Rigidity and Flexibility of Isometric Embeddings

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ABSTRACT

This thesis is dedicated to the study of a dichotomy concerning isometric embeddings of Riemannian manifolds into Euclidean space.

The classical question about the existence of such embeddings was answered affirmatively by J. Nash in the famous Nash–Kuiper theorem, revolutionary not only because of the mathematics invented for its proof but also due to its countertintuitive nature. It demonstrates a type of *flexibility* of the space of solutions to a formally overdetermined system of nonlinear partial differential equations. In particular, isometric embeddings are highly non-unique, even in low co-dimension. Contrasting this flexibility is the *rigidity* of closed convex surfaces. Classical results in differential geometry show that sufficiently regular isometric embeddings of positively curved surfaces are often uniquely determined. A particularly striking illustration of these dramatically different behaviours is the case of the standard 2-dimensional sphere $\mathcal{S}^2$: by a result due to Cohn-Vossen in 1927, the only isometric embbedding of $\mathcal{S}^2$ into $\mathbb{R}^3$ is (upto a translation and rotation) the standard inclusion. Yet, as a consequence of the Nash–Kuiper theorem, there exist isometric embeddings of $\mathcal{S}^2$ into arbitrarily small balls.

This apparent paradox is explained by a closer look at the regularity properties of the embeddings constructed in the Nash–Kuiper theorem. They are only C^1, whereas the surfaces considered in Cohn-Vossen's rigidity theorem are assumed to be at least C^2. Naturally, this leads to the question of whether there is a regularity threshold in between C^1 and C^2 which distinguishes these behaviours.

The first results in this direction were obtained by Yu. F. Borisov, who in the late fifties proved a rigidity theorem for $C^{1,2/3+\varepsilon}$ isometric embeddings. He also announced a version of the Nash–Kuiper theorem in $C^{1,\alpha}$ for $\alpha < 1/(1+n(n+1))$ which was then rigorously proved fifty years later by S. Conti, C. De Lellis and L. Székelyhidi Jr. In a recent note, M. Gromov conjectures $C^{1,1/2}$ to be a possible threshold distinguishing flexible isometric embeddings from rigid ones. It is the aim of this thesis to make progress on this question, which we will refer to as the *Borisov–Gromov problem*.

In Chapter 3 we present an improvement of Borisov's flexibility exponent in the special case of 2-dimensional disks. We show that in this case it can be raised from $1/7$ to $1/5$. We then turn to rigidity and describe how a short proof of Borisov's rigidity result given by Conti, De Lellis and Székelyhidi leads to questions about the integrability properties and the sign of the Brouwer degree, which we adress in Chapters 4 and 5 respectively. Inspired by a striking formal analogy to the famous Onsager conjecture in fluid dynamics, we then propose a relaxed version of the problem, and show in Chapter 6 that the Hölder space $C^{1,1/2}$ is critical in a suitable sense. We prove in particular

that for $\alpha > \frac{1}{2}$, the Levi-Civita connection of any isometric embedding is induced by the Euclidean connection. On the other hand, for any $\alpha < 1/2$, we construct $C^{1,\alpha}$ isometric embeddings of portions of the standard 2-dimensional sphere for which this property fails. Lastly, in the final Chapter 7 we introduce the notion of extrinsic parallel translation, and show that it coincides with the usual intrinsic parallel translation whenever $\alpha > \frac{1}{2}(\sqrt{5}-1)$.

Except for the contents of Chapter 7, all results in this thesis are either published or submitted for publication.

ZUSAMMENFASSUNG

In der vorliegenden Dissertation untersuchen wir eine Dichotomie, welche isometrische Einbettungen von Riemannschen Mannigfaltigkeiten in den Euklidschen Raum betrifft.

Die klassische Frage nach der Existenz solcher Einbettungen wurde im wohlbekannten Nash–Kuiper Theorem von J. Nash bejahend beantwortet. Dieses Theorem war nicht nur aufgrund der mathematischen Methoden revolutionär, welche für den Beweis entwickelt wurden, sondern ebenso wegen seiner kontraintuitiven Natur: Es garantiert eine Art *Flexibilität* des Lösungsraumes eines formell überdeterminierten Systems von nichtlinearen partiellen Differentialgleichungen. Insbesondere sind isometrische Einbettungen nicht eindeutig bestimmt, sogar in niedriger Kodimension. Im Kontrast zu dieser Flexibilität steht die *Starrheit* von geschlossenen konvexen Flächen. Klassische Resultate aus der Differentialgeometrie zeigen, dass genügend reguläre isometrische Einbettungen von positiv gekrümmten Flächen oftmals eindeutig bestimmt sind. Eine besonders bemerkenswerte Illustration dieses Verhaltens liefert die 2-dimensionale Einheitssphäre $\mathcal{S}^2$: Ein Resultat von Cohn-Vossen aus 1927 impliziert, dass die Inklusion bis auf eine Translation oder Rotation die einzige isometrische Einbettung von $\mathcal{S}^2$ in $\mathbb{R}^3$ ist. Andererseits folgt aus dem Nash–Kuiper Theorem, dass es isometrische Einbettungen von $\mathcal{S}^2$ in beliebig kleine Kugeln des Euklischen Raums gibt.

Dieses scheinbare Paradoxon löst sich auf, wenn man die Regularitätseigenschaften betrachtet. Die Einbettungen, welche im Nash–Kuiper Theorem konstruiert werden, sind bloss einmal stetig differenzierbar (also C^1), wohingegen der Starrheitssatz von Cohn-Vossen die zweifach stetige Differenzierbarkeit (d.h. C^2) der Flächen voraussetzt. Dies wirft die Frage auf, ob es eine Regularitätsgrenze zwischen C^1 und C^2 gibt, welche diese verschiedenen Verhaltensweisen trennt.

Die ersten Ergebnisse in diese Richtung gehen auf Yu. F. Borisov zurück, welcher in den späten Fünfzigerjahren einen Starrheitssatz für $C^{1,2/3+\varepsilon}$ isometrische Einbettungen bewies und eine Version des Nash–Kuiper Theorems in $C^{1,\alpha}$ für $\alpha < 1/(1+n(n+1))$ ankündigte. Fünfzig Jahre später wurde diese von S. Conti, C. De Lellis und L. Székelyhidi Jr. rigoros bewiesen. In einem kürzlich erschienen Artikel vermutet M. Gromov, dass $C^{1,1/2}$ eine mögliche Regularitätsgrenze im Problem darstellt. Ziel

der vorliegenden Arbeit ist es, bezüglich dieser Fragestellung Fortschritte zu machen, welche wir als *Borisov-Gromov Problem* bezeichnen werden.

Kapitel 3 beinhaltet eine Verbesserung von Borisovs Flexibilitätsexponenten: Wir zeigen, dass man den Exponenten im Spezialfall der 2-dimensionalen Kreisscheibe auf $1/5$ erhöhen kann. Danach wenden wir uns der Starrheit zu und beschreiben, wie ein kurzer Beweis von Borisovs Starrheitssatz von Conti, De Lellis und Székelyhidi Jr. zu Fragen über die Integrabilitätseigenschaften und das Vorzeichen des Brouwerschen Abbildungsgrades führt, welche wir dann in den Kapiteln 4 respektive 5 behandeln. Von einer bemerkenswerten formellen Analogie des Problems zur wohlbekannten Onsager Vermutung in der Fluiddynamik inspiriert, schlagen wir eine abgeschwächte Problemstellung vor und zeigen in Kapitel 6, dass der Hölderraum $C^{1,1/2}$ in folgendem Sinne kritisch ist: Wir beweisen insbesondere, dass der Levi-Civita Zusammenhang einer isometrischen Einbettung $u \in C^{1,\alpha}$ für $\alpha > \frac{1}{2}$ vom Euklidschen Zusammenhang induziert wird, wohingegen wir für alle $\alpha < \frac{1}{2}$ isometrische $C^{1,\alpha}$ Einbettungen von Teilen der 2-dimensionalen Einheitssphäre konstruieren, welche dieser Eigenschaft nicht genügen. Schlussendlich führen wir im letzten Kapitel 7 einen extrinsischen parallelen Transport ein, und zeigen, dass er mit dem klassischen (intrinsischen) parallelen Transport übereinstimmt, sofern $\alpha > \frac{1}{2}(\sqrt{5}-1)$.

Alle Resultate dieser Dissertation sind, mit Ausnahme des Kapitels 7, entweder bereits publiziert, oder für eine Veröffentlichung eingereicht.

CONTENTS

1 INTRODUCTION

In his habilitation lecture, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, in 1854, B. Riemann introduced the concept of a Riemannian manifold, an abstract manifold with an intrinsic metric structure. This contrasted the extrinsic approach of the Gaussian theory of surfaces, which studied curves and surfaces in 3-dimensional space or submanifolds of Euclidean space of higher dimension with their metrics induced from the ambient space, and led to a question of great conceptual importance: are Riemannian manifolds and submanifolds of Euclidean space the same? In other words, can every Riemannian manifold (M, g) be realized as a submanifold of some Euclidean space $\mathbb{R}^m$ of appropriate dimension m?

To answer this question affirmatively one has to find an isometric embedding of (M, g) into $\mathbb{R}^m$, i.e., a diffeomorphism $u : M \to u(M) \subset \mathbb{R}^m$ satisfying

$$g = u^\sharp e\,, \tag{1.1}$$

where $u^\sharp e$ denotes the pullback of the Euclidean metric e on $\mathbb{R}^m$ through u. From (1.1), one obtains, introducing local coordinates, the system

$$g_{ij} = \sum_{k=1}^{m} \frac{\partial u^k}{\partial x_i}\frac{\partial u^k}{\partial x_j} =: \partial_i u \cdot \partial_j u \tag{1.2}$$

of $n(n+1)/2$ partial differential equations in the unknowns $u = (u^1, \dots, u^m)$. Here, g_{ij} are the components of the (symmetric) metric tensor g in local coordinates.

Already in 1873, L. Schläfli conjectured the existence of *local* isometric embeddings (and hence the local solvability of the system (1.2)) if $m = n(n+1)/2$. The conjecture was proved in the analytic setting by Janet [40], Cartan [14] and Burstin [12].[1] Remarkably, in the smooth setting (i.e., C^∞) it is still open[2], even for $n = 2$.

The global problem seems even more difficult, and historically the first to tackle it was H. Weyl, who in 1916 stated what has become known as the Weyl problem.

Weyl's problem [57]: Every metric on the unit sphere with positive Gaussian curvature can be uniquely (modulo rigid motions) realized as a convex[3] surface in $\mathbb{R}^3$.

1 Janet only outlined a proof, which was then rigorously carried out by Burstin. Cartan, on the other hand, gave a completely different, independent proof.

2 Gromov and Rohklin [33] and (independently) Greene [30] proved the local solvability in the smooth category if $m = n(n+1)/2 + n$. In the special case $n = 2$, there are partial results (see [34] for a good survey), but the general case is still open.

3 A 2-dimensional submanifold of $\mathbb{R}^3$ is called *convex* if it lies on one side of each of its tangent planes.

In a more modern language, he wanted to show the existence and uniqueness of sufficiently smooth global solutions to the system (1.2) in the special case of (M, g) being an "ovaloid".[4]

Weyl proposed a strategy to tackle the problem and realized it in the case where the metric is close enough to the standard metric on the sphere. Finally, the following two theorems gave a complete solution to the problem.[5]

Theorem 1.1 (*Weyl's Problem: Existence*)**.** *For any metric $g \in C^2$ on the sphere $\mathcal{S}^2$ with positive Gaussian curvature, there exists an isometric embedding u such that $u(\mathcal{S}^2)$ is a convex surface in $\mathbb{R}^3$.*

Theorem 1.2 (*Weyl's Problem: Rigidity*)**.** *Assume $g \in C^2$ is a metric on the sphere with positive Gaussian curvature and $u \in C^2$ is an isometric embedding. Then u is uniquely determined up to a rigid motion.*

Theorem 1.1 provides a solution of the system (1.2) in the formally determined case $m = 3 = n(n+1)/2$. In the general case, in particular if the system is overdetermined, i.e., $m < n(n+1)/2$, there is seemingly little hope to solve the system. Yet, surprisingly, in 1954, J. Nash showed that plenty of global solutions exist, and that obstructions to the existence are of purely topological nature. To state his theorem we recall that a *short* embedding $u : M \to \mathbb{R}^m$ is an embedding such that

$$g_{ij} - \partial_i u \cdot \partial_j u \geq 0 \tag{1.3}$$

in the sense of quadratic forms. The famous Nash–Kuiper theorem is then the following.

Theorem 1.3 (Nash [45], Kuiper [42][6])**.** *Let (M, g) be a compact[7] n-dimensional Riemannian manifold with continuous metric g and let $m \geq n+1$. Then any short embedding $u : M \to \mathbb{R}^m$ can be uniformly approximated by isometric embeddings of class C^1.*

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4 Weyl's interest in this special case stemmed from the uniqueness part of the problem: by an (already then) classical result of A. L. Cauchy from 1813, isometric closed convex polyhedra in $\mathbb{R}^3$ are congruent, but at the time, little was known for more general surfaces.

5 Theorem 1.1 was proved for analytic metrics by Lewy [43] and a few years later for $g \in C^k$ for $k \geq 4$ by Nirenberg [47] and for $k = 3, 2$ by Heinz [35] (the resulting maps u are $C^{k-1,\alpha}$ for any $\alpha < 1$). The proofs followed Weyl's original approach. On the other hand, Alexandrov [1] investigated so-called *arbitrary convex surfaces* and showed that any metric on the sphere with positive Gaussian curvature can be realized by such a convex surface, although it was unclear if the resulting surface had the right regularity to be a solution of the problem. This was remedied by Pogorelov, who in [50] proved a suitable regularity theorem, thus giving an alternative proof of Theorem 1.1. A version of Pogorelov's regularity theorem in Hölder spaces was obtained by Sabitov [52]. From it one can conclude Theorem 1.1 with the mapping properties $g \in C^{k,\alpha} \Rightarrow u \in C^{k,\alpha}$ for $k \geq 2$, $0 < \alpha < 1$.
Theorem 1.2 is due to Herglotz [36], although an analytic version was already proved by Cohn-Vossen [15]. Moreover, it can also be deduced from a more general result by Porgorelov [51] stating that isometric closed convex surfaces are congruent.

6 Nash proved Theorem 1.3 for $m \geq n+2$, and indicated that, with "[...] a less easily controlled perturbation process", one could also show the case $m = n+1$. This was then carried out by Kuiper.

7 Nash's theorem is not limited to compact manifolds, but we will restrict our attention to this case.

By Whitney's embedding theorem, an n-dimensional manifold M can always be embedded in $\mathbb{R}^{2n}$, and if M is compact one can simply "shrink" this embedding by multiplying it with a small number to find a short embedding. Thus, Theorem 1.3 is not only an existence theorem; it also shows that the set of solutions is huge.

Strikingly, for $n \geq 3$ and $m = n + 1$ the system (1.2) is formally heavily overdetermined, and hence the flexibility of the solutions is very counter-intuitive. In fact, it took several years until it was realized that this behaviour is caused by the low regularity of the solutions constructed: building upon the new ideas introduced by Nash in the proof of Theorem 1.3, M. Gromov in [31] formulated the framework of *convex integration* and the so-called *h-principle*, linking the flexibility of isometric embeddings obtained above to a number of other counter-intuitive phenomena in geometry (cf. also Section 1.2).

In the case of isometric embeddings, the flexibility becomes ever so striking when directly contrasted with the aforementioned rigidity of the Weyl problem, leading to a dichotomy which lies at the heart of this thesis.

1.1 THE BORISOV–GROMOV PROBLEM

To illustrate this dichotomy, consider the unit sphere $\mathcal{S}^2$ of $\mathbb{R}^3$ with the standard metric induced by the inclusion $\iota : \mathcal{S}^2 \hookrightarrow \mathbb{R}^3$. Since $\mathcal{S}^2$ is compact, any small enough multiple of ι is a short map. Therefore, by the Nash–Kuiper theorem, there exist C^1 isometric embeddings of $(\mathcal{S}^2, \iota^\sharp e)$ into arbitrarily small balls $B_\varepsilon \subset \mathbb{R}^3$. Yet, by the rigidity theorem of the Weyl problem, in the C^2 category there is (modulo rigid motions) just one isometric embedding: the standard inclusion. Naturally, one is interested to know[8] if there is a regularity threshold, for example in the Hölder scale, dividing these two drastically different types of behaviours. More precisely,

(B–G): Does there exist $\alpha_0 \in]0,1[$ such that

(i) if $\alpha > \alpha_0$ and $u \in C^{1,\alpha}$ is an isometric embedding of a 2-dimensional closed Riemannian manifold with positive Gaussian curvature into $\mathbb{R}^3$, then u is uniquely determined up to a rigid motion;

(ii) if $\alpha < \alpha_0$, then the Nash–Kuiper theorem holds with C^1 replaced by $C^{1,\alpha}$?

It is the aim of this thesis to make progress on this question, which we will refer to as the *Borisov–Gromov problem*.

The first to investigate isometric embeddings of class $C^{1,\alpha}$ was Yu. F. Borisov, and in the late fifties, building upon works of Pogorelov, he proved in the series of short geometric articles [2]–[5] that the assumption $u \in C^2$ in the rigidity Theorem 1.2 can be replaced by $u \in C^{1,\alpha}$ for $\alpha > \frac{2}{3}$. Moreover, in [6] he claimed the validity of the Nash–Kuiper theorem for analytic metrics in $C^{1,\alpha}$ for $\alpha < 1/(1 + n(n+1))$, yet a proof

8 See, for example, question 27 in [58].

only appeared in [7] for the case of 2-dimensional disks and $\alpha < \frac{1}{13}$. In [17], S. Conti, C. De Lellis and L. Székelyhidi Jr. gave a proof of Borisov's claims for C^2 metrics. They showed in particular[9] that if $g \in C^2$ is a metric on a compact n-dimensional manifold M and $u : M \to \mathbb{R}^{n+1}$ is a short embedding, then it can be uniformly approximated by isometric embeddings of class $C^{1,\alpha}$, where $\alpha < 1/(1+n(n+1))$ in case the manifold is diffeomorphic to an n-ball and $\alpha < 1/(1+n(n+1)^2)$ in the general case (this decrepancy has been removed recently in a forthcoming work by W. Cao and Székelyhidi Jr.). In addition, they give a very short proof of Borisov's rigidity result, which we will briefly comment on in Chapter 5. Hence, if it exists, $1/7 \leq \alpha_0 \leq 2/3$.[10]

An improvement of Boriosov's exponent was found in the author's master's thesis: in the case where M is a 2-dimensional disk, we were able to raise the exponent from 1/7 to 1/5.

1.2 ONSAGER'S CONJECTURE: CONNECTION TO THE THEORY OF TURBULENT FLUIDS

Recently, De Lellis and Székelyhidi Jr. discovered a surprising connection of the Borisov–Gromov problem to the theory of turbulent fluids. Consider the incompressible Euler equations, which describe the motion of a perfect incompressible fluid,

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla p = 0 \\ \operatorname{div} v = 0, \end{cases} \tag{1.4}$$

where $v = v(x,t)$ is the velocity and $p = p(x,t)$ is the pressure. We will take the spatial domain to be the flat 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$.

For classical periodic solutions (i.e., if $v \in C^1(\mathbb{T}^3 \times I)$) the total kinetic energy,

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx \,,$$

is conserved by the flow induced by (1.4), so that $E(t) = E(0)$. However, for weak solutions this may not be true. Indeed, one of the cornerstones of 3-dimensional turbulence is so-called anomalous dissipation: it is an experimentally observed fact that the rate of energy dissipation in the vanishing viscosity limit (more precisely the infinite Reynolds number limit) stays above a certain non-zero constant.

Assuming that a turbulent fluid is represented by a solution of the incompressible Navier–Stokes equations, in the vanishing viscosity limit one obtains the system (1.4). Since classical solutions conserve the energy, in this (vaguely defined) limiting process one expects to find weak solutions of the Euler equations. It was L. Onsager in 1949 [49] who first formulated the corresponding mathematical problem: is there a threshold

9 The results are in fact more general, and include the case of rough metrics $g \in C^{0,\beta}$.
10 In [32], Gromov conjectures $\alpha_0 = \frac{1}{2}$.

between C^0 and C^1 regularity for energy conservation? Based on calculations in Fourier space, he stated the following conjecture.

Conjecture 1.4. *Consider periodic 3-dimensional weak solutions of the incompressible Euler equations, where the velocity v satisfies the uniform Hölder condition*

$$|v(x,t) - v(x',t)| \le C|x - x'|^{\theta}, \tag{1.5}$$

for constants C and θ independent of x, x' and t.

(a) If $\theta > \frac{1}{3}$, then the total kinetic energy of v is constant;

(b) For any $\theta < \frac{1}{3}$ there are v for which it is not *constant.*

Part (a) of the conjecture was fully resolved in [16] (see also the work [27]). On the other hand, part (b) was settled only very recently in the work [39] by P. Isett. The latter work concluded a series of partial results (cf. [8–11, 25, 38]), all started off by the work [24]. In that work De Lellis and Székelyhidi Jr., inspired by the methods pioneered by Nash in [45] in the isometric embedding problem, were able to introduce a new set of techniques to produce irregular continuous solutions of the Euler equations.

1.3 THE RELAXED PROBLEM

Both from the rigidity and from the flexibility side, the current state of the art is still far from reaching the conjectured threshold $\alpha_0 = \frac{1}{2}$ in the Borisov–Gromov problem, and despite the formal analogy between the two problems (and the similarities of the respective techniques involved), the current approaches which led to the solution of the Onsager conjecture do not seem to give new insight. That said, the Borisov–Gromov problem is a bit more stringent than its counter-part in fluid dynamics: the Euler equations come with an additional conservation law (the energy identity) which is valid for all solutions above a certain regularity threshold and involves only as many derivatives as there are in the equation. The rigidity of isometric embeddings, however, is a stronger property in the sense that a closer analogue in the case of weak solutions to the Euler equations would be their uniqueness. Moreover, it uses the Gauss identity, where second derivatives of u are involved (whereas in the equation only first derivatives appear).

It therefore seems sensible to consider a relaxed version of the Borisov–Gromov problem aimed at finding suitable geometric identities (analogous to the energy identity) which relate the intrinsic and extrinsic geometry of submanifolds, and showing that they are satisfied by isometric embeddings above a certain regularity threshold while producing convex integration solutions which violate them.

A geometric object well suited for this endeavor is the Levi-Civita connection. Intrinsically defined, it can be related to the extrinsic world via the Gauss formula: for (smooth) submanifolds of Euclidean space it coincides with the tangential connection (i.e., the one induced by the Euclidean connection). The equality of these two objects can be thought of as a weaker form of rigidity, and it can be made sense of in $C^{1,\alpha}$ for $\alpha > \frac{1}{2}$, thus making a good candidate for defining the relaxed problem.

Closely connected with the Levi-Civita connection is the notion of parallel translation, which provides another tool of measuring the intrinsic predictability of isometric embeddings. It is possible to define an extrinsic notion of parallel translation by taking a suitable limit of a discrete process consisting of a combination of parallel translation with respect to the ambient connection and projections onto the tangent bundle. For smooth embeddings these two notions agree, and, again, one hopes that this property remains true for isometric embeddings in $C^{1,\alpha}$ for $\alpha > \alpha_0$.

1.4 RESULTS AND OUTLINE OF THE THESIS

The first part of this thesis is dedicated to the Borisov–Gromov problem. In Chapter 3 we include a slightly improved version of the author's master's thesis, namely the joint work [22], where we prove the following thereom.

Theorem 1.5. *Let $g \in C^2$ be a metric on the closed unit disk $\overline{D}_1 \subset \mathbb{R}^2$ and $\delta > 0$ arbirarily small. Then any short embedding $u : \overline{D}_1 \to \mathbb{R}^3$ can be uniformly approximated by isometric embeddings of class $C^{1,1/5-\delta}$.*

Hence, in the case of 2-dimensional disks, we are able to raise Borisov's exponent from 1/7 to 1/5.

We then turn to the rigidity part of the problem. The proof of Borisov's rigidity result given in [17] yields a different, much simpler approach to the problem and reveals an interesting connection with the Brouwer degree, an object whose meaning for maps in various regularity classes has been extensively studied. The key observation used in this approach is that the $C^{1,2/3+\delta}$ regularity is enough for the following change of variables formula to hold:

$$\int_V f(N(x))\kappa_g(x)\, d\mathrm{Area}(x) = \int_{\mathcal{S}^2} f(y)\deg(N, V, y)\, dy\,. \tag{1.6}$$

Here, N is the normal map of the embedding, $\deg(N, V, \cdot)$ is its Brouwer degree with respect the compactly contained open set $V \subset\subset M$, $f \in C_c^\infty(\mathcal{S}^2 \setminus N(\partial V))$ is a test function and κ_g and dArea are the respective Gaussian curvature and volume element of the manifold (M, g). Observe that (1.6) relates the Gaussian curvature of the manifold and the normal map of the embedding. Therefore, it can be thought of as a surrogate, or weak version, of Gauss' theorem. As it turns out, this weak version is enough to guarantee the rigidity of the embedding.

In particular, if one could show the validity of (1.6) for embeddings of class $C^{1,1/2+\delta}$, then one would resolve the rigidity part of the Borisov–Gromov problem with the conjectured exponent $\alpha_0 = \frac{1}{2}$. This leads us to study the integrability properties of the Brouwer degree of Hölder continuous functions. Based on the work [48], we show in [20] that if α is big enough, the Brouwer degree of a function $N \in C^{0,\alpha}$ has some fractional Sobolev regularity. We present this result in Chapter 4. We do not know whether this *higher* integrability property can be used as an advantage in the Borisov–Gromov problem. However, as a first step towards the fractional Sobolev regularity we give a short, elementary proof of the fact that for large enough α, the degree of $N \in C^{0,\alpha}$ is an L^1 function (see Theorem 4.5).

The latter result is needed on the way towards a rigidity theorem in $C^{1,1/2+\delta}$. Indeed, as a consequence of Theorem 4.5, if the embedding is in $C^{1,1/2+\delta}$, both sides of (1.6) make sense also for the constant function $f \equiv 1$, and it is not difficult to show that the equality remains true if V is a Lipschitz set. Ideally one would like to show that this (weaker) change of variables formula is enough to conclude the rigidity, which leads to the following conjecture about the sign of the Brouwer degree.

Conjecture 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded open set and let $N : \overline{\Omega} \to \mathbb{R}^2$ be $C^{0,\alpha}$ with $\alpha > \frac{1}{2}$ and have the property that for all Lipschitz open sets $A \subset \Omega$,*

$$\int_{\mathbb{R}^2} \deg(N, A, y)\, dy \geq 0\,.$$

Then $\deg(N, A, y) \geq 0$ *for all such A and all $y \in R^2 \setminus N(\partial A)$.*

If this conjecture were true, one could use the arguments of [17] to conclude a rigidity theorem in $C^{1,1/2+\delta}$. In Chapter 5 we show that the conclusion of Conjecture 1.6 is correct when $\alpha > \frac{2}{3}$.

In the second part of this thesis we consider the relaxed problem. Chapter 6 represents the work [21], where we show that the Hölder exponent $\alpha_0 = \frac{1}{2}$ is indeed critical in the weak sense explained above: if $\alpha > \frac{1}{2}$, the equality between the Levi-Civita and the tangential connection remains true for isometric embeddings $u \in C^{1,\alpha}$, whereas for any $\alpha < \frac{1}{2}$ we construct isometric embeddings $u \in C^{1,\alpha}$ of portions of the 2-sphere which violate it (see Theorem 6.2).

Lastly, in Chapter 7 we define an *extrinsic* parallel translation and show that it preserves the lengths of vectors for $\alpha > \frac{1}{2}$ and, moreover, that it coincides with the usual (intrinsic) parallel translation whenever $u \in C^{1,\alpha}$ for $\alpha > \frac{1}{2}(\sqrt{5} - 1)$.

2 PRELIMINARIES

In this chapter, we fix the most important notation and gather some preliminary results which are needed across all chapters.

2.1 NOTATION

We will denote the Euclidean norm on $\mathbb{R}^n$ by $|\cdot|$ and the Euclidean scalarproduct between two vectors $X, Y \in \mathbb{R}^n$ by $\langle X, Y\rangle$ or $X \cdot Y$. For a set $A \subset \mathbb{R}^n$ we write $\mathring{A}$, $\overline{A}$, ∂A for its topological interior, closure and boundary respectively, except for Chapter 5, where we will use $\operatorname{int}(A)$ instead of $\mathring{A}$ due to notational convenience. The open ball of radius $r > 0$ and center $x_0 \in \mathbb{R}^n$ is denoted by $B_r(x_0)$ and the corresponding sphere by $\mathcal{S}_r^{n-1}(x_0) = \partial B_r(x_0)$. If the center x_0 corresponds to the origin we will often simply write B_r and $\mathcal{S}_r^{n-1}$, and in the case of the sphere we will omit the radius r if $r = 1$. The space of symmetric $n \times n$ matrices with real entries will be denoted by Sym_n and the subset of positive definite, symmetric matrices by Sym_n^+. We write $A^\intercal$ for the transpose of a matrix A, and it will be convenient to use the notation $\operatorname{sym}(A) = \frac{1}{2}(A + A^\intercal)$.

For an open set $\Omega \subset \mathbb{R}^n$, $C^0(\Omega, \mathbb{R}^m)$ stands for the space of continuous functions $u : \Omega \to \mathbb{R}^m$, and for $m = 1$, we simply write $C^0(\Omega)$. We use $C^0(\overline{\Omega}, \mathbb{R}^m)$ to denote the subset of functions $u \in C^0(\Omega, \mathbb{R}^m)$ which are uniformly continuous. For a positive integer k, we introduce the usual spaces

$$
\begin{aligned}
C^k(\Omega, \mathbb{R}^m) &= \{u : \Omega \to \mathbb{R} : u \text{ is } k\text{-times continuously differentiable}\}\,, \\
C^k(\overline{\Omega}, \mathbb{R}^m) &= \{u \in C^k(\Omega, \mathbb{R}^m) : D^\beta u \text{ is uniformly continuous for every } |\beta| \le k\}\,,
\end{aligned}
$$

where $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index and $|\beta| = \sum_i \beta_i$, and where, again, we omit the target space from the notation in case $m = 1$.

2.2 HÖLDER SPACES AND INTERPOLATION INEQUALITIES

In the following $k \in \mathbb{N}$, $\alpha \in]0,1][$, and β is a multi-index. The maps f can be real-valued, vector-valued, matrix-valued or generally tensor-valued. In all these cases we endow the targets with the standard Euclidean norms $|\cdot|$. We introduce the usual Hölder norms as follows. First of all, the supremum norm is denoted by $\|f\|_0 := \sup|f|$. We define the Hölder seminorms as

$$[f]_k = \max_{|\beta|=k} \|D^\beta f\|_0 \,,$$
$$[f]_{k+\alpha} = \max_{|\beta|=k} \sup_{x\neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \,.$$

The Hölder norms are then given by

$$\|f\|_k = \sum_{j=0}^{k} [f]_j \,,$$
$$\|f\|_{k+\alpha} = \|f\|_k + [f]_{k+\alpha}.$$

Sometimes we also use the notation $\|\cdot\|_{k,\alpha}$. If we want to put emphasis on the set Ω where the norm is intended, we write $\|\cdot\|_{k+\alpha,\Omega}$ or also $\|\cdot\|_{C^{k,\alpha}(\overline{\Omega})}$. For an open set $\Omega \subset \mathbb{R}^n$ the usual Hölder spaces are given by

$$C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^m) = \left\{ u \in C^k\left(\overline{\Omega}, \mathbb{R}^m\right) : \|u\|_{k,\alpha} < \infty \right\}.$$

We then recall the standard "Leibniz rule" to estimate norms of products

$$[fg]_r \leq C\big([f]_r\|g\|_0 + \|f\|_0[g]_r\big) \qquad \text{for any } 1 \geq r \geq 0 \tag{2.1}$$

and the interpolation inequalities (see, for example, [29])

$$[f]_s \leq C\|f\|_0^{1-\frac{s}{r}}[f]_r^{\frac{s}{r}} \qquad \text{for all } r \geq s \geq 0. \tag{2.2}$$

We also collect two classical estimates on the Hölder norms of compositions. These are also standard, for instance in applications of the Nash-Moser iteration technique.

Proposition 2.1. *Let $0 \leq \alpha < 1$, $\Psi : \Omega \to \mathbb{R}$ and $u : \mathbb{R}^n \supset U \to \Omega$ be two $C^{k,\alpha}$ functions, with $\Omega \subset \mathbb{R}^m$. Then there is a constant C (depending only on α, k, Ω and U) such that*

$$[\Psi \circ u]_{k+\alpha} \leq C[u]_{k+\alpha}\left([\Psi]_1 + \|u\|_0^{k-1}[\Psi]_k\right) + C[\Psi]_{k+\alpha}\left(\|u\|_0^{k-1}[u]_k\right)^{\frac{k+\alpha}{k}} \,, \tag{2.3}$$
$$[\Psi \circ u]_{k+\alpha} \leq C\left([u]_{k+\alpha}[\Psi]_1 + [u]_1^{k+\alpha}[\Psi]_{k+\alpha}\right) \,. \tag{2.4}$$

Let $f, g : \mathbb{R}^n \supset U \to \mathbb{R}$ two $C^{k,\alpha}$ functions. Then there is a constant C (depending only on α, k, n and U) such that

$$[fg]_{k+\alpha} \leq C(\|f\|_0[g]_{k+\alpha} + \|g\|_0[f]_{k+\alpha})\,. \tag{2.5}$$

Proof. The chain rule can be written as

$$D^k(\Psi \circ u) = \sum_{i=1}^{k} \left(D^i\Psi \circ u\right) \sum_j C_{i,j} (Du)^{j_1} \cdot \dots \cdot \left(D^k u\right)^{j_k} , \tag{2.6}$$

where $C_{i,j}$ are constants and $j = (j_1, \dots, j_k)$ is a multi-index with

$$\sum j_l = i, \quad \sum l\, j_l = k\,.$$

The claim then follows by the Leibniz rule (2.1) and a repeated application of the interpolation inequalities (2.2) to (2.6). Statement (2.5) is a straightforward consequence of the usual Leibniz rule, interpolation and the Young inequality. □

Remark 2.2. Observe that if $\alpha = 0$ we have the estimates

$$[\Psi \circ u]_k \le C[u]_k \left([\Psi]_1 + \|u\|_0^{k-1}[\Psi]_k\right), \tag{2.7}$$

$$[\Psi \circ u]_k \le C \left([u]_k[\Psi]_1 + [u]_1^k[\Psi]_k\right). \tag{2.8}$$

2.3 MOLLIFICATION

In the following chapters, except otherwise stated, φ will represent a non-negative, smooth function with compact support in the unit ball of $\mathbb{R}^n$, which is rotationally symmetric and has unit integral; in other words, a standard mollification kernel. We will then often use regularizations of maps f by convolution with $\varphi_\ell(y) := \ell^{-n}\varphi(\frac{y}{n})$. For functions f which do not have their support compactly contained in the domain, say Ω, we fix the convention that the mollified function f_ℓ, defined through

$$f_\ell(x) := f * \varphi_\ell(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\ell(y)\, dy\,,$$

is defined in $\Omega_\ell := \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) > \ell\}$. The following mollification estimates are crucial throughout the thesis.

Lemma 2.3. *For any $r, s \ge 0$ and $0 < \alpha \le 1$ we have*

$$[f * \varphi_\ell]_{r+s} \le C\ell^{-s}[f]_r, \tag{2.9}$$

$$[f - f * \varphi_\ell]_r \le C\ell^2[f]_{2+r}, \tag{2.10}$$

$$\|f - f * \varphi_\ell\|_r \le C\ell^{2-r}[f]_2, \qquad \text{if } 0 \le r \le 2 \tag{2.11}$$

$$\|(fg) * \varphi_\ell - (f * \varphi_\ell)(g * \varphi_\ell)\|_r \le C\ell^{2\alpha-r}\|f\|_\alpha\|g\|_\alpha\,, \tag{2.12}$$

where the constants C depend only upon s, r, α and φ.

Proof. For proof of estimates (2.9), (2.10) and (2.12) see, for example, [17, Lemma 1]. The additional estimate (2.11) can be seen as follows. Recall the estimate

$$\|f - f * \varphi_\ell\|_0 \le C\ell[f]_1\,,$$

which can be derived using the mean value theorem and an integration. In particular also

$$[f - f * \varphi_\ell]_1 \le C\ell[f]_2\,.$$

We combine this estimate with (2.2) and (2.10) to get

$$\begin{aligned}[f - f * \varphi_\ell]_r &\le C\|f - f * \varphi_\ell\|_0^{1-r}[f - f * \varphi_\ell]_1^r \\ &\le C\left(\ell^2\|D^2 f\|_0\right)^{1-r}\left(\ell\|D^2 f\|_0\right)^r \le C\ell^{2-r}[f]_2\,,\end{aligned}$$

whenever $0 \le r \le 1$. If however $1 \le r \le 2$, we invoke the trivial inequality

$$[f - f * \varphi_\ell]_2 \le C[f]_2$$

to deduce

$$[f - f * \varphi_\ell]_r \le C\|Df - Df * \varphi_\ell\|_0^{2-r}[Df - Df * \varphi_\ell]_1^{r-1} \le C\ell^{2-r}[f]_2\,,$$

from which the claim follows. $\square$

Part I

THE BORISOV-GROMOV PROBLEM

3 A NASH-KUIPER THEOREM FOR $C^{1,1/5-\delta}$ EMBEDDINGS OF SURFACES IN 3 DIMENSIONS

In this chapter we consider isometric immersions of 2-dimensional disks in $\mathbb{R}^3$. With D_r and $\overline{D}_r$ we denote, respectively, the open and closed disks in $\mathbb{R}^2$ with center at the origin and radius r. If g is a C^0 Riemannian metric on $\overline{D}_r$, an isometric immersion $u : \overline{D}_r \to \mathbb{R}^n$ is a C^1 immersion such that $u^\sharp e = g$, where e denotes the Euclidean metric on $\mathbb{R}^n$. In other words this means that

$$\partial_i u \cdot \partial_j u = g_{ij} \,. \tag{3.1}$$

We recall that if $\partial_i u \cdot \partial_j u \le g_{ij}$ in the sense of quadratic forms then u is called a short immersion. If moreover $\partial_i u \cdot \partial_j u < g$, then it is called stricly short. The main theorem of this chapter is the following approximation result which, using a popular terminology, is an "h-principle" statement, cf. [23, 31, 54].

Theorem 3.1. *Let g be a C^2 metric on $\overline{D}_2$ and $\bar{u} \in C^1(\overline{D}_2, \mathbb{R}^3)$ a short immersion. For every $\delta > 0$ and $\varepsilon > 0$ there is a $C^{1,1/5-\delta}$ isometric immersion u of $(\overline{D}_1, g)$ in $\mathbb{R}^3$ such that $\|\bar{u} - u\|_{C^0} < \varepsilon$. If in addition $\bar{u}$ is an embedding, then u can be chosen to be an embedding.*

Theorem 3.1 could be improved in several directions. In particular, with little additional technicalities, which we believe to be of secondary importance, we will also show the following

Theorem 3.2. *Let g be a C^2 metric on $\overline{D}_1$ and $\bar{u} \in C^1(\overline{D}_1, \mathbb{R}^3)$ a short immersion. For every $\delta > 0$ and $\varepsilon > 0$ there is a $C^{1,1/5-\delta}$ isometric immersion u of $(\overline{D}_1, g)$ in $\mathbb{R}^3$ such that $\|\bar{u} - u\|_{C^0} < \varepsilon$. If in addition $\bar{u}$ is an embedding, then u can be chosen to be an embedding.*

Note that here, in contrast to Theorem 3.1, the domain of the isometric embedding u is the same as the domain of the original short embedding $\bar{u}$. As mentioned in the introduction, in the case of 2-dimensional disks, this theorem improves the exponent claimed by Borisov for analytic metrics (and subsequently verified for C^2 metrics in [17]) from $1/7$ to $1/5$.

3.1 MAIN ITERATION

Theorem 3.1 is achieved via an iteration, which depends upon several parameters. We start introducing the main ones. The first parameter $\alpha > 0$ is an exponent, which is assumed to be rather small, in fact smaller than a geometric constant:

$$0 < \alpha < \bar{\alpha} \,. \tag{3.2}$$

Two further exponents will be called c and b, both assumed to be larger than 1, and a basis a, assumed to be very large. We then define the parameters

$$\delta_q := a^{-b^q} \qquad \lambda_q := a^{cb^{q+1}}, \tag{3.3}$$

where q is an arbitrary natural number. b can in fact be chosen rather close to 1: how much it is allowed to be close to 1 depends on how close is α to 0. c will be larger but rather close to $5/2$, depending on how close are $b-1$ and α to 0. More precisely, we summarize the conditions which b and c need to satisfy in the following two inequalities

$$\frac{3}{2} > b > \frac{2}{(2-\alpha)(1-2\alpha)} \tag{3.4}$$

$$c > \frac{2(2-\alpha)b^2-(3-2\alpha)b-1}{b((2-\alpha)(1-2\alpha)b-2)} = \frac{((4-2\alpha)b+1)(b-1)}{b((2-5\alpha+2\alpha^2)b-2)}. \tag{3.5}$$

Observe that when $\alpha \downarrow 0$, the right hand sides approach 1 respectively $5/2$ from above.

It is convenient to introduce the notation

$$g_q := g - \delta_{q+1} e, \tag{3.6}$$

which simplifies several formulas.

Proposition 3.3. *Fix a metric g as in Theorem 3.1. There is a positive constant $\bar\alpha$ such that for every α as in (3.2) we can choose positive numbers $\sigma_0(\alpha) < 1$ and C_0 with the following property. Assume b and c satisfy (3.4) and (3.5), fix any $\bar C \geq C_0$ and assume that λ_q and δ_q are defined as in (3.3), where a is sufficiently large depending on $\alpha, b, c, g, \bar C$, namely*

$$a > a_0(\alpha, b, c, g, \bar C). \tag{3.7}$$

If $q \in \mathbb{N}$ and $u_q : \overline{D}_{1+2^{-q-1}} \to \mathbb{R}^3$ is an immersion such that

$$\|g_q - u_q^\sharp e\|_\alpha \leq \sigma_0 \delta_{q+1} \tag{3.8}$$

$$\|D^2 u_q\|_0 \leq \bar C \delta_q^{1/2} \lambda_q, \tag{3.9}$$

then there is an immersion $u_{q+1} : \overline{D}_{1+2^{-q-2}} \to \mathbb{R}^3$ such that

$$\|g_{q+1} - u_{q+1}^\sharp e\|_0 \leq \frac{\sigma_0}{3} \delta_{q+2} \lambda_{q+1}^{-\alpha} \tag{3.10}$$

$$\|D(g_{q+1} - u_{q+1}^\sharp e)\|_0 \leq \frac{\sigma_0}{3} \delta_{q+2} \lambda_{q+1}^{1-\alpha} \tag{3.11}$$

$$\|u_q - u_{q+1}\|_0 \leq \delta_{q+1}^{1/2} \lambda_{q+1}^{-\gamma} \tag{3.12}$$

$$\|D(u_q - u_{q+1})\|_0 \leq C_0 \delta_{q+1}^{1/2} \tag{3.13}$$

$$\|D^2 u_{q+1}\|_0 \leq \bar C \delta_{q+1}^{1/2} \lambda_{q+1}, \tag{3.14}$$

where $\gamma = \gamma(\alpha, b, c) > 0$.

As already mentioned, Proposition 3.3 will be used in an iteration scheme to show Theorem 3.1. The reader will notice that the conclusions (3.10)-(3.11) do not exactly match the starting assumption (3.8). On the other hand, a simple interpolation shows that (3.10) and (3.11) together imply the estimate

$$\|g_{q+1} - u^\sharp_{q+1}e\|_\alpha \leq \sigma_0\delta_{q+2}\,,$$

which corresponds to (3.8) at the next step of the iteration. In particular, the conclusions are stronger and so they still allow to iterate the proposition. It is possible to state a version of Proposition 3.3 where the assumptions and conclusions look more homogeneous, but there would be no real simplification neither in the statement nor in the proof.

Observe that, by our condition upon the parameters, u_q is obviously a strictly short map, because we have

$$u^\sharp_q e \leq g_q + \sigma_0\delta_{q+1}e = g - (1-\sigma_0)\delta_{q+1}e < g\,,$$

where all the inequalities are understood in the sense of quadratic forms. Thus, as a simple corollary we know that

$$\|Du_q\|_{C^0} \leq C \tag{3.15}$$

for some constant C which only depends upon g.

As in the classical Nash-Kuiper theorem, the map u_{q+1} is obtained from the map u_q by adding a certain number of perturbations, each consisting of highly oscillatory functions. As it is clear from the arguments in [17], the threshold Hölder exponent that can be reached by a Nash–Kuiper type iteration is $\frac{1}{1+2n_\star}$, where $n_\star$ is the number of such perturbations. Each perturbation adds, modulo small error terms, a smooth symmetric rank-1 tensor, called "primitive metric", to $u^\sharp_q e$. $n_\star$ is then the smallest number of summands needed to write the metric error $g - u^\sharp_q e$ as a (positive) linear combination of such "primitive metrics".
We know by the inductive assumption that $(g - u^\sharp_q e)/\|g - u^\sharp_q e\|_0$ is close to e, which implies that $n_\star$ can be chosen to be the dimension of the space of symmetric matrices. Thus, if n is the dimension of the manifold, $n_\star = \frac{n(n+1)}{2}$: this explains the threshold $\frac{1}{1+2n_\star} = \frac{1}{1+n+n^2}$ reached in [17] and claimed originally by Borisov. In particular in dimension 2 the number $n_\star$ equals 3 and Borisov's threshold is $\frac{1}{7}$.

The starting point of this chapter is the simple observation that in 2 dimensions we can use a conformal change of coordinates to diagonalize $g - u^\sharp e$ and hence reduce the number $n_\star$ from 3 to 2: this justifies the new threshold $\frac{1}{5}$. However, the regularity of the change of coordinates needed to implement this idea deteriorates with q and thus it is not at all clear that the method really improves the regularity of the final map. In fact at first it is not even clear that the new iteration scheme yields any $C^{1,\alpha}$ regularity at all.

In order to overcome this difficulty we obviously need to estimate quite carefully several norms of the conformal change of coordinates, at each step: for this reason we need to keep track of some Hölder norm of $g - u_q^\sharp e$. However, to ensure convergence of the scheme, it does not seem enough to just combine the computations of [17] with the classical estimates on conformal mappings. In particular in order to close the argument we impose a much faster rate of convergence for $g - u_q^\sharp e$: in [17] it was sufficient to choose exponentially decaying δ_q (and exponentially growing λ_q), whereas in this chapter we take advantage of a double exponential Ansatz. This idea is in fact borrowed from [25], where a scheme with a double exponential decay was used to produce Hölder solutions to the Euler equations.

The rest of this chapter is organized as follows.

Section 3.2 collects the technical preliminary lemmas and propositions which will be used in the proofs of Proposition 3.3 and Theorem 3.1.

The proof of Proposition 3.3 is split into the Sections 3.3, 3.4, 3.5 and 3.6. Section 3.3 describes how to reach u_{q+1} from u_q and in particular it gives the precise formulas for the two oscillatory perturbations which we need to add. We will then collect in Section 3.4 the estimates concerning the first perturbation and in Section 3.5 the ones concerning the second perturbation. Section 3.6 will finally conclude the proof of Proposition 3.3.

Section 3.7 will prove Theorem 3.1 using Proposition 3.3. In fact the proof is not completely straightforward since we have to show the existence of a map u_0 which is C^0 close to the map $\bar{u}$ of Theorem 3.1 and at the same time satisfies the requirements of Proposition 3.3 (with $q = 0$), in order to be able to start the iterative procedure. Finally, in Section 3.8 we give briefly the necessary technical modifications to prove Theorem 3.2.

3.2 PRELIMINARIES

3.2.1 *Conformal coordinates*

The following proposition is a key technical point in the proof. It addresses rather well-known regularity properties of conformal changes of coordinates. However, it is crucial for us to have an explicit (linear) dependence of certain Hölder norms of the change of coordinates in terms of corresponding norms of the metric. Since we have not been able to find a precise reference in the literature, we include a proof in the appendix (see Appendix A.1).

Proposition 3.4. *For any N, α, β with $N \in \mathbb{N}, N \geq 1$, $0 < \beta \leq \alpha < 1$ there exist constants $C(N, \alpha, \beta), \sigma_1(N, \alpha, \beta) > 0$ and $\bar{C}(\alpha)$ such that the following holds. If $1 \leq r \leq 2$ and g is a $C^{N,\alpha}$ metric on $\overline{D}_r$ with*

$$\|g - e\|_\alpha \leq \sigma_1 \tag{3.16}$$

then there exists a $C^{N+1,\beta}$ *coordinate change* $\Phi : \overline{D}_r \to \mathbb{R}^2$ *and a* $C^{N,\beta}$ *function* $\rho : \overline{D}_r \to \mathbb{R}^+$ *satisfying*

$$g = \rho^2 \left(\nabla\Phi_1 \otimes \nabla\Phi_1 + \nabla\Phi_2 \otimes \nabla\Phi_2\right) \tag{3.17}$$

and the following estimates:

$$\|\rho - 1\|_\alpha + \|D\Phi - \mathrm{Id}\|_\alpha \leq \bar{C}\|g - e\|_\alpha \tag{3.18}$$

$$\|D^k\rho\|_\beta + \|D^{k+1}\Phi\|_\beta \leq C\|g - e\|_{k+\beta} \qquad \forall 1 \leq k \leq N\,. \tag{3.19}$$

3.2.2 *Oscillatory functions*

The construction of u_{q+1} is based on adding to the map u_q suitable "wrinkles", namely suitable perturbations. The basic model for this perturbation takes advantage of a pair of real-valued functions with very specific properties, which we will detail here.

Proposition 3.5. *There exists* $\delta_\star > 0$ *and a function* $\Gamma = (\Gamma^t, \Gamma^n) \in C^\infty([0, \delta_\star] \times \mathbb{R}, \mathbb{R}^2)$ *with the following properties*

(a) $\Gamma(s, \xi) = \Gamma(s, \xi + 2\pi)$ *for every* s, ξ*;*

(b) $(1 + \partial_\xi \Gamma^t)^2 + (\partial_\xi \Gamma^n)^2 = 1 + s^2$*;*

(c) The following estimates hold:

$$\|\partial_\xi^k \Gamma^n(s, \cdot)\|_0 \leq C(k)s \tag{3.20}$$

$$\|\partial_\xi^k \Gamma^t(s, \cdot)\|_0 \leq C(k)s^2 \tag{3.21}$$

$$\|\partial_s \partial_\xi^k \Gamma^t(s, \cdot)\|_0 \leq C(k)s\,. \tag{3.22}$$

Proof. Except for (3.21) the remaining claims are contained in [17, Lemma 2]. The idea is to let Γ have the form

$$\Gamma(s, \xi) := \int_0^\xi \left(\sqrt{1+s^2}\,(\cos(f(s)\sin(\tau)), \sin(f(s)\sin(\tau))) - (1, 0)\right) d\tau\,,$$

for an appropriately chosen function f such that (a), (3.20) and (3.22) are fulfilled. (b) is satisfied by construction. The additional statement (3.21) follows from integrating (3.22) in s. □

3.3 PROOF OF PROPOSITION 3.3, PART I

3.3.1 *Hierarchy of parameters*

A first ingredient in the construction of u_{q+1} is to smooth u_q suitably via a standard mollification. For this we introduce the mollification parameter ℓ, which is rather small: indeed it is defined by the relation

$$\ell^{2-\alpha} := \frac{1}{\tilde{C}} \frac{\delta_{q+1}}{\delta_q \lambda_q^2}, \tag{3.23}$$

where $\tilde{C}$ is a constant larger than 1 which depends only upon α, g, σ_0 and $\bar{C}$ and which will be specified in Section 3.3.3 below.

The map u_{q+1} will be obtained from (a suitable regularization of) the map u_q in two steps. First we will add an oscillatory perturbation whose frequency is

$$\mu := \hat{C} \frac{\delta_{q+1} \lambda_{q+1}^{\alpha}}{\delta_{q+2} \ell}, \tag{3.24}$$

where the constant $\hat{C}$, larger than 1, depends only upon α, g, and σ_0 (we specify its choice in Section 3.6). We will then choose a second perturbation whose frequency is λ_{q+1}.

We next record a few inequalities among the parameters which will be rather useful in simplifying some of our estimates in the remaining sections. Except for the very first inequality in (3.26), which requires a choice of a sufficiently large compared to the constant $\hat{C}$, all the others are immediate from the restrictions imposed so far on all the various parameters.

$$\delta_q \lambda_q^2 \geq 1, \tag{3.25}$$

$$\lambda_{q+1} \geq \mu \geq \ell^{-1} \geq \lambda_q, \tag{3.26}$$

$$\delta_q^{1/2} \lambda_q \leq \delta_q^{1/2} \lambda_q \ell^{-\alpha/2} \leq \delta_{q+1}^{1/2} \ell^{-1} \leq \delta_{q+1}^{1/2} \mu \leq \delta_{q+1}^{1/2} \lambda_{q+1}, \tag{3.27}$$

The first inequality (3.25) follows from $\delta_q \lambda_q^2 = a^{c^2 b^{2q+2} - b^q} \geq a^{b^2-1}$ (where we have used $c, b > 1$). Observe that this easily implies $\ell \leq 1$ (recall that δ_{q+2} and $\tilde{C}^{-1}$ are both smaller than 1), which in turn gives the first inequality in (3.27). Note also that the last inequality in (3.26) is weaker than the second inequality in (3.27):

$$\ell^{-1} \geq \ell^{-1+\alpha/2} \geq \frac{\delta_q^{1/2}}{\delta_{q+1}^{1/2}} \lambda_q \geq \lambda_q .$$

Coming to the second inequality in (3.27), observe that, by the definition of ℓ, this is just the requirement that $\tilde{C} \geq 1$. As for the last two inequalities in (3.27), they are equivalent to the first two in (3.26), which will be shown below. Moreover, since $\hat{C} > 1$, $\lambda_{q+1} > 1$ and $\delta_{q+1} \geq \delta_{q+2}$, the second inequality in (3.26) is obvious.

We are therefore left with showing the first inequality in (3.26) which, as already mentioned, needs a sufficiently large a. As it can be readily checked from the definition of μ, such inequality is in fact equivalent to $\delta_{q+2}\lambda_{q+1}^{1-\alpha} \geq \hat{C}\delta_{q+1}\ell^{-1}$. But we record in fact a much stronger inequality, which turns out to be the key relation to conclude the estimates in Proposition 3.3, as it will become apparent in Section 3.6. More precisely, given any constant $\underline{C}$ which depends upon α, g, σ_0 and $\bar{C}$, the following inequality holds provided a is chosen large enough:

$$\delta_{q+2}^2\lambda_{q+1}^{1-2\alpha} \geq \underline{C}\delta_{q+1}^2\ell^{-1}\,. \tag{3.28}$$

In fact such inequality is equivalent to

$$\delta_{q+2}^2\lambda_{q+1}^{1-2\alpha} \geq \underline{C}\tilde{C}^{1/(2-\alpha)}\delta_{q+1}^{2-1/(2-\alpha)}\delta_q^{1/(2-\alpha)}\lambda_q^{2/(2-\alpha)}\,.$$

Taking the logarithm in base a this is equivalent to

$$(c(1-2\alpha)-2)b^{q+2} \geq \left(\frac{1+2c}{2-\alpha}-2\right)b^{q+1} - \frac{1}{2-\alpha}b^q + \log_a \underline{C} + \frac{1}{2-\alpha}\log_a \tilde{C}\,.$$

The latter follows for a sufficiently large a (depending upon b, c, $\tilde{C}$ and $\underline{C}$) provided

$$\big(c(1-2\alpha)-2\big)b^2 > \left(\frac{1+2c}{2-\alpha}-2\right)b - \frac{1}{2-\alpha}\,,$$

which is equivalent to

$$cb((2-\alpha)(1-2\alpha)b-2) > 2(2-\alpha)b^2 + (1-2(2-\alpha))b - 1\,.$$

The latter inequality is however obviously implied by (3.4) and (3.5).

3.3.2 *Constants*

In the rest of the chapter we will deal with several estimates where we bound norms of various functions using the parameters introduced so far, namely $\delta_q, \lambda_q, \ell, \mu$ and λ_{q+1}. In front of the expressions involving such parameters there will always be some constants, independent of a, b and c. However it is important to distinguish between two types of such constants: the ones which depend only upon α, g and σ_0 will be denoted by C, whereas the ones which depend also upon the $\bar{C}$ of Proposition 3.3 will be denoted by $C^\star$. Note also that the parameter σ_0 will in fact be chosen as a function of α in Section 3.3.4. Therefore the constants denoted by C will depend only upon α and g, whereas those denoted by $C^\star$ will depend, additionally, also upon $\bar{C}$. Moreover, the values of C and $C^\star$ may change from line to line.

3.3.3 *Regularization*

Having fixed a standard mollifier φ, we then define

$$h_q := \frac{g * \varphi_\ell - (u_q * \varphi_\ell)^\sharp e}{\delta_{q+1}} - \frac{\delta_{q+2}}{\delta_{q+1}} e\,. \tag{3.29}$$

Observe that

$$(u_q * \varphi_\ell)^\sharp e + \delta_{q+1} h_q = g * \varphi_\ell - \delta_{q+2} e = g_{q+1} + (g * \varphi_\ell - g)\,.$$

So the strategy of the proof will be to perturb $u_q * \varphi_\ell$ to a map u_{q+1} such that

$$u_{q+1}^\sharp e = (u_q * \varphi_\ell)^\sharp e + \delta_{q+1} h_q + E = g_{q+1} + E + (g * \varphi_\ell - g)\,,$$

(cf. (3.48)) where the error term E is suitably small. Before coming to the construction of the map u_{q+1} we deal in this section with the smallness conditions to be imposed on ℓ.

First of all, by choosing $\tilde{C}$ larger than a geometric constant and a sufficiently large (depending upon b and c), we can assume that $\ell \leq 2^{-q-2}$, so that h_q is in fact defined on $\overline{D}_{1+2^{-q-2}}$. Next, using Lemma 2.3 we can estimate

$$\begin{aligned}
\|h_q - e\|_\alpha \leq\ & \frac{\delta_{q+2}}{\delta_{q+1}} + \frac{1}{\delta_{q+1}} \|g * \varphi_\ell - (u_q * \varphi_\ell)^\sharp e - \delta_{q+1} e\|_\alpha \\
\leq\ & a^{-(b-1)} + \frac{1}{\delta_{q+1}} \big(\|(u_q^\sharp e) * \varphi_\ell - (u_q * \varphi_\ell)^\sharp e\|_\alpha + \|(g_q - u_q^\sharp e) * \varphi_\ell\|_\alpha \\
& \qquad\qquad + \|g - g * \varphi_\ell\|_\alpha \big) \\
\leq\ & \sigma_0 + C^\star \frac{\ell^{2-\alpha} \delta_q \lambda_q^2}{\delta_{q+1}} + \sigma_0 + \frac{C}{\delta_{q+1}} \|D^2 g\|_0 \ell^{2-\alpha} \\
\overset{(3.25)}{\leq}\ & 2\sigma_0 + C^\star \frac{\ell^{2-\alpha} \delta_q \lambda_q^2}{\delta_{q+1}} \leq 3\sigma_0\,,
\end{aligned}$$

where the latter inequality specifies the condition needed on $\tilde{C}$ in (3.23).

Similarly, for $1 \leq k \leq 4$, we can bound

$$\begin{aligned}
\|D^k h_q\|_0 \leq & \frac{1}{\delta_{q+1}} \Big(\|D^k (g - u_q^\sharp e) * \varphi_\ell\|_0 \\
& \qquad + \|D^k ((u^\sharp e) * \varphi_\ell - (u_q * \varphi_\ell)^\sharp e)\|_0 \Big) \\
& \leq C \ell^{\alpha-k} \sigma_0 + C^\star \frac{\delta_q \lambda_q^2}{\delta_{q+1}} \ell^{2-k} \leq C \ell^{\alpha-k}\,,
\end{aligned} \tag{3.30}$$

where we have used (3.23) and Lemma 2.3. Interpolating, for any $0 \le k \le 3$ we then get

$$\|h_q - e\|_{k+\alpha} \le C\ell^{-k}\,. \tag{3.31}$$

We summarize the conclusions of the previous paragraphs in the following lemma.

Lemma 3.6. *If we choose $\tilde{C}$ sufficiently large, depending upon α, g and $\bar{C}$, we then have*

$$\|h_q - e\|_\alpha \le 3\sigma_0 \tag{3.32}$$
$$\|h_q - e\|_{k+\alpha} \le C\ell^{-k} \qquad \text{for } 1 \le k \le 3, \tag{3.33}$$

where the constant C depends only upon α and g.

3.3.4 *Conformal diffeomorphism*

We now wish to apply Proposition 3.4 with $\beta = \alpha > 0$ and $N = 3$. This requires to choose σ_0 such that $3\sigma_0 \le \sigma_1$, where σ_1 is the constant appearing in (3.16). We thus find maps Φ and ρ such that

$$h_q = \rho^2 \left(\nabla\Phi_1 \otimes \nabla\Phi_1 + \nabla\Phi_2 \otimes \nabla\Phi_2\right)\,.$$

Furthermore, if $\sigma_0(\alpha)$ is small enough we can assume in addition

$$\frac{1}{2} \le \rho \le 2 \qquad \|D\Phi - \mathrm{Id}\|_0 \le \frac{1}{2}\,, \tag{3.34}$$

thanks to (3.18) and the estimate (3.32). This exhausts the condition on σ_0: note that it depends only upon α, since N and β in Proposition 3.4 are fixed to be 3 and α.

Moreover, for any $1 \le k \le 3$ we apply (3.19) and (3.33) to estimate

$$\|D^k\rho\|_\alpha + \|D^{k+1}\Phi\|_\alpha \le C\ell^{-k}\,. \tag{3.35}$$

3.3.5 *Adding the first primitive metric*

We next set $w := u_q * \varphi_\ell$ and we define the following two three-dimensional vectors:

$$\tau_1 := Dw(Dw^T Dw)^{-1}\nabla\Phi_1 \tag{3.36}$$

and

$$\nu_1 := \frac{\partial_{x_1} w \times \partial_{x_2} w}{|\partial_{x_1} w \times \partial_{x_2} w|}\,. \tag{3.37}$$

Observe that ν_1 is in the kernel of Dw^T (or, in other words, $\nu_1(x)$ is a unit normal to the tangent plane $T_{w(x)}(\mathrm{I}m\,(w))$). Hence it follows easily that τ_1 and ν_1 are orthogonal.

We next normalize these vectors suitably, defining

$$t_1 := \frac{\tau_1}{|\tau_1|^2}\,, \tag{3.38}$$

$$n_1 := \frac{\nu_1}{|\tau_1|}\,. \tag{3.39}$$

Finally, we define the first perturbation of w, namely the map v given by the formula

$$v = w + \frac{1}{\mu}\Gamma^t\left(\delta_{q+1}^{1/2}|\tau_1|\rho, \mu\Phi_1\right) t_1 + \frac{1}{\mu}\Gamma^n\left(\delta_{q+1}^{1/2}|\tau_1|\rho, \mu\Phi_1\right) n_1\,, \tag{3.40}$$

whereas we define

$$E_1 := v^\sharp e - (w^\sharp e + \delta_{q+1}\rho^2\nabla\Phi_1 \otimes \nabla\Phi_1)\,. \tag{3.41}$$

3.3.6 *Adding the second primitive metric*

The map u_{q+1} is then obtained by adding a similar second perturbation to the map v. More precisely we define this time

$$\tau_2 := Dv(Dv^T Dv)^{-1}\nabla\Phi_2\,, \tag{3.42}$$

$$\nu_2 := \frac{\partial_{x_1}v \times \partial_{x_2}v}{|\partial_{x_1}v \times \partial_{x_2}v|}\,, \tag{3.43}$$

$$t_2 := \frac{\tau_2}{|\tau_2|^2}\,, \tag{3.44}$$

$$n_2 := \frac{\nu_2}{|\tau_2|}\,. \tag{3.45}$$

The map u_{q+1} is then given by the following formula (analogous to (3.40)):

$$u_{q+1} = v + \frac{1}{\lambda_{q+1}}\Gamma^t\left(\delta_{q+1}^{1/2}|\tau_2|\rho, \lambda_{q+1}\Phi_2\right) t_2 + \frac{1}{\lambda_{q+1}}\Gamma^n\left(\delta_{q+1}^{1/2}|\tau_2|\rho, \lambda_{q+1}\Phi_2\right) n_2\,. \tag{3.46}$$

Similarly we define

$$E_2 := u_{q+1}^\sharp e - (v^\sharp e + \delta_{q+1}\rho^2\nabla\Phi_2 \otimes \nabla\Phi_2)\,. \tag{3.47}$$

Observe that we have the following identity:

$$\begin{aligned} E :=& E_1 + E_2 = u_{q+1}^\sharp e - (w^\sharp e + \delta_{q+1}\rho^2(\nabla\Phi_1 \otimes \nabla\Phi_1 + \nabla\Phi_2 \otimes \nabla\Phi_2)) \\ &= u_{q+1}^\sharp e - w^\sharp e - \delta_{q+1}h_q = u_{q+1}^\sharp e + \delta_{q+2}e - g * \varphi_\ell \\ &= u_{q+1}^\sharp e - g_{q+1} + (g - g * \varphi_\ell)\,. \end{aligned} \tag{3.48}$$

Hence

$$\|g_{q+1} - u^\sharp_{q+1} e\|_0 \leq \|E\|_0 + \|g - g * \varphi_\ell\|_0 , \tag{3.49}$$

$$\|D(g_{q+1} - u^\sharp_{q+1} e)\|_0 \leq \|DE\|_0 + \|D(g - g * \varphi_\ell)\|_0 . \tag{3.50}$$

For α sufficiently small and a sufficiently big one can achieve

$$\|g - g * \varphi_\ell\|_0 \leq C\|D^2 g\|_0 \ell^2 \leq \frac{\sigma_0}{6}\delta_{q+2}\lambda_{q+1}^{-\alpha} , \tag{3.51}$$

$$\|D(g - g * \varphi_\ell)\|_0 \leq C\|D^2 g\|_0 \ell \leq \frac{\sigma_0}{6}\delta_{q+2}\lambda_{q+1}^{1-\alpha} . \tag{3.52}$$

To see this, note that (3.51) is implied by the condition

$$C^\star \frac{\delta_{q+1}}{\delta_q \lambda_q^2} \leq \delta_{q+2}\lambda_{q+1}^{-\alpha} ,$$

which for $a(\bar{C})$ big enough is guaranteed if

$$b^2 - b + 1 < (2 - \alpha b)cb ,$$

or equivalently

$$c > \frac{b^2 - b + 1}{b(2 - \alpha b)} . \tag{3.53}$$

Similarly (3.52) follows if

$$C^\star \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q} \leq \delta_{q+2}\lambda_{q+1}^{1-\alpha} ,$$

which (for $a(\bar{C})$ big enough) is satisfied whenever

$$c > \frac{2b^2 - b + 1}{2b(1 + (1 - \alpha)b)} . \tag{3.54}$$

Now for any $\alpha > 0$, $b > 1$ which satisfy the bounds of Proposition 3.3 we have

$$\frac{b^2 - b + 1}{b(2 - \alpha b)} > \frac{2b^2 - b + 1}{2b(1 + (1 - \alpha)b)} .$$

Indeed, since $b < \frac{3}{2}$ and $\alpha < \bar{\alpha}$, provided $\bar{\alpha}$ is small enough both denominators in the fractions above are positive. Hence the inequality is equivalent to

$$2b^2 + (\alpha - 4)b + (2 - \alpha) = (b - 1)(\alpha + 2b - 2) > 0 ,$$

which for $b > 1$ and $\alpha > 0$ is always true. Hence (3.53) implies (3.54).

Next, observe that the left hand side of (3.5) is larger than $g_\alpha(b) = \frac{(4-2\alpha)b+1}{2b}$, so (3.5) implies $c > g_\alpha(b)$. The bound (3.53) is instead $c > h_\alpha(b) = \frac{b^2-b+1}{b(2-\alpha b)}$. On the other hand on the interval $[1, \frac{3}{2}]$, g_α and h_α converge uniformly, as $\alpha \downarrow 0$, to the functions $g_0(b) = 2 + \frac{1}{2b}$ and $h_0(b) = \frac{b^2-b+1}{2b}$. Since on $[1, \frac{3}{2}]$ g_0 is strictly larger than h_0, we infer that for α small (3.5) guarantees (3.53). In particular we conclude that for a big enough (3.5) guarantees (3.51) and (3.52).
Thus, the goal of most of the remaining sections is to prove that the desired bounds hold for $\|E\|_0$, $\|DE\|_0$, $\|u_{q+1} - u_q\|_0$, $\|D(u_{q+1} - u_q)\|_0$ and $\|D^2 u_{q+1}\|_0$.

3.4 ESTIMATES ON v AND E_1

Our goal in this subsection is to estimate the C^0 norms of $v - u_q$, $D^k v$, E_1 and DE_1. To this aim we introduce the functions

$$A_1^t := \partial_\xi \Gamma^t \left(\delta_{q+1}^{1/2} |\tau_1| \rho, \mu\Phi_1\right) , \tag{3.55}$$

$$A_1^n := \partial_\xi \Gamma^n \left(\delta_{q+1}^{1/2} |\tau_1| \rho, \mu\Phi_1\right) , \tag{3.56}$$

$$B_1^t := \partial_s \Gamma^t \left(\delta_{q+1}^{1/2} |\tau_1| \rho, \mu\Phi_1\right) , \tag{3.57}$$

$$B_1^n := \partial_s \Gamma^n \left(\delta_{q+1}^{1/2} |\tau_1| \rho, \mu\Phi_1\right) , \tag{3.58}$$

$$C_1^t := \Gamma^t \left(\delta_{q+1}^{1/2} |\tau_1| \rho, \mu\Phi_1\right) , \tag{3.59}$$

$$C_1^n := \Gamma^n \left(\delta_{q+1}^{1/2} |\tau_1| \rho, \mu\Phi_1\right) , \tag{3.60}$$

and we decompose the derivative of v as

$$\begin{aligned} Dv = Dw + &\underbrace{A_1^t\, t_1 \otimes \nabla\Phi_1 + A_1^n\, n_1 \otimes \nabla\Phi_1}_{=:\mathbf{A}_1} \\ &+ \underbrace{\frac{\delta_{q+1}^{1/2}}{\mu} (B_1^t\, t_1 + B_1^n\, n_1) \otimes (\rho\nabla|\tau_1| + |\tau_1|\nabla\rho)}_{=:\mathbf{B}_1} + \underbrace{\frac{1}{\mu} \left(C_1^t\, Dt_1 + C_1^n Dn_1\right)}_{=:\mathbf{C}_1} . \end{aligned} \tag{3.61}$$

3.4.1 *First technical lemma*

In the next lemma we collect the estimates of the C^0 norm of the derivatives of the various quantities introduced above.

Lemma 3.7. *Let $\tilde{C}$ be fixed so that Lemma 3.6 holds and $\hat{C} \geq 1$. If $a \geq a_0(\alpha, g, b, c, \bar{C})$ for some a_0 sufficiently large, then there are constants C (depending upon α and g but not on $\bar{C}$) such that*

$$C^{-1} \leq |\tau_1| \leq C \tag{3.62}$$

and

$$\|w - u_q\|_0 \leq C\delta_{q+1}^{1/2}\ell\,, \tag{3.63}$$

$$\|D(w - u_q)\|_0 \leq C\delta_{q+1}^{1/2}\,, \tag{3.64}$$

$$\|Dw\|_0 \leq C\,, \tag{3.65}$$

$$\|D^k w\|_0 \leq C\delta_{q+1}^{1/2}\ell^{1-k} \qquad \text{for } 2 \leq k \leq 4, \tag{3.66}$$

$$\|D^k \nu_1\|_0 \leq C\delta_{q+1}^{1/2}\ell^{-k} \qquad \text{for } 1 \leq k \leq 3, \tag{3.67}$$

$$\|D^k t_1\|_0 + \|D^k \tau_1\|_0 + \|D^k n_1\|_0 \leq C\ell^{-k} \qquad \text{for } 0 \leq k \leq 3, \tag{3.68}$$

$$\|D^k A_1^t\|_0 + \|D^k C_1^t\|_0 \leq C\delta_{q+1}\mu^k \qquad \text{for } 0 \leq k \leq 3, \tag{3.69}$$

$$\|D^k A_1^n\|_0 + \|D^k B_1^t\|_0 + \|D^k C_1^n\|_0 \leq C\delta_{q+1}^{1/2}\mu^k \qquad \text{for } 0 \leq k \leq 3, \tag{3.70}$$

$$\|D^k B_1^n\|_0 \leq C\mu^k \qquad \text{for } 0 \leq k \leq 3. \tag{3.71}$$

Proof. **Estimates on $|\tau_1|$.** Since $\|D\Phi - \mathrm{Id}\|_0 \leq \frac{1}{2}$, we obviously have $\frac{1}{2} \leq |\nabla\Phi_1| \leq 2$. On the other hand the estimate (3.32) on h_q of the previous section implies

$$g + 5\delta_{q+1}e \geq w^\sharp e \geq g - 5\delta_{q+1}e\,.$$

If we assume a sufficiently large (depending only upon g, b and c), we conclude $2g \geq w^\sharp e \geq \frac{1}{2}g$. Since $w^\sharp e = Dw^T Dw$, this implies that

$$C|\nabla\Phi_1| \geq |\tau_1| \geq C^{-1}|\nabla\Phi_1|$$

for a constant C which depends only upon g, hence (3.62) follows.

Estimates on w. Observe that

$$\|w - u_q\|_0 \leq C\ell^2\|D^2 u_q\|_0 \leq C^\star \ell^2 \delta_q^{1/2}\lambda_q \tag{3.72}$$

$$\|D(w - u_q)\|_0 \leq C\ell\|D^2 u_q\|_0 \leq C^\star \ell\delta_q^{1/2}\lambda_q. \tag{3.73}$$

If we choose $a \geq a_0(\alpha, b, c, \bar{C})$ big enough such that $\bar{C} \leq \ell^{-\alpha/2}$, then (3.63) and (3.64) follow with the help of (3.27). Moreover, (3.64) implies (3.65) by (3.15). Finally, (3.66) is a consequence of (2.9), i.e. $\|D^k w\|_0 \leq C\ell^{2-k}\|D^2 u_q\|_0$, and $\bar{C} \leq \ell^{-\alpha/2}$.
Next, observe that $C \geq |\partial_{x_1} w \times \partial_{x_2} w| \geq C^{-1}$ (again due to $2g \geq Dw^T Dw \geq \frac{1}{2}g$). Hence (2.7) implies, for $k \geq 1$,

$$\|D^k \nu_1\|_0 \leq C[Dw]_k\|Dw\|_0 \leq C[Dw]_k \leq C\delta_{q+1}^{1/2}\ell^{-k}\,.$$

Estimates on τ_1, t_1 and n_1. The C^0 estimates in (3.68) are a trivial consequence of (3.62). Again by Proposition 2.1 we get

$$\begin{aligned}\|D^k \tau_1\|_0 \leq& C\|Dw\|_0\|D^{k+1}\Phi\|_0 + C\|D\Phi\|_0\big(\|D^{k+1}w\|_0 + \|D^2 w\|_0^k\big)\\ \leq& C\ell^{-k} + C\delta_{q+1}^{1/2}\ell^{-k} \leq C\ell^{-k}\,.\end{aligned}$$

A second application of Proposition 2.1 (combined with (3.62)) gives the estimates

$$\|D^k|\tau_1|\|_0 + \|D^k|\tau_1|^{-1}\|_0 \le C\ell^{-k}\,. \tag{3.74}$$

Combining (3.74) and (3.67), from (2.5) we infer

$$\|D^k n_1\|_0 \le C\delta_{q+1}^{1/2}\ell^{-k} + C\ell^{-k} \le C\ell^{-k}\,.$$

We argue similarly to conclude $\|D^k t_1\|_0 \le C\ell^{-k}$.

Remaining estimates. The cases $k = 0$ of (3.69), (3.70) and (3.71) are all simple consequences of Proposition 3.5 and $\||\tau_1||\rho|\|_0 \le C$. For the higher derivatives we consider first C_1^t. We introduce the function

$$\Psi(s,\xi) := \delta_{q+1}^{-1}\Gamma^t(\delta_{q+1}^{1/2}s,\xi)$$

and observe that $\|D^i\Psi\|_0 \le C(i)$ by the estimates in Proposition 3.5(c). If we introduce the map $U = (|\tau_1|\rho, \mu\Phi_1)$ we can then write

$$\|D^k C_1^t\|_0 = \delta_{q+1}\|D^k(\Psi \circ U)\|_0\,.$$

On the other hand observe that

$$\|D^k U\|_0 \le C\ell^{-k} + C\mu\ell^{1-k} \overset{(3.26)}{\le} C\mu\ell^{1-k}\,.$$

Hence, using (2.8) we infer

$$\|D^k C_1^t\|_0 \le C\delta_{q+1}(\mu\ell^{1-k} + \mu^k) \le C\delta_{q+1}\mu^k\,.$$

In case of A_1^t, A_n^1, B_1^t, C_1^n and B_1^n we apply the same argument, keeping the map U as defined above, but changing Ψ respectively to

$$\begin{aligned}
\Psi(s,\xi) :=& \delta_{q+1}^{-1}\partial_\xi\Gamma^t(\delta_{q+1}^{1/2}s,\xi)\\
\Psi(s,\xi) :=& \delta_{q+1}^{-1/2}\partial_\xi\Gamma^n(\delta_{q+1}^{1/2}s,\xi)\\
\Psi(s,\xi) :=& \delta_{q+1}^{-1/2}\partial_s\Gamma^t(\delta_{q+1}^{1/2}s,\xi)\\
\Psi(s,\xi) :=& \delta_{q+1}^{-1/2}\Gamma^n(\delta_{q+1}^{1/2}s,\xi)\\
\Psi(s,\xi) :=& \partial_s\Gamma^n(\delta_{q+1}^{1/2}s,\xi)\,.
\end{aligned}$$

□

3.4.2 *Estimates on $\|v - u_q\|_0$, $\|D(v-u_q)\|_0$ and $\|D^k v\|_0$.*

Taking into account Proposition 3.5 we obviously have

$$\|v - w\|_0 \le C\delta_{q+1}^{1/2}\mu^{-1}\,,$$

whereas by (3.63)

$$\|u_q - w\|_0 \le C\delta_{q+1}^{1/2}\ell \le C\delta_{q+1}^{1/2}\ell^{1-\alpha/2} \le C\frac{\delta_{q+1}}{\delta_q^{1/2}\lambda_q}\,.$$

We therefore conclude

$$\|u_q - v\|_0 \le C\delta_{q+1}^{1/2}\mu^{-1} + C\frac{\delta_{q+1}}{\delta_q^{1/2}\lambda_q}\,. \tag{3.75}$$

By Lemma 3.7 we easily see that

$$\|D(u_q - v)\|_0 \le C\delta_{q+1}^{1/2} \tag{3.76}$$

and

$$\|D^k v\|_0 \le C\delta_{q+1}^{1/2}\mu^{k-1} \qquad \text{for } k \in \{2,3\}\,. \tag{3.77}$$

Observe also that, by (3.15),

$$\|Dv\|_0 \le C\,. \tag{3.78}$$

3.4.3 *Estimates on $\|E_1\|_0$ and $\|DE_1\|_0$.*

Observe first that due to Proposition 3.5 (b) we have

$$(Dw + \mathbf{A}_1)^T(Dw + \mathbf{A}_1) = w^\sharp e + \delta_{q+1}\rho^2\nabla\Phi_1 \otimes \nabla\Phi_1\,,$$

where we recall that $\mathbf{A}$ (and also $\mathbf{B}$ and $\mathbf{C}$) are defined in (3.61). Using the notation $\operatorname{sym} P$ for the matrix $\frac{1}{2}(P + P^T)$ we can then write

$$E_1 = 2\operatorname{sym}\left(Dw^T(\mathbf{B}_1 + \mathbf{C}_1)\right) + 2\operatorname{sym}\left(\mathbf{A}_1^T(\mathbf{B}_1 + \mathbf{C}_1)\right) + (\mathbf{B}_1 + \mathbf{C}_1)^T(\mathbf{B}_1 + \mathbf{C}_1)\,.$$

We notice that, from Lemma 3.7 and the estimates (3.34) and (3.35) on ρ and Φ, we conclude

$$\|\mathbf{A}_1\|_0 + \mu^{-1}\|D\mathbf{A}_1\|_0 \le C\delta_{q+1}^{1/2}\,, \tag{3.79}$$

$$\|\mathbf{B}_1\|_0 + \|\mathbf{C}_1\|_0 + \mu^{-1}\big(\|D\mathbf{B}_1\|_0 + \|D\mathbf{C}_1\|_0\big) \le C\frac{\delta_{q+1}^{1/2}}{\ell\mu}\,. \tag{3.80}$$

It is therefore obvious that, since $\ell\mu \ge 1$,

$$\|E_1\|_0 \le \|Dw^T\mathbf{B}_1\| + \|Dw^T\mathbf{C}_1\|_0 + C\frac{\delta_{q+1}}{\ell\mu}\,, \tag{3.81}$$

$$\|DE_1\|_0 \le \|D(Dw^T\mathbf{B}_1)\|_0 + \|D(Dw^T\mathbf{C}_1)\|_0 + C\delta_{q+1}\ell^{-1}\,. \tag{3.82}$$

We next compute

$$Dw^T \mathbf{B}_1 = \frac{\delta_{q+1}^{1/2}}{\mu} B_1^t (Dw^T\, t_1) \otimes (\rho \nabla |\tau_1| + |\tau_1| \nabla \rho)\,.$$

Therefore we conclude from Lemma 3.7 that

$$\|Dw^T \mathbf{B}_1\|_0 \leq C \frac{\delta_{q+1}}{\ell\mu}\,, \tag{3.83}$$

$$\|D(Dw^T \mathbf{B}_1)\|_0 \leq C \delta_{q+1} \ell^{-1}\,. \tag{3.84}$$

Recalling moreover (3.39) we have

$$Dn_1 = \frac{D\nu_1}{|\tau_1|} - n_1 \otimes \frac{\nabla |\tau_1|}{|\tau_1|}$$

and we also conclude that

$$Dw^T \mathbf{C}_1 = \frac{C_1^t}{\mu} Dw^T D t_1 + \frac{C_1^n}{\mu} Dw^T \frac{D\nu_1}{|\tau_1|}\,.$$

In particular

$$\|Dw^T \mathbf{C}_1\|_0 \leq \frac{C\delta_{q+1}}{\mu\ell} + C \frac{\delta_{q+1}^{1/2}}{\mu} \delta_{q+1}^{1/2} \ell^{-1} \leq C \frac{\delta_{q+1}}{\mu\ell}\,.$$

Similarly we conclude

$$\|D(Dw^T \mathbf{C}_1)\|_0 \leq C \delta_{q+1} \ell^{-1}\,.$$

Thus we infer

$$\|E_1\|_0 \leq C \frac{\delta_{q+1}}{\ell\mu}\,, \tag{3.85}$$

$$\|DE_1\|_0 \leq C \delta_{q+1} \ell^{-1}\,. \tag{3.86}$$

3.5 ESTIMATES ON u_{q+1} AND E_2

Our goal in this section is to estimate the C^0 norms of $u_{q+1} - v$, Du_{q+1}, D^2u_{q+1}, E_2 and DE_2. We proceed in the same way as in the previous section and begin by defining the functions

$$A_2^t := \partial_\xi \Gamma^t \left(\delta_{q+1}^{1/2} |\tau_2| \rho, \lambda_{q+1} \Phi_2 \right) , \tag{3.87}$$

$$A_2^n := \partial_\xi \Gamma^n \left(\delta_{q+1}^{1/2} |\tau_2| \rho, \lambda_{q+1} \Phi_2 \right) , \tag{3.88}$$

$$B_2^t := \partial_s \Gamma^t \left(\delta_{q+1}^{1/2} |\tau_2| \rho, \lambda_{q+1} \Phi_2 \right) , \tag{3.89}$$

$$B_2^n := \partial_s \Gamma^n \left(\delta_{q+1}^{1/2} |\tau_2| \rho, \lambda_{q+1} \Phi_2 \right) , \tag{3.90}$$

$$C_2^t := \Gamma^t \left(\delta_{q+1}^{1/2} |\tau_2| \rho, \lambda_{q+1} \Phi_2 \right) , \tag{3.91}$$

$$C_2^n := \Gamma^n \left(\delta_{q+1}^{1/2} |\tau_2| \rho, \lambda_{q+1} \Phi_2 \right) \tag{3.92}$$

and decomposing the derivative of u_{q+1} as

$$\begin{aligned} Du_{q+1} &= Dv + \underbrace{A_2^t \, t_2 \otimes \nabla \Phi_2 + A_2^n \, n_2 \otimes \nabla \Phi_2}_{=:\mathbf{A}_2} \\ &+ \underbrace{\frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}} (B_2^t \, t_2 + B_2^n \, n_2) \otimes (\rho \nabla |\tau_2| + |\tau_2| \nabla \rho)}_{=:\mathbf{B}_2} + \underbrace{\frac{1}{\lambda_{q+1}} \left(C_2^t \, Dt_2 + C_2^n D n_2 \right)}_{=:\mathbf{C}_2} . \end{aligned} \tag{3.93}$$

3.5.1 *Second technical lemma*

As before we collect the estimates of the C^0 norm of the derivatives of the various quantities introduced above.

Lemma 3.8. *Assume $\tilde{C}$ is fixed so that Lemma 3.6 holds and $\hat{C} > 1$. If $a \geq a_0(\alpha, g, b, c, \bar{C}, \hat{C})$ for a sufficiently large a_0, then there are constants C (depending on α and g but not on $\bar{C}$) such that*

$$C^{-1} \leq |\tau_2| \leq C \tag{3.94}$$

$$\|D^k \nu_2\|_0 \leq C \delta_{q+1}^{1/2} \mu^k \qquad \text{for } k \in \{1, 2\} \tag{3.95}$$

and, for $k \in \{0, 1, 2\}$,

$$\|D^k t_2\|_0 + \|D^k \tau_2\|_0 + \|D^k n_2\|_0 \leq C \ell^{-k} + C \delta_{q+1}^{1/2} \mu^k \tag{3.96}$$

$$\|D^k A_2^t\|_0 + \|D^k C_2^t\|_0 \leq C \delta_{q+1} \lambda_{q+1}^k \tag{3.97}$$

$$\|D^k A_2^n\|_0 + \|D^k B_2^t\|_0 + \|D^k C_2^n\|_0 \leq C \delta_{q+1}^{1/2} \lambda_{q+1}^k \tag{3.98}$$

$$\|D^k B_2^n\|_0 \leq C \lambda_{q+1}^k . \tag{3.99}$$

Proof. The arguments are entirely similar to the ones of Lemma 3.7, where we only need to use the estimates (3.76) and (3.77) on $D^k v$ proved in the previous section and the fact that $\lambda_{q+1} \geq \mu$. □

3.5.2 *Estimates on $\|u_{q+1} - v\|_0$, $\|D(u_{q+1} - v)\|_0$ and $\|D^2 u_{q+1}\|_0$.*

The following estimates are straightforward consequences of Lemma 3.8:

$$\|u_{q+1} - v\|_0 \leq C\delta_{q+1}^{1/2}\lambda_{q+1}^{-1}, \tag{3.100}$$

$$\|Du_{q+1} - Dv\|_0 \leq C\delta_{q+1}^{1/2}, \tag{3.101}$$

$$\|D^2 u_{q+1}\|_0 \leq C\delta_{q+1}^{1/2}\lambda_{q+1}. \tag{3.102}$$

3.5.3 *Estimates on $\|E_2\|_0$ and $\|DE_2\|_0$.*

Arguing as in Section 3.4.3 we easily see that

$$\|E_2\|_0 \leq C\delta_{q+1}\frac{\mu}{\lambda_{q+1}}, \tag{3.103}$$

$$\|DE_2\|_0 \leq C\delta_{q+1}\mu. \tag{3.104}$$

3.6 PROOF OF PROPOSITION 3.3, CONCLUSION

Recall that

$$\mu := \hat{C}\frac{\delta_{q+1}\lambda_{q+1}^{\alpha}}{\delta_{q+2}\ell} \tag{3.105}$$

for an appropriately large constant $\hat{C}$, depending upon α and g (in particular not on a). It then follows that

$$\|E_1\|_0 + \lambda_{q+1}^{-1}\|DE_1\|_0 \leq \frac{\sigma_0}{12}\delta_{q+2}\lambda_{q+1}^{-\alpha}.$$

Hence, (recall (3.51) and (3.52)) to achieve the estimates (3.10) and (3.11) we need to verify

$$C\delta_{q+1}\frac{\mu}{\lambda_{q+1}} \leq \frac{\sigma_0}{12}\delta_{q+2}\lambda_{q+1}^{-\alpha},$$

which however is implied by (3.28), which is valid provided a is chosen sufficiently large. The three remaining inequalities (3.12), (3.13) and (3.14) are implied by (3.75)–(3.78) and (3.100)–(3.102).

3.7 PROOF OF THEOREM 3.1

3.7.1 *Step 1*

By using the compactness of the domain $\overline{D}$ we may assume without loss of generality that $\bar{u}$ is uniformly strictly short, that is, $g - \bar{u}^\sharp e \geq 2\bar{\delta}$ in $\overline{D}$ for some $\bar{\delta} > 0$. In a first step we will apply the classical Nash–Kuiper argument to obtain a good first approximation.

To this end recall (for a short proof see for example Proposition 2.3.1. in [19]) that there exist a finite number[1] of unit vectors $e_i \in \mathbb{R}^2$ and corresponding amplitudes $\phi_i \in C^\infty(\overline{D})$, $i = 1, \ldots, N$ such that

$$g - \bar{u}^\sharp e - \bar{\delta} e = \sum_{i=1}^{N} \phi_i^2 e_i \otimes e_i \quad \text{in } \overline{D}.$$

Define iteratively the smooth mappings $\bar{u}_0 := \bar{u}$, $\bar{u}_1, \ldots, \bar{u}_N =: \tilde{u}$ by setting, for $i = 1, \ldots, N$,

$$\tau_i := D\bar{u}_{i-1}(D\bar{u}_{i-1}^T D\bar{u}_{i-1})^{-1} e_i\,, \quad \nu_i := \frac{\partial_{x_1}\bar{u}_{i-1} \times \partial_{x_2}\bar{u}_{i-1}}{|\partial_{x_1}\bar{u}_{i-1} \times \partial_{x_2}\bar{u}_{i-1}|}\,,$$

$$t_i := \frac{\tau_i}{|\tau_i|^2}\,, \quad n_i := \frac{\nu_i}{|\tau_i|}\,.$$

and

$$\bar{u}_i(x) := \bar{u}_{i-1}(x) + \frac{1}{\mu_i}\Gamma^t\big(\varphi_i|\tau_i|, \mu_i e_i \cdot x\big) t_i + \frac{1}{\mu_i}\Gamma^n\big(\varphi_i|\tau_i|, \mu_i e_i \cdot x\big) n_i\,. \tag{3.106}$$

Here the frequencies $1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ will be inductively defined as follows. Let

$$E_i = \bar{u}_i^\sharp e - \bar{u}_{i-1}^\sharp e - \phi_i^2 e_i \otimes e_i$$

so that $\bar{u}_N^\sharp e = g - \bar{\delta} e + \sum_{i=1}^N E_i$. As in Section 3.4 we can estimate E_1 as

$$\|E_1\|_0 \leq \frac{C(\bar{u})}{\mu_1}\,, \quad \|E_1\|_1 \leq C(\bar{u}),$$

where $C(\bar{u})$ is a constant depending on $\bar{u}$. By interpolation we also have

$$\|E_1\|_\alpha \leq \frac{C(\bar{u})}{\mu_1^{1-\alpha}},$$

and moreover $\|\bar{u} - \bar{u}_1\|_0 \leq C\mu_1^{-1}$. Therefore we can choose μ_1 so that

$$\|E_1\|_\alpha \leq \frac{\sigma_1}{2N}\bar{\delta}, \quad \|\bar{u} - \bar{u}_1\|_0 \leq \frac{\varepsilon}{2N}.$$

1 Although the number N in this decomposition depends on $\bar{\delta} > 0$, there is a geometric constant N_* such that for any $x \in \overline{D}$ at most N_* of the functions ϕ_i are non-zero. Nevertheless, this information is not required for our purposes.

Continuing, analogously we obtain

$$\|E_2\|_0 \leq \frac{C(\bar{u},\mu_1)}{\mu_2}, \quad \|E_2\|_1 \leq C(\bar{u},\mu_1),$$

and hence choose μ_2 so that

$$\|E_2\|_\alpha \leq \frac{\sigma_1}{2N}\bar{\delta}, \quad \|\bar{u}_2 - \bar{u}_1\|_0 \leq \frac{\varepsilon}{2N}.$$

In a similar manner we can inductively choose μ_i, $i = 3, \ldots, N$ so that eventually we obtain

$$\|g - \bar{\delta}e - \tilde{u}^\sharp e\|_\alpha \leq \sum_{i=1}^N \|E_i\|_\alpha \leq \frac{\sigma_1}{2}\bar{\delta}$$

and

$$\|\bar{u} - \tilde{u}\|_0 \leq \frac{\varepsilon}{2}.$$

Remark 3.9. The construction above can be easily adapted to the case when $\bar{u}$ is an embedding, and in this case also $\tilde{u}$ will be an embedding. This is of course well-known and has been proved by Nash and Kuiper. In order to this thesis self-contained, we nevertheless include here a short proof.

Since the construction of $\tilde{u}$ from $\bar{u}$ involves finite number of steps, it suffices to ensure that at each step $\bar{u}_i$ remains an embedding, i.e. no self-intersections are introduced. To show this, we proceed by induction and assume that $\bar{u}_{i-1}$ is an embedding. By using Proposition 3.5 and the choice of vectors t_i, n_i we can write (3.106) as

$$\bar{u}_i(x) := \bar{u}_{i-1}(x) + \frac{1}{\mu_i} w_i(x, \mu_i x),$$

where $w_i = w_i(x, \xi)$ satisfies

$$\begin{aligned}\left[D\bar{u}_{i-1}(x) + \partial_\xi w_i(x,\mu_i z)\right]^T \left[D\bar{u}_{i-1}(x) + \partial_\xi w_i(x,\mu_i z)\right] = \\ = D\bar{u}_{i-1}(x)^T D\bar{u}_{i-1}(x) + \phi_i^2(x) e_i \otimes e_i.\end{aligned}$$

for any x, z. In particular, since $\bar{u}_{i-1}$ is an immersion, there exists $\omega_1 > 0$ so that

$$\left|(D\bar{u}_{i-1}(x) + \partial_\xi w_i(x,\mu_i z))e\right| \geq \left|D\bar{u}_{i-1}(x)e\right| \geq \omega_1 |e| \tag{3.107}$$

for any vector e.

Next, let $x, y \in \overline{D}$. By Taylor's theorem and the mean value theorem there exists z on the line segment $[x, y]$ such that

$$\bar{u}_i(x) - \bar{u}_i(y) = D\bar{u}_{i-1}(x)(x-y) + \partial_\xi w_i(x, \mu_i z)(x-y) + \tilde{E},$$

where

$$|\tilde{E}| \leq C\left(|x-y|^2 + \frac{1}{\mu_i}|x-y|\right),$$

and C is a constant depending on the functions $\bar{u}_{i-1}(x)$ and $w_i(x, \xi)$ but not on μ_i. Let $\rho = \frac{\omega_1}{4C}$ and choose $\mu_i > \rho^{-1}$. From (3.107) we deduce that if $|x - y| \leq \rho$, then

$$|\bar{u}_i(x) - \bar{u}_i(y)| \geq \frac{\omega_1}{2}|x - y|.$$

On the other hand, since $\bar{u}_{i-1}$ is assumed to be globally injective and $\overline{D}$ is compact, there exists $\omega_2 > 0$ such that

$$|\bar{u}_{i-1}(x) - \bar{u}_{i-1}(y)| \geq \omega_2|x - y| \quad \text{for all } |x - y| \geq \rho.$$

Since obviously $\|\bar{u}_i - \bar{u}_{i-1}\|_0 \leq C\mu_i^{-1}$, it follows that for sufficiently large μ_i we will also have

$$|\bar{u}_i(x) - \bar{u}_i(y)| \geq \omega_2|x - y| \quad \text{for all } |x - y| \geq \rho.$$

In summary, we have shown that, by choosing μ_i sufficiently large, we can ensure that $\bar{u}_i$ is also an embedding.

3.7.2 *Step 2*

In Step 1 we obtained a good approximation $\tilde{u}$ in the sense that (3.8) from Proposition 3.3 is satisfied. However, although $\tilde{u}$ is smooth, we have no information on the size of the second derivatives $D^2\tilde{u}$. Therefore in this step we obtain a further approximation u_0, where in addition second derivatives are controlled so that this second approximation can then be used as the starting point of an iteration with Proposition 3.3.

In this step we assume in addition[2]

$$c > \frac{2}{1 - 2\alpha} + \frac{1}{2b}. \tag{3.108}$$

We show that, no matter how large a is chosen, there is a map u_0 satisfying the assumptions (3.8) and (3.9) of Proposition 3.3, where the constant $\bar{C}$ in the latter estimate is however *independent* of a (because it depends only on g and $\tilde{u}$). We proceed as in Section 3.3, except no regularization step is necessary this time. We set

$$h := \frac{g - \tilde{u}^\sharp e}{\bar{\delta}} - \frac{\delta_1}{\bar{\delta}}e$$

and apply Proposition 3.4 to find (C^3) Φ_1, Φ_2 and ρ so that

$$h := \rho^2(\nabla\Phi_1 \otimes \nabla\Phi_1 + \nabla\Phi_2 \otimes \nabla\Phi_2)\,.$$

We then define

$$\tau_1 := D\tilde{u}(D\tilde{u}^T D\tilde{u})^{-1}\nabla\Phi_1\,,$$

2 Indeed it could be checked directly that (3.5) implies (3.108) and hence (3.108) is superfluous: however, proceeding as we do we can spare the reader a slightly tedious computation.

$$\nu_1 := \frac{\partial_{x_1}\tilde{u} \times \partial_{x_2}\tilde{u}}{|\partial_{x_1}\tilde{u} \times \partial_{x_2}\tilde{u}|},$$

and

$$t_1 := \frac{\tau_1}{|\tau_1|^2}, \quad n_1 := \frac{\nu_1}{|\tau_1|}.$$

Hence we set

$$v = \tilde{u} + \frac{1}{\mu}\Gamma^t\left(\bar{\delta}^{1/2}|\tau_1|\rho, \mu\Phi_1\right) t_1 + \frac{1}{\mu}\Gamma^n\left(\bar{\delta}^{1/2}|\tau_1|\rho, \mu\Phi_1\right) n_1 . \tag{3.109}$$

Then we define

$$\tau_2 := Dv(Dv^T Dv)^{-1}\nabla\Phi_2,$$

$$\nu_2 := \frac{\partial_{x_1}v \times \partial_{x_2}v}{|\partial_{x_1}v \times \partial_{x_2}v|},$$

and

$$t_2 := \frac{\tau_2}{|\tau_2|^2}, \quad n_2 := \frac{\nu_2}{|\tau_2|}.$$

The map u_0 is finally given by

$$u_0 = v + \frac{1}{\lambda}\Gamma^t\left(\bar{\delta}^{1/2}|\tau_2|\rho, \lambda\Phi_2\right) t_2 + \frac{1}{\lambda}\Gamma^n\left(\bar{\delta}^{1/2}|\tau_2|\rho, \lambda\Phi_2\right) n_2 . \tag{3.110}$$

Again we assume $\lambda \geq \mu \geq 1$. Analogous computations to the ones in Sections 3.4 and 3.5 lead to the estimates

$$\|g - (u_0^\sharp e + \delta_1 e)\|_\alpha \leq C\bar{\delta}^{1/2}\mu^{2\alpha-1} + C\bar{\delta}\mu\lambda^{\alpha-1}$$

$$\|D^2 u_0\|_0 \leq C\bar{\delta}^{1/2}\lambda,$$

where the constant C depends only on $\tilde{u}$ and g. We thus set

$$\mu := C_1\delta_1^{-1/(1-2\alpha)} \quad \text{and} \quad \lambda := C_2\mu^{1/(1-\alpha)}\delta_1^{-1/(1-\alpha)}.$$

For a sufficiently large choice of C_2 and C_1 we then achieve (3.8) (recall that $\bar{\delta} < 1$).

Clearly

$$\|D^2 u_0\|_0 \leq C_3\delta_1^{-2/(1-2\alpha)},$$

for a constant C_3 which depends only upon $\tilde{u}, g$ and α. In order to show that (3.9) is satisfied with a constant $\bar{C}$ independent of a, it suffices to show that

$$\delta_1^{-2/(1-2\alpha)} \leq \delta_0^{1/2}\lambda_0 .$$

Taking the logarithms in base a the latter inequality is implied by

$$cb \geq \frac{1}{2} + \frac{2}{1-2\alpha}b .$$

3.7.3 *Step 3*

Finally we are ready for the iteration based on Proposition 3.3. Fix any α, b and c which satisfies (3.4), (3.5) and (3.108). Then, for any sufficiently large a, we can construct a map u_0 as in the previous step which satisfies $\|\bar{u} - u_0\|_0 < \frac{\varepsilon}{2}$ and the assumptions of Proposition 3.3, with a constant $\bar{C}$ which does not depend on a. We can apply Proposition 3.3 to generate u_1. Observe the following explicit interpolation inequality, which follows easily from the definitions

$$\|f\|_\alpha \leq \|f\|_0 + 2\|f\|_0^{1-\alpha}[f]_1^\alpha .$$

With this we conclude

$$\begin{aligned}\|g_1 - u_1^\sharp e\|_\alpha \leq& \|g_1 - u_1^\sharp e\|_0 + 2\|g_1 - u_1^\sharp e\|_0^{1-\alpha}\|D(g_1 - u_1^\sharp e)\|_0^\alpha \\ \leq& \sigma_0\delta_2 .\end{aligned} \tag{3.111}$$

Hence u_1 satisfies again the assumptions of Proposition 3.3. More generally, the proposition can be applied inductively to generate a sequence $(u_q)_{q\geq 0}$. Observe that (3.12)-(3.14) imply that

- $(u_q)_{q\geq 0}$ converges uniformly to a map u which (assuming a sufficiently large) satisfies $\|u_0 - u\|_0 < \frac{\varepsilon}{2}$. By assumption on u_0 we therefore have $\|\bar{u} - u\|_0 < \varepsilon$.
- Interpolating $\|D(u_{q+1} - u_q)\| \leq C\delta_{q+1}^{1/2}$ and

  $$\begin{aligned}\|D^2(u_{q+1} - u_q)\|_0 \leq& \|D^2 u_{q+1}\|_0 + \|D^2 u_q\|_0 \leq \bar{C}\delta_{q+1}^{1/2}\lambda_{q+1} + \bar{C}\delta_q^{1/2}\lambda_q \\ \leq& 2\bar{C}\delta_{q+1}^{1/2}\lambda_{q+1}\end{aligned}$$

  shows

  $$\|D(u_{q+1} - u_q)\|_\beta \leq C^\star \delta_{q+1}^{1/2}\lambda_{q+1}^\beta ,$$

  for a constant $C^\star$ which depends on α, g and $\bar{C}$. Hence using the definitions (3.3) of δ_q and λ_q we can see that if $\beta < \frac{1}{2bc}$ then $(u_q)_{q\geq 0}$ is a Cauchy sequence on $C^{1,\beta}$.

We next show that, if α is chosen arbitrarily small, bc can be chosen arbitrarily close to $\frac{5}{2}$, which in turn implies that β can be made arbitrarily close to $\frac{1}{5}$. Indeed if we let $\alpha \downarrow 0$, the conditions (3.4), (3.5) and (3.108) become, respectively

$$b > 1 \tag{3.112}$$

$$c > \frac{4b^2 - 3b - 1}{2b(b-1)} = 2 + \frac{1}{2b} \tag{3.113}$$

$$c > 2 + \frac{1}{2b} . \tag{3.114}$$

This completes the proof in the case of immersions. We give the argument for the case of embeddings explicitly in the next section.

3.8 PROOF OF THEOREM 3.2

First of all we notice that, by classical extension theorems, the first statement can be reduced to Theorem 3.1: it suffices to extend both g and $\bar{u}$ smoothly from $\bar{D}_1$ to $\bar{D}_2$. The extended map is not necessarily short for the extended metric, but we can ensure this if we add to the extension of g a tensor of the form $\varphi(|x|)e$, where φ is a rapidly growing C^∞ function which vanishes identically on $[0,1]$.

Next, observe that the arguments of the Steps 2 and 3 in Section 3.7, combined with the extension trick outlined above give in fact the following corollary.

Corollary 3.10. *Let g be a C^2 metric on $\overline{D}_1$. Then there are positive constants $C_0, \bar{c}$ and $\bar{\eta}$ with the following properties. Assume that*

(i) $\underline{u} : \overline{D}_1 \to \mathbb{R}^3$ *is* C^∞,

(ii) $\|g - (\underline{u}^\sharp e + 2\eta e)\|_0 \leq \bar{c}\eta$ *for some* $\eta \in]0, \bar{\eta}[$.

Then for any $\varepsilon > 0$ and $\delta > 0$ there is an isometric map $u \in C^{1,1/5-\delta}(\overline{D}_1)$ such that $\|Du - D\underline{u}\|_0 \leq C_0\eta^{1/2}$ and $\|u - \underline{u}\|_0 \leq \varepsilon$.

With this corollary at hand we can prove Theorem 3.2 in two easy steps. In the proof we will restrict to the case of embeddings, the case of immersions can be obtained by easy modifications.

Proof of Theorem 3.2 and Theorem 3.1 for embeddings.

Let g be a C^2 metric on $\overline{D}_1$ and $\bar{u} \in C^1(\overline{D}_1, \mathbb{R}^3)$ a short embedding. By a simple rescaling and mollification we may assume without loss of generality that $\bar{u}$ is smooth and strictly short. Next, fix $\omega > 0$ such that $g \geq 16\omega^2 e$ and choose $\eta > 0$ such that $\eta \leq \min\{\omega^2, \bar{\eta}\}$ and $C_0\eta^{1/2} \leq \omega$.

As in Step 1 of the proof of Theorem 3.1 (including Remark 3.9) we first construct a smooth embedding $\underline{u}$ with

$$\|\underline{u} - \overline{u}\|_0 < \frac{\varepsilon}{2}$$

and such that

$$\|g - (\underline{u}^\sharp e + 2\eta e)\|_0 \leq \bar{c}\eta.$$

Then the assumptions of Corollary 3.10 are satisfied and we obtain $u \in C^{1,1/5-\delta}(\overline{D}_1)$ with $u^\sharp e = g$ and such that $\|Du - D\underline{u}\|_0 \leq C_0\eta^{1/2}$ and $\|u - \underline{u}\|_0 \leq \varepsilon/2$.

To complete the proof, it remains to show that the map u is an embedding. We again remark that this argument is well-known and is contained in the works of Nash and Kuiper. First of all, since $\underline{u}$ is C^1, there exists $\rho > 0$ such that $|D\underline{u}(z) - D\underline{u}(y)| \leq \omega$ if $|z - y| \leq \rho$. On the other hand, since $\underline{u}$ is an embedding, then there is $\zeta > 0$ such that $|\underline{u}(z) - \underline{u}(y)| \geq 3\zeta$ if $|z - y| \geq \rho$.

To show global injectivity, we now observe that

$$|u(z) - u(y)| \geq |\underline{u}(z) - \underline{u}(y)| - 2\varepsilon \geq 3\zeta - 2\zeta = \zeta \qquad \text{when } |z - y| \geq \rho.$$

On the other hand, if $|z-y| \leq \rho$ we know that

$$|Du(z) - Du(y)| \leq |D\underline{u}(z) - D\underline{u}(y)| + 2\omega \leq 3\omega\,,$$

and hence, using Taylor's formula

$$|u(z) - u(y) - Du(z)(z-y)| \leq 3\omega|z-y|\,.$$

We therefore can estimate

$$|u(z) - u(y)| \geq |Du(z)(z-y)| - 3\omega|z-y|$$

But $u^\sharp e = g \geq 16\omega^2 e$ implies $|Du(z)(z-y)|^2 \geq 16\omega^2|z-y|^2$, which in turn shows $|u(z) - u(y)| \geq \omega|z-y| > 0$.

This completes the proof of Theorem 3.2 and Theorem 3.1.

4 FRACTIONAL SOBOLEV REGULARITY FOR THE BROUWER DEGREE

In this chapter we want to investigate regularity and summability properties of the Brouwer degree of a Hölder continuous function $v \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$ defined on an open, bounded set $\Omega \subset \mathbb{R}^n$ with

$$n - 1 \leq d := \overline{\dim}_b(\partial\Omega) < n\,, \tag{4.1}$$

where $\overline{\dim}_b$ denotes the upper box-counting dimension. We recall that it is defined by

$$\overline{\dim}_b(\partial\Omega) = \limsup_{r\to 0} \frac{\log N_r}{-\log r} \tag{4.2}$$

where N_r can be chosen to be the number of closed cubes of a mesh of $\mathbb{R}^n$ of width $r > 0$ which intersect $\partial\Omega$. In the recent note [48], H. Olbermann showed that the Brouwer degree is an L^p function for every $1 \leq p < \frac{n\alpha}{d}$. A different proof of the L^1 summability when Ω has a Lipschitz boundary has been given independently by R. Züst in [59], and although it does not yield the range of summability exponents of Olbermann's proof, it allows to conclude the L^1 estimate when each component v^i has (possibly) different Hölder regularity C^{0,α_i} with $\frac{1}{n-1}\sum_i \alpha_i > 1$. Theorem 4.5 gives an extension of the latter L^1 summability to domains with fractal boundary. Finally, a more recent work by Züst unifies the previous results: in [60] it is shown that the degree of a map $v \in C^{0,\alpha_1} \times \ldots \times C^{0,\alpha_n}$ is in L^p for $1 \leq p < \frac{1}{d}\sum_i \alpha_i$.

In this chapter we show that Olbermann's idea can be improved to show higher (fractional) Sobolev regularity. In particular the following is our main theorem. As usual $[\cdot]_{C^{0,\alpha}}$ denotes the Hölder and $[\cdot]_{W^{\beta,p}}$ the Gagliardo seminorm when $\beta > 0$ and the L^p norm for $\beta = 0$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, d be as in (4.1) and $v \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$, where $\alpha \in]\frac{d}{n}, 1]$. Then the Brouwer degree $\deg(v, \Omega, \cdot)$ satisfies the estimate*

$$[\deg(v, \Omega, \cdot)]_{W^{\beta,p}} \leq C(\Omega, n, \alpha, \beta, p)[v]_{C^{0,\alpha}}^{\frac{n}{p}-\beta} \tag{4.3}$$

for any pair (β, p) with $\quad p \geq 1 \quad$ *and* $\quad 0 \leq \beta < \dfrac{n}{p} - \dfrac{d}{\alpha}\,.$ (4.4)

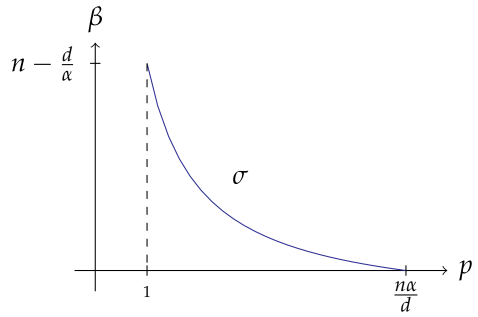

Figure 1: Range of exponents in (4.4)

Observe that the endpoints of (4.4) form the segment $\sigma = \{\beta = \frac{n}{p} - \frac{d}{\alpha}\}$ (see Figure 1) and if we let $(\beta_1, p_1) = (n - \frac{d}{\alpha}, 1)$ be the left extremum of the segment, then W^{β_1,p_1} embeds in $W^{\beta,p}$ for every $(\beta, p) \in \sigma$. In particular our theorem has the following obvious corollary.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, d be as in* (4.1) *and $\{v_k\} \subset C^{0,\alpha}(\Omega, \mathbb{R}^n)$ a bounded sequence converging uniformly to v, where $\alpha \in]\frac{d}{n}, 1]$. Then, for every pair (β, p) as in* (4.4)*, the sequence* $\deg(v_k, \Omega, \cdot)$ *converges to* $\deg(v, \Omega, \cdot)$ *strongly in $W^{\beta,p}$.*

As already mentioned above, our proof is built upon the ideas of Olbermann in [48]. However we report also a self-contained and more elementary argument for his result: the key simplification can be found in the direct elementary proof of Theorem 4.5 below. A part of this theorem is shown in [48] using tools from interpolation theory. We instead derive it directly and use our approach to extend Züst's L^1 result [59] in the sense mentioned above. For the reader's convenience we then show how to recover Olbermann's higher integrability in few lines, although the argument is already contained in [48]. From Theorem 4.5 we then derive Theorem 4.1 using heavier machinery from harmonic analysis.

It has already been shown in [48] that, when $\beta = 0$ and $d > n - 1$, the range of exponents in Theorem 4.1 cannot be extended beyond the endpoints: more precisely, [48, Theorem 1.2] proves that, if $p > \frac{n\alpha}{d}$, then there is a fixed open set Ω with $\overline{\dim}_b(\partial\Omega) = d$ and a bounded sequence $\{v_k\} \subset C^{0,\alpha}(\Omega)$ for which $\|\deg(v_k, \Omega, \cdot)\|_{L^p} \uparrow \infty$. Note however that the proof in [48] *does not* yield a $v \in C^{0,\alpha}(\Omega)$ for which $\deg(v, \Omega, \cdot) \notin L^p$, because the sequence produced by the argument converges to 0, cf. [48, Section 4.2]. In this note we discuss the optimality of the range in the case $d = n - 1$: our main conclusion is the following theorem, which, by Sobolev embedding, has the immediate Corollary 4.4.

Theorem 4.3. *For any $n \geq 2$, $p \geq 1$ and $\alpha < \frac{p(n-1)}{n}$ there is $v \in C^{0,\alpha}(B_1, \mathbb{R}^n)$ such that* $\deg(v, B_1, \cdot) \notin L^p$*, where $B_1 \subset \mathbb{R}^n$ is the unit ball.*

Corollary 4.4. *For any $n \geq 2$, $p \geq 1$, $\alpha \geq 0$ and $\beta > \frac{n}{p} - \frac{n-1}{\alpha}$ there is $v \in C^{0,\alpha}(B_1, \mathbb{R}^n)$ with* $\deg(v, B_1, \cdot) \notin W^{\beta,p}$.

The case of the endpoints is certainly more subtle. Indeed, if $v \in C^{0,1}$ and Ω is a bounded Lipschitz domain, then the area formula and elementary considerations in degree theory imply that $\deg v \in BV$ (the space of functions of bounded variation). In fact, with a little help from the theory of BV functions and Caccioppoli sets, the latter statement can be shown even under the more technical assumption that the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ is finite. Therefore:

- $\deg(v, \Omega, \cdot) \in L^{n/(n-1)}$, by the Sobolev embedding of $BV(\mathbb{R}^n)$, which shows that the endpoint $(\beta, p) = (0, \frac{n}{n-1})$ could be included if we assume that $\partial\Omega$ has finite $(n-1)$-dimensional measure;
- since the degree takes integer values and vanishes on $\mathbb{R}^n \setminus v(\overline{\Omega})$, it belongs to $W^{1,1}$ only if it vanishes identically: hence, even assuming that $\partial\Omega$ has finite $(n-1)$-dimensional measure, the endpoint $(\beta, p) = (1, 1)$ can be included only if we replace $W^{1,1}$ with BV.

4.1 FIRST ESTIMATE AND CHANGE OF VARIABLES

The starting point of Olbermann's proof is the classical change of variable formula

$$\int_{\mathbb{R}^n} \varphi(y) \deg(v, \Omega, y)\, dy = \int_{\Omega} \varphi(v(x)) \det Dv(x)\, dx\,, \tag{4.5}$$

which is valid if v is regular enough (compare e.g. [28]). By representing the integrand $\varphi(v(x))\det Dv(x)$ as a sum of weakly defined Jacobian determinants, using Stokes theorem and tools from interpolation theory Olbermann manages to bound the right hand side of (4.5) by a (suitable power of the) $C^{0,\alpha}$ norm of v and the $L^{p'}$ norm of φ, where α is as above and p' is conjugate to p. In fact, implicit in his proof is the estimate (4.7) below, which will play a crucial role for us as well. On the other hand our elementary argument yields immediately, as a byproduct, that the degree is an L^1 function and thus we do not have to resort to any weak notion of Jacobian determinant. Moreover, we also get a simple proof of Züst's L^1 result, together with the generalization to domains with fractal boundary.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$, n and d be as in Theorem 4.1. Assume that $v = (v^1, \dots, v^n)$ is a continuous map $v : \Omega \to \mathbb{R}^n$ for which $v^i \in C^{0,\alpha_i}$. If $\sum_i \alpha_i > d$, then* $\deg(v, \Omega, \cdot) \in L^1$ *and*

$$\|\deg(v, \Omega, \cdot)\|_{L^1} \le C(\Omega, n, \alpha_1, \dots, \alpha_n) \prod_{i=1}^{n} [v^i]_{C^{0,\alpha_i}}\,. \tag{4.6}$$

If in addition $\alpha = \min_i \alpha_i > \frac{d}{n}$, then for any $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\left| \int_{\mathbb{R}^n} \deg(v, \Omega, y)\, \mathrm{div}\, \psi(y)\, dy \right| \le C(\Omega, n, \alpha, \gamma) [v]^{n-1+\gamma}_{C^{0,\alpha}(\Omega)} [\psi]_{C^{0,\gamma}(B_R)}\,, \tag{4.7}$$

where $\gamma \in (0, 1)$ is such that $(n-1+\gamma)\alpha > d$ and $R > 0$ such that $\overline{v(\Omega)} \subset B_R(0)$.

4.1.1 *Two technical lemmas*

We record here two simple facts related to the dimension of $\partial\Omega$.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $d := \overline{\dim}_b(\partial\Omega) < n$. Then for any $\varepsilon > 0$ the function* $\operatorname{dist}(x,\partial\Omega)^{d+\varepsilon-n}$ *is integrable.*

Proof. Fix $0 < \varepsilon < n-d$ and let W be the Whitney decomposition of Ω and let $W_k := \{Q \in W : Q \text{ cube of sidelength } 2^{-k}\}$. Then

1. $\operatorname{dist}(Q,\partial\Omega) \geq 2^{-k}\sqrt{n}$ for any $Q \in W_k$ and
2. there exists $C \equiv C(\varepsilon) > 0$ such that $\#W_k \leq C2^{k(d+\varepsilon/2)}$ for any $k \in \mathbb{N}$ (cf. Theorem 3.12 in [44]).

Since $\mathring{Q} \cap \mathring{Q}' = \emptyset$ for any $Q \neq Q'$ we have

$$\begin{aligned}\int_\Omega \operatorname{dist}(x,\partial\Omega)^{d+\varepsilon-n}\,dx &= \sum_{k\geq 1}\sum_{Q\in W_k}\int_Q \operatorname{dist}(x,\partial\Omega)^{d+\varepsilon-n}\,dx \\ &\leq C(n)\sum_{k\geq 1}\sum_{Q\in W_k}\mathcal{L}^n(Q)\,2^{-k(d+\varepsilon-n)} \\ &\leq C(n,\varepsilon)\sum_{k\geq 1}2^{k(d+\varepsilon/2)}2^{-k(d+\varepsilon)} \leq C(n,\varepsilon) < +\infty\,. \qquad \square\end{aligned}$$

Lemma 4.7. *If v and Ω are as in Theorem 4.5 then $v(\partial\Omega)$ is a Lebesgue-null set.*

Proof. Fix a positive $\delta \leq \sum_{i=1}^n \alpha_i - d$. For any $\varepsilon > 0$ there is a covering of $\partial\Omega$ with balls $B_{r_i}(x_i)$ such that $\sum_i r_i^{d+\delta} \leq (\mathcal{H}^{d+\delta}(\partial\Omega)) + \varepsilon = \varepsilon$ and $r_i \leq 1$, where $\mathcal{H}^\omega$ denotes the ω-dimensional Hausdorff measure. Observe that $v(B_{r_i}(x_i))$ is contained in a box $Q_i = I_1^i \times \ldots \times I_n^i$, where each interval I_j^i has length at most $(2r_i)^{\alpha_j}[v^j]_{C^{0,\alpha_j}}$. Thus

$$|v(\partial\Omega)| \leq \sum_i |Q_i| \leq C\prod_{j=1}^n [v^j]_{C^{0,\alpha_j}}\sum_i r_i^{\alpha_1+\ldots+\alpha_n} \leq C(v)\varepsilon\sup_i r_i^{\alpha_1+\ldots+\alpha_n-d-\delta} \leq C\varepsilon\,.$$

Letting $\varepsilon \to 0$ we conclude the proof. $\square$

4.1.2 *Proof of Theorem 4.5*

First of all recall that the degree depends only upon the values of v at the boundary. We wish therefore to find a suitable extension $\tilde{v}$ of v which is smooth in the interior and satisfies suitable estimates on the derivatives. For $k = 0, 1, \ldots$ set

$$A_k := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > 2^{-k}\} \tag{4.8}$$

and define $D_0 := A_1$, $D_k := A_{k+1} \setminus \bar{A}_{k-1}$ for $k = 1, 2, \ldots$. Fix a partition of unity $\{\chi_k\}_{k\geq 1}$ subordinate to the cover $\{D_k\}_{k\geq 0}$, i.e.

$$0 \leq \chi_k \leq 1, \quad \mathrm{supp}\chi_k \subset D_k, \quad \sum_{k=0}^{+\infty} \chi_k = 1 \text{ on } \Omega\,.$$

Observe that each point $x \in \Omega$ has an open neighbourhood $U \subset \Omega$ on which at most three χ_k are non zero. Next fix a standard symmetric mollifier φ with support contained in the ball of radius 1 and define the functions $v_k : D_k \to \mathbb{R}^n$ by the convolution $v_k(x) := \varphi_{2^{-(k+1)}} * v(x)$. Finally, set

$$\tilde{v} := \sum_{k=0}^{+\infty} \chi_k v_k\,.$$

We have $\tilde{v} \in C^\infty(\Omega, \mathbb{R}^n)$ and we claim that for every $x \in \Omega$

$$|\nabla \tilde{v}^i(x)| \leq C\mathrm{dist}(x, \partial\Omega)^{\alpha_i - 1}[v^i]_{C^{0,\alpha_i}(\Omega)} \qquad \text{for all } i\,. \tag{4.9}$$

By standard estimates

$$|\nabla v_k^i(y)| \leq C\left(2^{-(k+1)}\right)^{\alpha_i - 1}[v^i]_{C^{0,\alpha_i}}\,, \quad \text{whenever } y \in D_k\,.$$

Moreover, since $\sum \nabla\chi_k = 0$ and $|\nabla\chi_k| \leq C2^k$ we get

$$\begin{aligned} |\nabla \tilde{v}^i(x)| &\leq \sum_{k=k_1}^{k_3} |\nabla\chi_k||v_k^i(x) - v^i(x)| + C\left(2^{-(k_3+1)}\right)^{\alpha_i - 1}[v^i]_{C^{0,\alpha_i}} \\ &\leq C\left(2^{-(k_3+1)}\right)^{\alpha_i - 1}[v^i]_{C^{0,\alpha_i}}\,. \end{aligned}$$

Next, notice that $|\deg(v, \Omega, y)| = |\deg(\tilde{v}, \Omega, y)|$ is bounded by the number of preimages $N(y)$ in Ω through $\tilde{v}$ whenever $y \notin v(\partial\Omega)$. Since $v(\partial\Omega)$ is a null set, by the area formula, (4.9) and Lemma 4.6 we have

$$\int_{\mathbb{R}^n} N(y)dy = \int_\Omega |\det D\tilde{v}(x)|\,dx \leq C\prod_{i=1}^n [v^i]_{C^{0,\alpha_i}} \int_\Omega \mathrm{dist}(x, \partial\Omega)^{\sum_i \alpha_i - n}\,dx \leq C\prod_{i=1}^n [v^i]_{C^{0,\alpha_i}}\,.$$

This estimate will be needed later on in Chapter 5 (compare (5.7)). Next, fix a C^1 test field ψ as in the second part of the statement and let $\alpha = \min_i \alpha_i$. Define the maps $\tilde{V}_j = (\tilde{v}^1, \ldots, \tilde{v}^{j-1}, \psi^j \circ \tilde{v}, \tilde{v}^{j+1}, \ldots, \tilde{v}^n)$ and the corresponding $V_j = (v^1, \ldots, v^{j-1}, \psi^j \circ v, v^{j+1}, \ldots, v^n)$ for $j = 1, \ldots, n$. In particular it follows $\sum_{j=1}^n \det D\tilde{V}_j = (\mathrm{div}\,\psi) \circ \tilde{v} \det D\tilde{v}$.

Let Ω_k be smooth domains compactly contained in Ω so that[1] $\Omega_k \uparrow \Omega$. By the smoothness of $\tilde{v}$ and ψ, we can apply the area formula and conclude

$$\begin{aligned}\int_{\mathbb{R}^n} \deg(\tilde{v}, \Omega_k, y) \operatorname{div} \psi(y)\, dy &= \int_{\Omega_k} \operatorname{div} \psi(\tilde{v}(x)) \det D\tilde{v}(x)\, dx \\ &= \sum_{j=1}^{n} \int_{\Omega_k} \det D\tilde{V}_j(x)\, dx = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \deg(\tilde{V}_j, \Omega_k, y)\, dy\,.\end{aligned}$$

Next, observe that the number $N(y)$ bounds $|\deg(\tilde{v}, \Omega_k, y)|$ for every y and k and thus, by the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \deg(\tilde{v}, \Omega_k, y) \operatorname{div} \psi(y)\, dy = \int_{\mathbb{R}^n} \deg(\tilde{v}, \Omega, y) \operatorname{div} \psi(y)\, dy\,.$$

The same argument can be applied to $\tilde{V}_j$, since $|\det D\tilde{V}_j| \leq |D\psi||D\tilde{v}|^n$ also belongs to $L^1(\Omega)$. Hence, passing into the limit in k and using the fact that $\tilde{v}$ agrees with v on $\partial\Omega$ we can conclude

$$\int_{\mathbb{R}^n} \deg(v, \Omega, y) \operatorname{div} \psi(y)\, dy = \sum_j \int \deg(V_j, \Omega, y)\, dy\,.$$

On the other hand for each V_j we have $[V_j^i]_{C^{0,\alpha}} \leq [v]_{C^{0,\alpha}}$ when $i \neq j$ and $[V_j^j]_{C^{0,\alpha\gamma}} \leq [\psi]_{C^{0,\gamma}} [v]^{\gamma}_{C^{0,\alpha}}$. Since by our choice of γ we have $(n-1+\gamma)\alpha > d$, we can apply (4.6) to conclude

$$\|\deg(V_j, \Omega, \cdot)\|_{L^1} \leq C(n, \Omega, \alpha, \gamma, d) [v]^{n-1+\gamma}_{C^{0,\alpha}} [\psi]_{C^{0,\gamma}}\,.$$

4.2 PROOFS OF THEOREM 4.1 AND OF COROLLARY 4.2

4.2.1 *Direct proof of Theorem 4.1 for $\beta = 0$*

This section follows essentially Olbermann's argument and is only added for the reader's convenience in order to show that the harmonic analysis of the next section is only needed for $\beta > 0$. The key is the following proposition.

Proposition 4.8. *Let $\Omega \subset \mathbb{R}^n$, n, d, α and v be as in Theorem 4.1 with $\|v\|_{C^0} \leq 1$ and fix $1 < p < \frac{n\alpha}{d}$. Then, if we denote by p' the dual exponent of p, we have the estimate*

$$\left| \int \deg(v, \Omega, y)\, \varphi(y)\, dy \right| \leq C(\Omega, n, d, \alpha, p.\beta) [v]^{\frac{n}{p}}_{C^{0,\alpha}} \|\varphi\|_{L^{p'}} \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^n)\,. \quad (4.10)$$

1 Let A_k be the sets in (4.8) and $\mathbf{1}_{A_k}$ their indicator functions, consider the mollifications $\eta_k := \mathbf{1}_{A_k} * \varphi_{2^{-k-1}}$ and set $\Omega_k = \{\eta_k > t_k\}$ for a suitably chosen $0 < t_k < 1$. The regularity of $\partial\Omega_k$ follows from Sard's Lemma.

The case $\beta = 0$ of Theorem 4.1 then follows easily when $\|v\|_{C^0} \le 1$: just take the supremum over $\varphi \in C_c^\infty \cap \{\|\varphi\|_{L^{p'}} \le 1\}$ in (4.10) and use the density of C_c^∞ in $L^{p'}$ together with the usual duality $(L^p)^* = L^{p'}$. To remove the assumption that $\|v\|_{C^0} \le 1$ it suffices, for a general nonzero v, to consider the normalization $v/\|v\|_{C^0}$ and compare its degree to that of v with an obvious scaling argument (cf. Section 4.2.2 below where this argument is repeated with more details). The extension to $p = 1$ follows because $\deg(v, \Omega, \cdot)$ is supported in the bounded set $v(\Omega)$, whose diameter can be estimated using the Hölder norm of the function v. We are thus left to show (4.10). Fix φ and consider the potential theoretic solution ζ of

$$-\Delta\zeta = \varphi .$$

By classical Calderon-Zygmund estimates we have $\|\zeta\|_{W^{2,p'}(B_2)} \le C\|\varphi\|_{L^{p'}}$. So, if we set $\psi = -\nabla\zeta$, we conclude $\operatorname{div}\psi = \varphi$ on B_2 and, from the Sobolev embedding, $[\psi]_{C^{0,\gamma}(B_2)} \le C\|\varphi\|_{L^{p'}}$, where $\gamma = 1 - \frac{n}{p'} = 1 - n + \frac{n}{p} > 1 - n + \frac{d}{\alpha}$. Since $\deg(v, \Omega, \cdot)$ is supported in B_2, we can apply Theorem 4.5 to conclude (4.10).

4.2.2 *Bessel potential spaces when $\beta > 0$*

Rather than showing estimate (4.3) we will show, for the exponents in the ranges $1 < p < \frac{n\alpha}{d}$ and $0 \le \beta < \frac{n}{p} - \frac{d}{\alpha}$, the slightly different estimate

$$\|\deg(v, \Omega, \cdot)\|_{\mathcal{H}^{\beta,p}} \le C\|v\|_{C^{0,\alpha}}^{\frac{n}{p}-\beta} \qquad \text{when } \|v\|_{C^0} \le 1, \tag{4.11}$$

where $\mathcal{H}^{\beta,p}(\mathbb{R}^n)$ is the Bessel potential space (see below for the relevant definition). Recall (see e.g. the classical textbook of Triebel [55]) that the spaces $W^{\beta,p}$ and $\mathcal{H}^{\beta,p}$ correspond, respectively, to the Triebel-Lizorkin spaces $F_\beta^{p,p}$ and $F_\beta^{p,2}$. Since we have the continuous embedding $F_\beta^{p,q} \subset F_{\beta-\varepsilon}^{p,q'}$ for every q, q' and every $\varepsilon > 0$, we get as a corollary of (4.11) the estimate

$$\|\deg(v, \Omega, \cdot)\|_{W^{\beta,p}} \le C\|v\|_{C^{0,\alpha}}^{\frac{n}{p}-\beta} \qquad \text{when } \|v\|_0 \le 1. \tag{4.12}$$

From (4.12) it follows by scaling that for any nonzero v as in Theorem 4.1 we have

$$[\deg(v, \Omega, \cdot)]_{W^{\beta,p}} = \|v\|_{C^0}^{\frac{n}{p}-\beta} \left[\deg\left(\frac{v}{\|v\|_{C^0}}, \Omega, \cdot\right)\right]_{W^{\beta,p}} \le C\|v\|_{C^{0,\alpha}}^{\frac{n}{p}-\beta} . \tag{4.13}$$

Apply the latter estimate to $\tilde{v} := v - v(x_0)$ for some $x_0 \in \Omega$. Since $\deg(\tilde{v}, \Omega, y) = \deg(v, \Omega, y + v(x_0))$ and $\|\tilde{v}\|_{C^0} \le C(\Omega, \alpha)[v]_{C^{0,\alpha}}$ we recover (4.3).

Recall that the Bessel potential of degree $\beta > 0$ is the L^1 function J_β such that $\hat{J}_\beta(\xi) = \left(1 + 4\pi^2|\xi|^2\right)^{-\beta/2}$ (where $\hat{h}$ denotes the Fourier transform of h). The convolution with J_β defines a continuous linear map $\mathcal{J}_\beta : L^p \to L^p$ and can be regarded as the pseudodifferential operator $(\mathrm{Id} - \Delta)^{-\beta/2}$. In particular

$$(\mathrm{Id} - \Delta)\mathcal{J}_2\varphi = \varphi \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n) . \tag{4.14}$$

Concerning the Bessel potential space $\mathcal{H}^{\beta,p}$ we will need the following facts (cf. again [55]):

(F1) $f \in \mathcal{H}^{\beta,p}$ if and only if there is $g \in L^p$ with $f = \mathcal{J}_\beta(g)$; such g is unique and $\|f\|_{\mathcal{H}^{\beta,p}} = \|g\|_{L^p}$;

(F2) $(\mathcal{H}^{\beta,p}, \|\cdot\|_{\mathcal{H}^{\beta,p}})$ is a separable reflexive Banach space for any $p \in]1,\infty[$ and $C_c^\infty(\mathbb{R}^n)$ is dense in it;

(F3) if $\beta p > n$ and $p \geq 2$ we have the continuous inclusion $\mathcal{H}^{\beta,p} \subset W^{\beta,p}$ and hence, by Morrey's embedding, $\mathcal{H}^{\beta,p} \subset C^{0,\gamma}$ with $\gamma = (\beta p - n)/p$.

The idea of the proof of Theorem 4.1 is to show that $\deg(v,\Omega,\cdot)$ is an element of the dual of $(\mathcal{H}^{\beta,p})^*$ and to use the reflexivity property in (F2). As usual, $(\mathcal{H}^{\beta,p})^*$ denotes the Banach space of bounded linear functionals $L : \mathcal{H}^{\beta,p} \to \mathbb{R}$ endowed with the dual norm $\|\cdot\|_{(\mathcal{H}^{\beta,p})^*}$. Moreover, since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}^{\beta,p}$, we clearly have

$$\|L\|_{(\mathcal{H}^{\beta,p})^*} := \sup\left\{L(u) : u \in C_c^\infty(\mathbb{R}^n) \text{ and } \|u\|_{\mathcal{H}^{\beta,p}} \leq 1\right\} . \tag{4.15}$$

Of course $\left(\mathcal{H}^{\beta,p}\right)^*$ is a subspace of the space of tempered distributions and we can consider C_c^∞ as a subset of $\left(\mathcal{H}^{\beta,p}\right)^*$ via the identification of any element $\varphi \in C_c^\infty$ with the linear functional $u \mapsto \int \varphi u$. We then have the following standard consequence of distribution theory

Lemma 4.9. *C_c^∞ is strongly dense in $(\mathcal{H}^{\beta,p})^*$ if $p \in]1,\infty[$.*

Proof. Let H be the closure of C_c^∞ in the norm $\|\cdot\|_{\left(\mathcal{H}^{\beta,p}\right)^*}$. If H were a strict subset of $\left(\mathcal{H}^{\beta,p}\right)^*$, then by Hahn-Banach there would be a nontrivial linear functional $L' : (\mathcal{H}^{\beta,p})^* \to \mathbb{R}$ wich vanishes on H. By reflexivity L' is given by an element $u \in \mathcal{H}^{\beta,p}$, which must therefore be nonzero. Since however L' vanishes on H, we conclude

$$\int u\varphi = 0 \qquad \forall \varphi \in C_c^\infty .$$

Since $u \in L^p$, the latter implies that $u \equiv 0$, which is a contradiction. □

(4.11) is then a consequence of the following natural generalization of Proposition 4.8.

Proposition 4.10. *Let $\Omega \subset \mathbb{R}^n$, n, d, α and v be as in Theorem 4.1 with the additional assumption $\|v\|_0 \leq 1$ and fix $1 < p < \frac{n\alpha}{d}$ and $0 < \beta < \frac{n}{p} - \frac{d}{\alpha}$. Then, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have the estimate*

$$\left|\int_{\mathbb{R}^n} \deg\left(v,\Omega,y\right)\varphi(y)\,dy\right| \leq C(\Omega,n,d,\alpha,p.\beta)[v]_{C^{0,\alpha}}^{\frac{n}{p}-\beta}\|\varphi\|_{(\mathcal{H}^{\beta,p})^*} . \tag{4.16}$$

We will prove Proposition 4.10 in the next section. Assuming it, we now show (4.11). Consider the linear functional $L'' : C_c^\infty \to \mathbb{R}$ given by

$$L''(\varphi) := \int_{\mathbb{R}^n} \deg(v, \Omega, y)\, \varphi(y)\, dy\,.$$

By Lemma 4.9 and (4.16), L'' extends to a unique bounded linear functional $\mathcal{L} : (\mathcal{H}^{\beta,p})^* \to \mathbb{R}$ and moreover

$$\|\mathcal{L}\|_{(\mathcal{H}^{\beta,p})^{**}} \le C\|v\|_{C^{0,\alpha}}^{\frac{n}{p}-\beta}\,.$$

By reflexivity $\mathcal{L}$ is represented by an element $u \in \mathcal{H}^{\beta,p}$ such that $\|u\|_{\mathcal{H}^{\beta,p}} = \|\mathcal{L}\|_{(\mathcal{H}^{\beta,p})^{**}}$. This means

$$\int_{\mathbb{R}^n} u(y)\varphi(y)\, dy = L''(\varphi) = \int_{\mathbb{R}^n} \deg(v, \Omega, y)\, \varphi(y)\, dy$$

for every $\varphi \in C_c^\infty$. Since however both $\deg(v, \Omega, \cdot)$ and u are L^p functions, they must coincide. Hence

$$\|\deg(v, \Omega, \cdot)\|_{\mathcal{H}^{\beta,p}} = \|u\|_{\mathcal{H}^{\beta,p}} = \|\mathcal{L}\|_{(\mathcal{H}^{\beta,p})^{**}} \le C\|v\|_{C^{0,\alpha}}^{\frac{n}{p}-\beta}\,.$$

4.2.3 *Proof of Proposition 4.10*

In order to prove the estimate (4.16), we will invoke property (4.7) after representing φ as the divergence of a suitable vector field, which is the purpose of the following lemma.

Lemma 4.11. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and assume $1 < p < \frac{n}{n-1}$ and $\beta \in]0,1[$ with $(1-\beta)p' > n$ (where p' is the dual exponent of p). Then there exists $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that*

$$\operatorname{div} \psi = \varphi \quad \text{on } B_2$$

and, setting $\gamma = 1 - \beta - n/p'$,

$$\|\psi\|_{C^{0,\gamma}(B_2)} \le C(n, \gamma, \beta, p)\|\varphi\|_{(\mathcal{H}^{\beta,p})^*}\,.$$

Proof. First of all observe that the condition $1 < p < \frac{n}{n-1}$ implies $p' > n$ so that the condition on β makes sense. Set $\zeta = \mathcal{J}_2\varphi$. Then $\zeta \in C^\infty(\mathbb{R}^n)$ satisfies

$$-\Delta\zeta + \zeta = \varphi \quad \text{on } \mathbb{R}^n \tag{4.17}$$

and we claim that

$$\|\zeta\|_{C^{1,\gamma}(\mathbb{R}^n)} \le C\|\varphi\|_{(\mathcal{H}^{\beta,p})^*}\,. \tag{4.18}$$

Indeed, set $f = \mathcal{J}_\beta \varphi \in L^{p'}(\mathbb{R}^n)$ with $\|f\|_{L^{p'}} \le C\|\varphi\|_{L^{p'}} < +\infty$, and $\mathcal{J}_{2-\beta} f = \mathcal{J}_2 \varphi = \zeta$. Observe that for any $g \in C_c^\infty(\mathbb{R}^n)$ with $\|g\|_{L^p} \le 1$ we have

$$\int_{\mathbb{R}^n} fg\,dx = \int_{\mathbb{R}^n} \varphi \mathcal{J}_\beta g\,dx \le \|\varphi\|_{(\mathcal{H}^{\beta,p})^*} \|\mathcal{J}_\beta g\|_{\mathcal{H}^{\beta,p}} \le \|\varphi\|_{(\mathcal{H}^{\beta,p})^*}\,.$$

Taking the supremum over such functions g yields $\|\zeta\|_{\mathcal{H}^{2-\beta,p'}} \le \|\varphi\|_{(\mathcal{H}^{\beta,p})^*}$. Claim (4.18) then follows by the continuous embedding (F3).

Now fix a cutoff function $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\eta \equiv 1$ on B_2 and $spt\,\eta \subset B_3$ and denote by $\bar\zeta$ the classical potential theoretic solution of $-\Delta\bar\zeta = \zeta\eta$. By classical estimates (cf. [29, Chapter 4]) we get

$$\|\nabla\bar\zeta\|_{C^{0,\gamma}(B_2)} \le C\|\zeta\eta\|_{C^{0,\gamma}(B_4)} \le C\|\zeta\|_{C^{1,\gamma}(\mathbb{R}^n)} \le C\|\varphi\|_{(\mathcal{H}^{\beta,p})^*}\,. \tag{4.19}$$

Finally we set $\psi := -\nabla(\bar\zeta + \zeta)$. Then by (4.17)

$$\operatorname{div}\psi = \zeta - \Delta\zeta = \varphi \quad \text{on } B_2\,,$$

and by (4.18) and (4.19)

$$\|\psi\|_{C^{0,\gamma}(B_2)} \le C\|\varphi\|_{(\mathcal{H}^{\beta,p})^*}\,. \qquad \square$$

The proof of (4.16) is now an immediate corollary of Theorem 4.5 and Lemma 4.11.

4.2.4 *Proof of Corollary 4.2*

Note that:

- $\deg(v_k,\Omega,\cdot)$ converges pointwise to $\deg(v,\Omega,\cdot)$ on $\mathbb{R}^n \setminus v(\partial\Omega)$;
- $v(\partial\Omega)$ is a Lebesgue null set;
- For any pair (β', p) as in (4.4) with $\beta' > \beta$ we have a uniform bound on $\|\deg(v_k,\Omega,\cdot)\|_{W^{\beta',p}}$;
- There is $R > 0$ such that $\|v\|_{C^0}, \sup_k \|v_k\|_{C^0} < R$ and thus the functions $\deg(v_k,\Omega,\cdot)$ and $\deg(v,\Omega,\cdot)$ all vanish outside $B_R(0)$.

Thus the strong convergence claimed in Corollary 4.2 follows from the compact embedding of $W^{\beta',p}(B_R(0))$ into $W^{\beta,p}(B_R(0))$.

4.3 PROOF OF THEOREM 4.3

To prove Theorem 4.3 we construct, for $p \in [1, \frac{n}{n-1}[$ and $\alpha < \frac{p(n-1)}{n}$, a map $v \in C^{0,\alpha}(B_1, \mathbb{R}^n)$ with $\deg(v, B_1, \cdot) \notin L^p(\mathbb{R}^n)$ by explicitly defining it on the boundary ∂B_1.

Since the support of the degree is bounded, clearly our map cannot belong to L^{p^*} for any p^* larger than such p. Any $C^{0,\alpha}$ extension of v to the whole $\overline{B}_1$ then does the job, since the degree only depends on the values on the boundary of the domain. The image $v(\partial B_1)$ will be the union of countably many spheres S_k with decreasing radii r_k. Each sphere S_k will be circled a certain c_k times in each direction. The goal is to choose the radii r_k and the number of circlings c_k in such a way that v is Hölder continuous with exponent $\alpha < \frac{p(n-1)}{n}$, but $\deg(v, B_1, \cdot) \notin L^p(\mathbb{R}^n)$.
Given $p \in [1, \frac{n}{n-1}[$ we define a partition $\{I_k\}_{k\geq 1}$ of the interval $[-\pi, \pi[$ as follows.
For $k \geq 1$ define the numbers

$$|I_k| = c(n,p)k^{-\left(\frac{n-1}{n}+\frac{1}{p(n-1)}\right)}, \tag{4.20}$$

where the constant $c(n,p)$ is determined by the condition $\sum_{k\geq 1} |I_k| = 2\pi$. The sets I_k are then defined by

$$I_1 = \left[\frac{-|I_1|}{2}, \frac{|I_1|}{2}\right[, \quad \text{and} \tag{4.21}$$

$$I_k = \left[\frac{-\sum_{i=1}^{k}|I_i|}{2}, \frac{-\sum_{i=1}^{k-1}|I_i|}{2}\right[\cup \left[\frac{\sum_{i=1}^{k-1}|I_i|}{2}, \frac{\sum_{i=1}^{k}|I_i|}{2}\right[, \quad \text{for } k \geq 2. \tag{4.22}$$

Note that in this way the length of the set I_k coincides with the number $|I_k|$.

For brevity (and clarity) we introduce the following map $\Phi : [-\pi, \pi[\times[0,\pi]^{n-2} \to \mathbb{R}^n$ which is the usual (almost) parametrization of the sphere:

$$\begin{aligned}\Phi(\theta_1, \dots, \theta_{n-1}) = (&\cos\theta_1, \sin\theta_1\cos\theta_2, \dots, \sin\theta_1 \cdot \dots \cdot \sin\theta_{n-2}\cos\theta_{n-1},\\ &\sin\theta_1 \cdot \dots \cdot \sin\theta_{n-2}\sin\theta_{n-1})\end{aligned}$$

The sets I_k naturally give a decomposition of the sphere ∂B_1 into

$$J_k := \Phi(I_k \times [0,\pi]^{n-2}).$$

In the rest of the proof by a slight abuse of notation we identify J_k with $I_k \times [0,\pi]^{n-2}$ and define v over the latter domains: the map Φ is a parametrization on $[-\pi,\pi[\times]0,\pi[^{n-2}$, however v will be constant on the set $[-\pi,\pi[\times\partial([0,\pi]^{n-2})$ and hence it will induce a well-defined map over the sphere.

For a given $\alpha < \frac{p(n-1)}{n}$ we then choose a number

$$q > \frac{\alpha p(n-1)^2}{n(p(n-1)-\alpha n)} \tag{4.23}$$

and define the radii

$$r_k = k^{-q} \quad \text{for } k \geq 1. \tag{4.24}$$

We then set the number of circlings to be

$$c_k = k^{\frac{qn-1}{p(n-1)}}\,, \tag{4.25}$$

which with an appropriate choice of q in (4.23) is a natural number for all k. For notational convenience we introduce the reparametrization

$$\Theta(\theta) = \begin{cases} \frac{2\pi}{|I_1|}\theta + \pi & \text{when } \theta \in I_1 \\ \frac{4\pi(c_k+\frac{1}{2})}{|I_k|}\theta + \phi_k(\theta) & \text{when } \theta \in I_k, k \geq 2\,, \end{cases}$$

where ϕ_k are phases defined by

$$\phi_k(\theta) = \pi + \pi(2c_k+1)\left(1 - \operatorname{sgn}(\theta)\frac{\sum_{i=1}^k |I_i|}{|I_k|}\right)\,, \tag{4.26}$$

which will ensure the continuity of the map.

We then introduce the centerpoints of the spheres

$$x_k = \begin{cases} (r_1, 0, \dots, 0) & \text{for } k = 1\,, \\ \left(r_k + 2\sum_{i=1}^{k-1} r_i, 0, \dots, 0\right) & \text{for } k \geq 2\,. \end{cases}$$

Finally we define

$$v(\theta_1, \dots, \theta_{n-1}) = x_k + r_k\Phi\left(\Theta(\theta_1), c_k\theta_2, \dots, c_k\theta_{n-1}\right) \quad \text{when } \theta_1 \in I_k\,. \tag{4.27}$$

The image $v(\partial B_1)$ decomposes into the union of countably many spheres $S_k = v(J_k)$ of radius r_k and centers x_k. The intersection of any S_k with S_{k+1} only contains the northpole of S_k (respectively the southpole of S_{k+1}), see Figure 2.

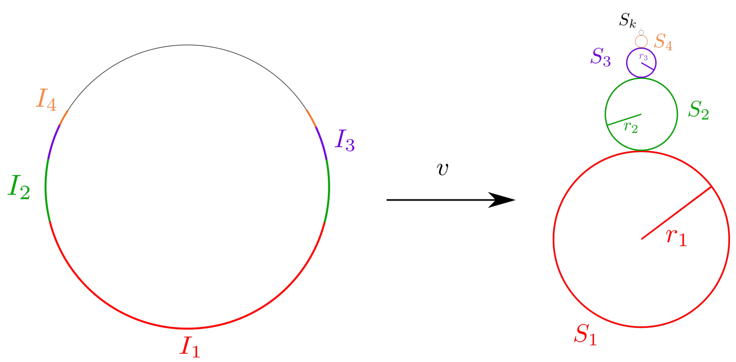

Figure 2: The map v for $n = 2$: it goes around S_1 once and traverses every S_k $2c_k + 1$ times ($c_k + 1/2$ times on each component of I_k)

We claim that $v \in C^{0,\alpha}(\partial B_1, \mathbb{R}^n)$. First observe that the choice of q in (4.23) implies

$$r_k \leq \left(\frac{|I_k|}{c_k}\right)^{\alpha}. \tag{4.28}$$

Indeed, this equation is equivalent to

$$k^{-q} \leq k^{-\alpha\left(\frac{n-1}{n}+\frac{qn}{p(n-1)}\right)},$$

which is satisfied whenever

$$q\left(1-\frac{\alpha n}{p(n-1)}\right) > \frac{\alpha(n-1)}{n},$$

i.e.

$$q > \frac{\alpha p(n-1)^2}{n(p(n-1)-\alpha n)}.$$

But inequality (4.28) guarantees the desired Hölder regularity. To see this, we first fix the angles $\theta_2, \ldots, \theta_{n-1}$ and consider variations only in the first variable. To this end we let

$$u(\theta) = v(\theta, \theta_2, \ldots, \theta_{n-1}) \quad \text{for } \theta \in [-\pi, \pi[,$$

fix $\theta, \tilde{\theta} \in [-\pi, \pi[$ and consider the following cases.

1. $\theta, \tilde{\theta} \in I_k$ for some $k \geq 1$. If $|\theta - \tilde{\theta}| \geq \frac{|I_k|}{2(c_k+1/2)} = \frac{|I_k|}{2c_k+1}$, then

   $$\frac{|u(\theta)-u(\tilde{\theta})|}{|\theta-\tilde{\theta}|^{\alpha}} \leq 2r_k\left(\frac{|I_k|}{2c_k+1}\right)^{-\alpha} \leq C,$$

   by (4.28). If however $|\theta - \tilde{\theta}| < \frac{|I_k|}{2c_k+1}$, then

$$|u(\theta)-u(\tilde{\theta})| \leq \frac{4\pi r_k(c_k+\frac{1}{2})}{|I_k|}|\theta-\tilde{\theta}| \leq \frac{4\pi r_k(c_k+\frac{1}{2})}{|I_k|}\left(\frac{|I_k|}{2c_k+1}\right)^{1-\alpha}|\theta-\tilde{\theta}|^{\alpha} \leq C|\theta-\tilde{\theta}|^{\alpha}.$$

2. $\theta \in I_{k+1}, \tilde{\theta} \in I_k$ for some $k \geq 1$. If $|\theta - \tilde{\theta}| \geq |I_k|$, then

   $$\frac{|u(\theta)-u(\tilde{\theta})|}{|\theta-\tilde{\theta}|^{\alpha}} \leq \frac{4r_k}{|I_k|^{\alpha}} \leq C.$$

   If however $|\theta - \tilde{\theta}| < |I_k|$ then they lie in adjacent intervals and we can compare with the endpoint $\theta_* = \frac{\operatorname{sgn}(\theta)\sum_{i=1}^{k}|I_i|}{2}$ to get

   $$\frac{|u(\theta)-u(\tilde{\theta})|}{|\theta-\tilde{\theta}|^{\alpha}} \leq \frac{|u(\theta)-u(\theta_*)|}{|\theta-\theta_*|^{\alpha}} + \frac{|u(\tilde{\theta})-u(\theta_*)|}{|\tilde{\theta}-\theta_*|^{\alpha}} \leq C.$$

3. $\theta \in I_{k+j}, \tilde\theta \in I_k$ for some $k \geq 1$ and $j \geq 2$. Clearly $|\theta - \tilde\theta| \geq \frac{1}{2}\sum_{i=1}^{j-1} |I_{k+i}|$ so

$$\frac{|u(\theta) - u(\tilde\theta)|}{|\theta - \tilde\theta|^\alpha} \leq 2^\alpha \frac{\sum_{i=k}^{k+j} 2r_i}{\left(\sum_{i=1}^{j-1} |I_{k+i}|\right)^\alpha} \leq 2^{1+\alpha} \left(\frac{r_k + r_{k+j}}{|I_{k+1}|^\alpha} + \sum_{i=1}^{j-1} \frac{r_i}{|I_{k+i}|^\alpha} \right)$$
$$\leq C \left(\frac{r_k}{|I_k|^\alpha} + \sum_{i=1}^{\infty} c_i^{-\alpha} \right) \leq C(p, \alpha)\,,$$

if q is chosen large enough.

The proof of the Hölder regularity is now complete in the case $n = 2$. In the more general case some extra care is needed: a similar computation yields the Hölder regularity in the variable θ_i for every $i = 2, \ldots, n-1$ but one must take into account that the map Φ is not really a parametrization of the sphere. We leave the details to the reader.

To compute the degree we introduce the natural extension $\tilde v : [0,1] \times [-\pi, \pi[\times [0, \pi]^{n-2} \to \mathbb{R}^n$ with

$$\tilde v(r, \theta_1, \ldots, \theta_{n-1}) = x_k + r \cdot r_k \Phi\left(\Theta(\theta_1), c_k\theta_2, \ldots, c_k\theta_{n-1}\right) \quad \text{when } \theta_1 \in I_k\,, \tag{4.29}$$

Then $\tilde v\left([0,1] \times J_k\right)$ is a ball B^k with boundary $\partial B^k = S_k$. Fix a $y \in \mathrm{Im}(\tilde v) \setminus \tilde v(\partial B_1)$. Then there exists a unique $k \in \mathbb{N}$ such that $y \in B^k$. We can therefore parametrize y by

$$y = x_k + r \cdot r_k \Phi(\phi_1, \ldots, \phi_{n-1}),$$

for some $r \in [0,1]$, $\phi_i \in [0, \pi]$ for $i = 1, \ldots, n-2$ and $\phi_{n-1} \in [0, 2\pi[$. By definition the degree is then given by

$$\deg(\tilde v, B_1, y) = \sum_{x \in \tilde v^{-1}(y)} \mathrm{sgn}\, \det D\tilde v(x)\,.$$

By the chain rule and the usual expression for the spherical volume element we get for a point $x = (\tilde r, \theta_1, \ldots, \theta_{n-1})$ with $\tilde v(x) = y$

$$\det D\tilde v(x) = r_k^n (r \cdot c_k)^{n-1} \frac{4\pi(c_k + \frac{1}{2})}{|I_k|} \sin^{n-2}(\Theta_1(\theta_1)) \sin^{n-3}\left(c_k\theta_2\right) \cdot \ldots \cdot \sin\left(c_k\theta_{n-2}\right)\,,$$

hence we have to investigate the sign of the sines. To this end we observe that $\tilde v(\tilde r, \theta_1, \ldots, \theta_{n-1}) = y$ if and only if

$$\begin{cases} \tilde r & = r \\ \Theta(\theta_1) & = \phi_1 + 2\pi m_1 \ \text{ for } m_1 \in \mathbb{N} \cap \left[\frac{1}{2} - \frac{\phi_1}{2\pi}, 2c_k + \frac{3}{2} - \frac{\phi_1}{2\pi}\right] \\ c_k\theta_2 & = \phi_2 + 2\pi m_2 \ \text{ for } m_2 = 1, \ldots, c_k \\ \ \vdots & \quad \vdots \\ c_k\theta_{n-1} & = \phi_{n-1} + 2\pi m_{n-1} \ \text{ for } m_{n-1} = 1, \ldots, c_k\,. \end{cases} \tag{4.30}$$

Since for $i = 1, \ldots, n-2$ the angles ϕ_i satisfy $0 \leq \phi_i \leq \pi$ this implies that sgn $\det Dv(x) = 1$ for any $x \in \tilde{v}^{-1}(y)$. Consequently, with the help of (4.30) we conclude

$$\deg(\tilde{v}, B_1, y) = \# v^{-1}(y) \geq 2c_k^{n-1}\,.$$

From this in turn we deduce

$$\int_{\mathbb{R}^n} |\deg(\tilde{v}, B_1, y)|^p \, dy \geq C \sum_{k\geq 1} r_k^n c_k^{p(n-1)} = C \sum_{k \geq 1} k^{-1} = +\infty\,,$$

by the choice of r_k and c_k in (4.24) and (4.25) respectively. To conclude the proof we extend v by keeping its $C^{0,\alpha}$ norm to the whole $\overline{B}_1$, and are left with a map $v \in C^{0,\alpha}\left(B_1, \mathbb{R}^n\right)$ such that $\deg(v, B_1, \cdot) = \deg(\tilde{v}, B_1, \cdot) \notin L^p\left(\mathbb{R}^n\right)$.

5 TOWARDS A RIGIDITY THEOREM IN $C^{1,1/2+\delta}$

As mentioned in the introduction, Borisov managed to prove the validity of the rigidity theorem of the Weyl problem (Theorem 1.2) for $C^{1,2/3+\delta}$ isometric embeddings. The key point of his proof is in fact the following statement.

Theorem 5.1. *Let (M^2, g) be a surface with C^2 metric with positive Gaussian curvature and let $u \in C^{1,\alpha}(M, \mathbb{R}^3)$ be an isometric embedding with $\alpha > \frac{2}{3}$. Then $u(M)$ is a surface with bounded extrinsic curvature in the sense of Pogorelov.*

Recall that, if u is regular enough, the Gaussian curvature of $u(M)$ can be defined as the area distortion of the Gauss map N. In a weak sense, this is a well-defined object even if the immersion is merely a C^1 map, because in this case N is continuous. The surface $u(M)$ is then said to have bounded extrinsic curvature in the sense of Pogorelov if this weak area distortion is bounded. More precisely (see p.590 in [50]):

Definition 5.2. Let $\Omega \subset \mathbb{R}^2$ be open and $u \in C^1(\Omega, \mathbb{R}^3)$ an immersion. The surface $u(\Omega)$ has *bounded extrinsic curvature in the sense of Pogorelov* if there exists a constant $C > 0$ such that

$$\sum_{i=1}^{M} |N(E_i)| \leq C$$

for every finite collection $\{E_i\}_{i=1}^M$ of pairwise disjoint closed subsets of Ω.

Using geometric arguments, in the series of papers [2–5] Borisov proved Theorem 5.1, which thanks to the works of Pogorelov and Sabitov (see [50], [52]) leads to the aforementioned extension of the rigidity theorem.

Note that if $u \in C^3$ is an isometric immersion of a 2-dimensional surface in $\mathbb{R}^3$ one can compute the area distortion of the Gauss map from the Riemann-curvature tensor, which in turn depends only on the metric. Even if the metric g is smooth, this identity is in general false if the isometry is not regular enough, as shown precisely by the Nash-Kuiper theorem. As described in the introduction, in [17] the authors show that, for $C^{1,2/3+\delta}$ isometric embeddings, the Gauss theorem can be expressed in the integral formula (1.6), which we recall here

$$\int_V f(N(x))\kappa_g(x)\, d\text{Area}(x) = \int_{S^2} f(y)\text{deg}(N, V, y)\, dy\,.$$

From this change of variables formula, Theorem 5.1 follows easily.

As mentioned in Chapter 4, Olbermann and Züst have recently (and independently) proved in [48] and [59] that the degree of a planar $C^{1/2+\delta}$ map (on a sufficiently regular domain) is in fact an L^1 function, which for the moment only allows a weaker version of (1.6), namely a version where V is assumed to be sufficiently regular and the test function f is identically 1:

$$\int_V \kappa(x)\, dA(x) \;=\; \int_{\mathbb{S}^2} \deg(N, V, y)\, d\sigma(y) \quad \forall \text{ Lipschitz open } V \subset M. \tag{5.1}$$

From here, to extend the validity of Theorem 5.1 to $C^{1,1/2+\delta}$ isometric embeddings, one is faced with the problem considered in Conjecture 1.6:

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded open set and let $N \in C^{0,\alpha}\left(\overline{\Omega}, \mathbb{R}^2\right)$ be a Hölder continuous function such that

$$\int_{\mathbb{R}^2} \deg(N, A, y)\, dy \geq 0 \quad \text{ for every open } A \subset \Omega \text{ with } \mathcal{H}^1\left(\partial A\right) < \infty\,. \tag{5.2}$$

Show that, if α is big enough, property (5.2) implies that $\deg(N, A, y) \geq 0$ for all open $A \subset \Omega$ and all $y \in \mathbb{R}^2 \setminus N(\partial A)$. Here, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. In this chapter, we show that the previous statement is true for $\alpha > \frac{2}{3}$, i.e. we prove the following theorem.

Theorem 5.3. *Let $\alpha \in]\frac{2}{3}, 1[$ and suppose $N \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ satisfies (5.2). Then $\deg(N, A, y) \geq 0$ for all open $A \subset \Omega$ and all $y \in \mathbb{R}^2 \setminus N(\partial A)$.*

We note here that in the setting where N is the Gauss map of an isometric embedding, Theorem 5.3 is a direct consequence of [17]. However, the proof heavily exploits this additional structure through the formula (1.6). Theorem 5.3 thus shows that the additional structure is not necessary.

5.1 PROOF OF THEOREM 5.3

The proof of Theorem 5.3 is based on the observation that for $\alpha > \frac{2}{3}$ we can find a change of variables formula analoguous to (1.6). Assume for a moment that $N \in C^1$ and consider the 1-form $\mu = N_1 dN_2$. The classical change of variables formula then gives

$$\int_U \psi(N)\, d\mu = \int_U \psi(N(x)) \det dN(x)\, dx = \int_{\mathbb{R}^2} \psi(y) \deg(N, U, y)\, dy\,. \tag{5.3}$$

In the following proposition we show that for $\alpha > \frac{1}{2}$ there exists a Radon measure μ which (distributionally) acts in the same way as $N_1 dN_2$ (cf. also Lemma 5.14).

Proposition 5.4. *If $\alpha > \frac{1}{2}$ and if $N \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ satisfies (5.2) then there exists a unique Radon measure $\mu : \mathcal{B}\left(\Omega\right) \to [0, +\infty[$ such that*

$$\mu(A) = \int_{\mathbb{R}^2} \deg(N, \operatorname{int}(A), y)\, dy \tag{5.4}$$

whenever $A \in \mathcal{B}(\Omega)$ *satisfies* $\overline{\dim}_b(\partial A) \leq 1$.

Here $\mathrm{int}(A)$ is the topological interior of the set A and we recall that the upper box-counting dimension is defined as

$$\overline{\dim}_b(\partial A) = \lim_{r \to 0} \frac{\log N_r}{-\log r}, \tag{5.5}$$

where N_r can be chosen to be the number of closed cubes of a mesh of $\mathbb{R}^2$ of width $r > 0$ which intersect ∂A. Moreover, we understand $\deg(f, \emptyset, \cdot) \equiv 0$.

The next proposition then shows that for $\alpha > \frac{2}{3}$ the change of variables formula (1.6) holds with respect to the measure μ.

Proposition 5.5. *Let* $\alpha > \frac{2}{3}$ *and* $U \subset \Omega$ *open. Then, for any* $\psi \in C_c^\infty\left(\mathbb{R}^2 \setminus N(\partial U)\right)$, *we have*

$$\int_U \psi(N)\, d\mu = \int_{\mathbb{R}^2} \psi(y) \deg(N, U, y)\, dy\,. \tag{5.6}$$

Theorem 5.3 now follows easily: assume, by contradiction, that there exists $U \subset \Omega$ open and $y_0 \in N(U) \setminus N(\partial U)$ with $\deg(N, U, y_0) < 0$. Let D be a disk centered at y_0 such that $\bar{D} \cap N(\partial U) = \emptyset$, and let $\psi \in C_c^\infty\left(\mathbb{R}^2\right)$ be a bump function with $\psi \equiv 0$ on $\mathbb{R}^2 \setminus D$. Since the degree is constant on connected components of $\mathbb{R}^2 \setminus N(\partial U)$ we have $\deg(N, U, y) < 0$ for all $y \in D$. However, since $\psi \in C_c^\infty\left(\mathbb{R}^2 \setminus N(\partial U)\right)$ we can apply Proposition 5.5 to get the contradiction

$$0 \leq \int_U \psi(N)\, d\mu = \int_{\mathbb{R}^2} \psi(y) \deg(N, U, y)\, dy < 0,$$

finishing the proof of Theorem 5.3. The rest of this chapter is therefore dedicated to proving Propositions 5.4 and 5.5.

5.2 PRELIMINARY RESULTS

In this section we gather some preliminary results needed in the proofs of Proposition 5.4 and 5.5.

For a continuous function $v : \Omega \to \mathbb{R}^2$ and a subset $V \subset \Omega$ the function $\deg(v, V, \cdot) : \mathbb{R}^2 \setminus v(\partial V) \to \mathbb{Z}$ denotes the Brouwer degree of $v|_V$. We recall that it is constant on connected components of $\mathbb{R}^2 \setminus v(\partial V)$ and invariant under homotopy; as a consequence $\deg(v_k, V, \cdot)$ converges pointwise to $\deg(v, V, \cdot)$ whenever $v_k \to v$ uniformly. We recall the following estimates of the L^1 norm of the degree, which follow from the proof of Theorem 4.5 and are needed in the proofs of both propositions.

Theorem 5.6. *Let* $V \subset \mathbb{R}^2$ *be a bounded, open set and* $v \in C^{0,\alpha'}\left(\overline{V}, \mathbb{R}^2\right)$. *Then*

$$\int_{\mathbb{R}^2 \setminus v(\partial V)} |\deg(v, V, y)|\, dy \leq C(\alpha') [v]^2_{C^{0,\alpha'}(\overline{V}, \mathbb{R}^2)} \int_V \mathrm{dist}(x, \partial V)^{2\alpha' - 2}\, dx\,. \tag{5.7}$$

In particular, if $\overline{\dim}_b(\partial V) < 2\alpha'$ *then* $\deg(v, V, \cdot) \in L^1\left(\mathbb{R}^2\right)$ *with*

$$\|\deg(v, V, \cdot)\|_{L^1(\mathbb{R}^2)} \leq C(\alpha', V)[v]^2_{C^{0,\alpha'}(\overline{V},\mathbb{R}^2)}. \tag{5.8}$$

The following proposition is needed to make sense of integrals of functions against the measure $\mu = N_1 dN_2$. Its proof is postponed to the appendix (see Section A.2).

Proposition 5.7. *Let* $\alpha \in]0,1[$ *and let* $U \subset \mathbb{R}^2$ *be an open, bounded set with* $d := \overline{\dim}_b(\partial U) < 2 - \alpha$. *Then the bilinear operator* $\mathcal{B}_U : C^{0,1}(\bar{U}) \times C^{0,\alpha}(\bar{U}) \to \mathbb{R}$ *defined by*

$$\mathcal{B}_U(f, g) = \int_U \frac{\partial f}{\partial x_i} g \, dx$$

satisfies the estimate

$$|\mathcal{B}_U(f, g)| \leq C(\alpha, \beta, d)[f]_{C^{0,\beta}(\bar{U})}\|g\|_{C^{0,\alpha}(\bar{U})}, \tag{5.9}$$

for any $\beta \in]1-\alpha, 1]$. *Hence, for any such* β, *it has a unique continuous extension to a bilinear operator* $\bar{\mathcal{B}}_U : C^{0,\beta}(\bar{U}) \times C^{0,\alpha}(\bar{U}) \to \mathbb{R}$. *If, in addition, U is a Lipschitz domain then also*

$$\left|\int_{\partial U} g \, df\right| \leq C(\alpha, \beta, U)[f]_{C^{0,\beta}(\partial U)}\|g\|_{C^{0,\alpha}(\partial U)}. \tag{5.10}$$

Lastly, we need the following technical estimate, which exploits the quadratic structure of the problem.

Lemma 5.8. *Let* $U \subset \mathbb{R}^2$ *be open and bounded,* $\alpha > \frac{1}{2}$, $\varepsilon > 0$ *and* $M \geq 1$. *Let* $f, g \in C^{0,\alpha}(\bar{U})$ *and assume* $\psi \in C^\infty_c(U)$ *is such that*

$$[\psi]_{C^{1,\gamma}(\bar{U})} \leq M\varepsilon^{\alpha-\gamma-1}$$

for every $\gamma \in [0, \alpha[$. *Then, for any* $\frac{1}{2} < \beta < \alpha$,

$$\left|\int_U d\psi * \varphi_\varepsilon \wedge g df - \int_U d\psi \wedge g * \varphi_\varepsilon df * \varphi_\varepsilon\right| \leq C(\alpha, \beta)M\varepsilon^{3\alpha-2\beta-1}\|f\|_{C^{0,\alpha}(\bar{U})}\|g\|_{C^{0,\alpha}(\bar{U})}, \tag{5.11}$$

where the first integral is understood in the sense of Proposition 5.7.

Also this lemma is proved in the appendix (see Section A.3). We are now ready for the proof of Proposition 5.4.

5.3 PROOF OF PROPOSITION 5.4

Consider the family of subsets

$$\mathcal{R} = \{A \subset \Omega : \overline{\dim}_b(\partial A) \leq 1\}\,.$$

Since $\mathcal{R}$ is closed with respect to set differences and finite unions it is a ring. We now define the set-function $\mu : \mathcal{R} \to [0, +\infty]$ by

$$\mu(A) = \int_{\mathbb{R}^2} \deg(N, \operatorname{int}(A), y)\, dy\,. \tag{5.12}$$

Observe that, since $\dim_{\mathcal{H}}(B) \leq \overline{\dim}_b(B)$ for all sets B for which the inequality makes sense, property (5.2) implies that $\mu \geq 0$. We need the following result (a proof of which is given in Lemma 4.7).

Lemma 5.9. *If $A \subset \Omega$ has $\overline{\dim}_b(\partial A) < 2\alpha$ then $N(\partial A)$ has Lebesgue measure zero.*

With this result at hand we can prove the following

Lemma 5.10. *For every $A, B \in \mathcal{R}$ we have*

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B)\,. \tag{5.13}$$

In particular, μ is monotone, finitely sub-additive and finitely additive on disjoint sets.

Proof. For any $A, B \in \mathcal{R}$, the sets $\operatorname{int}(A \cap B)$, $\operatorname{int}(A \setminus B)$ are open, disjoint subsets of $\operatorname{int}(A)$. Since

$$\bar{A} \setminus (\operatorname{int}(A \cap B) \cup \operatorname{int}(A \setminus B)) = \partial A \cup A \cap \partial B \subset \partial A \cup \partial B$$

it follows from Lemma 5.9 that

$$|N\,(\bar{A} \setminus (\operatorname{int}(A \cap B) \cup \operatorname{int}(A \setminus B)))\,| = 0\,.$$

Therefore, by the excision property of the degree,

$$\deg(N, \operatorname{int}(A), y) = \deg(N, \operatorname{int}(A \cap B), y) + \deg(N, \operatorname{int}(A \setminus B), y)\;,$$

almost everywhere, which gives (5.13). □

Proposition 5.4 readily follows from the following lemma and Carathéodory's extension theorem (see for example Theorem 3 in Chapter 5 of [18]).

Lemma 5.11. *If $A_i \in \mathcal{R}$ for $i \in \mathbb{N}$ are pairwise disjoint and such that $A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$, then*

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)\,.$$

To prove the latter statement we need the following

Lemma 5.12. *For every $A \in \mathcal{R}$ and every $\varepsilon > 0$ we can find an open set $O \in \mathcal{R}$ and a compact set $C \in \mathcal{R}$ such that $C \subset A \subset O$ and*

$$\mu(A \setminus C) + \mu(O \setminus A) < \varepsilon . \tag{5.14}$$

We give a proof of this lemma in the next subsection. With it we can now prove Lemma 5.11.

Proof of Lemma 5.11. For $i \in \mathbb{N}$, fix pairwise disjoint sets $A_i \in \mathcal{R}$ such that $A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$. By monotonicity and finite additivity it follows

$$\mu(A) \geq \sum_{i=1}^{M} \mu(A_i)$$

for every $M \in \mathbb{N}$ and hence also in the limit. We are therefore left to show the reverse inequality. Fix $\varepsilon > 0$. Since $A \in \mathcal{R}$ we can find a compact set $C \in \mathcal{R}$, contained in A, with $\mu(A \setminus C) < \varepsilon$. Moreover, for every $i \in \mathbb{N}$, we can choose an open set $O_i \in \mathcal{R}$, containing A_i and satisfying $\mu\left(O_i \setminus A_i\right) < \varepsilon 2^{-i}$. Consequently, $\{O_i\}_{i \in \mathbb{N}}$ is an open cover of the compact set C and therefore has a finite subcover $O_1, \ldots, O_M$. Using the finite subadditivity and the monotonicity of μ we can conclude

$$\mu(A) \leq \mu(C) + \varepsilon \leq \sum_{i=1}^{M} \mu(O_i) + \varepsilon \leq 3\varepsilon + \sum_{i=1}^{\infty} \mu(A_i) .$$

Letting $\varepsilon \to 0$ finishes the proof. □

5.3.1 *Proof of Lemma 5.12*

We need the following observation.

Lemma 5.13. *If $Q_r \subset \Omega$ is a cube of sidelength $r > 0$ then*

$$\mu(Q_r) \leq C(\Omega, \alpha, N) r^{2\alpha} .$$

Proof. This follows, by scaling, from estimate (5.8). Indeed, since Ω is open, we can find a largest cube $Q \subset \Omega$. Let $R > 0$ and $x_0 \in \Omega$ be the sidelength and the center of Q respectively. Let Q_r be any other cube in Ω and let x_1 be its center. We define the map $\tilde{N} : \bar{Q} \to \mathbb{R}^2$ by

$$\tilde{N}(x) = N\left(r \frac{x - x_0}{R} + x_1\right) .$$

It then follows that

$$\begin{aligned} \|\deg(N, \mathrm{int}(Q_r), \cdot)\|_{L^1(\mathbb{R}^2)} &= \|\deg(\tilde{N}, \mathrm{int}(Q), \cdot)\|_{L^1(\mathbb{R}^2)} \leq C(Q, \alpha)[\tilde{N}]^2_{C^{0,\alpha}(Q)} \\ &= C(Q, \alpha, R) r^{2\alpha} [N]^2_{C^{0,\alpha}(Q_r)} \leq C(\Omega, \alpha, N) r^{2\alpha} . \end{aligned}$$

□

We are now ready to prove Lemma 5.12. Fix any $A \in \mathcal{R}$. The case when $\operatorname{int}(A) = \emptyset$ is trivial, so assume $\operatorname{int}(A) \neq \emptyset$. Since Ω is open and $A \subset \Omega$ we have $\operatorname{dist}(A \cap \partial A, \partial\Omega) = \delta > 0$. Fix any $r < \frac{1}{\sqrt{2}}\delta$ and consider a mesh of $\mathbb{R}^2$ composed of closed cubes of sidelength r. Let $Q_1, \dots, Q_{M_r}$ be the cubes intersecting $A \cap \partial A$. By the choice of r we have $Q_i \subset \Omega$ for all $i = 1, \dots, M_r$. Moreover, since $\overline{\dim}_b(\partial A) = 1$ it follows that $M_r \leq \frac{1}{r}$. Now set

$$O_r = \operatorname{int}\left(A \cup \bigcup_{i=1}^{M_r} Q_i\right).$$

The set O_r is clearly open and belongs to $\mathcal{R}$. Moreover, it contains A. Indeed, every $x \in A \cap \partial A$ is by construction contained in at least one of the cubes Q_i. If $x \in \partial Q_i$ for some i then $x \in \partial Q_j$ for either one or three other cubes Q_j. In each case x is contained in the interior of the union of these cubes, hence $A \subset O_r$.

Now, since $O_r \setminus A \subset \operatorname{int}\left(\bigcup_{i=1}^{M_r} Q_i\right)$, we get, with repeated use of (5.13),

$$\mu(O_r \setminus A) \leq \mu\left(\bigcup_{i=1}^{M_r} Q_i\right) = \sum_{i=1}^{M_r} \mu(Q_i).$$

Lemma 5.13 then gives

$$\mu(O_r \setminus A) \leq C(\Omega, \alpha, N) M_r r^{2\alpha} \leq C r^{2\alpha - 1}.$$

Since $\alpha > \frac{1}{2}$, $O := O_r$ for r suitably small satisfies the required properties.
The set compact set C is constructed entirely similar:

$$C_r = \overline{A \setminus \bigcup_{i=1}^{M_r} Q_i},$$

where now, however, $Q_1, \dots, Q_{M_r}$ are the closed cubes of an $r-$width mesh of $\mathbb{R}^2$ intersecting all of ∂A. C_r is clearly compact, contained in A and, arguing as before, for a given $\varepsilon > 0$ we find $\mu(A \setminus C_r) < \varepsilon$ for r small enough. Setting $C := C_r$ finishes the proof.

5.4 PROOF OF PROPOSITION 5.5

Now that the existence of the measure μ is proved we want to show that it behaves distributionally in the same way as $N_1 dN_2$.

Lemma 5.14. *Assume U is an open subset of Ω and $f \in C_c^\infty(\Omega \setminus \partial U)$. Then, if $\alpha > \frac{1}{2}$,*

$$\int_U f \, d\mu = -\int_U df \wedge N_1 dN_2. \tag{5.15}$$

Proof. We first prove (5.15) for a standard, radially symmetric mollifier $\varphi \in C_c^\infty(U)$. Let $N^k \in C^\infty(\Omega)$ be a sequence converging to N in $C^{0,\beta}$ for some $\frac{1}{2} < \beta < \alpha$.
By Proposition 5.7 (observe that we don't need any restriction on $\overline{\dim}_b(\partial U)$ since $\operatorname{supp}(\varphi) \subset U$) it holds

$$\int_U d\varphi \wedge N_1 dN_2 = \lim_{k\to\infty} \int_U d\varphi \wedge N_1^k dN_2^k .$$

Rewriting the latter integral and using the co-area formula (more specifically Theorem 3.11 in [26]) gives

$$\int_U d\varphi \wedge N_1^k dN_2^k = \int_U N_1^k \langle \nabla N_2^k, \nabla^\perp \varphi \rangle \, dx = \int_0^{\max\varphi} \left(\int_{\{\varphi = s\}} N_1^k \left\langle \nabla N_2^k, \frac{\nabla^\perp \varphi}{|\nabla \varphi|} \right\rangle d\mathcal{H}^1 \right) ds ,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and $\nabla^\perp \varphi = \left(-\frac{\partial \varphi}{\partial x^2}, \frac{\partial \varphi}{\partial x^1} \right)$ is the rotated gradient of φ. Considering the set $\{\varphi = s\}$ as the boundary of $\{\varphi > s\}$ we can see that $\nu = \frac{\nabla \varphi}{|\nabla \varphi|}$ is the interior unit normal to $\{\varphi = s\}$. Consequently, $T = -\frac{\nabla^\perp \varphi}{|\nabla \varphi|}$ is the tangent field to $\{\varphi = s\}$ respecting orientation induced by $\{\varphi > s\}$. Therefore

$$\begin{aligned} \int_U d\varphi \wedge N_1^k dN_2^k &= \int_0^{\max\varphi} \left(\int_{\{\varphi = s\}} N_1^k \left\langle \nabla N_2^k, \frac{\nabla^\perp \varphi}{|\nabla \varphi|} \right\rangle d\mathcal{H}^1 \right) ds \\ &= -\int_0^{\max\varphi} \left(\int_{\{\varphi = s\}} N_1^k dN_2^k \right) ds . \end{aligned}$$

Next observe that if $V \subset \Omega$ is an open Lipschitz subset then, by Stokes' theorem and because of the L^1-convergence of the degree granted by Theorem 4.5,

$$\begin{aligned} \lim_{k\to\infty} \int_{\partial V} N_1^k dN_2^k &= \lim_{k\to\infty} \int_V dN_1^k \wedge dN_2^k = \lim_{k\to\infty} \int_{\mathbb{R}^2} \deg(N^k, V, y) \, dy \\ &= \int_{\mathbb{R}^2} \deg(N, V, y) \, dy = \mu(V) . \end{aligned}$$

Since $\{\varphi > s\}$ are open Lipschitz sets (they are in fact open disks of some radius r_s), we would therefore like to use the dominated convergence theorem to conclude

$$\begin{aligned} \int_U d\varphi \wedge N_1 dN_2 &= -\lim_{k\to\infty} \int_0^{\max\varphi} \left(\int_{\{\varphi = s\}} N_1^k dN_2^k \right) ds = -\int_0^{\max\varphi} \mu(\{\varphi > s\}) \, ds \\ &= -\int_U \varphi \, d\mu . \end{aligned}$$

To do this we apply estimate (5.7) to $V = \{\varphi > s\}$ for $s \in]0, \max\varphi[$, $v = N^k$ and $\alpha' = \beta$ to find

$$\begin{aligned} \|\deg(N^k, \{\varphi > s\}, \cdot)\|_{L^1(\mathbb{R}^2)} &\leq C[N^k]^2_{C^{0,\beta}(\bar{B}_{r_s})} \int_{B_{r_s}} \operatorname{dist}(x, \partial B_{r_s})^{2(\beta-1)} \, dx \\ &\leq C[N]^2_{C^{0,\alpha}(\bar{\Omega})} r_s^{2\beta} \left(\frac{1}{2\beta - 1} - \frac{1}{2\beta} \right) \\ &\leq C(\alpha, \beta, \Omega)[N]^2_{C^{0,\alpha}(\bar{\Omega})} . \end{aligned}$$

This uniform bound in s and k allows us to use dominated convergence to infer (5.15) for radially symmetric mollifiers φ which are compactly supported in U.
Now let $\varphi_\varepsilon \in C_c^\infty(B_\varepsilon)$ be the standard radially symmetric mollifier and consider $f \in C_c^\infty(\Omega \setminus \partial U)$. If ε is small enough then also $f * \varphi_\varepsilon \in C_c^\infty(\Omega \setminus \partial U)$. Using Fubini we find

$$\begin{aligned}
\int_U f * \varphi_\varepsilon \, d\mu &= \int_U \left(\int_{B_\varepsilon} f(x-z)\varphi_\varepsilon(z)\, dz \right) d\mu(x) \\
&= \int_U \left(\int_U f(y)\varphi_\varepsilon(x-y)\, dy \right) d\mu(x) \\
&= \int_U f(y) \left(\int_U \varphi_\varepsilon(x-y)\, d\mu(x) \right) dy\,.
\end{aligned}$$

If ε is small enough then for every $y \in \operatorname{supp}(f) \cap U$ we have $\varphi_\varepsilon(\cdot - y) \in C_c^\infty(U)$, and hence (5.15) holds. Therefore

$$\int_U f * \varphi_\varepsilon \, d\mu = -\int_U f(y) \left(\int_U d\varphi_\varepsilon(\cdot - y) \wedge N_1 \, dN_2 \right) dy\,.$$

Let again $N^k \in C^\infty(\Omega)$ be a sequence converging to N in $C^{0,\beta}$ for some $\frac{1}{2} < \beta < \alpha$ and define, for $y \in \operatorname{supp}(f) \cap U$,

$$g_k(y) = \int_U d\varphi_\varepsilon(\cdot - y) \wedge N_1^k \, dN_2^k\,.$$

Observe that we can replace the integration domain by a smooth, open set $\tilde{U}$ with $\operatorname{supp}(f) \cap U \subset \tilde{U} \subset U$. Proposition 5.7 then implies

$$|g_k(y)| \leq C(\alpha, \beta) \|d\varphi_\varepsilon(\cdot - y) N_1^k\|_{C^{0,\beta}(\bar{U})} [N_2^k]_{C^{0,\beta}(\bar{U})} \leq C\left(\alpha, \beta, \varepsilon, \|N\|_{C^{0,\alpha}(\bar{\Omega})}\right)\,.$$

Consequently, we can apply the dominated convergence theorem and find

$$\int_U f * \varphi_\varepsilon \, d\mu = -\int_U \lim_{k\to\infty} f(y) g_k(y)\, dy = -\lim_{k\to\infty} \int_U f(y) \left(\int_U d\varphi_\varepsilon(\cdot - y) \wedge N_1^k \, dN_2^k \right) dy\,.$$

Applying Fubini once again yields

$$\begin{aligned}
\int_U f(y) \left(\int_U d\varphi_\varepsilon(\cdot - y) \wedge N_1^k \, dN_2^k \right) dy &= \int_U d \left(\int_U f(y)\varphi_\varepsilon(\cdot - y)\, dy \right) \wedge N_1^k \, dN_2^k \\
&= \int_U d\left(f * \varphi_\varepsilon\right) \wedge N_1^k \, dN_2^k\,,
\end{aligned}$$

so that

$$\int_U f * \varphi_\varepsilon \, d\mu = -\int_U d\left(f * \varphi_\varepsilon\right) \wedge N_1 \, dN_2\,.$$

Finally, thanks to Proposition 5.7, taking the limit $\varepsilon \to 0$ shows (5.15). □

With the integration by parts formula (5.15) we can now prove Proposition 5.5. Let $U \subset \mathbb{R}^2$ open, $\psi \in C_c^\infty\left(\mathbb{R}^2 \setminus N(\partial U)\right)$ and set $N^\varepsilon = N * \varphi_\varepsilon$. If $\varepsilon > 0$ is small enough then $\operatorname{dist}\left(\operatorname{supp}\left(\psi(N^\varepsilon) * \varphi_\varepsilon\right), \partial U\right) > 0$ so that $\psi\left(N^\varepsilon\right) * \varphi_\varepsilon \in C_c^\infty\left(\Omega \setminus \partial U\right)$ is a valid test function in Lemma 5.14. Moreover, for every $\gamma \in [0, \alpha[$, we can compute

$$[\psi\left(N^\varepsilon\right)]_{C^{1,\gamma}(\bar{U})} \leq M\varepsilon^{\alpha-\gamma-1}$$

for some constant $M \geq 1$ depending on α, $\|\psi\|_{C^2(\mathbb{R}^2)}$ and $\|N\|_{C^{0,\alpha}(\bar{U},\mathbb{R}^2)}$. Because $\alpha > \frac{2}{3}$ we can find $\frac{1}{2} < \beta < \alpha$ such that $3\alpha - 2\beta - 1 > 0$. Therefore, invoking also Lemma 5.8, we have

$$\begin{aligned}\int_U \psi\left(N\right)\, d\mu &= \lim_{\varepsilon\to 0}\int_U \psi\left(N^\varepsilon\right) * \varphi_\varepsilon \, d\mu \overset{(5.15)}{=} -\lim_{\varepsilon\to 0}\int_U d\left(\psi\left(N^\varepsilon\right)\right) * \varphi_\varepsilon \wedge N_1\, dN_2 \\ &\overset{(5.11)}{=} -\lim_{\varepsilon\to 0}\int_U d\left(\psi\left(N^\varepsilon\right)\right) \wedge N_1 * \varphi_\varepsilon\, dN_2 * \varphi_\varepsilon \\ &= \lim_{\varepsilon\to 0}\int_U \psi\left(N^\varepsilon\right) dN_1^\varepsilon \wedge dN_2^\varepsilon \,.\end{aligned}$$

Finally, the change of variables formula (cf Theorem 3.9. in [26]) and the local uniform convergence of the degree imply

$$\int_U \psi\left(N\right)\, d\mu = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \psi(y)\deg(N^\varepsilon, U, y)\, dy = \int_{\mathbb{R}^2} \psi(y)\deg(N, U, y)\, dy\,.$$

Part II

THE RELAXED PROBLEM

6

$C^{1,\alpha}$ ISOMETRIC EMBEDDINGS OF POLAR CAPS

In this chapter we investigate a relaxed version of the Borisov–Gromov problem based on the classical equality of the Levi-Civita connection of a smooth submanifold of $\mathbb{R}^m$ and its tangential connection (i.e. the connection induced by the ambient Euclidean space). The outcome of our investigations is that, when we consider $C^{1,\alpha}$ isometric embeddings, the Hölder exponent $\alpha_0 = \frac{1}{2}$ is a threshold in the following sense. When $\alpha > \frac{1}{2}$ and v is a $C^{1,\alpha}$ isometric immersion of a C^2 Riemannian manifold (Σ, g), the Levi-Civita connection of (Σ, g) agrees with the tangential connection. Instead, for any $\alpha < \frac{1}{2}$ we can produce isometric immersions for which the tangential connection differs from the Levi-Civita connection. While we prove the first statement in full generality, cf. Proposition 6.5, concerning the second statement we focus instead on a particular case which, in our opinion, provides the cleanest illustration of the criticality of the exponent $\alpha = \frac{1}{2}$ in Theorem 6.2 below.

Consider the standard 2-dimensional sphere as the subset $\mathcal{S}^2 := \{x : |x|^2 = 1\} \subset \mathbb{R}^3$ and for $a \in]-1, 1[$ denote by (Σ_a, σ) the Riemannian manifold (with boundary) given by

$$\Sigma_a = \mathcal{S}^2 \cap \{x_3 \geq a\} = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^3 = 1 \text{ and } x_3 \geq a\}\,, \tag{6.1}$$

equipped with the standard metric σ as submanifold of $\mathbb{R}^3$.

Definition 6.1. We denote by $\mathscr{I}_k^\alpha(\Sigma_a)$ the space of isometric immersions $v : \Sigma_a \to \mathbb{R}^{2+k}$ of class $C^{1,\alpha}$ with the property that $v(x_1, x_2, a) = (x_1, x_2, 0, \ldots, 0)$ for all $(x_1, x_2, a) \in \partial\Sigma_a$. Moreover we denote by γ_a the circle $v(\partial\Sigma_a)$.

We recall that $\langle x, y\rangle$ denotes the scalar product of vectors $x, y \in \mathbb{R}^m$.

Theorem 6.2. *Let X be the interior unit normal to $\partial\Sigma_a$ in Σ_a and $Z : \gamma_a \to \mathbb{R}^{2+k}$ the unit vector field $Z(x_1, x_2, 0, \ldots, 0) = -(1-a^2)^{-1/2}(x_1, x_2, 0, \ldots, 0)$. For any element $v \in \mathscr{I}_k^\alpha(\Sigma_a)$ let $Y : \gamma_a \to \mathbb{R}^{2+k}$ be the vector field $v_* X$. Then the following holds*

(a) If $\alpha > \frac{1}{2}$, $-1 < a < 1$, $k \geq 1$ and $v \in \mathscr{I}_k^\alpha(\Sigma_a)$, then $\langle Y, Z\rangle = a$.

(b) For any $\alpha < \frac{1}{2}$, $0 < a < 1$ and $k \geq 12$ there is $v \in \mathscr{I}_k^\alpha(\Sigma_a)$ such that $\langle Y, Z\rangle > a$.

The proof of part (b) follows a suitable modification of the Nash–Kuiper construction, and hence (a) is an obstruction to the implementation of such methods, at least in our context where a boundary condition is imposed. Note indeed that without such restriction Källen in [41] is able to reach the threshold $C^{1,1}$: our theorem implies thus that the Nash–Kuiper construction and Källen's iteration differ in a rather nontrivial way.

We do not expect the codimension 12 for part (b) in Theorem 6.2 to have any geometric meaning, but we conjecture that the same holds in any codimension:

Conjecture 6.3. *For any $\alpha < \frac{1}{2}$ and any $0 < a < 1$ there is $v \in \mathscr{I}_1^\alpha(\Sigma_a)$ such that $\langle Y, Z\rangle > a$.*

It is possible to use the same ideas of this chapter to show that indeed conclusion (b) of Theorem 6.2 holds for every $\alpha < \alpha_0(k)$, where $\alpha_0(k)$ is an explicitely computable number. For $k = 1$ such threshold is $\frac{1}{5}$ and this can be shown quickly using some of the results of Chapter 3. We also mention here that while we were completing our work we learned that the authors in [13] were dealing with Nash–Kuiper constructions of $C^{1,\alpha}$ isometric embeddings of Riemannian manifolds which are prescribed at the boundary, although with a different purpose. The C^1 case was first settled in [37] and it was a source of inspiration for this work.

Concerning part (a) of Theorem 6.2, recall that in the codimension one case a much stronger conclusion holds if $\alpha > \frac{2}{3}$: in that case any $v \in \mathscr{I}_1^\alpha(\Sigma_a)$ must be the standard isometric embedding, namely $v(\Sigma_a) = \Sigma_a$, up to translations and rotations. This follows from Borisov's rigidity theorem (Theorem 5.1) and Pogorelov's work (in particular Theorem 8 on p. 650 of [50]).

6.1 RIGIDITY: PROOF OF THEOREM 6.2 (A)

6.1.1 *Preliminaries*

We start by recalling some well known facts in the theory of distributions. Given a closed interval $[a,b]$ we will denote by $C^{1,\alpha}_0([a,b])$ the Banach space which is the closure of $C^{1,\alpha}_c(]a,b[)$ in $C^{1,\alpha}([a,b])$. Thus $C^{1,\alpha}_0([a,b])$ is the subspace of $C^{1,\alpha}$ functions φ for which $\varphi(a) = \varphi'(a) = \varphi(b) = \varphi'(b) = 0$. If h is a continuous function, we then regard h as an element of the dual space $(C^{1,\alpha}_0([a,b]))^*$ after identifying it with the linear map

$$\varphi \mapsto \int h\varphi\,.$$

Lemma 6.4. *Let $\alpha > \frac{1}{2}$ and $[a,b] \subset \mathbb{R}$ a closed interval. Then the bilinear map*

$$\mathcal{B} : C^\alpha([a,b]) \times C^1([a,b]) \ni (f,g) \mapsto fg' \in C([a,b])$$

extends to a unique continuous bilinear map $\mathscr{B} : C^\alpha([a,b]) \times C^\alpha([a,b]) \to (C^{1,\alpha}_0([a,b]))^*$.

Proof. First of all, by translating and dilating we can assume that $[a,b] = [0,\pi]$. Secondly, every C^α function on $[0,\pi]$ can be extended to a C^α periodic function on $[-\pi,\pi]$ by reflection, whereas every $C^{1,\alpha}_0$ function on $[0,\pi]$ can be extended to a $C^{1,\alpha}$ periodic function on $[-\pi,\pi]$ by setting it equal to 0 on $[-\pi,0]$. The first extension maps C^1

functions into Lipschitz maps. If $f \in L^\infty(\mathbb{S}^1)$ and $g \in Lip(\mathbb{S}^1)$, then fg' is a well defined L^∞ function on $[-\pi, \pi]$ by Rademacher's theorem, which in turn we can identify with an element of $(C^{1,\alpha}(\mathbb{S}^1))^*$ by integration. On the other hand for maps $\varphi \in C^{1,\alpha}(\mathbb{S}^1)$ which vanish on $[-\pi, 0]$ the integral $\int fg'\varphi$ takes place only on $[0, \pi]$. We have thus reduced to prove that the bilinear map

$$C^\alpha(\mathbb{S}^1) \times \mathrm{Lip}(\mathbb{S}^1) \ni (f, g) \mapsto fg' \in (C^{1,\alpha}(\mathbb{S}^1))^*$$

extends to a unique continuous bilinear operator $\mathscr{B} : C^\alpha(\mathbb{S}^1) \times C^\alpha(\mathbb{S}^1) \to \left(C^{1,\alpha}(\mathbb{S}^1)\right)^*$. The uniqueness part is a consequence of the fact that for every $\psi \in C^\alpha(\mathbb{S}^1)$ we can find a sequence of Lipschitz maps $\{\psi_k\}$ which converge to ψ in C^β for every $\beta < \alpha$ and such that $\|\psi_k\|_{C^\alpha} \leq \|\psi\|_{C^\alpha}$. We thus just need to show the existence of a constant C such that the estimate

$$\left|\int fg'\varphi\right| \leq C\|f\|_{C^\alpha}\|g\|_{C^\alpha}\|\varphi\|_{C^{1,\alpha}} \tag{6.2}$$

holds for every triple $f \in C^\alpha$, $g \in \mathrm{Lip}$ and $\varphi \in C^{1,\alpha}(\mathbb{S}^1)$. Taking the supremum over $\varphi \in C^{1,\alpha}$ with $\|\varphi\|_{C^{1,\alpha}} \leq 1$ the latter estimate gives indeed the bound

$$\|\mathcal{B}(f,g)\|_{(C^{1,\alpha})^*} \leq C\|f\|_{C^\alpha}\|g\|_{C^\alpha} \qquad \forall (f,g) \in \mathrm{Lip} \times C^\alpha\,. \tag{6.3}$$

In turn this implies the local uniform continuity of the bilinear map $\mathcal{B}$, since we can simply use the bilinearity and the triangle inequality to estimate

$$\|\mathcal{B}(f,g) - \mathcal{B}(h,k)\|_{(C^{1,\alpha})^*} \leq \|f\|_{C^\alpha}\|g-k\|_{C^\alpha} + \|f-h\|_{C^\alpha}\|k\|_{C^\alpha}\,.$$

The existence and uniqueness of the continuous extension $\mathscr{B}$ is then an obvious fact.

We next observe that, by a standard approximation procedure, it suffices to prove the estimate (6.2) for a triple of smooth periodic functions. Indeed we remind the reader that, although C^∞ is not dense in the strong topology of C^α (nor in that of Lip), given a triple $(f, g, \varphi) \in C^\alpha \times \mathrm{Lip} \times C^{1,\alpha}$ we can find a sequence $(f_k, g_k, \varphi_k) \in C^\infty \times C^\infty \times C^\infty$ such that:

- $\lim_k \|f_k - f\|_{C^0} = 0$ and $\|f_k\|_{C^\alpha} \leq \|f\|_{C^\alpha}$;
- $g_k' \rightharpoonup^* g'$ in L^∞ and $\|g_k\|_{C^\alpha} \leq \|g\|_{C^\alpha}$;
- $\lim_k \|\varphi_k - \varphi\|_{C^0} = 0$ and $\|\varphi_k\|_{C^{1,\alpha}} \leq \|\varphi\|_{C^{1,\alpha}}$.

The conditions above are enough to infer

$$\lim_{k\to\infty} \int f_k g_k' \varphi_k = \int fg'\varphi$$

and thus it suffices to show that

$$\left|\int f_k g_k' \varphi_k\right| \leq \|f_k\|_{C^\alpha}\|g_k\|_{C^\alpha}\|\varphi_k\|_{C^{1,\alpha}}\,.$$

Fix therefore a triple $f, g, \varphi \in C^\infty(\mathbb{S}^1)$ and let

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ik\cdot x} \tag{6.4}$$

$$g(x) = \sum_{k\in\mathbb{Z}} \hat{g}_k e^{ik\cdot x} \tag{6.5}$$

$$\varphi(x) = \sum_{k\in\mathbb{Z}} \hat{\varphi}_k e^{ik\cdot x} \tag{6.6}$$

be their Fourier expansions.

We then know that the Fourier coefficients are necessarily real and that

$$\int f g' \varphi = \sum_{(k,\ell)\in\mathbb{Z}^2} i(k-\ell)\hat{f}_\ell \hat{g}_{k-\ell}\hat{\varphi}_k\,. \tag{6.7}$$

Recall next that, by Bernstein's inequality, $C^\alpha \subset H^\beta$ for every $\beta < \alpha$, where H^β denotes the fractional Sobolev space $W^{\beta,2}$. Thus

$$\sum_k (1+|k|^{2\beta})|\hat{f}_k|^2 \le C(\alpha,\beta)\|f\|^2_{C^\alpha} \qquad \forall \beta < \alpha \tag{6.8}$$

$$\sum_k (1+|k|^{2\beta})|\hat{g}_k|^2 \le C(\alpha,\beta)\|g\|^2_{C^\alpha} \qquad \forall \beta < \alpha\,. \tag{6.9}$$

We finally need the simple estimate

$$|\hat{\varphi}_k| \le C\|\varphi\|_{C^{1,\alpha}}(1+|k|)^{-1-\alpha} \tag{6.10}$$

We are now ready to conclude and we start observing

$$\begin{aligned}
\left|\sum_\ell i(k-\ell)\hat{f}_\ell \hat{g}_{k-\ell}\right| &\le |2k|^{1-\beta} \sum_{-k\le\ell\le k} |k-\ell|^\beta |\hat{g}_{k-\ell}||\hat{f}_\ell| \\
&\quad + \sqrt{2}\sum_{\ell\le -k,\ell\ge k} \sqrt{|k-\ell|}\sqrt{|\ell|}|\hat{g}_{k-\ell}||\hat{f}_\ell| \\
&\le |2k|^{1-\beta}\left(\sum_j |j|^{2\beta}|\hat{g}_j|^2\right)^{1/2}\left(\sum_j |\hat{f}_j|^2\right)^{1/2} \\
&\quad + \sqrt{2}\left(\sum_j |j||\hat{g}_j|^2\right)^{1/2}\left(\sum_j |j||\hat{f}_j|^2\right)^{1/2} \\
&\le C(1+|k|)^{1-\beta}\|f\|_{C^\alpha}\|g\|_{C^\alpha}\,.
\end{aligned} \tag{6.11}$$

Combining (6.7), (6.10) and (6.11) we then conclude

$$\begin{aligned}
\left|\int f g'\varphi\right| &\le C\|f\|_{C^\alpha}\|g\|_{C^\alpha}\|\varphi\|_{C^{1,\alpha}}\sum_k (1+|k|)^{-\alpha-\beta} \\
&\le C\|f\|_{C^\alpha}\|g\|_{C^\alpha}\|\varphi\|_{C^{1,\alpha}}\,,
\end{aligned} \tag{6.12}$$

where we have used that, since we are free to choose any $\beta < \alpha$ and $\alpha > \frac{1}{2}$, we can impose $\alpha + \beta > 1$, which ensures the convergence of the series $\sum_k (1+|k|)^{-\alpha-\beta}$. □

6.1.2 *Connection*

Consider now a C^2 Riemannian manifold (Σ, g) with C^2 boundary, a C^2 curve $\gamma : [a,b] \to \Sigma$ and a C^1 vector field W along γ. In local coordinates we can write

$$W(t) = \sum_i W^i(t) \frac{\partial}{\partial x_i}, \tag{6.13}$$

$$\dot\gamma(t) = \sum_i \dot\gamma^i(t) \frac{\partial}{\partial x_i}. \tag{6.14}$$

We then know that $\nabla_{\dot\gamma} W$ is given by the formula

$$\frac{dW^i}{dt}\frac{\partial}{\partial x_i} + \sum_{j,k} \Gamma^i_{jk}(\gamma)\, \dot\gamma^j\, W^k \frac{\partial}{\partial x_i}, \tag{6.15}$$

where the C^1 functions Γ^i_{jk} are the Christoffel symbols of the metric g.

Let $u : \Sigma \to \mathbb{R}^m$ be a $C^{1,\alpha}$ isometric immersion. The vector field $u_* W = \sum W^i \frac{\partial u}{\partial x_i}$ can thus be seen as a C^α map $u_* W : [a,b] \to \mathbb{R}^m$. In particular, if $\alpha > \frac{1}{2}$ we can use Lemma 6.4 to make sense of the scalar product

$$\left\langle \frac{d}{dt} u_* W, \frac{\partial u}{\partial x_\ell} \right\rangle . \tag{6.16}$$

For smooth isometric immersions (6.16) and (6.15) are then related by the identity

$$\left\langle \frac{d}{dt}(u_* W(\gamma)), \frac{\partial u}{\partial x_\ell}(\gamma) \right\rangle = \sum_i \left(\frac{d}{dt}(W^i(\gamma)) + \sum_{j,k} \Gamma^i_{jk}(\gamma)\, \dot\gamma^j\, W^k(\gamma) \right) g_{i\ell}(\gamma) . \tag{6.17}$$

The latter is just the classical relation between the Levi-Civita connection and the tangential connection. Lemma 6.4 allows not only to make sense of the left hand side of the identity for $C^{1,\alpha}$ immersions when $\alpha > \frac{1}{2}$, but it also implies that, under the same regularity assumption, the identity (6.17) remains valid.

Proposition 6.5. *Let (Σ, g) be a C^2 Riemannian manifold with C^2 boundary, let $\gamma : [a,b] \to \Sigma$ be a C^2 curve, let W be a C^1 vector field along γ and let $u : \Sigma \to \mathbb{R}^m$ be an isometric immersion of class $C^{1,\alpha}$ for some $\alpha > \frac{1}{2}$. Then* (6.17) *holds.*

The proof of the proposition is postponed to the end of the section. We now show how Theorem 6.2(a) follows from it.

Proof of Theorem 6.2(a). The proposition implies part (a) of Theorem 6.2 right away. Indeed, fix a point $p \in \partial\Sigma_a$ and choose local coordinates in a neighborhood U of p so that $X = \frac{\partial}{\partial x_2}$ on U and $\frac{\partial}{\partial x_1}$ is tangent to Σ_a. Choose then W tangent to Σ_a and parametrize the curve $\gamma = \Sigma_a$ so that $\frac{d}{dt} v_* W = Z$. If we first use (6.17) for the standard embedding, we easily see that

$$\sum_i \left(\frac{d}{dt}(W^i(\gamma)) + \sum_{j,k} \Gamma^i_{jk}(\gamma)\, \dot\gamma^j\, W^k \right) g_{i2}(\gamma) = a\,.$$

If we then use it for $u = v$ we conclude

$$\langle Y, Z \rangle = \left\langle \frac{\partial v}{\partial x_2}(\gamma), \frac{d}{dt}(v_* W(\gamma)) \right\rangle = a\,. \qquad \square$$

In order to prove the above proposition we recall the quadratic estimate in [17, Proposition 1.6], which follows from estimate (2.12):

Lemma 6.6 (Quadratic estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $v \in C^{1,\alpha}(\Omega, \mathbb{R}^m)$ with $v^\sharp e \in C^2$ and $\varphi \in C^\infty(\mathbb{R}^n)$ a standard symmetric convolution kernel. Then, for every compact set $K \subset \Omega$*

$$\|(v * \varphi_\epsilon)^* e - v^* e\|_{C^1(K)} = O(\epsilon^{2\alpha - 1})\,.$$

Proof of Proposition 6.5. First observe that without loss of generality we can assume that W is defined on the whole manifold. Secondly, observe that it suffices to prove the identity for curves γ which lie in the interior. Consider indeed a C^2 curve γ which touches the boundary of the manifold and approximate it in C^2 with a sequence of curves γ_j which are contained in the interior. Then the maps $W(\gamma_j)$ converge in C^1 to $W(\gamma)$. As such, the maps $u_* W(\gamma_j)$ are uniformly bounded in C^α and converge in $C^{\bar\alpha}$ to $u_* W(\gamma)$ for every $\bar\alpha < \alpha$. Since we can choose $\bar\alpha > \frac{1}{2}$, Lemma 6.4 implies that the distributions

$$\left\langle \frac{d}{dt}(u_* W(\gamma_j)), \frac{\partial u}{\partial x_\ell}(\gamma_j) \right\rangle$$

converge to the distribution

$$\left\langle \frac{d}{dt}(u_* W(\gamma)), \frac{\partial u}{\partial x_\ell}(\gamma) \right\rangle\,. \tag{6.18}$$

Moreover, obviously

$$\frac{d}{dt}(W^i(\gamma_j)) + \sum_{k,\ell} \Gamma^i_{k\ell}(\gamma_j)\dot\gamma_j^k W^\ell(\gamma_j)$$

converge uniformly to

$$\frac{d}{dt}(W^i(\gamma)) + \sum_{k,\ell} \Gamma^i_{k\ell}(\gamma)\dot\gamma^k W^\ell(\gamma) \tag{6.19}$$

Fix now a curve γ in the interior and a coordinate patch U compactly contained in another coordinate patch V, both not intersecting the boundary of the manifold. We can smooth u by convolution with a standard kernel by $u * \varphi_\varepsilon$. For ε small enough the convolution is well defined on the coordinate patch U. Clearly the maps $(u * \varphi_\varepsilon)_* W$ and $(u * \varphi_\varepsilon)_* \frac{\partial}{\partial x_i}$ are uniformly bounded in C^α and converge, as $\varepsilon \downarrow 0$, to $u_* W$ and $u_* \frac{\partial}{\partial x_i}$ in C^β for every $\beta < \alpha$. Choosing a $\beta > \frac{1}{2}$ we apply Lemma 6.4 to conclude that the distributions

$$\left\langle \frac{d}{dt}(((u * \varphi_\varepsilon)_* W)(\gamma)), \frac{\partial (u * \varphi_\varepsilon)}{\partial x_i}(\gamma) \right\rangle \tag{6.20}$$

converge (weakly in the sense of distributions) to (6.18). On the other hand, from Lemma 6.6, if $\Gamma^i_{\varepsilon,k,\ell}$ denote the Christoffel symbols of the metric $(u * \varphi_\varepsilon)^* e$, then we conclude that they converge uniformly to $\Gamma^i_{k,\ell}$. Thus

$$\frac{d}{dt}(W^i(\gamma)) + \sum_{k,\ell} \Gamma^i_{\varepsilon,k,\ell}(\gamma)\dot{\gamma}^k W^\ell(\gamma) \tag{6.21}$$

converge uniformly to (6.18) and $[(u * \varphi_\varepsilon)^* e]_{ij}$ converges uniformly to g_{ij}. In particular,

$$\sum_i \left(\frac{d}{dt}(W^i(\gamma)) + \sum_{k,\ell} \Gamma^i_{\varepsilon,k,\ell}(\gamma)\dot{\gamma}^k W^\ell(\gamma) \right) [(u * \varphi_\varepsilon)^* e]_{i\ell}(\gamma) \tag{6.22}$$

converge uniformly to the right hand side of (6.17). However, since u_ε is smooth, (6.20) and (6.22) are equal by classical differential geometry. Letting $\varepsilon \downarrow 0$ we then conclude (6.17). □

6.2 FLEXIBILITY: PROOF OF THEOREM 6.2 (B)

The maps v violating the rigidity are produced by convex integration. Their construction relies on the following more general theorem, the proof of which is the content of most of the remaining sections. Recall that an immersion u is called *strictly short* if $g - u^\sharp e$ is positive definite.

Theorem 6.7. *Fix two integers $n \geq 2$, $m \geq n(n+2)$ and a metric $g \in C^2$ on $\bar{B}_1 \subset \mathbb{R}^n$. There exists $\bar{\sigma}_0 > 0$ such that if $u \in C^\infty(\bar{B}_1, \mathbb{R}^m)$ and $h \in C^\infty(\bar{B}_1)$ are such that*

$$h \equiv h(|x|) > 0 \text{ on } B_1,\ h(1) = 0 \text{ and } h'(1) \neq 0 \tag{6.23}$$

$$u \text{ is strictly short in } B_1 \text{ and} \tag{6.24}$$

$$(1 - \bar{\sigma}_0)he \leq g - u^\sharp e \leq (1 + \bar{\sigma}_0)he \text{ in a neighborhood of } \partial B_1\,, \tag{6.25}$$

then for every $\alpha < \frac{1}{2}$, *every constant* $x_0 \in \mathbb{R}^{n(n+1)}$ *and every* $\varepsilon > 0$ *there exists a map* $v \in C^{1,\alpha}(\bar{B}_1, \mathbb{R}^{m+n(n+1)})$ *such that*

$$\|v - (u, x_0)\|_{C^0(\bar{B}_1, \mathbb{R}^{m+n(n+1)})} < \varepsilon\,,$$
$$v = (u, x_0) \text{ and } \nabla v = (\nabla u\, 0)^{\mathsf{T}} \text{ on } \partial B_1$$
$$g = v^\sharp e\,.$$

In addition, if u is injective then v can be chosen to be injective as well.

If we manage to construct h and u satisfying (6.23)–(6.25) and, in addition, violating the rigidity at the boundary then we are done since the derivatives of v and u agree at the boundary.
Fix $R > 1$ and consider the scaled spherical cap $\bar{\Sigma}_R \subset \mathbb{R}^3$ given as the image of $\Phi : \bar{B}_1 \to \mathbb{R}^3$, where $\Phi(x_1, x_2) = (x_1, x_2, \sqrt{R^2 - x_1^2 - x_2^2} - \sqrt{R^2 - 1})$. We use polar coordinates to define the map $u : \bar{B}_1 \to \mathbb{R}^8$ by

$$u(r, \theta) = (\varphi(r)\cos\theta, \varphi(r)\sin\theta, 0, \dots, 0)\,, \tag{6.26}$$

where $\varphi \in C^\infty([0,1])$ is a suitable reparametrization such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(1) = \frac{R}{\sqrt{R^2-1}}$, and, for every $r \in]0,1[$,

$$\frac{R^2}{R^2 - r^2} - \varphi'(r)^2 > 0\,, \tag{6.27}$$
$$r^2 - \varphi(r)^2 > 0\,. \tag{6.28}$$

Observe that, once we produce such a φ, the map u is strictly short in $\mathring{B}_1$ (except maybe in the origin, where the polar coordinates are not suited to the problem) and isometric on the boundary. Indeed, the metric induced by u is given in polar coordinates by

$$u^\sharp e = \varphi'^2 dr^2 + \varphi^2 d\theta^2\,,$$

whereas the metric on Σ_R which is induced by the inclusion into $\mathbb{R}^3$ reads

$$g = \frac{R^2}{R^2 - r^2} dr^2 + r^2 d\theta^2\,.$$

Hence, the shortness away from the origin is given by (6.27) and (6.28) whereas the isometry on the boundary is apparent from the values $\varphi(1)$ and $\varphi'(1)$. In the following, we construct a piecewise smooth function $\tilde{\varphi}$ satisfying the above assumptions; smoothing out the corners will then provide φ. We abbreviate $\gamma := \frac{R}{\sqrt{R^2-1}}$. Because $R > 1$ we can fix a positive $\eta \in]2 - \gamma, 1[$. Since $\eta + \gamma > 2$ we can then find $\varepsilon > 0$ small enough such that

$$0 < 1 - \varepsilon(\eta + \gamma) + \frac{\varepsilon^2}{2}(1 - \frac{1}{\gamma} + \gamma^3 R^{-2}) \leq (1 - 2\varepsilon)\left(1 - \left(\varepsilon R^{-1}\right)^2\right)^{-1/2}\,, \tag{6.29}$$

as one can see by expanding $(1+x^2)^{-1/2}$ around $x=0$. Set

$$\beta := \frac{1-\varepsilon(\eta+\gamma)+\frac{\varepsilon^2}{2}(1-\gamma^{-1}+\gamma^3R^{-2})}{1-2\varepsilon},$$

and define the piecewise continous

$$\phi(r) = \begin{cases} \eta\,, & \text{for } r\in[0,\varepsilon[\\ \beta\,, & \text{for } r\in[\varepsilon,1-\varepsilon[\\ \gamma-(1-\gamma^{-1}+\gamma^3R^{-2})(1-r)\,, & \text{for } r\in[1-\varepsilon,1]\,. \end{cases}$$

The definition of β ensures that

$$\begin{aligned}\int_0^1 \phi(r)dr &= \eta\varepsilon+\beta(1-2\varepsilon)+\varepsilon(\gamma-(1-\gamma^{-1}+\gamma^3R^{-2})) \\ &\quad + \frac{1}{2}(1-\gamma^{-1}+\gamma^3R^{-2})\varepsilon(2-\varepsilon) \\ &= \varepsilon(\eta+\gamma)+(1-2\varepsilon)\beta-\frac{1}{2}(1-\gamma^{-1}+\gamma^3R^{-2})\varepsilon^2 = 1\,.\end{aligned}$$

Consequently, setting $\tilde\varphi(r)=\int_0^r\phi(s)ds$ yields a continuous, piecewise smooth function with $\tilde\varphi(1)=1$ and $\tilde\varphi'(1)=\gamma=\frac{R}{\sqrt{R^2-1}}$. We claim that $\tilde\varphi$ satisfies (6.27) and (6.28). Indeed, on $]0,\varepsilon[$ this is provided by the fact that $\eta<1$. Moreover, if ε is small enough then $\beta<1$ which, together with (6.29), shows the inequalites on $[\varepsilon,1-\varepsilon[$. If ε is small enough, (6.27) holds on $]1-\varepsilon,1]$ since

$$\begin{aligned}\frac{d}{dr}\bigg|_{r=1}\left(\frac{R^2}{R^2-r^2}-\tilde\varphi'(r)^2\right) &= \frac{2R^2}{(R^2-1)^2}-2\phi(1)\phi'(1) \\ &= 2(\gamma^4R^{-2}-\gamma(1-\gamma^{-1}+\gamma^3R^{-2}))<0\,,\end{aligned}$$

and

$$\frac{R^2}{R^2-1}-\tilde\varphi'(1)^2=0\,.$$

Finally, on $[1-\varepsilon,1]$ we have

$$\tilde\varphi' \geq \gamma-\varepsilon(1-\gamma^{-1}+\gamma^3R^{-2})\,.$$

In particular, for ε small enough we have $\tilde\varphi'>1$ on $[1-\varepsilon,1]$. Since $\tilde\varphi(1)=1$, the latter implies that $\tilde\varphi(r)<r$ on $[1-\varepsilon,1[$, thus concluding the proof of (6.28).

Consequently, if u is defined by (6.26) then it is isometric on ∂B_1 and strictly short in $B_1\setminus\{0\}$. To show that it is also strictly short in the origin we switch to euclidean coordinates and observe that $u(x_1,x_2)=(\eta x_1,\eta x_2,0)$ if $|x|<\varepsilon$. Hence

$$\begin{aligned}g-u^\sharp e &= \left(1-\eta^2+\frac{x_1^2}{R^2-|x|^2}\right)dx_1^2+\left(1-\eta^2+\frac{x_2^2}{R^2-|x|^2}\right)dx_2^2 \\ &\quad +2\frac{x_1x_2}{R^2-|x|^2}dx_1dx_2\,.\end{aligned}$$

The shortness around the origin then again follows from $\eta < 1$. Lastly, we define

$$h(r) = 2(\gamma - 1)(1 - r)\,.$$

Obviously, (6.23) is satisfied and we claim that, sufficiently close to ∂B_1, also (6.25) holds. For this we again consider the terms in polar coordinates. Expanding around $r = 1$ gives

$$1 - \left(\frac{\varphi}{r}\right)^2 = 2(\gamma - 1)(1 - r) + o(|1 - r|)\,,$$

and

$$\begin{aligned}\frac{R^2}{R^2 - r^2} - \varphi'^2 &= \gamma^2 + 2\gamma^4 R^{-2}(r-1) - \gamma^2 - 2\gamma(1 - \gamma^{-1} + \gamma^3 R^{-2})(r-1) \\ &\quad + o(|r-1|) \\ &= 2\gamma(r-1)(\gamma^3 R^{-2} - (1 - \gamma^{-1} + \gamma^3 R^{-2})) + o(|r-1|) \\ &= 2(\gamma - 1)(1 - r) + o(|r-1|)\,.\end{aligned}$$

This shows that

$$\begin{aligned} g - u^\sharp e - he &= \left(\frac{R^2}{R^2 - r^2} - \varphi'^2 - h\right) dr^2 + r^2\left(1 - \left(\frac{\varphi}{r}\right)^2 - h\right) d\theta^2 \\ &= o(|r-1|)e\,,\end{aligned}$$

hence (6.25) is satisfied. Now fix $\alpha < \frac{1}{2}$. Then Theorem 6.7 can be applied to find an isometric immersion $v = (\underline{v}, w) \in C^{1,\alpha}\left(\bar{B}_1, \mathbb{R}^{8+6}\right)$ such that on ∂B_1 we have $\nabla \underline{v} = \nabla u$, $w = 0$ and $\nabla w = 0$.

We now consider the appropriate rescaling of the map v by R, namely $\frac{v}{R}$, which induces an isometric embedding of Σ_a for $a = \sqrt{1 - R^{-2}}$. Since the map is an isometry, the vector $Y = v_* X$ has the same length as the vector X, namely $|X| = 1$. Observe, moreover, that by construction such vector field is in fact parallel to the vector field Z and it has positive scalar product with it. In particular we conclude that $\langle Y, Z\rangle = 1$.

6.3 TOWARDS A PROOF OF THEOREM 6.7: MAIN ITERATION

The proof of Theorem 6.7 is based on the iteration scheme developed by J. Nash in [45] to prove his counterintuitive result about the existence of C^1 isometric embeddings of n-dimensional manifolds into Euclidean space with suprisingly low codimension $n+1$ (Theorem 1.3). We need to adapt the scheme in two ways. First of all, in its original state it only produces maps which are C^1. As mentioned above, the first improvement is due to the work [17] (with which one is able to construct $C^{1,1/1+n(n+1)-\delta}$ isometric embeddings), while in Chapter 3 we showed how to improve the threshold further in the case of two dimensional disks. As realised by A. Källén in [41], more regular isometric embeddings can be produced at the expense of increasing the codimension.

Secondly, the iteration process needs to keep the boundary values fixed. This can be achieved, as done in [37], by multiplying the perturbations by cutoff functions which are suited to the iteration scheme (see Lemma 6.11). The following proposition is the main building block of the iteration.

Proposition 6.8. *Let $n \geq 2$, $m \geq n(n+2)$, $\lambda > 0$ and fix an embedding $\tilde{u} \in C^\infty(\bar{B}_1, \mathbb{R}^m)$. There exist constants $\sigma_0 \in]0, \frac{1}{2}[$, $R(\lambda) \geq 1$, $\Lambda(R) \geq 1$ and $C_0(\tilde{u}, \Lambda) \geq 1$ such that the following holds. Fix $c > b > 1$ and*

$$a > a_0(b, c, \sigma_0, \tilde{u}, \lambda, R, \Lambda, C_0),$$

and define

$$\delta_q = a^{-b^q}, \quad \lambda_q = a^{cb^{q+1}}.$$

Assume $\tilde{g} \in C^2$ is a metric on $\bar{B}_1$ with

$$[\tilde{g}]_k \leq C_0(1 + \delta_1^{1-k}) \qquad \text{for } k = 0, 1, 2, \tag{6.30}$$

and suppose $v_q \in C^\infty(\bar{B}_1, \mathbb{R}^m)$ and $h_q \in C^\infty(\bar{B}_1)$ are such that

$$v_q = \tilde{u} \text{ on } \bar{B}_1 \setminus B_{1-R\delta_{q+1}}, \ \|v_q - \tilde{u}\|_1 < C_0 \sum_{k=1}^{q} \delta_k^{1/2}, \ [v_q]_2 \leq C_0 \delta_q^{1/2} \lambda_q, \tag{6.31}$$

$$h_q \text{ is linear on } \bar{B}_1 \setminus B_{1-R\delta_{q+1}} \text{ with } h_q(1) = 0,\ h_q'(1) = -\lambda$$
$$\text{and } \Lambda^{-1}\delta_{q+1} \leq h_q \leq \Lambda\delta_{q+1} \text{ on } \bar{B}_{1-R\delta_{q+1}}, \tag{6.32}$$

$$[h_q]_k \leq C_0 \delta_{q+1}^{1-k} \text{ for } k = 0, 1, 2, 3, \text{ and} \tag{6.33}$$

$$(1 - \sigma_0(1+\eta_q))h_q e \leq \tilde{g} - v_q^\sharp e \leq (1 + \sigma_0(1+\eta_q))h_q e \text{ on } \bar{B}_1, \tag{6.34}$$

where $\eta_q \in C_c^\infty(\bar{B}_1)$ is a radially symmetric cutoff function with $\eta_q \equiv 0$ on $\bar{B}_1 \setminus B_{1-R\delta_{q+1}}$, $\eta_q \equiv 1$ on $\bar{B}_{1-(R+1)\delta_{q+1}}$ and taking values between 0 and 1 (cf. Lemma 6.11 for the definition of the cutoffs). We can then find $v_{q+1}, h_{q+1}, \eta_{q+1}$ satisfying (6.31)–(6.34) *with q replaced by $q+1$ and, in addition, the following estimates hold:*

$$\|v_{q+1} - v_q\|_0 \leq C_0 \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}}, \tag{6.35}$$

$$[v_{q+1} - v_q]_1 \leq C_0 \delta_{q+1}^{1/2}. \tag{6.36}$$

6.4 PROOF OF PROPOSITION 6.8: PRELIMINARIES

6.4.1 *Existence of normals*

The following proposition claims the existence of an orthonormal family of normal vectorfields to the embedded surface together with the appropriate estimates (6.39). It is already contained in [41], but our condition on the co-dimension is less restrictive ($d \geq 1$ as opposed to $d \geq n+1$). The reason for this is that in the proof we use Lemma A.11 in the appendix instead of Lemma 2.5 of [41]. The rest of the proof is essentially unchanged. For the reader's convenience we provide the details in Section A.4.1 of the appendix.

Proposition 6.9. *Let $n \geq 2$, $d \geq 1$, B a set diffeomorphic to the closed unit ball of $\mathbb{R}^n$ and $u \in C^\infty\left(B, \mathbb{R}^{n+d}\right)$ an immersion. There exists $\rho_0 \equiv \rho_0(d,n,u) > 0$ and constants C_k depending only on u such that the following holds. If $v \in C^\infty\left(B, \mathbb{R}^{n+d}\right)$ is such that*

$$\|v-u\|_{C^1} < \rho_0\,,$$

then there exist $\zeta_1(v), \dots, \zeta_d(v) \in C^\infty\left(B, \mathbb{R}^{n+d}\right)$ such that for all $1 \leq i,j \leq d$ we have

$$\langle \zeta_i(v), \zeta_j(v)\rangle = \delta_{ij} \quad \text{on } B \tag{6.37}$$

$$\nabla v \cdot \zeta_i(v) = 0 \qquad \text{on } B \tag{6.38}$$

and

$$[\zeta_i(v)]_k \leq C_k(1 + \|v\|_{k+1})\,. \tag{6.39}$$

6.4.2 *Decomposition of the metric error*

We use the following decomposition of the metric error, in the spirit of Lemma 2.3 in [41]. The proof is a simple application of the implicit function theorem and is provided in the appendix. Recall that $n_* = n(n+1)/2$.

Proposition 6.10. *There exists $r_0 > 0$ and $\nu_1, \dots, \nu_{n_*} \in \mathbb{S}^{n-1}$ with the following property. If $\tau : \bar{B}_1 \to Sym_n^+$ and $\{M_i\}_{i=1,\dots,n_*}$, $\{G_{ij}\}_{i,j=1,\dots,n_*} \subset C^\infty(\bar{B}_1, Sym_n)$ are such that*

$$\|\tau - Id\|_0 + \sum_{i=1}^{n_*} \|M_i\|_0 + \sum_{i,j=1}^{n_*} \|G_{ij}\|_0 < r_0\,,$$

then there exist smooth functions $c_1, \dots, c_{n_*} : \bar{B}_1 \to \mathbb{R}$ with

$$\tau(x) = \sum_{i=1}^{n_*} c_i^2(x)\nu_i \otimes \nu_i + \sum_{i=1}^{n_*} c_i(x) M_i(x) + \sum_{i,j=1}^{n_*} c_i(x)c_j(x)G_{ij}(x) \tag{6.40}$$

and $c_i(x) > r_0$ on $\bar{B}_1$, and for any $\Omega \subset \bar{B}_1$

$$\|c_i\|_{k,\Omega} \leq C_k \left(1 + \|\tau\|_{k,\Omega} + \sum_{i=1}^{n_*} \|M_i\|_{k,\Omega} + \sum_{i,j=1}^{n_*} \|G_{ij}\|_{k,\Omega}\right)\,. \tag{6.41}$$

6.4.3 *Cutoff functions*

In order to keep the boundary values the same along the iteration we will multiply the perturbations with a suitable cutoff function. The following lemma clarifies the type of cutoff we will use and its most important properties.

Lemma 6.11. *There exist universal constants $\varepsilon > 0$, $C \geq 1$ and a sequence of radially symmetric cutoff functions $(\eta_q)_{q\in\mathbb{N}} \subset C_c^\infty(\bar{B}_1)$ such that for any $q \in \mathbb{N}$ we have*

$$\eta_q \equiv 1 \text{ on } \bar{B}_{1-(R+1)\delta_{q+1}} \text{ and } \eta_q \equiv 0 \text{ on } \bar{B}_1 \setminus B_{1-R\delta_{q+1}}, \tag{6.42}$$

$$[\eta_q]_k \leq C\delta_{q+1}^{-k} \text{ for } k \geq 0, \tag{6.43}$$

$$\eta_q \leq \varepsilon \Rightarrow |\nabla\eta_q^\intercal \nabla\eta_q| \leq C\delta_{q+1}^{-2}\eta_q. \tag{6.44}$$

Proof. Define $f \in C^0(\mathbb{R})$ by $f \equiv 0$ on $]-\infty, \frac{1}{4}]$, $f \equiv 1$ on $[\frac{3}{4}, +\infty[$ and linear in between. Smoothing out the corners by mollifying f with a standard mollifying kernel φ_ℓ with parameter $\ell < \frac{1}{4}$ we find a function $h = f * \varphi_\ell \in C^\infty(\mathbb{R})$ satisfying $h \equiv 0$ on $]-\infty, 0]$ and $h \equiv 1$ on $[1, +\infty[$. Also, since $h''(r) \to 0$ as $r \to 0$, we can find $\varepsilon > 0$ such that

$$h \leq \varepsilon \Rightarrow (h')^2 \leq h.$$

The sequence η_q is then easily constructed by setting, for $x \in \bar{B}_1$,

$$\eta_q(x) := h\left(\delta_{q+1}^{-1}\left(1 - R\delta_{q+1} - |x|\right)\right). \qquad \square$$

6.4.4 *Parameters*

To counteract the loss of derivatives appearing along the iteration we mollify the map by convolution with a standard kernel so that we can control higher derivatives with the mollification parameter ℓ. However, we have to make sure that this parameter is chosen small enough to keep the metric error (6.34) of the same size. It turns out that the right choice is

$$\ell := \frac{1}{\tilde{C}} \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q}, \tag{6.45}$$

where $\tilde{C} \geq 1$ is a universal constant, depending additionally on $\tilde{u}$, $\tilde{g}$, R, Λ and C_0, which will be chosen in Lemma 6.12. In the course of the proof we will need the following hierarchy of the parameters

$$\delta_{q+1}^{-1} \leq \delta_{q+2}^{-1} \leq \ell^{-1} \leq \lambda_{q+1}. \tag{6.46}$$

The first inequality is true by definition, while the second follows from

$$\begin{aligned}\log_a(\delta_{q+2}\ell^{-1}) = \log_a\left(\tilde{C}\delta_q\delta_{q+1}^{-1/2}\delta_{q+2}\lambda_q\right) &> -\frac{1}{2}b^q + \left(c+\frac{1}{2}\right)b^{q+1} - b^{q+2}\\ &= b^q\left(\frac{1}{2}(b-1) + b(c-b)\right) > 0\,.\end{aligned}$$

In particular, we also have

$$\delta_{q+1}^{-1/2} \le \delta_q^{1/2}\lambda_q\,. \tag{6.47}$$

The last inequality in (6.46) is a consequence of the following stronger estimate, which will be needed in Section 6.7. Fix any constant $\hat{C}(b, c, \sigma_0, \tilde{u}, g, \lambda, R, \Lambda, C_0)$. Then, if $a \ge a_0(\hat{C})$ is chosen large enough, we have

$$\hat{C}\frac{\delta_{q+1}}{\ell^2\lambda_{q+1}^2} \le \delta_{q+2}\,. \tag{6.48}$$

Indeed, inserting the definition of ℓ we see that the inequality is satisfied if

$$\hat{C}^{-1}\tilde{C}^{-2}\delta_q^{-1}\lambda_q^{-2}\delta_{q+2}\lambda_{q+1}^2 \ge 1\,.$$

Taking the logarithms gives

$$b^q\left(b^2(2c-1) - 2bc + 1\right) - \log_a\left(\hat{C}\tilde{C}^2\right) \ge 0\,.$$

Rewriting the first term, we find

$$b^q(b-1)\left(b(2c-1)-1\right) - \log_a\left(\hat{C}\tilde{C}^2\right) \ge 0\,.$$

This inequality is satisfied if a is chosen large enough, so that (6.48) holds.

6.5 Proof of Proposition 6.8: Setup

6.5.1 *Mollification*

Fix a standard, symmetric mollifier, i.e. a radially symmetric, nonnegative function $\varphi \in C_c^\infty(B_1)$ on $\mathbb{R}^n$ with unit integral and set $\varphi_\ell(x) = \ell^{-n}\varphi(x/\ell)$. We define the mollification parameter ℓ by (6.45) and set

$$\bar{v}_q := (v_q - \tilde{u}) * \varphi_\ell + \tilde{u}\,, \tag{6.49}$$

which mollifies the map v_q while keeping the boundary value: since $\delta_q^{1/2}\lambda_q > \delta_{q+1}^{-1/2}$ we have $\ell < \frac{1}{2}R\delta_{q+1}$ if $\tilde{C}$ is chosen large enough, so that, thanks to (6.31), we have

$$\bar{v}_q = \tilde{u} \text{ on } \bar{B}_1 \setminus B_{1-\frac{1}{2}R\delta_{q+1}}\,.$$

Lastly, we set

$$\tau := \frac{\tilde{g} - \bar{v}_q^\sharp e}{h_q} - \frac{\delta_{q+2}}{h_q} e \,. \tag{6.50}$$

Observe that τ is well defined and smooth on every open Ω which is compactly contained in B_1. We gather a few important estimates on $\bar{v}_q$ and τ in the next

Lemma 6.12. *If $\tilde{C}(\tilde{u}, \Lambda, C_0)$, $a_0(C_0, \Lambda)$ and $R(\lambda)$ are chosen large enough and if $\sigma_0 > 0$ is chosen small enough, then, for $k = 0, 1, 2$, we have*

$$[\bar{v}_q]_{k+1} \le C(1 + \delta_{q+1}^{1/2} \ell^{-k}) \,, \tag{6.51}$$

$$[\bar{v}_q^\sharp e - v_q^\sharp e * \varphi_\ell]_k \le C\ell^{2-k} [v_q]_2^2 \,, \tag{6.52}$$

$$|\tau - e| \le \frac{r_0}{2} \text{ on } \bar{B}_{1-R\delta_{q+2}} \,, \tag{6.53}$$

$$|D^k \tau| \le C\ell^{-k} \text{ on } \bar{B}_{1-R\delta_{q+2}} \,, \tag{6.54}$$

for some constant C depending on $\tilde{u}$ and Λ.

Proof. First observe that if $a_0(C_0)$ is large enough we get $\|v_q\|_1 \le C(\tilde{u})$. Therefore, using again (6.31) and the mollification estimates from Lemma 2.3,

$$\begin{aligned} [\nabla \bar{v}_q]_k &= [\nabla v_q * \varphi_\ell]_k + [\nabla(\tilde{u} - \tilde{u} * \varphi_\ell)]_k \le C(\tilde{u})(1 + \ell^{1-k}[v_q]_2) + C\ell^{1-k}[\tilde{u}]_2 \\ &\le C(\tilde{u})(1 + \delta_{q+1}^{1/2} \ell^{-k}) \,, \end{aligned}$$

if $\tilde{C}(C_0)$ is large enough. For the second estimate we compute

$$\begin{aligned} \nabla \bar{v}_q^\intercal \nabla \bar{v}_q = {} & \nabla(v_q * \varphi_\ell)^\intercal \nabla(v_q * \varphi_\ell) + \nabla(\tilde{u} - \tilde{u} * \varphi_\ell)^\intercal \nabla(\tilde{u} - \tilde{u} * \varphi_\ell) \\ & + 2\mathrm{sym}\left(\nabla(v_q * \varphi_\ell)^\intercal \nabla(\tilde{u} - \tilde{u} * \varphi_\ell)\right) \,, \end{aligned}$$

where we recall the notation $\mathrm{sym}(A) = \frac{1}{2}(A + A^\intercal)$. This gives

$$\begin{aligned} [\bar{v}_q^\sharp e - v_q^\sharp e * \varphi_\ell]_k &\le C(\tilde{u}) \Big([(v_q * \varphi_\ell)^\sharp e - v_q^\sharp e * \varphi_\ell]_k + [\tilde{u} - \tilde{u} * \varphi_\ell]_{k+1} \\ &\qquad + \left([\tilde{u} - \tilde{u} * \varphi_\ell]_{k+1} + [v_q * \varphi_\ell]_{k+1}\right) [\tilde{u} - \tilde{u} * \varphi_\ell]_1 \Big) \\ &\le C(\tilde{u}) \left(\ell^{2-k}[v_q]_2^2 + \ell^{2-k}[\tilde{u}]_3 + (1 + \ell^{1-k}[v_q]_2)\ell^2[\tilde{u}]_3 \right) \\ &\le C(\tilde{u})\ell^{2-k}[v_q]_2^2 \,. \end{aligned}$$

We will prove the estimates (6.53) and (6.54) separately on $\bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}}$ and on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$. Since on the former we have $\bar{v}_q = \tilde{u} = v_q$, and consequently

$$\tau - e = \frac{\tilde{g} - v_q^\sharp e - h_q e}{h_q} - \frac{\delta_{q+2}}{h_q} e \,,$$

it follows with (6.34) and $h_q \geq \lambda R \delta_{q+2}$ that

$$|\tau - e| \leq C\sigma_0 + C\frac{1}{\lambda R} \leq \frac{r_0}{2}$$

if σ_0 is small and $R(\lambda)$ large enough. By (6.34) we have the pointwise estimate

$$|\tilde{g} - v_q^\sharp e| \leq C|h_q|\,,$$

so that with the help of (6.30) and (6.33)

$$|\nabla\tau| \leq C\left(\frac{|\nabla(\tilde{g} - v_q^\sharp e)|}{h_q} + \frac{|\nabla h_q|}{h_q}\right) \leq C(\tilde{u})C_0\delta_{q+2}^{-1}\,,$$

and similarly

$$\begin{aligned}|D^2\tau| &\leq C\left(\frac{|D^2 h_q|}{h_q} + \frac{|\nabla h_q|\left(|\nabla h_q| + |\nabla(\tilde{g} - v_q^\sharp e)|\right)}{h_q^2} + \frac{|D^2(\tilde{g} - v_q^\sharp e)|}{h_q}\right)\\ &\leq C(\tilde{u})C_0\left(\delta_{q+1}^{-1}\delta_{q+2}^{-1} + \delta_{q+2}^{-2} + \delta_{q+1}^{-1}\delta_{q+2}^{-1}\right) \leq C(\tilde{u})C_0\delta_{q+2}^{-2}\,.\end{aligned}$$

Observe that, if $\tilde{C} \geq C_0$ then $C_0\delta_{q+2}^{-k} \leq \ell^{-k}$ for $k = 1, 2$, thanks to (6.46). This shows (6.54) on $\bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}}$. To show the estimates on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$ we write

$$\begin{aligned}|\tau - e| &\leq C\frac{\delta_{q+2}}{\Lambda^{-1}\delta_{q+1}} + \frac{1}{h_q}\big|(\tilde{g} - v_q^\sharp e - h_q e) * \varphi_\ell + (v_q^\sharp e * \varphi_\ell - \bar{v}_q^\sharp e)\\ &\qquad\qquad + (h_q * \varphi_\ell - h_q)e + (\tilde{g} - \tilde{g} * \varphi_\ell)\big|\\ &\leq \frac{r_0}{8} + \frac{C}{h_q}\left(\sigma_0|h_q * \varphi_\ell| + \ell^2([v_q]_2^2 + [h_q]_2 + [\tilde{g}]_2)\right)\\ &\leq \frac{r_0}{8} + C\sigma_0 + \frac{C\ell^2}{h_q}\left(C_0^2\delta_q\lambda_q^2 + C_0\delta_{q+1}^{-1} + C_0(2+\sigma_0)\delta_{q+1}^{-1}\right)\,.\end{aligned}$$

Hence we have

$$|\tau - e| \leq \frac{r_0}{4} + C\frac{C_0^2}{\tilde{C}^2\Lambda^{-1}} \leq \frac{r_0}{2}$$

if σ_0 is chosen small and $\tilde{C}(\Lambda, C_0)$ as well as $a(\Lambda)$ are large enough. This fixes the choice of $\tilde{C}$. For (6.54) we estimate

$$\begin{aligned}[\tilde{g} - \bar{v}_q^\sharp e]_k &\leq [(\tilde{g} - v_q^\sharp e) * \varphi_\ell]_k + [\tilde{g} - \tilde{g} * \varphi_\ell]_k + [v_q^\sharp e * \varphi_\ell - \bar{v}_q^\sharp e]_k\\ &\leq C(\tilde{u})\left(\ell^{-k}\|\tilde{g} - v_q^\sharp e\|_0 + \ell^{2-k}([\tilde{g}]_2 + [v_q]_2^2)\right) \leq C(\tilde{u}, \Lambda)\delta_{q+1}\ell^{-k}\,.\end{aligned}$$

Hence, with the help of (2.7) we get the following estimate on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$

$$\begin{aligned}|D^k\tau| &\leq C\Big(\Lambda\delta_{q+1}^{-1}[\tilde{g}-\bar{v}_q^\sharp e]_k \\ &\quad + [h_q]_k\left(\Lambda^2\delta_{q+1}^{-2}+(\Lambda\delta_{q+1})^{k-1}(\Lambda^{-1}\delta_{q+1})^{-k-1}\right)\left(\|\tilde{g}-\bar{v}_q^\sharp e\|_0+\delta_{q+2}\right)\Big) \\ &\leq C(\tilde{u},\Lambda)\left(\ell^{-k}+C_0\delta_{q+1}^{-k}\right)\leq C(\tilde{u},\Lambda)\ell^{-k}. \qquad \square\end{aligned}$$

6.5.2 *Decomposition*

Our goal in constructing v_{q+1} is to add the (rescaled) metric error τ by an ansatz of the form

$$v_{q+1}=\bar{v}_q+\sum_{k=1}^{n_*}\frac{a_k}{\lambda_{q+1}}\left(\sin(\lambda_{q+1}\nu_k\cdot x)\zeta_k^1+\cos(\lambda_{q+1}\nu_k\cdot x)\zeta_k^2\right), \tag{6.55}$$

where $\nu_k\in\mathcal{S}^{n-1}$, a_k are smooth coefficients and where ζ_k^1,ζ_k^2 are smooth, mutually orthogonal unit vector fields which are normal to $\bar{v}_q$. We compute

$$\begin{aligned}\nabla v_{q+1} &= \nabla\bar{v}_q+\sum_{k=1}^{n_*}a_k\underbrace{\left(\cos(\lambda_{q+1}\nu_k\cdot x)\zeta_k^1\otimes\nu_k-\sin(\lambda_{q+1}\nu_k\cdot x)\zeta_k^2\otimes\nu_k\right)}_{=:A_k} \\ &\quad+\sum_{k=1}^{n_*}\frac{a_k}{\lambda_{q+1}}\underbrace{\left(\sin(\lambda_{q+1}\nu_k\cdot x)\nabla\zeta_k^1+\cos(\lambda_{q+1}\nu_k\cdot x)\nabla\zeta_k^2\right)}_{=:B_k} \\ &\quad+\sum_{k=1}^{n_*}\frac{1}{\lambda_{q+1}}\underbrace{\left(\sin(\lambda_{q+1}\nu_k\cdot x)\zeta_k^1+\cos(\lambda_{q+1}\nu_k\cdot x)\zeta_k^2\right)}_{=:C_k}\nabla a_k,\end{aligned} \tag{6.56}$$

so that (in coordinates) the induced metric is

$$\begin{aligned}\nabla v_{q+1}^\intercal\nabla v_{q+1} &= \nabla\bar{v}_q^\intercal\nabla\bar{v}_q+\sum_{k=1}^{n_*}a_k^2\nu_k\otimes\nu_k+\boxed{2\sum_{k=1}^{n_*}\frac{a_k}{\lambda_{q+1}}\mathrm{sym}(\nabla\bar{v}_q^\intercal B_k)} \\ &\quad+\boxed{2\sum_{i,j=1}^{n_*}\frac{a_ia_j}{\lambda_{q+1}}\mathrm{sym}(A_i^\intercal B_j)}+2\sum_{i,j=1}^{n_*}\frac{a_ia_j}{\lambda_{q+1}^2}\mathrm{sym}\left(B_i^\intercal B_j\right) \\ &\quad+2\sum_{i,j=1}^{n_*}\frac{a_i}{\lambda_{q+1}^2}\mathrm{sym}(B_i^\intercal C_j\nabla a_j)+\sum_{k=1}^{n_*}\frac{1}{\lambda_{q+1}^2}\nabla a_k^\intercal\nabla a_k.\end{aligned} \tag{6.57}$$

The usual practice is to decompose the metric error $\tilde{g}-\bar{v}_q^\sharp e$ into a sum of the form $\sum_{k=1}^{n_*}a_k^2\nu_k\otimes\nu_k$ and hence the ansatz (6.55) allows the addition of the metric error up to remainders which are (if λ_{q+1} is chosen large) very small. However, as realized in [41], a better convergence rate is achieved if only the last three terms of (6.57) are treated as error terms. Consequently, one needs a slightly subtler decomposition, which is provided by Proposition 6.10 once we know that the boxed terms are small enough. This is the content of Lemma 6.13 once we have found suitable normal vectors ζ_k^1, ζ_k^2. But this is an easy task thanks to Proposition 6.9, once we require $a(\tilde{u}, C_0)$ to be so large that $C_0 \sum_{k=1}^{q} \delta_q^{1/2} < \rho_0(\tilde{u})$, where ρ_0 is given by Proposition 6.9. Then, since

$$\|\bar{v}_q - \tilde{u}\|_1 = \|(v_q - \tilde{u}) * \varphi_\ell\|_1 \le \|v_q - \tilde{u}\|_1 < \rho_0(\tilde{u}),$$

Proposition 6.9 provides an orthonormal family $\{\xi_i(\bar{v}_q)\}_{i=1}^{m-n} \subset C^\infty(\bar{B}_1, \mathbb{R}^m)$ of vector fields which are normal to $\bar{v}_q$ and enjoy the estimates

$$|D^k \xi_i| \le C(\tilde{u}) \text{ on } \bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}} \tag{6.58}$$

$$|D^k \xi_i| \le C(\tilde{u})(1 + \delta_{q+1}^{1/2} \ell^{-k}) \text{ on } \bar{B}_{1-\frac{1}{2}R\delta_{q+1}}, \tag{6.59}$$

for $k = 0, 1, 2$, thanks to (6.51). We next define

$$\zeta_i^1 := \xi_i, \zeta_i^2 := \xi_{n_*+i}, \quad \text{for } i = 1, \dots, n_*, \tag{6.60}$$

which is possible in view of $m - n \ge n(n+2) - n = 2n_*$.

Finally, we let $\nu_1, \dots, \nu_{n_*}$ be the vectors given by Proposition 6.10, define A_k, B_k and C_k as in (6.56), let $\eta := \eta_{q+1}$ be one of the cutoff functions constructed in Lemma 6.11 and set

$$M_i := \frac{2}{h_q^{1/2} \lambda_{q+1}} \operatorname{sym}\left(\nabla \bar{v}_q^\intercal B_i\right) \tag{6.61}$$

$$\begin{aligned} G_{ij} := {} & \frac{2}{\lambda_{q+1}} \operatorname{sym}\left(A_i^\intercal B_j\right) + \frac{2}{\lambda_{q+1}^2} \operatorname{sym}\left(B_i^\intercal (B_j + C_j \nabla \eta)\right) \\ & + \frac{2}{h_q^{1/2} \lambda_{q+1}^2} \operatorname{sym}\left(B_i^\intercal C_j \nabla h_q^{1/2}\right) + \frac{\delta_{ij}}{\lambda_{q+1}^2} \nabla \eta^\intercal \nabla \eta + \frac{2\delta_{ij}}{h_q^{1/2} \lambda_{q+1}^2} \operatorname{sym}\left(\nabla \eta^\intercal \nabla h_q^{1/2}\right) \\ & + \frac{\delta_{ij}}{h_q \lambda_{q+1}^2} \nabla (h_q^{1/2})^\intercal \nabla h_q^{1/2}. \end{aligned} \tag{6.62}$$

We are now ready to estimate the various terms.

Lemma 6.13. *For $a(b, c, \tilde{u}, \lambda, R, C_0)$ large enough there exists a constant $C > 0$ (depending only on $\tilde{u}$ and Λ) such that for $k = 0, 1, 2$*

$$|D^k A_i| + |D^k C_i| \le C \lambda_{q+1}^k \text{ on } \bar{B}_{1-R\delta_{q+2}}, \tag{6.63}$$

$$\begin{aligned} & |D^k B_i| \le C \delta_{q+1}^{1/2} \ell^{-1} \lambda_{q+1}^k \text{ on } \bar{B}_{1-\frac{1}{2}R\delta_{q+1}} \text{ and} \\ & |D^k B_i| \le C \lambda_{q+1}^k \text{ on } \bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}}, \end{aligned} \tag{6.64}$$

$$|D^k M_i| + |D^k G_{ij}| \le C \ell^{-1} \lambda_{q+1}^{k-1} \text{ on } \bar{B}_{1-R\delta_{q+2}}. \tag{6.65}$$

Proof. Since the vectors v_k are constant, the estimate for A_i and C_i is (up to a constant) the same:

$$|D^k C_i| \leq C\left(\lambda_{q+1}^k + [\zeta_i^j]_k\right) \leq C\left(\lambda_{q+1}^k + C(\tilde{u})\delta_{q+1}^{1/2}\ell^{-k}\right) \leq C\lambda_{q+1}^k\,,$$

where we have used $\lambda_{q+1} \geq \ell^{-1}$ and $a(\tilde{u})$ large enough. The estimate for B_i follows from

$$|D^k B_i| \leq C(\lambda_{q+1}^k[\zeta_i^j]_1 + [\zeta_i^j]_{k+1})$$

using (6.58) and (6.59) respectively. Since $h_q \geq R\lambda\delta_{q+2} \geq \delta_{q+2}$ on $\bar{B}_{1-R\delta_{q+2}}$ and $h_q \geq \Lambda^{-1}\delta_{q+1}$ on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$ we get, using (2.7),

$$|D^{k+1}h_q^{1/2}| \leq C(\Lambda)C_0\delta_{q+1}^{-k}\left(\delta_{q+1}^{-1/2} + \delta_{q+1}^k\delta_{q+1}^{-1/2-k}\right) \leq C(\Lambda)C_0\delta_{q+1}^{-1/2-k}$$

on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$, and

$$|D^{k+1}h_q^{1/2}| \leq C(\lambda,R)C_0\delta_{q+1}^{-k}\left(\delta_{q+2}^{-1/2} + \delta_{q+1}^k\delta_{q+2}^{-1/2-k}\right) \leq C(\lambda,R)C_0\delta_{q+2}^{-1/2-k}$$

on $\bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}}$. Now, combining (6.64) and the previous two estimates,

$$\begin{aligned}
|D^k M_i| &\leq \frac{C}{\lambda_{q+1}}\left([h_q^{-1/2}]_k\|B_i\|_0 + |h_q^{-1/2}|\left([\bar{v}_q]_{k+1}\|B_i\|_0 + |D^k B_i|\right)\right) \\
&\leq \frac{C(\tilde{u},\Lambda)}{\lambda_{q+1}}\left(C_0\delta_{q+1}^{-1/2-k}\delta_{q+1}^{1/2}\ell^{-1} + \ell^{-1}(1+\delta_{q+1}^{1/2}\ell^{-k}) + \ell^{-1}\lambda_{q+1}^k\right) \\
&\leq \frac{C(\tilde{u},\Lambda)}{\lambda_{q+1}\ell}\left(C_0\delta_{q+1}^{-k} + \delta_{q+1}^{1/2}\ell^{-k} + \lambda_{q+1}^k\right) \leq C(\tilde{u},\Lambda)\ell^{-1}\lambda_{q+1}^{k-1}
\end{aligned}$$

on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$, where we used that $C_0\delta_{q+1}^{-k} \leq \lambda_{q+1}^k$ for $a(b,c,C_0)$ big enough. On the other hand, on $\bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}}$ we have

$$|D^k M_i| \leq \frac{C(\tilde{u},g)}{\lambda_{q+1}}\left(C(\lambda,R)C_0\delta_{q+2}^{-1/2-k} + \delta_{q+2}^{-1/2}\lambda_{q+1}^k\right) \leq C(\tilde{u},g)\delta_{q+2}^{-1/2}\lambda_{q+1}^{k-1}\,,$$

where again, $a(b,c,\lambda,R,C_0)$ is chosen so large that $C(\lambda,R)C_0\delta_{q+2}^{-k} \le \lambda_{q+1}^k$. Similarly, on $\bar{B}_{1-\frac{1}{2}R\delta_{q+1}}$, we find

$$\begin{aligned}
|D^k G_{ij}| &\le C\delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{k-1} + C\delta_{q+1}\ell^{-2}\lambda_{q+1}^{k-2} \\
&\quad + \frac{C(\tilde{u},\Lambda,C_0)}{\lambda_{q+1}^2}\left(\delta_{q+1}^{-1/2-k}\delta_{q+1}^{1/2}\ell^{-1} + \ell^{-1}\left(\lambda_{q+1}^k\delta_{q+1}^{-1/2} + \delta_{q+1}^{-1/2-k}\right)\right) \\
&\quad + \frac{C(\tilde{u},\Lambda,C_0)}{\lambda_{q+1}^2}\left(\delta_{q+1}^{-1-k}\delta_{q+1}^{-1} + \delta_{q+1}^{-1}\delta_{q+1}^{-1-k}\right) \\
&\le C\delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{k-1} + \frac{C(\tilde{u},\Lambda,C_0)}{\lambda_{q+1}^2}\left(\delta_{q+1}^{-k}\ell^{-1} + \delta_{q+1}^{-1/2}\ell^{-1}\lambda_{q+1}^k + \delta_{q+1}^{-2-k}\right) \\
&\le C(\tilde{u},\Lambda)\delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{k-1},
\end{aligned}$$

where we used that $\nabla\eta = 0$ in this region and that $C(C_0)\delta_{q+1}^{-1} \le C(C_0)\ell^{-1} \le \lambda_{q+1}$ for $a(b,c,C_0)$ large enough. Lastly, we check the region $\bar{B}_{1-R\delta_{q+2}} \setminus B_{1-\frac{1}{2}R\delta_{q+1}}$:

$$\begin{aligned}
|D^k G_{ij}| &\le C(\tilde{u})\lambda_{q+1}^{k-1} + \frac{C(\tilde{u})}{\lambda_{q+1}^2}\left(\lambda_{q+1}^k\delta_{q+2}^{-1} + \delta_{q+2}^{-k-1}\right) \\
&\quad + \frac{C(\lambda,R,C_0)}{\lambda_{q+1}^2}\left(\delta_{q+2}^{-1-k} + \delta_{q+2}^{-1/2}\left(\delta_{q+2}^{-1/2}\lambda_{q+1}^k + \delta_{q+2}^{-1/2-k}\right)\right) \\
&\quad + C\delta_{q+2}^{-k-2}\lambda_{q+1}^{-2} + \frac{C(\lambda,R,C_0)}{\lambda_{q+1}^2}\left(\delta_{q+2}^{-1/2-k}\delta_{q+2}^{-3/2} + \delta_{q+2}^{-1/2}\delta_{q+2}^{-k-3/2}\right) \\
&\quad + \frac{C(\lambda,R,C_0)}{\lambda_{q+1}^2}\left(\delta_{q+2}^{-k-1}\delta_{q+2}^{-1} + \delta_{q+2}^{-1}\delta_{q+2}^{-1-k}\right) \\
&\le C(\tilde{u})\lambda_{q+1}^{k-1} + C(\lambda,R,C_0)\delta_{q+2}^{-2-k}\lambda_{q+2}^{-2} \le C(\tilde{u})\ell^{-1}\lambda_{q+1}^{k-1},
\end{aligned}$$

where we used $C(\lambda,R,C_0)\delta_{q+2}^{-1} \le C(\lambda,R,C_0)\ell^{-1} \le \lambda_{q+1}$. □

Hence, if a is chosen large enough, we have

$$\|\tau - e\|_0 + \sum_i \|M_i\|_0 + \sum_{i,j}\|G_{ij}\|_0 < r_0\,,$$

where the norms are intended on $\bar{B}_{1-R\delta_{q+2}}$. Proposition 6.10 thus yields smooth functions $c_1,\ldots,c_{n_*} : \bar{B}_{1-R\delta_{q+2}} \to \mathbb{R}$, such that

$$\tau = \sum_i c_i^2 \nu_i \otimes \nu_i + \sum_i c_i M_i + \sum_{i,j} c_i c_j G_{ij}\,, \tag{6.66}$$

$c_i > r_0$ on $\bar{B}_{1-R\delta_{q+2}}$ and for $k = 0,1,2$

$$\|c_i\|_k \le C(\tilde{u},\Lambda)\left(1 + \ell^{-k} + \ell^{-1}\lambda_{q+1}^{k-1}\right) \le C(\tilde{u},\Lambda)\left(1 + \ell^{-1}\lambda_{q+1}^{k-1}\right). \tag{6.67}$$

6.6 PROOF OF PROPOSITION 6.8: PERTURBATION

Finally, we pick $\eta := \eta_{q+1}$ from Lemma 6.11, set $a_k := \eta h_q^{1/2} c_k$ where the c_k's are the functions found in the previous step, and define v_{q+1} as in (6.55). Observe that, although c_k is only defined in $\bar{B}_{1-R\delta_{q+2}}$, a_k can be continued smoothly to $\bar{B}_1$ by setting it equal to zero. Also, $v_{q+1} = \bar{v}_q = \tilde{u}$ on $\bar{B}_1 \setminus B_{1-R\delta_{q+2}}$. Then, by (6.57) we find

$$
\begin{aligned}
\nabla v_{q+1}^{\mathsf{T}} \nabla v_{q+1} = \nabla \bar{v}_q^{\mathsf{T}} \nabla \bar{v}_q &+ \eta^2 h_q \sum_{k=1}^{n_*} c_k^2 \nu_k \otimes \nu_k + 2\eta h_q \sum_{k=1}^{n_*} \frac{c_k}{h_q^{1/2} \lambda_{q+1}} \operatorname{sym}\left(\nabla \bar{v}_q^{\mathsf{T}} B_k\right) \\
&+ 2\eta^2 h_q \sum_{i,j=1}^{n_*} \frac{c_i c_j}{\lambda_{q+1}} \left(\operatorname{sym}\left(A_i^{\mathsf{T}} B_j\right) + \frac{1}{\lambda_{q+1}} \operatorname{sym}\left(B_i^{\mathsf{T}} B_j\right) \right) \\
&+ 2\eta h_q \sum_{i,j=1}^{n_*} \frac{c_i c_j}{\lambda_{q+1}^2} \operatorname{sym}\left(B_i^{\mathsf{T}} C_j \nabla \eta\right) + h_q \sum_{k=1}^{n_*} \frac{c_k^2}{\lambda_{q+1}^2} \nabla \eta^{\mathsf{T}} \nabla \eta \\
&+ 2\eta^2 h_q \sum_{i,j=1}^{n_*} \frac{c_i c_j}{h_q^{1/2} \lambda_{q+1}} \operatorname{sym}\left(B_i^{\mathsf{T}} C_j \nabla h_q^{1/2}\right) \\
&+ \eta^2 h_q \sum_{k=1}^{n_*} \frac{c_k^2}{h_q \lambda_{q+1}^2} \left(\nabla h_q^{1/2}\right)^{\mathsf{T}} \nabla h_q^{1/2} \\
&+ 2\eta h_q \sum_{k=1}^{n_*} \frac{c_k^2}{h_q^{1/2} \lambda_{q+1}^2} \operatorname{sym}\left(\nabla \eta^{\mathsf{T}} \nabla h_q^{1/2}\right) + E_1 \,,
\end{aligned}
$$

where we have set

$$
\begin{aligned}
E_1 := {} & 2\eta^2 h_q \sum_{i,j=1}^{n_*} \frac{c_i}{\lambda_{q+1}^2} \operatorname{sym}\left(B_i^{\mathsf{T}} C_j \nabla c_j\right) + 2\eta h_q^{1/2} \sum_{k=1}^{n_*} \frac{c_i}{\lambda_{q+1}^2} \operatorname{sym}\left(\nabla \left(\eta h_q^{1/2}\right)^{\mathsf{T}} \nabla c_i\right) \\
&+ \eta^2 h_q \sum_{k=1}^{n_*} \frac{1}{\lambda_{q+1}^2} \nabla c_k^{\mathsf{T}} \nabla c_k \,.
\end{aligned}
$$

Hence we can write

$$
\begin{aligned}
\nabla v_{q+1}^{\mathsf{T}} \nabla v_{q+1} = {} & \nabla \bar{v}_q^{\mathsf{T}} \nabla \bar{v}_q + \eta^2 h_q \left(\sum_{k=1}^{n_*} c_k \nu_k \otimes \nu_k + \sum_{k=1}^{n_*} c_k M_k + \sum_{i,j=1}^{n_*} c_i c_j G_{ij} \right) \\
&+ E_1 + E_2 \,,
\end{aligned}
$$

with

$$
\begin{aligned}
E_2 := {} & \eta(1-\eta) h_q \left(\sum_{k=1}^{n_*} c_k M_k + 2 \sum_{k=1}^{n_*} \frac{c_k^2}{h_q^{1/2} \lambda_{q+1}^2} \operatorname{sym}\left(\nabla \eta^{\mathsf{T}} \nabla h_q^{1/2}\right) \right) \\
&+ 2\eta(1-\eta) h_q \sum_{i,j=1}^{n_*} \frac{c_i c_j}{\lambda_{q+1}^2} \operatorname{sym}\left(B_i^{\mathsf{T}} C_j \nabla \eta\right) + (1-\eta^2) h_q \sum_{k=1}^{n_*} \frac{c_k^2}{\lambda_{q+1}^2} \nabla \eta^{\mathsf{T}} \nabla \eta \,.
\end{aligned}
$$

Recalling (6.66) and the definition of τ in (6.50), we can see that

$$v_{q+1}^\sharp e = \bar{v}_q^\sharp e + \eta^2 \left(\tilde{g} - \bar{v}_q^\sharp e - \delta_{q+2} e\right) + E_1 + E_2\,,$$

and consequently

$$\begin{aligned}\tilde{g} - v_{q+1}^\sharp e &= \tilde{g} - \bar{v}_q^\sharp e - \eta^2(\tilde{g} - \bar{v}_q^\sharp e - \delta_{q+2}e) - E_1 - E_2 \\ &= (1-\eta^2)(\tilde{g} - v_q^\sharp e) + \eta^2 \delta_{q+2} e - E_1 - E_2\,,\end{aligned}$$

where we used that $\bar{v}_q = v_q$ whenever $1 - \eta^2 > 0$. We now define

$$h_{q+1} := \frac{1-\sigma_0^2(1+\eta)}{1-\sigma_0^2(1+\eta)^2}(1-\eta^2)h_q + \frac{\eta^2}{1-\sigma_0^2(1+\eta)^2}\delta_{q+2}\,. \tag{6.68}$$

We have $h_{q+1} = h_q$ on $\bar{B}_1 \setminus B_{1-R\delta_{q+2}}$ granting linearity and $|h'_{q+1}(1)| = \lambda$. Since $\sigma_0 < \frac{1}{2}$ we find that on $\bar{B}_{1-R\delta_{q+2}} \setminus B_{1-(R+1)\delta_{q+2}}$ we have

$$\begin{aligned}h_{q+1} &\geq \frac{1}{2}(1-\eta^2)h_q + \eta^2\delta_{q+2} \geq \frac{1}{2}\lambda R \delta_{q+2}(1-\eta^2) + \eta^2 \delta_{q+2} \\ &= \frac{1}{2}\lambda R\delta_{q+2} + \eta^2\delta_{q+2}(1 - \frac{1}{2}\lambda R) =: f(|x|)\,.\end{aligned}$$

The function f is monotonically increasing since $\lambda R > 2$. Hence $h_{q+1} \geq f \geq f(0) = \delta_{q+2}$. This bound holds obviously also on $\bar{B}_{1-(R+1)\delta_{q+2}}$. Moreover, a rough estimate gives

$$\begin{aligned}h_{q+1} &\leq (1-\eta^2)h_q + \frac{1}{1-4\sigma_0^2}\delta_{q+2} \leq (R+1)\lambda\delta_{q+2} + 2\delta_{q+2} \leq 2(R+1)\lambda\delta_{q+2} \\ &\leq \Lambda \delta_{q+2}\end{aligned}$$

provided σ_0 is small enough and $\Lambda(R)$ big enough, which settles (6.32). To show (6.33) we define

$$\Phi(x) = \frac{1-\sigma_0^2(1+x)}{1-\sigma_0^2(1+x)^2}(1-x^2)\,, \Psi(x) = \frac{x^2}{1-\sigma_0^2(1+x)^2}\,,$$

and write

$$h_{q+1} = \Phi(\eta)h_q + \Psi(\eta)\delta_{q+2}\,.$$

Since $\sigma_0 < \frac{1}{2}$ one finds constants C_k such that

$$[\Phi]_k + [\Psi]_k \leq C_k\,, k \in \mathbb{N}\,.$$

Then (6.33) is a consequence of Proposition 2.1 and estimates (6.43).

6.7 PROOF OF PROPOSITION 6.8: CONCLUSION

6.7.1 *Error estimation*

Lastly, we need to check if, once a is chosen large enough, (6.34) is satisfied with q replaced by $q+1$. First of all, we show that the upper bound is true by using (6.34) to write

$$\begin{aligned}\tilde{g} - v^\sharp_{q+1} e &\leq (1-\eta^2)(1+\sigma_0)h_q e + \eta^2 \delta_{q+2} e - E_1 - E_2 \\ &= (1+\sigma_0(1+\eta))h_{q+1} e \\ &\quad + \underbrace{(1-\eta^2)(1+\sigma_0)h_q e + \eta^2\delta_{q+2} e - E_1 - E_2 - (1+\sigma_0(1+\eta))h_{q+1} e}_{=:E}\,.\end{aligned}$$

Hence, the task is to show that $E \leq 0$. First of all, on $\bar{B}_1 \setminus B_{1-R\delta_{q+2}}$ we have $\eta \equiv 0$ and $h_{q+1} = h_q$ resulting in $E = 0$. On $\bar{B}_{1-R\delta_{q+2}}$ we compute

$$\begin{aligned}E &= (1-\eta^2)(1+\sigma_0)h_q e + \eta^2\delta_{q+2} e - E_1 - E_2 \\ &\quad - \left(\frac{1-\sigma_0^2(1+\eta)}{1-\sigma_0(1+\eta)}(1-\eta^2)h_q e + \frac{\eta^2}{1-\sigma_0(1+\eta)}\delta_{q+2} e\right) \\ &= \left(1+\sigma_0 - \frac{1-\sigma_0^2(1+\eta)}{1-\sigma_0(1+\eta)}\right)(1-\eta^2)h_q e + \left(1 - \frac{1}{1-\sigma_0(1+\eta)}\right)\eta^2\delta_{q+2} e \\ &\quad - E_1 - E_2 \\ &= \frac{-\sigma_0\eta}{1-\sigma_0(1+\eta)}(1-\eta^2)h_q e - \frac{\sigma_0(1+\eta)\eta^2}{1-\sigma_0(1+\eta)}\delta_{q+2} e - E_1 - E_2\,.\end{aligned}$$

Since $h_q \geq \lambda R \delta_{q+2}$ whenever $1-\eta^2 > 0$ we can conclude that

$$\frac{-\sigma_0(1-\eta^2)}{2(1-\sigma_0(1+\eta))}h_q e - \frac{\sigma_0(1+\eta)\eta}{1-\sigma_0(1+\eta)}\delta_{q+2} e \leq -C(\sigma_0,\lambda,R)\delta_{q+2} e\,,$$

for some $C(\sigma_0,\lambda,R) > 0$. Using the estimates of Lemma 6.13 and (6.67) we find the pointwise estimate

$$|E_1| \leq C(\lambda,\Lambda,R)\frac{\delta_{q+1}}{\lambda_{q+1}^2\ell^2}\eta\,.$$

For a large enough it therefore follows from (6.48) that

$$\begin{aligned}E &\leq \eta\left(C(\lambda,\Lambda,R)\frac{\delta_{q+1}}{\lambda_{q+1}^2\ell^2} e - C(\sigma_0,\lambda,R)\delta_{q+2} e\right) - \frac{\sigma_0\eta(1-\eta^2)}{2(1-\sigma_0(1+\eta))}h_q e - E_2 \\ &\leq -\frac{\sigma_0\eta(1-\eta^2)}{2(1-\sigma_0(1+\eta))}h_q e - E_2\,.\end{aligned}$$

To estimate this final term we recall from (6.44) that there exists $\varepsilon > 0$ such that $|\nabla\eta^\intercal\nabla\eta| \leq C\delta_{q+2}^{-2}\eta$ whenever $\eta \leq \varepsilon$. Consequently, when $\eta \leq \varepsilon$ we can estimate

$$|E_2| \leq C(\lambda, R)\eta(1-\eta)h_q\left(\ell^{-1}\lambda_{q+1}^{-1} + \delta_{q+2}^{-2}\lambda_{q+1}^{-2}\right) \leq C(\lambda, R)\eta(1-\eta)\frac{h_q}{\ell^2\lambda_{q+1}^2},$$

so that

$$E \leq \eta(1-\eta)h_q\left(\frac{C(\lambda, R)}{\ell^2\lambda_{q+1}^2}e - \frac{\sigma_0}{2}e\right) \leq 0,$$

if $a(\sigma_0, \lambda, R)$ is large enough. On the other hand, when $\eta \geq \varepsilon$, then

$$\begin{aligned} E &\leq \eta(1-\eta)h_q\left(\frac{C(\lambda, R)}{\ell^2\lambda_{q+1}^2}e - \frac{\sigma_0}{4}e\right) \\ &\quad + (1-\eta^2)h_q\left(\sum_{k=1}^{n_*}\frac{c_k^2}{\lambda_{q+1}^2}|\nabla\eta^\intercal\nabla\eta|e - \frac{\sigma_0\eta}{4(1-\sigma_0(1+\eta))}e\right) \\ &\leq C(1-\eta^2)h_q\left(\delta_{q+2}^{-2}\lambda_{q+1}^{-2}e - \frac{\sigma_0\varepsilon}{4}e\right) \leq 0, \end{aligned}$$

if $a(\sigma_0, \varepsilon)$ is large enough. Recall in particular that ε does not depend on q, hence we can choose a depending on ε. This proves the upper bound in (6.34). The lower bound is proven analoguously.

6.7.2 *Estimates on v_{q+1}*

First of all, on $\bar{B}_1 \setminus B_{1-R\delta_{q+2}}$ we have $v_{q+1} = \tilde{u} = v_q$. On the other hand, on $\bar{B}_{1-R\delta_{q+2}}$ we can estimate, for $k = 0, 1, 2$,

$$[\bar{v}_q - v_q]_k \leq C\ell^{2-k}[v_q]_2 + C\ell^{2-k}[\tilde{u}]_2 \leq \delta_{q+1}^{1/2}\ell^{1-k},$$

if $\tilde{C}$ in the definition (6.45) of ℓ is large enough. Moreover, combining the estimates of Lemma 6.13 with estimates (6.43), (6.59) and (6.67) we can estimate

$$\begin{aligned} [v_{q+1} - \bar{v}_q]_k &\leq \frac{C}{\lambda_{q+1}}\left([\eta h_q^{1/2}c_i]_k + C(\tilde{u}, \Lambda)\delta_{q+1}^{1/2}\lambda_{q+1}^k\right) \\ &\leq \frac{C(\tilde{u}, \Lambda)\delta_{q+1}^{1/2}}{\lambda_{q+1}}\left(\delta_{q+2}^{-k} + C_0\delta_{q+2}^{-k} + \ell^{-1}\lambda_{q+1}^{k-1} + \lambda_{q+1}^k\right) \\ &\leq C(\tilde{u}, \Lambda)\delta_{q+1}^{1/2}\lambda_{q+1}^{k-1} \leq C_0\delta_{q+1}^{1/2}\lambda_{q+1}^{k-1}. \end{aligned}$$

This concludes the proof of the proposition.

6.8 PROOF OF THEOREM 6.7

6.8.1 *First approximation*

Let $\sigma_0 > 0$ from Proposition 6.8 be given and assume that $\bar{\sigma}_0 < \frac{1}{2}\sigma_0$. Suppose that g, u satisfy (6.23) and (6.25) and fix an $\alpha < \frac{1}{2}$ and a constant $x_0 \in \mathbb{R}^{n(n+1)}$. We choose $c > b > 1$ such that $\alpha < \frac{1}{2bc}$. For any a big enough we now want to construct maps v_0, h_0 satisfying the assumptions (6.31)-(6.34) for the metric $\tilde{g} = g - w^\sharp e$, where $w \in C^\infty\left(\bar{B}_1, \mathbb{R}^{n(n+1)}\right)$ is a suitable map constructed below in (6.74). Then Proposition 6.8 can be applied iteratively to generate a sequence $v_q \in C^\infty\left(\bar{B}_1, \mathbb{R}^m\right)$ converging in $C^{1,\alpha}$ to a map $\underline{v}$ inducing the metric $\tilde{g}$. Setting $v = (\underline{v}, w)$ will then yield the wanted isometric map. First of all, we need to do a first approximation to get into the range of assumption (6.34).

Lemma 6.14. *Let $m \geq n+2$, $\tilde{\sigma}_0 \in]0, \frac{1}{4}[$ and assume $u \in C^\infty(\bar{B}_1, \mathbb{R}^m)$ and $h \in C^\infty(\bar{B}_1)$ satisfy* (6.23)–(6.25) *with $\bar{\sigma}_0$ replaced by $\tilde{\sigma}_0$. There exist $\bar{\delta} > 0$ and $\bar{\Lambda} > 1$ (depending only on $\tilde{\sigma}_0$ and h) such that for any positive $\delta < \bar{\delta}$ there exist $\tilde{u} \in C^\infty(\bar{B}_1, \mathbb{R}^m)$, $\tilde{h} \in C^\infty(\bar{B}_1)$ with*

$$(1-\tilde{\sigma}_0(2+\eta))\tilde{h}e \leq g - \tilde{u}^\sharp e \leq (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e\,, \tag{6.69}$$

$$\tilde{u} = u \text{ on } \bar{B}_1 \setminus B_{1-\delta}\,, \tag{6.70}$$

$$\tilde{h}(1) = 0 \text{ and } \tilde{h} \text{ is linear on } \bar{B}_1 \setminus B_{1-\delta}\,, \tag{6.71}$$

$$\bar{\Lambda}^{-1}\delta \leq \tilde{h} \leq \bar{\Lambda}\delta \text{ on } \bar{B}_{1-\delta}\,, \tag{6.72}$$

$$\|D^k\tilde{h}\|_{C^0(\bar{B}_1)} \leq C\delta^{1-k} \text{ for } k = 0,1,2,3\,, \tag{6.73}$$

where η is a suitable, radially symmetric, smooth cutoff function with $\eta \equiv 1$ on $\bar{B}_{1-2\delta}$ and $\eta \equiv 0$ on $\bar{B}_1 \setminus B_{1-\delta}$ and the constant C in (6.73) *depends only on $|h'(1)|$. In addition, $\tilde{u}$ can be chosen to be arbitrarily close to u in C^0.*

We postpone the proof of this lemma until the end of this section and now show how to conclude the Theorem 6.7 from it. Firstly, choose $\tilde{\sigma}_0 = \bar{\sigma}_0$ and fix some $\delta < \bar{\delta}$ to find first approximations $\tilde{u}, \tilde{h}$ satisfying (6.69)–(6.73). We then set $\lambda := |\tilde{h}'(1)|$, choose some $a > a_0(b, c, \tilde{u}, \sigma_0, \lambda, R, \Lambda, \delta)$ big enough to satisfy $(R+1)\delta_1 < \delta$, where we recall $\delta_q = a^{-b^q}$. To start the iterative process we now would like to find maps v_0, h_0 satisfying (6.31)–(6.34). In particular, v_0 will have to satisfy $\|v_0 - \tilde{u}\|_1 < \rho_0(\tilde{u})$ in order to find the normal vectorfields with the help of Proposition 6.9. A perturbation like the one used in the proof of Proposition 6.8 would produce a map v_0 satisfying most of the needed conditions, however we could only control $\|v_0 - \tilde{u}\|_1 \leq C\delta^{1/2}$. Since $C\delta^{1/2}$ might be bigger than $\rho_0(\tilde{u})$ such a perturbation is not sufficient. The solution, which unfortunately comes at the expense of increasing the codimension, is to perturb the metric instead: we set $v_0 = \tilde{u}$ and find a metric $\tilde{g}$ of the form $\tilde{g} = g - w^\sharp e$ such that $\tilde{g} - \tilde{u}^\sharp e$ is very small. It is then not difficult to find h_0 such that v_0, h_0 and $\tilde{g}$ satisfy

(6.30)–(6.34). Note that this idea is used in Nash's second work [46] on isometric embeddings.

To construct the map w we define

$$\tau = \frac{g - \tilde{u}^\sharp e}{\tilde{h}} - \frac{\delta_1}{\tilde{h}} e\,.$$

If R is big and $\tilde{\sigma}_0$ is small enough we can decompose τ on $\bar{B}_{1-R\delta_1}$, since

$$|\tau - e| \le C\tilde{\sigma}_0 + \frac{C}{R\lambda} < r_0\,.$$

Here, we assumed that $a(\bar{\Lambda})$ is taken large enough to guarantee $\bar{\Lambda}^{-1}\delta \ge \lambda R \delta_1$. We can then also compute

$$|D^k \tau| \le C(g, \tilde{u})\delta_1^{-k}\,,$$

for $k = 1, 2, 3$. Hence, by Proposition 6.10 we find $\nu_1, \dots, \nu_{n*} \in \mathcal{S}^{n-1}$ and $c_1, \dots, c_{n_*} \in C^\infty\left(\bar{B}_{1-R\delta_1}\right)$ with

$$\tau = \sum c_i^2 \nu_i \otimes \nu_i\,,$$

and, for $k = 0, 1, 2, 3$,

$$|D^k c_i| \le C|D^k \tau| \le C(g, \tilde{u})\delta_1^{-k}$$

as well as the improved estimates, for $k = 1, 2, 3$,

$$|\tilde{h}^{1/2} D^k c_i| \le C(g, \tilde{u})\delta_1^{1/2-k}\,.$$

6.8.2 *Perturbation*

Fix a cutoff η_0 given by Lemma 6.11, pick a constant $x_0 \in \mathbb{R}^{n(n+1)}$ and define

$$w = x_0 + \sum_{k=1}^{n_*} \frac{\eta_0 \tilde{h}^{1/2} c_k}{\mu} \left(\sin(\mu x \cdot \nu_k) e_k + \cos(\mu x \cdot \nu_k) e_{n_*+k}\right)\,, \tag{6.74}$$

where $e_i \in \mathbb{R}^{n(n+1)}$ is the $i-$th standard basis vector and $\mu > 1$ will be chosen later. We compute

$$\begin{aligned}\nabla w &= \sum_{k=1}^{n_*} \eta_0 \tilde{h}^{1/2} c_k \left(\cos(\mu x \cdot \nu_k) e_k \otimes \nu_k - \sin(\mu x \cdot \nu_k) e_{n_*+k} \otimes \nu_k\right) \\ &\quad + \frac{1}{\mu} \sum_{k=1}^{n_*} \nabla\left(\eta_0 \tilde{h}^{1/2} c_k\right) \left(\sin(\mu x \cdot \nu_k) e_k + \cos(\mu x \cdot \nu_k) e_{n_*+k}\right)\,,\end{aligned}$$

so that

$$\nabla w^{\mathsf{T}} \nabla w = \eta_0^2 \tilde{h} \sum_{k=1}^{n_*} c_k^2 \nu_k \otimes \nu_k + \frac{1}{\mu^2} \sum_{k=1}^{n_*} \nabla \left(\eta_0 \tilde{h}^{1/2} c_k \right)^{\mathsf{T}} \nabla \left(\eta_0 \tilde{h}^{1/2} c_k \right) .$$

Now we define $\tilde{g} = g - w^\sharp e$,

$$h_0 = \frac{1 - \sigma_0^2 (2 + \eta_0)}{1 - \sigma_0^2 (2 + \eta_0)^2} (1 - \eta_0^2) \tilde{h} + \frac{\eta_0^2}{1 - \sigma_0^2 (2 + \eta_0)^2} \delta_1 ,$$

and we claim that $\tilde{g}, v_0$ and h_0 satisfy the assumptions of Proposition 6.8.

6.8.3 *Final estimates to start the iteration*

First of all, since $v_0 = \tilde{u}$ the assumptions (6.31) are trivially satisfied once $a(\tilde{u}, C_0)$ is large enough. Now since $|g - \tilde{u}^\sharp e| \leq C\delta_1$ whenever $\nabla \eta_0 \neq 0$ (thanks to (6.69)), we can estimate for $k = 1, 2, 3$

$$|D^k \left(\eta_0 \tilde{h}^{1/2} c_k \right)| \leq C(g, \tilde{u}, \Lambda) \delta_1^{1/2 - k} ,$$

so that for $k = 1, 2$

$$|D^k \left(w^\sharp e \right)| \leq C(g, \tilde{u}) \delta_1^{1-k} + \frac{C(g, \tilde{u}, \Lambda)}{\mu^2} \delta_1^{-k-1} \leq C(g, \tilde{u}, \Lambda) \delta_1^{1-k} ,$$

if $\mu \geq \delta_1^{-1}$. Consequently, (6.30) is satisfied. With the same reasoning as in the proof of Proposition 6.8 we can conclude (6.32) and (6.33) and also (6.34) if

$$\mu = \hat{C} \delta_1^{-1}$$

for a large enough constant $\hat{C}$ depending on $g, \tilde{u}, \varepsilon$ and σ_0. Moreover, we can achieve

$$\|w - x_0\|_0 < \frac{\varepsilon}{2} ,$$

if $\hat{C}$ is large enough.

6.8.4 *Conclusion*

We can now apply Proposition 6.8 iteratively to generate the sequence v_q. Because of the estimate (6.36) the sequence converges in C^1 to a map $\underline{v}$ which satisfies, since we can pass to the limit in (6.34), $\underline{v}^\sharp e = \tilde{g}$. Lastly, we can estimate

$$\|v_{q+1} - v_q\|_{1,\alpha} \leq C \|v_{q+1} - v_q\|_1^{1-\alpha} [v_{q+1} - v_q]_2^\alpha \leq C \delta_{q+1}^{1/2} \lambda_{q+1}^\alpha = C a^{-1/2 b^q (1 - 2\alpha bc)} .$$

Since $\alpha < \frac{1}{2bc}$ the sequence converges in $C^{1,\alpha}$ and consequently $\underline{v} \in C^{1,\alpha}$. Setting $v = (\underline{v}, w)$ then concludes the proof of the main theorem. We are therefore left to proving Lemma 6.14.

6.8.5 *Proof of Lemma 6.14*

Let $r > 0$ be such that

$$(1-2\tilde{\sigma}_0)h'(1)(|x|-1)e \le (g-u^\sharp e)_x \le (1+2\tilde{\sigma}_0)h'(1)(|x|-1)e \tag{6.75}$$

for all $x \in \bar{B}_1 \setminus B_{1-r}$. Since u is strictly short and $\bar{B}_{1-r}$ is compact we can find $\bar{\rho} > 0$ such that

$$g - u^\sharp e > \bar{\rho} e \quad \text{on } \bar{B}_{1-r}\,.$$

Fix ρ such that

$$2\rho \max\{1, ((2\tilde{\sigma}_0 - 1)h'(1))^{-1}\} < \min\{r, \bar{\rho}\}\,.$$

With this choice we have

$$g - u^\sharp e \ge \rho e \;\text{ on } \bar{B}_{1-\delta}\,,$$

where we set $\delta = \rho \max\{1, ((2\tilde{\sigma}_0 - 1)h'(1))^{-1}\}$. Since $(g - u^\sharp e - \frac{\rho}{2}e)(\bar{B}_{1-\delta})$ is compact there exist M nonnegative smooth functions $a_1, \ldots, a_M \in C^\infty(\bar{B}_{1-\delta})$ and unit vectors $\nu_1, \ldots, \nu_M \in \mathcal{S}^{n-1}$ such that

$$g - u^\sharp e - \frac{\rho}{2}e = \sum_{i=1}^{M} a_i^2 \nu_i \otimes \nu_i\,, \tag{6.76}$$

on $\bar{B}_{1-\delta}$ (see for example Lemma 1 in [54]). Fix a radially symmetric cutoff $\eta \in C^\infty(\bar{B}_1)$ such that

$$\eta \equiv 1 \;\text{ on } \bar{B}_{1-2\delta}\,, \tag{6.77}$$
$$\eta \equiv 0 \;\text{ on } \bar{B}_1 \setminus B_{1-\delta}\,, \tag{6.78}$$
$$\|\eta^{(k)}\|_0 \le C_k \delta^{-k} \;\text{ for } k \ge 0\,, \tag{6.79}$$
$$(\eta')^2 = o(\eta) \;\text{ as } \eta \to 0\,, \tag{6.80}$$

Such a function can be constructed in the same way as in Lemma 6.11. We now use a Nash twist to construct $\tilde{u}$, i.e. for $k = 0, \ldots, M$ we define iteratively $u_0 := u$ and

$$u_k = u_{k-1} + \frac{\eta a_k}{\lambda_k}(\sin(\lambda_k x \cdot \nu_k)\zeta_k^1 + \cos(\lambda_k x \cdot \nu_k)\zeta_k^2)\,,$$

where $\lambda_k > 1$ are large frequencies to be chosen and $\zeta_k^1, \zeta_k^2 \in C^\infty(\bar{B}_1, \mathbb{R}^m)$ are orthogonal unit vector fields which are normal to u_{k-1} and are provided by Lemma A.11. Finally we set $\tilde{u} := u_M$. $\tilde{u}$ is smooth and because of the properties of η we certainly have $\tilde{u} = u$ on $\bar{B}_1 \setminus B_{1-\delta}$. To compute the induced metric we note that

$$\begin{aligned}\nabla u_k = \nabla u_{k-1} &+ \eta a_k(\cos(\lambda_k x \cdot \nu_k)\zeta_k^1 \otimes \nu_k - \sin(\lambda_k x \cdot \nu_k)\zeta_k^2 \otimes \nu_k) \\ &+ O\left(\lambda_k^{-1}\right)(\eta + \nabla\eta)\end{aligned} \tag{6.81}$$

Consequently

$$\nabla u_k^\intercal \nabla u_k = \nabla u_{k-1}^\intercal \nabla u_{k-1} + \eta^2 a_k^2 \nu_k \otimes \nu_k + O\left(\lambda_k^{-1}\right)(\eta + \nabla\eta^\intercal\nabla\eta)\,. \tag{6.82}$$

Remembering (6.76), we therefore find

$$\begin{aligned} g - \tilde{u}^\sharp e &= g - u^\sharp e + \sum_{k=1}^{M}\left(u_{k-1}^\sharp e - u_k^\sharp e\right) = (1-\eta^2)(g - u^\sharp e) + \eta^2\frac{\rho}{2}e \\ &\quad - (\eta + \nabla\eta^\intercal\nabla\eta)\underbrace{\sum_{k=1}^{M} O\left(\lambda_k^{-1}\right)}_{=:E}\,. \end{aligned}$$

We now set

$$\tilde{h}(x) = \frac{1-2\tilde{\sigma}_0^2(2+\eta)}{1-\tilde{\sigma}_0^2(2+\eta)^2}(1-\eta^2)h'(1)(|x|-1) + \frac{\eta^2}{1-\tilde{\sigma}_0^2(2+\eta)^2}\frac{\rho}{2}\,.$$

Then $\tilde{h} \in C^\infty(\bar{B}_1)$ and (6.71) follows directly. Moreover , one can write

$$\tilde{h}(x) = \Phi(\eta)h'(1)(|x|-1) + \Psi(\eta)\rho\,,$$

for the two rational functions

$$\Phi(x) = \frac{1-2\tilde{\sigma}_0^2(2+x)}{1-\tilde{\sigma}_0^2(2+x)^2}(1-x^2)\,,\quad \Psi(x) = \frac{x^2}{2-2\tilde{\sigma}_0^2(2+x)^2}\,.$$

Since $\tilde{\sigma}_0 \in]0, \frac{1}{4}[$, one easily finds a constant $C \geq 1$ such that

$$[\Phi]_{C^k([0,1])} + [\Psi]_{C^k([0,1])} \leq C\,, k = 0,1,2,3\,. \tag{6.83}$$

Hence,

$$\tilde{h} \leq C(|h'(1)|\delta + \rho) \leq \bar{\Lambda}\delta\,,$$

everywhere and

$$\tilde{h} \geq (1-\eta^2)h'(1)(|x|-1) + \eta^2\frac{\rho}{2} \geq |h'(1)|\delta + \eta^2(\frac{\rho}{2} - |h'(1)|\delta) \geq \frac{\rho}{2} \geq \bar{\Lambda}^{-1}\delta$$

on $\bar{B}_{1-\delta}$ for a suitably chosen $\bar{\Lambda}$ depending only on h and $\tilde{\sigma}_0$. Hence (6.72) is satisfied as well, while (6.73) follows with the help of Proposition 2.1 in view of (6.79) and (6.83). It therefore remains to show (6.69). On $\bar{B}_1 \setminus B_{1-\delta}$ it is implied by (6.75). If we choose λ_k so big that $\|E\|_0 < \tilde{\sigma}_0\rho$, then on $\bar{B}_{1-2\delta}$ one finds

$$g - \tilde{u}^\sharp e - \tilde{h}e = \frac{9\tilde{\sigma}_0^2\rho}{2(1-9\tilde{\sigma}_0^2)}e - E \leq \frac{2-9\tilde{\sigma}_0^2}{2(1-9\tilde{\sigma}_0^2)}\tilde{\sigma}_0\rho e \leq 2\tilde{\sigma}_0\tilde{h}e\,,$$

and similarly

$$g - \tilde{u}^\sharp e - \tilde{h}e \geq -2\tilde{\sigma}_0\tilde{h}e\,.$$

We are left with the set $\bar{B}_{1-\delta} \setminus B_{1-2\delta}$. Observe that

$$\begin{aligned}(1-\tilde{\sigma}_0(2+\eta))\tilde{h} - &(1-2\tilde{\sigma}_0)(1-\eta^2)h'(1)(|x|-1)\\ &= \left(\frac{1-2\tilde{\sigma}_0^2(2+\eta)}{1+\tilde{\sigma}_0(2+\eta)} - (1-2\tilde{\sigma}_0)\right)(1-\eta^2)h'(1)(|x|-1)\\ &\quad + \frac{\eta^2}{1+\tilde{\sigma}_0(2+\eta)}\frac{\rho}{2}\\ &= \frac{-\tilde{\sigma}_0\eta}{1+\tilde{\sigma}_0(2+\eta)}(1-\eta^2)h'(1)(|x|-1) + \frac{\eta^2}{1+\tilde{\sigma}_0(2+\eta)}\frac{\rho}{2}\,,\end{aligned}$$

and similarly

$$\begin{aligned}(1+\tilde{\sigma}_0(2+\eta))\tilde{h} - &(1+2\tilde{\sigma}_0)(1-\eta^2)h'(1)(|x|-1)\\ &= \frac{\tilde{\sigma}_0\eta}{1-\tilde{\sigma}_0(2+\eta)}(1-\eta^2)h'(1)(|x|-1) + \frac{\eta^2}{1-\tilde{\sigma}_0(2+\eta)}\frac{\rho}{2}\,,\end{aligned}$$

Remembering (6.75) we find

$$\begin{aligned}g - \tilde{u}^\sharp e &\leq (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e - (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e\\ &\quad + (1+2\tilde{\sigma}_0)(1-\eta^2)h'(1)(|x|-1)e + \eta^2\frac{\rho}{2}e + C(\eta+|\eta'|^2)|E|e\\ &= (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e + C(\eta+|\eta'|^2)|E|e\\ &\quad - \eta\left(\frac{\tilde{\sigma}_0(1-\eta^2)}{1-\tilde{\sigma}_0(2+\eta)}h'(1)(|x|-1)e + \frac{\tilde{\sigma}_0(2+\eta)}{1-\tilde{\sigma}_0(2+\eta)}\eta\frac{\rho}{2}e\right)\end{aligned}$$

and also

$$\begin{aligned}g - \tilde{u}^\sharp e &\geq (1-\tilde{\sigma}_0(2+\eta))\tilde{h}e - C(\eta+|\eta'|^2)|E|e\\ &\quad + \eta\left(\frac{\tilde{\sigma}_0(1-\eta^2)}{1+\tilde{\sigma}_0(2+\eta)}h'(1)(|x|-1)e + \frac{\tilde{\sigma}_0(2+\eta)}{1+\tilde{\sigma}_0(2+\eta)}\eta\frac{\rho}{2}e\right)\,.\end{aligned}$$

Now, because of (6.80) we can find ε such that

$$|\eta'|^2 \leq \eta \qquad \text{for } \eta \leq \varepsilon\,.$$

Then, on the region where $\eta > \varepsilon$, we have

$$\eta\left(\frac{\tilde{\sigma}_0(1-\eta^2)}{1-\tilde{\sigma}_0(2+\eta)}h'(1)(|x|-1)e + \frac{\tilde{\sigma}_0(2+\eta)}{1-\tilde{\sigma}_0(2+\eta)}\eta\frac{\rho}{2}e\right) \geq C(\varepsilon)e\,,$$

and consequently, choosing λ_k big enough, we find

$$(1-\tilde{\sigma}_0(2+\eta))\tilde{h}e \le g - \tilde{u}^\sharp e \le (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e\,.$$

On the other hand, when $\eta \le \varepsilon$, it holds

$$\begin{aligned} g - \tilde{u}^\sharp e &\le (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e - \eta\Big(\frac{\tilde{\sigma}_0(1-\eta^2)}{1-\tilde{\sigma}_0(2+\eta)} h'(1)(|x|-1)e \\ &\qquad\qquad + \frac{\tilde{\sigma}_0(2+\eta)}{1-\tilde{\sigma}_0(2+\eta)}\eta\frac{\rho}{2}e - C|E|e\Big) \\ &\le (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e - \eta\left(C(\varepsilon)e - C|E|e\right) \le (1+\tilde{\sigma}_0(2+\eta))\tilde{h}e \end{aligned}$$

if the λ_k's are chosen large enough. The lower bound follows in the same way, concluding the proof of the lemma.

7 INTRINSIC VS. EXTRINSIC PARALLEL TRANSLATION

In this short chapter we investigate another notion which could be helpful for a relaxed version of the Borisov–Gromov problem: parallel translation. We define an extrinsic version of parallel translation on n-dimensional submanifolds of $\mathbb{R}^m$ (where $m \geq n+1$) and show that, if the submanifold is smooth enough, it coincides with the usual (intrinsic) notion. In [2], Borisov introduces similar notions of parallel translation and shows that, under certain geometric conditions, such process preserves the lengths of vectors. In our case, it turns out that this is true whenever the submanifold is the image of an embedding $v \in C^{1,\alpha}$ with $\alpha > \frac{1}{2}$, see (7.1).

To simplify the presentation we work in coordinates. Hence, let $\Omega \subset \mathbb{R}^n$ be an open, bounded set equipped with a smooth Riemannian metric g. Assume that $v \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ is an isometric embedding and let $\gamma : [0,T] \to \overline{\Omega}$ be a smooth curve. Fix a vector $\bar{X}_0 \in T_{\gamma(0)}\overline{\Omega}$ and call $X : [0,T] \to T\overline{\Omega}$ the parallel translate of $\bar{X}_0$ along γ, i.e. $X(0) = \bar{X}_0$, $X(t) \in T_{\gamma(t)}\overline{\Omega}$ and $D_t X = 0$, where D_t is the covariant derivative along the curve γ, i.e. $D_t X(t) = \nabla_{\dot{\gamma}(t)}\tilde{X}$ whenever $\tilde{X}$ is an extension of X and ∇ is the Levi-Civita connection induced by g. We now define a notion of parallel translation on the image $v(\overline{\Omega})$ by discretisation. Fix $k \in \mathbb{N}$ and a partition $P_k = (0 = t_0 < t_1 < \ldots < t_k = T)$ of $[0,T]$. For any point $p \in \overline{\Omega}$ we call $\pi_p : \mathbb{R}^m \to T_{v(p)}v(\overline{\Omega})$ the orthogonal (with respect to the euclidean metric on $\mathbb{R}^m$) projection.

Definition 7.1. The *discrete parallel translate with respect to the partition P_k* of the vector $\bar{X}_0$ along γ is the vectorfield $P_k\bar{X}_0 : [0,T] \to Tv(\overline{\Omega})$ defined for $t \in]t_i, t_{i+1}]$ by

$$P_k\bar{X}_0(t) := \pi_{\gamma(t)}X_i \,,$$

where X_i is iteratively given by $X_i = \pi_{\gamma(t_i)}X_{i-1}$, $X_0 = dv(\bar{X}_0)$.

The main theorem of this chapter is then the following.

Theorem 7.2. *Let $\alpha > \frac{1}{2}$ and suppose $P_k = (0 = t_0 < \ldots < t_k = T)$ is a partition such that $\Delta(P_k) = \max_i |t_i - t_{i-1}| \to 0$ as $k \to \infty$. Then, for any $t \in [0,T]$, we have*

$$|dv(\bar{X}_0)| = \lim_{k\to\infty} |P_k\bar{X}_0(t)| \,. \tag{7.1}$$

If in addition $\alpha > \frac{1}{2}(\sqrt{5}-1)$ and $P_k = (0, \frac{1}{k}T, \ldots, \frac{k-1}{k}T, T)$, then

$$\|P_k\bar{X}_0 - dv(X)\|_{C^0([0,T])} \to 0 \text{ as } k \to \infty \,. \tag{7.2}$$

We will give a short proof of the first assertion in the next subsection; the rest of the chapter is then devoted to proving the second claim. The difficulty in getting a good estimate for (7.2) is that, since v is only $C^{1,\alpha}$, the normal to the surface $v(\overline{\Omega})$ is in general not differentiable, which prevents us from differentiating the expression and using the equation for the parallel translate X. Instead, we mollify the embedding v by convolution with a standard kernel φ_ℓ and derive the theorem on the approximated surface $v * \varphi_\ell(\overline{\Omega})$. Then we show that when the mollification parameter ℓ goes to zero, the discrete parallel translate on the mollified surface (denoted by $P_k^\ell \bar{X}_0$) converges to the corresponding $P_k \bar{X}_0$, and moreover, the parallel translates with respect to the metrics $g_\ell := (v * \varphi_\ell)^\sharp e$ converge to X.

7.0.1 *Invariance of lengths: Proof of* (7.1)

In this subsection we show (7.1). Fix therefore $\alpha > \frac{1}{2}$ and a partition P_k with $\Delta(P_k) \to 0$, let $\bar{X}_0 \in T_{\gamma(0)}\overline{\Omega}$ and fix a $t \in]0, T]$. We first observe that, for any $Y \in \mathbb{R}^m$ the projection $\pi_p Y$ is given by

$$\pi_p Y = Y - \sum_{j=1}^{m-n} \langle Y, \zeta_j|_p \rangle \zeta_j|_p \,, \tag{7.3}$$

where $\zeta_j = \zeta_j(v) : \bar{\Omega} \to \mathbb{R}^m$ are the normal vectorfields provided by Proposition 6.9. We recall that they constitute an orthonormal family which is normal to v, and satisfy

$$[\zeta_j]_\alpha \leq C(1 + [v]_{1,\alpha}) \,. \tag{7.4}$$

Hence, for any $i = 1, \dots, k$ we can write

$$X_i = \pi_{\gamma(t_i)} X_{i-1} = X_{i-1} - \sum_{j=1}^{m-n} \langle X_{i-1}, \zeta_j|_{\gamma(t_i)} \rangle \zeta_j|_{\gamma(t_i)}$$

and thus, as a consequence of the orthonormality,

$$|X_i|^2 = |X_{i-1}|^2 - \sum_{j=1}^{m-n} \left| \langle X_{i-1}, \zeta_j|_{\gamma(t_i)} \rangle \right|^2 = |X_{i-1}|^2 - \sum_{j=1}^{m-n} \left| \langle X_{i-1}, \zeta_j|_{\gamma(t_i)} - \zeta_j|_{\gamma(t_{i-1})} \rangle \right|^2 .$$

By the properties of the projection we clearly have $|X_i| \leq |X_0|$ for any i, and with the help of (7.4) we can estimate

$$\left| |X_i|^2 - |X_{i-1}|^2 \right| \leq C(v) \, |X_0| \, |t_i - t_{i-1}|^{2\alpha} .$$

Similarly, if we fix i such that $t \in]t_i, t_{i+1}]$ then

$$\left| |P_k \bar{X}_0(t)|^2 - |X_i|^2 \right| \leq C(v) |X_0| |t - t_i|^{2\alpha} .$$

From this we can infer

$$\big||P_k\bar{X}_0(t)|^2 - |X_0|^2\big| \leq C(v)|X_0| \sum_{j=0}^{i} |t_{j+1} - t_j|^{2\alpha} \leq C(v)|X_0| T\Delta(P_k)^{2\alpha-1} \,,$$

which shows (7.1).

7.1 DISCRETE PARALLEL TRANSLATE

Fix a radially symmetric, nonnegative function $\varphi \in C_c^\infty(B_1)$ with unit integral and define $\varphi_\ell(x) := \ell^{-n}\varphi(\frac{x}{\ell})$. We extend the function v to $\bar{v} \in C^{1,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$ with $\|\bar{v}\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq C\|v\|_{C^{1,\alpha}(\overline{\Omega})}$ and set $v_\ell : \mathbb{R}^n \to \mathbb{R}^m$ to be $v_\ell = \bar{v} * \varphi_\ell$. Since $\bar{\Omega}$ is compact we have $g > \delta e$ for some $\delta > 0$ and we can choose ℓ so small that $v_\ell^\sharp e > \frac{\delta}{2} e$ on $\bar{\Omega}$.

Analogously to Definition 7.1, given any partition $P_k = (0 = t_0 < \ldots < t_k = T)$ and $t \in]t_i, t_{i+1}]$ we

$$P_k^\ell \bar{X}_0(t) = \pi^\ell_{\gamma(t)} X_i^\ell \,,$$

where now $\pi_p^\ell : \mathbb{R}^m \to T_{v_\ell(p)} v_\ell(\overline{\Omega})$ is the orthogonal projection, and X_i^ℓ are iteratively given by $X_i^\ell = \pi^\ell_{\gamma(t_i)} X^\ell_{i-1}$, $X_0^\ell = dv_\ell(\bar{X}_0)$. Moreover, we let $X^\ell : [0,T] \to T\overline{\Omega}$ be the parallel translate of the vector $\bar{X}_0 \in T_{\gamma(0)}\overline{\Omega}$ along the curve γ with respect to the connection induced by the metric $g_\ell := v_\ell^\sharp e|_{\bar{\Omega}}$. Set $M := \|v\|_{C^{1,\alpha}(\bar{\Omega})}$. We then get the following.

Lemma 7.3. *For any $\alpha > 0$ and any subdivision P_k we have*

$$\|P_k^\ell \bar{X}_0 - dv_\ell\big(X^\ell\big)\|_{C^0([0,T])} \leq C(\gamma, g, M)|\bar{X}_0| \ell^{\alpha-1}\Delta(P_k)^\alpha \,,$$

where, as above, $\Delta(P_k) = \max_i |t_i - t_{i-1}|$.

Proof. Let $\ell > 0$. We fix global coordinates $(x^1, \ldots, x^n)$ on $\overline{\Omega}$ and write the parallel translate X^ℓ in these coordinates as $X^\ell(t) = X^i_\ell(t)\frac{\partial}{\partial x^i}|_{\gamma(t)}$ (using the summation convention). The coefficients X^i_ℓ are the solutions of the system of ODE's

$$\dot{Y}^l(t) = -Y^i(t)\dot{\gamma}^j(t)\Gamma^{l,\ell}_{ij}(\gamma(t))$$

with initial condition $X^i_\ell(0) = \bar{X}^i_0$. Here, $\Gamma^{l,\ell}_{ij}$ are the Christoffel symbols of the connection induced by the metric $g_\ell = v_\ell^\sharp e|_{\bar{\Omega}}$.

We now fix $m - n$ normal vectorfields to the surface $v_\ell(\overline{\Omega})$, provided by Proposition 6.9, namely $\zeta_i = \zeta_i(v_\ell)$. Recall once again that they satisfy the estimates

$$[\zeta_i]_k \leq C_k(1 + \|v_\ell\|_{k+1}) \,. \tag{7.5}$$

In order to improve readibility, in the rest of this proof we identify the vectors $X^\ell(t)$ with their images $dv_\ell\big(X^\ell(t)\big)$. Using an analogous decomposition as in (7.3) we can write, fixing $\bar{t} \in [0, T]$,

$$\frac{d}{dt}\pi^\ell_{\gamma(t)} X^\ell(\bar{t}) = -\sum_{i=1}^{m-n} \langle X^\ell(\bar{t}), \frac{d}{dt}\zeta_i|_{\gamma(t)}\rangle \zeta_i|_{\gamma(t)} - \sum_{i=1}^{m-n} \langle X^\ell(\bar{t}), \zeta_i|_{\gamma(t)}\rangle \frac{d}{dt}\zeta_i|_{\gamma(t)} \,.$$

We use the fact that X^ℓ is parallel along γ and that $X^\ell(t)$ is orthogonal to any $\zeta_i|_{\gamma(t)}$ to write

$$\begin{aligned}
\frac{d}{dt}\left(X^\ell(t) - \pi^\ell_{\gamma(t)} X^\ell(\bar{t})\right) &= \pi^\ell_{\gamma(t)} \frac{d}{dt} X^\ell(t) + \sum_{i=1}^{m-n} \langle \frac{d}{dt} X^\ell(t), \zeta_i|_{\gamma(t)}\rangle \zeta_i|_{\gamma(t)} \\
&\quad - \frac{d}{dt}\pi^\ell_{\gamma(t)} X^\ell(\bar{t}) \\
&= \sum_{i=1}^{m-n} \langle X^\ell(\bar{t}) - X^\ell(t), \frac{d}{dt}\zeta_i|_{\gamma(t)}\rangle \zeta_i|_{\gamma(t)} \\
&\quad + \sum_{i=1}^{m-n} \langle X^\ell(\bar{t}) - X^\ell(t), \zeta_i|_{\gamma(t)}\rangle \frac{d}{dt}\zeta_i|_{\gamma(t)} \,.
\end{aligned}$$

This implies the estimate

$$\left| X^\ell(t) - \pi^\ell_{\gamma(t)} X^\ell(\bar{t}) \right| \le 2 \sum_{i=1}^{m-n} \int_{\bar{t}}^{t} \left| X^\ell(\bar{t}) - X^\ell(s) \right| \left| \frac{d}{dt}\Big|_{t=s} \zeta_i|_{\gamma(t)} \right| ds \,.$$

Now

$$|X^\ell(\bar{t}) - X^\ell(s)|^2 = \sum_{j=1}^{m} |X^i_\ell(\bar{t}) \frac{\partial v^j_\ell}{\partial x^i}(\gamma(\bar{t})) - X^i_\ell(s) \frac{\partial v^j_\ell}{\partial x^i}(\gamma(s))|^2 \,.$$

This yields the estimate

$$\left| X^\ell(\bar{t}) - X^\ell(s) \right| \le C(g, \gamma, M) |\bar{X}_0| |\bar{t} - s|^\alpha \,,$$

where we used the fact that the length of X^ℓ is invariant and that γ is C^1. With the help of (7.5) and the mollification estimate (2.9) we also find

$$\left| \frac{d}{dt}\Big|_{t=s} \zeta_i|_{\gamma(t)} \right| \le C(\gamma) [\nabla v]_\alpha \ell^{\alpha - 1} \,.$$

Consequently

$$\left| X^\ell(t) - \pi^\ell_{\gamma(t)} X^\ell(\bar{t}) \right| \le C(\gamma, g, M) |\bar{X}_0| |t - \bar{t}|^{1+\alpha} \ell^{\alpha - 1} \,.$$

Finally, fix a $t \in]t_i, t_{i+1}]$. Then

$$\begin{aligned}
\left|X^\ell(t) - P_k^\ell \bar{X}_0(t)\right| = \left|X^\ell(t) - \pi^\ell_{\gamma(t)} X_i^\ell\right| &\leq \left|X^\ell(t) - \pi^\ell_{\gamma(t)} X^\ell(t_i)\right| \\
&\quad + \left|\pi^\ell_{\gamma(t)} X^\ell(t_i) - \pi^\ell_{\gamma(t)} X_i^\ell\right| \\
&\leq C|\bar{X}_0||t - t_i|^{1+\alpha}\ell^{\alpha-1} + \left|X^\ell(t_i) - \pi^\ell_{\gamma(t_i)} X^\ell_{i-1}\right| \\
&\leq C|\bar{X}_0|\ell^{\alpha-1}\left(|t - t_i|^{1+\alpha} + \sum_{j=1}^{i} |t_i - t_{i-1}|^{1+\alpha}\right) \\
&\leq C|\bar{X}_0|\ell^{\alpha-1}\Delta(P_k)^\alpha(t - t_0)\,,
\end{aligned}$$

proving the lemma. □

Now we show that the discrete parallel translate on the approximate surface converges to the one on the original surface.

Lemma 7.4. *It holds*

$$\|P_k^\ell \bar{X}_0 - P_k \bar{X}_0\|_{C^0([0,T])} \leq C(\gamma, g, M)|\bar{X}_0|k\ell^\alpha\,.$$

Proof. Fix $\ell > 0$, $k \in \mathbb{N}$, a partition $P_k = (0 = t_0 < \ldots < t_k = T)$ and a vector $Y \in \mathbb{R}^m$. We claim the estimate

$$|\pi^\ell_{\gamma(t)} Y - \pi_{\gamma(t)} Y| \leq C(\gamma, g, M)|Y|\ell^\alpha\,.$$

To see this, consider once again the normal vectorfields $\zeta_i(v_\ell)$ from the proof of the last lemma and abbreviate $\zeta_i^\ell := \zeta_i(v_\ell)$. We then also need normal vectorfields to the original surface $v(\Omega)$, which are also provided by Proposition 6.9. We denote them by $\zeta_i := \zeta_i(v)$. Now, in addition to the estimate (7.5), the normal vectorfields satisfy

$$\left|\zeta_i^\ell - \zeta_i\right| \leq C\|v_\ell - v\|_1 \leq C\|v\|_{1,\alpha}\ell^\alpha\,,$$

as can be seen in (A.32). Therefore, it follows

$$\begin{aligned}
\left|\pi^\ell_{\gamma(t)} Y - \pi_{\gamma(t)} Y\right| &\leq \sum_{i=1}^{m-n} \left|\langle Y, \zeta_i^\ell\rangle\zeta_i^\ell - \langle Y, \zeta_i\rangle\zeta_i\right| \\
&= \sum_{i=1}^{m-n} \left|\langle Y, \zeta_i^\ell - \zeta_i\rangle\zeta_i^\ell + \langle Y, \zeta_i\rangle\left(\zeta_i^\ell - \zeta_i\right)\right| \\
&\leq 2|Y| \sum_{i=1}^{m-n} \left|\zeta_i^\ell - \zeta_i\right| \leq C|Y|\ell^\alpha\,.
\end{aligned}$$

Hence for $t \in]t_i, t_{i+1}]$ we get

$$\begin{aligned}
\left|P_k^\ell \bar{X}_0(t) - P_k \bar{X}_0(t)\right| &\le |\pi^\ell_{\gamma(t)} X_i^\ell - \pi_{\gamma(t)} X_i^\ell| + |\pi_{\gamma(t)}(X_i^\ell - X_i)| \\
&\le C|X_i^\ell|\ell^\alpha + |\pi^\ell_{\gamma(t_i)} X_{i-1}^\ell - \pi_{\gamma(t_i)} X_{i-1}| \\
&\le C\ell^\alpha(|X_i^\ell| + |X_{i-1}^\ell|) + |\pi_{\gamma(t)}(X_{i-1}^\ell - X_{i-1})| \\
&\le C\ell^\alpha \sum_{j=1}^{i} |X_j^\ell| + |X_0^\ell - X_0| \,.
\end{aligned}$$

By construction, we have $|X_j^\ell| \le |X_0^\ell|$ for any j. Combining with

$$|X_0^\ell - X_0|^2 = \sum_{j=1}^{m} |X_0^i \frac{\partial v_\ell^j}{\partial x^i}(\gamma(0)) - X_0^i \frac{\partial v^j}{\partial x^i}(\gamma(0))|^2 \le C[\nabla v]_\alpha^2 \ell^{2\alpha} \sum_{i=1}^{n} |X_0^i|^2 \,.$$

one gets the wanted estimate

$$|P_k^\ell \bar{X}_0(t) - P_k \bar{X}_0(t)| \le C(\gamma, v)|\bar{X}_0| k \ell^\alpha \,. \qquad \square$$

Lastly, we observe that, if $\alpha > \frac{1}{2}$, the parallel translates X^ℓ with respect to g_ℓ converge to the parallel translate X with respect to g.

Lemma 7.5. *It holds*

$$\|X^\ell - X\|_{C^0([0,T])} \le C(\gamma, g, M)|\bar{X}_0|\ell^{2\alpha-1} \,.$$

For this we recall the quadratic estimate of Lemma 6.6 in Chapter 6.

Proposition 7.6 (quadratic estimate). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $v \in C^{1,\alpha}(\Omega, \mathbb{R}^m)$ with $v^\sharp e \in C^2$ and $\varphi \in C^\infty(\mathbb{R}^n)$ a standard symmetric convolution kernel. Then, for every compact set $K \subset \Omega$*

$$\|(v * \varphi_\ell)^\sharp e - v^\sharp e\|_{C^1(K)} = O(\ell^{2\alpha-1}) \,.$$

In particular, this estimate implies the uniform convergence of the Christoffel symbols when $\alpha > \frac{1}{2}$:

$$\|\Gamma_{ij}^{l,\ell} - \Gamma_{ij}^l\|_{C^0(K)} \le C\ell^{2\alpha-1} \,. \tag{7.6}$$

Proof of Lemma 7.5. The estimate follows from a Gronwall argument. Fix real numbers $\bar{X}_0^1, \ldots, \bar{X}_0^n$. The coefficients of X^ℓ and X are the solutions of the system of ODE's

$$\dot{Y}^l = -Y^i \dot{\gamma}^j \Gamma_{ij}^{l,\ell}(\gamma) \quad \text{and} \quad \dot{Y}^l = -Y^i \dot{\gamma}^j \Gamma_{ij}^l(\gamma)$$

respectively, with the same initial condition $Y^l(0) = \bar{X}_0^l$. If we define the $n \times n$ matrices $M_{ij}^\ell := \dot{\gamma}^l \Gamma_{lj}^{i,\ell}(\gamma)$ and $M_{ij} := \dot{\gamma}^l \Gamma_{lj}^i(\gamma)$, then

$$\dot{X}^\ell = -M^\ell \cdot X^\ell \,, \quad \dot{X} = -M \cdot X \,.$$

Set $\beta(t) := |X^\ell(t) - X(t)|^2 = \sum_{i=1}^n |X^i_\ell(t) - X^i(t)|^2$. Then

$$\begin{aligned}
\frac{d}{dt}\beta(t) &= 2\langle X(t) - X^\ell(t), M(t)\cdot X(t) - M^\ell(t)\cdot X^\ell(t)\rangle \\
&= 2\langle X(t) - X^\ell(t), M(t)\cdot (X(t) - X^\ell(t))\rangle \\
&\quad + 2\langle X(t) - X^\ell(t), (M(t) - M^\ell(t))\cdot X^\ell(t)\rangle \\
&\le 2\beta(t)\|M(t)\|_O + C|\bar{X}_0|\|M^\ell(t) - M(t)\|_O\sqrt{\beta(t)} \\
&\le C(\gamma,g)\beta(t) + |\bar{X}_0|^2\|M^\ell(t) - M(t)\|_O^2
\end{aligned}$$

From this we see

$$\begin{aligned}
e^{-C(\gamma,g)t}\beta(t) &= \beta(0) + \int_0^t \left(\frac{d}{dt}\Big|_{t=s} e^{-C(\gamma,g)t}\beta(t)\right) ds \\
&\le \int_0^t e^{-C(\gamma,g)s}|\bar{X}_0|^2\|M^\ell(s) - M(s)\|_O^2\, ds \le C(\gamma,g,M)|\bar{X}_0|^2\ell^{2(2\alpha-1)}t
\end{aligned}$$

thanks to (7.6). From here the estimate follows since

$$\begin{aligned}
|X^i_\ell(t)\frac{\partial v^j_\ell}{\partial x^i}(\gamma(t)) - X^i(t)\frac{\partial v^j}{\partial x^i}(\gamma(t))| &\le |X^i_\ell(t) - X^i(t)|[\nabla v]_0 + |X^i(t)|\ell^\alpha[\nabla v]_\alpha \\
&\le C(\gamma,g,M)|\bar{X}_0|\ell^{2\alpha-1} + C|\bar{X}_0|\ell^\alpha \\
&\le C(\gamma,g,M)|\bar{X}_0|\ell^{2\alpha-1}\,. \qquad \square
\end{aligned}$$

7.2 PROOF OF THEOREM 7.2

The proof of Theorem 7.2 now follows easily. For $\alpha > \frac{1}{2}(\sqrt{5}-1)$ we can choose $\frac{1}{\alpha} < \beta < \frac{\alpha}{1-\alpha}$. Set $\ell(k) = k^{-\beta}$ and fix the subdivision $P_k = (0, \frac{1}{k}T, \frac{2}{k}T, \ldots, \frac{k-1}{k}T, T)$. Then we find

$$\begin{aligned}
\|P_k\bar{X}_0 - dv(X)\|_{C^0([0,T])} &\le \|P_k\bar{X}_0 - P_k^{\ell(k)}\bar{X}_0\|_{C^0} + \|P_k^{\ell(k)}\bar{X}_0 - dv_{\ell(k)}\left(X^{\ell(k)}\right)\|_{C^0} \\
&\quad + \|dv_{\ell(k)}\left(X^{\ell(k)} - X\right)\|_{C^0} + \|dv_{\ell(k)}(X) - dv(X)\|_{C^0} \\
&\le C(\gamma,g,M)\left(k^{1-\alpha\beta} + k^{-\beta(1-\alpha)-\alpha} + k^{-\alpha\beta(2\alpha-1)} + k^{-\alpha\beta}\right),
\end{aligned}$$

which converges to zero as $k \to \infty$ because of the choice of β.

Part III

APPENDIX

A PROOFS OF TECHNICAL LEMMAS AND PROPOSITIONS

A.1 PROOF OF PROPOSITION 3.4

A.1.1 *Beurling and Cauchy transforms*

We will need the following two classical integral operators to construct the coordinate transformation of Proposition 3.4. In this section we use the standard notation $z = x + iy$ for complex numbers. Moreover, we recall two standard differential operators $\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right)$ and $\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$.

Definition A.1. Suppose $G \subset \mathbb{C}$ is a bounded smooth open set and $f : G \to \mathbb{C}$ a function. For $z_0 \in \mathbb{C}$ we define the *Cauchy transform*

$$\mathscr{C}_G[f](z_0) := -\frac{1}{\pi}\int_G \frac{f(z)}{z - z_0}\, dx\, dy$$

and the *Beurling transform*

$$\mathscr{S}_G[f](z_0) := -\frac{1}{\pi}\int_G \frac{f(z)}{(z - z_0)^2}\, dx\, dy\,.$$

The latter integral must be understood as a Cauchy principal value, in case it exists, i.e.

$$\mathscr{S}_G[f](z_0) = \lim_{\varepsilon \to 0} -\frac{1}{\pi}\int_{G\setminus D_\varepsilon(z_0)} \frac{f(z)}{(z - z_0)^2}\, dx\, dy\,.$$

As it is easy to check, the Hölder continuity of f is enough to guarantee its existence at every point. Note that such property uses the fact that

$$\int_{\partial D_\varepsilon(z_0)} \frac{f(z_0)}{(z - z_0)^2}\, dx\, dy\,.$$

Remark A.2. In the literature the terms Cauchy and Beurling transforms are often used only for the operators $\mathscr{C}_{\mathbb{C}}$ and $\mathscr{S}_{\mathbb{C}}$.

In the book of I. N. Vekua [56] one can find the following important properties of the operators $\mathscr{C}_G$ and $\mathscr{S}_G$ (cf. Theorem 1.32 in [56]).

Lemma A.3. *Let $N \in \mathbb{N}$, $0 < \alpha < 1$, $G \subset \mathbb{C}$ bounded and $f \in C^{N,\alpha}(\overline{G})$. Then we have*

(i) $\mathscr{C}_G[f] \in C^{N+1,\alpha}(\overline{G})$ and $\mathscr{S}_G[f] \in C^{N,\alpha}(\overline{G})$;

(ii) $\frac{\partial}{\partial \bar{z}}\mathscr{C}_G[f](z) = f(z)$ *and* $\frac{\partial}{\partial z}\mathscr{C}_G[f](z) = \mathscr{S}_G[f](z)\ \forall z \in G$*;*

(iii) There exists a constant $C_{N,\alpha}$ *such that*

$$\|\mathscr{S}_G[f]\|_{N+\alpha} \le \|\mathscr{C}_G[f]\|_{N+1+\alpha} \le C_{N,\alpha}\|f\|_{N+\alpha}\,.$$

Property (iii) will be key in order to prove Proposition 3.4. Observe that we can easily find solutions of equations of the type $f_{\bar{z}} = g$ by setting $f = \mathscr{C}_G[g]$. Moreover, we have $\partial_{\bar{z}}\mathscr{S}_G[f] = f_z$, so $\mathscr{S}_G$ links the two operators $\partial_{\bar{z}}$ and ∂_z. To prove regularity and get good estimates we need one more thing, namely that under suitable circumstances the transforms commute with differentiation. This will be the content of Corollary A.7, for which we will first need the following lemma.

Lemma A.4. *Let* $r > 0$ *and* $f \in C^1(\overline{D}_r)$*. Then for any* $z_0 \in D_r$ *we have the identities*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D_r}\frac{f(z)}{z-z_0}\,dz - \frac{1}{\pi}\int_{D_r}\frac{f_{\bar{z}}(z)}{z-z_0}\,dx\,dy\,, \tag{A.1}$$

$$\frac{1}{\pi}\int_{D_r}\frac{f(z)}{(z-z_0)^2}\,dx\,dy = \frac{1}{\pi}\int_{D_r}\frac{f_z(z)}{z-z_0}\,dx\,dy + \frac{1}{2\pi i}\int_{\partial D_r}\frac{f(z)}{z-z_0}\,d\bar{z}\,. \tag{A.2}$$

Proof. Take a fixed $z_0 \in D_r$ and consider the differential one-form $\omega = \frac{dz}{z-z_0}$. We can see that

$$d(\omega f) = \frac{f_{\bar{z}}}{z-z_0}\,d\bar{z}\wedge dz = 2i\frac{f_{\bar{z}}}{z-z_0}\,dx\wedge dy\,,$$

hence by Stoke's theorem we have

$$2i\int_{D_r\setminus D_\varepsilon}\frac{f_{\bar{z}}(z)}{z-z_0}\,dx\,dy = \int_{\partial D_r}\frac{f(z)}{z-z_0}\,dz - \int_{\partial D_\varepsilon}\frac{f(z)}{z-z_0}\,dz\,. \tag{A.3}$$

We can easily compute

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)\,,$$

and therefore passing to the limit $\varepsilon \to 0$ in (A.3) yields the first statement; the same reasoning applied to the one-form $\tilde{\omega} = \frac{d\bar{z}}{z-z_0}$ shows the second. □

Remark A.5. Observe that if we define $\Psi(z_0) = \frac{1}{2\pi i}\int_{\partial D_r}\frac{f(z)}{z-z_0}\,dz$ then the statements of the previous lemma can be rewritten as

$$f(z_0) = \Psi(z_0) + \mathscr{C}_{D_r}[f_{\bar{z}}](z_0)\,,$$

$$\mathscr{S}_{D_r}[f](z_0) = \mathscr{C}_{D_r}[f_z](z_0) - \frac{1}{2\pi i}\int_{\partial D_r}\frac{f(z)}{z-z_0}\,d\bar{z}\,.$$

Remark A.6. It follows from Lemma A.4 that if $f \in C^1_0(\overline{D}_r)$, then

(i) $\mathscr{C}_{D_r}[f_{\bar{z}}] = f$,

(ii) $\mathscr{C}_{D_r}[f_z] = \mathscr{S}_{D_r}[f]$.

Combining these two identities with Lemma A.3 we can derive

$$(\mathscr{C}_{D_r}[f])_z = \mathscr{S}_{D_r}[f] = \mathscr{C}_{D_r}[f_z],$$
$$(\mathscr{C}_{D_r}[f])_{\bar{z}} = f = \mathscr{C}_{D_r}[f_{\bar{z}}],$$

and

$$(\mathscr{S}_{D_r}[f])_z = (\mathscr{C}_{D_r}[f_z])_z = \mathscr{S}_{D_r}[f_z],$$
$$(\mathscr{S}_{D_r}[f])_{\bar{z}} = (\mathscr{C}_{D_r}[f_z])_{\bar{z}} = \mathscr{C}_{D_r}[(f_z)_{\bar{z}}] = \mathscr{C}_{D_r}[(f_{\bar{z}})_z] = \mathscr{S}_{D_r}[f_{\bar{z}}].$$

This shows that for (sufficiently regular) functions with compact support in D_r, the operators $\mathscr{C}_{D_r}$ and $\mathscr{S}_{D_r}$ commute with any linear differential operator $\mathscr{D}$ with constant coefficients. The regularity needed on the function is only linked to the order of the operator $\mathscr{D}$.

We summarize the latter discussion in the following

Corollary A.7. *Let $r > 0$ and let $\mathscr{D}$ be a linear differential operator with constant coefficients of order k. Then we have the following identities on $C^k_c(D_r)$:*

(i) $\mathscr{D} \circ \mathscr{C}_{D_r} = \mathscr{C}_{D_r} \circ \mathscr{D}$ *and* $\mathscr{D} \circ \mathscr{S}_{D_r} = \mathscr{S}_{D_r} \circ \mathscr{D}$;

(ii) $\partial_{\bar{z}} \circ \mathscr{C}_{D_r} = \mathscr{C}_{D_r} \circ \partial_{\bar{z}} = Id$ *and* $\partial_z \circ \mathscr{C}_{D_r} = \mathscr{C}_{D_r} \circ \partial_z = \mathscr{S}_{D_r}$;

(iii) $\partial_{\bar{z}} \circ \mathscr{S}_{D_r} = \mathscr{S}_{D_r} \circ \partial_{\bar{z}} = \partial_z$.

A.1.2 *Beltrami's equation*

Using the various properties established above, we take a fundamental step to the proof of Proposition 3.4. As usual we denote by $C^{N,\alpha}_0(\overline{D}_r)$ the closure of $C^{N,\alpha}_c(D_r)$ in the Hölder space $C^{N,\alpha}(\overline{D}_r)$.

Lemma A.8. *Let $r \geq 1$, $N \in \mathbb{N}, N \geq 1$, $0 < \beta \leq \alpha < 1$, $\mu, h \in C^{N,\alpha}_0(\overline{D}_r)$. Then there exist constants $C(N, r, \alpha, \beta)$, $c(N, r, \alpha, \beta)$ and $\bar{C}(\alpha)$ such that if $\|\mu\|_\alpha \leq c$ there exists a solution $\Phi \in C^{N+1,\alpha}(\overline{D}_r)$ to*

$$\Phi_{\bar{z}} - \mu \Phi_z = h \tag{A.4}$$

with

$$\|\Phi\|_{1+\alpha} \leq \bar{C}\|h\|_\alpha, \tag{A.5}$$
$$\|D^k\Phi\|_{1+\beta} \leq C\left(\|D^k h\|_\beta + \|D^k\mu\|_\beta\|h\|_\beta\right), \tag{A.6}$$

for any $1 \leq k \leq N$.

Proof. By a standard approximation argument, it suffices to prove the lemma under the assumption that the supports of μ and h are compactly contained in D_r.

In order to simplify our notation we will use $\mathscr{S}$ and $\mathscr{C}$ in place of $\mathscr{S}_{D_r}$ and $\mathscr{C}_{D_r}$. We know (thanks to Lemma A.3) that $\mathscr{S} : C^{0,\alpha}(\overline{D}_r) \to C^{0,\alpha}(\overline{D}_r)$ as well as $\mathscr{C} : C^{0,\alpha}(\overline{D}_r) \to C^{1,\alpha}(\overline{D}_r)$ and that there exist two constants C_α, C_β (wlog $C_\alpha, C_\beta > 1$) such that

$$\|\mathscr{S}[f]\|_\alpha \leq \|\mathscr{C}[f]\|_{1+\alpha} \leq C_\alpha \|f\|_\alpha \,,$$
$$\|\mathscr{S}[f]\|_\beta \leq \|\mathscr{C}[f]\|_{1+\beta} \leq C_\beta \|f\|_\beta \,.$$

Consider the operator

$$\mathscr{L}_\alpha : C^{0,\alpha}(\overline{D_r}) \to C^{0,\alpha}(\overline{D_r}), \quad f \mapsto h + \mu\mathscr{S}[f]$$

We have

$$\|\mathscr{L}_\alpha(f_1) - \mathscr{L}_\alpha(f_2)\|_\alpha \leq \|\mu\|_\alpha C_\alpha \|f_1 - f_2\|_\alpha \,.$$

So, if

$$\|\mu\|_\alpha \leq \frac{1}{2C_\alpha}$$

then $\mathscr{L}_\alpha$ has a unique fixpoint $f \in C^{0,\alpha}(\overline{D}_r)$. This means

$$f = h + \mu\mathscr{S}[f] \,,$$

from which we deduce

$$\|f\|_\alpha \leq \frac{\|h\|_\alpha}{1 - \|\mu\|_\alpha C_\alpha} \leq 2\|h\|_\alpha$$

and

$$f = (Id - \mu\mathscr{S})^{-1} h = \sum_{n \geq 0} (\mu\mathscr{S})^n h =: \sum_{n \geq 0} \omega_n \,.$$

This shows in particular that f is compactly supported. Using Corollary A.7 one can show by induction that for any $1 \leq k \leq N$ and any $n \geq 1$

$$\|D^k \omega_n\|_\alpha \leq \tilde{C} C_\alpha (2\tilde{C} C_\alpha \|\mu\|_\alpha)^{n-1} \left(\|\mu\|_\alpha \|D^k h\|_\alpha + \|D^k \mu\|_\alpha \|h\|_\alpha \right) \,, \tag{A.7}$$

where $\tilde{C}$ is the constant in (2.5). Therefore, if we require

$$\|\mu\|_\alpha \leq \left(4\tilde{C} C_\alpha C_\beta (2r)^{\alpha-\beta} \right)^{-1} \,, \tag{A.8}$$

then the series

$$\sum_{n \geq 0} D^k \omega_n$$

converges uniformly in $C^{0,\alpha}(\overline{D_r})$ to $D^k f$, hence $f \in C^{N,\alpha}(\overline{D_r})$. Moreover, by the same argument

$$\|D^k f\|_\beta \leq \tilde{C} C_\beta (\|\mu\|_\beta \|D^k h\|_\beta + \|D^k \mu\|_\beta \|h\|_\beta) \sum_{n\geq 1} \left(2\tilde{C} C_\beta (2r)^{\alpha-\beta} \|\mu\|_\alpha\right)^{n-1} \\ + \|D^k h\|_\beta \leq C \left(\|D^k h\|_\beta + \|D^k \mu\|_\beta \|h\|_\beta\right) , \tag{A.9}$$

with the help of (A.8), where the constant C depends only on N, r, α and β. Now we define

$$\Phi(z) = \mathscr{C}[f](z), \quad z \in D_r .$$

By property (iii) of Lemma A.3 we have

$$\Phi_{\bar{z}} = f, \Phi_z = \mathscr{S}[f] ,$$

hence

$$\Phi_{\bar{z}} - \mu \Phi_z = f - \mu \mathscr{S}[f] = (Id - \mu \mathscr{S}) f = h ,$$

so the function Φ solves (A.4) and satisfies

$$\|\Phi\|_{1+\alpha} \leq C_\alpha \|f\|_\alpha \leq 2C_\alpha \|h\|_\alpha .$$

Since $D^k \Phi = \mathscr{C}[D^k f]$ by Corollary A.7 we get by recalling (A.9)

$$\|D^k \Phi\|_{1+\beta} \leq C_\beta \|D^k f\|_\beta \leq C \left(\|D^k h\|_\beta + \|D^k \mu\|_\beta \|h\|_\beta\right) .$$

This shows the claim. □

We immediately get the following

Corollary A.9. *Let $r \geq 1$, $N \in \mathbb{N}, N \geq 1$, $0 < \beta \leq \alpha < 1$, $\mu \in C_0^{N,\alpha}(\overline{D}_r)$. Then there exist constants $C(N, r, \alpha, \beta)$, $c(N, r, \alpha, \beta)$ and $\bar{C}(\alpha)$ such that if $\|\mu\|_\alpha \leq c$ there exists a solution $\Phi \in C^{N+1,\alpha}(\overline{D}_r)$ to the Beltrami equation*

$$\Phi_{\bar{z}} = \mu \Phi_z \tag{A.10}$$

with

$$\|\Phi(z) - z\|_{1+\alpha} \leq \bar{C} \|\mu\|_\alpha , \tag{A.11}$$

$$\|D^k (\Phi(z) - z)\|_{1+\beta} \leq C \|D^k \mu\|_\beta , \tag{A.12}$$

for any $1 \leq k \leq N$.

Proof. In the Lemma A.8 choose $h = \mu$ to recover a constant c such that, if $\|\mu\|_\alpha \leq c$, then we find ϕ solving

$$\phi_{\bar{z}} - \mu\phi_z = \mu\,.$$

Set $\Phi(z) = z + \phi(z)$. Then obviously

$$\Phi_{\bar{z}} = \mu\Phi_z$$

and using Lemma A.8 we find

$$\|\Phi(z) - z\|_{1+\alpha} = \|\phi\|_{1+\alpha} \leq \bar{C}\|\mu\|_\alpha\,,$$

and

$$\|D^k(\Phi(z) - z)\|_{1+\beta} = \|D^k\phi\|_{1+\beta} \leq C\|D^k\mu\|_\beta$$

for any $1 \leq k \leq N$, which is what we wanted. □

A.1.3 *Proof of Proposition 3.4*

Given the estimates of the previous paragraphs, Proposition 3.4 can be proved following the classical approach, see for instance [53, Addendum 1 to Chapter 9]. We report however the argument for the reader's convenience.

With a simple scaling argument we can assume $r = 1$. Let x, y be global coordinates on $\overline{D}_1$. Then g takes the form

$$g = \xi dx^2 + 2\zeta dxdy + \omega dy^2\,,$$

for some functions $\xi, \zeta, \omega \in C^{N,\alpha}(D_1)$. We want to find a function $\Phi : D_1 \to \mathbb{R}^2, (x, y) \mapsto (\Phi_1(x, y), \Phi_2(x, y)) =: (s, t)$ such that in these new coordinates we have

$$g = \rho^2 \circ \Phi^{-1}(s, t)\left(ds^2 + dt^2\right)\,,$$

hence

$$g = \rho^2\left(\left(\Phi_{1x}^2 + \Phi_{2x}^2\right)dx^2 + 2\left(\Phi_{1x}\Phi_{1y} + \Phi_{2x}\Phi_{2y}\right)dxdy + \left(\Phi_{1y}^2 + \Phi_{2y}^2\right)dy^2\right)\,,$$

or

$$g = \rho^2\left(\nabla\Phi_1 \otimes \nabla\Phi_1 + \nabla\Phi_2 \otimes \nabla\Phi_2\right)\,. \tag{A.13}$$

A comparison yields

$$\xi\omega - \zeta^2 = \rho^4\left(\Phi_{1x}\Phi_{2y} - \Phi_{1y}\Phi_{2x}\right)^2 = \rho^4 J\Phi^2\,,$$

with $J\Phi = \det \nabla\Phi$. Consequently

$$\rho^2 = \frac{\sqrt{\Delta}}{J\Phi}\,, \tag{A.14}$$

where $\Delta = \xi\zeta - \omega^2$.

It is convenient to switch to complex notation. Consider $z = x + iy$, $\Phi(z) = \Phi_1(x, y) + i\Phi_2(x, y)$. A computation shows that (A.13) is equivalent to the Beltrami equation for Φ:

$$\Phi_{\bar{z}}(z) = \mu(z)\Phi_z(z), \quad z \in D_1\,, \tag{A.15}$$

with the coefficient

$$\mu = \frac{\xi - \omega + 2i\zeta}{\xi + \omega + 2\sqrt{\Delta}}\,. \tag{A.16}$$

Now we extend g to a symmetric 2×2 tensor to $\mathbb{R}^2$ so that

$$\begin{aligned} \|g - e\|_{\alpha;\mathbb{R}^2} &\leq \bar{C}(\alpha)\|g - e\|_{\alpha;\overline{D}_1}\,, \\ \|g - e\|_{k+\beta;\mathbb{R}^2} &\leq C(N, \alpha, \beta)\|g - e\|_{k+\beta;\overline{D}_1}\,, \end{aligned}$$

for $1 \leq k \leq N$. In particular note that if σ_1 is chosen sufficiently small, then $g \geq \frac{1}{2}e$ on the whole $\mathbb{R}^2$. Repeated applications of (2.3) and (2.5) to the expression (A.16) then yield

$$\|\mu\|_{\alpha;\mathbb{R}^2} \leq C\|g - e\|_{\alpha;D_1}\,, \tag{A.17}$$

$$\|\mu\|_{k+\beta;\mathbb{R}^2} \leq C\|g - e\|_{k+\beta;D_1}\,, \tag{A.18}$$

where the constant in (A.17) is a universal one and the constant in (A.18) depends only on α, β and N. Hence $\mu \in C^{N,\alpha}\left(\mathbb{R}^2\right)$. Next we choose a C^∞ cutoff function η such that

$$\eta(z) = \begin{cases} 1, & \text{if } z \in \overline{D}_1 \\ 0, & \text{if } z \in \mathbb{C} \setminus D_{\frac{3}{2}}\,. \end{cases}$$

With this define a new function

$$\tilde{\mu} = \eta\mu\,.$$

By definition we have $\tilde{\mu} \in C_c^{N,\alpha}(D_2)$, thus by Corollary A.9 there exist constants C, c and $\overline{C}$ such that, if $\|\tilde{\mu}\|_{\alpha;D_2} \leq c$ then there exists $\Phi \in C^{N+1,\alpha}(\overline{D}_2)$ with

$$\Phi_{\bar{z}}(z) = \tilde{\mu}(z)\Phi_z(z), \quad z \in D_2\,,$$

and

$$\begin{aligned} \|\Phi(z) - z\|_{1+\alpha;D_2} &\le \overline{C}\|\tilde{\mu}\|_{\alpha;D_2}\,, \\ \|D^k(\Phi(z) - z)\|_{1+\beta;D_2} &\le C\|\tilde{\mu}\|_{k+\beta;D_2}\,, \end{aligned} \tag{A.19}$$

for any $1 \le k \le N$. Observe that in particular Φ solves (A.15). Moreover,

$$\|\tilde{\mu}\|_{\alpha;D_2} \le \|\mu\|_{\alpha;D_2}\|\eta\|_{\alpha;D_2} \le C\|\mu\|_{\alpha;D_2} \le C\|g - e\|_{\alpha;D_1}\,,$$

and similarly

$$\|\tilde{\mu}\|_{k+\beta;D_2} \le C\|\mu\|_{k+\beta;D_2} \le C\|g - e\|_{k+\beta;D_1}\,,$$

by (A.17) and (A.18). This shows that if $\|g - e\|_{\alpha;D_1} \le \sigma_1$ with σ_1 small enough, we recover a coordinate change Φ solving (A.15). The estimates for Φ follow immediately. For the estimates of ρ we use the fact that due to (A.19) we have

$$(1 - C\|g - e\|_{\alpha,\overline{D}_1})^2 \le J\Phi \le (1 + C\|g - e\|_{\alpha,\overline{D}_1})^2\,,$$

which together with the expression (A.14), the bounds on Φ, (2.3) and (2.5) imply

$$\|D^k\rho\|_\beta \le C\|g - e\|_{k+\beta;\overline{D}_1}$$

for $1 \le k \le N$. This proves the claim.

A.2 PROOF OF PROPOSITION 5.7

The following lemma will imply Proposition 5.7 for cubes. We will then infer the general case with the help of a Whitney decomposition.

Lemma A.10. *Let $\alpha \in]0, 1[$ and fix $f \in C^{0,1}([0, \pi])$ and $g \in C^{0,\alpha}([0, \pi])$. Then*

$$\left|\int_0^\pi f'(t)g(t)\,dt\right| \le C(\alpha, \beta)[f]_{C^{0,\beta}([0,\pi])}\|g\|_{C^{0,\alpha}([0,\pi])}\,, \tag{A.20}$$

for all $\beta \in]1 - \alpha, 1]$.

Proof. Observe that it suffices to prove (A.20) under the additional assumption $f(0) = f(\pi) = 0$. Indeed, consider

$$\tilde{f}(t) = f(t) - f(0) - \frac{t}{\pi}(f(\pi) - f(0))$$

and observe that $\tilde{f}(0) = \tilde{f}(\pi) = 0$ and $[\tilde{f}]_{C^{0,\gamma}([0,\pi])} \le 2[f]_{C^{0,\gamma}([0,\pi])}$ for all $\gamma \in]0, 1]$. Moreover

$$\begin{aligned} \left|\int_0^\pi f'(t)g(t)\,dt\right| &= \left|\int_0^\pi \tilde{f}'(t)g(t)\,dt + \frac{f(\pi) - f(0)}{\pi}\int_0^\pi g(t)\,dt\right| \\ &\le \pi^\beta[f]_{C^{0,\beta}}\|g\|_{C^0} + \left|\int_0^\pi \tilde{f}'(t)g(t)\,dt\right|\,. \end{aligned}$$

Therefore we assume from now on $f(0) = f(\pi) = 0$. Extend f and g to $[-\pi, \pi]$ by setting

$$\bar{f}(t) = \begin{cases} f(t) & \text{for } t \in [0,\pi] \\ -f(-t) & \text{for } t \in [-\pi, 0] \end{cases} \qquad \bar{g}(t) = \begin{cases} g(t) & \text{for } t \in [0,\pi] \\ g(-t) & \text{for } t \in [-\pi, 0] \,. \end{cases}$$

Because of the assumption on f, $\bar{f}$ is continuous and in fact still Lipschitz (and hence almost everywhere differentiable) with $\bar{f}'(t) = \bar{f}'(-t)$ almost everywhere. Also, obviously, $\bar{g}(-t) = \bar{g}(t)$. In particular, we can understand $\bar{f}$ and $\bar{g}$ as functions on $\mathcal{S}^1$ with

$$[\bar{g}]_{C^{0,\alpha}(\mathcal{S}^1)} \leq [g]_{C^{0,\alpha}([0,\pi])}\,,\ [\bar{f}]_{C^{0,\gamma}(\mathcal{S}^1)} \leq 2[f]_{C^{0,\gamma}([0,\pi])} \text{ for every } \gamma \in]0,1]\,.$$

We can expand $\bar{f}$ and $\bar{g}$ in Fourier series with real coeffiecients as

$$\bar{f}(t) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikt}\,,\ \bar{g}(t) = \sum_{k\in\mathbb{Z}} \hat{g}_k e^{ikt}\,.$$

Fix now a $\beta \in]1-\alpha, 1]$ and $\varepsilon \in]0, \alpha+\beta-1[$. Using that $\bar{g}$ and $\bar{f}'$ are even, we can estimate

$$\begin{aligned} \left| \int_0^\pi f'(t) g(t)\, dt \right| &= \frac{1}{2} \left| \int_{-\pi}^\pi \bar{f}'(t)\bar{g}(t)\, dt \right| = \frac{1}{2} \left| \sum_{k\in\mathbb{Z}} ik \hat{f}_k \hat{g}_k \right| \\ &\leq \frac{1}{2} \left(\sum_{k\in\mathbb{Z}} |k|^{2(\beta-\varepsilon)} |\hat{f}_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} |k|^{2(1-(\beta-\varepsilon))} |\hat{g}_k|^2 \right)^{\frac{1}{2}} \\ &= C(\beta,\varepsilon) [\bar{f}]_{H^{\beta-\varepsilon}(\mathcal{S}^1)} [\bar{g}]_{H^{1-(\beta-\varepsilon)}(\mathcal{S}^1)}\,, \end{aligned}$$

where we recall that $[\cdot]_{H^\gamma(\mathcal{S}^1)}$ is the Gagliardo-seminorm of the fractional Sobolev space $H^\gamma(\mathcal{S}^1) = W^{\gamma,2}(\mathcal{S}^1)$ and is given by

$$[u]^2_{H^\gamma(\mathcal{S}^1)} = \int_{\mathcal{S}^1} \int_{\mathcal{S}^1} \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\gamma}}\, dt\, ds\,.$$

The equivalence (up to a constant) of the Gagliardo seminorm and the Fourier-type norm follows from Plancharel's formula. Moreover one can infer immediately that, for any $0 < \gamma' < \gamma < 1$,

$$[u]_{H^{\gamma'}(\mathcal{S}^1)} \leq C(\gamma,\gamma') [u]_{C^{0,\gamma}(\mathcal{S}^1)}\,.$$

This implies

$$\left| \int_0^\pi f'(t) g(t)\, dt \right| \leq C(\alpha,\beta) [f]_{C^{0,\beta}([0,\pi])} [g]_{C^{0,\alpha}([0,\pi])}\,,$$

which completes the proof. □

By scaling it follows easily from (A.20) that

$$\left| \int_a^b f'(t)g(t)\,dt \right| \leq C(\alpha,\beta)|b-a|^\beta [f]_{C^{0,\beta}([a,b])} \|g\|_{C^{0,\alpha}([a,b])}$$

whenever one of the sides of the inequality makes sense. Therefore, with Fubini, we get (5.9) for cubes Q with sidelength $L > 0$:

$$\left| \int_Q \frac{\partial f}{\partial x_i}\, g\, dx \right| \leq C(\alpha,\beta) L^{1+\beta} [f]_{C^{0,\beta}(\bar{Q})} \|g\|_{C^{0,\alpha}(\bar{Q})} . \tag{A.21}$$

Finally, for an open, bounded $U \subset \mathbb{R}^2$ with $d := \dim_b(\partial U) < 2 - \alpha$ we consider its Whitney decomposition W. Let $W_k = \{Q \in W : Q \text{ has sidelength } 2^{-k}\}$. By Theorem 3.12 in [44] there exists $C > 0$ such that $\# W_k \leq C 2^{kd}$ for any $k \in \mathbb{N}$. We can then estimate with the help of (A.21)

$$\begin{aligned} \left| \int_U \frac{\partial f}{\partial x_i} g dx \right| &= \left| \sum_{k\in\mathbb{N}} \sum_{Q \in W_k} \int_Q \frac{\partial f}{\partial x_i} g dx \right| \leq C(\alpha,\beta)[f]_{C^{0,\beta}(\bar{U})} \|g\|_{C^{0,\alpha}(\bar{U})} \sum_{k\in\mathbb{N}} 2^{kd} (2^{-k})^{1+\beta} \\ &= C(\alpha,\beta)[f]_{C^{0,\beta}(\bar{U})} \|g\|_{C^{0,\alpha}(\bar{U})} \sum_{k\in\mathbb{N}} 2^{k(d-(1+\beta))} \\ &= C(\alpha,\beta,d)[f]_{C^{0,\beta}(\bar{U})} \|g\|_{C^{0,\alpha}(\bar{U})} , \end{aligned}$$

where the convergence of the sum follows from $d < 2 - \alpha < 1 + \beta$. This proves the estimate (5.9). We then immediately get

$$|B_U(f,g) - B_U(\tilde{f},\tilde{g})| \leq C(\alpha,\beta,d) \left([f]_{C^{0,\beta}} \|g - \tilde{g}\|_{C^{0,\alpha}} + [f - \tilde{f}]_{C^{0,\beta}} \|\tilde{g}\|_{C^{0,\alpha}} \right) , \tag{A.22}$$

which shows local uniform continuity and also the existence of the unique extension: for $\beta \in]1-\alpha, 1]$ we can find $1 - \alpha < \beta' < \beta$ and, for any $f \in C^{0,\beta}(\bar{U})$, a smooth sequence f_k converging to f in $C^{0,\beta'}$. Thanks to (A.22) the sequence $B_U(f_k, g)$ then converges and the limit doesn't depend on the approximating sequence. The estimate (5.10) follows at once from (A.20) by writing ∂U as the finite union of Lipschitz curves. This concludes the proof.

A.3 PROOF OF LEMMA 5.8

First observe that it suffices to prove (5.11) for smooth f, g. Indeed, in the general case approximate f, g by sequences $f_k, g_k \in C^\infty(U)$ with $\|f_k - f\|_{C^{0,\alpha'}(\bar{U})} \to 0$ for every $\alpha' < \alpha$ and $\|f_k\|_{C^{0,\alpha}(\bar{U})} \leq \|f\|_{C^{0,\alpha}(\bar{U})}$ (and analoguously for g_k). Because of the uniform convergence $f_k, g_k \to f, g$ and the boundedness of U we get by dominated convergence and Proposition 5.7, assuming (5.11) for smooth functions,

$$\begin{aligned}
&\left| \int_U d\psi * \varphi_\varepsilon \wedge g df - \int_U d\psi \wedge g * \varphi_\varepsilon df * \varphi_\varepsilon \right| \\
&\qquad \le \limsup_{k\to\infty} \left(\left| \int_U d\psi * \varphi_\varepsilon \wedge g_k df_k - \int_U d\psi \wedge g_k * \varphi_\varepsilon df_k * \varphi_\varepsilon \right| \right. \\
&\qquad\qquad \left. + \left| \int_U d\psi \wedge (g_k * \varphi_\varepsilon df_k * \varphi_\varepsilon - g * \varphi_\varepsilon df * \varphi_\varepsilon) \right| \right) \\
&\qquad \le C(\alpha,\beta) M \varepsilon^{3\alpha - 2\beta - 1} \|f\|_{C^{0,\alpha}(\bar{U})} \|g\|_{C^{0,\alpha}(\bar{U})} .
\end{aligned}$$

Therefore we can assume from now on that f and g are smooth. Since ψ has compact support in U we have

$$\int_U d\psi * \varphi_\varepsilon \wedge g df = \int_U d\psi \wedge (g df) * \varphi_\varepsilon .$$

Now observe that, for any $x \in \operatorname{supp}(\psi)$, we have

$$\begin{aligned}
(gdf) * \varphi_\varepsilon(x) - g * \varphi_\varepsilon(x) df * \varphi_\varepsilon(x) = ((g - g(x))(df - df(x)) * \varphi_\varepsilon(x) \\
- (g - g(x))\,(df - df(x)) * \varphi_\varepsilon(x) ,
\end{aligned}$$

so that

$$\begin{aligned}
\int_U d\psi \wedge ((gdf) * \varphi_\varepsilon - g * \varphi_\varepsilon df * \varphi_\varepsilon) &= \int_U d\psi(x) \wedge ((g - g(x))\,(df - df(x))) * \varphi_\varepsilon(x) \\
&\quad - \int_U d\psi(x) \wedge (g - g(x)) * \varphi_\varepsilon(x)\,(df - df(x)) * \varphi_\varepsilon(x) \\
&=: \mathrm{I} - \mathrm{II} .
\end{aligned}$$

To estimate the integral I we observe that it consists (modulo sign) of terms

$$\mathrm{I}_{ij} = \int_U \partial_i \psi(x) \left((g - g(x)) \left(\partial_j f - \partial_j f(x)\right)\right) * \varphi_\varepsilon(x)\, dx ,$$

for $i \neq j$. Let $V = \operatorname{supp}(\psi)$ and fix $\frac{1}{2} < \beta < \alpha$. Using Fubini and Proposition 5.7 we can estimate

$$\begin{aligned}
\left|\mathrm{I}_{ij}\right| &\le \sup_{y \in B_\varepsilon(0)} \left| \int_U \partial_i \psi(x) \left(g(x-y) - g(x)\right) \left(\partial_j f(x-y) - \partial_j f(x)\right) \varphi_\varepsilon(y)\, dx \right| \\
&\le \sup_{y \in B_\varepsilon(0)} \|\partial_i \psi \Delta_y g\|_{C^{0,\beta}(V)} [\Delta_y f]_{C^{0,\beta}(V)} ,
\end{aligned}$$

where we used the notation $\Delta_y f$ for the function $\Delta_y f(x) = f(x-y) - f(x)$. We use the interpolation inequalities (2.2) to estimate

$$[\Delta_y f]_{C^{0,\beta}(V)} \le C(\alpha,\beta) \|\Delta_y f\|_{C^0(V)}^{1-\beta/\alpha} [\Delta_y f]_{C^{0,\alpha}(V)}^{\beta/\alpha} \le C(\alpha,\beta) |y|^{\alpha-\beta} [f]_{C^{0,\alpha}(\bar{U})} ,$$

if $\varepsilon < \operatorname{dist}(\operatorname{supp}(\psi), \partial U)$. On the other hand

$$\begin{aligned}\|\partial_i\psi\Delta_y g\|_{C^{0,\beta}(V)} &\leq C\left([\psi]_{C^1(V)}[\Delta_y g]_{C^{0,\beta}(V)} + [\psi]_{C^{1,\beta}(V)}\|\Delta_y g\|_{C^0(V)}\right)\\ &\leq C(\alpha,\beta)M[g]_{C^{0,\alpha}(\bar{U})}\left(\varepsilon^{\alpha-1}|y|^{\alpha-\beta} + \varepsilon^{\alpha-\beta-1}|y|^{\alpha}\right).\end{aligned}$$

Combining the estimates and $|y| < \varepsilon$ leads to the wanted bound on I. The bound for II is shown analogously.

A.4 PROOFS OF PROPOSITIONS 6.9 AND 6.10

A.4.1 *Proof of Proposition 6.9*

To prove Proposition 6.9 we need the following well known lemma, an elementary proof of which is contained, for example, in [19].

Lemma A.11. *Let n, d, B, u be as in the assumptions of Proposition 6.9. For every $1 \leq k \leq d$ there exist $\zeta_1, \ldots, \zeta_k \in C^\infty\left(B, \mathbb{R}^{n+d}\right)$ such that for all $1 \leq i, j \leq d$ we have*

$$\langle \zeta_i, \zeta_j\rangle = \delta_{ij} \quad \text{on } B\,, \tag{A.23}$$

$$\nabla u \cdot \zeta_i = 0 \quad \text{on } B\,. \tag{A.24}$$

Proof of Proposition 6.9. In the proof all the constants appearing may depend on the embedding u. Fix $0 < \rho_0 < 1$ and let $v \in C^\infty(B, \mathbb{R}^{n+d})$ be such that $\|v - u\| < \rho_0$. Since B is compact and u is an embedding there exists a constant $C > 0$ such that

$$C^{-1}\mathrm{Id} \leq \nabla u^\intercal \nabla u \leq C\,\mathrm{Id}$$

in the sense of quadratic forms. Hence if ρ_0 is small enough we have

$$(2C)^{-1}\mathrm{Id} \leq \nabla v^\intercal \nabla v \leq 2C\,\mathrm{Id}\,, \tag{A.25}$$

and consequently also

$$(2C)^{-n} \leq \det(\nabla v^\intercal \nabla v) \leq (2C)^n\,. \tag{A.26}$$

Let $\zeta_1, \ldots, \zeta_m \in C^\infty(B, \mathbb{R}^{n+d})$ be the maps from Lemma A.11 and define

$$\nu_i(v) := \zeta_i - \sum_{j=1}^n r_{ij}(v)\partial_j v\,, \tag{A.27}$$

where $r_{ij}(v)$ are such that $\langle \nu_i(v), \partial_k v\rangle = 0$ for every k. We claim that the functions $r_{ij}(v) \in C^\infty(B, \mathbb{R}^{n+d})$ depend smoothly on ∇v and satisfy the estimates

$$\|r_{ij}(v)\|_k \leq C_k\|v - u\|_{k+1} \quad \text{for } k \geq 0\,. \tag{A.28}$$

To see this, denote $b_{ik}(v) = \langle \zeta_i, \partial_k v \rangle$ and observe that

$$0 = \langle \nu_i(v), \partial_k v \rangle = b_{ik}(v) - \sum_{j=1}^{n} r_{ij}(v) \langle \partial_j v, \partial_k v \rangle \,,$$

i.e.

$$R(v) \cdot \nabla v^{\mathsf{T}} \nabla v = B(v) \,,$$

where $R(v)$ and $B(v)$ are the $m \times n$ matrices with entries $r_{ij}(v)$ and $b_{ij}(v)$ respectively. By (A.25), $R(v)$ is uniquely determined. We write

$$(\nabla v^{\mathsf{T}} \nabla v)^{-1}_{ij} = (\det \nabla v^{\mathsf{T}} \nabla v)^{-1} P_{ij}(\nabla v) \,,$$

where $P_{ij}(\nabla v)$ is a polynomial in the arguments $\partial_k v^l$. Since by assumption $[v]_1 \leq [u]_1 + 1$, Proposition 2.1 yields

$$[P_{ij}(\nabla v)]_k \leq C_k [v]_{k+1} \,.$$

Moreover, (A.26) implies

$$[(\det \nabla v^{\mathsf{T}} \nabla v)^{-1}]_k \leq C_k [v]_{k+1} \,,$$

so that

$$[(\nabla v^{\mathsf{T}} \nabla v)^{-1}_{ij}]_k \leq C_k [v]_{k+1} \,. \tag{A.29}$$

For the other factor we observe that $b_{ij}(v) = \langle \zeta_i, \partial_j v - \partial_j u \rangle$, since ζ_i is orthogonal to $Tu(B)$ at any point. Whence, by the Leibnitz rule

$$[b_{ij}(v)]_k \leq C_k([v-u]_1 + [v-u]_{k+1}) \leq C_k \|v-u\|_{k+1} \,. \tag{A.30}$$

Combining (A.29) and (A.30) leads to the estimate (A.28).

As a consequence, we can deduce

$$\delta_{ij} - \frac{1}{2d} \leq \langle \nu_i(v), \nu_j(v) \rangle \leq \delta_{ij} + \frac{1}{2d} \tag{A.31}$$

for ρ_0 small enough. This implies that the family $\{\nu_i(v)\}_{i=1,\dots,d}$ is linearly independent at every point and thus (being in addition orthogonal to $Tv(B)$) constitutes a frame for the normal bundle $Nv(B)$. The wanted vectorfields ζ_i are then produced by a Gram-Schmidt normalization procedure. To get the estimates (6.37) we carry out the procedure in details.

Therefore, we set

$$\zeta_1(v) := \frac{\nu_1(v)}{|\nu_1(v)|} \,.$$

If ρ_0 is small enough, then $|\nu_i(v)| \geq \frac{1}{2}$ for every i (thanks to (A.28)), and so $\zeta_1(v)$ is a smooth function with

$$[\zeta_1(v)]_k \leq C_k[\nu_1(v)]_k \leq C_k(1 + \|v-u\|_{k+1}) \leq C_k(1 + \|v\|_{k+1})\,.$$

Moreover

$$|\zeta_1(v) - \zeta_1| \leq \frac{2|\nu_1(v) - \zeta_1|}{|\nu_1(v)|} \leq C\|v-u\|_1\,.$$

We now assume that $\zeta_1(v), \ldots, \zeta_{l-1}(v)$ are already constructed, satisfying (6.37)-(6.39) and in addition

$$\|\zeta_i(v) - \zeta_i\|_0 \leq C\|v-u\|_1\,. \tag{A.32}$$

We then set

$$\theta_l(v) = \nu_l(v) - \sum_{j=1}^{l-1} \langle \nu_l(v), \zeta_j(v)\rangle \zeta_j(v)$$

and $\zeta_l(v) = \dfrac{\theta_l(v)}{|\theta_l(v)|}$. It remains to show that $\zeta_l(v)$ satisfies (6.37)-(6.39) and (A.32). Observe that

$$\langle \nu_l(v), \zeta_j(v)\rangle = \langle \nu_l(v) - \zeta_l, \zeta_j(v)\rangle + \langle \zeta_l, \zeta_j(v) - \zeta_j\rangle$$

so that $\|\langle \nu_l(v), \zeta_j(v)\rangle\|_0 \leq C\|v-u\|_1$ and

$$\begin{aligned}
[\langle \nu_l(v), \zeta_j(v)\rangle]_k &\leq C_k\big(1 + [r_{ij}(v)]_k + \|r_{ij}(v)\|_0[v]_{k+1} + \|v-u\|_1(1 + \|v\|_{k+1}) \\
&\qquad\qquad + [\zeta_j(v) - \zeta_j]_k\big) \\
&\leq C_k(1 + \|v\|_{k+1})\,.
\end{aligned}$$

In particular $|\theta_l(v)| \geq \frac{1}{4}$ for ρ_0 small enough and

$$[\theta_l(v)]_k \leq C_k(1 + \|v\|_{k+1})\,.$$

Therefore $\zeta_l(v)$ satisfies (6.37)-(6.39). Since moreover

$$\begin{aligned}
|\zeta_l(v) - \zeta_l| &\leq \frac{2|\theta_l(v) - \zeta_l|}{|\theta_l(v)|} \leq C(|\theta_l(v) - \nu_l(v)| + |\nu_l(v) - \zeta_l|) \\
&\leq C\|v-u\|_1
\end{aligned}$$

the proposition is proved. $\square$

A.4.2 *Proof of Proposition 6.10*

For the proof of Proposition 6.10 we need the following lemma from [17].

Lemma A.12. *Let $g_0 \in Sym_n^+$. There exists $r \equiv r(g_0, n) > 0$, $\nu_1, \ldots, \nu_{n_*} \in \mathcal{S}^{n-1}$, and linear maps $L_1, \ldots, L_{n_*} : Sym_n \to \mathbb{R}$ such that*

$$g = \sum_{k=1}^{n_*} L_k(g) \nu_k \otimes \nu_k \,,$$

for every $g \in Sym_n$. Moreover, if $g \in Sym_n$ is such that $|g - g_0| < r$, then $L_k(g) > r$ for every k.

Now the proposition is an easy consequence of the classical implicit function theorem.

Proof of Proposition 6.10. Let $r > 0$ be the radius and $\nu_1, \ldots, \nu_{n_*} \in \mathcal{S}^{n-1}$ be the vectors given by Lemma A.12 when $g_0 = \mathrm{Id}_n$ and define the map

$$\begin{aligned} \Psi : (\mathrm{Sym}_n)^{n_*^2} \times (\mathrm{Sym}_n)^{n_*} \times \mathbb{R}^{n_*} \times \mathbb{R}^{n_*} &\to \mathrm{Sym}_n \\ \big(\{G_{ij}\}, \{M_i\}, g, \{c_i\}\big) &\mapsto \sum_{i}^{n*} c_i^2 \nu_i \otimes \nu_i + \sum_{i=1}^{n_*} c_i M_i + \sum_{i,j=1}^{n_*} c_i c_j G_{ij} - g \,. \end{aligned}$$

Ψ is smooth and by Lemma A.12 there exist $\bar{c}_1, \ldots, \bar{c}_{n*} \in \mathbb{R}$ with $\bar{c}_j > r$ for every j and

$$\Psi(0, 0, \mathrm{Id}_n, \{\bar{c}_j\}) = 0 \,, \quad \partial_{c_i} \Psi|_{(0,0,\mathrm{Id}_n,\{\bar{c}_j\})} = 2\bar{c}_i \nu_i \otimes \nu_i \,.$$

Since the family $\{\nu_i \otimes \nu_i\}$ is linearly independent the differential of Ψ with respect to the variable $c = (c_1, \ldots, c_{n*})$ has full rank at $(0, 0, \mathrm{Id}_n, \bar{c})$. Consequently, by the implicit function theorem, there exist neighborhoods V of $(0, 0, \mathrm{Id}_n)$ and U of $\bar{c}$ respectively and a diffeomorphism $\Phi : V \to U$ such that

$$\begin{aligned} &\{\Psi = 0\} \cap (V \times \mathbb{R}^{n_*}) \\ &\qquad = \{(\{G_{ij}\}, \{M_i\}, g, \Phi(\{G_{ij}\}, \{M_i\}, g)) : (\{G_{ij}\}, \{M_i\}, g) \in V\} \,. \end{aligned}$$

Therefore, if r_0 is small enough $c_k(x) := \Phi(\{G_{ij}(x)\}, \{M_i(x)\}, \tau(x))_k$ will satisfy (6.40). The estimates (6.41) are then a consequence of Proposition 2.1. $\square$

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