Dissipative Euler flows and Onsager's conjecture

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Abstract

Building upon the techniques introduced in [15], for any $\theta < \frac{1}{10}$ we construct periodic weak solutions of the incompressible Euler equations which dissipate the total kinetic energy and are Hölder-continuous with exponent θ. A famous conjecture of Onsager states the existence of such dissipative solutions with any Hölder exponent $\theta < \frac{1}{3}$. Our theorem is the first result in this direction.

Keywords. Euler equations, Onsager's conjecture, turbulence

1 Introduction

The Euler equations for the motion of an inviscid perfect fluid are

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases}, \tag{1.1}$$

where $v(x,t)$ is the velocity vector and $p(x,t)$ is the internal pressure. In this paper we consider the equations in 3 dimensions and assume the domain to be periodic, i.e. the 3-dimensional torus $\mathbb{T}^3 = \mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$. Multiplying (1.1) by v itself and integrating, we obtain the formal energy balance

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} |v(x,t)|^2\,dx = -\int_{\mathbb{T}^3} [((v\cdot\nabla)v)\cdot v](x,t)\,dx.$$

If v is continuously differentiable in x, we can integrate the right hand side by parts and conclude that

$$\int_{\mathbb{T}^3} |v(x,t)|^2\,dx = \int_{\mathbb{T}^3} |v(x,0)|^2\,dx \quad \text{for all } t>0. \tag{1.2}$$

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On the other hand, in the context of 3-dimensional turbulence it is important to consider *weak solutions*, where v and p are not necessarily differentiable. If (v, p) is merely continuous, one can define weak solutions (see e.g. [27, 24]) by integrating (1.1) over simply connected subdomains $U \subset \mathbb{T}^3$ with C^1 boundary, to obtain the identities

$$\begin{aligned}\int_U v(x,0)\,dx = \int_U v(x,t)\,dx + \int_0^t \int_{\partial U} [v(v\cdot\nu) + p\nu](x,s)\,dA(x)\,ds \\ \int_{\partial U} [v\cdot\nu](x,t)\,dA(x) = 0\end{aligned} \tag{1.3}$$

for all t. In these identities ν denotes the unit outward normal to U on ∂U and dA denotes the usual area element. Indeed, the formulation (1.3) appears *first* in the derivation of the Euler equations from Newton's laws in continuum mechanics, and (1.1) is then *deduced* from (1.3) for sufficiently regular (v, p). It is also easy to see that pairs of continuous functions (v, p) satisfy (1.3) for all fluid elements U and all times t if and only if they solve (1.1) in the "modern" distributional sense (rewriting the first line as $\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0$).

For weak solutions, the energy conservation (1.2) might be violated, and indeed, this possibility has been considered for a rather long time in the context of 3 dimensional turbulence. In his famous note [26] about statistical hydrodynamics, Onsager considered weak solutions satisfying the Hölder condition

$$|v(x,t) - v(x',t)| \le C|x - x'|^\theta, \tag{1.4}$$

where the constant C is independent of $x, x' \in \mathbb{T}^3$ and t. He conjectured that

(a) Any weak solution v satisfying (1.4) with $\theta > \frac{1}{3}$ conserves the energy;

(b) For any $\theta < \frac{1}{3}$ there exist weak solutions v satisfying (1.4) which do not conserve the energy.

This conjecture is also very closely related to Kolmogorov's famous K41 theory [23] for homogeneous isotropic turbulence in 3 dimensions. We refer the interested reader to [19, 28, 18], see also Section 1.1 below.

Part (a) of the conjecture is by now fully resolved: it has first been considered by Eyink in [17] following Onsager's original calculations and proved by Constantin, E and Titi in [10]. Slightly weaker assumptions on v (in Besov spaces) were subsequently shown to be sufficient for energy conservation in [16, 6]. In contrast, until now part (b) of the conjecture remained widely open. In this paper we address specifically this question by proving the following theorem:

Theorem 1.1. *Let $e : [0, 1] \to \mathbb{R}$ be a smooth positive function. For every $\theta < \frac{1}{10}$ there is a pair $(v, p) \in C(\mathbb{T}^3 \times [0, 1])$ with the following properties:*

- (v, p) *solves the incompressible Euler equations in the sense* (1.3)*;*
- v *satisfies* (1.4)*;*
- *the energy satisfies*

$$e(t) = \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx \qquad \forall t \in [0,1] \, . \tag{1.5}$$

This is the first result in the direction of part (b) of Onsager's conjecture, where Hölder-continuous solutions are constructed. Prior to this result, there have been several constructions of weak solutions violating (1.2) in [29, 30, 31, 12, 13], but the solutions constructed in these papers are not continuous. The ones of [29, 30] are just square summable functions of time and space, whereas the example of [31] was the first to be in the energy space and the constructions of [12, 13] gave bounded solutions. Recently, in [15] we have constructed *continuous* weak solutions, but no Hölder exponent was given.

Remark 1.2. Since completion of this work, our technique for getting Hölder continuity has been refined in [21] to improve the regularity exponent in Theorem 1.1 to $\theta < \frac{1}{5}$, see also the works [5], [4] and [3].

Remark 1.3. In fact our proof of Theorem 1.1 yields some further regularity properties of the pair (v, p). First of all, our solutions v are Hölder-continuous in *space and time*, i.e. there is a constant C such that

$$|v(x,t) - v(x',t')| \leq C \left(|x - x'|^\theta + |t - t'|^\theta \right)$$

for all pairs $(x, t), (x', t') \in \mathbb{T}^3 \times [0, 1]$.

From the equation $\Delta p = -\operatorname{div}\operatorname{div}(v \otimes v)$ (after normalizing the pressure so that $\int p(x,t)\, dx = 0$) and standard Schauder estimates one can easily derive Hölder regularity in space for p as well, with Hölder exponent θ. A more careful estimate[3] improves the exponent to 2θ. It is interesting to observe that in fact our scheme produces pressures p which have that very Hölder regularity in *time and space*, namely

$$|p(x,t) - p(x',t')| \leq C \left(|x - x'|^{2\theta} + |t - t'|^{2\theta} \right) \, .$$

1.1 The energy spectrum

The energy spectrum $E(\lambda)$ gives the decomposition of the total energy by wavenumber, i.e.

$$\int |v|^2 \, dx = \int_0^\infty E(\lambda) d\lambda.$$

[3] personal communication with L. Silvestre

One of the cornerstones of the K41 theory is the famous Kolmogorov spectrum

$$E(\lambda) \sim \epsilon^{2/3}\lambda^{-5/3}$$

for wave numbers λ in the inertial range for fully developed 3-dimensional turbulent flows, where ϵ is the energy dissipation rate. For dissipative weak solutions of the Euler equations as conjectured by Onsager, this would be the expected energy spectrum for all $\lambda \in (\lambda_0, \infty)$.

Our construction, based on the scheme and the techniques introduced in [15], allows for a rather precise analysis of the energy spectrum. In a nutshell the scheme can be described as follows. We construct a sequence of (smooth) approximate solutions to the Euler equations v_k, where the error is measured by the (traceless part of the) Reynolds stress tensor $\mathring{R}_k$, cf. (2.1) and (3.5). The construction is explicitly given by a formula of the form

$$v_{k+1}(x,t) = v_k(x,t) + W\big(v_k(x,t), R_k(x,t); \lambda_k x, \lambda_k t\big) + \text{corrector}. \tag{1.6}$$

The corrector is to ensure that v_{k+1} remains divergence-free. The vector field W consists of periodic Beltrami flows in the fast variables (at frequency λ_k), which are modulated in amplitude and phase depending on v_k and R_k. More specifically, the amplitude is determined by the error R_k from the previous step, so that

$$\|v_{k+1} - v_k\|_0 \lesssim \delta_k^{1/2}, \tag{1.7}$$

$$\|v_{k+1} - v_k\|_1 \lesssim \delta_k^{1/2}\lambda_k, \tag{1.8}$$

where $\delta_k = \|\mathring{R}_k\|_{C^0}$.

The frequencies λ_k are therefore the active modes in the Fourier spectrum of the velocity field in the limit. Since the sequence λ_k diverges rather fast, it is natural to think of (1.6) as iteratively defining the Littlewood-Paley pieces at frequency λ_k. Following [9] we can then estimate the (Littlewood-Paley-) *energy spectrum* in the limit as

$$E(\lambda_k) \sim \frac{\langle |v_{k+1} - v_k|^2 \rangle}{\lambda_k}$$

for the active modes λ_k, where $\langle\cdot\rangle$ denotes the average over the space-time domain. Since W is the superposition of finitely many Beltrami modes, we can estimate $\langle |v_{k+1} - v_k|^2\rangle \sim \delta_k$. Thus, both the regularity of the limit and its energy spectrum are determined by the rates of convergence $\delta_k \to 0$ and $\lambda_k \to \infty$.

In [15] it was shown (cp. Proposition 2.2 and its proof) that W can be chosen so that

$$\|\mathring{R}_{k+1}\|_{C^0} \leq C(v_k, \mathring{R}_k)\lambda_k^{-\gamma} \tag{1.9}$$

for some fixed $0 < \gamma \leq 1$. By choosing the frequencies $\lambda_k \to \infty$ sufficiently fast, C^0 convergence of this scheme follows easily. However, in order to obtain a rate on the divergence of λ_k we need to obtain an estimate on the error in (1.9) with an explicit dependence on v_k and $\mathring{R}_k$. This is achieved in Proposition 8.1 and forms a key part of the paper. Roughly speaking, our estimate has the form

$$\|\mathring{R}_{k+1}\|_{C^0} \lesssim \frac{\delta_k^{1/2}\|v_k\|_{C^1}}{\lambda_k^\gamma}, \tag{1.10}$$

with $\gamma \sim \frac{1}{2}$. A first attempt (based on experience with the isometric embedding problem, see below) at obtaining a rate on λ_k would then go as follows: in order to decrease the error in (1.10) by a fixed factor $K > 1$ (i.e. $\delta_{k+1} \leq \frac{1}{K}\delta_k$), we choose λ_k accordingly, so that

$$\lambda_k^\gamma \sim K\|v_k\|_{C^1}\delta_k^{-1/2}. \tag{1.11}$$

Using (1.8) we can then obtain an estimate on $\|v_{k+1}\|_{C^1}$ and iterate. However, it is easy to see that this leads to super-exponential growth of λ_k whenever $\gamma < 1$. From this one can only deduce the energy spectrum $E(\lambda) \sim \lambda^{-1}$ and no Hölder regularity.

Our solution to this problem is to force a double-exponential convergence of the scheme, see Section 2. In this way the finite Hölder regularity in Theorem 1.1 as well as the energy spectrum

$$E(\lambda_k) \lesssim \lambda_k^{-(6/5-\varepsilon)} \tag{1.12}$$

can be achieved, see Remark 2.3. It is quite remarkable, and much akin to the Nash-Moser iteration, that the more rapid (super-exponential) convergence of the scheme leads to a better regularity in the limit.

An underlying physical intuition in the turbulence theory is that the flux in the energy cascade should be controlled by local interactions, see [23, 26, 17, 6]. A consequence for part (b) of Onsager's conjecture is that in a dissipative solution the active modes, among which the energy transfer takes place, should be (at most) exponentially distributed. Indeed, Onsager explicitly states in [26] (cp. also [18]) that this should be the case.

For the scheme (1.6) in this paper the interpretation is that λ_k should increase at most exponentially. As seen in the discussion above, this would only be possible with $\gamma = 1$ in the estimate (1.10). On the other hand, it is also easy to see that with $\gamma = 1$ the estimate indeed leads to Onsager's critical $\frac{1}{3}$ Hölder exponent as well as the Kolmogorov spectrum. Indeed, from (1.11) together with (1.10) and (1.8) we would obtain $\delta_k \sim K^{-k}$ and $\lambda_k \sim K^{3/2k}$, leading to $E(\lambda_k) \sim \lambda_k^{-5/3}$. Thus, our scheme provides yet another route towards understanding the necessity of local interactions as well as towards the Kolmogorov spectrum, albeit one that *does not involve considerations on the energy cascade* but is rather based on the *ansatz* (1.6).

Onsager's conjecture has also been considered on shell-models [22, 7, 8], whose derivation is motivated by the intuition on locality of interactions. Roughly speaking, the Euler equations is considered in the Littlewood-Paley decomposition, but only nearest neighbor interactions in frequency space are retained in the nonlinear term, leading to an infinite system of coupled ODEs. The analogue of both part (a) and (b) of Onsager's conjecture has been proven in [7, 8], in the sense that the ODE system admits a unique fixed point which exhibits a decay of (Fourier) modes consistent with the Kolmogorov spectrum.

Although our Theorem 1.1 and the corresponding spectrum (1.12) falls short of the full conjecture, it highlights an important feature of the Euler equations that cannot be seen on such shell models: the critical $\frac{1}{3}$ exponent of Onsager is not just the borderline between energy conservation and dissipation in the sense of part (a) and (b) above. For exponents $\theta < \frac{1}{3}$ one should expect an entirely different behavior of weak solutions altogether, namely the type of non-uniqueness and flexibility that usually comes with the h-principle of Gromov [20].

1.2 h-principle and convex integration

Our iterative scheme is ultimately based on the convex integration technique introduced by Nash in [25] to produce C^1 isometric embeddings of Riemannian manifolds in low codimension, and vastly generalized by Gromov [20], although several modifications of this technique are required (see the Introduction of [15]). Nevertheless, in line with other results proved using a convex integration technique, our construction again adheres to the usual features of the h-principle. In particular, as in [15] we are concerned in this paper with the *local* aspects of the h-principle. For the Euler equations this means that we only treat the case of a periodic space-time domain instead of an initial/boundary value problem. Also, it should be emphasized that although in Theorem 1.1 the existence of *one* solution is stated, the method of construction leads to an *infinite number* of solutions, as indeed any instance of the h-principle does. We refer the reader to the survey [14] for the type of (global) results that could be expected even in the current Hölder-continuous setting.

It is of certain interest to notice that in the isometric embedding problem a phenomenon entirely analogous to the Onsager's conjecture occurs. Namely, if we consider $C^{1,\alpha}$ isometric embeddings in codimension 1, then it is possible to prove the h-principle for sufficiently small exponents α, whereas one can show the absence of the h-principle (and in fact even some rigidity statements) if the Hölder exponent is sufficiently large. This phenomenon was first observed by Borisov (see [1] and [2]) and proved in greater generality and with different techniques in [11]. In particular the proofs given in [11] of both the h-principle and the rigidity statements share many similarities with the analogous results for the Euler equations.

The connection between the existence of dissipative weak solutions of Euler and the convex integration techniques used to prove the h-principle in geometric problems (and unexpected solutions to differential inclusions) was first observed in [12]. Since then these techniques have been used successfully in other equations of fluid dynamics: we refer the interested reader to the survey article [14].

1.3 Loss of derivatives and regularization

Finally, let us make a technical remark. Since the negative power of λ in estimate (1.9) comes from a stationary-phase type argument (Proposition 4.4 in Section 4), the constant $C(v_k, \mathring{R}_k)$ will then depend on higher derivatives of v_k (and of $\mathring{R}_k$). In fact, with $\theta \to \frac{1}{10}$ the number of derivatives m required in the estimates converges to ∞. To overcome this *loss of derivative* problem, we use the well-known device from the Nash-Moser iteration to mollify v_k and $\mathring{R}_k$ at some appropriate scale ℓ_k. Although we are chiefly interested in derivative bounds in space, due to the nature of the equation such bounds are connected to derivative bounds in time, necessitating a mollification in space and time. To simplify the presentation we will therefore treat time also as a periodic variable and we will therefore construct solutions on $\mathbb{T}^3 \times \mathbb{S}^1$ rather than on $\mathbb{T}^3 \times [0, 1]$.

2 Iteration with double exponential decay

2.1 Notation in Hölder norms

In the following $m = 0, 1, 2, \ldots$, $\alpha \in (0, 1)$, and β is a multiindex. We introduce the usual (spatial) Hölder norms as follows. First of all, the supremum norm is denoted by $\|f\|_0 := \sup_{\mathbb{T}^3} |f|$. We define the Hölder seminorms as

$$[f]_m = \max_{|\beta|=m} \|D^\beta f\|_0 \,,$$
$$[f]_{m+\alpha} = \max_{|\beta|=m} \sup_{x \neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \,.$$

The Hölder norms are then given by

$$\begin{aligned} \|f\|_m &= \sum_{j=0}^m [f]_j \\ \|f\|_{m+\alpha} &= \|f\|_m + [f]_{m+\alpha}. \end{aligned}$$

For functions depending on *space and time*, we define spatial Hölder norms as

$$\|v\|_r = \sup_t \|v(\cdot, t)\|_r \,,$$

whereas the Hölder norms in space *and time* will be denoted by $\|\cdot\|_{C^r}$.

We also remark that we use the convention $0 \in \mathbb{N}$: therefore estimates stated for the norms $\|\cdot\|_m$ with $m \in \mathbb{N}$ include the C^0 norm as well.

2.2 The iterative scheme

We follow here [15] and introduce the Euler-Reynolds system (cp. with Definition 2.1 therein). We also establish the following common notation: if u is a 3×3 matrix with entries u_{ij}, we let $\operatorname{div} u$ be the (column) vector field whose components are given by the divergences of the rows of u, namely $(\operatorname{div} u)_i = \sum_j \partial_j u_{ij}$. We will mostly deal with symmetric matrices, however we will in some place take divergences of nonsymmetric ones and it is useful to notice that, according to our convention, if a and b are smooth vector fields, then $\operatorname{div}(a \otimes b) = (b \cdot \nabla)a + (\operatorname{div} b)a$.

Definition 2.1. Assume $v, p, \mathring{R}$ are C^1 functions on $\mathbb{T}^3 \times \mathbb{S}^1$ taking values, respectively, in $\mathbb{R}^3, \mathbb{R}, \mathcal{S}_0^{3\times 3}$. We say that they solve the Euler-Reynolds system if

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R} \\ \\ \operatorname{div} v = 0\,. \end{cases} \tag{2.1}$$

The next proposition is the main building block of our construction: the proof of Theorem 1.1 is achieved by applying it inductively to generate a suitable sequence of solutions to (2.1) where the right hand side vanishes in the limit.

Proposition 2.2. *Let e be a smooth positive function on $\mathbb{S}^1$. There exist positive constants η, M depending on e with the following property.*

Let $\delta \leq 1$ be any positive number and $(v, p, \mathring{R})$ a solution of the Euler-Reynolds system (2.1) *in $\mathbb{T}^3 \times \mathbb{S}^1$ such that*

$$\tfrac{3\delta}{4} e(t) \leq e(t) - \int |v|^2(x,t)\,dx \leq \tfrac{5\delta}{4} e(t) \qquad \forall t \in \mathbb{S}^1\,, \tag{2.2}$$

$$\|\mathring{R}\|_0 \leq \eta\delta \tag{2.3}$$

and

$$D := \max\{1, \|\mathring{R}\|_{C^1}, \|v\|_{C^1}\}\,. \tag{2.4}$$

For every $\bar{\delta} \leq \frac{1}{2}\delta^{\frac{3}{2}}$ and every $\varepsilon > 0$ there exists a second triple $(v_1, p_1, \mathring{R}_1)$ which solves as well the Euler-Reynolds system and satisfies the following estimates:

$$\tfrac{3\bar{\delta}}{4} e(t) \leq e(t) - \int |v_1|^2(x,t)\,dx \leq \tfrac{5\bar{\delta}}{4} e(t) \qquad \forall t \in \mathbb{S}^1\,, \tag{2.5}$$

$$\|\mathring{R}_1\|_0 \leq \eta\bar{\delta}\,, \tag{2.6}$$

$$\|v_1 - v\|_0 \le M\sqrt{\delta}\,, \tag{2.7}$$

$$\|p_1 - p\|_0 \le M^2\delta\,, \tag{2.8}$$

and

$$\max\{\|v_1\|_{C^1}, \|\mathring{R}_1\|_{C^1}\} \le A\delta^{\frac{3}{2}}\left(\frac{D}{\bar{\delta}^2}\right)^{1+\varepsilon} \tag{2.9}$$

where the constant A depends on e, $\varepsilon > 0$ and $\|v\|_0$.

We next show how to conclude Theorem 1.1 from Proposition 2.2: the rest of the paper is then devoted to prove the Proposition.

Proofs of Theorem 1.1. Let e be as in the statement, i.e. smooth and positive. Without loss of generality we can assume that e is defined on $\mathbb{R}$, with period 2π, and it is smooth and positive on the entire real line.

Step 1. Fix any arbitrarily small number $\varepsilon > 0$ and let $a, b \ge \frac{3}{2}$ be numbers whose choice will be specified later and will depend only on ε. We define $(v_0, p_0, \mathring{R}_0)$ to be identically 0 and we apply Proposition 2.2 inductively with

$$\delta_n = a^{-b^n}$$

to produce a sequence $(v_n, p_n, \mathring{R}_n)$ of solutions of the Euler-Reynolds system and numbers D_n satisfying the following requirements:

$$\tfrac{3\delta_n}{4}e(t) \le e(t) - \int |v_1|^2(x,t)\,dx \le \tfrac{5\delta_n}{4}e(t) \qquad \forall t \in \mathbb{S}^1\,, \tag{2.10}$$

$$\|\mathring{R}_n\|_0 \le \eta\delta_n\,, \tag{2.11}$$

$$\|v_n - v_{n-1}\|_0 \le M\sqrt{\delta_{n-1}}\,, \tag{2.12}$$

$$\|p_n - p_{n-1}\|_0 \le M^2\delta_{n-1}\,. \tag{2.13}$$

$$D_n = \max\{1, \|v_n\|_{C^1}, \|\mathring{R}_n\|_{C^1}\} \quad \text{and} \quad \delta_{n+1} \le \frac{1}{2}\delta_n^{3/2}\,. \tag{2.14}$$

Observe that with this choice of δ_n and since $a, b \ge \frac{3}{2}$, (v_n, p_n) converges uniformly to a continuous pair (v, p) and in particular

$$\|v_n\|_0 \le M\sum_{j=0}^{\infty} a^{-\frac{1}{2}b^j} \le M\sum_{j=0}^{\infty}\left(\frac{3}{2}\right)^{-\frac{1}{2}\left(\frac{3}{2}\right)^j}\,.$$

Therefore, $\|v_n\|_0$ is bounded uniformly, with a constant depending only on e. By Proposition 2.2 we have

$$D_{n+1} \le A\delta_n^{\frac{3}{2}}\left(\frac{D_n}{\delta_{n+1}^2}\right)^{1+\varepsilon}\,.$$

Since A is depending only on e, ε and $\|v_n\|_0$, which in turn can be estimated in terms of e, we can assume that A depends only on ε and e.

We claim that, for a suitable choice of the constants a, b there is a third constant $c > 1$ for which we inductively have the inequality

$$D_n \leq a^{cb^n}.$$

Indeed, for $n = 0$ this is obvious. Assuming the bound for D_n, we obtain for D_{n+1}

$$D_{n+1} \leq A \frac{a^{-\frac{3}{2}b^n} a^{c(1+\varepsilon)b^n}}{a^{-2(1+\varepsilon)b^{n+1}}} = A a^{(-\frac{3}{2}+(1+\varepsilon)(c+2b))b^n}.$$

We impose $\varepsilon < \frac{1}{4}$ and set

$$b = \frac{3}{2} \quad \text{and} \quad c = \frac{3(1+2\varepsilon)}{1-2\varepsilon} + \varepsilon.$$

This choice leads to

$$cb - \left(-\tfrac{3}{2} + (1+\varepsilon)(c+2b)\right) = \frac{\varepsilon}{2}(1-2\varepsilon) > \frac{\varepsilon}{4}.$$

Since $b^n \geq 1$, we conclude

$$D_{n+1} \leq \left(A a^{-\varepsilon/4}\right) a^{cb^{n+1}}$$

Choosing $a = A^{4/\varepsilon}$ we conclude $D_{n+1} \leq a^{cb^{n+1}}$.

Step 2. Consider now the sequence v_n provided in the previous step. By (2.10), (2.11), (2.12) and (2.13) we conclude that (v_n, p_n) converges uniformly to a solution (v, p) of the Euler equations such that $e(t) = \int |v|^2(x,t)dx$ for every $t \in \mathbb{S}^1$. On the other hand, observe that

$$\|v_{n+1} - v_n\|_0 \leq M\sqrt{\delta_n} \leq M a^{-\frac{1}{2}b^n}$$

and

$$\|v_{n+1} - v_n\|_{C^1} \leq D_n + D_{n+1} \leq 2a^{cb^{n+1}}.$$

Therefore

$$\begin{aligned} \|v_{n+1} - v_n\|_{C^\theta} &\leq \|v_{n+1} - v_n\|_0^{1-\theta} \|v_{n+1} - v_n\|_{C^1}^{\theta} \\ &\leq 2M a^{\left(\theta cb - \frac{(1-\theta)}{2}\right)b^n}. \end{aligned}$$

If

$$\theta < \frac{1}{1+2cb} = \frac{1-2\varepsilon}{10+19\varepsilon-6\varepsilon^2},$$

then $\theta cb - \frac{(1-\theta)}{2} < 0$ and therefore $\{v_n\}$ is a Cauchy sequence on C^θ, which implies that it converges in the C^θ norm.

We have shown that, for every $\varepsilon < \frac{1}{4}$ and every $\theta < \frac{1-2\varepsilon}{10+19\varepsilon-6\varepsilon^2}$ there is a pair $(v, p) \in C^\theta(\mathbb{T}^3 \times \mathbb{S}^1, \mathbb{R}^3) \times C(\mathbb{T}^3 \times \mathbb{S}^1)$ as in Theorem 1.1. Letting $\varepsilon \downarrow 0$ we obtain the conclusions of Theorem 1.1 (and indeed even the Hölder regularity in time). $\square$

Remark 2.3. Using the bounds on δ_n and D_n in the proof above, we can obtain an estimate on the energy spectrum of v. First of all we observe (cp. Section 3) that in Fourier space $v_{n+1} - v_n$ is essentially supported in a frequency band around wavenumber λ_n. For λ_n we then have the relation

$$\|v_{n+1} - v_n\|_{C^1} \sim \|v_{n+1} - v_n\|_{C^0} \, \lambda_n.$$

Therefore, Step 2 of the proof above implies

$$\lambda_n \sim a^{(bc+\frac{1}{2})b^n},$$

and consequently the energy spectrum satisfies

$$E(\lambda_n) \sim \frac{\delta_n}{\lambda_n} \sim a^{-(\frac{3}{2}+bc)b^n} \sim \lambda_n^{-\frac{3+2bc}{1+2bc}}.$$

Plugging in the choice of b, c from Step 1 of the proof yields in the limit $\varepsilon \to 0$

$$E(\lambda_n) \sim \lambda_n^{-6/5}.$$

2.3 Plan of the remaining sections

Except for Section 10, in which we prove the side Remark 1.3, the remaining sections are all devoted to the proof of Proposition 2.2.

Section 3 contains the precise definition of the maps $(v_1, p_1, \mathring{R}_1)$ of Proposition 2.2. The maps will depend upon various parameters, which will be specified only at the end.

Section 4 contains some preliminaries on classical estimates for the Hölder norms of products and compositions of functions, some classical Schauder estimates for the elliptic operators involved in the construction and a ”stationary phase lemma” (Proposition 4.4) for the Hölder norms of highly oscillatory functions. This last lemma is also a quite classical fact, but it plays a key role in our estimates.

In Section 5 we prove the key estimates on the main building blocks of the construction in terms of the relevant parameters: all these estimates are collected in the technical Proposition 5.1.

The various tools introduced in the Sections 4 and 5 are then used in Section 6, 7 and 8 to derive the fundamental estimates on the Hölder norms of v_1 and $\mathring{R}_1$ in terms of the relevant parameters. In particular:

- Section 6 contains the estimates on v_1;
- Section 7 the estimate on the kinetic energy $\int |v_1|^2$;
- Section 8 the estimates on the Reynolds stress $\mathring{R}_1$.

Finally, in Section 9 the estimates of the Sections 6, 7 and 8 are used to tune the parameters and prove Proposition 2.2.

3 Definition of the maps v_1, p_1 and $\mathring{R}_1$

From now on we fix a triple $(v, p, \mathring{R})$ and numbers $\delta, \bar{\delta}, \varepsilon > 0$ as in Proposition 2.2. As in [15] the new velocity v_1 is obtained by adding two perturbations, w_o and w_c:

$$v_1 = v + w_o + w_c = v_1 + w\,, \tag{3.1}$$

where w_c is a *corrector* to ensure that v_1 is divergence-free. Thus, w_c is defined as

$$w_c := -\mathcal{Q} w_o \tag{3.2}$$

where $\mathcal{Q} = \mathrm{Id} - \mathcal{P}$ and $\mathcal{P}$ is the Leray projection operator, see [15, Definition 4.1].

3.1 Conditions on the parameters

The main perturbation w_o is a highly oscillatory function which depends on three parameters: a (small) length scale $\ell > 0$ and (large) frequencies μ, λ such that

$$\lambda,\ \mu,\ \frac{\lambda}{\mu} \in \mathbb{N}.$$

In the subsequent sections we will assume the following inequalities:

$$\mu \geq \delta^{-1} \geq 1, \quad \ell^{-1} \geq \frac{D}{\eta\delta} \geq 1, \quad \lambda \geq \max\left\{(\mu D)^{1+\omega}, \ell^{-(1+\omega)}\right\}\,. \tag{3.3}$$

Here $\omega := \frac{\varepsilon}{2+\varepsilon} > 0$ so that

$$1 + \varepsilon = \frac{1+\omega}{1-\omega}\,.$$

Of course, at the very end, the proof of Proposition 2.2 will use a specific choice of the parameters, which will be shown to respect the above conditions. However, at this stage the choices in (3.3) seem rather arbitrary. We could leave the parameters completely free and carry all the relevant estimates in general, but this would give much more complicated and lengthy formulas in all of them. It turns out that the conditions (3.3) above greatly simplifies many computations.

3.2 Definition of w_o

In order to define w_o we draw heavily upon the techniques introduced in [15].

- First of all we let $r_0 > 0$, $N, \lambda_0 \in \mathbb{N}$, $\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \lambda_0\}$ and $\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id}))$ be as in [15, Lemma 3.2].

- Next we let $\mathcal{C}_j \subset \mathbb{Z}^3$, $j \in \{1, \dots, 8\}$ and the functions α_k be as in [15, Section 4.1]; as in that section, we define the functions

$$\phi_{k,\mu}^{(j)}(v,\tau) := \sum_{l\in\mathcal{C}_j} \alpha_l(\mu v) e^{-i\frac{k\cdot l}{\mu}\tau} .$$

Next, we let $\chi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ be a smooth standard nonnegative radial kernel supported in $[-1,1]^4$ and we denote by

$$\chi_\ell(x,t) := \frac{1}{\ell^4}\chi\left(\frac{x}{\ell}, \frac{t}{\ell}\right)$$

the corresponding family of mollifiers. We define

$$\begin{aligned} v_\ell(x,t) &= \int_{\mathbb{T}^3\times\mathbb{S}^1} v(x-y,t-s)\chi_\ell(y,s)\,dy\,ds \\ \mathring{R}_\ell(x,t) &= \int_{\mathbb{T}^3\times\mathbb{S}^1} \mathring{R}(x-y,t-s)\chi_\ell(y,s)\,dy\,ds. \end{aligned}$$

Similarly to [15, Section 4.1], we define the function

$$\rho_\ell(t) := \frac{1}{3(2\pi)^3}\left(e(t)(1-\bar{\delta}) - \int_{\mathbb{T}^3} |v_\ell|^2(x,t)\,dx\right) \tag{3.4}$$

and the symmetric 3×3 matrix field

$$R_\ell(x,t) = \rho_\ell(t)\mathrm{Id} - \mathring{R}_\ell(x,t) . \tag{3.5}$$

Finally, w_o is defined by

$$w_o(x,t) := \sqrt{\rho_\ell(t)}\sum_{j=1}^{8}\sum_{k\in\Lambda_j}\gamma_k^{(j)}\left(\frac{R_\ell(x,t)}{\rho_\ell(t)}\right)\phi_{k,\mu}^{(j)}\left(v_\ell(x,t),\lambda t\right)B_k e^{i\lambda k\cdot x} , \tag{3.6}$$

where $B_k \in \mathbb{C}^3$ are vectors of unit length satisfying the assumptions of [15, Proposition 3.1]. Recall that the maps $\gamma_k^{(j)}$ are defined only in $B_{r_0}(\mathrm{Id})$. The function w_o is nonetheless well defined: the fact that the arguments of $\gamma_k^{(j)}$ are contained in $B_{r_0}(\mathrm{Id})$ will be ensured by the choice of η in Section 3.3 below.

3.3 The constants η and M

We start by observing that, by standard estimates on convolutions

$$\|v_\ell\|_r + \|\mathring{R}_\ell\|_r \le C(r)D\ell^{-r} \quad \text{for any } r \ge 1, \tag{3.7}$$

$$\|v_\ell - v\|_0 + \|\mathring{R}_\ell - \mathring{R}\|_0 \le CD\ell \ , \tag{3.8}$$

where the first constant depends only on r and the second is universal. By writing $\big||v_\ell|^2 - |v|^2\big| \leq |v-v_\ell|^2 + 2|v||v-v_\ell|$ we deduce

$$\int_{\mathbb{T}^3} \big||v_\ell|^2 - |v|^2\big|\, dx \leq C(D\ell)^2 + Ce(t)^{1/2} D\ell \tag{3.9}$$

$$\leq C\eta\delta \left(\max_t e(t)^{1/2} + 1\right), \tag{3.10}$$

where the last inequality follows from (3.3). This leads to the following lower bound on ρ_ℓ:

$$\begin{aligned} \rho_\ell(t) &\geq \frac{1}{3(2\pi)^3}\left(e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |v|^2\, dx - \int_{\mathbb{T}^3} \big||v_\ell|^2 - |v|^2\big|\, dx\right) \\ &\overset{(3.10)}{\geq} \frac{1}{3(2\pi)^3}\left(\frac{\delta}{4}\min_t e(t) - C\eta\delta\left(\max_t e(t)^{1/2} + 1\right)\right) \end{aligned} \tag{3.11}$$

We then choose $0 < \eta < 1$ so that the quantity on the right hand side is greater than $\frac{2\eta\delta}{r_0}$. This is clearly possible with a choice of η only depending on e. In turn, this leads to

$$\left\|\frac{R_\ell}{\rho_\ell} - \mathrm{Id}\right\|_0 \leq \frac{\|\mathring{R}_\ell\|_0}{\min_t \rho_\ell(t)} \leq \frac{r_0}{2}. \tag{3.12}$$

Therefore w_o in (3.6) is well-defined.

In an analogous way we estimate ρ_ℓ from above as

$$\begin{aligned} \rho_\ell(t) &\leq \frac{1}{3(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3} |v|^2\, dx + \int_{\mathbb{T}^3} \big||v_\ell|^2 - |v|^2\big|\, dx\right) \\ &\leq \frac{1}{3(2\pi)^3}\left(\frac{5\delta}{4}\max_t e(t) + C\delta\left(\max_t e(t)^{1/2} + 1\right)\right) \\ &\leq C\delta\left(1 + \max_t e(t)\right). \end{aligned} \tag{3.13}$$

Since $|w_o|$ can be estimated as

$$|w_o(x,t)| \leq C\sqrt{\rho_\ell(t)}\,,$$

we can choose the constant M, depending only on e, in such a way that

$$\|w_o\|_0 \leq \frac{M}{2}\sqrt{\delta}\,. \tag{3.14}$$

This is essentially the major point in the definition of M: the remaining terms leading to (2.7) and (2.8) will be shown to be negligible thanks to an appropriate choice of the parameters λ, μ and ℓ. We will therefore require that, in addition to (3.14), $M \geq 1$

3.4 The pressure p_1

The pressure p_1 differs slightly from the corresponding one chosen in [15]. It is given by

$$p_1 = p - \frac{|w_o|^2}{2} - \frac{2}{3}\langle v - v_\ell, w\rangle\,. \tag{3.15}$$

Observe that, by (3.14), we have

$$\|p_1 - p\|_0 \le \frac{M^2}{4}\delta + \|v - v_\ell\|_0 \|w\|_0\,. \tag{3.16}$$

3.5 The Reynolds stress $\mathring{R}_1$

The Reynolds stress $\mathring{R}_1$ is defined by a slightly more complicated formula than the corresponding one in [15, Section 4.5]. Recalling the operator $\mathcal{R}$ from [15, Definition 4.2] we define $\mathring{R}_1$ as

$$\begin{aligned}\mathring{R}_1 &= \mathcal{R}[\partial_t w + \mathrm{div}(w \otimes v_\ell + v_\ell \otimes w)] \\ &\quad + \mathcal{R}[\mathrm{div}(w \otimes w + \mathring{R}_\ell - \tfrac{|w_o|^2}{2}\mathrm{Id})] \\ &\quad + [w \otimes (v - v_\ell) + (v - v_\ell) \otimes w - \tfrac{2\langle (v - v_\ell), w\rangle}{3}\mathrm{Id}] \\ &\quad + [\mathring{R}_\ell - \mathring{R}]\,.\end{aligned} \tag{3.17}$$

The summands in the third and fourth line are obviously trace-free and symmetric. The summands in the first and second line are symmetric and trace-free because of the properties of the operator $\mathcal{R}$ (cp. with [15, Lemma 4.3]). Moreover, the expressions to which the operator $\mathcal{R}$ is applied have average 0. For the second line this is obvious because the expression is the divergence of a matrix field. As for the first line, since $w = \mathcal{P}w_o$, its average is zero by the definition of the operator $\mathcal{P}$. Therefore the average of $\partial_t w$ is also zero. The remaining term is a divergence and hence its average equals 0.

We now check that the triple $(v_1, p_1, \mathring{R}_1)$ satisfies the Euler-Reynolds system. First of all, recall that $\nabla g = \mathrm{div}(g\mathrm{Id})$ for any smooth function g and that $\mathrm{div}\mathcal{R}F = F$ for any smooth F with average 0. Since we already observed that the expressions to which $\mathcal{R}$ is applied average to 0, we can compute

$$\mathrm{div}\,\mathring{R}_1 - \nabla p_1 = \partial_t w + \mathrm{div}(w \otimes w) + \mathrm{div}(w \otimes v + v \otimes w) - \nabla p + \mathrm{div}\,\mathring{R}\,.$$

But recalling that $\mathrm{div}\,\mathring{R} = \partial_t v + \mathrm{div}\,(v \otimes v) + \nabla p$ we also get

$$\mathrm{div}\,\mathring{R}_1 - \nabla p_1 = \partial_t(v + w) + \mathrm{div}\,[w \otimes w + v \otimes v + w \otimes v + v \otimes w]\,.$$

Since $v_1 = v + w$ we then conclude the desired identity.

In order to complete the proof of Proposition 2.2 we need to show that the (minor) estimates (2.7), (2.8) and the (major) estimates (2.5), (2.6), (2.9) hold: essentially all the rest of the paper is devoted to prove them.

3.6 Constants in the estimates

The rest of the paper is devoted to estimating several Hölder norms of the various functions defined so far. The constants appearing in the estimates will always be denoted by the letter C, which might be followed by an appropriate subscript. First of all, by this notation we will throughout understand that the value may change from line to line. In order to keep track of the quantities on which these constants depend, we will use subscripts to make the following distinctions.

- C: without a subscript will denote universal constants;

- C_h: will denote constants in estimates concerning standard functional inequalities in Hölder spaces C^r (such as (4.1), (4.2)). These constants depend only on the specific norm used and therefore only on the parameter $r \geq 0$: however we keep track of this dependence because the number r will be chosen only at the end of the proof of Proposition 2.2 and its value may be very large;

- C_e: throughout the rest paper the prescribed energy density $e = e(t)$ of Theorem 1.1 and Proposition 2.2 will be assumed to be a fixed smooth function bounded below and above by positive constants; several estimates depend on these bounds and the related constants will be denoted by C_e;

- C_v: in addition to the dependence on e, there will be estimates which depend also on the supremum norm of the velocity field $\|v\|_0$: such constants increase with $\|v\|_0$ (this explains the origin of the constant A in (2.9));

- C_s, $C_{e,s}$, $C_{v,s}$: will denote constants which are typically involved in Schauder estimates for $C^{m+\alpha}$ norms of elliptic operators, when $m \in \mathbb{N}$ and $\alpha \in]0,1[$; these constants not only depend on the specific norm used, but they also degenerate as $\alpha \downarrow 0$ and $\alpha \uparrow 1$; the ones denoted by $C_{e,s}$ and $C_{v,s}$ depend also, respectively, upon e and upon e and $\|v\|_0$.

Observe in any case that, no matter which subscript is used, such constants *never* depend on the parameters $\mu, \ell, \delta, \lambda$ and D; they are, however, allowed to depend on ω and ε.

4 Preliminary Hölder estimates

In this section we collect several estimates which will be used throughout the rest of the paper.

We start with the following elementary inequalities:

$$[f]_s \le C_h\big(\varepsilon^{r-s}[f]_r + \varepsilon^{-s}\|f\|_0\big) \tag{4.1}$$

for $r \ge s \ge 0$ and $\varepsilon > 0$, and

$$[fg]_r \le C_h\big([f]_r\|g\|_0 + \|f\|_0[g]_r\big) \tag{4.2}$$

for any $1 \ge r \ge 0$, where the constants depend only on r and s. From (4.1) with $\varepsilon = \|f\|_0^{1/r}[f]_r^{-1/r}$ we obtain the standard interpolation inequalities

$$[f]_s \le C_h\|f\|_0^{1-s/r}[f]_r^{s/r}. \tag{4.3}$$

Next we collect two classical estimates on the Hölder norms of compositions. These are also standard, for instance in applications of the Nash-Moser iteration technique. For the convenience of the reader we recall the short proof.

Proposition 4.1. *Let $\Psi : \Omega \to \mathbb{R}$ and $u : \mathbb{R}^n \to \Omega$ be two smooth functions, with $\Omega \subset \mathbb{R}^N$. Then, for every $m \in \mathbb{N}\setminus\{0\}$ there is a constant C_h (depending only on m, N and n) such that*

$$[\Psi\circ u]_m \le C_h\sum_{i=1}^m [\Psi]_i\|u\|_0^{i-1}[u]_m \tag{4.4}$$

$$[\Psi\circ u]_m \le C_h\sum_{i=1}^m [\Psi]_i[u]_1^{(i-1)\frac{m}{m-1}}[u]_m^{\frac{m-i}{m-1}}\,. \tag{4.5}$$

Proof. Denoting by D^j any partial derivative of order j, the chain rule can be written symbolically as

$$D^m(\Psi\circ u) = \sum_{l=1}^m (D^l\Psi)\circ u \sum_\sigma C_{l,\sigma}(Du)^{\sigma_1}(D^2u)^{\sigma_2}\dots(D^mu)^{\sigma_m} \tag{4.6}$$

for some constants $C_{l,\sigma}$, where the inner sum is over $\sigma = (\sigma_1,\dots,\sigma_m) \in \mathbb{N}^m$ such that

$$\sum_{j=1}^m \sigma_j = l, \quad \sum_{j=1}^m j\sigma_j = m.$$

From (4.3) we have

(a) $[u]_j \le C_h\|u\|_0^{1-\frac{j}{m}}[u]_m^{\frac{j}{m}}$ for $j \ge 0$;

(b) $[u]_j \le C_h[u]_1^{1-\frac{j-1}{m-1}}[u]_m^{\frac{j-1}{m-1}}$ for $j \ge 1$.

Then (4.4) and (4.5) follow from applying (a) and (b) to (4.6), respectively. □

4.1 Estimates on $\phi_{k,\mu}^{(j)}$

Recall that $\phi_{k,\mu}^{(j)} = \phi_{k,\mu}^{(j)}(v,\tau)$ are defined on $\mathbb{R}^3 \times \mathbb{S}^1$ and they are smooth (here v is treated as an independent variable). Because the τ-derivatives are not bounded in v, we introduce the seminorms

$$[f]_{m,R} = \max_{|\beta|=m} \|D_v^\beta f\|_{C^0(B_R(0)\times\mathbb{S}^1)}$$

and

$$[f]_{m+\alpha,R} = \max_{|\beta|=m} \sup_{v\neq w\in B_R(0),\tau\in\mathbb{S}^1} \frac{|D_v^\beta f(v,\tau) - D_v^\beta f(w,\tau)|}{|v-w|^\alpha}\,,$$

where D_v^β denotes partial derivatives in the v variable with multiindex $\beta = (\beta_1,\beta_2,\beta_3)$.

Proposition 4.2. *There are constants C_h depending only on $m \in \mathbb{N}$ and such that the following estimates hold:*

$$\left[\phi_{k,\mu}^{(j)}\right]_{m,R} + R^{-1}\left[\partial_\tau \phi_{k,\mu}^{(j)}\right]_{m,R} + R^{-2}\left[\partial_{\tau\tau}\phi_{k,\mu}^{(j)}\right]_{m,R} \le C_h\mu^m \tag{4.7}$$

$$\left[\partial_\tau\phi_{k,\mu}^{(j)} + i(k\cdot v)\phi_{k,\mu}^{(j)}\right]_m \le C_h\mu^{m-1} \tag{4.8}$$

$$R^{-1}\left[\partial_\tau\left(\partial_\tau\phi_{k,\mu}^{(j)} + i(k\cdot v)\phi_{k,\mu}^{(j)}\right)\right]_{m,R} \le C_h\mu^{m-1} \tag{4.9}$$

Proof. We recall briefly the definition of the maps $\phi_{k,\mu}^{(j)}$ from [15, Section 4.1]. First of all we fix two constants c_1 and c_2 such that $\frac{\sqrt{3}}{2} < c_1 < c_2 < 1$ and then $\varphi \in C_c^\infty(B_{c_2}(0))$ which is nonnegative and identically 1 on the ball $B_{c_1}(0)$. We then set

$$\psi(v) := \sum_{k\in\mathbb{Z}^3} (\varphi(v-k))^2 \qquad \text{and} \qquad \alpha_k(v) := \frac{\varphi(v-k)}{\sqrt{\psi(v)}}\,.$$

By the choice of c_1 we easily conclude that $\psi^{-\frac{1}{2}} \in C^\infty$. On the other hand it is also obvious that $\psi(v-k) = \psi(v)$. Thus there is a function $\alpha \in C_c^\infty(B_1(0))$ such that $\alpha_k(v) = \alpha(v-k)$.

We next consider the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ and its quotient by $(2\mathbb{Z})^3$ and we denote by $\mathcal{C}_j$, $j = 1,\dots,8$ the 8 equivalence classes of $\mathbb{Z}^3/\sim$. Finally, in [15, Section 4.1] we set

$$\phi_k^{(j)}(v,\tau) := \sum_{l\in\mathcal{C}_j} \alpha_l(\mu v) e^{-i(k\cdot\frac{l}{\mu})\tau}. \tag{4.10}$$

Observe that, for each fixed j, the functions $\{\alpha_l : l \in \mathcal{C}_j\}$ have pairwise disjoint supports. Therefore the estimate

$$\left[\phi_{k,\mu}^{(j)}\right]_m \le C[\alpha]_m\mu^m \le C_h\mu^m$$

follows trivially. Next,

$$\partial_\tau\phi_k^{(j)}(v,\tau) := \sum_{l\in\mathcal{C}_j} -i\left(k\cdot\frac{l}{\mu}\right)\alpha_l(\mu v)e^{-i(k\cdot\frac{l}{\mu})\tau}\,.$$

On the other hand, if $|v| \le R$, then $\alpha_l(\mu v) = 0$ for any l with $|l| \ge \mu R + 2$: hence

$$\left[\partial_\tau \phi_k^{(j)}\right]_{m,R} \le |k| \left(R + 2\mu^{-1}\right) [\varphi]_m \mu^m \le C_h R\, \mu^m$$

(in principle the constant C_h depends on k, but on the other hand k ranges in $\cup_j \Lambda_j$, which is a finite set). A similar argument applies to $\partial_{\tau\tau}\phi_{k,\mu}^{(j)}$ and hence concludes the proof of (4.7).

We finally compute

$$\begin{aligned} D_v^m \left(\partial_\tau \phi_{k,\mu}^{(j)} + i(k\cdot v)\phi_{k,\mu}^{(j)}\right) &= \sum_{l\in\mathcal{C}_j} ik\cdot\left(v - \frac{l}{\mu}\right)\mu^m [D^m\alpha](\mu(v-l)) e^{-i(k\cdot\frac{l}{\mu})\tau} \\ &\quad + \mu^{m-1}\sum_{l\in\mathcal{C}_j} ik\otimes [D^{m-1}\alpha](\mu(v-l))e^{-i(k\cdot\frac{l}{\mu})\tau}\,. \end{aligned}$$

Recall however that $\alpha \in C_c^\infty(B_1(0))$: thus $|v - \frac{l}{\mu}| \le \mu^{-1}$ if $[D^m\alpha](\mu(v-l)) \neq 0$. It follows easily that

$$\left[\partial_\tau \phi_{k,\mu}^{(j)} + i(k\cdot v)\phi_{k,\mu}^{(j)}\right]_m \le C\mu^{m-1}\left([\alpha]_m + |k|[\alpha]_{m-1}\right) \le C_h \mu^{m-1}\,,$$

which proves (4.8). On the other hand, differentiating once more the identities in τ, (4.9) follows from the same arguments used above for $[\partial_\tau\phi]_{m,R}$. □

4.2 Schauder estimates for elliptic operators

We now recall some classical Schauder estimates for the various operators involved in the construction. These estimates were already collected in [15, Proposition 5.1] and will be used several times in what follows. We state them again for the readers convenience and because of the convention on constants as set in Section 3.3, and refer to [15, Definitions 4.1, 4.2] for the precise definition of the operators $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$.

Proposition 4.3. *For any $\alpha \in (0,1)$ and any $m \in \mathbb{N}$ there exists a constant $C_s(m,\alpha)$ so that the following estimates hold:*

$$\|\mathcal{Q}v\|_{m+\alpha} \le C_s(m,\alpha)\|v\|_{m+\alpha} \tag{4.11}$$

$$\|\mathcal{P}v\|_{m+\alpha} \le C_s(m,\alpha)\|v\|_{m+\alpha} \tag{4.12}$$

$$\|\mathcal{R}v\|_{m+1+\alpha} \le C_s(m,\alpha)\|v\|_{m+\alpha} \tag{4.13}$$

$$\|\mathcal{R}(\operatorname{div} A)\|_{m+\alpha} \le C_s(m,\alpha)\|A\|_{m+\alpha} \tag{4.14}$$

$$\|\mathcal{R}\mathcal{Q}(\operatorname{div} A)\|_{m+\alpha} \le C_s(m,\alpha)\|A\|_{m+\alpha}\,. \tag{4.15}$$

4.3 Stationary phase lemma

Finally, we state a key ingredient of our construction, which yields estimates for highly oscillatory functions. Though this proposition is also essentially contained in [15], it is nowhere explicitly stated in this form. Since it will be used several more times and in a more subtle way than in [15], it is useful to isolate it from the rest.

Proposition 4.4. *Let $k \in \mathbb{Z}^3 \setminus \{0\}$ and $\lambda \geq 1$.*

(i) For any $a \in C^\infty(\mathbb{T}^3)$ and $m \in \mathbb{N}$ we have

$$\left|\int_{\mathbb{T}^3} a(x) e^{i\lambda k\cdot x}\, dx\right| \leq \frac{[a]_m}{\lambda^m}. \tag{4.16}$$

(ii) Let $k \in \mathbb{Z}^3 \setminus \{0\}$. For a smooth vector field $F \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ let $F_\lambda(x) := F(x)e^{i\lambda k\cdot x}$. Then we have

$$\begin{aligned}\|\mathcal{R}(F_\lambda)\|_\alpha &\leq \frac{C_s}{\lambda^{1-\alpha}}\|F\|_0 + \frac{C_s}{\lambda^{m-\alpha}}[F]_m + \frac{C_s}{\lambda^m}[F]_{m+\alpha},\\ \|\mathcal{R}\mathcal{Q}(F_\lambda)\|_\alpha &\leq \frac{C_s}{\lambda^{1-\alpha}}\|F\|_0 + \frac{C_s}{\lambda^{m-\alpha}}[F]_m + \frac{C_s}{\lambda^m}[F]_{m+\alpha},\end{aligned}$$

where $C_s = C_s(m,\alpha)$ (i.e. the constant does not depend on λ nor on k).

Proof. For $j = 0, 1, \ldots$ define

$$\begin{aligned}A_j(y,\xi) &:= -i\left[\frac{k}{|k|^2}\left(i\frac{k}{|k|^2}\cdot\nabla\right)^j a(y)\right]e^{ik\cdot\xi}\,,\\ B_j(y,\xi) &:= \left[\left(i\frac{k}{|k|^2}\cdot\nabla\right)^j a(y)\right]e^{ik\cdot\xi}\,.\end{aligned}$$

Direct calculation shows that

$$B_j(x,\lambda x) = \frac{1}{\lambda}\mathrm{div}\left[A_j(x,\lambda x)\right] + \frac{1}{\lambda}B_{j+1}(x,\lambda x).$$

In particular, for any $m \in \mathbb{N}$

$$a(x)e^{i\lambda k\cdot x} = B_0(x,\lambda x) = \frac{1}{\lambda}\sum_{j=0}^{m-1}\frac{1}{\lambda^j}\mathrm{div}\left[A_j(x,\lambda x)\right] + \frac{1}{\lambda^m}B_m(x,\lambda x)$$

Integrating this over $\mathbb{T}^3$ and using that $|k| \geq 1$ we obtain (4.16).

Next, using (4.1) and (4.2) we conclude

$$\begin{aligned}\|A_j(\cdot,\lambda\cdot)\|_\alpha &\leq C\left(\lambda^\alpha[a]_j + [a]_{j+\alpha}\right)\\ &\leq C\lambda^{j+\alpha}\left(\lambda^{-m}[a]_m + \|a\|_0\right) \qquad \text{for any } j \leq m-1\end{aligned}$$

and similarly

$$\|B_m(\cdot,\lambda\cdot)\|_\alpha \le C\left(\lambda^\alpha [a]_m + [a]_{m+\alpha}\right) .$$

Applying the previous computations to each component of the vector field F we then get the identity

$$F(x)e^{i\lambda k\cdot x} = G_0(x,\lambda x) = \frac{1}{\lambda}\sum_{j=0}^{m-1}\frac{1}{\lambda^j}\mathrm{div}\left[H_j(x,\lambda x)\right] + \frac{1}{\lambda^m}G_m(x,\lambda x)$$

where the H_j are matrix-valued functions (not necessarily symmetric) and G_m is a vector field. H_j and G_m enjoy the same estimates of A_j and B_m respectively. Thus, using (4.13), (4.14) and (4.16) we conclude

$$\begin{aligned}\|\mathcal{R}(F_\lambda)\|_\alpha &\le C_s\left(\frac{1}{\lambda}\sum_{j=0}^{m-1}\frac{1}{\lambda^j}\|H_j(\cdot,\lambda\cdot)\|_\alpha + \frac{1}{\lambda^m}\|G_m(\cdot,\lambda\cdot)\|_\alpha\right)\\ &\le C_s\left(\frac{1}{\lambda^{1-\alpha}}\|F\|_0 + \frac{1}{\lambda^{m-\alpha}}[F]_m + \frac{1}{\lambda^m}[F]_{m+\alpha}\right) .\end{aligned}$$

Finally, using (4.11), (4.13) and (4.15) we get

$$\|\mathcal{R}\mathcal{Q}(F_\lambda)\|_\alpha \le C_s\left(\frac{1}{\lambda^{1-\alpha}}\|F\|_0 + \frac{1}{\lambda^{m-\alpha}}[F]_m + \frac{1}{\lambda^m}[F]_{m+\alpha}\right)$$

as well. □

5 Doubling the variables and corresponding estimates

It will be convenient to write w_o as

$$w_o(x,t) = W(x,t,\lambda t,\lambda x),$$

where

$$\begin{aligned}W(y,s,\tau,\xi) &:= \sum_{|k|=\lambda_0} a_k(y,s,\tau)B_k e^{ik\cdot\xi} && (5.1)\\ &= \sqrt{\rho_\ell(s)}\sum_{j=1}^{8}\sum_{k\in\Lambda_j}\gamma_k^{(j)}\left(\frac{R_\ell(y,s)}{\rho_\ell(s)}\right)\phi^{(j)}_{k,\mu}\left(v_\ell(y,s),\tau\right)B_k e^{ik\cdot\xi} && (5.2)\end{aligned}$$

(cp. with [15, Section 6]). The following Proposition corresponds to [15, Proposition 6.1], with an important difference: the estimates stated here keep track of not only the dependence of the constants on the parameter μ, but also on the parameter ℓ and the functions v and $\mathring{R}$ (as it can be easily observed, these estimates do not depend on p): more precisely we will make explicit their dependence on δ and D (for the constants recall the convention stated in Section 3.3). Observe that all the estimates claimed below are in *space only*!

Proposition 5.1. *(i) Let* $a_k \in C^\infty(\mathbb{T}^3 \times \mathbb{S}^1 \times \mathbb{R})$ *be given by* (5.1). *Then for any* $r \geq 1$ *and any* $\alpha \in [0,1]$ *we have the following estimates:*

$$\begin{aligned} \|a_k(\cdot, s, \tau)\|_r &\leq C_e \sqrt{\delta} \left(\mu^r D^r + \mu D \ell^{1-r}\right) && (5.3)\\ \|\partial_\tau a_k(\cdot, s, \tau)\|_r + \|\partial_{\tau\tau} a_k(\cdot, s, \tau)\|_r &\leq C_v \sqrt{\delta} \left(\mu^r D^r + \mu D \ell^{1-r}\right) && (5.4)\\ \|(\partial_\tau a_k + i(k \cdot v_\ell) a_k)(\cdot, s, \tau)\|_r &\leq C_e \sqrt{\delta} \left(\mu^{r-1} D^r + D \ell^{1-r}\right) && (5.5)\\ \|\partial_\tau(\partial_\tau a_k + i(k \cdot v_\ell) a_k)(\cdot, s, \tau)\|_r &\leq C_v \sqrt{\delta} \left(\mu^{r-1} D^r + D \ell^{1-r}\right) && (5.6) \end{aligned}$$

$$\begin{aligned} \|a_k(\cdot, s, \tau)\|_\alpha &\leq C_e \sqrt{\delta} \mu^\alpha D^\alpha && (5.7)\\ \|\partial_\tau a_k(\cdot, s, \tau)\|_\alpha + \|\partial_{\tau\tau} a_k(\cdot, s, \tau)\|_\alpha &\leq C_v \sqrt{\delta} \mu^\alpha D^\alpha && (5.8)\\ \|(\partial_\tau a_k + i(k \cdot v_\ell) a_k)(\cdot, s, \tau)\|_\alpha &\leq C_e \sqrt{\delta} \mu^{\alpha-1} D^\alpha && (5.9)\\ \|\partial_\tau(\partial_\tau a_k + i(k \cdot v_\ell) a_k)(\cdot, s, \tau)\|_\alpha &\leq C_v \sqrt{\delta} \mu^{\alpha-1} D^\alpha && (5.10) \end{aligned}$$

The following estimates hold for any $r \geq 0$*:*

$$\begin{aligned} \|\partial_s a_k(\cdot, s, \tau)\|_r &\leq C_e \sqrt{\delta} \left(\mu^{r+1} D^{r+1} + \mu D \ell^{-r}\right) && (5.11)\\ \|\partial_{s\tau} a_k(\cdot, s, \tau)\|_r &\leq C_v \sqrt{\delta} \left(\mu^{r+1} D^{r+1} + \mu D \ell^{-r}\right) && (5.12)\\ \|\partial_{ss} a_k(\cdot, s, \tau)\|_r &\leq C_e \sqrt{\delta} \left(\mu^{r+2} D^{r+2} + \mu D \ell^{-1-r}\right) && (5.13)\\ \|\partial_s(\partial_\tau a_k + i(k \cdot v_\ell) a_k)(\cdot, s, \tau)\|_r &\leq C_v \sqrt{\delta} \left(\mu^r D^{r+1} + \mu D \ell^{-r}\right) && (5.14) \end{aligned}$$

(ii) The matrix-function $W \otimes W$ *can be written as*

$$(W \otimes W)(y, s, \tau, \xi) = R_\ell(y, s) + \sum_{1 \leq |k| \leq 2\lambda_0} U_k(y, s, \tau) e^{ik\cdot\xi}, \qquad (5.15)$$

where the coefficients $U_k \in C^\infty(\mathbb{T}^3 \times \mathbb{S}^1 \times \mathbb{R}; \mathcal{S}^{3\times 3})$ *satisfy*

$$U_k k = \frac{1}{2}(\operatorname{tr} U_k) k\,. \qquad (5.16)$$

Moreover, we have the following estimates for any $r \geq 1$ *and any* $\alpha \in [0,1]$*:*

$$\begin{aligned} \|U_k(\cdot, s, \tau)\|_r &\leq C_e \delta \left(\mu^r D^r + \mu D \ell^{1-r}\right) && (5.17)\\ \|\partial_\tau U_k(\cdot, s, \tau)\|_r &\leq C_v \delta \left(\mu^r D^r + \mu D \ell^{1-r}\right) && (5.18)\\ \|U_k(\cdot, s, \tau)\|_\alpha &\leq C_e \delta \mu^\alpha D^\alpha && (5.19)\\ \|\partial_\tau U_k(\cdot, s, \tau)\|_\alpha &\leq C_v \delta \mu^\alpha D^\alpha && (5.20) \end{aligned}$$

and the following estimate for any $r \geq 0$*:*

$$\|\partial_s U_k(\cdot, s, \tau)\|_r \leq C_e \delta \left(\mu^{r+1} D^{r+1} + \mu D \ell^{-r}\right)\,. \qquad (5.21)$$

Proof. The arguments for (5.15) and (5.16) are analogous to those in the proof of [15, Proposition 6.1]. Moreover, precisely as argued there, the estimates for the U_k terms follow easily from the estimates for the a_k coefficients, since each U_k is the sum of finitely many terms of the form $a_{k'}a_{k''}$. Here we focus, therefore, on the estimates (5.3)-(5.14).

First of all observe that it suffices to prove the cases $r \in \mathbb{N}$, since the remaining ones can be obtained by interpolation. Recall now the formula for a_k: if $k \in \bigcup_j \Lambda_j$, then

$$a_k = \sqrt{\rho_\ell(s)}\gamma_k^{(j)}\left(\frac{R_\ell(y,s)}{\rho_\ell(s)}\right)\phi_{k,\mu}^{(j)}\left(v_\ell(y,s),\tau\right) , \tag{5.22}$$

otherwise a_k vanishes identically.

Observe that the functions a_k depend on the variables y, s and τ. We introduce the notation $[\![\cdot]\!]_m$ for the Hölder seminorms in y *and* s

$$[\![a_k(\cdot,\cdot,\tau)]\!]_m = \sum_{j+|\beta|=m}\left\|\partial_s^j D_y^\beta a_k\right\|_0$$

and the notation $|||a_k(\cdot,\cdot,\tau)|||_m$ for the Hölder norm in y *and* s:

$$|||a_k(\cdot,\cdot,\tau)|||_m = \sum_{i=0}^m [\![a_k(\cdot,\cdot,\tau)]\!]_i \,.$$

We next introduce the functions

$$\Gamma(y,s) = \gamma_k^{(j)}\left(\frac{R_\ell(y,s)}{\rho_\ell(s)}\right) \qquad \text{and} \qquad \Phi(y,s,\tau) = \phi_{k,\mu}^{(j)}\left(v_\ell(y,s),\tau\right)$$

and observe that

$$a_k = \sqrt{\rho_\ell}\,\Gamma\,\Phi\,.$$

Recall that $\|\rho_l\|_0 \le C_e\delta$ by (3.13). Therefore the claimed estimate for $r = \alpha = 0$ follows trivially. Thus, we assume $r \in \mathbb{N}\setminus\{0\}$ and we focus on the estimates (5.3)-(5.6) and (5.11)-(5.14).

Proof of the estimates (5.3)**,** (5.11) **and** (5.13)**.** Recalling (4.2), we estimate

$$\begin{aligned}|||a_k|||_r &\le C_h\|\sqrt{\rho_\ell}\|_0\|\Gamma\|_0[\![\Phi]\!]_r + C_h\|\sqrt{\rho_\ell}\|_0\|\Phi\|_0[\![\Gamma]\!]_r + C_h\|\Phi\|_0\|\Gamma\|_0[\![\sqrt{\rho_\ell}]\!]_r\\ &\le C_e\left(\sqrt{\delta}\left([\![\Phi]\!]_r + [\![\Gamma]\!]_r\right) + [\![\sqrt{\rho_\ell}]\!]_r\right)\,.\end{aligned} \tag{5.23}$$

Next, by (3.7), for any $j \ge 1$ we have $[v_\ell]_j \le C_h D\ell^{1-j}$ for every $j \ge 1$. Applying (4.5) in Proposition 4.1 and Proposition 4.2 we conclude

$$\begin{aligned}[\![\Phi]\!]_r &\le C_h\sum_{i=1}^r\left[\phi_{k,\mu}^{(j)}\right]_i [v_\ell]_1^{(i-1)\frac{r}{r-1}}[v_\ell]_r^{\frac{r-i}{r-1}} \le C_h\sum_{i=1}^r\left[\phi_{k,\mu}^{(j)}\right]_i D^i\ell^{i-r}\\ &\stackrel{(4.7)}{\le} C_h\sum_{i=1}^r C_h\mu^i D^i\ell^{-r+i} \le C_h\left(\mu^r D^r + \mu D\ell^{1-r}\right)\,.\end{aligned} \tag{5.24}$$

Applying (4.4) of Proposition 4.1 we also conclude

$$[\![\Gamma]\!]_r \le C_h \sum_{i=1}^{r} \left[\gamma_k^{(j)}\right]_i \left\|\frac{R_\ell}{\rho_\ell}\right\|_0^{i-1} \left[\!\!\left[\frac{R_\ell}{\rho_\ell}\right]\!\!\right]_r \tag{5.25}$$

Now, by (3.12) we have

$$\left\|\frac{R_\ell}{\rho_\ell}\right\|_0 \le \frac{r_0}{2} + 1\,.$$

Moreover $[\gamma_k^{(j)}]_r \le C_h$: indeed recall that, because of our choice of η in Section 3.3, the range of $\frac{R_\ell}{\rho_\ell}$ is contained in $B_{\frac{r_0}{2}}(\mathrm{Id})$, whereas the $\gamma_k^{(j)}$ are defined on the open ball $B_{r_0}(\mathrm{Id})$; since the $\gamma_k^{(j)}$ are smooth and finitely many, obviously we can bound their norms uniformly on the range of the function $\frac{R_\ell}{\rho_\ell}$.

Using these estimates in (5.25) we thus get

$$[\![\Gamma]\!]_r \le C_h \left[\!\!\left[\frac{R_\ell}{\rho_\ell}\right]\!\!\right]_r \stackrel{(4.2)}{\le} \|\rho_\ell^{-1}\|_0 [\![R_\ell]\!]_r + \|R_\ell\|_0 [\![\rho_\ell^{-1}]\!]_r\,. \tag{5.26}$$

Recall next that, by (3.11), $\rho_\ell(s) \ge C_e\delta$ for every s. Moreover, by (3.4), for $r \ge 1$ we have

$$\partial_s^r \rho_\ell(s) = \frac{1}{3(2\pi)^3}\left((1-\bar\delta)\partial_s^r e(s) - \sum_{j=0}^{r} \binom{r}{j} \int_{\mathbb{T}^3} \left(\partial_s^j v_\ell \cdot \partial_s^{r-j} v_\ell\right)(x,s)\,dx\right)\,.$$

Thus, we conclude

$$\begin{aligned}
[\rho_\ell]_r &\le C_e + C\|v_\ell\|_{C_t^0 L_x^2}[v_\ell]_r + C_h \sum_{j=1}^{r-1} [v_\ell]_j [v_\ell]_{r-j} \\
&\le C_e + C_e [v_\ell]_r + C_h \sum_{j=1}^{r-1} [v_\ell]_j [v_\ell]_{r-j} \\
&\stackrel{(3.7)}{\le} C_e D \ell^{1-r} + C_h D^2 \ell^{r-2} \stackrel{(3.3)}{\le} C_e D \ell^{1-r}\,.
\end{aligned} \tag{5.27}$$

Set $\Psi(\zeta) = \zeta^{-1}$. On the domain $[\delta, \infty[$, we have the estimate $[\Psi]_i \le C_h \delta^{-i-1}$. Therefore, applying again (4.4) we conclude

$$[\![\rho_\ell^{-1}]\!]_r \le C_h \sum_{i=1}^{r} \delta^{-i-1} \|\rho_\ell\|_0^{i-1} [\rho_\ell]_r \le C_h \delta^{-2} [\rho_\ell]_r \le C_e \delta^{-2} D \ell^{r-1}\,. \tag{5.28}$$

It follows from (5.26), (5.28) and (3.7) that

$$[\![\Gamma]\!]_r \le C_e \delta^{-1} D \ell^{r-1}\,. \tag{5.29}$$

Next, set $\Psi(\zeta) = \zeta^{\frac12}$. In this case, on the domain $[\delta, C_e\delta[$ we have the estimates $[\Psi]_i \le C_e \delta^{\frac12 - i}$. Thus, by (4.4) and (5.27):

$$[\![\sqrt{\rho_\ell}]\!]_r \le C_h \sum_{i=1}^{r} C_e \delta^{\frac12 - i} \|\rho_\ell\|_0^{i-1} [\rho_\ell]_r \le C_e \delta^{-\frac12} D \ell^{1-r}\,. \tag{5.30}$$

Inserting (5.24), (5.29) and (5.30) into (5.23) we conclude

$$|||a_k|||_r \leq C_e \delta^{-\frac{1}{2}} D\ell^{1-r} + C_e \delta^{\frac{1}{2}} \mu^r D^r + C_e \delta^{\frac{1}{2}} \mu D\ell^{1-r} .$$

Recall, however, that $\mu \geq \delta^{-1}$ and hence

$$|||a_k|||_r \leq C_e \sqrt{\delta} \left(\mu^r D^r + \mu D\ell^{1-r}\right) .$$

From this we derive the claimed estimates for $\|a_k\|_r$ for any $r \geq 1$ and for $\|\partial_s a_k\|_r$ and $\|\partial_{ss} a_k\|_r$ for any $r \geq 0$.

Proof of the estimates (5.4) **and** (5.12)**.** Differentiating in τ we obtain the identities

$$\partial_\tau a_k(\cdot,\cdot,\tau) = \sqrt{\rho_\ell}\, \Gamma\, \partial_\tau \phi^{(j)}_{k,\mu}(v_\ell,\tau)$$
$$\partial_{\tau\tau} a_k(\cdot,\cdot,\tau) = \sqrt{\rho_\ell}\, \Gamma\, \partial_{\tau\tau} \phi^{(j)}_{k,\mu}(v_\ell,\tau) .$$

Thus, arguing precisely as above, we achieve the desired estimates for the quantities $\|\partial_\tau a_k\|_r$, $\|\partial_{\tau s} a_k\|_r$ and $\|\partial_{\tau\tau} a_k\|_r$. However, note that we use the estimate (4.7) with $R := \|v\|_0$ and for $[\partial_t \phi^{(j)}_{k,\mu}]_{m,R}$ and $[\partial_{\tau\tau} \phi^{(j)}_{k,\mu}]_{m,R}$. It turns out, therefore, that the constants in the estimates (5.4) and (5.12) depend also on $\|v\|_0$.

Proof of the estimates (5.5)**,** (5.6) **and** (5.14)**.** Finally, we introduce the function

$$\chi^{(j)}_{k,\mu}(v,\tau) := \partial_\tau \phi^{(j)}_{k,\mu} + i(k\cdot v)\phi^{(j)}_{k,\mu}$$

and $\chi(y,s,\tau) = \chi^{(j)}_{k,\mu}(v_\ell(y,s),\tau)$. Then

$$\partial_\tau a_k + i(k\cdot v_\ell) a_k = \sqrt{\rho_\ell}\chi\Gamma .$$

Applying the same computations as above and using the estimates in Proposition 4.2 we achieve the desired estimates for $\|\partial_\tau a_k + i(k\cdot v_\ell)a_k\|_r$ and $\|\partial_s(\partial_\tau a_k + i(k\cdot v_\ell)a_k)\|_r$. Finally,

$$\partial_\tau(\partial_\tau a_k + i(k\cdot v_\ell)a_k) = \sqrt{\rho_\ell}\, \Gamma \left[\partial_\tau \chi^{(j)}_{k,\mu}\right](v_\ell,\tau)$$

and hence the arguments above carry over to estimate also the quantity $\|\partial_\tau(\partial_\tau a_k + i(k\cdot v_\ell)a_k)\|_r$. □

6 Estimates on w_o, w_c and v_1

Proposition 6.1. *Under assumption* (3.3)*, the following estimates hold for any* $r \geq 0$

$$\|w_o\|_r \leq C_e \sqrt{\delta}\lambda^r , \tag{6.1}$$
$$\|\partial_t w_o\|_r \leq C_v \sqrt{\delta}\lambda^{r+1} \tag{6.2}$$

and the following for any $r > 0$*:*

$$\|w_c\|_r \leq C_e\sqrt{\delta} D\mu\, \lambda^{r-1} \tag{6.3}$$

$$\|\partial_t w_c\|_r \leq C_v\sqrt{\delta} D\mu\lambda^r \,. \tag{6.4}$$

In particular

$$\|w\|_0 \leq C_e\sqrt{\delta}\,, \tag{6.5}$$

$$\|w\|_{C^1} \leq C_v\sqrt{\delta}\lambda\,. \tag{6.6}$$

Proof. First of all observe that it suffices to prove (6.1) when $r = m \in \mathbb{N}$, since the remaining inequalities can be obtained by interpolation. By writing

$$\begin{aligned}
w_o(x,t) &= \sum_{|k|=\lambda_0} a_k(x,t,\lambda t) B_k e^{i\lambda k\cdot x} =: \sum_{|k|=\lambda_0} a_k(x,t,\lambda t)\Omega_k(\lambda x),\\
\partial_t w_o(x,t) &= \lambda \sum_{|k|=\lambda_0} \partial_\tau a_k(x,t,\lambda t)\Omega_k(\lambda x) + \sum_{|k|=\lambda_0} \partial_s a_k(x,t,\lambda t)\Omega_k(\lambda x),
\end{aligned}$$

from (4.2) we obtain

$$\begin{aligned}
\|w_o\|_m &\leq C_h \sum_{|k|=\lambda_0} \left(\|\Omega_k\|_0 [a_k]_m + \lambda^m \|a_k\|_0 [\Omega_k]_m\right),\\
\|\partial_t w_o\|_m &\leq C_h\lambda \sum_{|k|=\lambda_0} \left(\|\Omega_k\|_0 [\partial_\tau a_k]_m + \lambda^m \|\partial_\tau a_k\|_0 [\Omega_k]_m\right)\\
&\quad + C_h \sum_{|k|=\lambda_0} \left(\|\Omega_k\|_0 [\partial_s a_k]_m + \lambda^m \|\partial_s a_k\|_0 [\Omega_k]_m\right).
\end{aligned}$$

When $m = 0$, we then use (5.7) to conclude (6.1) and (5.8) and (5.11) to conclude (6.2). For $m \geq 1$ we use, respectively, (5.3) and the estimates (5.4) and (5.11) to get:

$$\begin{aligned}
\|w_o\|_m &\leq C_e\sqrt{\delta}\left(\mu^m D^m + \mu D\ell^{1-m} + \lambda^m\right)\\
\|\partial_t w_o\|_m &\leq C_v\sqrt{\delta}\Big(\lambda\mu^m D^m + \lambda\mu D\ell^{1-m} + \lambda^{m+1}\\
&\qquad + \mu^{m+1} D^{m+1} + \mu D\ell^{-m} + \lambda^m \mu D\Big)
\end{aligned}$$

However, recall from (3.3) that $\lambda \geq (D\mu)^{1+\omega} \geq D\mu$ and $\lambda \geq \ell^{-1}$. Thus (6.1) and (6.2) follow easily.

As for the estimates on w_c we argue as in [15, Lemma 6.2] and start with the observation that, since $k\cdot B_k = 0$,

$$\begin{aligned}
w_o(x,t) =& \frac{1}{\lambda}\nabla\times\left(\sum_{|k|=\lambda_0} -i a_k(x,t,\lambda t)\frac{k\times B_k}{|k|^2} e^{i\lambda x\cdot k}\right) +\\
&+ \frac{1}{\lambda}\sum_{|k|=\lambda_0} i\nabla a_k(x,t,\lambda t)\times\frac{k\times B_k}{|k|^2} e^{i\lambda x\cdot k}.
\end{aligned}$$

Hence

$$w_c(x,t) = \frac{1}{\lambda}\mathcal{Q}u_c(x,t), \tag{6.7}$$

where

$$u_c(x,t) = \sum_{|k|=\lambda_0} i\nabla a_k(x,t,\lambda t) \times \frac{k\times B_k}{|k|^2}e^{i\lambda x\cdot k}. \tag{6.8}$$

The Schauder estimate (4.11) gives then

$$\|w_c\|_{m+\alpha} \le \frac{C_s}{\lambda}\|u_c\|_{m+\alpha} \tag{6.9}$$

for any $m\in\mathbb{N}$ and $\alpha\in(0,1)$. We next wish to estimate $\|u_c\|_r$. For integer m we can argue as for the estimate of $\|w_o\|$ to get

$$\begin{aligned}\|u_c\|_m &\le C_e\left([a_k]_1\lambda^m + [a_k]_{m+1}\right) \le C_e\sqrt{\delta}\left(\mu D\lambda^m + \mu D\ell^{-m}\right)\\ &\le C_e\sqrt{\delta}\mu D\lambda^m\,.\end{aligned}$$

Hence, by interpolation, we reach the estimate $\|u_c\|_{m+\alpha}\le C_e\sqrt{\delta}\mu D\lambda^{m+\alpha}$ for any m,α. Combining this with (6.9), for $r>0$ which is not an integer we conclude $\|w_c\|_r \le C_{e,s}\sqrt{\delta}\mu D\lambda^{r-1}$. On the other hand the corresponding estimates for any integer $r>0$ can then be reached by interpolation.

Similarly, for $\partial_t w_c$ we have

$$\partial_t w_c = \frac{1}{\lambda}\mathcal{Q}\partial_t u_c\,.$$

Differentiating (6.8) we achieve

$$\begin{aligned}\partial_t u_c(x,t) = \lambda&\sum_{|k|=\lambda_0} i\nabla\partial_\tau a_k(x,t,\lambda t)\times\frac{k\times B_k}{|k|^2}e^{i\lambda x\cdot k}\\ &+\sum_{|k|=\lambda_0} i\nabla\partial_s a_k(x,t,\lambda t)\times\frac{k\times B_k}{|k|^2}e^{i\lambda x\cdot k}\,.\end{aligned}$$

Using Proposition 5.1 and (3.3) we deduce, analogously to above

$$\|\partial_t u_c\|_r \le C_v\sqrt{\delta}\mu D\lambda^{r+1}\,.$$

Using (6.9) once more we arrive at (6.3).

To obtain (6.5) and (6.6), recall that $w = w_o + w_c$. For any $\alpha>0$ we therefore have

$$\|w\|_0 \le \|w_o\|_0 + \|w_c\|_\alpha \le C_e\sqrt{\delta} + C_{e,s}\sqrt{\delta}D\mu\lambda^{\alpha-1}. \tag{6.10}$$

We now use (6.10) with $\alpha = \frac{\omega}{1+\omega}$: since by (3.3) we have $\lambda^{1-\alpha} = \lambda^{\frac{1}{1+\omega}} \ge D\mu$, (6.5) follows. In the same way

$$\begin{aligned}\|w\|_{C^1} &\le \|w_o\|_1 + \|\partial_t w_o\|_0 + \|w_c\|_{1+\alpha} + \|\partial_t w_c\|_\alpha\\ &\le C_v\sqrt{\delta}\lambda + C_{v,s}\sqrt{\delta}D\mu\lambda^\alpha\,.\end{aligned}$$

Again choosing $\alpha = \frac{\omega}{1+\omega}$ and arguing as above we conclude (6.6). $\square$

7 Estimate on the energy

Proposition 7.1. *For any $\alpha \in (0, \frac{\omega}{1+\omega})$ there is a constant $C_{v,s}$, depending only on α, e and $\|v\|_0$, such that, if the parameters satisfy* (3.3)*, then*

$$\left| e(t)(1-\bar\delta) - \int |v_1|^2(x,t)\,dx \right| \le C_e D\ell + C_{v,s}\sqrt{\delta}\mu D\lambda^{\alpha-1} \qquad \forall t\,. \tag{7.1}$$

Proof. We write

$$|v_1|^2 = |v|^2 + |w_o|^2 + |w_c|^2 + 2w_o\cdot v + 2w_o\cdot w_c + 2w_c\cdot v\,. \tag{7.2}$$

Since

$$\left|\int w_c\cdot v\right| \le \|w_c\|_0\|v(\cdot,t)\|_{L^2} \le \sqrt{e(t)}\|w_c\|_0\,,$$

integrating the identity (7.2) we then reach the inequality

$$\left|\int (|v_1|^2 - |w_o|^2 - |v|^2)\,dx\right| \le C_e\|w_c\|_0(1+\|w_c\|_0+\|w_o\|_0) + 2\left|\int w_o\cdot v\right|\,.$$

By Proposition 6.1 we then have

$$\begin{aligned}\left|\int (|v_1|^2 - |w_o|^2 - |v|^2)\,dx\right| &\le C_{e,s}\sqrt{\delta}D\mu\lambda^{\alpha-1}\left(1 + C_e\sqrt{\delta}D\mu\lambda^{\alpha-1} + C_e\sqrt{\delta}\right)\\ &\quad + 2\left|\int w_o\cdot v\right|\end{aligned}$$

and hence, recalling that $\lambda \ge (D\mu)^{1+\omega}$ we reach

$$\left|\int (|v_1|^2 - |w_o|^2 - |v|^2)\,dx\right| \le C_{e,s}\sqrt{\delta}D\mu\lambda^{\alpha-1} + 2\left|\int w_o\cdot v\right|$$

Applying Proposition 4.4(i) and Proposition 5.1 we obtain

$$\left|\int w_o\cdot v\right| \le C_e \sum_{k=|\lambda_0|} \frac{[va_k]_1}{\lambda} \le C_e\|v\|_0\sqrt{\delta}D\mu\lambda^{-1} + C_e D\sqrt{\delta}\lambda^{-1}\,,$$

and hence

$$\left|\int (|v_1|^2 - |w_o|^2 - |v|^2)\right| \le C_{v,s}\sqrt{\delta}D\mu\lambda^{\alpha-1}\,. \tag{7.3}$$

Next, taking the trace of identity (5.15) in Proposition 5.1 we have

$$|W(y,s,\tau,\xi)|^2 = \operatorname{tr} R_\ell(y,s) + \sum_{1\le|k|\le 2\lambda_0} c_k(y,s,\tau)e^{ik\cdot\xi}$$

for the coefficients $c_k = \operatorname{tr} U_k$. Recall that

$$\int_{\mathbb{T}^3} \operatorname{tr} R_\ell(x,t)\,dx = 3(2\pi)^3\rho_\ell(t) = e(t)(1-\bar\delta) - \int_{\mathbb{T}^3} |v_\ell|^2\,dx.$$

Moreover, by Proposition 4.4(i) with $m = 1$ we have

$$\left|\int (|w_o|^2(x,t) - \operatorname{tr} R_\ell(x,t))\, dx\right| \le \sum_{1\le|k|\le 2\lambda_0} \left|\int c_k(x,t,\lambda t) e^{ik\cdot\lambda x}\, dx\right|$$
$$\le C\lambda^{-1} \sum_{1\le|k|\le 2\lambda_0} [c_k]_1 \overset{(5.17)}{\le} C_e \delta D\mu\lambda^{-1}\,. \tag{7.4}$$

Thus we conclude

$$\left|\int \left(|w_o|^2 + |v_\ell|^2\right)\, dx - e(t)(1-\bar\delta)\right| \le C_e\delta D\mu\lambda^{-1}\,. \tag{7.5}$$

Finally, recall from (3.9) that

$$\left|\int (|v|^2 - |v_\ell|^2)\right| \le C_e D\ell\,. \tag{7.6}$$

Putting (7.3), (7.5) and (7.6) together, we achieve (7.1). □

8 Estimates on the Reynolds stress

Proposition 8.1. *For every $\alpha \in (0, \frac{\omega}{1+\omega})$, there is a constant $C_{v,s}$, depending only on α, ω, e and $\|v\|_0$, such that, if the conditions* (3.3) *are satisfied, then the following estimates hold:*

$$\|\mathring{R}_1\|_0 \le C_{v,s}\left(D\ell + \sqrt{\delta}D\mu\lambda^{2\alpha-1} + \sqrt{\delta}\mu^{-1}\lambda^\alpha\right) \tag{8.1}$$
$$\|\mathring{R}_1\|_{C^1} \le C_{v,s}\lambda\left(\sqrt{\delta}D\ell + \sqrt{\delta}D\mu\lambda^{2\alpha-1} + \sqrt{\delta}\mu^{-1}\lambda^\alpha\right)\,. \tag{8.2}$$

Proof. We split the Reynolds stress into seven parts:

$$\mathring{R}_1 = \mathring{R}_1^1 + \mathring{R}_1^2 + \mathring{R}_1^3 + \mathring{R}_1^4 + \mathring{R}_1^5 + \mathring{R}_1^6 + \mathring{R}_1^7$$

where

$$\begin{aligned}
\mathring{R}_1^1 &= \mathring{R}_\ell - \mathring{R}\\
\mathring{R}_1^2 &= [w\otimes(v-v_\ell) + (v-v_\ell)\otimes w - \tfrac{2\langle(v-v_\ell),w\rangle}{3}\mathrm{Id}]\\
\mathring{R}_1^3 &= \mathcal{R}[\operatorname{div}(w_o\otimes w_o + \mathring{R}_\ell - \tfrac{|w_o|^2}{2}\mathrm{Id})]\\
\mathring{R}_1^4 &= \mathcal{R}\partial_t w_c\\
\mathring{R}_1^5 &= \mathcal{R}\operatorname{div}((v_\ell+w)\otimes w_c + w_c\otimes(v_\ell+w) - w_c\otimes w_c)\\
\mathring{R}_1^6 &= \mathcal{R}\operatorname{div}(v_\ell\otimes w_o)\\
\mathring{R}_1^7 &= \mathcal{R}[\partial_t w_o + \operatorname{div}(w_o\otimes v_\ell)] = \mathcal{R}[\partial_t w_o + v_\ell\cdot\nabla w_o]\,.
\end{aligned}$$

In what follows we will estimate each term separately in the order given above.

Step 1. Recalling (3.8):

$$\begin{aligned} \|\mathring{R}_1^1\|_0 &\leq CD\ell && (8.3)\\ \|\mathring{R}_1^1\|_{C^1} &\leq 2D \leq 2D\sqrt{\delta}\mu\lambda^{2\alpha}\,, && (8.4)\end{aligned}$$

where in the last inequality we have used (3.3).

Step 2. Again by (3.8) and (3.7):

$$\begin{aligned} \|v - v_\ell\|_0 &\leq CD\ell\\ \|v - v_\ell\|_{C^1} &\leq 2D\,. \end{aligned}$$

Moreover, Proposition 6.1 gives

$$\|w\|_0 \leq C_e\sqrt{\delta}$$
$$\|w\|_{C^1} \leq C_v\sqrt{\delta}\lambda\,.$$

Using this and (4.2) we conclude

$$\begin{aligned} \|\mathring{R}_1^2\|_0 &\leq C_e\sqrt{\delta}D\ell \leq C_eD\ell && (8.5)\\ \|\mathring{R}_1^2\|_{C^1} &\leq C_e\sqrt{\delta}D + C_v\sqrt{\delta}\lambda D\ell \leq C_v\sqrt{\delta}\lambda D\ell\,, && (8.6)\end{aligned}$$

where in the last inequality we have used (3.3).

Step 3. We next argue as in the proof of [15, Lemma 7.2]. Recall the formula (5.15) from Proposition 5.1. Since ρ_ℓ is a function of t only, we can write $\mathring{R}_1^3$ as

$$\begin{aligned} \mathrm{div}\,(w_o\otimes w_o - &\tfrac12(|w_o|^2-\rho_\ell)\mathrm{Id} + \mathring{R}_\ell)\\ &= \mathrm{div}\,\left(w_o\otimes w_o - R_\ell - \tfrac12(|w_o|^2 - \mathrm{tr}\,R_\ell)\mathrm{Id}\right)\\ &= \mathrm{div}\,\left[\sum_{1\leq|k|\leq 2\lambda_0}(U_k - \tfrac12(\mathrm{tr}\,U_k)\mathrm{Id})(x,t,\lambda t)e^{i\lambda k\cdot x}\right]\\ &\overset{(5.16)}{=} \sum_{1\leq|k|\leq 2\lambda_0}\mathrm{div}\,_y[U_k - \tfrac12(\mathrm{tr}\,U_k)\mathrm{Id}](x,t,\lambda t)e^{i\lambda k\cdot x}\,. && (8.7)\end{aligned}$$

We can therefore apply Proposition 4.4 with

$$m = \left\lfloor\frac{1+\omega}{\omega}\right\rfloor + 1 \tag{8.8}$$

and $\alpha\in(0,\frac{\omega}{1+\omega})$. Combining the corresponding estimates with Proposition 5.1 we get

$$\begin{aligned} \|\mathring{R}_1^3\|_0 &\leq C_s(m,\alpha)\sum_{1\leq|k|\leq 2\lambda_0}\left(\lambda^{\alpha-1}[U_k]_1 + \lambda^{\alpha-m}[U_k]_{m+1} + \lambda^{-m}[U_k]_{m+1+\alpha}\right)\\ &\leq C_s(m,\alpha)C_e\Big(\lambda^{\alpha-1}\delta\mu D + \lambda^{\alpha-m}\delta\left(\mu^{m+1}D^{m+1} + \mu D\ell^{-m}\right)\\ &\qquad + \lambda^{-m}\delta\left(\mu^{m+1+\alpha}D^{m+1+\alpha} + \mu D\ell^{-m-\alpha}\right)\Big)\\ &\overset{(3.3)}{\leq} C_{e,s}\delta\mu D\lambda^{\alpha-1}\,. && (8.9)\end{aligned}$$

Observe that in the last inequality we have used (3.3): indeed, since $m \geq \frac{1+\omega}{\omega}$ by (8.8), we get

$$\lambda \geq \max\left\{\ell^{-(1+\omega)},\, (\mu D)^{1+\omega}\right\} \geq \max\left\{\ell^{-\frac{m}{m-1}},\, (\mu D)^{\frac{m}{m-1}}\right\}. \tag{8.10}$$

Next, differentiating (8.7) in space and using the same argument:

$$\begin{aligned}\|\mathring{R}_1^3\|_1 \leq\, & C_e\lambda\|\mathring{R}_1^3\|_0 \\ & + C_s \sum_{1\leq|k|\leq 2\lambda_0}\left(\lambda^{\alpha-1}[U_k]_2 + \lambda^{\alpha-m}[U_k]_{m+2} + \lambda^{-m}[U_k]_{m+2+\alpha}\right) \\ \leq\, & C_{e,s}\delta\mu D\lambda^{\alpha}.\end{aligned}$$

Finally, differentiating (8.7) in time:

$$\begin{aligned}\partial_t \mathrm{div}\,(w_o\otimes w_o - \tfrac{1}{2}(|w_o|^2-\rho_\ell)\mathrm{Id} + \mathring{R}_\ell) \\ = \sum_{1\leq|k|\leq 2\lambda_0} \mathrm{div}\,_y[\partial_s U_k - \tfrac{1}{2}(\mathrm{tr}\,\partial_s U_k)\mathrm{Id}](x,t,\lambda t)e^{i\lambda k\cdot x} \\ + \lambda\sum_{1\leq|k|\leq 2\lambda_0} \mathrm{div}\,_y[\partial_\tau U_k - \tfrac{1}{2}(\mathrm{tr}\,\partial_\tau U_k)\mathrm{Id}](x,t,\lambda t)e^{i\lambda k\cdot x}.\end{aligned}$$

Thus, applying the same argument as above,

$$\begin{aligned}\|\partial_t\mathring{R}_1^3\|_0 \leq\, & C_s\sum_{1\leq|k|\leq 2\lambda_0}\left(\lambda^{\alpha-1}[\partial_s U_k]_1 + \lambda^{\alpha-m}[\partial_s U_k]_{m+1} + \lambda^{-m}[\partial_s U_k]_{m+1+\alpha}\right) \\ & + C_s\lambda\sum_{1\leq|k|\leq 2\lambda_0}\left(\lambda^{\alpha-1}[\partial_\tau U_k]_1 + \lambda^{\alpha-m}[\partial_\tau U_k]_{m+1}\right. \\ & \qquad\qquad \left. + \lambda^{-m}[\partial_\tau U_k]_{m+1+\alpha}\right) \\ \leq\, & C_{v,s}(\mu D + \ell^{-1} + \lambda)\delta\mu D\lambda^{\alpha-1} \\ \leq\, & C_{v,s}\delta\mu D\lambda^{\alpha}.\end{aligned}$$

Finally, putting these last two estimates together:

$$\|\mathring{R}_1^3\|_{C^1} \leq \|\mathring{R}_1^3\|_1 + \|\partial_t\mathring{R}_1^3\|_0 \leq C_{v,s}\delta\mu D\lambda^{\alpha}. \tag{8.11}$$

Step 4. In this case we argue as in [15, Lemma 7.3]. Differentiate in t the identity (6.7) to get

$$\partial_t w_c = \tfrac{1}{\lambda}\mathcal{Q}\partial_t u_c,$$

where

$$\begin{aligned}\partial_t u_c(x,t) =& \lambda\sum_{|k|=\lambda_0} i(\nabla\partial_\tau a_k)(x,t,\lambda t)\times\frac{k\times B_k}{|k|^2}e^{i\lambda x\cdot k} + \\ & + \sum_{|k|=\lambda_0} i(\nabla\partial_s a_k)(x,t,\lambda t)\times\frac{k\times B_k}{|k|^2}e^{i\lambda x\cdot k}.\end{aligned}$$

Choose again m as in (8.8) and apply the Propositions 4.4 and 5.1 to get

$$
\begin{aligned}
\|\mathring{R}_1^4\|_0 \le C_s & \sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_\tau a_k]_1 + \lambda^{\alpha-m}[\partial_\tau a_k]_{m+1} + \lambda^{-m}[\partial_\tau a_k]_{m+1+\alpha}\right)\\
&+ \frac{C_s}{\lambda}\sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_s a_k]_1 + \lambda^{\alpha-m}[\partial_s a_k]_{m+1} + \lambda^{-m}[\partial_s a_k]_{m+1+\alpha}\right)\\
\le & C_v(\lambda^{-1}\mu D + \lambda^{-1}\ell^{-1} + 1)\sqrt{\delta}\mu D\lambda^{\alpha-1}\,,
\end{aligned} \tag{8.12}
$$

where in the last inequality we have again used (8.10) for the two rightmost summands in the corresponding parantheses (cp. with the argument given for (8.9) in the paragraph right after). Using then (3.3) we conclude $\|\mathring{R}_1^4\| \le C_v\sqrt{\delta}\mu D\lambda^{\alpha-1}$.

Following the same strategy as in Step 3:

$$
\begin{aligned}
\|\mathring{R}_1^4\|_1 \le C_e\lambda\|\mathring{R}_1^4\|_0 &\\
+ C_s & \sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_\tau a_k]_2 + \lambda^{\alpha-m}[\partial_\tau a_k]_{m+2} + \lambda^{-m}[\partial_\tau a_k]_{m+2+\alpha}\right)\\
+ \frac{C_s}{\lambda} & \sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_s a_k]_2 + \lambda^{\alpha-m}[\partial_s a_k]_{m+2} + \lambda^{-m}[\partial_s a_k]_{m+2+\alpha}\right)\\
\le C_{v,s}\sqrt{\delta}\mu D\lambda^{\alpha}\,. &
\end{aligned} \tag{8.13}
$$

Differentiating in time

$$
\begin{aligned}
&\|\partial_t\mathring{R}_1^4\|_0\\
&\le C_s\lambda\sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_{\tau\tau} a_k]_1 + \lambda^{\alpha-m}[\partial_{\tau\tau} a_k]_{m+1} + \lambda^{-m}[\partial_{\tau\tau} a_k]_{m+1+\alpha}\right)\\
&\quad + C_s\sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_{\tau s} a_k]_1 + \lambda^{\alpha-m}[\partial_{\tau s} a_k]_{m+1} + \lambda^{-m}[\partial_{\tau s} a_k]_{m+1+\alpha}\right)\\
&\quad + \frac{C_s}{\lambda}\sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_{ss} a_k]_1 + \lambda^{\alpha-m}[\partial_{ss} a_k]_{m+1} + \lambda^{-m}[\partial_{ss} a_k]_{m+1+\alpha}\right)\\
&\le C_{v,s}\sqrt{\delta}\mu D\lambda^{\alpha}\,.
\end{aligned} \tag{8.14}
$$

Putting (8.13) and (8.14) together we obtain

$$
\|\mathring{R}_1^4\|_{C^1} \le C_{v,s}\sqrt{\delta}\mu D\lambda^{\alpha}\,. \tag{8.15}
$$

Step 5. In this step we argue as in [15, Lemma 7.4]. We first estimate

$$
\begin{aligned}
\|(v_\ell + w)\otimes w_c + & w_c\otimes(v_\ell + w) - w_c\otimes w_c\|_\alpha \le\\
&\le C(\|v_\ell + w\|_0\|w_c\|_\alpha + \|v_\ell + w\|_\alpha\|w_c\|_0 + \|w_c\|_0\|w_c\|_\alpha)\,.\\
&\le C\|w_c\|_\alpha\left(\|v\|_0 + \|w_o\|_\alpha + \|w_c\|_\alpha\right)\,.
\end{aligned}
$$

From Proposition 6.1 we then conclude

$$\|(v_\ell + w) \otimes w_c + w_c \otimes (v_\ell + w) - w_c \otimes w_c\|_\alpha \le C_{v,s}\sqrt{\delta} D\mu\lambda^{2\alpha-1}\,.$$

By the Schauder estimate (4.14), we get

$$\|\mathring{R}_1^5\|_0 \le C_{v,s}\sqrt{\delta} D\mu\lambda^{2\alpha-1}. \tag{8.16}$$

As for $\|\mathring{R}_1^5\|_1$ the same argument yields

$$\|\mathring{R}_1^5\|_1 \le C_{v,s}\sqrt{\delta} D\mu\lambda^{2\alpha}\,.$$

Next we estimate

$$\begin{aligned}
&\|\partial_t((v_\ell + w) \otimes w_c + w_c \otimes (v_\ell + w) - w_c \otimes w_c)\|_\alpha \\
\le&\|w_c\|_\alpha \left(\|\partial_t v_\ell\|_\alpha + \|\partial_t w_o\|_\alpha + \|\partial_t w_c\|_\alpha\right) + \|\partial_t w_c\|_\alpha \left(\|v_\ell\|_\alpha + \|w_o\|_\alpha\right)
\end{aligned}$$

Observe that $\|\partial_t v_\ell\|_\alpha \le C_h \|\partial_t v\|_0 \ell^{-\alpha} \le C_h D\ell^{-\alpha}$ and $\|v_\ell\|_\alpha \le C_h\|v\|_0\ell^{-\alpha} \le C_h\sqrt{\delta}\ell^{-\alpha}$. Thus, recalling Proposition 6.1 we conclude

$$\begin{aligned}
&\|\partial_t((v_\ell + w) \otimes w_c + w_c \otimes (v_\ell + w) - w_c \otimes w_c)\|_\alpha \\
\le&C_{e,s}\sqrt{\delta} D\mu\lambda^{\alpha-1}\left(C_h D\ell^{-\alpha} + C_v\sqrt{\delta}\lambda^{1+\alpha} + C_{v,s}\sqrt{\delta}D\mu\lambda^\alpha\right) \\
&+ C_{v,s} D\mu\lambda^\alpha\left(C_h\sqrt{\delta}\ell^{-\alpha} + C_e\sqrt{\delta}\lambda^\alpha\right) \le C_{v,s}\sqrt{\delta} D\mu\lambda^{2\alpha}\,,
\end{aligned} \tag{8.17}$$

where in the last inequality we have used (3.3). Applying (4.14) we then achieve

$$\|\mathring{R}_1^5\|_{C^1} \le C_{v,s}\sqrt{\delta} D\mu\lambda^{2\alpha}\,. \tag{8.18}$$

Step 6. In this step we argue as in [15, Lemma 7.5]. Since $B_k \cdot k = 0$, we can write

$$\begin{aligned}
\operatorname{div}(v_\ell \otimes w_o) &= (w_o \cdot \nabla)v_\ell + (\operatorname{div} w_o)v_\ell \\
&= \sum_{|k|=\lambda_0} \left[a_k(B_k\cdot\nabla)v_\ell + v_\ell(B_k\cdot\nabla)a_k\right] e^{i\lambda k\cdot x}\,.
\end{aligned}$$

Choose m as in (8.8), apply Propositions 4.4 and 5.1 and use (8.10) to get

$$\begin{aligned}
\|\mathring{R}_1^6\|_0 \le{}& C_s \sum_{|k|=\lambda_0} \lambda^{\alpha-1}\left(\|a_k\|_0[v_\ell]_1 + \|v_\ell\|_0[a_k]_1\right) \\
&+ C_s \sum_{|k|=\lambda_0} \lambda^{-m+\alpha}\left(\|a_k\|_0[v_\ell]_{m+1} + \|v_\ell\|_0[a_k]_{m+1}\right) \\
&+ C_s \sum_{|k|=\lambda_0} \lambda^{-m}\left(\|a_k\|_0[v_\ell]_{m+1+\alpha} + \|v_\ell\|_0[a_k]_{m+1+\alpha}\right) \\
\le{}& C_{v,s}\lambda^{\alpha-1}\sqrt{\delta}(D + D\mu) + C_{v,s}\lambda^{-m+\alpha}\sqrt{\delta}\left(D\ell^{-m} + D^{m+1}\mu^{m+1}\right) \\
&+ C_{v,s}\lambda^{-m}\sqrt{\delta}\left(D\ell^{-m-\alpha} + D^{m+1+\alpha}\mu^{m+1+\alpha}\right) \\
\le{}& C_{v,s}\sqrt{\delta}D\mu\lambda^{\alpha-1}\,.
\end{aligned} \tag{8.19}$$

As in the Steps 3 and 4:

$$
\begin{aligned}
\|\mathring{R}_1^6\|_1 \leq C_e\lambda\|\mathring{R}_1^6\|_0 &+ C_s\sum_{|k|=\lambda_0}\lambda^{\alpha-1}\left(\|a_k\|_0[v_\ell]_2+\|v_\ell\|_0[a_k]_2\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{\alpha-m}\left(\|a_k\|_0[v_\ell]_{m+2}+\|v_\ell\|_0[a_k]_{m+2}\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{-m}\left(\|a_k\|_0[v_\ell]_{m+2+\alpha}+\|v_\ell\|_0[a_k]_{m+2+\alpha}\right)\\
&\leq C_{v,s}\sqrt{\delta}D\mu\lambda^\alpha\,. \qquad (8.20)
\end{aligned}
$$

As for the time derivative, we can estimate

$$
\|\partial_t\mathring{R}_1^6\|_0 \leq (\mathrm{I})+(\mathrm{II})+(\mathrm{III})\,,
$$

where

$$
\begin{aligned}
(\mathrm{I}) = C_s\sum_{|k|=\lambda_0}\lambda^{\alpha}&\left(\|\partial_\tau a_k\|_0[v_\ell]_1+\|v_\ell\|_0[\partial_\tau a_k]_1\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{\alpha-1}\left(\|\partial_s a_k\|_0[v_\ell]_1+\|v_\ell\|_0[\partial_s a_k]_1\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{\alpha-1}\left(\|a_k\|_0[\partial_t v_\ell]_1+\|\partial_t v_\ell\|_0[a_k]_1\right)\,, \qquad (8.21)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{II}) = C_s\sum_{|k|=\lambda_0}\lambda^{\alpha+1-m}&\left(\|\partial_\tau a_k\|_0[v_\ell]_{m+1}+\|v_\ell\|_0[\partial_\tau a_k]_{m+1}\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{\alpha-m}\left(\|\partial_s a_k\|_0[v_\ell]_{m+1}+\|v_\ell\|_0[\partial_s a_k]_{m+1}\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{\alpha-m}\left(\|a_k\|_0[\partial_t v_\ell]_{m+1}+\|\partial_t v_\ell\|_0[a_k]_{m+1}\right) \qquad (8.22)
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{III}) = C_s\sum_{|k|=\lambda_0}\lambda^{1-m}&\left(\|\partial_\tau a_k\|_0[v_\ell]_{m+1+\alpha}+\|v_\ell\|_0[\partial_\tau a_k]_{m+1+\alpha}\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{-m}\left(\|\partial_s a_k\|_0[v_\ell]_{m+1+\alpha}+\|v_\ell\|_0[\partial_s a_k]_{m+1+\alpha}\right)\\
&+ C_s\sum_{|k|=\lambda_0}\lambda^{-m}\left(\|a_k\|_0[\partial_t v_\ell]_{m+1+\alpha}+\|\partial_t v_\ell\|_0[a_k]_{m+1+\alpha}\right)\,. \qquad (8.23)
\end{aligned}
$$

Again using Proposition 5.1 and the conditions (3.3) we can see that

$$
\|\partial_t\mathring{R}_1^6\|_0 \leq C_{v,s}\sqrt{\delta}D\mu\lambda^\alpha\,. \qquad (8.24)
$$

Thus,

$$\|\mathring{R}_1^6\|_{C^1} \le \|\mathring{R}_1^6\|_1 + \|\partial_t \mathring{R}_1^6\|_0 \le C_{v,s}\sqrt{\delta} D\mu\lambda^\alpha \,. \tag{8.25}$$

Step 7. Finally, to bound the last term we argue as in [15, Lemma 7.1]. We write

$$\mathring{R}_1^7 = \mathcal{R}(\partial_t w_o + v_\ell \cdot \nabla w_o) = \mathring{R}_1^8 + \mathring{R}_1^9 + \mathring{R}_1^{10},$$

where

$$\begin{aligned}
\mathring{R}_1^8 &:= \lambda \mathcal{R}\left(\sum_{|k|=\lambda_0} (\partial_\tau a_k + i(k\cdot v_\ell)a_k)(x,t,\lambda t) B_k e^{i\lambda k\cdot x}\right)\\
\mathring{R}_1^9 &:= \mathcal{R}\left(\sum_{|k|=\lambda_0} (\partial_s a_k)(x,t,\lambda t) B_k e^{i\lambda k\cdot x}\right)\\
\mathring{R}_1^{10} &:= \mathcal{R}\left(\sum_{|k|=\lambda_0} (v_\ell\cdot\nabla_y a_k)(x,t,\lambda t) B_k e^{i\lambda k\cdot x}\right) .
\end{aligned}$$

The arguments of Step 6 have already shown

$$\|\mathring{R}_1^{10}\|_0 \le C_{v,s}\sqrt{\delta} D\mu\lambda^{\alpha-1} \tag{8.26}$$

$$\|\mathring{R}_1^{10}\|_{C^1} \le C_{v,s}\sqrt{\delta} D\mu\lambda^{\alpha} \,. \tag{8.27}$$

As for $\mathring{R}_1^9$, we apply Proposition 4.4 with m as in (8.8) to get

$$\begin{aligned}
\|\mathring{R}_1^9\|_0 &\le C_s \sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}\|\partial_s a_k\|_0 + \lambda^{-m+\alpha}[\partial_s a_k]_m + \lambda^{-m}[\partial_s a_k]_{m+\alpha}\right)\\
&\le C_{e,s}\sqrt{\delta} D\mu\lambda^{\alpha-1} \,.
\end{aligned} \tag{8.28}$$

Analogously

$$\begin{aligned}
\|\mathring{R}_1^9\|_1 &\le C_e\lambda\|\mathring{R}_1^9\|_0\\
&\quad + C_s \sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}[\partial_s a_k]_1 + \lambda^{-m+\alpha}[\partial_s a_k]_{m+1} + \lambda^{-m}[\partial_s a_k]_{m+1+\alpha}\right)\\
&\le C_{e,s}\sqrt{\delta} D\mu\lambda^{\alpha}
\end{aligned} \tag{8.29}$$

and

$$\begin{aligned}
\|\partial_t \mathring{R}_1^9\|_0 &\le C_s \sum_{|k|=\lambda_0} \left(\lambda^{\alpha-1}\|\partial_{ss} a_k\|_0 + \lambda^{-m+\alpha}[\partial_{ss} a_k]_m + \lambda^{-m}[\partial_{ss} a_k]_{m+\alpha}\right)\\
&\quad + C_s \sum_{|k|=\lambda_0} \left(\lambda^{\alpha}\|\partial_{s\tau} a_k\|_0 + \lambda^{1-m+\alpha}[\partial_{s\tau} a_k]_m + \lambda^{1-m}[\partial_{s\tau} a_k]_{m+\alpha}\right)\\
&\le C_{v,s}\sqrt{\delta} D\mu\lambda^{\alpha} \,,
\end{aligned} \tag{8.30}$$

which in turn imply

$$\|\mathring{R}_1^9\|_{C^1} \le C_{v,s}\sqrt{\delta} D\mu\lambda^\alpha\,. \tag{8.31}$$

For the term $\mathring{R}_1^8$ define the functions

$$b_k(y,s,\tau) := (\partial_\tau a_k + i(k\cdot v_\ell)a_k)(y,s,\tau)\,.$$

Applying Proposition 4.4 with m as in (8.8) then yields

$$\begin{aligned}
\|\mathring{R}_1^8\|_0 &\le C_s \sum_{|k|=\lambda_0} \left(\lambda^\alpha \|b_k\|_0 + \lambda^{\alpha+1-m}[b_k]_m + \lambda^{1-m}[b_k]_{m+\alpha}\right)\\
&\le C_{e,s}\sqrt{\delta}\mu^{-1}\lambda^\alpha + C_{e,s}\sqrt{\delta}\left(\mu^{m-1}D^m + D\ell^{1-m}\right)\lambda^{\alpha+1-m}\\
&\quad + C_{e,s}\sqrt{\delta}\left(\mu^{m-1+\alpha}D^{m+\alpha} + D\ell^{1-m-\alpha}\right)\lambda^{1-m}\\
&\le C_{e,s}\sqrt{\delta}\mu^{-1}\lambda^\alpha\,,
\end{aligned} \tag{8.32}$$

where we have used (5.5) and (5.9) in Proposition 5.1 to bound $\|b_k\|_0$, $[b_k]_m$ and $[b_k]_{m+\alpha}$. Similarly,

$$\begin{aligned}
\|\mathring{R}_1^8\|_1 &\le C_e\lambda\|\mathring{R}_1^8\|_0\\
&\quad + C_s \sum_{|k|=\lambda_0} \left(\lambda^\alpha [b_k]_1 + \lambda^{\alpha+1-m}[b_k]_{m+1} + \lambda^{1-m}[b_k]_{m+1+\alpha}\right)\\
&\le C_{e,s}\sqrt{\delta}\mu^{-1}\lambda^{1+\alpha}\,.
\end{aligned} \tag{8.33}$$

Finally, differentiating $\mathring{R}_1^8$ in time and using the same arguments:

$$\begin{aligned}
\|\partial_t\mathring{R}_1^8\|_0 &\le C_s\lambda \sum_{|k|=\lambda_0} \left(\lambda^\alpha \|\partial_\tau b_k\|_0 + \lambda^{\alpha+1-m}[\partial_\tau b_k]_m + \lambda^{1-m}[\partial_\tau b_k]_{m+\alpha}\right)\\
&\quad + C_s \sum_{|k|=\lambda_0} \left(\lambda^\alpha \|\partial_s b_k\|_0 + \lambda^{\alpha+1-m}[\partial_s b_k]_m + \lambda^{1-m}[\partial_s b_k]_{m+\alpha}\right)\\
&\le C_{v,s}\sqrt{\delta}\mu^{-1}\lambda^{1+\alpha}\,.
\end{aligned} \tag{8.34}$$

Therefore

$$\|\mathring{R}_1^8\|_{C^1} \le C_{v,s}\sqrt{\delta}\mu^{-1}\lambda^{1+\alpha}\,. \tag{8.35}$$

Summarizing

$$\|\mathring{R}_1^7\|_0 \le C_{v,s}\sqrt{\delta}\left(D\mu\lambda^{\alpha-1} + \mu^{-1}\lambda^\alpha\right) \tag{8.36}$$

$$\|\mathring{R}_1^7\|_{C^1} \le C_{v,s}\sqrt{\delta}\left(D\mu\lambda^{\alpha} + \mu^{-1}\lambda^{\alpha+1}\right)\,. \tag{8.37}$$

Conclusion. From (8.3), (8.5), (8.9), (8.12), (8.16), (8.19) and (8.36), we conclude

$$\begin{aligned}
\|\mathring{R}_1\|_0 &\le C_{v,s}\Big(D\ell + \sqrt{\delta}D\ell + \delta D\mu\lambda^{\alpha-1} + \sqrt{\delta}D\mu\lambda^{\alpha-1}\\
&\qquad + \sqrt{\delta}D\mu\lambda^{2\alpha-1} + \sqrt{\delta}\mu^{-1}\lambda^\alpha\Big)\\
&\le C_{v,s}\left(D\ell + \sqrt{\delta}D\mu\lambda^{2\alpha-1} + \sqrt{\delta}\mu^{-1}\lambda^\alpha\right)\,.
\end{aligned} \tag{8.38}$$

From (8.4), (8.6), (8.11), (8.15), (8.18), (8.25) and (8.37), we conclude

$$\begin{aligned}\|\mathring{R}_1\|_{C_1} &\le C_{v,s}\Big(D+\sqrt{\delta}\lambda D\ell+\delta D\mu\lambda^\alpha+\sqrt{\delta}D\mu\lambda^\alpha \\ &\qquad +\sqrt{\delta}D\mu\lambda^{2\alpha}+\sqrt{\delta}\mu^{-1}\lambda^{1+\alpha}\Big) \\ &\le C_{v,s}\left(\sqrt{\delta}D\ell\lambda+\sqrt{\delta}D\mu\lambda^{2\alpha}+\sqrt{\delta}D\mu^{-1}\lambda^{\alpha+1}\right). \end{aligned} \tag{8.39}$$

In the last inequality we have used (3.3) once more: $\sqrt{\delta}\mu D \ge D\delta^{-1/2} \ge D$. □

9 Proof of Proposition 2.2

Step 1. We now specify the choice of the parameters, in the order in which they are chosen. Recall that ε is a fixed positive number, given by the proposition. The exponent ω has already been chosen according to

$$1+\varepsilon=\frac{1+\omega}{1-\omega}. \tag{9.1}$$

Next we choose a suitable exponent α for which we can apply the Propositions 7.1 and 8.1. To be precise we set

$$\alpha=\frac{\omega}{2(1+\omega)}. \tag{9.2}$$

The reason for these choices will become clear in the following. For the moment we just observe that both α and ω depend only on ε and that $\alpha\in(0,\frac{\omega}{1+\omega})$, i.e. both Propositions 7.1 and 8.1 are applicable.

We next choose:

$$\ell=\frac{1}{L_v}\frac{\bar{\delta}}{D} \tag{9.3}$$

with L_v being a sufficiently large constant, which depends only on $\|v\|_0$ and e.

Next, we impose

$$\mu^2 D=\lambda \tag{9.4}$$

and

$$\lambda=\Lambda_v\left(\frac{D\delta}{\bar{\delta}^2}\right)^{\frac{1}{1-4\alpha}}=\Lambda_v\left(\frac{D\delta}{\bar{\delta}^2}\right)^{\frac{1+\omega}{1-\omega}}=\Lambda_v\left(\frac{D\delta}{\bar{\delta}^2}\right)^{1+\varepsilon}, \tag{9.5}$$

where Λ_v is a sufficiently large constant, which depends only on $\|v\|_0$. Concerning the constants L_v and Λ_v we will see that they will be chosen in this order in Step 3 below. Observe also that μ, λ and $\frac{\lambda}{\mu}$ must be integers. However, this can be reached by imposing the less stringent constraints

$$\frac{\lambda}{2}\le\mu^2 D\le\lambda$$

and

$$\Lambda_v \left(\frac{D\delta}{\bar{\delta}^2}\right)^{1+\varepsilon} \leq \lambda \leq 2\Lambda_v \left(\frac{D\delta}{\bar{\delta}^2}\right)^{1+\varepsilon},$$

provided Λ_v is larger than some universal constant. This would require just minor adjustments in the rest of the argument.

Step 2. Compatibility conditions. We next check that all the conditions in (3.3) are satisfied by our choice of the parameters.

First of all, since $\bar{\delta} \leq \delta$, the inequality $\ell^{-1} \geq \frac{D}{\eta\delta}$ is for sure achieved if we impose

$$L_v \geq \eta^{-1}. \tag{9.6}$$

Next, (9.5) and $\Lambda_v \geq 1$ implies

$$\mu = \sqrt{\frac{\lambda}{D}} \geq \frac{\sqrt{\delta}}{\bar{\delta}} \geq \delta^{-1}$$

because by assumption $\bar{\delta} \leq \delta^{\frac{3}{2}}$.

Also,

$$\frac{\lambda}{(\mu D)^{1+\omega}} \overset{(9.4)}{=} \frac{\lambda^{\frac{1-\omega}{2}}}{D^{\frac{1+\omega}{2}}} \overset{(9.5)}{=} \Lambda_v^{\frac{1-\omega}{2}} \left(\frac{\delta}{\bar{\delta}^2}\right)^{\frac{1+\omega}{2}}.$$

Since $\omega < 1$, $\Lambda_v \geq 1$ and $\bar{\delta} \leq \delta$, we conclude $\lambda \geq (\mu D)^{1+\omega}$. Finally

$$\lambda \ell^{1+\omega} \overset{(9.3)\&(9.5)}{=} \Lambda_v \left(\frac{D\delta}{\bar{\delta}^2}\right)^{\frac{1+\omega}{1-\omega}} \left(L_v^{-1}\bar{\delta}D^{-1}\right)^{1+\omega} = \frac{\Lambda_v}{L_v^{1+\omega}} \left(D^\omega \frac{\delta}{\bar{\delta}^{1+\omega}}\right)^{\frac{1+\omega}{1-\omega}}.$$

Thus, by requiring

$$\Lambda_v \geq L_v^{1+\omega} \tag{9.7}$$

we satisfy $\lambda \geq \ell^{-(1+\omega)}$. Hence, all the requirements in (3.3) are satisfied provided that the constants L_v and Λ_v are chosen to satisfy (9.6) and (9.7).

Step 3. C^0 estimates. Having verified that $\alpha \in (0, \frac{\omega}{1+\omega})$ and that (3.3) holds, we can apply the Propositions 6.1, 7.1 and 8.1. Proposition 8.1 implies

$$\begin{aligned} \|\mathring{R}_1\|_0 &\leq C_v \left(D\ell + \sqrt{\delta} D^{\frac{1}{2}} \lambda^{2\alpha - \frac{1}{2}} + \sqrt{\delta} D^{\frac{1}{2}} \lambda^{\alpha - \frac{1}{2}}\right) \\ &\leq \frac{C_v}{L_v}\bar{\delta} + \frac{C_v}{\Lambda_v^{\frac{1+\varepsilon}{2}}}\bar{\delta} \end{aligned} \tag{9.8}$$

(since now the exponent α has been fixed, we can forget about the α-dependence of the constants in the estimates of Proposition 7.1 and 8.1). Choosing first L_v and, then, Λ_v sufficiently large, we can achieve the desired inequalities (9.6)-(9.7) together with

$$\|\mathring{R}_1\|_0 \leq \eta\bar{\delta}.$$

Next, using Proposition 7.1, it is also easy to check that, by this choice, (2.5) is satisfied as well. Furthermore, recall that, by Proposition 6.1,

$$\|v_1 - v\|_0 = \|w\|_0 \le C_e\sqrt{\delta}\,.$$

If we impose M to be larger than this particular constant C_e (which depends only on e), we then achieve (2.7).

Finally, as already observed in (3.16),

$$\|p_1 - p\|_0 = \frac{M^2}{4}\delta + \|v - v_\ell\|_0\|w\|_0\,.$$

Since $\|v - v_\ell\|_0 \le CD\ell \le C\bar{\delta}$ and $\|w\|_0 \le C_e\sqrt{\delta}$, we easily conclude the inequality (2.8). This completes the proof of all the conclusions of Proposition 2.2 except for the estimate of $\max\{\|v_1\|_{C^1}, \|\mathring{R}_1\|_{C^1}\}$.

Step 4. C^1 **estimates.** By Proposition 8.1 and the choices specified above we also have

$$\|\mathring{R}_1\|_{C^1} \le \bar{\delta}\lambda$$

whereas Proposition 6.1 shows

$$\|v_1\|_{C^1} \le D + \|w\|_{C^1} \le D + C_e\sqrt{\delta}\lambda.$$

Thus, we conclude

$$\begin{aligned}\max\left\{\|v_1\|_{C^1}, \|\mathring{R}_1\|_{C^1}\right\} &\le D + C_e\sqrt{\delta}\lambda \le D + C_e\sqrt{\delta}\Lambda_v\left(\frac{D\delta}{\bar{\delta}^2}\right)^{1+\varepsilon}\\ &\le D + C_e\Lambda_v\delta^{\frac{3}{2}}\left(\frac{D}{\bar{\delta}^2}\right)^{1+\varepsilon}\,.\end{aligned}$$

Since $\delta^{\frac{3}{2}} \ge \bar{\delta}^2$, we obtain

$$\max\left\{\|v_1\|_{C^1}, \|\mathring{R}_1\|_{C^1}\right\} \le 2C_e\Lambda_v\delta^{\frac{3}{2}}\left(\frac{D}{\bar{\delta}^2}\right)^{1+\varepsilon}\,.$$

Setting $A = 2C_e\Lambda_v$, we conclude estimate (2.9).

10 Proof of Remark 1.3

Step 1. Estimate on the C^1 **norm.** We claim that the proof of Proposition 2.2 yields also the estimate

$$\|p_1\|_{C^1} \le \|p\|_{C^1} + A\delta^{2+\varepsilon}\left(\frac{D}{\bar{\delta}^2}\right)^{1+\varepsilon}\,, \tag{10.1}$$

where, as in Proposition 2.2, A is a constant which depends only on e, $\varepsilon > 0$ and $\|v\|_0$. Indeed, recall the formula for the pressure:

$$p_1 = p - \frac{|w_o|^2}{2} - \langle v - v_\ell, w \rangle .$$

Therefore we estimate, using Proposition 6.1

$$\begin{aligned}
\|p_1\|_{C^1} - \|p\|_{C^1} &\leq \|w_o\|_0 \|w_o\|_{C^1} + \|w\|_0 \|v - v_\ell\|_{C^1} + \|w\|_{C^1} \|v - v_\ell\|_0 \\
&\leq C_e \delta \lambda + C_e D \sqrt{\delta} + C_e D \ell \sqrt{\delta} \lambda .
\end{aligned}$$

As before, (3.3) implies $\lambda \geq \mu D \geq D\delta^{-1}$ and $D\ell \leq \delta$. Therefore, we conclude

$$\begin{aligned}
\|p_1\|_{C^1} &\leq \|p\|_{C^1} + C_e \delta \lambda \leq \|p\|_{C^1} + C_e \Lambda_v \delta \left(\frac{D\delta}{\bar{\delta}^2} \right)^{1+\varepsilon} \\
&\leq \|p\|_{C^1} + A \delta^{2+\varepsilon} \left(\frac{D}{\bar{\delta}^2} \right)^{1+\varepsilon} .
\end{aligned}$$

Step 2. Iteration. We now proceed as in the proof of Theorem 1.1. We construct the sequence $(p_n, v_n, \mathring{R}_n)$ of solutions to the Euler-Reynolds system, starting from

$$(p_0, v_0, \mathring{R}_0) = (0, 0, 0)$$

and applying Proposition 2.2 with $\delta_n = a^{-b^n}$. As in the proof of Theorem 1.1, we set

$$b = \frac{3}{2}, \quad c = \frac{3(1+2\varepsilon)}{1-2\varepsilon} + \varepsilon$$

and choose a sufficiently large so to guarantee the inequality

$$D_n = \max\{\|v_n\|_{C^1}, \|\mathring{R}_n\|_{C^1}\} \leq a^{cb^n} .$$

We then use (10.1) to conclude

$$\|p_{n+1}\|_{C^1} \leq \|p_n\|_{C^1} + A a^{(1+2\varepsilon)(c+1)b^n} .$$

Since A depends only on $\|v_n\|_0$ which turns out to be uniformly bounded, we can assume that A does not depend on n. Therefore, if we choose a sufficiently large, we can then write

$$\|p_{n+1}\|_{C^1} \leq \|p_n\|_{C^1} + a^{(1+3\varepsilon)(c+1)b^n}$$

Since $p_0 = 0$, we inductively get the estimate

$$\|p_{n+1}\|_{C^1} \leq (n+1) a^{(1+3\varepsilon)(c+1)b^n} \leq a^{[(1+4\varepsilon)(c+1)]b^n}$$

(again the last inequality is achieved choosing a sufficiently large). Summarizing, if we set $\vartheta = (1+4\varepsilon)(c+1)$, we have

$$\|p_{n+1} - p_n\|_0 \le C_e \delta_n \le C_e a^{-b^n}$$
$$\|p_{n+1} - p_n\|_{C^1} \le a^{\vartheta b^n}$$

Interpolating we get $\|p_{n+1} - p_n\|_{C^\varrho} \le C_e a^{(\varrho(1+\vartheta)-1)b^n}$ for every $\varrho \in (0,1)$. Thus the limiting pressure p belongs to C^ϱ for every

$$\varrho < \frac{1}{1+\vartheta} = \frac{1}{1+(1+4\varepsilon)(c+1)}\,.$$

As $\varepsilon \downarrow 0$, we have $c \downarrow 3$ and hence

$$\frac{1}{1+\vartheta} \uparrow \frac{1}{5}\,.$$

Therefore, for every $\theta < \frac{1}{10}$, if the ε in Proposition 2.2 is chosen sufficiently small, we construct a pair (p,v) which satisfies the conclusion of Theorem 1.1 and belongs to $C^\theta(\mathbb{T}^3 \times \mathbb{S}^1, \mathbb{R}^3) \times C^{2\theta}(\mathbb{T}^3 \times \mathbb{S}^1)$.

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