

Chapter 1

h -principle and rigidity for $C^{1,\alpha}$ isometric embeddings

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Abstract In this paper we study the embedding of Riemannian manifolds in low codimension. The well-known result of Nash and Kuiper [21, 20] says that any short embedding in codimension one can be uniformly approximated by C^1 isometric embeddings. This statement clearly cannot be true for C^2 embeddings in general, due to the classical rigidity in the Weyl problem. In fact Borisov extended the latter to embeddings of class $C^{1,\alpha}$ with $\alpha > 2/3$ in [3, 5]. On the other hand he announced in [6] that the Nash-Kuiper statement can be extended to local $C^{1,\alpha}$ embeddings with $\alpha < (1 + n + n^2)^{-1}$, where n is the dimension of the manifold, provided the metric is analytic. Subsequently a proof of the 2-dimensional case appeared in [7]. In this paper we provide analytic proofs of all these statements, for general dimension and general metric.

1.1 Introduction

Let M^n be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g . An isometric immersion of (M^n, g) into \mathbb{R}^m is a map $u \in C^1(M^n; \mathbb{R}^m)$ such that the induced metric agrees with g . In local coordinates this amounts to the system

$$\partial_i u \cdot \partial_j u = g_{ij} \tag{1.1}$$

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consisting of $n(n+1)/2$ equations in m unknowns. If in addition u is injective, it is an isometric embedding. Assume for the moment that $g \in C^\infty$. The two classical theorems concerning the solvability of this system are:

- (A) if $m \geq (n+2)(n+3)/2$, then any short embedding can be uniformly approximated by isometric embeddings of class C^∞ (Nash [22], Gromov [16]);
- (B) if $m \geq n+1$, then any short embedding can be uniformly approximated by isometric embeddings of class C^1 (Nash [21], Kuiper [20]).

Recall that a short embedding is an injective map $u : M^n \rightarrow \mathbb{R}^m$ such that the metric induced on M by u is shorter than g . In coordinates this means that $(\partial_i u \cdot \partial_j u) \leq (g_{ij})$ in the sense of quadratic forms. Thus, (A) and (B) are not merely existence theorems, they show that there exists a huge (essentially C^0 -dense) set of solutions. This type of abundance of solutions is a central aspect of Gromov's h -principle, for which the isometric embedding problem is a primary example (see [16, 12]).

Naively, this type of flexibility could be expected for high codimension as in (A), since then there are many more unknowns than equations in 1.1. The h -principle for C^1 isometric embeddings is on the other hand rather striking, especially when compared to the classical rigidity result concerning the Weyl problem: if (S^2, g) is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into \mathbb{R}^3 , then u is uniquely determined up to a rigid motion ([8, 17], see also [30] for a thorough discussion). Thus it is clear that isometric immersions have a completely different qualitative behaviour at low and high regularity (i.e. below and above C^2).

This qualitative difference is further highlighted by the following optimal mapping properties in the case when m is allowed to be sufficiently high:

- (C) if $g \in C^{l,\beta}$ with $l + \beta > 2$ and m is sufficiently large, then there exists a solution $u \in C^{l,\beta}$ (Nash [22], Jacobowitz [18]);
- (D) if $g \in C^{l,\beta}$ with $0 < l + \beta < 2$ and m is sufficiently large, then there exists a solution $u \in C^{1,\alpha}$ with $\alpha < (l + \beta)/2$ (Källen [19]).

These results are optimal in the sense that in both cases there exists $g \in C^{l,\beta}$ to which no solution u has better regularity than stated.

The techniques are also different: whereas the proofs of (A) and (C) rely on the Nash-Moser implicit function theorem, the proofs of (B) and (D) involve an iteration technique called convex integration. This technique was developed by Gromov [15, 16] into a very powerful tool to prove the h -principle in a wide variety of geometric problems (see also [12, 32]). In general the regularity of solutions obtained using convex integration agrees with the highest derivatives appearing in the equations (see [31]). Thus, an interesting question raised in [16] p219 is how one could extend the methods to produce more regular solutions. Essentially the same question, in the case of isometric embeddings, is also mentioned in [33] (see Problem 27). For high codimension this is resolved in (D).

Our primary aim in this paper is to consider the low codimension case, i.e. when $m = n + 1$. This range was first considered by Borisov. In [6] it was announced that if g is analytic, then the h -principle holds for local isometric embeddings $u \in C^{1,\alpha}$

for $\alpha < \frac{1}{1+n+n^2}$. A proof for the case $n = 2$ appeared in [7]. Our main result is to provide a proof of the h -principle in this range for g which is not necessarily analytic and general $n \geq 2$ (see Section 1.1.1 for precise statements). Moreover, at least for $l = 0$ and sufficiently small $\beta > 0$, we recover the optimal mapping range corresponding to (D). Thus, there seems to be a direct trade-off between codimension and regularity.

The novelty of our approach, compared to Borisov's, is that only a finite number of derivatives need to be controlled. This is achieved by introducing a smoothing operator in the iteration step, analogous to the device of Nash used to overcome the loss of derivative problem in [22]. A similar method was used by Källen in [19]. See Section 1.3 for an overview of the iteration procedure. In addition, the errors coming from the smoothing operator are controlled by using certain commutator estimates on convolutions. These estimates are in Section 1.2.

Concerning rigidity in the Weyl problem, it is known from the work of Pogorelov and Sabitov that

1. closed C^1 surfaces with positive Gauss curvature and bounded extrinsic curvature are convex (see [25]);
2. closed convex surfaces are rigid in the sense that isometric immersions are unique up to rigid motion [24];
3. a convex surface with metric $g \in C^{l,\beta}$ with $l \geq 2, 0 < \beta < 1$ and positive curvature is of class $C^{l,\beta}$ (see [25, 26]).

Thus, extending the rigidity in the Weyl problem to $C^{1,\alpha}$ isometric immersions can be reduced to showing that the image of the surface has bounded extrinsic curvature (for definitions see Section 1.7). Using geometric arguments, in a series of papers [1, 2, 3, 4, 5] Borisov proved that for $\alpha > 2/3$ the image of surfaces with positive Gauss curvature has indeed bounded extrinsic curvature. Consequently, rigidity holds in this range and in particular $2/3$ is an upper bound on the range of Hölder exponents that can be reached using convex integration.

Using the commutator estimates from Section 1.2, at the end of this paper (in Section 1.7) we provide a short and self-consistent analytic proof of this result.

1.1.1 The h -principle for small exponents

In this subsection we state our main existence results for $C^{1,\alpha}$ isometric immersions. One is of local nature, whereas the second is global. Note that for the local result the exponent matches the one announced in [6]. In what follows, we denote by sym_n^+ the cone of positive definite symmetric $n \times n$ matrices. Moreover, given an immersion $u : M^n \rightarrow \mathbb{R}^m$, we denote by $u^\sharp e$ the pullback of the standard Euclidean metric through u , so that in local coordinates

$$(u^\sharp e)_{ij} = \partial_i u \cdot \partial_j u.$$

Finally, let

$$n_* = \frac{n(n+1)}{2}.$$

Theorem 1 (Local existence). *Let $n \in \mathbb{N}$ and $g_0 \in \text{sym}_n^+$. There exists $r > 0$ such that the following holds for any smooth bounded open set $\Omega \subset \mathbb{R}^n$ and any Riemannian metric $g \in C^\beta(\overline{\Omega})$ with $\beta > 0$ and $\|g - g_0\|_{C^0} \leq r$. There exists a constant $\delta_0 > 0$ such that, if $u \in C^2(\overline{\Omega}; \mathbb{R}^{n+1})$ and α satisfy*

$$\|u^\sharp e - g\|_0 \leq \delta_0^2 \quad \text{and} \quad 0 < \alpha < \min \left\{ \frac{1}{1+2n_*}, \frac{\beta}{2} \right\},$$

then there exists a map $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n+1})$ with

$$v^\sharp e = g \quad \text{and} \quad \|v - u\|_{C^1} \leq C \|u^\sharp e - g\|_{C^0}^{1/2}.$$

Corollary 1 (Local h-principle). *Let $n, g_0, \Omega, g, \alpha$ be as in Theorem 1. Given any short map $u \in C^1(\overline{\Omega}; \mathbb{R}^{n+1})$ and any $\varepsilon > 0$ there exists an isometric immersion $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n+1})$ with $\|u - v\|_{C^0} \leq \varepsilon$.*

Theorem 2 (Global existence). *Let M^n be a smooth, compact manifold with a Riemannian metric $g \in C^\beta(M)$ and let $m \geq n + 1$. There is a constant $\delta_0 > 0$ such that, if $u \in C^2(M; \mathbb{R}^m)$ and α satisfy*

$$\|u^\sharp e - g\|_{C^0} \leq \delta_0^2 \quad \text{and} \quad 0 < \alpha < \min \left\{ \frac{1}{1+2(n+1)n_*}, \frac{\beta}{2} \right\},$$

then there exists a map $v \in C^{1,\alpha}(M; \mathbb{R}^m)$ with

$$v^\sharp e = g \quad \text{and} \quad \|v - u\|_{C^1} \leq C \|u^\sharp e - g\|_{C^0}^{1/2}.$$

Corollary 2 (Global h-principle). *Let (M^n, g) and α be as in Theorem 2. Given any short map $u \in C^1(M; \mathbb{R}^m)$ with $m \geq n + 1$ and any $\varepsilon > 0$ there exists an isometric immersion $v \in C^{1,\alpha}(M; \mathbb{R}^m)$ with $\|u - v\|_{C^0} \leq \varepsilon$.*

Remark 1. In both corollaries, if u is an embedding, then there exists a corresponding v which in addition is an embedding.

1.1.2 Rigidity for large exponents

The following is a crucial estimate on the metric pulled back by standard regularizations of a given map.

Proposition 1 (Quadratic estimate). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $v \in C^{1,\alpha}(\Omega, \mathbb{R}^m)$ with $v^\sharp e \in C^2$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ a standard symmetric convolution kernel. Then, for every compact set $K \subset \Omega$,*

$$\|(v * \varphi_\ell)^\sharp e - v^\sharp e\|_{C^1(K)} = O(\ell^{2\alpha-1}). \quad (1.2)$$

In particular, fix a map u and a kernel φ satisfying the assumptions of the Proposition with $\alpha > 1/2$. Then the Christoffel symbols of $(v * \varphi_\ell)^\sharp e$ converge to those of $v^\sharp e$. This corresponds to the results of Borisov in [1, 2], and hints at the absence of h -principle for $C^{1, \frac{1}{2} + \varepsilon}$ immersions. Relying mainly on this estimate we can give a fairly short proof of Borisov's theorem:

Theorem 3. *Let (M^2, g) be a surface with C^2 metric and positive Gauss curvature, and let $u \in C^{1,\alpha}(M^2; \mathbb{R}^3)$ be an isometric immersion with $\alpha > 2/3$. Then $u(M)$ is a surface of bounded extrinsic curvature.*

This leads to the following corollaries, which follow from the work of Pogorelov and Sabitov.

Corollary 3. *Let (S^2, g) be a closed surface with $g \in C^2$ and positive Gauss curvature, and let $u \in C^{1,\alpha}(S^2; \mathbb{R}^3)$ be an isometric immersion with $\alpha > 2/3$. Then, $u(S^2)$ is the boundary of a bounded convex set and any two such images are congruent. In particular if the Gauss curvature is constant, then $u(S^2)$ is the boundary of a ball $B_r(x)$.*

Corollary 4. *Let $\Omega \subset \mathbb{R}^2$ be open and $g \in C^{2,\beta}$ a metric on Ω with positive Gauss curvature. Let $u \in C^{1,\alpha}(\Omega; \mathbb{R}^3)$ be an isometric immersion with $\alpha > 2/3$. Then $u(\Omega)$ is $C^{2,\beta}$ and locally uniformly convex (that is, for every $x \in \Omega$ there exists a neighborhood V such that $u(\Omega) \cap V$ is the graph of a $C^{2,\beta}$ function with positive definite second derivative).*

1.1.3 Connections to the Euler equations

There is an interesting analogy between isometric immersions in low codimension (in particular the Weyl problem) and the incompressible Euler equations. In [10] a method, which is very closely related to convex integration, was introduced to construct highly irregular energy-dissipating solutions of the Euler equations. Being in conservation form, the "expected" regularity space for convex integration for the Euler equations should be C^0 . This is still beyond reach, and in [10] a weak version of convex integration was applied instead, to produce solutions in L^∞ (see also [11] for a slightly better space) and, moreover, to show that a weak version of the h -principle holds.

Nevertheless, just like for isometric immersions, for the Euler equations there is particular interest to go beyond C^0 : in [23] L. Onsager, motivated by the phenomenon of anomalous dissipation in turbulent flows, conjectured that there exist weak solutions of the Euler equations of class C^α with $\alpha < 1/3$ which dissipate energy, whereas for $\alpha > 1/3$ the energy is conserved. The latter was proved in [13, 9], but on the construction of energy-dissipating weak solutions nothing is known beyond L^∞ (for previous work see [27, 28, 29]). It should be mentioned that the critical

exponent $1/3$ is very natural - it agrees with the scaling of the energy cascade predicted by Kolmogorov's theory of turbulence (see for instance [14]).

For the analogous problem for isometric immersions there does not seem to be a universally accepted critical exponent (c.f. Problem 27 of [33]), even though $1/2$ seems likely (c.f. section 1.1.2 and the discussion in [7]). In fact, the regularization and the commutator estimates used in our proof of Proposition 1 and Theorem 3 have been inspired by (and are closely related to) the arguments of [9].

1.2 Estimates on convolutions: Proof of Proposition 1

As usual, we denote the norm on the Hölder space $C^{k,\alpha}(\overline{\Omega})$ by

$$\|f\|_{k,\alpha} := \sup_{x \in \Omega} \sum_{|a| \leq k} |\partial^a f(x)| + \sup_{x,y \in \Omega, x \neq y} \sum_{|a|=k} \frac{|\partial^a f(x) - \partial^a f(y)|}{|x-y|^\alpha}.$$

Here $k = 0, 1, 2, \dots$, $a = (a_1, \dots, a_n)$ is a multi-index with $|a| = a_1 + \dots + a_n$ and $\alpha \in [0, 1[$. For simplicity we will also use the abbreviation $\|f\|_k = \|f\|_{k,0}$ and $\|f\|_\alpha = \|f\|_{0,\alpha}$.

Recall the following interpolation inequalities for these norms:

$$\|f\|_{k,\alpha} \leq C \|f\|_{k_1,\alpha_1}^\lambda \|f\|_{k_2,\alpha_2}^{1-\lambda},$$

where C depends on the various parameters, $0 < \lambda < 1$ and

$$k + \alpha = \lambda(k_1 + \alpha_1) + (1 - \lambda)(k_2 + \alpha_2).$$

The following estimates are well known and play a fundamental role in both the constructions and the proof of rigidity.

Lemma 1. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be symmetric and such that $\int \varphi = 1$. Then for any $r, s \geq 0$ and $\alpha \in]0, 1[$ we have*

$$\|f * \varphi_\ell\|_{r+s} \leq C \ell^{-s} \|f\|_r, \quad (1.3)$$

$$\|f - f * \varphi_\ell\|_r \leq C \ell^2 \|f\|_{r+2}, \quad (1.4)$$

$$\|(fg) * \varphi_\ell - (f * \varphi_\ell)(g * \varphi_\ell)\|_r \leq C \ell^{2\alpha-r} \|f\|_\alpha \|g\|_\alpha. \quad (1.5)$$

Proof. For any multi-indices a, b with $|a| = r, |b| = s$ we have $\partial^{a+b}(f * \varphi_\ell) = \partial^a f * \partial^b \varphi_\ell$, hence

$$|\partial^{a+b}(f * \varphi_\ell)| \leq C_s \ell^{-s} \|f\|_r.$$

This proves 1.3.

Next, by considering the Taylor expansion of f at x we see that

$$f(x-y) - f(x) = f'(x)y + r_x(y),$$

where $\sup_x |r_x(y)| \leq C|y|^2 \|f\|_2$. Moreover, since φ is symmetric,

$$\int \varphi_\ell(y)y dy = 0.$$

Thus,

$$|f - f * \varphi_\ell| = \left| \int \varphi_\ell(y)(f(x-y) - f(x))dy \right| \quad (1.6)$$

$$\leq C\|f\|_2 \int \ell^{-n} \left| \varphi\left(\frac{y}{\ell}\right) \right| |y|^2 dy = C\ell^2 \|f\|_2. \quad (1.7)$$

This proves 1.4 for the case $r = 0$. To obtain the estimate for general r , repeat the same argument for the partial derivatives $\partial^a f$ with $|a| = r$.

For the proof of estimate 1.5 let a be any multi-index with $|a| = r$. By the product rule

$$\partial^a [\varphi_\ell * (fg) - (\varphi_\ell * f)(\varphi_\ell * g)] \quad (1.8)$$

$$= \partial^a \varphi_\ell * (fg) - \sum_{b \leq a} \binom{a}{b} (\partial^b \varphi_\ell * f)(\partial^{a-b} \varphi_\ell * g) \quad (1.9)$$

$$= \partial^a \varphi_\ell * (fg) - (\partial^a \varphi_\ell * f)(\varphi_\ell * g) + (\varphi_\ell * f)(\partial^a \varphi_\ell * g) \quad (1.10)$$

$$- \sum_{0 < b < a} \binom{a}{b} [\partial^b \varphi_\ell * (f - f(x))][\partial^{a-b} \varphi_\ell * (g - g(x))] \quad (1.11)$$

$$= \partial^a \varphi_\ell * [(f - f(x))(g - g(x))] \quad (1.12)$$

$$- \sum_{b \leq a} \binom{a}{b} \partial^b \varphi_\ell * (f - f(x)) \cdot \partial^{a-b} \varphi_\ell * (g - g(x)), \quad (1.13)$$

where we have used the fact that

$$\partial^a \varphi_\ell * f(x) = \begin{cases} f(x) & \text{if } a = 0, \\ 0 & \text{if } a \neq 0. \end{cases}$$

Now observe that

$$|\partial^a \varphi_\ell * [(f - f(x))(g - g(x))]| \quad (1.14)$$

$$= \left| \int \partial^a \varphi_\ell(y)(f(x-y) - f(x))(g(x-y) - g(x))dy \right| \quad (1.15)$$

$$\leq \int |\partial^a \varphi_\ell(y)| |y|^{2\alpha} dy \|f\|_\alpha \|g\|_\alpha = C_r \ell^{2\alpha-r} \|f\|_\alpha \|g\|_\alpha. \quad (1.16)$$

Similarly, all the terms in the sum over b obey the same estimate. This concludes the proof of 1.5. \square

Proof (of Proposition 1). Set $g := v^\sharp e$ and $g^\ell := (v * \varphi_\ell)^\sharp e$. We have

$$\|g_{ij}^\ell - g_{ij}\|_1 \leq \|g_{ij}^\ell - g_{ij} * \varphi_\ell\|_1 + \|g_{ij} * \varphi_\ell - g_{ij}\|_1.$$

The first term can be written as

$$\|g_{ij}^\ell - g_{ij} * \varphi_\ell\|_1 = \|\partial_j v * \varphi_\ell \cdot \partial_i v * \varphi_\ell - (\partial_j v \cdot \partial_i v) * \varphi_\ell\|_1, \quad (1.17)$$

so that 1.5 applies, to yield the bound $\ell^{2\alpha-1} \|v\|_{1,\alpha}^2$. For the second term 1.4 gives the bound $\ell \|g\|_2$. Combining these two we obtain

$$\|g_{ij}^\ell - g_{ij}\|_k \leq C(\ell^{2\alpha-1} \|v\|_{1,\alpha}^2 + \ell \|g\|_2),$$

from which 1.2 readily follows. \square

1.3 h -principle: The general scheme

The general scheme of our construction follows the method of Nash and Kuiper [21, 20]. For convenience of the reader we sketch this scheme in this section. Assume for simplicity that g is smooth.

The existence theorems are based on an iteration of *stages*, and each *stage* consists of several *steps*. The purpose of a *stage* is to correct the error $g - u^\sharp e$. In order to achieve this correction, the error is decomposed into a sum of primitive metrics as

$$g - u^\sharp e = \sum_{k=1}^{n_*} a_k^2 v_k \otimes v_k \quad (\text{locally})$$

$$g - u^\sharp e = \sum_j \sum_{k=1}^{n_*} (\psi_j a_{j,k})^2 v_{j,k} \otimes v_{j,k} \quad (\text{globally})$$

The natural estimates associated with this decomposition are

$$\|a_k\|_0 \sim \|g - u^\sharp e\|_0^{1/2} \quad (1.18)$$

$$\|a_k\|_{N+1} \sim \|u\|_{N+2} \quad \text{for } N = 0, 1, 2, \dots \quad (1.19)$$

A *step* then involves adding one primitive metric. In other words the goal of a *step* is the metric change

$$u^\sharp e \mapsto u^\sharp e + a^2 v \otimes v.$$

Nash used spiralling perturbations (also known as the Nash twist) to achieve this; for the codimension one case Kuiper replaced the spirals by corrugations. Using the same ansatz (see formula 1.36) one easily checks that addition of a primitive metric is possible with the following estimates (see Proposition 2):

$$C^0\text{-error in the metric} \sim \|g - u^\sharp e\|_0 \frac{1}{K}$$

$$\text{increase of } C^1\text{-norm of } u \sim \|g - u^\sharp e\|_0^{1/2}$$

$$\text{increase of } C^2\text{-norm of } u \sim \|u\|_2 K$$

for any $K \geq 1$. Observe that the first two of these estimates is essentially the same as in [21, 20]. Furthermore, the third estimate is only valid modulo a loss of derivative (see Remark 2).

The low codimension forces the steps to be performed serially. This is in contrast with the method of Källén in [19], where the whole *stage* can be performed in one step due to the high codimension. Thus the number of *steps* in a *stage* equals the number of primitive metrics in the above decomposition which interact. This equals n_* for the local construction and $(n+1)n_*$ for the global construction. To deal with the *loss of derivative* problem we mollify the map u at the start of every stage, in a similar manner as is done in a Nash-Moser iteration. Because of the quadratic estimate 1.5 in Lemma 1 there will be no additional error coming from the mollification. Therefore, iterating the estimates for one step over a single stage (that is, over N_* steps) leads to

$$\begin{aligned} C^0\text{-error in the metric} &\sim \|g - u^\sharp e\|_0 \frac{1}{K} \\ \text{increase of } C^1\text{-norm of } u &\sim \|g - u^\sharp e\|_0^{1/2} \\ \text{increase of } C^2\text{-norm of } u &\sim \|u\|_2 K^{N_*} \end{aligned}$$

With these estimates, iterating over the *stages* leads to exponential convergence of the metric error, leading to a controlled growth of the C^1 norm and an exponential growth of the C^2 norm of the map. In particular, interpolating between these two norms leads to convergence in $C^{1,\alpha}$ for $\alpha < \frac{1}{1+2N_*}$.

1.4 h -principle: Construction step

The main step of our construction is given by the following proposition.

Proposition 2 (Construction step). *Let $\Omega \subset \mathbb{R}^n$, $v \in S^{n-1}$ and $N \in \mathbb{N}$. Let $u \in C^{N+2}(\overline{\Omega}; \mathbb{R}^{n+1})$ and $a \in C^{N+1}(\overline{\Omega})$. Assume that $\gamma \geq 1$ and $\ell, \delta \leq 1$ are constants such that*

$$\frac{1}{\gamma} I \leq u^\sharp e \leq \gamma I \quad \text{in } \Omega, \quad (1.20)$$

$$\|a\|_0 \leq \delta, \quad (1.21)$$

$$\|u\|_{k+2} + \|a\|_{k+1} \leq \delta \ell^{-(k+1)} \quad \text{for } k = 0, 1, \dots, N. \quad (1.22)$$

Then, for any

$$\lambda \geq \ell^{-1} \quad (1.23)$$

there exists $v \in C^{N+1}(\overline{\Omega}; \mathbb{R}^{n+1})$ such that

$$\left\| v^\sharp e - (u^\sharp e + a^2 v \otimes v) \right\|_0 \leq C \frac{\delta^2}{\lambda \ell} \quad (1.24)$$

and

$$\|u - v\|_j \leq C \delta \lambda^{j-1} \quad \text{for } j = 0, 1, \dots, N+1, \quad (1.25)$$

where C is a constant depending only on n, N and γ .

Remark 2. Observe that if 1.25 would hold for $j = N+2$, then the conclusion of the proposition would say essentially (with $N = 0$) that the equation

$$v^\sharp e = u^\sharp e + a^2 v \otimes v$$

admits approximate solutions in C^2 with estimates

$$\begin{aligned} \|v^\sharp e - (u^\sharp e + a^2 v \otimes v)\|_0 &\leq C \delta^2 \frac{1}{K}, \\ \|u - v\|_2 &\leq C \|u\|_2 K. \end{aligned}$$

Here $K = \lambda \ell \geq 1$. The fact that 1.25 holds only for $j \leq N+1$ amounts to a loss of derivative in the estimate.

In the higher codimension case we need an additional technical assumption in order to carry on the same result. As usual the oscillation $\text{osc } u$ of a vector-valued map u is defined as $\sup_{x,y} |u(x) - u(y)|$.

Proposition 3 (Step in higher codim.). *Let $m, n, N \in \mathbb{N}$ with $n, N \geq 1$ and $m \geq n+1$. Then there exist a constant $\eta_0 > 0$ with the following property. Let Ω , g , a , v and $u \in C^{2+N}(\overline{\Omega}, \mathbb{R}^m)$ satisfy the assumptions of Proposition 2 and assume in addition $\text{osc } \nabla u \leq \eta_0$. Then there exists a map $v \in C^{1+N}(\overline{\Omega}, \mathbb{R}^m)$ satisfying the same conclusion as in Proposition 2.*

1.4.1 Basic building block

In order to prove the Proposition we need the following lemma. The function Γ will be our corrugation.

Lemma 2. *There exists $\delta_* > 0$ and a function $\Gamma \in C^\infty([0, \delta_*] \times \mathbb{R}; \mathbb{R}^2)$ with $\Gamma(\delta, t + 2\pi) = \Gamma(\delta, t)$ and having the following properties:*

$$|\partial_t \Gamma(s, t) + e_1|^2 = 1 + s^2, \quad (1.26)$$

$$|\partial_s \partial_t^k \Gamma_1(s, t)| + |\partial_t^k \Gamma(s, t)| \leq C_k s \quad \text{for } k \geq 0. \quad (1.27)$$

Proof. Define $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $H(\tau, t) = (\cos(\tau \sin t), \sin(\tau \sin t))$. Then

$$\int_0^{2\pi} H_2(\tau, t) dt = \int_0^{2\pi} \sin(\tau \sin t) dt = \int_{-\pi}^{\pi} \sin(\tau \sin t) dt = 0 \quad (1.28)$$

by the symmetry of the sine function. Set

$$J_0(\tau) := \frac{1}{2\pi} \int_0^{2\pi} H_1(\tau, t) dt = \frac{1}{2\pi} \int_0^{2\pi} \cos(\tau \sin t) dt. \quad (1.29)$$

Note that $J_0 \in C^\infty(\mathbb{R})$ with $J_0(0) = 1$, $J_0'(0) = 0$ and $J_0''(0) < 0$. We claim that there exists $\delta > 0$ and a function $f \in C^\infty(-\delta, \delta)$ such that $f(0) = 0$ and

$$J_0(f(s)) = \frac{1}{\sqrt{1+s^2}}. \quad (1.30)$$

This is a consequence of the implicit function theorem. To see this, set

$$F(s, r) = J_0(r^{1/2}) - (1+s^2)^{-1/2}.$$

Then $F \in C^\infty(\mathbb{R}^2)$. Indeed, since the Taylor expansion of $\cos x$ contains only even powers of x , $J_0(r^{1/2})$ is obviously analytic. Moreover,

$$J_0(r^{1/2}) = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \frac{r}{2} \sin^2 t\right) dt + O(r^2).$$

In particular $\partial_r F(0, 0) = -1/4$. Since also $F(0, 0) = 0$, the implicit function theorem yields $\delta > 0$ and $g \in C^\infty(-\delta, \delta)$ such that $g(0) = 0$ and

$$F(s, g(s)) = 0.$$

Next, observe that $\partial_s F(0, 0) = 0$ and $\partial_s^2 F(0, 0) = 1$. Therefore

$$g'(0) = 0 \text{ and } g''(0) = 4.$$

This implies that $f(s) := g(s)^{1/2}$ is also a smooth function, with

$$f(0) = 0 \text{ and } f'(0) = \sqrt{2},$$

thus proving our claim.

Having found $f \in C^\infty(-\delta, \delta)$ with $f(0) = 0$ and 1.30, we finally set

$$\Gamma(s, t) := \int_0^t \left[\sqrt{1+s^2} H(f(s), t') - e_1 \right] dt'. \quad (1.31)$$

By construction $|\partial_t \Gamma(s, t) + e_1|^2 = 1 + s^2$. Moreover

$$\begin{aligned} \Gamma(s, t + 2\pi) - \Gamma(s, t) &= \int_t^{t+2\pi} \left[\sqrt{1+s^2} H(f(s), t') - e_1 \right] dt' \\ &= \sqrt{1+s^2} \int_0^{2\pi} H(f(s), t') dt' - 2\pi e_1 \\ &\stackrel{1.281.29}{=} 2\pi e_1 \left[\sqrt{1+s^2} J_0(f(s)) - 1 \right] \stackrel{1.30}{=} 0. \end{aligned}$$

Thus the function Γ is 2π -periodic in the second argument.

We now come to the estimates. Fix $\delta_* < \delta$. Then $\Gamma \in C([0, \delta_*] \times \mathbb{R}; \mathbb{R}^2)$, and since it is periodic in the second variable, Γ and all its partial derivatives are uniformly bounded. Straightforward computations show that for any $k = 0, 1, \dots$

$$\partial_t^k \Gamma(0, t) = 0 \text{ and } \partial_s \partial_t^k \Gamma_1(0, t) = 0 \quad \text{for all } t.$$

Hence, integrating in s , we conclude that

$$\begin{aligned} |\partial_t^k \Gamma(s, t)| &\leq s \|\partial_s \partial_t^k \Gamma\|_0, \\ |\partial_s \partial_t^k \Gamma_1(s, t)| &\leq s \|\partial_s^2 \partial_t^k \Gamma_1\|_0, \end{aligned}$$

which give the desired estimates. \square

1.4.2 Proof of Proposition 2

Throughout the proof the letter C will denote a constant, whose value might change from line to line, but otherwise depends only on n, N and γ . Fix a choice of orthonormal coordinates in \mathbb{R}^n . In these coordinates the pullback metric can be written as $(u^\#e)_{ij} = \partial_i u \cdot \partial_j u$ or, denoting the matrix differential of u by $\nabla u = (\partial_j u^i)_{ij}$, as

$$u^\#e = \nabla u^T \nabla u.$$

From now on we will work with this notation.

Let

$$\xi = \nabla u \cdot (\nabla u^T \nabla u)^{-1} \cdot v, \quad \zeta = \partial_1 u \wedge \partial_2 u \wedge \dots \wedge \partial_n u. \quad (1.32)$$

Because of 1.20 the vectorfields ξ, ζ are well-defined and satisfy

$$\frac{1}{C} \leq |\xi(x)|, |\zeta(x)| \leq C \quad \text{for } x \in \Omega \quad (1.33)$$

with some $C \geq 1$. Now let

$$\xi_1 = \frac{\xi}{|\xi|^2}, \quad \xi_2 = \frac{\zeta}{|\xi||\zeta|}, \quad \Psi(x) = \xi_1(x) \otimes e_1 + \xi_2(x) \otimes e_2,$$

and

$$\tilde{a} = |\xi|a.$$

Then

$$\nabla u^T \Psi = \frac{1}{|\xi|^2} v \otimes e_1, \quad \Psi^T \Psi = \frac{1}{|\xi|^2} I, \quad (1.34)$$

and

$$\begin{aligned} \|\Psi\|_j &\leq C \|u\|_{j+1}, \\ \|\tilde{a}\|_j &\leq C (\|a\|_j + \|a\|_0 \|u\|_{j+1}), \end{aligned} \quad (1.35)$$

for $j = 0, 1, \dots, N+1$. Finally, let

$$v(x) := u(x) + \frac{1}{\lambda} \Psi(x) \Gamma(\tilde{a}(x), \lambda x \cdot v), \quad (1.36)$$

where $\Gamma = \Gamma(s, t)$ is the function constructed in Lemma 2.

Proof of 1.24. First we compute $\nabla v^T \nabla v$. We have

$$\nabla v = \underbrace{\nabla u + \Psi \cdot \partial_t \Gamma \otimes v}_A + \underbrace{\lambda^{-1} \Psi \cdot \partial_s \Gamma \otimes \nabla \tilde{a}}_{E_1} + \underbrace{\lambda^{-1} \nabla \Psi \cdot \Gamma}_{E_2}. \quad (1.37)$$

Using the notation $\text{sym}(A) = (A + A^T)/2$ one has

$$\nabla v^T \nabla v = A^T A + 2\text{sym}(A^T E_1 + A^T E_2) + (E_1 + E_2)^T (E_1 + E_2). \quad (1.38)$$

Using 1.34 and 1.26:

$$\begin{aligned} A^T A &= \nabla u^T \nabla u + \frac{1}{|\xi|^2} (2\partial_t \Gamma_1 + |\partial_t \Gamma|^2) v \otimes v \\ &= \nabla u^T \nabla u + \frac{1}{|\xi|^2} \tilde{a}^2 v \otimes v = \nabla u^T \nabla u + a^2 v \otimes v. \end{aligned} \quad (1.39)$$

Next we estimate the error terms. First of all

$$\begin{aligned} A^T E_1 &= \frac{1}{\lambda} (\nabla u^T \Psi) (\partial_s \Gamma \otimes \nabla \tilde{a}) + \frac{1}{\lambda} (v \otimes \partial_t \Gamma) (\Psi^T \Psi) (\partial_s \Gamma \otimes \nabla \tilde{a}) \\ &= \frac{1}{\lambda |\xi|^2} (\partial_s \Gamma_1 + \partial_t \Gamma \cdot \partial_s \Gamma) (v \otimes \nabla \tilde{a}). \end{aligned} \quad (1.40)$$

Note that 1.27 together with 1.35 implies:

$$\|\Gamma\|_0, \|\partial_t \Gamma\|_0, \|\partial_s \Gamma_1\|_0 \leq C \|a\|_0.$$

Therefore

$$\|\text{sym}(A^T E_1)\|_0 \leq \frac{C}{\lambda} \|a\|_0 \|\tilde{a}\|_1 \leq C \frac{\delta^2}{\lambda \ell},$$

and similarly

$$\|\text{sym}(A^T E_2)\|_0 \leq \frac{C}{\lambda} \|a\|_0 \|u\|_2 \leq C \frac{\delta^2}{\lambda \ell}. \quad (1.41)$$

Finally,

$$\|E_1 + E_2\|_0 \leq \frac{C}{\lambda} (\|\tilde{a}\|_1 + \|a\|_0 \|u\|_2) \leq \frac{C}{\lambda} (\|a\|_1 + \delta \|u\|_2) \leq C \frac{\delta}{\lambda \ell}. \quad (1.42)$$

In particular $\|E_1 + E_2\|_0 \leq C \delta$ and hence

$$\|(E_1 + E_2)^T (E_1 + E_2)\|_0 \leq C \frac{\delta^2}{\lambda \ell}. \quad (1.43)$$

Putting these estimates together we obtain 1.24 as required.

Proof of 1.25. In fact

$$\|u - v\|_0 \leq C\delta \frac{1}{\lambda}$$

is obvious, whereas the estimates for $j = 1, \dots, N$ will follow by interpolation, provided the case $j = N + 1$ holds. Therefore, we now prove this case. A simple application of the product rule and interpolation yields

$$\begin{aligned} \|v - u\|_{N+1} &\leq \frac{C}{\lambda} (\|\Psi\|_{N+1} \|\Gamma\|_0 + \|\Psi\|_0 \|\Gamma\|_{N+1}) \\ &\leq \frac{C}{\lambda} (\|u\|_{N+2} \|\tilde{a}\|_0 + \|\Gamma\|_{N+1}). \end{aligned} \quad (1.44)$$

Denoting by D_x^j any partial derivative in the variables x_1, \dots, x_n of order j , the chain rule can be written symbolically as

$$D_x^{N+1} \Gamma = \sum_{i+j \leq N+1} \left(\partial_s^i \partial_t^j \Gamma \right) \lambda^j \sum_{\sigma} C_{i,j,\sigma} (D_x \tilde{a})^{\sigma_1} (D_x^2 \tilde{a})^{\sigma_2} \dots (D_x^{N+1} \tilde{a})^{\sigma_{N+1}},$$

where the inner sum is over all σ with

$$\begin{aligned} \sigma_1 + \dots + \sigma_{N+1} &= i, \\ \sigma_1 + 2\sigma_2 + \dots + (N+1)\sigma_{N+1} + j &= N+1. \end{aligned}$$

These relations can be checked by counting the order of differentiation. Therefore, by using 1.21, 1.22 and 1.23

$$\begin{aligned} \|D_x^{N+1} \Gamma\|_0 &\leq C \sum_{i+j \leq N+1} \left\| \partial_s^i \partial_t^j \Gamma \right\|_0 \lambda^j \delta^i \ell^{-(N+1-j)} \\ &\leq C \sum_{i+j \leq N+1} \left\| \partial_s^i \partial_t^j \Gamma \right\|_0 \delta^i \lambda^{N+1} \leq C\delta \lambda^{N+1}. \end{aligned} \quad (1.45)$$

In particular, since $\|\Gamma\|_0 \leq \delta$, we deduce that $\|\Gamma\|_{N+1} \leq C\delta \lambda^{N+1}$. Therefore

$$\|v - u\|_{N+1} \leq \frac{C}{\lambda} (\delta \|u\|_{N+2} + \delta \lambda^{N+1}) \leq C\delta \lambda^N. \quad (1.46)$$

This concludes the proof of the proposition.

1.4.3 Proof of Proposition 3

The proof of Proposition 2 would carry over to this case if we can choose an appropriate normal vector field ζ as at the beginning of the proof of Proposition 2, enjoying the estimate 1.33 with a fixed constant.

To obtain $\zeta(x)$ let $T(x)$ be the tangent plane to $u(\mathbb{R}^n)$ at the point $u(x)$, i.e. the plane generated by $\{\partial_1 u, \dots, \partial_n u\}$. Denote by π_x the orthogonal projection of \mathbb{R}^m onto $T(x)$. Assuming that ∇u has oscillation smaller than η_0 , there exists a vector $w \in S^{n-1}$ such that $|\pi_x w| \leq 1/2$ for every $x \in \overline{\Omega}$. Hence, we can define

$$\zeta(x) := w - \pi_x w.$$

It is straightforward to see that this choice of ζ gives a map enjoying the same estimates as the ζ used in the proof of Proposition 2.

1.5 h -principle: stage

Proposition 4 (Stage, local). *For all $g_0 \in \text{sym}_n^+$ there exists $0 < r < 1$ such that the following holds for any $\Omega \subset \mathbb{R}^n$ and $g \in C^\beta(\overline{\Omega})$ with $\|g - g_0\|_0 \leq r$. There exists a $\delta_0 > 0$ such that, if $K \geq 1$ and $u \in C^2(\overline{\Omega}, \mathbb{R}^{n+1})$ satisfies*

$$\|u^\sharp e - g\|_0 \leq \delta^2 \leq \delta_0^2 \quad \text{and} \quad \|u\|_2 \leq \mu,$$

then there exists $v \in C^2(\overline{\Omega}, \mathbb{R}^{n+1})$ with

$$\|v^\sharp e - g\|_0 \leq C\delta^2, \left(\frac{1}{K} + \delta^{\beta-2}\mu^{-\beta} \right) \quad (1.47)$$

$$\|v\|_2 \leq C\mu K^{n_*}, \quad (1.48)$$

$$\|u - v\|_1 \leq C\delta. \quad (1.49)$$

Here C is a constant depending only on n, g_0, g and Ω .

The Proposition above is the basic stage of the iteration scheme which will prove Theorem 1. A similar proposition, to be used in the proof of Theorem 2 will be stated later.

1.5.1 Decomposing a metric into primitive metrics

Lemma 3. *Let $g_0 \in \text{sym}_n^+$. Then there exists $r > 0$, vectors $v_1, \dots, v_{n_*} \in \mathbb{S}^{n-1}$ and linear maps $L_k : \text{sym}_n \rightarrow \mathbb{R}$ such that*

$$g = \sum_{k=1}^{n_*} L_k(g) v_k \otimes v_k \quad \text{for every } g \in \text{sym}_n$$

and, moreover, $L_k(g) \geq r$ for every k and every $g \in \text{sym}_n^+$ with $|g - g_0| \leq r$.

Proof. Consider the set $S := \{(e_i + e_j) \otimes (e_i + e_j), i \leq j\}$, where $\{e_i\}$ is the standard basis of \mathbb{R}^n . Since the span of S contains all matrices of the form $e_i \otimes e_j + e_j \otimes e_i$,

clearly S generates sym_n . On the other hand S consists of n_* matrices with $n_* = \dim(\text{sym}_n)$. So S is a basis for sym_n . Let us relabel the vectors $e_i + e_j$ ($i \leq j$) as f_1, \dots, f_{n_*} , and let

$$h = \sum_{k=1}^{n_*} f_k \otimes f_k.$$

Then $h \in \text{sym}_n^+$ and hence there exists an invertible linear transformation L such that $LhL^T = g_0$. In particular, writing $v_k = Lf_k/|Lf_k| \in \mathbb{S}^{n-1}$, we have

$$g_0 = \sum_{k=1}^{n_*} Lf_k \otimes Lf_k = \sum_{k=1}^{n_*} |Lf_k|^2 v_k \otimes v_k.$$

Note that the set $\{v_k \otimes v_k\}$ is also a basis for sym_n and therefore there exist linear maps $L_k : \text{sym}_n \rightarrow \mathbb{R}$ such that $\sum L_k(A) v_k \otimes v_k$ is the unique representation of $A \in \text{sym}_n$ as linear combination of $v_k \otimes v_k$. In particular, $L_i(g_0) = |Lf_k|^2 > 0$. The existence of $r > 0$ satisfying the claim of the lemma follows easily. \square

1.5.2 Proof of Proposition 4

Choose $r > 0$ and $\gamma > 1$ so that the statement of Lemma 3 holds with g_0 and $2r$, and so that

$$\frac{1}{\gamma} I \leq h \leq \gamma I \quad \text{for any } h \in \text{sym}_n^+ \text{ with } |h - g_0| < 2r.$$

Moreover, extend u and g to \mathbb{R}^n so that

$$\|u\|_{C^2(\mathbb{R}^n)} \leq C \|u\|_{C^2(\bar{\Omega})}, \quad \|g\|_{C^\beta(\mathbb{R}^n)} \leq C \|g\|_{C^\beta(\bar{\Omega})}.$$

The procedure of such an extension is well known, with the constant C depending on n, β and Ω . In what follows, the various constants will be allowed to depend in addition on r and γ .

Step 1. Mollification. We set

$$\ell = \frac{\delta}{\mu},$$

and let

$$\tilde{u} = u * \varphi_\ell, \quad \tilde{g} = g * \varphi_\ell, \tag{1.50}$$

where $\varphi \in C_c^\infty(B_1(0))$ is a symmetric nonnegative convolution kernel with $\int \varphi = 1$. Lemma 1 implies

$$\|\tilde{u} - u\|_1 \leq C \|u\|_2 \ell \leq C \delta, \tag{1.51}$$

$$\|\tilde{g} - g\|_0 \leq C \|g\|_\beta \ell^\beta, \tag{1.52}$$

$$\|\tilde{u}\|_{k+2} \leq C \|u\|_2 \ell^{-k} \leq C \delta \ell^{-(k+1)}, \tag{1.53}$$

and

$$\begin{aligned} \|\tilde{u}^\sharp e - \tilde{g}\|_k &\leq \|\tilde{u}^\sharp e - (u^\sharp e) * \varphi_\ell\|_k + \|(u^\sharp e) * \varphi_\ell - g * \varphi_\ell\|_k \\ &\leq C \ell^{2-k} \|u\|_2^2 + C \ell^{-k} \|u^\sharp e - g\|_0 \leq C \delta^2 \ell^{-k}, \end{aligned} \quad (1.54)$$

where $k = 0, 1, \dots, n_*$. Moreover, since the set $\{h \in \text{sym}_n^+ : |h - g_0| \leq r\}$ is convex, \tilde{g} also satisfies $\|\tilde{g} - g_0\|_0 \leq r$.

Step 2. Rescaling. First of all, observe that

$$\tilde{h} := \tilde{g} + \frac{r}{C\delta^2} (\tilde{g} - \tilde{u}^\sharp e)$$

satisfies the condition $|\tilde{h}(x) - g_0| \leq \frac{r}{C\delta^2} \|\tilde{g} - \tilde{u}^\sharp e\|_0 + r \leq 2r$. Therefore, using Lemma 3 we have

$$(1 + Cr^{-1}\delta^2)\tilde{g} - \tilde{u}^\sharp e = \frac{C\delta^2}{r} \tilde{h} = \sum_{i=1}^{n_*} \tilde{a}_i^2 v_i \otimes v_i,$$

where $\tilde{a}_i(x) = \left(C\frac{\delta^2}{r} L_i(\tilde{h}(x))\right)^{1/2}$. In particular \tilde{a}_i is smooth and

$$\begin{aligned} \|\tilde{a}_i\|_k &\leq C\delta \frac{\|L_i(\tilde{h})\|_k}{\|L_i(\tilde{h})\|_0^{1/2}} \leq C\delta \|\tilde{h}\|_k \\ &\leq C\delta \left(\|\tilde{g}\|_k + \frac{1}{\delta^2} \|\tilde{g} - \tilde{u}^\sharp e\|_k \right) \leq C\delta \ell^{-k} \end{aligned}$$

for $k = 0, 1, 2, \dots, n_*$ (note that the first inequality is achieved through interpolation). Let

$$u_0 = \frac{1}{(1 + Cr^{-1}\delta^2)^{1/2}} \tilde{u}, \quad a_i = \frac{1}{(1 + Cr^{-1}\delta^2)^{1/2}} \tilde{a}_i.$$

Then we have

$$\tilde{g} - u_0^\sharp e = \sum_{i=1}^{n_*} a_i^2 v_i \otimes v_i,$$

with

$$\|\tilde{u} - u_0\|_1 \leq C\delta, \quad (1.55)$$

$$\|a_i\|_0 \leq C\delta, \quad (1.56)$$

$$\|u_0\|_{k+2} + \|a_i\|_{k+1} \leq C\delta \ell^{-(k+1)}, \quad (1.57)$$

for $k = 0, 1, \dots, n_*$. Notice that the constants above depend also on k , but since we will only use these estimates for $k \leq n_*$, this dependence can be suppressed.

Finally, using 1.54 we have $\|u_0^\sharp e - g_0\|_0 \leq r + C\delta^2$, so that $\gamma^{-1}I \leq u_0^\sharp e \leq \gamma I$, provided δ_0 is sufficiently small.

Step 3. Iterating one-dimensional oscillations. We now apply n_* times successively Proposition 2, with

$$\ell_j = \ell K^{-j}, \lambda_j = K^{j+1} \ell^{-1}, N_j = n_* - j$$

for $j = 0, 1, \dots, n_*$. In other words we construct a sequence of immersions u_j such that $\frac{1}{\gamma}I \leq u_j^\sharp e \leq \gamma I$ and

$$\|u_j\|_{k+2} \leq C\delta \ell_j^{-(k+1)} \quad \text{for } k = 0, 1, \dots, N_j. \quad (1.58)$$

To see that Proposition 2 is applicable, observe that $\lambda_j = K \ell_j^{-1}$. Therefore it suffices to check inductively the validity of 1.58. This follows easily from 1.25. The constants will depend on j , but this can again be suppressed because $j \leq n_*$.

In this way we obtain the functions u_1, u_2, \dots, u_{n_*} with estimates

$$\begin{aligned} \|u_j\|_2 &\leq C\delta \ell^{-1} K^j, \\ \|u_{j+1}^\sharp e - (u_j^\sharp e + a_{j+1}^2 v_{j+1} \otimes v_{j+1})\|_0 &\leq C \frac{\delta^2}{\lambda_j \ell_j} = C\delta^2 \frac{1}{K}, \end{aligned}$$

and moreover

$$\|u_{j+1} - u_j\|_1 \leq C\delta. \quad (1.59)$$

Observe also that $\|u_j^\sharp e - g_0\|_0 \leq r + C\delta^2$, so that, provided δ_0 is sufficiently small, $\gamma^{-1}I \leq u_j^\sharp e \leq \gamma I$ for all j .

Thus $v := u_{n_*}$ satisfies the estimates

$$\begin{aligned} \|v^\sharp e - \tilde{g}\|_0 &\leq C\delta^2 \frac{1}{K}, \\ \|v\|_2 &\leq C\mu K^{n_*}, \\ \|v - u_0\|_1 &\leq C\delta. \end{aligned}$$

The estimates 1.47, 1.48 and 1.49 follow from the above combined with 1.51, 1.52 and 1.55.

1.5.3 Stage for general manifolds

Given M as in Theorem 2 we fix a finite atlas of M with charts Ω_i and a corresponding partition of unity $\{\phi_i\}$, so that $\sum \phi_i = 1$ and $\phi_i \in C_c^\infty(\Omega_i)$. Furthermore, on each Ω_i we fix a choice of coordinates.

Using the partition of unity we define the space $C^k(M)$. In particular, let

$$\|u\|_k := \sum_i \|\phi_i u\|_k.$$

Similarly, we define mollification on M via the partition of unity. In other words we fix $\varphi \in C_c^\infty(B_1(0))$, and for a function u on M we define

$$u * \varphi_\ell := \sum_i (\phi_i u) * \varphi_\ell. \quad (1.60)$$

It is not difficult to check that the estimates in Lemma 1 continue to hold on M with these definitions.

Next, let g be a metric on M as in Theorem 2. Since M is compact and g is continuous, there exists $\gamma > 0$ such that

$$\frac{1}{\gamma} I \leq g \leq \gamma I \quad \text{in } M. \quad (1.61)$$

Moreover, also by compactness, there exists $r_0 > 0$ such that Lemma 3 holds with $r = 2r_0$ for any g_0 satisfying $\frac{1}{\gamma} I \leq g_0 \leq \gamma I$. Therefore there exists $\rho_0 > 0$ so that

$$\begin{aligned} U \subset \Omega_i \text{ for some } i \text{ and } \operatorname{osc}_U g < r_0 \\ \text{whenever } U \subset M \text{ with } \operatorname{diam} U < \rho_0. \end{aligned} \quad (1.62)$$

Here $\operatorname{osc}_U g$ is to be evaluated in the coordinates of the chart Ω_i .

In the following we will need coverings of M with the following property:

Definition 1 (Minimal cover of M). For $\rho > 0$ a finite open covering \mathcal{C} of M is a minimal cover of diameter ρ if:

1. the diameter of each $U \in \mathcal{C}$ is less than ρ ;
2. \mathcal{C} can be subdivided into $n + 1$ subfamilies \mathcal{F}_i , each consisting of pairwise disjoint sets.

The existence of such coverings is a well-known fact. For the convenience of the reader we give a short proof at the end of this section.

We are now ready to state the iteration stage needed for the proof of Theorem 2. Recall that $\eta_0 > 0$ is the constant from Proposition 3.

Proposition 5 (Stage, global). *Let (M^n, g) be a smooth, compact Riemannian manifold with $g \in C^\beta(M)$, and let \mathcal{C} be a minimal cover of M of diameter $\rho < \rho_0$, where ρ_0 is as in 1.62. There exists $\delta_0 > 0$ such that, if $K \geq 1$ and $u \in C^2(M, \mathbb{R}^m)$ satisfies*

$$\|u^\sharp e - g\|_0 \leq \delta^2 < \delta_0^2, \quad (1.63)$$

$$\|u\|_2 \leq \mu, \quad (1.64)$$

$$\operatorname{osc}_U \nabla u \leq \eta_0/2 \text{ for all } U \in \mathcal{C}, \quad (1.65)$$

then there exists $v \in C^2(M, \mathbb{R}^m)$ with

$$\|v^\sharp e - g\|_0 \leq C\delta^2 \left(\frac{1}{K} + \delta^{\beta-2} \mu^{-\beta} \right), \quad (1.66)$$

$$\|v\|_2 \leq C\mu K^{(n+1)n_*}, \quad (1.67)$$

$$\|u - v\|_1 \leq C\delta. \quad (1.68)$$

The constants C depend only (M^n, g) and \mathcal{C} .

1.5.4 Proof of Proposition 5

We proceed as in the proof of Proposition 4. Enumerate the covering as $\mathcal{C} = \{U_j\}_{j \in J}$, and for each j choose a matrix $g_j \in \text{sym}_n^+$ such that

$$|g(x) - g_j| \leq r_0 \text{ for } x \in U_j.$$

Furthermore, fix a partition of unity $\{\psi_j\}$ for \mathcal{C} in the sense that $\psi_j \in C_c^\infty(U_j)$ and $\sum_j \psi_j^2 = 1$ on M .

Step 1. Mollification. The mollification step is precisely as in Proposition 4. We set

$$\ell = \frac{\delta}{\mu},$$

and let

$$\tilde{u} = u * \varphi_\ell, \quad \tilde{g} = g * \varphi_\ell, \quad (1.69)$$

where now the convolution is defined in 1.60 above. Then, as before,

$$\|\tilde{u} - u\|_1 \leq C\delta, \quad (1.70)$$

$$\|\tilde{g} - g\|_0 \leq C\|g\|_\beta \ell^\beta, \quad (1.71)$$

$$\|\tilde{u}\|_{k+2} \leq C\delta \ell^{-(k+1)}, \quad (1.72)$$

$$\|\tilde{u}^\sharp e - \tilde{g}\|_k \leq C\delta^2 \ell^{-k}, \quad (1.73)$$

for $k = 0, 1, \dots, (n+1)n_*$. In particular, for any $j \in J$ and any $x \in U_j$

$$|\tilde{g}(x) - g_j| \leq r_0 + C\ell^\beta \leq r_0 + C\delta_0^\beta \leq \frac{3}{2}r_0$$

provided $\delta_0 > 0$ is sufficiently small.

Step 2. Rescaling. We rescale the map analogously to Step 2 in Proposition 4. Accordingly,

$$\tilde{h} := \tilde{g} + \frac{r_0}{2C\delta^2} (\tilde{g} - \tilde{u}^\sharp e)$$

satisfies

$$|\tilde{h}(x) - g_j| \leq \frac{r_0}{2C\delta^2} \|\tilde{g} - \tilde{u}^\sharp e\|_0 + \frac{3}{2}r_0 \leq 2r_0 \text{ in } U_j.$$

Therefore, using Lemma 3 for each g_j and introducing

$$u_0 = \frac{1}{(1 + Cr_0^{-1}\delta^2)^{1/2}} \tilde{u}$$

we obtain (as in Proposition 4)

$$\tilde{g} - u_0^\sharp e = \sum_{i=1}^{n_*} a_{i,j}^2 v_{i,j} \otimes v_{i,j} \text{ in } U_j$$

for some functions $a_{i,j} \in C^\infty(U_j)$ satisfying the estimates

$$\|a_{i,j}\|_{C^{k+1}(U_j)} \leq C\delta\ell^{-(k+1)} \text{ for } j \in J \text{ and } k = 0, 1, \dots, (n+1)n_*.$$

In particular, using the partition of unity $\{\psi_j\}$ we obtain

$$\tilde{g} - u_0^\sharp e = \sum_{j \in J} \sum_{i=1}^{n_*} (\psi_j a_{i,j})^2 v_{i,j} \otimes v_{i,j}, \quad (1.74)$$

with

$$\|u - u_0\|_1 \leq C\delta, \quad (1.75)$$

$$\|\psi_j a_{i,j}\|_0 \leq C\delta, \quad (1.76)$$

$$\|u_0\|_{k+2} + \|\psi_j a_{i,j}\|_{k+1} \leq C\delta\ell^{-(k+1)} \quad (1.77)$$

for $k = 0, 1, \dots, (n+1)n_*$.

Step 3. Iterating one-dimensional oscillations We now argue as in the Step 3 of the proof of Proposition 4. However, there are two differences. First of all we apply Proposition 3 in place of Proposition 2. This requires an additional control of the oscillation of ∇u in each U_j . Second, the number of steps is $(n+1)n_*$. Indeed, observe that 1.74 can be written as

$$\tilde{g} - u_0^\sharp e = \sum_{\sigma=1}^{n+1} \sum_{i=1}^{n_*} \sum_{j \in J_\sigma} (\psi_j a_{i,j})^2 v_{i,j} \otimes v_{i,j}, \quad (1.78)$$

where the index set J is decomposed as $J = J_1 \cup \dots \cup J_{n+1}$ so that $U_j \in \mathcal{F}_\sigma$ if and only if $j \in J_\sigma$. The point is that the sum in j consists of functions with disjoint supports, and hence for this sum Proposition 3 can be performed in parallel, in one step. Thus, the number of steps to be performed serially is the number of summands in σ and i , which is precisely $(n+1)n_*$.

To deal with the restriction on the oscillation of u_k in each step, observe that $\text{osc}_{U_j} \nabla u \leq \eta_0/2$ by assumption, and clearly the same holds for u_0 . Also, at each step we have the estimate $\|u_{k+1} - u_k\|_1 \leq C\delta \leq C\delta_0$. Therefore, choosing $\delta_0 > 0$ sufficiently small (only depending on the constants and on η_0), we ensure that the condition remains satisfied inductively $(n+1)n_*$ times.

Thus, proceeding as in the proof of Proposition 4 we apply Proposition 3 successively with $\ell_k = \ell K^{-k}$, $\lambda_k = K^{k+1} \ell^{-1}$, and $N_k = (n+1)n_* - k$. In this way we obtain a final map $v := u_{(n+1)n_*}$ such that

$$\begin{aligned} \|v^\sharp e - \tilde{g}\|_0 &\leq C\delta^2 \frac{1}{K}, \\ \|v\|_2 &\leq C\mu K^{(n+1)n_*}, \\ \|v - u_0\|_1 &\leq C\delta. \end{aligned}$$

The above inequalities combined with 1.70, 1.71 and 1.75 imply the estimates 1.66, 1.67 and 1.68. This concludes the proof.

1.5.5 Existence of minimal covers

We fix a triangulation T of M with simplices having diameter smaller than $\rho/3$. We let S_0 be the vertices of the triangulation, S_1 be the edges, S_k be the k -faces. \mathcal{F}_0 is made by pairwise disjoint balls centered on the elements of S_0 , with radius smaller than $\rho/2$. We let M_0 be the union of these balls. Next, for any element $\sigma \in S_1$, we consider $\sigma' = \sigma \setminus M_0$. The σ' are therefore pairwise disjoint compact sets and we let \mathcal{F}_1 be a collection of pairwise disjoint neighborhoods of σ' , each with diameter less than ρ . We define M_1 to be the union of the elements of \mathcal{F}_1 and \mathcal{F}_0 . We proceed inductively. At the step k , for every k -dim. face $F \in S_k$ we define $F' = F \setminus A_{k-1}$. Clearly, the F' are pairwise disjoint compact sets and hence we can find pairwise disjoint neighborhoods of the F' with diameter smaller than ρ . Figure 1.1 below shows the elements of \mathcal{F}_i for a 2-d triangulation.

Clearly, the collection $\mathcal{F}_0 \cup \dots \cup \mathcal{F}_n$ covers any simplex of T , and hence is a covering of M .

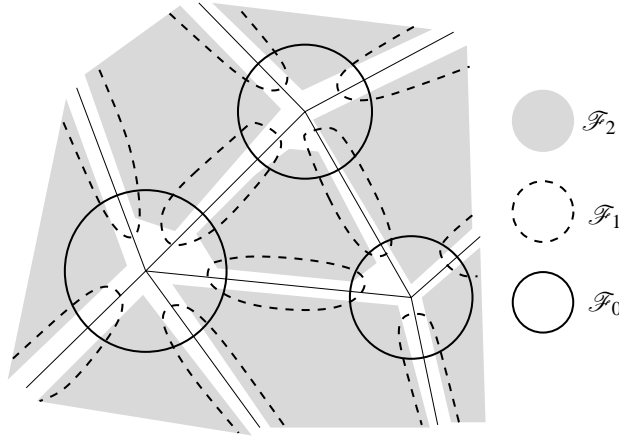


Fig. 1.1 The triangulation T and the covering for a 2-dimensional manifold.

1.6 h -principle: iteration

1.6.1 Proof of Theorem 1

Let $\mu_0, \delta_0 > 0$ be such that

$$\begin{aligned}\|u^\sharp e - g\|_0 &\leq \delta_0^2 \\ \|u\|_2 &\leq \mu_0.\end{aligned}$$

Let also $K \geq 1$. Later on we are going to adjust the parameters μ_0 and K in order to achieve the required convergence in $C^{1,\alpha}$. Applying Proposition 4 successively, we obtain a sequence of maps $u_k \in C^2(\overline{\Omega}, \mathbb{R}^{n+1})$ such that

$$\begin{aligned}\|u_k^\sharp e - g\|_0 &\leq \delta_k^2 \\ \|u_k\|_2 &\leq \mu_k \\ \|u_{k+1} - u_k\|_1 &\leq C\delta_k,\end{aligned}$$

where

$$\delta_{k+1}^2 = C\delta_k^2 \left(\frac{1}{K} + \delta_k^{\beta-2} \mu_k^{-\beta} \right), \quad (1.79)$$

$$\mu_{k+1} = C\mu_k K^{n_*}. \quad (1.80)$$

Substituting K with $\max\{C^{1/n_*} K, K\}$ we can absorb the constant in 1.80 to achieve $\mu_{k+1} = \mu_k K^{n_*}$, at the price of getting a possibly worse constant in 1.79. In particular $\mu_k = \mu_0 K^{kn_*}$. Next, we show by induction that for any

$$a < \min\left\{ \frac{1}{2}, \frac{\beta n_*}{2 - \beta} \right\} \quad (1.81)$$

there exists a suitable initial choice of K and μ_0 so that

$$\delta_k \leq \delta_0 K^{-ak}.$$

The case $k = 0$ is obvious. Assuming the inequality to hold for k , we have

$$\delta_{k+1}^2 \leq C\delta_0^2 K^{-2ak-1} + C\delta_0^\beta \mu_0^{-\beta} K^{-\beta k(a+n_*)}.$$

Therefore $\delta_{k+1} \leq \delta_0 K^{-a(k+1)}$ provided

$$2C \leq K^{1-2a} \quad \text{and} \quad 2C \leq \mu_0^\beta \delta_0^{2-\beta} K^{k[\beta(a+n_*)-2a]-2a}.$$

By choosing first K and then $\mu_0 \geq \|u\|_2$ sufficiently large, these two inequalities can be satisfied for any given a in the range prescribed in 1.81. This proves our claim.

Next we show that for any

$$\alpha < \min \left\{ \frac{1}{1+2n_*}, \frac{\beta}{2} \right\} \quad (1.82)$$

the parameters μ_0 and K can be chosen so that the sequence u_k converges in $C^{1,\alpha}(\Omega; \mathbb{R}^{n+1})$. To this end observe that to any α satisfying 1.82 there exists an a satisfying 1.81 such that

$$\alpha < \frac{a}{a+n_*}.$$

Then, choosing μ_0 and K sufficiently large as above, we obtain a sequence u_k such that

$$\begin{aligned} \|u_{k+1} - u_k\|_1 &\leq C\delta_0 K^{-ak} \\ \|u_{k+1} - u_k\|_2 &\leq \mu_{k+1} + \mu_k \leq 2\mu_0 K^{(k+1)n_*}. \end{aligned}$$

Therefore, by interpolation

$$\begin{aligned} \|u_{k+1} - u_k\|_{1,\alpha} &\leq \|u_{k+1} - u_k\|_1^{1-\alpha} \|u_{k+1} - u_k\|_2^\alpha \\ &\leq \tilde{C} K^{-[(1-\alpha)a - \alpha n_*]k}. \end{aligned} \quad (1.83)$$

Thus the sequence converges in $C^{1,\alpha}$ to some limit map $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n+1})$. Since $\delta_k \rightarrow 0$, the limit satisfies $v^\sharp e = g$ in Ω .

Finally, choosing K so large that $K^{-a} \leq 1/2$, we have

$$\|v - u\|_1 \leq C\delta_0 \sum_k K^{-ak} \leq 2C\delta_0.$$

1.6.2 Proof of Theorem 2

Recall from Section 1.5.3 that for the whole construction we work with a fixed atlas $\{\Omega_i\}$ of the manifold M , and that to the given metric $g \in C^\beta(M)$ there exist constants $\gamma > 1$ and $\rho_0 > 0$ such that 1.61 and 1.62 hold.

Since $u \in C^2(M; \mathbb{R}^m)$ and there are a finite number of charts Ω_i , there exists $\rho < \rho_0$ such that

$$\text{osc}_U \nabla u < \eta_0/4 \quad \text{whenever } U \subset M \text{ with } \text{diam } U < \rho.$$

Fix a minimal cover \mathcal{C} of M with diameter ρ and let $\mu_0, \delta_0 > 0$ be such that

$$\begin{aligned} \|u^\sharp e - g\|_0 &\leq \delta_0^2 \\ \|u\|_2 &\leq \mu_0. \end{aligned}$$

The iteration now proceeds with respect to this fixed cover, parallel to the proof of Theorem 1. More precisely, arguing as in in Theorem 1, Proposition 5 yields a sequence $u_k \in C^2(M; \mathbb{R}^m)$ with

$$\begin{aligned} \|u_k^\sharp e - g\|_0 &\leq \delta_k^2 \\ \|u_k\|_2 &\leq \mu_0 K^{k(n+1)n_*} \\ \|u_{k+1} - u_k\|_1 &\leq C\delta_k, \end{aligned}$$

where

$$\delta_{k+1}^2 = C\delta_k^2 \left(\frac{1}{K} + \delta_k^{\beta-2} K^{-\beta k(n+1)n_*} \right). \quad (1.84)$$

The proof that μ_0 and K can be chosen so that u_k converges in $C^{1,\alpha}$ for

$$\alpha < \min \left\{ \frac{1}{1+2(n+1)n_*}, \frac{\beta}{2} \right\} \quad (1.85)$$

follows entirely analogously. Recall that this argument yields in particular

$$\delta_k \leq \delta_0 K^{-ak}.$$

The only difference is that the estimates 1.63 and 1.65 need to be fulfilled at each stage. To this end note that $\delta_k \leq \delta_0$, so that 1.63 will hold at stage k if it holds at the initial stage. Moreover,

$$\text{osc}_U \nabla u_k \leq \text{osc}_U \nabla u + \sum_{j=0}^{k-1} 2\|u_{j+1} - u_j\|_1 \leq \frac{\eta_0}{4} + 2C\delta_0 \sum_j K^{-aj} \leq \frac{\eta_0}{4} + 4C\delta_0,$$

so that 1.65 is fulfilled by u_k provided δ_0 is sufficiently small (depending only on the various constants).

1.6.3 Proof of Corollaries 1 and 2

The corollaries are a direct consequence of the Nash-Kuiper theorem combined with Theorems 1 and 2 respectively. For simplicity, we allow M to be either $\overline{\Omega}$ for a smooth bounded open set $\Omega \subset \mathbb{R}^n$ or a compact Riemannian manifold of dimension n , and assume that $g \in C^\beta(M)$ is satisfying either the assumptions of Theorem 1 or those of Theorem 2. We then set $\alpha_0 = \min\{(2n_* + 1)^{-1}, \beta/2\}$ in the first case, and $\alpha_0 = \min\{(2(n+1)n_* + 1)^{-1}, \beta/2\}$ in the second.

Let $u \in C^1(M; \mathbb{R}^m)$ be a short map and $\varepsilon > 0$. We may assume without loss of generality that $\varepsilon < \delta_0$. Using the Nash-Kuiper theorem together with a standard regularization, there exists $u_0 \in C^2(M; \mathbb{R}^m)$ such that

$$\begin{aligned} \|u - u_0\|_1 &\leq \varepsilon/2, \\ \|u_0^\sharp e - g\|_0 &\leq \left(\frac{\varepsilon}{2C} \right)^2, \end{aligned}$$

where C is the constant in Theorems 1 and 2 respectively. Then the theorem, applied to u_0 , yields an isometric immersion $v \in C^{1,\alpha}(M; \mathbb{R}^m)$ for any $\alpha < \alpha_0$, such that $\|v - u_0\|_1 \leq \varepsilon/2$, so that $\|v - u\|_1 \leq \varepsilon$. This proves the corollaries.

We now come to Remark 1. This follows immediately from the fact that the Nash-Kuiper theorem also works for embeddings, and that the set of embeddings of a compact manifold is an open set in $C^1(M; \mathbb{R}^m)$. Indeed, if u is an embedding, the Nash-Kuiper theorem gives the existence of an embedding u_0 with the estimates above. Ensuring in addition that ε is so small that any map $v \in C^1(M; \mathbb{R}^m)$ with $\|v - u\|_1 \leq \varepsilon$ is an embedding, we reach the required conclusion.

1.7 Rigidity: Proof of Theorem 3

1.7.1 Curvature and Brouwer degree

Let (M, g) be as in Theorem 3. As usual, we denote by dA the area element in M and by κ the Gauss curvature of (M, g) . Consider next a C^2 isometric embedding $v: M \rightarrow \mathbb{R}^3$. The unit normal $N(p)$ to $v(M)$ is the unique vector of \mathbb{R}^3 such that, given a positively oriented basis e_1, e_2 for $T_p(M)$, the triple $(dv_p(e_1), dv_p(e_2), N(p))$ is an orthonormal positively oriented frame of \mathbb{R}^3 .

As it is well known, if $d\sigma$ denotes the area element in \mathbb{S}^2 , then $N^\#d\sigma = \kappa dA$. Therefore, for every open set $V \subset\subset M$ and for every $f \in C^1(\mathbb{S}^2)$, the usual change of variable formula yields

$$\int_V f(N(x)) \kappa(x) dA(x) = \int_{\mathbb{S}^2} f(y) \deg(y, V, N) d\sigma(y), \quad (1.86)$$

where $\deg(y, V, N)$ denotes the Brouwer degree of the map N . Though the differential definition of \deg makes sense only for regular values of N , it is a classical observation that \deg is constant on connected components of $\mathbb{S}^2 \setminus N(\partial V)$. Thus it has a unique continuous extension to $\mathbb{S}^2 \setminus N(\partial V)$, which will be denoted as well by \deg .

Consider next an isometric embedding $v \in C^1$. In this case $N \in C^0$. The Brouwer degree $\deg(y, V, N)$ can still be defined and we recall the following well-known theorem.

Theorem 4. *Let $N \in C(V, \mathbb{S}^2)$ and $\{N_k\} \subset C^\infty(V, \mathbb{S}^2)$ be a sequence converging uniformly to N . Let $K \subset \mathbb{S}^2 \setminus N(\partial V)$ be a closed set. For any k sufficiently large, $\deg(\cdot, V, N_k) \equiv \deg(\cdot, V, N)$ on K .*

Thus $\deg(\cdot, V, N) \in L^1_{loc}(\mathbb{S}^2 \setminus N(\partial V))$. A key step to the proof of Theorem 3 is to show that formula 1.86 holds for $v \in C^{1,\alpha}$ with $\alpha > 2/3$.

Proposition 6. *Let $v \in C^{1,\alpha}(M, \mathbb{R}^3)$ be an isometric embedding with $\alpha > 2/3$. Then 1.86 holds for every open set $V \subset\subset M$ diffeomorphic to a subset of \mathbb{R}^2 and every $f \in L^\infty$ with $\text{supp}(f) \subset \mathbb{S}^2 \setminus N(\partial V)$.*

In order to deal with $N(\partial V)$ we recall the following elementary fact.

Lemma 4. *Let M and \tilde{M} be 2-dimensional Riemannian manifolds, $\beta > \frac{1}{2}$ and $N \in C^{0,\beta}(M, \tilde{M})$. If $E \subset M$ has Hausdorff dimension 1, then the area of $N(E)$ is 0.*

The following is then a corollary of Proposition 6 and Lemma 4.

Corollary 5. *Let (M, g) and v be as in Proposition 6, with $\kappa \geq 0$. For any open $V \subset\subset M$, $\deg(\cdot, V, N)$ is a nonnegative L^1 function and 1.86 holds for every $f \in L^\infty(\mathbb{S}^2 \setminus N(\partial V))$.*

1.7.2 Proof of Proposition 6

By a standard approximation argument, it suffices to prove the statement when f is smooth. Under this additional assumption the proof is a direct consequence of Theorem 4 and of the convergence result below, which is a consequence of Proposition 1. Since V is diffeomorphic to an open set of the euclidean plane, we can consider global coordinates x_1, x_2 on it. Fix a symmetric kernel $\varphi \in C_c^\infty(\mathbb{R}^2)$, set $\varphi_\varepsilon(x) = \varepsilon^{-2}\varphi(x/\varepsilon)$ and let $v^\varepsilon := (v\mathbf{1}_V) * \varphi_\varepsilon$ (we consider here the convolution of the two functions in \mathbb{R}^2 using the coordinates x_1, x_2 and the corresponding Lebesgue measure).

Proposition 7. *Let v and v^ε be defined as above and denote by N^ε , g^ε , A^ε and κ^ε respectively, the normal to $v^\varepsilon(M)$, the pull-back of the metric on $v^\varepsilon(M)$, and the corresponding area element and Gauss curvature. Then,*

$$\lim_{\varepsilon \downarrow 0} \int_V f(N^\varepsilon) \kappa^\varepsilon dA^\varepsilon = \int_V f(N) \kappa dA \quad \forall f \in C_c^\infty(\mathbb{S}^2 \setminus N(\partial V)). \quad (1.87)$$

Proof. In coordinates, our aim is to show that

$$\lim_{\varepsilon \downarrow 0} \int_V f(N^\varepsilon(x)) \kappa^\varepsilon(x) (\det g^\varepsilon(x))^{\frac{1}{2}} dx = \int_V f(N(x)) \kappa(x) (\det g(x))^{\frac{1}{2}} dx. \quad (1.88)$$

We recall the formulas for the Christoffel symbols, the Riemann tensor and the Gauss curvature in V , in the system of coordinates already fixed:

$$\Gamma_{jk}^i = \frac{1}{2} g^{im} (\partial_k g_{jm} + \partial_j g_{mk} - \partial_m g_{kj}), \quad (1.89)$$

$$R_{iljk} = g_{lm} (\partial_k \Gamma_{ij}^m - \partial_j \Gamma_{ik}^m + \Gamma_{ij}^l \Gamma_{kl}^m - \Gamma_{ik}^l \Gamma_{jl}^m), \quad (1.90)$$

$$\kappa = \frac{R_{1212}}{\det(g_{ij})}. \quad (1.91)$$

After obvious computations we conclude that

$$\kappa = (\det g)^{-1} (c_{ijkl} \partial_{kl} g_{ij} + d_{ijklmn}(g) \partial_k g_{ij} \partial_l g_{mn}) \quad (1.92)$$

where c_{ijkl} are constant coefficients and the functions d_{ijklmn} are smooth.

Proposition 1 implies that $\partial_k g_{ij}^\varepsilon$ and g_{ij}^ε converge locally uniformly to $\partial_k g_{ij}$ and g_{ij} respectively. Moreover, N^ε converges locally uniformly to N . Since there is a compact set containing $f(N^\varepsilon)$ and $f(N)$, we only need to show that

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_V f(N^\varepsilon(x)) (\det g^\varepsilon(x))^{-\frac{1}{2}} \partial_{kl} g_{ij}^\varepsilon(x) dx \\ &= \int_V f(N(x)) (\det g(x))^{-\frac{1}{2}} \partial_{kl} g_{ij}(x) dx. \end{aligned} \quad (1.93)$$

Denote by ψ^ε the function $f(N^\varepsilon(x)) (\det g^\varepsilon(x))^{-\frac{1}{2}}$. Since $f(N^\varepsilon)$ is smooth and compactly supported in V we can integrate by parts to get

$$\int_V \psi^\varepsilon \partial_{kl} g_{ij}^\varepsilon = \int_V \partial_k \psi^\varepsilon \partial_l g_{ij}^\varepsilon. \quad (1.94)$$

Note that $\|\partial_k \psi^\varepsilon\| \leq C\varepsilon^{\alpha-1}$ by obvious estimates on convolutions. Hence, 1.2 gives

$$\int_V \partial_k \psi^\varepsilon (\partial_l g_{ij}^\varepsilon - \partial_l g_{ij}) = O(\varepsilon^{3\alpha-2}) \quad (1.95)$$

which converges to 0 because $\alpha > 3/2$. Integrating again by parts, we get

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_V f(N^\varepsilon(x)) (\det g^\varepsilon(x))^{-\frac{1}{2}} \partial_{kl} g_{ij}^\varepsilon(x) dx \\ &= \lim_{\varepsilon \downarrow 0} \int_V f(N^\varepsilon(x)) (\det g^\varepsilon(x))^{-\frac{1}{2}} \partial_{kl} g_{ij}(x) dx. \end{aligned}$$

Using the uniform convergence of N^ε to N and of g^ε to g we then conclude 1.93 and hence the proof of the Proposition. \square

1.7.3 Proof of Lemma 4 and Corollary 5

Proof (of Lemma 4). By the definition of Hausdorff dimension, for every $\varepsilon > 0$ and $\eta > 1$ there exists a covering of E with closed sets E_i such that

$$\sum_i (\text{diam}(E_i))^\eta \leq \varepsilon. \quad (1.96)$$

On the other hand, $\text{diam}(g(E_i)) \leq C(\text{diam}(E_i))^\beta$ and hence the area $|g(E_i)|$ can be estimated with $C(\text{diam}(E_i))^{2\beta}$. Since $\beta > 1/2$, we can pick $\eta = 2\beta$ to conclude that

$$|g(E)| \leq C \sum_i (\text{diam}(E_i))^\eta \leq C\varepsilon.$$

The arbitrariness of ε implies $|g(E)| = 0$. \square

Proof (of Corollary 5). First of all, we know from Proposition 6 that the formula 1.86 is valid for any open set V which is diffeomorphic to an open set of \mathbb{R}^2 , and any $f \in L^\infty$ compactly supported in $\mathbb{S}^2 \setminus N(\partial V)$. Since κ is nonnegative, we conclude that $\deg(\cdot, N, V) \geq 0$. Testing 1.86 with a sequence of compactly supported functions $f_k \uparrow \mathbf{1}_{\mathbb{S}^2 \setminus N(\partial V)}$ we derive that

$$\int \deg(y, N, V) d\sigma(y) = \int_V \kappa dA < \infty,$$

which implies $\deg(\cdot, N, V) \in L^1$.

Next, consider a V with smooth boundary. We decompose it into the union of finitely many nonoverlapping Lipschitz open sets V_i diffeomorphic to open sets of the euclidean plane. Then

$$\deg(y, N, V) = \sum_i \deg(y, N, V_i) \quad \text{for every } y \notin \bigcup_i N(\partial V_i).$$

On the other hand, by Lemma 4, $\bigcup_i N(\partial V_i)$ is a negligible set, and hence we conclude the formula for V from the previous step.

Finally, fix a generic V and an $f \in L^\infty$ with $\text{supp}(f) \subset \mathbb{S}^2 \setminus N(\partial V)$. Choose an open set V' with smooth boundary $\partial V'$ sufficiently close to ∂V . Then $\deg(\cdot, V, N)$ and $\deg(\cdot, V', N)$ coincide on the support of f , whereas the support of $f(N(\cdot))$ is contained in V' . From the formula for V' and f we conclude then the validity of the formula for V and f . Arguing again as above, we conclude that $\deg(\cdot, N, V)$ is summable and nonnegative and that the formula 1.86 holds for any V and any $f \in L^\infty(\mathbb{S}^2 \setminus N(\partial V))$. \square

1.7.4 Bounded extrinsic curvature. The proof of Theorem 3

We recall the notion of bounded extrinsic curvature for a C^1 immersed surface (see p. 590 of [25]).

Definition 2. Let $\Omega \subset \mathbb{R}^2$ be open and $u \in C^1(\Omega, \mathbb{R}^3)$ an immersion. The surface $u(\Omega)$ has *bounded extrinsic curvature* if there is a C such that

$$\sum_{i=1}^N |N(E_i)| \leq C \tag{1.97}$$

for any finite collection $\{E_i\}$ of pairwise disjoint closed subsets of Ω .

The proof of Theorem 3 follows now from Corollary 5.

Proof (of Theorem 3). The theorem follows easily from the claim:

$$\deg(\cdot, V, N) \geq \mathbf{1}_{N(V) \setminus N(\partial V)} \quad \text{for every open } V \subset \Omega. \tag{1.98}$$

In fact, given disjoint closed sets E_1, \dots, E_N , we can cover them with disjoint open sets V_1, \dots, V_N with smooth boundaries. By 1.98 and Corollary 5,

$$\sum_i |N(E_i) \setminus N(\partial V_i)| \leq \sum_i |N(V_i) \setminus N(\partial V_i)| \leq \sum_i \int_{V_i} \kappa \leq \int_{\Omega} \kappa. \quad (1.99)$$

On the other hand, by Lemma 4, $|N(\partial V_i)| = 0$. Thus, 1.99 shows 1.97.

We now come to the proof of 1.98. Obviously $\deg(y, V, N) = 0$ if $y \notin N(V)$. Moreover, by Corollary 5, $\deg(\cdot, V, N) \geq 0$. Therefore, fix $y_0 \in N(V) \setminus N(\partial V)$ and assume, by contradiction, that $\deg(y_0, V, N) = 0$. Consider a small open disk D centered at y_0 such that $N^{-1}(D) \cap \partial V = \emptyset$ and let $W := N^{-1}(D) \cap V$. Then $N(\partial W) \subset \partial D$ and $N(W) \subset D$. So, $\deg(\cdot, W, N)$ vanishes on $\mathbb{S}^2 \setminus \bar{D}$ and is a constant integer k on D . On the other hand $k = \deg(y_0, W, N) = \deg(y_0, V, N) - \deg(y_0, V \setminus \bar{W}, N) = -\deg(y_0, V \setminus \bar{W}, N)$. Since $y_0 \notin N(V \setminus \bar{W})$, we conclude $k = 0$ and hence

$$0 = \int \deg(y, W, N) dy = \int_W \kappa dA.$$

which is a contradiction because $W \neq \emptyset$ and $\kappa > 0$. \square

Corollary 3 follows from Theorem 3 and the results of Pogorelov cited in the introduction. More precisely, by Theorem 9 on p650 [25], $u(\mathbb{S}^2)$ is a closed convex surface, which by [24] is rigid.

Corollary 4 also follows from the results in [25] and [26]. However, we were unable to find an exact reference for open surfaces, and therefore, for the reader's convenience, we have included a proof in the appendix.

Appendix

Proof (of Corollary 4).

First of all, since the theorem is local, without loss of generality we can assume that:

1. $\Omega = B_r(0)$, $u \in C^{1,\alpha}(\bar{B}_r(x))$, $g \in C^{2,\beta}(\bar{B}_r(x))$ and u is an embedding;
2. $u(\Omega)$ has bounded extrinsic curvature.

Step 1. Density of regular points. For any point $z \in \mathbb{S}^2$ we let $n(z)$ be the cardinality of $N^{-1}(z)$. It is easy to see that, for a surface of bounded extrinsic curvature, $\int_{\mathbb{S}^2} n < \infty$ (cp. with Theorem 3 of p. 590 in [25]). Therefore, the set $E := \{n = \infty\}$ has measure zero. Let $\Omega_r := N^{-1}(\mathbb{S}^2 \setminus E)$. Observe that

$$\Omega_r \text{ is dense in } \Omega. \quad (1.100)$$

Otherwise there is a nontrivial smooth open set V such that $N(V) \subset E$. But then, $\deg(\cdot, V, N) = 0$ for every $y \notin N(\bar{V})$, and since $|N(V)| = |N(\partial V)| = 0$, it follows that $\deg(\cdot, V, N) = 0$ a.e.. By Corollary 5, $\int_V \kappa = 0$, which contradicts $\kappa > 0$.

Step 2. Convexity around regular points. Note next that, for every $x \in \Omega_r$ there is a neighborhood U of x such that $N(y) \neq N(x)$ for all $y \in U \setminus \{x\}$, i.e. x is regular in the sense of [25] p. 582. Recalling 1.98, $\deg(\cdot, V, N) \geq \mathbf{1}_{V \setminus \partial V}$ for every V : therefore the index of the map N at every point $x \in \Omega_r$ is at least 1. So, by the Lemma of page 594 in [25], any point $x \in \Omega_r$ is an elliptic point relative to the mapping N (that is, there is a neighborhood U of x such that the tangent plane π to $u(\Omega)$ in x intersects $U \cap u(\Omega)$ only in $u(x)$; cp. with page 593 of [25]).

By the discussion of page 650 in [25], $u(\Omega)$ has nonnegative extrinsic curvature as defined in IX.5 of [25]. Then, Lemma 2 of page 612 shows that, for every elliptic point $y \in u(\Omega)$ there is a neighborhood where $u(\Omega)$ is convex. This conclusion applies, therefore, to any $y \in \Omega_r$. We next claim the existence of a constant C with the following property. Set $\rho(y) := C^{-1} \min\{1, \text{dist}(u(y), u(\partial\Omega))\}$. Then

$$u(\Omega) \cap B_{\rho(y)}(y) \text{ is convex for all } y \in \Omega_r. \quad (1.101)$$

Recall that u is an embedding and hence $\text{dist}(u(y), u(\partial\Omega)) > 0$ for every $y \in \Omega$. By 1.100, 1.101 gives for any $y \in \Omega$ there is a neighborhood where $u(\Omega)$ is convex. This would complete the proof.

Step 3. Proof of 1.101. First of all, since u is an embedding and $\|u\|_{C^{1,\alpha}}$ is finite, there is a constant c_0 such that, for any point x , $B_{c_0}(x) \cap u(\Omega)$ is the graph of a $C^{1,\alpha}$ function with $\|\cdot\|_{C^{1,\alpha}}$ norm smaller than 1. In order to prove 1.101 we assume, without loss of generality, that $y = 0$ and that the tangent plane to $u(\Omega)$ at y is $\{x_3 = 0\}$. Denote by π the projection on $\{x_3 = 0\}$. By [26] there is a constant $\lambda > 0$ (depending only on $\|g\|_{C^{2,\beta}}$, $\|\kappa\|_{C^0}$ and $\|\kappa^{-1}\|_{C^0}$) with the following property.

(Est) Let U be an open convex set such that $U \cap u(\partial\Omega) = \emptyset$, $\text{diam}(U) \leq c_0$ and $U \cap u(\Omega)$ is locally convex. Then $U \cap u(\Omega)$ is the graph of a function $f : \pi(u(\Omega) \cap U) \rightarrow \mathbb{R}$ with $\|f\|_{C^{2,1/2}} \leq \lambda^{-1}$ and $D^2 f \geq \lambda \text{Id}$.

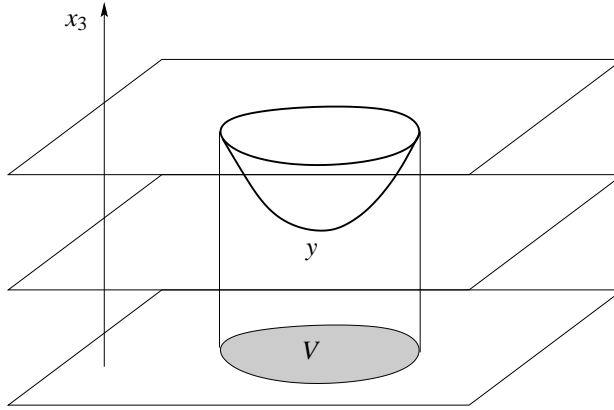


Fig. 1.2 The convex sets of type $V \times]-a, a[$ among which we choose the maximal one U_m .

We now look for sets U as in (Est) with the additional property that $U = V \times]-a, a[$ and $f|_{\partial V} = a$ (see Figure 1.2). Let U_m be the maximal set of this form for which the assumptions of (Est) hold. We claim that, either $\partial U_m \cap u(\partial \Omega) \neq \emptyset$, or $\text{diam}(U_m) = c_0$. By (Est), this claim easily implies 1.101. To prove the claim, assume by contradiction that it is wrong and let $U_m = W_m \times]-a_m, a_m[$ be the maximal set. Let $\gamma = \partial U_m \cap u(\Omega)$. By the choice of c_0 , γ is necessarily the curve $\partial W_m \times \{a\}$. On the other hand, by the estimates of (Est), it follows that every tangent plane to $u(\Omega)$ at a point of γ is transversal to $\{x_3 = 0\}$. So, for a sufficiently small $\varepsilon > 0$, the intersection $\{x_3 = a_m + \varepsilon\} \cap u(\Omega)$ contains a curve γ' bounding a connected region $D \subset u(\Omega)$ which contains $u(\Omega) \cap U_m$. By Theorem 8 of page 650 in [25], D is a convex set. This easily shows that U_m was not maximal. \square

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