hypersurface $\Sigma_c := \Sigma - c$ is still a graph over $\mathcal{W}$, and its radius u_c satisfies*

$$\|u_c\|_{C^1(\mathcal{W})} \leqslant C_2\varepsilon.$$

We consider ε so small that Σ_c is still in the 2ε-tubular neighborhood of $\mathcal{W}$. Again, we argue by proving that the projection map

$$p_c \colon \Sigma_c \longrightarrow \mathcal{W}, \quad p_c \colon q = x + u_c\nu_{\mathcal{W}}(x) \longmapsto x$$

is a diffeomorphism. Following the same strategy of the proof of proposition 4.18, we just need to show that $\nu_{\mathcal{W}}(p_c(q)) \notin T_q\Sigma_c$ for every $q \in \Sigma_c$. Let then $q \in \Sigma_c$ be given. By the very definition of Σ_c, we have that $\tilde{q} := q - c \in \Sigma$. Moreover, since Σ is a graph over $\mathcal{W}$ with radius u, there exists $x \in \mathcal{W}$ such that $\tilde{q} = x + u(x)\nu_{\mathcal{W}}(x)$. By the computation made in [12, App. B], we deduce

$$|\nu_\Sigma(\tilde{q}) - \nu_{\mathcal{W}}(x)| \leqslant C(\mathcal{W})\varepsilon. \tag{4.50}$$

Since $\Sigma_c = \Sigma + c$, we know that $\nu_\Sigma(\tilde{q}) = \nu_{\Sigma_c}(\tilde{q} + c) = \nu_{\Sigma_c}(q)$. On the other hand,

$$|\nu_{\mathcal{W}}(p_c(q)) - \nu_{\mathcal{W}}(x)| \leqslant \varepsilon. \tag{4.51}$$

Combining (4.50) with (4.51), we deduce that

$$\begin{aligned} |\nu_{\Sigma_c}(q) - \nu_{\mathcal{W}}(p_c(q))| &= |\nu_\Sigma(\tilde{q}) - \nu_{\mathcal{W}}(p_c(q))| \\ &\leqslant |\nu_\Sigma(\tilde{q}) - \nu_{\mathcal{W}}(x)| + |\nu_{\mathcal{W}}(x) - \nu_{\mathcal{W}}(p_c(q))| \overset{(4.50),(4.51)}{\leqslant} C\varepsilon. \end{aligned}$$

This shows that for ε sufficiently small, $\nu_{\mathcal{W}}(p_c(q)) \notin T_q\Sigma_c$, and thus we can conclude as in the proof of Theorem 4.18.

STEP 2 *We consider the map*

$$\Phi\colon \mathbb{B}_{C_1\varepsilon} \longrightarrow \mathbb{R}^{n+1}, \quad \Phi(c) := \sum_{i=1}^{n} \langle u_c, \varphi_i \rangle_{L^2} w_i \tag{4.52}$$

where φ_i, w_i *are defined as in* (3.11). *Then there exists a constant* C_3 *depending on* C_1 *such that the following estimate holds:*

$$|\Phi(c) - \Phi(0) - c| \leqslant C_3 \varepsilon^2. \tag{4.53}$$

Indeed, for every c such that $|c| < C_1\, \varepsilon$ we find

$$d_{\mathrm{HD}}(\Sigma - c, \mathcal{W}) \leqslant d_{\mathrm{HD}}(\Sigma - c, \Sigma) + d_{\mathrm{HD}}(\Sigma, \mathcal{W}) \leqslant (C_1 + 1)\varepsilon.$$

Arguing as in the Step 1 it is easy to see that also the function u_c satisfies the estimates

$$\|u_c\|_{C^1} \leqslant C(n, \mathcal{W})\varepsilon, \tag{4.54}$$

We start the linearisation with the following simple consideration: for every $z \in \mathcal{W}$ there exists $x_c = x_c(z) \in \mathcal{W}$ so that

$$\psi_c(z) = \psi(x_c(z)) - c.$$

We expand this equality and find

$$z + u_c(z)\, \nu_{\mathcal{W}}(z) = x_c(z) + u(x_c(z))\nu_{\mathcal{W}}(x_c(z)) - c. \tag{4.55}$$

Using the C^0-smallness of u and u_c, we can easily see that $x_c = x_c(z)$ satisfies the relation

$$|x_c(z) - z| \leqslant C(n, \mathcal{W})\varepsilon. \tag{4.56}$$

This approximation, combined with (4.54), gives an estimate of u close to z:

$$|u(x_c(z)) - u(z)| \leqslant C(n, \mathcal{W})\varepsilon^2. \tag{4.57}$$

We evaluate F^* in the point in (4.55):

$$\underbrace{F^*(z + u_c(z)\, \nu_{\mathcal{W}}(z))}_{=1+u_c(z)\, dF^*|_z[\nu_{\mathcal{W}}(z)]+\mathcal{R}} \quad = \quad \underbrace{F^*(x_c(z) + u(x_c(z))\, \nu_{\mathcal{W}}(x_c(z)) - c)}_{=1+u(x_c(z))\, dF^*|_{x_c(z)}[\nu_{\mathcal{W}}(x_c(z))]-\, dF^*|_{x_c(z)}[c]+\mathcal{R}} \quad ,$$

where

$$|\mathcal{R}| \leqslant C(n, \mathcal{W})\varepsilon^2.$$

Plugging in the previous equality the gauge property (0.3), we obtain

$$|u_c(z)\langle \nu_{\mathcal{W}}(z), \nu_{\mathcal{W}}(z)\rangle - u(x_c(z))\langle \nu_{\mathcal{W}}(x_c), \nu_{\mathcal{W}}(x_c)\rangle + \langle c, \nu_{\mathcal{W}}(x_c)\rangle| \leqslant C(n, \mathcal{W})\varepsilon^2,$$

which by (4.57) reads

$$|u_c(z) - u(z) + \underbrace{\langle c, \nu_{\mathcal{W}}(z)\rangle}_{=\varphi_c(z)}| \leqslant C(n, \mathcal{W})\varepsilon^2. \tag{4.58}$$

Integrating over $\mathcal{W}$ and using (4.58), we conclude the proof of Step 2.

STEP 3 *Conclusion*. We argue by contradiction, and choose C_1 so that the map

$$\tilde{\Phi}\colon \mathbb{B}_1 \longrightarrow \mathbb{R}^{n+1}, \tilde{\Phi}(c) := \frac{\Phi(C_1 \varepsilon c)}{C_1 \varepsilon}$$

satisfies

$$\left|\tilde{\Phi}(0)\right| \leqslant \frac{1}{10}, \quad \left|\tilde{\Phi}(c) - \tilde{\Phi}(0) - c\right| \leqslant \varepsilon.$$

Since inequality (4.53) holds, such C_1 exists thanks to the very same computations given in Proposition 3.7. If 0 does not belong to the image of Φ, then we are allowed to define the map $\varphi := \frac{\tilde{\Phi}}{|\tilde{\Phi}|}$. Then, with the very same computations done in the proof of Proposition 3.7 we restrict φ to $\mathbb{S}^n = \partial\mathbb{B}^{n+1}$ and find a map with the following property:

$$\varphi\colon \mathbb{S}^n \longrightarrow \mathbb{S}^n, |\varphi(x) - x| < 2. \tag{4.59}$$

The thesis follows as in 3.7. □

5 MISCELLANEA

In this chapter we include here some result obtained throughout the development of other theorems. We think that there may still be some interest about, and report them.

5.1 The non convex case for sphere parametrisations

One of the main problems for the non-convex generalisation of theorem 1.1 concern the existence of a radial parametrisations. In order to find it, we had to assume a control on the the L^p-norm of the second fundamental form, and perform a fine analysis. One may ask whether assuming that the hypersurface is already radially parametrized can relax the hypothesis. This is actually the case, and we see it in the following proposition.

Proposition 5.1. *Let* $2 \leqslant n$, $1 < p < \infty$ *be given, and let* $\Sigma = \psi(\mathbb{S}^n)$ *be a closed hypersurface in* $\mathbb{R}^{n+1}$*, where* ψ *is defined as in* (1.4)*. Assume that* Σ *satisfies the following conditions:*

$$\mathrm{Vol}_n(\Sigma) = \mathrm{Vol}_n(\mathbb{S}^n), \tag{5.1}$$

$$\|\nu - \mathrm{id}\|_{C^0(\mathbb{S}^n)} \leqslant \Lambda, \text{ where } 0 < \Lambda < \sqrt{2}, \tag{5.2}$$

where ν *denotes the outer normal of* Σ*. Then there is a number* $0 < \delta_0(n, p, \Lambda)$ *with the following property. If*

$$\|\mathring{A}\|_{L^p(\Sigma)} \leqslant \delta_0, \tag{5.3}$$

then there exists a vector $c = c(\Sigma) \in \mathbb{R}^{n+1}$ *such that* $\Sigma - c$ *still admits a radial parametrisation* $\psi_c = e^{f_c}\,\mathrm{id}$ *and*

$$\|f_c\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p, \Lambda)\|\mathring{A}\|_{L^p(\Sigma)}. \tag{5.4}$$

Proof. Most of the work has already been made in Chapters 1 and 4. We show how the proposition follows from the following lemma, that we shall prove at the end.

Lemma 5.2. *Let* $\Sigma = \psi(\mathbb{S}^n)$ *satisfy* (5.1) *and* (5.2)*. Then there exists a constant* $C = C(n, p, \Lambda)$ *such that*

$$\|f\|_{C^1(\mathbb{S}^n)} \leqslant C. \tag{5.5}$$

Let us show how Lemma 5.2 allows us to conclude. Firstly, it grants us the oscillation proposition 1.12. Indeed, as clarified in Lemma 1.14 we can easily refine Proposition 1.12 and obtain a constant $C(n, p, \Lambda)$ such that

$$\|H - \overline{H}\|_{L^p} \leqslant C\|\mathring{A}\|_{L^p}. \tag{5.6}$$

Then, expanding (5.6) with Lemma 1.9 we obtain that f satisfies the following equation:

$$H(f) := -\operatorname{div}_\sigma\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)e^{-f} + \frac{ne^{-f}}{\sqrt{1+|\nabla f|^2}} = \overline{H} + \mathcal{R}, \tag{5.7}$$

where $\mathcal{R}$ is a quantity whose L^p-norm is controlled by $\|\mathring{A}\|_{L^p}$. See Proposition 1.15 and Appendix A.1 for the relevant computations. Integrating (5.7) w.r.t. the measure dV_σ and using the divergence theorem we obtain

$$|\overline{H}| \leqslant C(n, p, \Lambda) + \|\mathring{A}\|_{L^p} \leqslant C(n, p, \Lambda). \tag{5.8}$$

Now we prove the C^1-closeness, and this will conclude the proof. Indeed, the computations leading to inequality (1.8) do not require convexity, and the non convex version of the centering proposition has been given in Chapter 4.

The C^1-closeness follows by a compactness argument: as previously used, let $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of closed hypersurfaces $\Sigma_k = \psi_k(\mathbb{S}^n)$ satisfying (5.1), (5.3) and the decay

$$\lim_k \|\mathring{A}\|_{L^p} = 0.$$

Then from the (5.6), (5.7) and (5.8) we obtain the functions f_k satisfy

$$\|f_k\|_{W^{2,p}(\mathbb{S}^n)} \leqslant C(n, p, \Lambda)\ \forall k, \quad \lim_k \|H(f) - \overline{H}_k\| = 0. \tag{5.9}$$

Plus, we can assume that $\overline{H}_k$ converges to a number λ. Thus, we can choose a (not relabelled) subsequence $\psi_k(x) = e^{f_k(x)}x$ such that $f_k \rightharpoonup f$ in $W^{2,p}(\mathbb{S}^n)$ and strongly in C^0 for the Ascoli-Arzelà theorem (see [6] for a precise formulation). The limit hypersurface $\Sigma = \psi(\mathbb{S}^n)$, with $\psi(x) = e^{f(x)}x$, is a $W^{2,p}$, Lipschitz hypersurface. Since the radii f_k satisfy (5.9), then the limit f that gives the parametrisation satisfies

$$H(f) := -\operatorname{div}_\sigma\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)e^{-f} + \frac{e^{-f}}{\sqrt{1+|\nabla f|^2}} = \lambda. \tag{5.10}$$

From (5.10) and classical elliptic regularity theory (see [20]) we can easily infer that f is smooth, thus Σ is a smooth hypersurface with constant mean curvature and diagonal second fundamental form. This shows that Σ is the round sphere and $f = 0$, and since the chosen subsequence is arbitrary, all the sequence $(f_k)_{k\in\mathbb{N}}$ converges to 0. One can also notice with a bit of work that this convergence is actually strong in $W^{2,p}$, but we do not need this additional convergence for the subsequent arguments.

Now that we have obtained the C^1-closeness of f to 0, we proceed as in Chapter 4 and obtain the main result.

It remains to prove Lemma 5.5 and conclude. The lemma follows from combining inequality (5.2) with formula (1.13). Indeed,

$$|\nu(x) - x|^2 \leqslant \left|\frac{x - \nabla f}{\sqrt{1+|\nabla f|^2}} - x\right|^2$$

$$= \left(1 - \frac{1}{\sqrt{1+|\nabla f|^2}}\right) + \frac{|\nabla f|^2}{1+|\nabla f|^2} = 2\left(1 - \frac{1}{\sqrt{1+|\nabla f|^2}}\right) \leqslant \Lambda^2.$$

This latter estimate easily implies

$$|\nabla f| \leqslant \sqrt{4-\Lambda^2}\frac{\Lambda}{2-\Lambda} =: L(\Lambda). \tag{5.11}$$

We show how (5.11), combined with the volume control leads to the conclusion. Indeed, (5.1) united with formula (1.16) for the volume dV_g gives us

$$\fint_{\mathbb{S}^n} e^{nf}\sqrt{1+|\nabla f|^2}\, dV_\sigma = 1 \Rightarrow c(n, \Lambda) \leqslant \int_{\mathbb{S}^n} e^{nf}\, dV_\sigma \leqslant C(n, \Lambda).$$

This latter estimate allows us to conclude. Indeed, assume there exists a sequence of hypersurfaces $\Sigma_k = \psi_k(\mathbb{S}^n)$ satisfying (5.1), (5.11) and such that $l_k := f_k(x_k) \uparrow \infty$. Since the round sphere is a complete measure space one easily obtains

$$f(y) = f(x_k) + (f(y) - f(x_k) \geqslant f(x_k) - \pi L(\Lambda).$$

Thus,

$$\fint_{\mathbb{S}^n} e^{nf(y)}\, dV_\sigma(y) \geqslant e^{-n\pi L(\Lambda)+l_k} \uparrow +\infty.$$

As shown above, this integral is however bounded, thus there cannot be convergence of the maximum to ∞. The same idea holds with the minimum. □

5.2 A WEAKER CONVERGENCE PROPOSITION

In Chapter 2 we proved the qualitative closeness under the strong assumption of a C^0-control of the second fundamental form. As already remarked, this hypothesis is somewhat innatural, because morally speaking we are assuming a $W^{2,\infty}$-control in order to achieve a $W^{2,p}$-quantitative closeness proposition. Here we try to levy the hypothesis.

Proposition 5.3. *Let $3 \leqslant n$, $2 \leqslant p < \infty$ be given, $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of closed, convex hypersurfaces satisfying* (5.1) *and the following the conditions:*

$$\|A\|_{L^p(\Sigma)} \leqslant c_0, \tag{5.12}$$

$$\lim_k \|\mathring{Ric}\|_{L^p} = 0. \tag{5.13}$$

Then up to translating $\Sigma_k = \psi_k(\mathbb{S}^n)$ for $\psi_k = e^{f_k}$ radial parametrisation as usual, and the functions f_k converge to 0 weakly in $W^{2,p}$ and strongly in C^1.

This proposition is still far from being optimal because it requires a $W^{2,p}$-control in order to achieve a weak $W^{2,p}$-decay, but it may still be useful. After having reached the qualitative result, we cannot however proceed to make it quantitative because without the C^0- control on A we would have to linearise the equation $\mathrm{Ric} - \lambda g$, and obtain

$$|\nabla^2 f|^2 - (\Delta f)^2 - 2(n-1)\big(\Delta f + n\big(f - \bar{f}\big)\big) = \mathcal{R}, \text{ where} \|\mathcal{R}\|_{L^p} \leqslant C\|\mathring{Ric}\|_{L^p}.$$

Insofar we do not know any useful estimate that can be conclusive in this case.

Proof. The proposition follows by applying the same technique of Chapter 4. We consider a sequence of closed, convex hypersurfaces $(\Sigma_k)_{k\in\mathbb{N}}$ satisfying (5.12) and (5.13). The integral control of the second fundamental form and the convexity grant us as usual the oscillation proposition

$$\|\mathrm{Ric} - (n-1)\kappa_k g\|_{L^p(\Sigma_k)} \leqslant C(n, p, c_0)\|\mathring{\mathrm{Ric}}\|_{L^p(\Sigma_k)},$$

where as usual $n(n-1)\kappa_k = \overline{\mathrm{Scal}}$. Now we notice that $(\kappa_k)_{k\in\mathbb{N}}$ is bounded. Indeed,

$$n(n-1)|\kappa| = |\overline{\mathrm{Scal}}| \leqslant \fint_\Sigma |\mathrm{Scal}|\, dV_g \leqslant \fint_\Sigma \left|H^2 - |A|^2\right| dV_g \leqslant C(n, p, c_0).$$

Then $\kappa_k \to \kappa$. Notice that the sets U_k bounded by Σ_k are uniformly bounded and contain a ball with non decaying radius, thus Blaschke selection theorem applies: we obtain that a (not relabelled) subsequence $\overline{U_k}$ converges to U in the Hausdorff distance, and U is a bounded, convex open set in $\mathbb{R}^{n+1}$. Therefore we obtain that the radii $f_k \to f$ in C^0, and are equilipschitz, thus f is also a Lipschitz function. Since (5.12) holds, we obtain that the functions $(f_k)_{k\in\mathbb{N}}$ are uniformly bounded in $W^{2,p}$, and thus converge weakly to f. Then we notice that the limit hypersurface Σ is of class $W^{2,p}$, and its weak Ricci tensor satisfies

$$\mathrm{Ric} = (n-1)\kappa g,$$

for a certain $\kappa \in \mathbb{R}$. The study of harmonic coordinates already made in Section 4.2 shows that Σ is the round sphere as in the proof of 5.1, and we conclude. □

5.3 A COUNTEREXAMPLE IN THE NON CONVEX CASE

Here we show how for $3 \leqslant n$ the convexity assumption on the hypersurface Σ is not an artificial hypothesis. The counterexample is given in the isotropic case, showing how, even for the simplest surface energy, the convexity plays a crucial role. Our counterexample is not new. A precise construction of it has been given in [11, Prop 4.1], and other counterexamples can be found in [41, Chapter 4]. In this short section we limit ourselves just to show the main idea of it.

Proposition 5.4. *For every $3 \leqslant n$ and every $1 < p < n-1$ there exists a sequence $\{\Sigma_k\}_{k\in\mathbb{N}}$ of smooth hypersurfaces in $\mathbb{R}^{n+1}$ satisfying the following conditions:*

$$2\,\mathrm{Vol}_n(\mathbb{S}^n) \leqslant \mathrm{Vol}(\Sigma_k) \leqslant 2\,\mathrm{Vol}(\mathbb{S}^n) + C_n, \tag{5.14}$$

$$\lim_k \|\mathring{A}_k\|_{L^p(\Sigma_k)} = 0, \tag{5.15}$$

$$d_{HD}(\Sigma_k, \mathbb{S}^n + c) \geqslant \varepsilon_0 > 0 \text{ for every } c \in \mathbb{R}^{n+1}. \tag{5.16}$$

Proof. Our counterexample is given by the union of two disjoint hyperspheres smoothly connected by a j-cylinder $\mathrm{Cyl} = \mathbb{S}^j_{r_k} \times [0, 1]^{n-j}$. We recall that in the j-cylinder the second fundamental form A_k has the expression

$$A_k = \begin{bmatrix} \frac{1}{r_k}\mathrm{Id}_j & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.17}$$

therefore we easily obtain the expression for the traceless second fundamental form $\mathring{A}_k$, which is given by

$$\mathring{A}_k = \frac{1}{r_k}\begin{bmatrix} \frac{n-j}{n}\,\mathrm{Id}_j & 0 \\ 0 & -\frac{j}{n}\,\mathrm{Id}_{n-j} \end{bmatrix}. \tag{5.18}$$

It is easy to see that we can choose Σ_k so that $\mathring{A}_k$ is 0 in the two hyperspheres, it is given by (5.18) in the j-cylinder and it is L^p arbitrarily small near the region where the hyperspheres and the j-cylinder are connected. Hence, in order to prove condition (5.15) we just have to show how $A_k \to 0$ in L^p-norm on the j-cylinder, but this is trivial since, choosing $1 < p < j < n$, then

$$\|\mathring{A}_k\|^p_{L^p(Cyl)} = r_k^{j-p} \to 0.$$

Conditions (5.14) and (5.16) are obviously satisfied by construction, getting the conclusion of the proof. □

A

APPENDIX

The appendix is devoted to proving technical and computational propositions used throughout all the thesis.

A.1 COMPUTATIONAL LEMMAS

Proof of Lemma 1.9

Firstly, we compute the differential of ψ:

$$d\psi|_x : T_x\mathbb{S}^n \longrightarrow T_{\psi(x)}\Sigma, \;\; d\psi|_x\,[z] = e^{f(x)}(z + \nabla_z f\, x). \tag{A.1}$$

In order to compute the expression for g in $\mathbb{S}^n$, we fix x in $\mathbb{S}^n$ and use the usual polar coordinates $\{\frac{\partial}{\partial\vartheta^1}\dots\frac{\partial}{\partial\vartheta^n}\}$ for the sphere. We find

$$\begin{aligned} g = g_{ij}\,d\vartheta^i\,d\vartheta^j &= \psi^*\,\delta|_\Sigma\left(\frac{\partial}{\partial\vartheta^i}, \frac{\partial}{\partial\vartheta^j}\right)d\vartheta^i\,d\vartheta^j \\ &= e^{2f}\left(\frac{\partial}{\partial\vartheta^i} + \nabla_i f\,x, \frac{\partial}{\partial\vartheta^j} + \nabla_j f\,x\right)d\vartheta^i\,d\vartheta^j \\ &= e^{2f}\big(\sigma_{ij} + \nabla_i f\,\nabla_j f\big)d\vartheta^i\,d\vartheta^j. \end{aligned}$$

The expression for g^{-1} follows from a direct computation.
Now we compute the normal $\nu = \nu_\Sigma$. Fix $x \in \mathbb{S}^n$ and consider the system $\{\frac{\partial}{\partial\vartheta^1}\dots\frac{\partial}{\partial\vartheta^n}, x\}$ which is orthogonal in $\mathbb{R}^{n+1}$. By the definition of ν we have the relation $(\nu(x),\; d\psi|_x)[z] = 0$ for every $z \in <x>^\perp$. Now we write $\nu = \nu^j\frac{\partial}{\partial\vartheta^j} + \nu^x\,x$ and obtain

$$\left|\frac{\partial}{\partial\vartheta^j}\right|^2 \nu_j + \nabla_j f\,\nu^x = 0 \text{ for every } j.$$

Normalizing we have

$$\nu(x) = \frac{1}{\sqrt{1+|\nabla f|^2}}(x - \nabla f(x)),$$

which is exactly (1.13).
The expression for A is more complex to compute. Firstly, we easily compute the differential of ν:

$$d\nu\left[\frac{\partial}{\partial\vartheta^j}\right] = \nabla_j\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\right)(x - \nabla f(x)) + \frac{1}{\sqrt{1+|\nabla f|^2}}\left(\frac{\partial}{\partial\vartheta^j} - \nabla_j(\nabla f)\right),$$

and now we can make our computation

$$A_{ij} := \left\langle d\psi\left[\frac{\partial}{\partial\vartheta^i}\right],\, d\nu\left[\frac{\partial}{\partial\vartheta^j}\right]\right\rangle = \frac{e^f}{\sqrt{1+|\nabla f|^2}}\left\langle \frac{\partial}{\partial\vartheta^i} + \nabla_i f,\, \frac{\partial}{\partial\vartheta^j} - \nabla_j \nabla f\right\rangle$$
$$= \frac{e^f}{\sqrt{1+|\nabla f|^2}}\left(\sigma_{ij} - \nabla_i f\,\langle\, \nabla_j\nabla f,\, \underbrace{x}_{\nu_{S^n}}\,\rangle - \left\langle \nabla_j\nabla f,\, \frac{\partial}{\partial\vartheta^i}\right\rangle\right).$$

We compute $\nabla_j\nabla f$ in the orthogonal system $\{\frac{\partial}{\partial\vartheta^1}\dots\frac{\partial}{\partial\vartheta^n}\}$.

$$\langle\, \nabla_j\nabla f,\, x\,\rangle = \nabla_j\underbrace{\langle\nabla f,\, x\rangle}_{=0} - \langle\nabla f,\, \nabla_j\nu_{S^n}\rangle = -A_{S^n}\left(\nabla f,\, \frac{\partial}{\partial\vartheta^j}\right) = -\nabla_j f$$

$$\left\langle \nabla_j\nabla f,\, \frac{\partial}{\partial\vartheta^i}\right\rangle = \nabla_j\underbrace{\left\langle\nabla f,\, \frac{\partial}{\partial\vartheta^i}\right\rangle}_{=\partial_i f} - \left\langle\nabla f,\, \nabla_i\frac{\partial}{\partial\vartheta^j}\right\rangle = \partial^2_{ij}f - \Gamma^k_{ij}\partial_k f = \nabla^2_{ij}f.$$

We finally write

$$A_{ij} = \frac{e^f}{\sqrt{1+|\nabla f|^2}}\big(\sigma_{ij} + \nabla_i f\nabla_j f - \nabla^2_{ij}, f\big)$$

which is exactly (1.14), and we are done. Equality (1.15) follows from a direct computation after writing $A^i_j = g^{li}A_{lj}$ and we do not report it.
Formula (1.16) follows from the area formula (see [1]):

$$\int_\Sigma h(y)\, dV_g(y) = \int_{S^n} h(\psi(x))J_{d\psi}(x)\, dV_\sigma \text{ for any } h \in C(\Sigma),$$

where

$$J_{d\psi}(x)^2 = \det\, d^*\psi|_x \circ d\psi|_x\,,$$

and $d^*\psi$ is the adjoint differential, whose representative matrix is simply the transpose of the $d\psi$ representative matrix. Taking $\{\frac{\partial}{\partial\vartheta^1}\dots\frac{\partial}{\partial\vartheta^n}, x\}$ as frame for $\mathbb{R}^{n+1}$ we easily find the expression

$$\det\, d^*\psi|_x \circ d\psi|_x = e^{2nf}\big(1+|\nabla f|^2\big),$$

and the result follows simply by taking the square root.

Lastly we deal with the Christoffel symbols. We recall the formula

$${}_g\Gamma^k_{ij} = \frac{1}{2}g^{ks}\big(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}\big),$$

and now we expand it:

$${}_g\Gamma^k_{ij} = \frac{1}{2}\left(\sigma^{ks} - \frac{\nabla^k f\nabla^s f}{1+|\nabla f|^2}\right)\big(\partial_i\sigma_{js} + \partial_j\sigma_{is} - \partial_s\sigma_{ij}\big) +$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\sigma^{ks}-\frac{\nabla^k f\nabla^s f}{1+|\nabla f|^2}\right)\big(\partial_i(\partial_j f\partial_s f)+\partial_j(\partial_i f\partial_s f)-\partial_s(\partial_i f\partial_j f)\big)+\\
&+\frac{1}{2}g^{ks}\big(\nabla_i f\, g_{js}+\nabla_j f\, g_{is}-\nabla_s f\, g_{ij}\big)\\
&=\frac{1}{2}\left(\sigma^{ks}-\frac{\nabla^k f\nabla^s f}{1+|\nabla f|^2}\right)\big(\partial_i\sigma_{js}+\partial_j\sigma_{is}-\partial_s\sigma_{ij}+2\,\partial^2_{ij}f\,\partial_s f\big)+\\
&+\frac{1}{2}\big(\nabla_i f\,\delta^k_j+\nabla_j f\,\delta^k_i-_g\nabla_k f\, g_{ij}\big)\\
&=\Gamma^k_{ij}+\partial^2_{ij}f\,\partial^k f-\frac{|\nabla f|^2}{1+|\nabla f|^2}\partial^2_{ij}f\,\partial^k f-\frac{1}{1+|\nabla f|^2}\partial^k f\,\partial_s f\,\frac{\sigma^{ls}}{2}\big(\partial_i\sigma_{js}+\partial_j\sigma_{is}-\partial_s\sigma_{ij}\big)+\\
&+\frac{1}{2}\big(\nabla_i f\,\delta^k_j+\nabla_j f\,\delta^k_i-_g\nabla_k f\, g_{ij}\big)\\
&=\Gamma^k_{ij}+\frac{1}{1+|\nabla f|^2}\big(\partial^2_{ij}f-\partial_s f\,\Gamma^s_{ij}\big)\partial^k f+\frac{1}{2}\big(\nabla_i f\,\delta^k_j+\nabla_j f\,\delta^k_i-_g\nabla_k f\, g_{ij}\big)\\
&=\Gamma^k_{ij}+\frac{1}{1+|\nabla f|^2}\nabla^2_{ij}f\nabla^k f+\frac{1}{2}\big(\nabla_i f\,\delta^k_j+\nabla_j f\,\delta^k_i-_g\nabla_k f\, g_{ij}\big).
\end{aligned}
$$

Proof of Lemma 1.13

We firstly recall the Codazzi equation for the second fundamental form (see [19, p. 250] for a proof):

$$ {}_g\nabla_k A^i_j = {}_g\nabla_j A^i_k \tag{A.2}$$

Equation (A.2) however holds for the Levi-Civita connection ${}_g\nabla$ taken with respect to the metric g, while we need to find a formula for the σ connection ∇. So we firstly expand ${}_g\nabla A$:

$$ {}_g\nabla_k A^i_j = D_k A^i_j +{}_g\Gamma^i_{kl}A^l_j -{}_g\Gamma^l_{kj}A^i_l $$

Now we plug this expression into (A.2), and use the expression (1.17) for the Christoffel symbols obtaining

$$
\begin{aligned}
\nabla_k A^i_j &= \nabla_j A^i_k+\frac{\nabla^i f}{1+|\nabla f|^2}\big(\nabla^2_{jl}fA^l_k-\nabla^2_{jk}fA^i_l\big)+\big(\nabla_j f\,\delta^i_l+\nabla_l\, f\delta^i_j-_g\nabla^i f\, g_{jl}\big)A^l_k+\\
&\quad-\big(\nabla_k f\,\delta^i_l+\nabla_l f\,\delta^i_k-_g\nabla^i f\, g_{kl}\big)A^l_j.
\end{aligned}
$$

We now notice that $\nabla^2_{jl}fA^l_k=\nabla^2_{kl}fA^l_j$. Expanding the term in fact we have

$$
\begin{aligned}
\nabla^2_{jl}fA^l_k &= \frac{e^{-f}\,\nabla^2_{jl}f}{1+|\nabla f|^2}\left(\delta^l_k-\nabla^l\nabla_k f+\frac{1}{1+|\nabla f|^2}\nabla^2 f[\nabla f]_j\,\nabla^l f\right)\\
&=\frac{e^{-f}}{1+|\nabla f|^2}\left(\nabla^2_{jk}f-\big(\nabla^2 f\,\nabla^2 f\big)_{jk}+\frac{\nabla^2 f[\nabla f]_j}{1+|\nabla f|^2}\,\nabla^2 f[\nabla f]_k\right)\\
&=\frac{e^{-f}}{1+|\nabla f|^2}\left(\nabla^2_{jk}f-\big(\nabla^2 f\,\nabla^2 f\big)_{kj}+\frac{\nabla^2 f[\nabla f]_k}{1+|\nabla f|^2}\,\nabla^2 f[\nabla f]_j\right)
\end{aligned}
$$

$$= \nabla^2_{kl} f A^l_j$$

This allows us to simplify the equation, obtaining

$$\nabla_k A^i_j = \nabla_j A^i_k + \left(\nabla_j f\, \delta^i_l + \nabla_l\, f \delta^i_j\right) A^l_k - \left(\nabla_k f\, \delta^i_l + \nabla_l f\, \delta^i_k\right) A^l_j$$

We contract the indices i and j:

$$\begin{aligned}\nabla_k A^i_i &= \nabla_i A^i_k + \left(\nabla_i f\, \delta^i_l + \nabla_l\, f \delta^i_i\right) A^l_k - \left(\nabla_k f\, \delta^i_l + \nabla_l f\, \delta^i_k\right) A^l_i \\ &= \nabla_i A^i_k + n A^l_k\, \nabla_l f - A^i_i\, \nabla_k f\end{aligned}$$

Finally we complete the proof. Indeed we write

$$A^i_j = \mathring{A}^i_j + \frac{1}{n} A^l_l\, \delta^i_j = \mathring{A}^i_j + H\, \delta^i_j$$

ith this expression we obtain

$$(n-1)\nabla_k H = \nabla_i \mathring{A}^i_k + n\mathring{A}^l_k\, \nabla_l f$$

The thesis follows dividing by $n - 1$.

Anisotropic computations

Proof of Lemma 3.9. Let x be in Σ, and let $\{z_1, \dots z_n\}$ be a frame for $T_x\mathcal{W}$. We compute the differential $d\psi$ in these coordinates, obtaining

$$\nabla_i \psi = z_i + \nabla_i u\, \nu + u \nabla_i \nu. \tag{A.3}$$

We use A.3 to compute the metric g.

$$\begin{aligned} g_{ij} &= \langle \nabla_i \psi,\, \nabla_j \psi\rangle = \langle z_i + \nabla_i u\, \nu + u\nabla_i \nu,\, z_j + \nabla_j u\, \nu + u \nabla_j \nu\rangle \\ &= \omega_{ij} + 2uA_{ij} + \nabla_i u\, \nabla_j u + u^2 \underbrace{\langle \nabla_i \nu,\, \nabla_j \nu\rangle}_{=A^k_i A_{kj}}. \end{aligned}$$

Now we search for a vector $V = \nu + a^i z_i$ which satisfies the condition $\langle V, \nabla_j \psi\rangle = 0$ for every $j = 1, \dots, n$ and we will recover $\nu^\Sigma = \frac{V}{|V|}$. We compute

$$0 = \langle \nu^\Sigma,\, \nabla_j \psi\rangle = \langle \nu + a^i z_i,\, z_j + \nabla_j u\, \nu + u \nabla_j \nu\rangle = \nabla_j u + (\omega + uA)^i_j a_i.$$

Normalising, we obtain the expression for ν^Σ, as desired.

$$\nu^\Sigma = \frac{\nu - (\omega + uA)^{-1}[\nabla u]}{|\nu - (\omega + Ah)^{-1}[\nabla u]|}.$$

Now we compute the approximated formula (3.15). Notice that, using the C^0 smallness of u we obtain

$$\nu^\Sigma = \frac{\nu - \nabla u}{\sqrt{1 + |(\mathrm{Id} + uh)^{-1}[\nabla u]|^2}} + \mathcal{R}, \tag{A.4}$$

where $\mathcal{R}$ is a combination of product of u and ∇u. We use this expression to compute A^{Σ}.

$$\begin{aligned} A^{\Sigma}_{ij} &= \langle \nabla_i \nu^{\Sigma}, \nabla_j \psi \rangle \\ &= \frac{\langle \nabla_i \nu + \nabla_i \nabla u, z_j + u \nabla_j \nu + \nabla_j u \nu \rangle + \langle \nabla_i \mathcal{R}, \nabla_j \psi \rangle}{\sqrt{1 + |(\mathrm{Id} + uA)^{-1}[\nabla u]|^2}} \\ &= \frac{A_{ij} - \nabla^2_{ij} u + u A^2_{ij} - u \langle \nabla_i (\nabla u), \nabla_j \nu \rangle + A[\nabla u]_i \nabla_j u + \langle \nabla_i \mathcal{R}, \nabla_j \psi \rangle}{\sqrt{1 + |(\mathrm{Id} + uA)^{-1}[\nabla u]|^2}}, \end{aligned} \tag{A.5}$$

where in (A.5) every element in $\nabla \mathcal{R}$ must be either a product of ∇u and ∇u or a product of u and $\nabla^2 u$, and every element is controlled by constants depending only on $\mathcal{W}$. Therefore, since u is small in the C^0-norm we can absorb the products of u and $\nabla^2 u$ into $\nabla^2 u$. Since Σ is convex, we know that $0 \leqslant A^{\Sigma}$, we easily obtain (3.16). □

A.2 OSCILLATION PROPOSITIONS

Proof of Lemmas 1.14, 2.12, 3.10

We recall the equations we are going to study.

$$\nabla u = \operatorname{div} \mathfrak{f} + \mathfrak{f}[\mathfrak{h}], \tag{A.6}$$

$$\nabla \operatorname{Scal} = \frac{2n}{n-2} \operatorname{div} \mathring{\operatorname{Ric}}, \tag{A.7}$$

$$\nabla H_F = \operatorname{div} A_F. \tag{A.8}$$

The first equation takes place in the round sphere S^n, the other two are studied in a closed hypersurface that satisfies condition $a)$ or $b')$ as in Proposition 0.1. These equations present clear similarities, since they are all variations of the equation

$$\nabla u = \operatorname{div} \mathfrak{f}$$

in a closed manifold. In all the three cases an immediate but naive covering argument may show the existence of a number λ such that

$$\|u - \lambda\|_{L^p(M)} \leqslant C(M) \|\mathfrak{f}\|_{L^p(M)}. \tag{A.9}$$

The problem in such argument is that we do not only have to obtain an estimate, but also to keep an eye on the constant C, which in our case has to depend only on general parameters. We are going to show an improved estimate which is basically (A.9), but gives a better control on the bounding constant. The technique we are going to use has been used and developed in [41], where the author deals with the isotropic version of equation (A.8). Considered the massive use we are making of this type of estimates and ideas throughout the thesis, we have decided to report the proof. We split it into the following steps.

- We show by direct computation in graph parametrisation how the three equations can be written as particular cases of a more general proposition.
- We obtain a local estimate of our desired inequality, with the bounding constant depending on determined parameters.

- We show how to make the local estimate global without losing the information on the bounding constant.

Let us start the proof.

Unifying the equations.

We recall Lemma 4.6:

Lemma A.1. *Let* $M = \mathrm{Graph}(u, \mathbb{B}^n)$ *be a smooth graph. Then the following formulas hold.*

$$g_{ij} = \delta_{ij} + \partial_i u\, \partial_j u \tag{A.10}$$

$$g^{ij} = \delta^{ij} - \frac{\partial^i u\, \partial^j u}{1 + |\partial u|^2} \tag{A.11}$$

$$dV_g = \sqrt{1 + |\partial u|^2}\, dx \tag{A.12}$$

$${}_g\Gamma^k_{ik} = \nu^k A_{ij}, \text{ where } \nu = \frac{\partial u}{\sqrt{1 + |\partial u|^2}}. \tag{A.13}$$

We compute the divergence term of equations (A.6), (A.7), (A.8) in graph parametrisation, and notice how this does not depend on Christoffel symbols.

(A.6) Let $\mathfrak{f}$ be a $(1, 1)$-tensor, $\mathfrak{f}^i_j = \sigma^{ik}\mathfrak{f}_{kj}$, where $\mathfrak{f}_{ij}$ is a symmetric tensor. By formula (A.13) and equality $A_{ij} = \sigma_{ij}$ for the sphere we obtain

$$\begin{aligned} \operatorname{div} \mathfrak{f}_k &= \nabla_i \mathfrak{f}^i_k = \partial_i \mathfrak{f}^i_k - \Gamma^i_{il}\, \mathfrak{f}^l_k - \Gamma^l_{ik}\, \mathfrak{f}^i_l = \partial_i \mathfrak{f}^i_k - \nu^i\, \sigma_{il}\, \mathfrak{f}^l_k - \nu^l \sigma_{ik} \mathfrak{f}^i_l \\ &= \partial_i \mathfrak{f}^i_k - \nu^l(\mathfrak{f}_{lk} - \mathfrak{f}_{kl}) = \partial_i \mathfrak{f}^i_k. \end{aligned}$$

(A.7) We compute the divergence term in equation (A.7). Firstly we compute the divergence of the Ricci tensor.

$$\begin{aligned} \nabla_i \mathrm{Ric}^i_k &= \partial_i \mathrm{Ric}^i_k + \Gamma^i_{ip} \mathrm{Ric}^p_k - \Gamma^p_{ik} \mathrm{Ric}^i_p = \partial_i \mathrm{Ric}^i_k + \nu^i A_{ip} \mathrm{Ric}^p_k - \nu^p A_{ik} \mathrm{Ric}^i_p \\ &= \partial_i \mathrm{Ric}^i_k + \nu^i A_{ip} \left(H h^p_k - A^p_q A^q_k\right) - \nu^p A_{ik} \left(H h^i_p - A^i_q A^q_p\right) \\ &= \partial_i \mathrm{Ric}^i_k + H \underbrace{\left(\nu^i A_{ip} A^p_k - \nu^p A_{ik} A^i_p\right)}_{=\nu^i A_{ip} A^p_k - \nu^i A_{pk} A^p_i = 0} + \underbrace{\left(\nu^p A_{ik} A^i_q h^q_p - \nu^i A_{ip} A^p_q A^q_k\right)}_{=\nu^p (A_{ik} A^i_q A^q_p - A_{pq} A^q_i h^i_k) = 0} \\ &= \partial_i \mathrm{Ric}^i_k. \end{aligned}$$

Now we write $\mathring{\mathrm{Ric}}^i_j = \mathrm{Ric}^i_j - \frac{\mathrm{Scal}}{n}\delta^i_j$, and notice that δ is a symmetric tensor. The computation of it is identical to the previous one, and we are done.

(A.8) Firstly, we need to prove that equation (A.8) holds. This follows from the computation below. Here we denote $(A_F)^i_j = A^i_j$ for the anisotropic second fundamental form and $A = h_{ij}$ for the isotropic one, with a little abuse of notation.

$$\begin{aligned} \operatorname{div}_g A_k &= \nabla_i A^i_k = \nabla_i \left(S^i_p\big|_\nu\, h^p_k\right) = \nabla_i \left(S^i_p\big|_\nu\right) h^p_k + S^i_p\big|_\nu\, \nabla_i h^p_k \\ &= D_q S^i_p\big|_\nu\, h^q_i h^p_k + S^i_p\big|_\nu\, \nabla_i h^p_k = D_p S^i_q\big|_\nu\, h^q_i h^p_k + S^i_p\big|_\nu\, \nabla^p h_{ik} \end{aligned}$$

$$= \nabla_k \left(S^i_p \big|_\nu h^p_i \right) = \nabla_k H_F.$$

Now we notice how also the last divergence term can be written as a flat divergence. We find

$$\begin{aligned} \operatorname{div}_g A_k &= \nabla_i A^i_k = \partial_i A^i_k + \Gamma^i_{ip} A^p_k - \Gamma^p_{ik} A^i_p = \partial_i A^i_k + \nu^i h_{ip} A^p_k - \nu^p h_{ik} A^i_p \\ &= \partial_i A^i_k + \nu^i h_{ip} S^p_q h^q_k - \nu^p h_{ik} S^q_p h^i_q = \partial_i A^i_k + \nu^i S^{pq} \left(h_{ip} h_{qk} - h_{pk} h_{qi} \right) \\ &= \partial_i A^i_k + \nu^i h_{ip} h_{qk} (S^{pq} - S^{qp}) = \partial_i S^i_k. \end{aligned}$$

Lastly we write in graph chart $\nabla f = \partial f$, since at the first order the Levi-Civita coincides with the classical derivations. These computations show how we have reduced the three problems to the following lemma:

Lemma : Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface. Assume Σ has fixed volume V and satisfies the conclusions of Lemma 0.2, i.e. admits two numbers L and R such that around every $q \in \Sigma$ we can find a chart defined on the ball $\mathbb{B}^n_R$, which is the graph of a smooth, L-Lipschitz function u_q. Assume there are $u \colon M \longrightarrow \mathbb{R}$, $\mathfrak{f} \in \Gamma(T^*M \otimes T^*M)$ that satisfy a differential relation which in every graph parametrisation at every point admits the following form:

$$\partial_k u = \partial_i \mathfrak{f}^i_k + T[\mathfrak{f}]_k, \quad \text{in } \mathbb{B}_R. \tag{A.14}$$

Here T is a linear operator satisfying $|T[\mathfrak{f}]| \leqslant C|\mathfrak{f}|$ and C a universal constant. Then there exists a $\lambda \in \mathbb{R}$, such that the following estimate holds.

$$\|u - \lambda\|_{L^p(M)} \leqslant C(n, p, V, R, L) \|\mathfrak{f}\|_{L^p(M)}.$$

Notice that in all the cases the manifold M satisfies condition $a)$ or $b')$ as in Proposition 0.1, and thus it allows Lemma 0.2 and Lemma 0.3 to hold. These will be crucial in the proof. We now prove the lemma..

Obtaining local estimates.

We begin by working in the graph, and write $u = \mathfrak{v} + \mathfrak{w}$, with $\mathfrak{v}$ and $\mathfrak{w}$ satisfying the conditions:

$$\begin{cases} \Delta_\delta \mathfrak{v} = \partial^k \partial_i \mathfrak{f}^i_k \\ \mathfrak{v}|_{\partial \mathbb{B}_R} = u|_{\partial \mathbb{B}_R}, \end{cases}$$

and

$$\begin{cases} \Delta_\delta \mathfrak{w} = \partial^k T[\mathfrak{f}]_k \\ \mathfrak{w}|_{\partial \mathbb{B}_R} = 0, \end{cases}$$

where Δ_δ is the flat laplacian. The estimate for the first systems follows by applying the classic Calderon-Zygmund theorem (See [41, Prop. 1.11] for a detailed proof in this particular case). We find a number λ such that

$$\|\mathfrak{v} - \lambda\|_{L^p(\mathbb{B}_{R/2})} \leqslant C(n, p) \|\mathfrak{f}\|_{L^p(\mathbb{B}_R)}$$

The second system is well known. In [20] the following inequality is shown:

$$\|\mathfrak{w}\|_{L^p(\mathbb{B}_{R/2})} \leqslant C(n, p)\|T[\mathfrak{f}]\|_{L^p(B_R)} \leqslant C_0\|\mathfrak{f}\|_{L^p(\mathbb{B}_R)}$$

The last constant C_0 depends on n, p and the control constant associated to the operator T. This does not appear in equations (A.7) and (A.8), it appears only in (A.6) and depends on n, p and $\|\mathfrak{h}\|_{C^0}$. We patch together the two estimates, and obtain

$$\|u - \lambda\|_{L^p(\mathbb{B}_{R/2})} \leqslant C_0\|\mathfrak{f}\|_{L^p(\mathbb{B}_R)}. \tag{A.15}$$

Estimate (A.15) is almost what we want. It is indeed a local estimate, but it concerns all Euclidean quantities. We show how to swap Euclidean measures with manifold metrics, and how to substitute Euclidean balls with geodesic balls.

The first follows easily from equation (A.12) and Lemma (0.2). Since $\mathrm{Lip}(u) \leqslant L$, we obtain indeed

$$dx \leqslant \sqrt{1 + |\partial u|^2}\, dx = dV_g \leqslant \sqrt{1 + L^2}\, dx.$$

Thus the measures are equivalent, and the control constants depend only on L. The same constant L control the switch from the Euclidean metric δ to the metric g.

Now Lemma 0.3 allows us to pass from Euclidean to geodesic balls and grants our privileged covering of balls. In particular, we obtain the existence of radius R such that

$$\min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\mathbb{B}^g_r(q))} \leqslant C(n, p, V, L, R)\|\mathfrak{f}\|_{L^p(M)},$$

for every $0 < r \leqslant R$.

Making the estimate global.

Now we make the estimate global. We follow the technique used in [41, p. 6-7] and prove the following lemma.

Lemma A.2. *Let* M *be a closed manifold, with fixed volume* $\mathrm{Vol}_n(M) = V$. *Suppose* $u \in C^\infty(M)$ *has the following property. There is a radius* ρ *such that for every* $x \in M$ *the following local estimate is satisfied:*

$$\|u - \lambda(x)\|_{L^p(\mathbb{B}_r(x))} \leqslant \beta, \tag{A.16}$$

where $\lambda(x)$ *is a real number depending on* x, $r \leqslant 2\rho$ *and* β *does not depend on* x. *Then there exists* $\lambda \in \mathbb{R}$ *such that*

$$\|u - \lambda\|_{L^p(M)} \leqslant C(n, \rho, V)\beta.$$

Proof. We choose a finite covering of balls $\{(\mathbb{B}_j, \lambda_j)\}_{j=1}^N$ which satisfies the following properties. Every ball $\mathbb{B}_j$ has radius 2ρ, estimate (A.16) holds with λ_j, and for every j, k there exists a ball of radius ρ contained in $\mathbb{B}_j \cap \mathbb{B}_k$.

Therefore, given two balls $\mathbb{B}_j$ and $\mathbb{B}_k$ whose intersection is non empty, we have:

$$|\lambda_j - \lambda_k| = \frac{1}{\mathrm{Vol}_n(\mathbb{B}_j \cap \mathbb{B}_k)^{\frac{1}{p}}}\|\lambda_j - \lambda_k\|_{L^p(\mathbb{B}_j \cap \mathbb{B}_k)}$$

$$
\begin{aligned}
&= \frac{1}{\mathrm{Vol}_n(\mathbb{B}_j \cap \mathbb{B}_k)^{\frac{1}{p}}} \|\lambda_j - u + u - \lambda_k\|_{L^p(\mathbb{B}_j \cap \mathbb{B}_k)} \\
&\leqslant \frac{1}{\mathrm{Vol}_n(\mathbb{B}_j \cap \mathbb{B}_k)^{\frac{1}{p}}} \left(\|u - \lambda_k\|_{L^p(\mathbb{B}_j \cap \mathbb{B}_k)} + \|u - \lambda_k\|_{L^p(\mathbb{B}_j \cap \mathbb{B}_k)} \right) \\
&\leqslant \frac{2\beta}{\mathrm{Vol}_n(\mathbb{B}_j \cap \mathbb{B}_k)^{\frac{1}{p}}}.
\end{aligned}
$$

Using the properties of the covering we obtain

$$
|\lambda_j - \lambda_k| \leqslant 2\,\mathrm{Vol}_n(\mathbb{B}_\rho)^{-\frac{1}{p}} \beta.
$$

Define $\lambda_{\min} := \min_{1 \leqslant j \leqslant n} \lambda_j$ and $\lambda_{\max} := \max_{1 \leqslant j \leqslant n} \lambda_j$. Consider a path joining the ball in the cover with $\lambda_{\min}$ to the one with $\lambda_{\max}$. Since this path can cross at most N different balls, we obtain

$$
|\lambda_{\max} - \lambda_{\min}| \leqslant 2N\,\mathrm{Vol}_n(\mathbb{B}_\rho)^{-\frac{1}{p}} \beta = C(n,\, p,\, \rho)\, \beta.
$$

For every $\lambda_{\min} \leqslant \lambda \leqslant \lambda_{\max}$ we have

$$
\begin{aligned}
\|u - \lambda\|_{L^p_\sigma(S^n)} &\leqslant \sum_{j=1}^{N} \|u - \lambda\|_{L^p_\sigma(\mathbb{B}_j)} \leqslant \sum_{j=1}^{N} \|u - \lambda_j + \lambda_j - \lambda\|_{L^p_\sigma(\mathbb{B}_j)} \\
&\leqslant \sum_{j=1}^{N} \|u - \lambda_j\|_{L^p_\sigma(\mathbb{B}_j)} + |\lambda_j - \lambda|\, \mathrm{Vol}_n(\mathbb{B}_j)^{-\frac{1}{p}} \\
&\leqslant \sum_{j=1}^{N} \|u - \lambda_j\|_{L^p_\sigma(\mathbb{B}_j)} + |\lambda_{\max} - \lambda_{\min}|\, \mathrm{Vol}_n(\mathbb{B}_j)^{-\frac{1}{p}} \leqslant C_2(n,\, p,\, \rho)\, \beta
\end{aligned}
$$

and the proof of Lemma A.2 is completed. □

A.3 SPARSE RESULTS

Proof of Proposition 3.15. Let $c \in \mathbb{R}^{n+1}$ be given. We easily obtain

$$
\begin{aligned}
\|u - \varphi_u\|_{W^{2,p}} &\leqslant \|u - \varphi_c\|_{W^{2,p}} + \|\varphi_c - \varphi_u\|_{W^{2,p}} \\
&\leqslant \|u - \varphi_c\|_{W^{2,p}} + \left\| \sum_{i=1}^{n+1} \langle u - \varphi_u,\, \varphi_i \rangle_{L^2} \varphi_i \right\|_{W^{2,p}} \\
&\leqslant \|u - \varphi_c\|_{W^{2,p}} + \sum_{i=1}^{n+1} \|\langle u - \varphi_c,\, \varphi_i \rangle_{L^2} \varphi_i\|_{W^{2,p}} \\
&\leqslant \|u - \varphi_c\|_{W^{2,p}} + \|u - \varphi_c\|_{L^1} \sum_{i=1}^{n+1} \|\varphi_i\|_{L^1} \|\varphi_i\|_{W^{2,p}} \\
&\leqslant C(n, p, F) \|u - \varphi_u\|_{W^{2,p}}
\end{aligned}
$$

and since c is arbitrary, we obtain the thesis. □

Proof of Proposition 3.11. As already noticed in the proof, the first inequality in (3.18) is an easy consequence of the fact that A_F is obtained multiplying A by the positive definite matrix S_F. Then we obtain

$$|A| \leqslant \left|(S_F)^{-1}\right||A_F| \leqslant C(n, F)|A_F|.$$

Thus, we focus on the second inequality. Firstly, we notice that a closed, convex hypersurface has non-negative anisotropic principal curvatures. Although this result seems to be known, we did not find its proof in the literature and we report it for the reader's convenience.

Lemma A.3. *Let Σ be a closed hypersurface, and let $\{\kappa_1, \dots \kappa_n\}$ be the spectrum of A_F. If Σ is convex, then $\kappa_i \geqslant 0$ for every $i = 1, \dots, n$.*

Proof. We recall that $(A_F)^i_j = (S_F)^i_k A^k_j$, where S_F is positive definite by hypothesis and A is non-negative definite by convexity (see [41, Prop. 3.2]).

Let $(S_F)^{\frac{1}{2}}$ be the square root of S_F. By standard linear algebra, $(S_F)^{\frac{1}{2}}$ exists and it is the unique symmetric matrix M with positive eigenvalues such that $M^2 = S_F$. Then we find

$$S_F A = (S_F)^{\frac{1}{2}}\Big((S_F)^{\frac{1}{2}} A (S_F)^{\frac{1}{2}}\Big)(S_F)^{-\frac{1}{2}}.$$

By this simple decomposition, we deduce that $S_F A$ has the same eigenvalues of $(S_F)^{\frac{1}{2}} A (S_F)^{\frac{1}{2}}$. This completes the proof: indeed, for every vector $v \in \mathbb{R}^n$, since A is non-negative definite, we can compute

$$\Big((S_F)^{\frac{1}{2}} A (S_F)^{\frac{1}{2}}[v], v\Big) = \Big(A (S_F)^{\frac{1}{2}}[v], (S_F)^{\frac{1}{2}}[v]\Big) \geqslant 0,$$

which is the thesis. □

If we look carefully at the proof of Lemma A.3, we can also notice that we have found the existence of a constant $c_1 = c_1(n, p, F)$ such that

$$\|A\|_{L^p(\Sigma)} \leqslant c_1 \|A_F\|_{L^p(\Sigma)}.$$

In order to conclude the proof of (3.11), we just have to focus on showing the remaining inequality

$$\|A_F\|_{L^p(\Sigma)} \leqslant c_2\Big(1 + \|\mathring{A}_F\|_{L^p(\Sigma)}\Big). \tag{A.17}$$

This follows by generalizing the isotropic result shown in [41]. Firstly, we notice that, for every couple of indices (i, j), we have:

$$\left(\int_\Sigma |\kappa_i - \kappa_j|^p\right)^{\frac{1}{p}} \leqslant \left(\int_\Sigma |\kappa_i - H_F|^p\right)^{\frac{1}{p}} + \left(\int_\Sigma |\kappa_j - H_F|^p\right)^{\frac{1}{p}} \leqslant c(n, p)\|\mathring{A}_F\|_p.$$

Consequently, we can estimate:

$$
\begin{aligned}
\|A_F\|_{L^p(\Sigma)} &\leqslant \left(\int_\Sigma \left(\sum_{i=1}^n \kappa_i^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leqslant \left(\int_\Sigma \left(\sum_{i=1}^n \kappa_i\right)^{p}\right)^{\frac{1}{p}} \\
&= \left(\int_\Sigma \left(n\kappa_1 + \sum_{i=2}^n (\kappa_i - \kappa_1)\right)^{p}\right)^{\frac{1}{p}} \leqslant c(n,\, p)\left(\left(\int_\Sigma \kappa_1^p\right)^{\frac{1}{p}} + \|\mathring{A}_F\|_p\right) \qquad \text{(A.18)} \\
&\leqslant c(n,\, p)\left(\left(\int_\Sigma \kappa_1^n\right)^{\frac{1}{n}} + \|\mathring{A}_F\|_p\right) \leqslant c(n,\, p)\left(\left(\int_\Sigma \det A_F\right)^{\frac{1}{n}} + \|\mathring{A}_F\|_p\right).
\end{aligned}
$$

We observe furthermore that the integral of the determinant of A_F does not depend on Σ but only on n, p and F. Indeed, by convexity of Σ (see [47, eq. (2.5.29), p.112]) we find

$$\int_\Sigma \det A_F = \int_\Sigma \det S_F|_\nu \det d\nu = \int_{S^n} \det S_F = c(n,\, p,\, F).$$

Plugging the previous equality in (A.18), we deduce (A.17) and conclude the proof. □

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