NOTES ON HYPERBOLIC SYSTEMS OF CONSERVATION LAWS AND TRANSPORT EQUATIONS

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Contents

1. Introduction

The aim of these notes is to give an account of some recent results about transport equations with variable BV coefficients, and their applications to a class of hyperbolic systems of conservation laws in several space dimensions. Besides collecting results which are scattered in the literature, it has been my intention to give a self–contained and more readable reference, and to provide details, remarks, and connections barely mentioned in the original papers.

1.1. **The Keyfitz and Kranzer system.** We start by considering the following system of equations:

$$\begin{cases} \partial_t u^i + \sum_{\alpha=1}^m \partial_{x_\alpha}(g^\alpha(|u|)u^i) \;=\; 0 \\ \\ u^i(0,\cdot) \;=\; \overline{u}^i(\cdot) \end{cases} \tag{1}$$

where $u = (u^1,\ldots,u^k) : \mathbb{R}^+_t \times \mathbb{R}^m_x \to \mathbb{R}^k$ is the unknown vector map, $\overline{u} = (\overline{u}^1,\ldots,\overline{u}^k)$ the initial data, and $g^\alpha : \mathbb{R}\to\mathbb{R}$ are given (sufficiently smooth) scalar functions. In one space dimension (1) was first studied by Keyfitz and Kranzer in [34] and later on by several other authors, as a prototypical example of a non-strict hyperbolic system, see for instance [28], [29], [30], [31], and [35]. Indeed, in the 1–dimensional terminology, the hyperbolicity of (1) degenerates at the origin (see for instance [22], Section 7.2).

However, the Keyfitz and Kranzer system has many features. In particular it can be formally reduced to a scalar conservation law and a system of transport equations with variable coefficients. More precisely, if u is smooth and solves (1), then $\rho := |u|$ solves

$$\begin{cases} \partial_t \rho + D_x \cdot (\rho g(\rho)) = 0 \\ \rho(0,\cdot) = |\overline{u}|(\cdot)\,, \end{cases} \tag{2}$$

and, if in addition $|u| > 0$, then $\theta := u/|u|$ solves

$$\begin{cases} \partial_t \theta + g(\rho)\cdot D_x \theta = 0 \\ \theta(0,\cdot) = \overline{u}/|\overline{u}|(\cdot)\,, \end{cases} \tag{3}$$

One can use this observation to produce solutions to (1). However, as it is well known, even starting from extremely regular initial data, solutions of (2) develop singularities in finite time, and one cannot hope to get better than BV regularity. Thus, in order to construct solutions in the way described above, one has to face the problem of solving transport equations

$$\begin{cases} \partial_t \theta(t,x) + b(t,x)\cdot D_x \theta(t,x) = 0 \\ \theta(0,x) = \overline{\theta}(x)\,, \end{cases} \tag{4}$$

when b is quite irregular.

From now on, we will say that a distributional solution u of (1) is a *renormalized entropy solution* if $\rho := |u|$ solves, in the sense of Kruzkov, the scalar law (2) (see Definition 5.1 and 5.4).

1.2. **Bressan's compactness conjecture.** In [17] Bressan showed that in 2 space dimensions renormalized entropy solutions might lead to an ill posed Cauchy problem for bounded initial data. However he conjectured that this does not happen when the absolute value of the initial data are in BV_{loc}. In particular, in order to show the existence of renormalized entropy solutions to (1) when $|\overline{u}| \in L^\infty \cap BV$ and $|\overline{u}|^{-1} \in L^\infty$, he advanced the following

Conjecture 1.1 (Bressan's compactness conjecture)**.** *Let $b_n : \mathbb{R}_t \times \mathbb{R}^m_x \to \mathbb{R}^m$, $n \in \mathbb{N}$, be smooth maps and denote by Φ_n the solutions of the ODEs:*

$$\begin{cases} \dfrac{d}{dt}\Phi_n(t,x) = b_n(t,\Phi_n(t,x)) \\ \Phi_n(0,x) = x\,. \end{cases} \tag{5}$$

Assume that $\|b_n\|_\infty + \|\nabla b_n\|_{L^1}$ is uniformly bounded and that the fluxes Φ_n are nearly incompressible, i.e. that

$$C^{-1} \le \det(\nabla_x \Phi_n(t,x)) \le C \qquad \text{for some constant } C > 0. \tag{6}$$

Then the sequence $\{\Phi_n\}$ is strongly precompact in $L^1_{\rm loc}$.

An affirmative answer to this conjecture leads immediately to the existence of renormalized entropy solutions of (1) when $C \geq |\overline{u}| \geq c > 0$ and $\overline{u} \in BV$. Indeed, assume that these assumptions hold and consider the Kruzkov solution ρ of (2). It is well known that $\rho \in BV_{loc}$ and $C \geq \rho \geq c > 0$. Thus, $g(\rho)$ is also $BV_{loc} \cap L^\infty$. It is not difficult to see that we can approximate $b := g(\rho)$ and ρ with two sequences $\{b_n\}$ and $\{\rho_n\}$ of smooth functions such that

(i) $\|b_n\|_{BV} + \|b_n\|_\infty$ is uniformly bounded;
(ii) $C_1 \geq \rho_n \geq c_1 > 0$ for some constant c_1;
(iii) $\partial_t \rho_n + D_x \cdot (b_n \rho_n) = 0$.

If we set $\overline{\theta} := \overline{u}/\overline{\rho}$, then we can solve

$$\begin{cases} \partial_t \theta_n(t,x) + b_n(t,x) \cdot D_x \theta_n(t,x) \;=\; 0 \\ \\ \theta(0,x) \;=\; \overline{\theta}(x) \end{cases} \tag{7}$$

with the classical method of characteristics. If we let Φ_n be as in (5), then the continuity equations of (iii), condition (ii) and the standard maximum principle for transport equations with smooth coefficients imply the existence of a constant C such that (6) holds. At this stage we could use Conjecture 1.1 to show that θ_n converges locally strongly to a function θ (up to subsequences). This strong convergence implies that $u := \theta\rho$ is a renormalized entropy solution.

1.3. **Ambrosio's renormalization Theorem.** In the recent ground-breaking paper [2], Ambrosio has shown well–posedness of

$$\begin{cases} \partial_t \theta(t,x) + b(t,x) \cdot D_x \theta(t,x) \;=\; 0 \\ \\ \theta(0,x) \;=\; \overline{\theta}(x)\,, \end{cases} \tag{8}$$

under the assumptions that $b \in BV$ and $D_x \cdot b$ is a bounded function.

The result of Ambrosio uses the theory of renormalized solutions, first introduced by DiPerna and Lions in [27] (in that paper the authors proved, among other results, the well–posedness of (8) under the assumptions $b \in L^\infty \cap W^{1,1}$ and $D_x \cdot b \in L^\infty$).

The core of Ambrosio's well–posedness theorem is a new "renormalization lemma". In order to understand its content, consider first a smooth vector field B in $\Omega \subset \mathbb{R}^d$ and a smooth scalar function u such that $B \cdot Du = 0$. For any smooth function β the classical chain rule yields

$$B \cdot D(\beta(u)) \;=\; B \cdot [\beta'(u) Du] \;=\; 0\,.$$

Next assume that $B \in BV$, that the divergence $D \cdot B$ is an absolutely continuous measure, and that $u \in L^\infty$. Then, the expression

$$D \cdot (uB) - uD \cdot B$$

makes sense distributionally, and can be taken as a definition of $B \cdot Du$. Ambrosio's renormalization Theorem states that the conclusion

$$0 \;=\; B \cdot D(\beta(u)) \;:=\; D \cdot (\beta(u)B) - \beta(u) D \cdot B \qquad \forall \beta \in C^1(\mathbb{R})$$

holds even under these much weaker assumptions.

Assume now that $b \in BV$, $D_x \cdot b \in L^1$ and u is a bounded weak solution of the transport equation $\partial_t u + b \cdot Du = 0$ with initial data $\overline{u}$. More precisely, assume that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^m} u(t,x) \big\{ \partial_t \varphi(t,x) + b(t,x) \cdot D\varphi(t,x) - [D_x \cdot b](t,x) \varphi(t,x) \big\} \, dt \, dx \\ = & \; -\int_{\mathbb{R}^m} \overline{u}(x) \varphi(0,x) \, dx \end{aligned}$$

for every smooth compactly supported test function φ. Applying Ambrosio's renormalization Theorem to the field $B = (1, b) : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m$, we infer that $\beta(u)$ solves the corresponding Cauchy problem with initial data $\beta(\overline{u})$ (actually a technical step is needed in order to conclude that $\beta(u)$ has initial data $\beta(\overline{u})$; see Sections 3 and 4). If in addition we have the bounds $b \in L^\infty$ and $D_x \cdot b \in L^\infty$, the equation satisfied by $\beta(\overline{u})$ can be used (for special choices of β) to derive estimates and comparison principles, via standard Gronwall–type arguments. These comparison principles are indeed enough to show uniqueness and stability for weak solutions of (8).

A byproduct of the renormalization property is that solutions of (8) are stable even under approximation of the coefficient b. In the DiPerna–Lions theory this is used to conclude existence, stability, and compactness properties for the ODEs with coefficients b. Therefore Ambrosio's result can be used to infer that Bressan's compactness Conjecture holds when we replace the bound (6) with the stronger assumption

$$-C \;\leq\; D_x \cdot B \;\leq\; C \, . \tag{9}$$

1.4. **Well–posedness for the Keyfitz and Kranzer system.** Though presently there is no general proof of Bressan's compactness conjecture, it is still possible to use Ambrosio's renormalization Theorem to show existence of renormalized entropy solutions when $|\overline{u}| \in BV_{loc}$. The difference with respect to Bressan's compactness conjecture is that in this specific case one can take advantage of an additional information. Indeed, if ρ is a Kruzhkov solution of the scalar law (2), then the coefficient $b := g(\rho)$ has a solution of the continuity equation which, besides being bounded from above and from below, also enjoys BV regularity. This information is missing in the assumptions of Conjecture 1.1.

Basically Ambrosio's renormalization lemma is powerful enough to provide a DiPerna–Lions theory for transport equations with $BV \cap L^\infty$ coefficients which possess a BV nonnegative solution ρ of the continuity equation. As shown in [4], this yields well–posedness for the Keyfitz and Kranzer system when $|\overline{u}| \in BV_{loc} \cap L^\infty$ (in particular it also allows to drop the unnatural assumption $|\overline{u}| \geq c > 0$). More precisely, for every $\overline{u}$ with $|\overline{u}| \in BV_{loc} \cap L^\infty$ there exists a unique renormalized entropy solution of (1). Moreover if a sequence of initial data

$\overline{u}_n$ converges to $\overline{u}$ and $\|\overline{u}_n\|_\infty + \||\overline{u}_n|\|_{BV_{loc}}$ is uniformly bounded, then the corresponding renormalized entropy solutions converge.

This result raises the following natural question: Is system (1) well posed in BV? In other words, when the whole initial data $\overline{u}$ (and not only its absolute value $|\overline{u}|$) is in BV, does the renormalized entropy solution enjoy BV regularity? The answer to this question is no to a large extent. More precisely, in [25] it has been shown that, in 3 space dimensions, for every g which is not constant there exist bounded renormalized entropy solutions of (1) which are not in BV_{loc} but have BV initial data. These examples can be produced by starting from initial data which are arbitrarily close (both in L^∞ and BV norm) to a constant different from 0. Thus, the lack of BV regularity nor is a "large data" effect, neither is due to the degeneracy of the hyperbolicity of the system at the origin. In 2 space dimensions similar examples can be produced for a large class of fluxes g.

The same "irregularity" also holds for general entropy solutions. Indeed in [25] it is shown that, when the convex hull of the essential image of $\overline{u}$ does not contain the origin, any bounded admissible solution of (1) with BV regularity necessarily coincides with the renormalized entropy solution.

1.5. **Renormalization conjecture for nearly incompressible BV fields.** Though we can prove the wellposedness of (1) bypassing Conjecture 1.1, this conjecture remains a challenging and interesting open problem in the theory of transport equations with non–smooth coefficients. Presently we are able to show it only under some technical assumptions (the most general result concerning Bressan's compactness Conjecture is contained in [10]). One interesting case in which we are able to show Conjecture 1.1 is when we assume that the singular part of the measure $D_x \cdot b$ is concentrated on a set of codimension 1.

Our approach to Conjecture 1.1 is again through the theory of renormalized solutions à la DiPerna-Lions. Indeed, though we drop the assumption $D_x \cdot b \in L^1$, it is possible to use nonnegative solutions of the continuity equation $\partial_t \rho + D_x \cdot (\rho b) = 0$ to build a theory of renormalized solutions. In this framework, in [4] we proposed a renormalization lemma for "nearly incompressible BV coefficients" which is a natural generalization of Ambrosio's renormalization theorem. More precisely

Conjecture 1.2 (Renormalization Conjecture)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set. Assume $B \in BV \cap L^\infty(\Omega, \mathbb{R}^d)$ and $\rho \in L^\infty(\Omega)$ satisfy $D \cdot (\rho B) = 0$ and $\rho \geq C > 0$. Then, for every $u \in L^\infty(\Omega)$ such that $D \cdot (\rho u B) = 0$ and for every $\beta \in C^1$, we have $D \cdot (\rho \beta(u) B) = 0$.*

This conjectured chain rule leads naturally to investigate coupling between bounded functions and measures. Recently, in [6] the authors have shown trace theorems and regularity properties for ρ and u, coming from the equations $D \cdot (\rho B) = 0$ and $D \cdot (\rho u B) = 0$. In particular, it turns out that ρ and u possess a suitably strong notion of trace on hypersurfaces which are transversal to B. In [10] we combine these trace properties with Ambrosio's renormalization theorem to show Conjecture 8.2 when the singular part of the measure $D \cdot B$ is concentrated on a set of codimension 1.

In the general case, we decompose the measure $D \cdot B$ into the part which is absolutely continuous with respect to the Lebesgue measure and the singular part, denoted respectively by $D^a \cdot B$ and $D^s \cdot B$. Further, we follow [24] and decompose $D^s \cdot B$ into a "jump part" $D^j \cdot B$, concentrated on a set of codimension 1, and a "Cantor part" $D^c \cdot B$ (see Section 2 and [11] for the details). It turns out that $D^j \cdot B$ is concentrated on the set where the BV field B has jump–singularities (the jump set J_B), whereas the measure $D^c \cdot B$ is a singular measure of "fractal type" which is "less singular" than $D^j \cdot B$: More precisely, $|D^c \cdot B|(\Sigma) = 0$ for every set Σ of codimension 1 with finite Hausdorff measure. In this framework, the result mentioned in the previous paragraph can be restated as

- Conjecture 1.2 has a positive answer when $D^c \cdot B = 0$.

However, the results of [6] and [10] allow to handle a more general case. Indeed, one can define a notion of "transversality" between the measure $D^c \cdot B$ and the field B. In [6] the authors showed that, when $D^c \cdot B$ and B are transversal, ρ and u are approximately continuous $|D^c \cdot B|$–almost everywhere. In [10] we prove a new renormalization result, showing that Conjecture 1.2 holds whenever ρ and u are approximately continuous $|D^c \cdot B|$–a.e.. Thus we conclude that Conjecture 1.2 holds whenever $D^c \cdot B$ and B are transversal.

Unfortunately it is possible to show BV fields for which $D^c \cdot B$ and B are *not* transversal (see Section 9 and [10]). However it is not clear whether this can happen under the additional hypothesis that B is nearly incompressible.

1.6. **Plan of the paper.** In Section 2 we collect facts about measure theory and BV functions which will be relevant to our purposes, together with appropriate references on where to find their proofs. In Section 3 we develop the DiPerna–Lions theory for nearly incompressible fields. In Section 4 we prove Ambrosio's renormalization theorem and in Section 5 we use this theorem and the DiPerna–Lions theory to address the existence, uniqueness and stability of renormalized entropy solutions to the Keyfitz and Kranzer system. In Section 6 we show that the BV norm of renormalized entropy solutions blow up in a large number of cases.

In the last three sections we address the most recent results on the Renormalization Conjecture. Section 7 contains the trace properties and partial regularity of solutions to transport equations proved in [6]. Section 8 follows [10] and shows Conjecture 1.2 under the assumption that ρ and u are approximately continuous $|D^c \cdot B|$–a.e.. Finally, Section 9 contains an example of [10]: A planar BV vector field for which $D^c \cdot B$ and B are not transversal.

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2. Preliminaries

In this section we will collect some preliminary facts about measure theory and BV functions. Most of them can be found in the monograph [11].

2.1. **Notation.** When $\Omega \subset \mathbb{R}^d$, we will denote by id the identity map id $: \Omega \ni x \to x \in \mathbb{R}^d$. If $x_1, \ldots, x_d$ is a standard system of coordinates on $\mathbb{R}^d$ we denote by $\{e_i\}_{i=1,\ldots,d}$ the standard unit orthonormal vector fields such that $x = \sum_i x_i e_i$. If A and B are $k \times n$ and $n \times m$ matrices, $A \cdot B$ will denote the usual product $(k \times m)$ matrix, whereas A^t will denote the transpose of the matrix A. Vectors will usually be considered as $n \times 1$ matrices and therefore, if a and b are vectors, $a^t \cdot b$ is the usual scalar product. With a slight abuse of notation we will simply write $a \cdot b$, and similarly, if a and b are vectors and A is a matrix, we will use $a \cdot A \cdot b$ in place of $a^t \cdot A \cdot b$.

Given a vector valued map $B : \Omega \to \mathbb{R}^k$ and some system of coordinates on $\mathbb{R}^k$, with $\{e_i\}_{i=1,\ldots,k}$ orthonormal vectors, we will denote by B^i the scalar function given by $e_i \cdot B$. Whereas the subscript B_j will be always used to denote the element of a sequence $\{B_j\}_{j\in\mathbb{N}}$ of maps.

If $E \subset \mathbb{R}^d$ then we denote by $\mathbf{1}_E$ the function given by

$$\mathbf{1}_E(x) \;:=\; \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Given $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B_r(x)$ the ball of $\mathbb{R}^d$ centered at x of radius r. $\mathscr{L}^d$ denotes the Lebesgue d–dimensional measure, $\mathscr{H}^k$ denotes the usual Hausdorff k–dimensional measure, and we set $\omega_d := \mathscr{L}^d(B_1(0))$.

When μ is a measure and A a μ–measurable set, we denote by $\mu \llcorner A$ the measure given by

$$\mu \llcorner A(B) \;=\; \mu(A \cap B)\,.$$

In many case, we will deal with the Lebesgue measure $\mathscr{L}^d$ restricted on some measurable set $\Omega \subset \mathbb{R}^d$. When it will be clear from the context, to simplify the notation we will use $\mathscr{L}^d$ in place of $\mathscr{L}^d \llcorner \Omega$.

If μ on A is a measure and $f : A \to B$ is a measurable function, then we denote by $f_\# \mu$ the usual push–forward of μ, that it is, the measure on B defined by

$$\int \varphi d[f_\# \mu] \;=\; \int \varphi(f(x))\, d\mu(x) \qquad \text{for every } \varphi \in C_c(A).$$

When μ is Radon (vector–valued) measure, $|\mu|$ denotes its total variation measure. Moreover, if $E \subset \Omega$ is a Borel set and μ a Radon measure on Ω such that $|\mu|(\Omega \setminus E) = 0$, then we say that μ is concentrated on E

We say that $\eta \in C^\infty_c(\mathbb{R}^d)$ is a standard kernel if $\int \eta = 1$. Moreover, for any $\varepsilon > 0$ we denote by η_ε the function defined by $\eta_\varepsilon(x) := \varepsilon^{-d}\eta(x/\varepsilon)$. If Ω is an open subset of $\mathbb{R}^d$ and $f \in L^1(\Omega)$, then we denote by $f * \eta_\varepsilon$ the function $(f\mathbf{1}_\Omega) * \eta_\varepsilon$.

If $T \in \mathcal{D}'(\Omega)$, then we denote by $\langle T, \varphi \rangle$ the value of T on the test function $\varphi \in C_c^\infty(\Omega)$. Moreover, if η is as above, we set

$$T * \eta_\delta(y) \ := \ \langle T, \eta_\delta(\cdot - y) \rangle$$

for every $y \in \Omega$ such that $\eta_\delta(\cdot - y)$ is compactly supported in Ω. In particular, if $\tilde{\Omega} \subset\subset \Omega$ and δ is sufficiently small, $T * \eta_\delta$ defines a distribution in $\mathcal{D}'(\tilde{\Omega})$.

2.2. **Measure theory.** We now recall the following elementary results in Measure Theory (see for instance Proposition 1.62(b) of [11]):

Proposition 2.1. *Let $\{\mu_n\}_n$ be a sequence of Radon measures on $\Omega \subset \mathbb{R}^d$, which converge weakly* to μ and assume that $|\mu_n|$ converge weakly* to λ. Then $\lambda \geq |\mu|$. Moreover if E is a compact set or a bounded open set such that $\lambda(\partial E) = 0$, then $\mu_n(E) \to \mu(E)$.*

Proposition 2.2. *Let μ be a Radon measure on Ω, $\eta \in C_c^\infty(\mathbb{R}^d)$ be a standard kernel supported in the unit ball, and $\{\eta_\delta\}_\delta$ the corresponding standard family of mollifiers. Then, for any $\tilde{\Omega} \subset\subset \Omega$, $\mu * \eta_\delta$ converges weakly* to μ in $\tilde{\Omega}$ and $|\mu * \eta_\delta|$ converges weakly* to $|\mu|$ in $\tilde{\Omega}$.*

Let μ be a Radon $\mathbb{R}^k$–valued measure on Ω. By the Lebesgue decomposition theorem, μ has a unique decomposition into *absolutely continuous part* μ^a and *singular part* μ^s with respect to Lebesgue measure $\mathscr{L}^d$. Further, by the Radon-Nikodym theorem there exists a unique $f \in L^1_{\rm loc}(\Omega, \mathbb{R}^k)$ such that $\mu^a = f\mathscr{L}^d$.

One can further decompose μ^s as follows:

Proposition 2.3 (Decomposition of the singular part)**.** *If $|\mu^s|$ vanishes on any $\mathscr{H}^{d-1}$-negligible set, then μ^s can be uniquely written as a sum $\mu^c + \mu^j$ of two measures such that*
 (a) *$\mu^c(A) = 0$ for every Borel set A with $\mathscr{H}^{d-1}(A) < +\infty$;*
 (b) *$\mu^j = f\mathscr{H}^{d-1} \llcorner J_\mu$ for some Borel set J_μ σ-finite with respect to $\mathscr{H}^{d-1}$.*

The proof of this Proposition is analogous to the proof of decomposition of derivatives of BV functions (and indeed in this case the decompositions coincide), see Proposition 3.92 of [11]. In this proof, the Borel set J_μ is defined as

$$J_\mu \ := \ \left\{ x \in \Omega \ \middle| \ \limsup_{r \downarrow 0} \frac{|\mu|(B_r(x))}{r^{d-1}} \ > \ 0 \right\}. \tag{10}$$

These measures will be called, respectively, *jump part* and *Cantor part* of the measure μ. Sometimes we will use the notation μ^d for the measure $\mu^a + \mu^c$ (here the superscript d stays for "diffused").

For $B \in L^1_{\rm loc}(\Omega, \mathbb{R}^k)$ we denote by $DB = (D_i B^j)_{ij}$ the derivative in the sense of distributions of B, i.e. the $\mathbb{R}^{k \times d}$-valued distribution defined by

$$\langle D_i B^j, \varphi \rangle \ := \ -\int_\Omega B^j \frac{\partial \varphi}{\partial x_i} \, dx \quad \forall \, \varphi \in C_c^\infty(\Omega), \quad 1 \leq i \leq d, 1 \leq j \leq k\,.$$

When $\Omega \subset \mathbb{R}^d$ and $k = d$, we denote by $D \cdot B$ the distribution $\sum_i D_i B^i$. We have the following

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ and let $B \in L^\infty(\Omega, \mathbb{R}^d)$ be such that $D \cdot B$ is a Radon measure. Then $D \cdot B << \mathscr{H}^{d-1}$.*

Thanks to this lemma, for any bounded vector field B such that $D\cdot B$ is a Radon measure, we can apply the decomposition of Definition 2.3 to $D\cdot B$. Therefore we will denote by $D^a\cdot B$, $D^c\cdot B$, and $D^j\cdot B$ respectively the absolutely continuous part, Cantor part and jump part of $D\cdot B$. Moreover we will sometimes use $D^s\cdot B$ for $D^c\cdot B+D^j\cdot B$ and $D^d\cdot B$ for $D^a\cdot B+D^c\cdot B$.

Proof of Lemma 2.4. We will show that $|[D\cdot B]\,(B_r(x))| \le \|B\|_\infty \omega_{d-1} r^{d-1}$ for every ball $B_r(x) \subset\subset \Omega$. This implies the claim by a standard covering argument (see for instance Theorem 2.56 of [11]). Therefore let $x\in\Omega$ be given and fix a smooth nonnegative kernel $\eta \in C^\infty_c(\mathbb{R}^d)$. Consider $\mu_\varepsilon := D\cdot(B*\eta_\varepsilon) = (D\cdot B)*\eta_\varepsilon$. Then $\mu_\varepsilon \rightharpoonup^* D\cdot B$ on any set $\tilde\Omega \subset\subset \Omega$. Note that for any fixed $B_r(x)\subset\subset\Omega$ we have

$$
\begin{aligned}
|\mu_\varepsilon(B_r(x))| &= \left|\int_{B_r(x)} D_x\cdot(B*\eta_\varepsilon)(x)\,dx\right| \\
&= \left|\int_{\partial B_r(x)} B*\eta_\varepsilon\cdot\nu\right| \;\le\; \|B*\eta_\varepsilon\|_\infty \omega_{d-1} r^{d-1} \;\le\; \|B\|_\infty \omega_{d-1} r^{d-1}\,.
\end{aligned}
$$

Define $S \subset]0, \operatorname{dist}(x,\partial\Omega)[$ as the set of radii ρ such that $|D\cdot B|(\partial B_r(x)) > 0$, which is at most countable. Since $\mu_\varepsilon \rightharpoonup^* D\cdot B$, for any $r\in]0,\operatorname{dist}(x,\partial\Omega)[\,\setminus S$ we have

$$
|[D\cdot B]\,(B_r(x))| \;=\; \lim_{\varepsilon\downarrow 0} |\mu_\varepsilon(B_r(x))| \;\le\; |B\|_\infty \omega_{d-1} r^{d-1}\,.
$$

Moreover, since S is at most countable, for any $r\in S$ there exists $\{r_n\}\subset\,]0,\operatorname{dist}(x,\partial\Omega)[\,\setminus S$ such that $r_n\uparrow\rho$. Therefore

$$
|[D\cdot B]\,(B_r(x))| \;=\; \lim_{r_n\uparrow r} |[D\cdot B]\,(B_{r_n}(x))| \;\le\; \|B\|_\infty \omega_{d-1} r^{d-1}\,.
$$

□

2.3. **Approximate continuity and approximate jumps.** The L^1*–approximate discontinuity set* $S_B \subset \Omega$ of a locally summable $B:\Omega\to\mathbb{R}^k$ and the *Lebesgue limit* are defined as follows: $x\notin S_B$ if and only if there exists $z\in\mathbb{R}^k$ satisfying

$$
\lim_{r\downarrow 0} r^{-d}\int_{B_r(x)} |B(y)-z|\,dy = 0.
$$

The vector z, if it exists, is unique and denoted by $\tilde B(x)$, the Lebesgue limit of B at x. It is easy to check that the set S_B is Borel and that $\tilde B$ is a Borel function in its domain (see §3.6 of [11] for details). By Lebesgue differentiation theorem the set S_B is Lebesgue negligible and $\tilde B = B$ $\mathscr{L}^d$-a.e. in $\Omega\setminus S_B$.

In a similar way one can define the L^1*–approximate jump set* $J_B\subset S_B$, by requiring the existence of a, $b\in\mathbb{R}^k$ with $a\neq b$ and of a unit vector ν such that

$$
\lim_{r\downarrow 0} r^{-d}\int_{B_r^+(x,\nu)} |B(y)-a|\,dy = 0, \qquad \lim_{r\downarrow 0} r^{-d}\int_{B_r^-(x,\nu)} |B(y)-b|\,dy = 0,
$$

where

$$B_r^+(x,\nu) := \{y \in B_r(x): \ \langle y-x,\nu\rangle > 0\}\,,$$
$$B_r^-(x,\nu) := \{y \in B_r(x): \ \langle y-x,\nu\rangle < 0\}\,. \tag{11}$$

The triplet (a,b,ν), if it exists, is unique up to a permutation of a and b and a change of sign of ν, and denoted by $(B^+(x),B^-(x),\nu(x))$, where $B^\pm(x)$ are called *Lebesgue one-sided limits* of B at x. It is easy to check that the set J_B is Borel and that $B^\pm$ and ν can be chosen to be Borel functions in their domain (see again §3.6 of [11] for details).

2.4. BV functions.

Definition 2.5 (BV functions)**.** *We say that $B \in L^1(\Omega;\mathbb{R}^k)$ has bounded variation in Ω, and we write $B \in BV(\Omega;\mathbb{R}^k)$, if DB is representable by a $\mathbb{R}^{k\times d}$-valued measure, still denoted by DB, with finite total variation in Ω.*

It is a well known fact that for $B \in BV$ one has $D_iB^j << \mathscr{H}^{d-1}$ (for instance it follows directly from Lemma 2.4 applied to the vector field $U = B^je_i$). Therefore we can apply the decomposition of Section 2.1 to the measure DB and we will use the notation D^aB, D^cB, and D^jB, respectively for the absolutely continuous part, Cantor part, and jump part of DB. Moreover we will denote by D^sB and D^dB respectively the measures $D^cB + D^jB$ and $D^aB + D^cB$.

Next we recall the fine properties of $\mathbb{R}^k$-valued BV functions defined in an open set $\Omega \subset \mathbb{R}^d$.

First of all we need the definition of rectifiable sets.

Definition 2.6 (Countably $\mathscr{H}^{d-1}$-rectifiable sets)**.** *We say that $\Sigma \subset \mathbb{R}^d$ is countably $\mathscr{H}^{d-1}$-rectifiable if there exist (at most) countably many C^1 embedded hypersurfaces $\Gamma_i \subset \mathbb{R}^d$ such that*

$$\mathscr{H}^{d-1}\left(\Sigma \setminus \bigcup_i \Gamma_i\right) = 0.$$

A Borel map $\nu : \Sigma \to \mathbf{S}^{d-1}$ is normal to Σ if $\nu(x)$ is normal to Γ_i for $\mathscr{H}^{d-1}$–a.e. $x \in \Gamma_i \cap \Sigma$.

Denoting by $\zeta \otimes \xi$ the linear map from $\mathbb{R}^d$ to $\mathbb{R}^k$ defined by $v \mapsto \zeta\langle \xi, v\rangle$, the following structure theorem holds (see for instance Theorem 3.77 and Proposition 3.92 of [11]):

Theorem 2.7 (BV structure theorem)**.** *If $B \in BV_{\mathrm{loc}}(\Omega,\mathbb{R}^k)$, then $\mathscr{H}^{d-1}(S_B \setminus J_B) = 0$ and J_B is a countably $\mathscr{H}^{d-1}$–rectifiable set. Moreover*

$$D^jB \ = \ (B^+ - B^-) \otimes \nu \mathscr{H}^{d-1} \llcorner J_B\,, \tag{12}$$

and ν is normal to Σ.

As a corollary, since D^aB and D^cB are both concentrated on $\Omega \setminus S_B$, we conclude that $|D^aB| + |D^cB| = |D^dB|$–a.e. x is a Lebesgue point for B, with value $\tilde{B}(x)$. The space of functions of special bounded variation (denoted by SBV) is defined as follows:

Definition 2.8 (SBV)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set. The space $SBV(\Omega,\mathbb{R}^m)$ is the set of all $u \in BV(\Omega,\mathbb{R}^m)$ such that $D^cu = 0$.*

2.5. **Caccioppoli sets and Coarea formula.** We say that $A \subset \Omega$ is a Caccioppoli set if $\mathbf{1}_A \in BV(\Omega)$. Then, as a particular case of Theorem 2.7, we conclude that there exists a rectifiable set F such that:

- For $\mathscr{H}^{d-1}$–a.e. $x \in F$ the Lebesgue limit of $\mathbf{1}_A$ is either 0 or 1;
- Every $x \in F$ is an approximate jump point for $\mathbf{1}_A$ such that $\mathbf{1}_A^+(x) = 1$, $\mathbf{1}_A^-(x) = 0$ and ν is normal to F;
- $D^j \mathbf{1}_A = \nu \mathscr{H}^{d-1} \llcorner F$.

F is called the reduced boundary of A and denoted by $\mathcal{F}A$ (see Section 3.5 of [11]). ν is called the approximate exterior unit normal to A. An additional important fact is that $D^c \mathbf{1}_A = D^a \mathbf{1}_A = 0$. More precisely we have (cp. with Theorem 3.59 of [11])

Theorem 2.9 (De Giorgi's rectifiability Theorem). *If A is a Caccioppoli set, then $D\mathbf{1}_A = D^j \mathbf{1}_A = \nu \mathscr{H}^{d-1} \llcorner \partial^* A$.*

Thus, $\mathscr{H}^{d-1}(A) = |D\mathbf{1}_A|(\Omega) < \infty$.

A second important tool of the theory of BV functions is the coarea formula. Before stating it, we introduce the following notation. Assume that $[a, b] \ni t \mapsto \mu_t$ is a map which takes values on the space of $\mathbb{R}^k$–valued measures. We say that this map is weakly* measurable if for every test function $\varphi \in C_c(\Omega, \mathbb{R}^k)$, the map $t \mapsto \int \varphi \cdot d\mu_t$ is measurable. If $\int |\mu_t|(\Omega)\, dt$ is finite, then we denote by $\int \mu_t\, dt$ the measure μ defined by

$$\int \varphi \cdot d\mu \;:=\; \int \left(\int \varphi \cdot d\mu_t \right) dt\,.$$

Then we have (cp. with Theorem 3.40 of [11])

Theorem 2.10 (Coarea formula). *Let $u \in BV(\Omega)$ be a scalar BV function. For $t \geq 0$ we set $\Omega_t := \{u > t\}$ and for $t < 0$ we set $\Omega_t := \{u < t\}$. Then Ω_t is a Caccioppoli set for $\mathscr{L}^1$–a.e. t, $t \mapsto D\mathbf{1}_{\Omega_t}$ is a weakly* measurable, and $\int |D\mathbf{1}_{\Omega_t}|(\Omega)\, dt < \infty$. Moreover*

$$Du \;=\; \int_0^\infty D\mathbf{1}_{\Omega_t} - \int_0^\infty D\mathbf{1}_{\Omega_{-t}} \tag{13}$$

$$|Du| \;=\; \int_{-\infty}^\infty \mathscr{H}^{d-1} \llcorner \Omega_t\, dt\,. \tag{14}$$

2.6. **The Volpert Chain rule.** Next, note that, if $B \in BV(\Omega, \mathbb{R}^k)$ and $H \in W^{1,\infty}(\mathbb{R}^k, \mathbb{R}^m)$, then $H \circ B \in BV_{loc}(\Omega, \mathbb{R}^m)$. Indeed, let $\{B_n\}_n$ be any sequence of smooth functions such that $B_n \to B$ strongly in L^1 and

$$\limsup_{n\uparrow\infty} \int_\Omega |\nabla B^n(x)|\, dx \;<\; \infty\,.$$

Clearly, $H \circ B_n \to H \circ B$ strongly in L^1 and

$$\begin{aligned}\limsup_{n\uparrow\infty} \int_\Omega |\nabla[H \cdot B_n](x)|\,dx &= \limsup_{n\uparrow\infty} \int_\Omega |\nabla H(B_n(x)) \cdot \nabla B(x)|\,dx \\ &\le \|\nabla H\|_\infty \limsup_{n\uparrow\infty} \int_\Omega |\nabla B_n(x)|\,dx \;<\; \infty\,.\end{aligned}$$

Therefore $D[H \cdot B]$ is a Radon measure. In addition, if $H \in C^1$, then the following chain rule, first proved by Vol'pert, holds (see Theorem 3.96 of [11]).

Theorem 2.11. *Let $u \in BV(\Omega, \mathbb{R}^k)$ and $H \in C^1(\mathbb{R}^k, \mathbb{R}^m)$. Then,*

$$D[H \circ u] \;=\; [\nabla H \circ \tilde{u}] \cdot D^d u \;+\; \left\{[H(u^+) - H(u^-)] \otimes \nu\right\} \mathscr{H}^{d-1} \llcorner J_u\,. \tag{15}$$

Remark 2.12. *In [7] the authors proved a suitable extension of Theorem 2.11 to $H \in W^{1,\infty}$. In what follows we will sometimes consider the measure $D[H \circ u]$ for H which indeed are $W^{1,\infty}$ but not C^1. However we will not need the general result of [7], since in all the cases considered in this paper we will be able to use some "ad hoc" considerations.*

2.7. **Alberti's Rank–one Theorem.** In [1] Alberti proved the following deep result:

Theorem 2.13 (Alberti's rank one theorem)**.** *Let $B \in BV_{\rm loc}(\Omega, \mathbb{R}^k)$. Then there exist Borel functions $\xi : \Omega \to \mathbf{S}^{d-1}, \zeta : \Omega \to \mathbf{S}^{k-1}$ such that*

$$D^s B \;=\; \zeta \otimes \xi |D^s B|\,. \tag{16}$$

Clearly, if we replace $D^s B$ with $D^j B$ in (16), this conclusion can be easily drawn from Theorem 2.7. However, in order to prove the same for the *full* singular part of DB, many new interesting ideas were introduced in [1] (see also [26] for a recent description of Alberti's proof).

3. DiPerna–Lions theory for nearly incompressible flows

In this section we develop a theory à la DiPerna–Lions for transport equations and ordinary differential equations, in which the usual assumption of boundedness of the divergence of the coefficients is replaced by a control on the Jacobian (or by the existence of a solution of the continuity equation which is bounded away from 0 and ∞).

3.1. **Lagrangian flows.**

Definition 3.1. *Let $b \in L^\infty([0,\infty[\times\mathbb{R}^m, \mathbb{R}^m)$. A map $\Phi : [0,\infty[\times\mathbb{R}^m \to \mathbb{R}^m$ is a regular Lagrangian flow for b if*

(a) *For $\mathscr{L}^1$–a.e. t we have $|\{x : \Phi(t,x) \in A\}| = 0$ for every Borel set A with $|A| = 0$;*
(b) *The following identity is valid in the sense of distributions*

$$\begin{cases} \partial_t \Phi(t,x) \;=\; b(t, \Phi(t,x)) \\[2ex] \Phi(0,x) = x\,. \end{cases} \tag{17}$$

The identity (17) in the sense of distributions means that for every $\psi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^m, \mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} \psi(0,x)\cdot x\,dx + \int_0^\infty \int_{\mathbb{R}^m} \Phi(t,x)\cdot \partial_t\psi(t,x)\,dt\,dx \;=\; -\int_0^\infty\int_{\mathbb{R}^m} \psi(t,x)\cdot b(t,\Phi(t,x))\,dt\,dx\,. \tag{18}$$

Note that assumption (a) guarantees that $b(t,\Phi(t,x))$ is well defined. More precisely, if $\hat{b}(t,x) = b(t,x)$ for $\mathscr{L}^{m+1}$–a.e. (t,x), then $\hat{b}(t,\Phi(t,x)) = b(t,\Phi(t,x))$ for $\mathscr{L}^{m+1}$–a.e. (t,x).

Moreover, it is easy to check that if Φ is a regular Lagrangian flow and $\Psi(t,x) = \Phi(t,x)$ for $\mathscr{L}^{m+1}$–a.e. (t,x), then Ψ is as well a regular Lagrangian flow.

The following Lemma has a standard proof:

Lemma 3.2. *Let Φ be a regular Lagrangian flow. Then, $\Phi(\cdot,x)\in W^{1,\infty}_{loc}([0,\infty[)$ for $\mathscr{L}^m$–a.e. x and, if we denote by Φ_x the Lipschitz function such that $\Phi_x(t) = \Phi(t,x)$ for $\mathscr{L}^1$–a.e. t, then:*

- $\mathrm{Lip}\,(\Phi_x) \leq \|b\|_\infty$.
- $\Phi_x(0) = x$.
- $\Phi'_x(t) = b(t,\Phi_x(t))$ *for $\mathscr{L}^1$–a.e. t.*

As an easy corollary we get

Corollary 3.3. *Let Φ be a regular Lagrangian flow. Then, for any Borel set A and $\mathscr{L}^1$–a.e. $T>0$ we have*

$$\int_A |\Phi(T,x) - x|\,dx \;\leq\; \|b\|_\infty T|A|\,. \tag{19}$$

From now on we denote by μ_Φ the measure $(\mathrm{id}\,,\Phi)_\#\mathscr{L}^{m+1}\llcorner([0,\infty[\times\mathbb{R}^m)$, that is the push forward via the map $(t,x)\mapsto(t,\Phi(t,x))$ of the Lebesgue $m+1$–dimensional measure on $[0,\infty[\times\mathbb{R}^m$. Thus,

$$\int_{[0,\infty[\times\mathbb{R}^m} \psi(t,x)\,d\mu_\Phi(t,x) \;=\; \int_{[0,\infty[\times\mathbb{R}^m} \psi(t,\Phi(t,x))\,d\mathscr{L}^{m+1}(t,x)$$

for every $\psi\in C^c(\mathbb{R}\times\mathbb{R}^m)$.

Having introduced μ_Φ, (a) is equivalent to

$$\mu_\Phi \;<<\; \mathscr{L}^{m+1}\,. \tag{20}$$

Thus for every regular Lagrangian flow Φ there exists a $\rho\in L^1_{loc}([0,\infty[\times\mathbb{R}^n)$ such that $\mu_\Phi = \rho\mathscr{L}^{m+1}$.

Definition 3.4. *This ρ will be called the* density *of the flow Φ, and by definition it satisfies the following "change of variables" identity*

$$\int \psi(t,\Phi(t,x))\,dt\,dx \;=\; \int \psi(t,x)\rho(t,x)\,dt\,dx \tag{21}$$

for every test function $\psi\in L^\infty$ and with bounded support.

The next proposition shows the connections between regular Lagrangian flows and solutions of transport and continuity equations with coefficient b.

Proposition 3.5. *Let Φ be a regular Lagrangian flow for a field b.*

(i) *Let $\overline{\zeta} \in L^\infty(\mathbb{R}^n)$ and consider the measure μ on $[0,\infty[\times T$ given by $(\mathrm{id}\,,\Phi)_\#(\overline{\zeta}\mathscr{L}^{m+1})$, that is*

$$\int \varphi(t,x)\,d\mu(t,x) \;=\; \int_A \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx \qquad \textit{for every Borel set } A.$$

Then there exists $\zeta \in L^1([0,\infty[\times\mathbb{R}^m)$ such that $\mu = \zeta\mathscr{L}^{m+1}$. Moreover, ζ satisfies the following equation in the sense of distributions:

$$\left\{\begin{array}{rcl} \partial_t\zeta + D_x\cdot(\zeta b) &=& 0 \\ \\ \zeta(0,\cdot) &=& \overline{\zeta}\,. \end{array}\right. \tag{22}$$

(ii) *Let ρ be the density of the flow Φ. If $u \in L^\infty([0,T[\times\mathbb{R}^m)$ and $\overline{u} \in L^\infty(\mathbb{R}^m)$ satisfy the identity*

$$u(t,\Phi(t,x)) \;=\; \overline{u}(x) \qquad \textit{for } \mathscr{L}^{m+1}\textit{–a.e. } (t,x), \tag{23}$$

then the following equation holds in the sense of distributions

$$\left\{\begin{array}{rcl} \partial_t(\rho u) + D_x\cdot(u\rho b) &=& 0 \\ \\ u(0,\cdot)\rho(0,\cdot) &=& \overline{u}\,. \end{array}\right. \tag{24}$$

Thus, as a particular case of this proposition, we get the usual continuity equation satisfied by the density ρ of flows of regular vector fields:

$$\left\{\begin{array}{rcl} \partial_t\rho + D_x\cdot(\rho b) &=& 0 \\ \\ \rho(0,\cdot) &=& 1\,. \end{array}\right. \tag{25}$$

Proof. First of all note that (ii) follows from (i). Indeed, let u and $\overline{u}$ be given as in (ii). Set $\overline{\zeta} := \overline{u}$ and $\zeta := u\rho$. For every L^∞ function with bounded support φ we have

$$\begin{aligned} \int u(t,x)\rho(t,x)\varphi(t,x)\,dt\,dx \;&=\; \int u(t,\Phi(t,x))\varphi(t,\Phi(t,x))\,dt\,dx \\ &=\; \int \overline{u}(x)\varphi(t,\Phi(t,x))\,dt\,dx\,. \end{aligned}$$

Thus, if μ is defined as in (i), then $\zeta\mathscr{L}^{m+1} = \mu$. Therefore (i) gives (22), from which we get (24).

We now come to the proof of (i). First of all set note that

$$\begin{aligned} |\mu(A)| &= \left|\int \overline{\zeta}(x)\mathbf{1}_A(t,\Phi(t,x))\,dt\,dx\right| \\ &\leq \|\overline{\zeta}\|_\infty \int \mathbf{1}_A(t,\Phi(t,x))\,dt\,dx \;\leq\; \|\overline{\zeta}\|_\infty \int_A \rho(t,x)\,dt\,dx\,. \end{aligned}$$

Since $\rho \in L^1$, this means that μ is absolutely continuous. Therefore there exists an L^1 function ζ such that $\mu = \zeta\mathscr{L}^{m+1}$. Now, let $\psi \in C^\infty(\mathbb{R}\times\mathbb{R}^m)$ be any given test function. Our goal is to show that

$$-\int_{[0,\infty[\times\mathbb{R}^n} \zeta(t,x)\big(\partial_t\psi(t,x) + b(t,x)\cdot\nabla_x\psi(t,x)\big)\,dx\,dt \;=\; \int_{\mathbb{R}^n} \overline{\zeta}(x)\psi(0,x)\,dx\,. \tag{26}$$

By definition, the left hand side of (26) is equal to

$$-\int_{\mathbb{R}^n} \overline{\zeta}(x)\left[\int_0^\infty \big(\partial_t\psi(t,\Phi(t,x)) + \nabla_x\psi(t,\Phi(t,x))\cdot b(t,\Phi(t,x))\big)\,dt\right]\,dx\,. \tag{27}$$

We conclude the proof by showing that, for any x for which the conclusion of Lemma 3.2 applies, we have

$$-\psi(0,x) \;=\; \int_0^\infty \big(\partial_t\psi(t,\Phi_x(t)) + \nabla_x\psi(t,\Phi_x(t))\cdot\Phi_x'(t)\big)\,dt\,.$$

For such x the integral in t in (27) is given by

$$\int_0^\infty \big(\partial_t\psi(t,\Phi_x(t)) + \nabla_x\psi(t,\Phi_x(t))\cdot\Phi_x'(t)\big)\,dt\,.$$

Since Φ_x is Lipschitz and ψ is a smooth function, $\psi(\cdot,\Phi_x(\cdot))$ is a Lipschitz function of t. Therefore, $\psi(\cdot,\Phi_x(\cdot))$ and $\Phi_x(\cdot)$ are both differentiable at $\mathscr{L}^1$–a.e. t, and the identity given by the usual chain rule

$$\partial_t\psi(t,\Phi_x(t)) + \nabla_x\psi(t,\Phi_x(t))\cdot\Phi_x'(t) \;=\; \frac{d}{dt}\big(\psi(t,\Phi_x(t))\big)$$

is valid for a.e. t. Moreover, note that

- $\psi(0,\Phi_x(0)) = \psi(0,x)$;
- $\psi(T,\Phi_x(T)) = 0$ for T large enough, since η has bounded support.

Therefore we conclude

$$\int_0^\infty \big(\partial_t\psi(t,\Phi(t,x)) + \nabla_x\psi(t,\Phi(t,x))\cdot b(t,\Phi(t,x))\big)\,dt \;=\; -\psi(0,x)\,. \tag{28}$$

□

3.2. Nearly incompressible fields and fields with the renormalization property.

Definition 3.6. *We say that a field* $b \in L^\infty([0,\infty[\times\mathbb{R}^m,\mathbb{R}^m)$ *is nearly incompressible if there exists a function* $\rho \in L^\infty([0,\infty[\times\mathbb{R}^m)$ *and a positive constant* C *such that* $C^{-1} \le \rho \le C$ *and*

$$\partial_t \rho + D_x \cdot (\rho b) \;=\; 0 \tag{29}$$

in the sense of distributions.

The following lemma has a standard proof.

Lemma 3.7. *If* ρ *is bounded and satisfies* (29)*, then, after possibly modifying it on a set of measure zero,* $[0,1] \ni t \mapsto \rho(t,\cdot) \in L^\infty$ *is a weakly* continuous map*

Remark 3.8. *As a consequence of Lemma 3.7 we get the following useful fact. Given any* $\zeta \in C^\infty_c(]0,\infty[)$ *with* $\int \zeta = 1$*, if we denote by* $\{\zeta_\varepsilon\}$ *the standard family of mollifiers generated by* ζ*, then the functions*

$$\int_0^\infty \zeta_\varepsilon(t)\rho(t,x)\,dt$$

converge weakly in* L^∞ *to* $\rho(0,\cdot)$*.*

Proof. We claim that

(Cl) For every $\varphi \in C^\infty_c(\mathbb{R}^m)$ the functions

$$f^T_\varphi(t) \;:=\; \int_{\mathbb{R}^m} \frac{1}{T}\int_t^{T+t} \rho(s,x)\varphi(x)\,ds\,dx$$

are uniformly continuous.

This claim implies the Lemma. Indeed, let $\varphi \in C^\infty_c(\mathbb{R}^m)$. Then from (Cl) we conclude that $\{f^T_\varphi\}_{0<T<1}$ is precompact in $C([0,R])$ for every $R > 0$. Let f denote any limit of a subsequence $\{f^{T_k}_\varphi\}$ with $T_k \downarrow 0$. Then we have

$$\int f(t)\psi(t)\,dt \;=\; \int \rho(t,x)\varphi(x)\psi(t)\,dt\,dx$$

for every $\psi \in C^\infty_c(\mathbb{R})$. Therefore we conclude that f^T_φ is converging (uniformly on compact sets) to a unique $f^0_\varphi \in C([0,\infty[)$, as $T \to 0$.

It is clear that $|f^0_\varphi(t)| \le \|\rho\|_\infty \|\varphi\|_{L^1}$ and that $f^0_{a\varphi+b\psi}(t) = af^0_\varphi(t) + bf^0_\psi(t)$. Therefore for each t there exists a unique $\rho_t \in L^\infty$ such that

$$\int \rho_t(x)\varphi(x)\,dx \;=\; f^0_\varphi(t) \qquad \text{for every } \varphi \in C^\infty_c(\mathbb{R}^m).$$

Since $C^\infty_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, the map $t \mapsto \rho_t$ is weakly* continuous. Moreover, for any test function $\psi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^m)$ we have

$$\int \rho_t(x)\psi(t,x)\,dt\,dx \;=\; \int \rho(t,x)\psi(t,x)\,dt\,dx\,.$$

It remains to show (Cl). Therefore, let $\varphi \in C^\infty_c(\mathbb{R}^n)$ be any given test function. For every $0 < T < 1$ consider

$$\chi_T(t) := \begin{cases} t/T & \text{for } t \in [0,T] \\ 1 & \text{for } t \in [T,1] \\ 2-t & \text{for } t \in [1,2] \\ 0 & \text{for } t \geq 2. \end{cases}$$

Set $\psi_T(t,\tau,x) := \chi_T(\tau - t)\varphi(x)$. It is not difficult to see that

$$\int \rho(\tau,x)(\partial_t \psi_T(t,\tau,x) + b(\tau,x)\cdot\nabla_x \psi_T(t,\tau,x))\,d\tau\,dx = 0\,,$$

from which we get

$$\begin{aligned} f_T(t) &= \frac{1}{T}\int_t^{t+T}\int_{\mathbb{R}^m} \rho(\tau,x)\varphi(x)\,dx\,d\tau = \int_{t+1}^{t+2}\int_{\mathbb{R}^n} \rho(\tau,x)\varphi(x)\,dx\,d\tau \\ &\quad - \int_0^\infty \int_{\mathbb{R}^m} \rho(\tau,x)\chi_T(\tau-t)\nabla\varphi(x)\cdot b(\tau,x)\,dx\,d\tau\,. \end{aligned}$$

From this identity we easily conclude that $\{f_T\}_{0<T<1}$ is uniformly continuous. □

Definition 3.9. *We say that a pair $b \in L^\infty([0,\infty[\times\mathbb{R}^m,\mathbb{R}^m)$, $\rho \in L^\infty([0,\infty[\times\mathbb{R}^m)$ have the renormalization property if ρ satisfies* (29) *and the following property holds:*

(R) *For every $T > 0$ and for every bounded u which solves*

$$\begin{cases} \partial_t(\rho u) + D_x\cdot(u\rho b) = 0 \\ [u\rho](0,\cdot) = \rho(0,\cdot)\overline{u} \\ [u\rho](T,\cdot) = \rho(T,\cdot)\hat{u}\,. \end{cases} \tag{30}$$

$v := u^2$ solves

$$\begin{cases} \partial_t(\rho v) + D_x\cdot(v\rho b) = 0 \\ [v\rho](0,\cdot) = \rho(0,\cdot)\overline{u}^2 \\ [v\rho](T,\cdot) = \rho(T,\cdot)\hat{u}^2\,. \end{cases} \tag{31}$$

In the previous definition $\rho(0,\cdot)$ and $\rho(T,\cdot)$ are the traces of ρ given by Lemma 3.7, and the identity (30) means that for every test function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^m)$ we have

$$\begin{aligned} &\int_{[0,\infty\times\mathbb{R}^m} \rho(t,x)u(t,x)\big(\partial_t\varphi(t,x) + b(t,x)\cdot\nabla\varphi(t,x)\big)\,dt\,dx \\ &= \int_{\mathbb{R}^m} \big(\rho(T,x)\hat{u}(x)\varphi(T,x) - \rho(0,x)\overline{u}(x)\varphi(0,x)\big)\,dx\,. \end{aligned}$$

The following proposition holds

Proposition 3.10. *Assume that (b,ρ) have the renormalization property. Then:*

(GR) *For every finite family of bounded solutions $\{u^i\}_{i=1,\dots,N}$ of*

$$\begin{cases} \partial_t(\rho u^i) + D_x \cdot (u^i \rho b) &= 0 \\ [u^i\rho](0,\cdot) &= \rho(0,\cdot)\overline{u}^i \\ [u^i\rho](T,\cdot) &= \rho(T,\cdot)\hat{u}^i\,, \end{cases} \tag{32}$$

and any $H \in C(\mathbb{R}^N)$, $v := H(u)$ solves

$$\begin{cases} \partial_t(\rho v) + D_x \cdot (v \rho b) &= 0 \\ [v\rho](0,\cdot) &= \rho(0,\cdot)H(\overline{u}) \\ [v\rho](T,\cdot) &= \rho(T,\cdot)H(\hat{u})\,. \end{cases} \tag{33}$$

Proof. Note that the claim is always true when H is a linear function. Moreover, since $u^1u^2 = ((u^1+u^2)^2 - (u^1)^2 - (u^2)^2)/2$, from the renormalization property (R) we conclude that

$$\text{(GR) holds for } N = 2 \text{ and } H(u^1,u^2) = u^1u^2. \tag{34}$$

Using inductively (34) we get that

$$\text{(GR) holds whenever } H \text{ is a polynomial.} \tag{35}$$

In order to prove the general case, let u and H be given as in the statement of the proposition. By Stone-Weierstrass there exists a sequences of polynomials $H_k : \mathbb{R}^N \to \mathbb{R}$ such that $H_k \to H$ uniformly on $\overline{B}_{\|u\|_\infty}(0) \subset \mathbb{R}^N$. From (35) we get

$$\begin{cases} \partial_t(\rho H_k(u)) + D_x \cdot (H_k(u)\rho b) &= 0 \\ [H_k(u)\rho](0,\cdot) &= \rho(0,\cdot)H_k(\overline{u}) \\ [H_k(u)\rho](T,\cdot) &= \rho(T,\cdot)H_k(\hat{u})\,, \end{cases} \tag{36}$$

and letting $k \uparrow \infty$ we conclude (33). □

Corollary 3.11. *Let b a bounded nearly incompressible vector field with the renormalization property, and assume that ρ is as in Definitions 3.6 and 3.9. If ζ is any other function such that $0 < C^{-1} \le \zeta \le C$ and $\partial_t\zeta + D_x \cdot (\zeta b) = 0$, then (GR) also holds with ζ in place of ρ.*

This corollary justifies the following

Definition 3.12. *We say that a bounded nearly incompressible vector field b has the* renormalization property *if there exists a ρ as in Definition 3.6 such that the pair (b,ρ) has the renormalization property of Definition 3.9*

Proof of Corollary 3.11. Let $\{u^i\}_{i=1,\dots,N}$ be any given solutions of

$$\begin{cases} \partial_t(\overline{\zeta} u^i) + D_x\cdot(u^i\zeta b) \;=\; 0 \\[2mm] [u^i\zeta]\,(0,\cdot) \;=\; \zeta(0,\cdot)\overline{u}^i \\[2mm] [u^i\zeta]\,(T,\cdot) \;=\; \zeta(T,\cdot)\hat{u}^i \end{cases} \tag{37}$$

Next, let $v^{n+1} := \zeta/\rho$, $\overline{v}^{n+1} := \zeta(0,\cdot)/\rho(0,\cdot)$, and $\hat{v}^{n+1} := \zeta(T,\cdot)/\rho(T,\cdot)$. Then define $v^i := u^i/v^{n+1}$, $\overline{v}^i := \overline{u}^i/\overline{v}^{n+1}$, and $\hat{v}^i := \hat{u}^i/\hat{v}^{n+1}$. Note that

$$\begin{cases} \partial_t(\rho v^i) + D_x\cdot(v^i\rho b) \;=\; 0 \\[2mm] [v^i\rho]\,(0,\cdot) \;=\; \rho(0,\cdot)\overline{v}^i \\[2mm] [v^i\zeta]\,(T,\cdot) \;=\; \zeta(T,\cdot)\hat{v}^i,. \end{cases} \tag{38}$$

Given $H \in C(\mathbb{R}^N)$, we define $\hat{H} \in C(\mathbb{R}^{N+1})$ by $\hat{H}(v) := v^{n+1}H(v^1v^{n+1},\dots,v^nv^{n+1})$. Since (GR) holds, we conclude

$$\begin{cases} \partial_t(\rho\hat{H}(v)) + D_x\cdot(\hat{H}(v)\rho b) \;=\; 0 \\[2mm] \left[\hat{H}(v)\rho\right](0,\cdot) \;=\; \rho(0,\cdot)\hat{H}(\overline{v}) \\[2mm] \left[\hat{H}(v)\rho\right](T,\cdot) \;=\; \rho(T,\cdot)\hat{H}(\hat{v})\,. \end{cases} \tag{39}$$

On the other hand, from the definitions of v and $\hat{H}$, we have

$$\rho\hat{H}(v) \;=\; \zeta H(u)\,, \qquad \rho(0,\cdot)\hat{H}(\overline{v}) \;=\; \zeta(0,\cdot)H(\overline{u}) \qquad \text{and} \qquad \rho(T,\cdot)\hat{H}(\hat{v}) \;=\; \zeta(T,\cdot)H(\hat{u})\,.$$

□

3.3. **Existence and uniqueness of solutions to transport equations.**

Proposition 3.13. *Assume b is a bounded vector field and ρ is a nonnegative function which satisfies* (29)*. Then for every bounded $\overline{u}$ there exists a solution of*

$$\begin{cases} \partial_t(\rho u) + D_x\cdot(u\rho b) \;=\; 0 \\[2mm] [u\rho]\,(0,\cdot) \;=\; \overline{u\rho}\,. \end{cases} \tag{40}$$

Assume, moreover that the pair (b,ρ) has the renormalization property. If u_1 and u_2 solve

$$\begin{cases} \partial_t(\rho u_i) + D_x\cdot(u_i\rho b) \;=\; 0 \\[2mm] [u_i\rho]\,(0,\cdot) \;=\; \overline{u}_i\rho(0,\cdot)\,, \end{cases} \tag{41}$$

and $\overline{u}_1 \geq \overline{u}_2$, then $\rho u_1 \geq \rho u_2$.

The following are easy corollaries of Proposition 3.13.

Corollary 3.14. *If b is a bounded nearly incompressible vector field with the renormalization property and ρ is as in Definition 3.6, then for every bounded $\overline{u}$ there exists a unique bounded solution u of (40). Moreover, after possibly changing u on a set of measure zero, the map $t \mapsto u(t,\cdot)$ is continuous in the strong topology of L^1_{loc}.*

Corollary 3.15. *Let $\overline{\zeta} \in L^\infty(\mathbb{R}^m)$. If b is a bounded nearly incompressible vector field with the renormalization property, then there exists a unique bounded distributional solution ζ of*

$$\begin{cases} \partial_t \zeta + D_x \cdot (\zeta b) \;=\; 0 \\ \\ \zeta(0,\cdot) \;=\; \overline{\zeta}\,. \end{cases} \tag{42}$$

Moreover, if $\overline{\zeta}$ is bounded away from zero, so is ζ.

This justifies the following

Definition 3.16. *Let b be a bounded nearly incompressible vector field with the renormalization property. Then the* density *generated by b is the unique solution of*

$$\begin{cases} \partial_t \rho + D_x \cdot (\rho b) \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; 1\,. \end{cases} \tag{43}$$

Moreover note that, if Φ is a regular Lagrangian flow for b, then the density of Φ coincides with the density generated by b.

The proof of the comparison principle of Proposition 3.13 is an easy consequence of the following lemma.

Lemma 3.17. *Let $w \in L^\infty([0,T]\times\mathbb{R}^m)$ and $g \in L^\infty([0,T]\times\mathbb{R}^m,\mathbb{R}^m)$ be such that*

$$\begin{cases} \partial_t w + D_x \cdot g \le 0 \\ \\ w(0,\cdot) \;=\; \overline{w} \end{cases} \tag{44}$$

and $|g| \le Cw$. Then, for $\mathscr{L}^1$–a.e. $\tau \in]0,T]$, we have that

$$\int_{B_R(x_0)} w(\tau,)\,dx \;\le\; \int_{B_{R+C\tau}(x_0)} \overline{w}(x)\,dx \qquad \text{for every } x_0 \in \mathbb{R}^n \text{ and } R>0. \tag{45}$$

Proof. Let $\tau \in]0,T]$ be such that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau+\varepsilon}\int_K |w(t,x)-w(\tau,x)|\,dx\,dt \;=\; 0\,, \tag{46}$$

for every compact set $K \subset \mathbb{R}^m$. We will prove the statement of the lemma for any such τ.

Without loss of generality we assume $x_0 = 0$. Let $\chi_\varepsilon \in C^\infty\big(\mathbb{R}^+\big)$ be such that

$$\chi_\varepsilon = 1 \text{ on } [0,1]\,, \qquad \chi_\varepsilon = 0 \text{ on } [1+\varepsilon,+\infty[\,, \quad \text{and} \quad \chi'_\varepsilon \le 0\,.$$

Define the test function $\varphi(t,x) := \chi_\varepsilon\left(\frac{|x|}{R+C(\tau-t)}\right)$. Note that φ is nonnegative and belongs to $C^\infty([0,\tau]\times\mathbb{R}^m)$. Note that we can test (44) with $\varphi(t,x)\mathbf{1}_{[-1,\tau]}(t)$. Indeed let μ be the measure $\partial_t w + D_x\cdot g$. Consider a standard family of nonnegative mollifiers $\xi^\delta \in C^\infty(\mathbb{R})$ and set $\zeta^\delta := \mathbf{1}_{[-1,\tau]} * \xi^\delta$. Testing (44) with $\varphi(t,x)\zeta^\delta(t)$ we get

$$\begin{aligned}\int w(s,y)\varphi(s,y)\xi^\delta(\tau-s)\,ds\,dy - \int_{\mathbb{R}^m} \overline{w}(y)\varphi(0,y)\,dy \\ = \int \zeta^\delta\big[w\,\partial_t\varphi + g\cdot\nabla_x\varphi\big] + \int \zeta^\delta \varphi d\mu\,. \end{aligned} \tag{47}$$

Note that $\int \zeta^\delta d\mu \leq 0$. Moreover, by (46) the integral

$$\int w(s,y)\varphi(s,y)\xi^\delta(\tau-s)\,ds\,dy$$

converge to $\int \varphi(\tau,x)w(\tau,x)\,dx$ as $\delta\downarrow 0$. Hence, in the limit we get

$$\begin{aligned}\int_{[0,\tau]\times\mathbb{R}^n}\big[w\,\partial_t\varphi + g\cdot\nabla_x\varphi\big] \;\geq\; & \int_{\mathbb{R}^n}\varphi(\tau,x)w(\tau,x)\,dx \\ & -\int_{\mathbb{R}^n}\varphi(0,x)\overline{w}(x)\,dx\,. \end{aligned} \tag{48}$$

We compute $w(s,y)\partial_t\varphi(s,y) + g(s,y)\cdot\nabla_x\varphi(s,y)$ as

$$\chi'_\varepsilon\left(\frac{|y|}{R+C(\tau-s)}\right)\left[\frac{C|y|w(s,y)}{(R+C(\tau-s))^2} + \frac{y\cdot g(s,x)}{|y|(R+C(\tau-s))}\right]. \tag{49}$$

Letting $\alpha := |y|/((R+C(\tau-s))$, the expression in (49) becomes

$$\frac{\chi'_\varepsilon(\alpha)}{R+C(\tau-s)}\left[Cw\alpha + g\cdot\frac{y}{|y|}\right].$$

For $\alpha\leq 1$ we have $\chi'_\varepsilon(\alpha)=0$, whereas for $\alpha\geq 1$ we have $\chi'_\varepsilon(\alpha)\leq 0$ and $Cw\alpha \geq |g|$. Thus we conclude that the integrand of the left hand side of (48) is nonpositive. Hence

$$\int_{\mathbb{R}^m}\chi_\varepsilon\left(\frac{|x|}{R}\right)w(\tau,y)\,dx \;\leq\; \int_{\mathbb{R}^m}\chi_\varepsilon\left(\frac{|x|}{R+C\tau}\right)\overline{w}(y)\,dy\,.$$

Letting $\varepsilon\downarrow 0$ we get (45). □

Proof of Proposition 3.13. **Existence** Let $\overline{u}\in L^\infty(\mathbb{R}^m)$ be given and consider a standard family of mollifiers $\{\eta_\varepsilon\}$ in $\mathbb{R}^m$ and a standard family of mollifiers ζ_ε in $\mathbb{R}$, the latter generated by a kernel $\zeta\in C^\infty_c(]0,\infty[)$. Then consider the functions $\rho_\varepsilon\in C^\infty([0,\infty[\times\mathbb{R}^m)$ and $b_\varepsilon\in C^\infty([0,\infty[\times\mathbb{R}^m,\mathbb{R}^m)$ given by

$$\overline{u}_\varepsilon \;:=\; \overline{u}*\eta_\varepsilon\,, \qquad \rho_\varepsilon \;:=\; \varepsilon + \rho*(\eta_\varepsilon\zeta_\varepsilon) \qquad \text{and} \qquad b_\varepsilon \;:=\; \frac{(b\rho)*(\eta_\varepsilon\zeta_\varepsilon)}{\rho_\varepsilon}\,.$$

Note that

(i) b_ε is Lipschitz for every ε;

(ii) $\|b_\varepsilon\|_\infty + \|\rho_\varepsilon\|_\infty + \|\overline{u}_\varepsilon\|_\infty$ is uniformly bounded;
(iii) $b_\varepsilon \to b$ and $\rho_\varepsilon \to \rho$ strongly in L^1_{loc};
(iv) $\partial_t \rho_\varepsilon + D_x \cdot (\rho_\varepsilon b_\varepsilon) = 0$ in the classical sense;
(v) $\rho_\varepsilon(0,\cdot)$ converges weakly* in L^∞ to $\overline{\rho}$, see Lemma 3.7 and Remark 3.8.

Since b_ε is Lipschitz we can solve globally in time

$$\begin{cases} \partial_t \Phi_\varepsilon(t,x) \;=\; b_\varepsilon(t,\Phi_\varepsilon(t,x)) \\ \\ \Phi_\varepsilon(0,x) \;=\; x\,. \end{cases}$$

Each $\Phi_\varepsilon(t,\cdot)$ is a diffemorphism of $\mathbb{R}^m$. Thus, $u_\varepsilon(t,x) := \overline{u}([\Phi_\varepsilon(t,\cdot)]^{-1}(x))$ solves the equation

$$\begin{cases} \partial_t u_\varepsilon + b_\varepsilon \cdot \nabla_x u_\varepsilon \;=\; 0 \\ \\ u_\varepsilon(0,\cdot) \;=\; \overline{u}_\varepsilon\,. \end{cases}$$

Using the chain rule and (iv) we conclude that

$$\begin{cases} \partial_t(u_\varepsilon \rho_\varepsilon) + D_x \cdot\, (\rho_\varepsilon b_\varepsilon u_\varepsilon) \;=\; 0 \\ \\ [\rho_\varepsilon u_\varepsilon]\,(0,\cdot) \;=\; \rho_\varepsilon(0,\cdot)\overline{u}_\varepsilon\,. \end{cases} \tag{50}$$

Due to (ii) we can extract a subsequence $\varepsilon_n \downarrow 0$ such that u_{ε_n} converges weakly* in L^∞ to some $u \in L^\infty$. From (ii), (iii), and (v), we conclude that:

- $u_{\varepsilon_n}\rho_{\varepsilon_n} \rightharpoonup^* u\rho$ and $b_{\varepsilon_n}\rho_{\varepsilon_n}u_{\varepsilon_n} \rightharpoonup^* b\rho u$ in $L^\infty([0,\infty[\times\mathbb{R}^m)$;
- $u_{\varepsilon_n}\rho_{\varepsilon_n}(0,\cdot) \rightharpoonup^* \overline{u\rho}$ in $L^\infty(\mathbb{R}^m)$.

Passing into the limit in the distributional formulation of (50) we conclude that u solves (40) in the sense of distributions.

Comparison principle Let u_i and $\overline{u}_i$ be given as in the statement of the second part of the proposition. We apply the renormalization property to $v := (u_2 - u_1)_+$ to get

$$\begin{cases} \partial_t(\rho v) + D_x \cdot (\rho v b) \;=\; 0 \\ \\ [v\rho]\,(0,\cdot) \;=\; 0\,. \end{cases} \tag{51}$$

Then we apply Lemma 3.17 with $w = \rho v$ and $g = \rho v b$ and we conclude that for $\mathscr{L}^1$–a.e. t we have

$$\int_{\mathbb{R}^n} \rho(t,x)v(t,x)\,dx \;=\; 0\,.$$

Since $v \geq 0$ and $\rho \geq 0$, we conclude $\rho v = 0$, and hence $\rho u_1 \geq \rho u_2$. □

Proof of Corollary 3.14. The existence has been proved in the previous proposition. Moreover, from the comparison principle proved above, the uniqueness of solutions of (40) for b and ρ as in the statement readily follows.

Next, recalling Lemma 3.7, up to changing their value on a set of measure zero, we have that $t \mapsto \rho(t,\cdot)$ and $t \mapsto \rho(t,\cdot)u(t,\cdot)$ are weakly* continuous. Consider $\zeta = \rho u^2$. Similarly, we conclude from Lemma 3.7 that there exists a $\hat{\zeta}$ such that $\hat{\zeta} = \zeta$ a.e. and $t \mapsto \hat{\zeta}(t,\cdot)$ is weakly* continuous. Therefore, for every $T > 0$, $\hat{\zeta}$ solves

$$\begin{cases} \partial_t \hat{\zeta} + D_x \cdot (\hat{\zeta} b) \;=\; 0 \\ \\ \hat{\zeta}(0,\cdot) \;=\; \hat{\zeta}(0,\cdot) \\ \\ \hat{\zeta}(T,\cdot) \;=\; \hat{\zeta}(T,\cdot) \end{cases}$$

in the sense of distributions. On the other hand, from the renormalization property we have

$$\begin{cases} \partial_t \hat{\zeta} + D_x \cdot (\hat{\zeta} b) \;=\; 0 \\ \\ \hat{\zeta}(0,\cdot) \;=\; \rho(0,\cdot)[u(0,\cdot)]^2 \\ \\ \hat{\zeta}(T,\cdot) \;=\; \rho(T,\cdot)[u(T,\cdot)]^2 \,. \end{cases}$$

Thus, we conclude that $\rho(T,\cdot)[u(T,\cdot)]^2 = \zeta(T,\cdot)$ for every T and hence $t \mapsto \rho(t,\cdot)[u(t,\cdot)]^2$ is weakly* continuous. For any $\tau \geq 0$ consider

$$\rho(\tau,\cdot)(u(t,\cdot) - u(\tau,\cdot))^2 \;=\; \rho(\tau,\cdot)[u(t,\cdot)]^2 - 2\,[\rho(\tau,\cdot)u(\tau,\cdot)]\,u(t,\cdot) + \rho(\tau,\cdot)[u(\tau,\cdot)]^2 \,.$$

It follows that, for $\tau \to t$, $\rho(\tau,\cdot)(u(t,\cdot) - u(\tau,\cdot))^2 \rightharpoonup^* 0$ in L^∞. Since $\rho(\tau,\cdot) \geq C > 0$ for every τ, we conclude that $u(\tau,\cdot) \to u(t,\cdot)$ strongly in L^1_{loc}. This proves that $u \mapsto u(t,\cdot)$ is strongly continuous in L^1_{loc}. □

Corollary 3.15 follows trivially from Proposition 3.13

Remark 3.18. *Clearly, the proof of the previous proposition can be used to solve transport and continuity equations even when we drop one of the boundary conditions. Namely, under the same assumptions, for every $T \in \mathbb{R}$ and every bounded $\hat{u}$ and $\overline{u}$ there exist unique solutions to both the forward and the backward transport equations:*

$$\begin{cases} \partial_t(\rho u) + D_x \cdot (\rho u b) \;=\; 0 & \text{in }]\infty, T] \times \mathbb{R}^n \\ \\ [\rho u]\,(T,\cdot) \;=\; \rho(T,\cdot)\hat{u} \end{cases} \tag{52}$$

$$\begin{cases} \partial_t(\rho u) + D_x \cdot (\rho u b) \;=\; 0 & \text{in } [T, \infty[\times \mathbb{R}^n \\ \\ [\rho u]\,(T,\cdot) \;=\; \rho(T,\cdot)\overline{u} \end{cases} \tag{53}$$

3.4. **Stability of solutions to transport equations.** The uniqueness results proved in the previous section have the following easy corollary.

Corollary 3.19. *Let $\{b_n\} \subset L^\infty([0,\infty[\times\mathbb{R}^m)$ be a sequence of vector fields converging strongly in L^1_{loc} to a bounded nearly incompressible vector field b with the renormalization* *property. Let ζ_n be solutions of*

$$\begin{cases} \partial_t\zeta_n + D_x\cdot(\zeta_n b_n) \;=\; 0 \\ \\ \zeta_n(0,\cdot) \;=\; \overline{\zeta}_n\,. \end{cases} \tag{54}$$

If $\|\zeta_n\|_\infty$ is uniformly bounded and $\overline{\zeta}_n \rightharpoonup^* \overline{\zeta}$ in L^∞, then ζ_n converges weakly in L^∞ to the unique solution ζ of*

$$\begin{cases} \partial_t\zeta + D_x\cdot(\zeta b) \;=\; 0 \\ \\ \zeta(0,\cdot) \;=\; \overline{\zeta} \end{cases} \tag{55}$$

Proof. If $\tilde{\zeta}$ is the weak* limit of any subsequence of $\{\zeta_n\}$, then $\tilde{\zeta}$ solves (55). Since the solution to such equation is unique, it follows that the whole sequence converges weakly* to ζ. □

Corollary 3.20. *Let $\{b_n\}, b \subset L^\infty([0,\infty[\times\mathbb{R}^m,\mathbb{R}^m)$, $\{\zeta_n\},\zeta,\{u_n\},u \subset L^\infty([0,\infty[\times\mathbb{R}^m)$ and $\overline{\rho}_n,\overline{\rho},\overline{u}_n,\overline{u} \subset L^\infty(\mathbb{R}^m)$ be such that*

(a) *$\zeta,\zeta_n > 0$, $\zeta^{-1},\zeta_n^{-1} \in L^\infty$ and $\|\zeta_n\|_\infty + \|\zeta_n^{-1}\|_\infty + \|\overline{u}_n\|_\infty$ is uniformly bounded;*
(b) *$\{b_n\}$ and b have the renormalization property and $b_n \to b$ in L^1_{loc};*
(c) *$\partial_t\zeta + D_x\cdot(\zeta b) = \partial_t\zeta_n + D_x\cdot(\zeta_n b_n) = 0$;*
(d) *u_n and u solve*

$$\begin{cases} \partial_t(\zeta_n u_n) + D_x\cdot(\zeta_n u_n b_n) \;=\; 0 \\ \\ [\zeta_n u_n](0,\cdot) \;=\; \zeta_n(0,\cdot)\overline{u}_n\,, \end{cases} \tag{56}$$

$$\begin{cases} \partial_t(\zeta u) + D_x\cdot(\zeta u b) \;=\; 0 \\ \\ [\zeta u](0,\cdot) \;=\; \zeta(0,\cdot)\overline{u}\,. \end{cases} \tag{57}$$

If $\zeta_n(0,\cdot)\rightharpoonup^*\zeta(0,\cdot)$ in L^∞ and $\overline{u}_n \to \overline{u}$ in L^1_{loc}, then $u_n \to u$ in L^1_{loc}.

Proof. From the comparison principle of Proposition 3.13 it follows that $\|u_n\|_\infty \le \|\overline{u}_n\|_\infty$. Moreover, from Corollary 3.19 it follows that $\zeta_n \rightharpoonup^* \zeta$.

Set $\beta_n := \zeta_n u_n$ and $\overline{\beta}_n := \zeta_n(0,\cdot)\overline{u}_n$. We conclude from Corollary 3.19 that β_n converges weakly* in L^∞ to the unique solution β of

$$\begin{cases} \partial_t\beta + D_x\cdot(\beta b) \;=\; 0 \\ \\ \beta(0,\cdot) \;=\; \zeta(0,\cdot)\overline{u}\,. \end{cases} \tag{58}$$

Therefore, by Corollary 3.14, $\beta/\zeta = u$. Applying the renormalization property, we conclude that $v_n := u_n^2$ and $v := u^2$ solve

$$\begin{cases} \partial_t(\zeta_n v_n) + D_x\cdot(\zeta_n v_n b_n) \;=\; 0 \\ \\ [\zeta_n v_n](0,\cdot) \;=\; \zeta_n(0,\cdot)\overline{u}_n^2\,, \end{cases} \tag{59}$$

$$\begin{cases} \partial_t(\zeta v) + D_x \cdot (\zeta v b) \;=\; 0 \\ \\ [\zeta v]\,(0,\cdot) \;=\; \zeta(0,\cdot)\overline{u}^2\,. \end{cases} \tag{60}$$

Therefore, applying the argument above we conclude that $\zeta_n u_n^2 \rightharpoonup^* \zeta u^2$. Note that

$$\zeta_n(u_n-u)^2 = \zeta_n u_n^2 + \zeta_n u^2 - 2\zeta_n u_n u \rightharpoonup^* \zeta u^2 + \zeta u^2 - 2\zeta u u \;=\; 0\,.$$

Since for some constant C we have $\zeta_n \geq C$ for every n, we conclude that $(u_n-u)^2 \to 0$ strongly in L^1_{loc}. □

In the same way we can prove the following more refined version of the previous corollary, which will be used in studying the well–posedness for the Keyfitz and Kranzer system.

Corollary 3.21. *Assume that*

- *The pairs* $\{(b_n,\rho_n)\}_n, (b,\rho)$ *have the renormalization property and* $\rho_n \geq 0$*;*
- $(b_n,\rho_n) \to (b,\rho)$ *in* L^1_{loc} *and* $\|b_n\|_\infty + \|\rho_n\|_\infty$ *is uniformly bounded;*
- *The traces* $\rho_n(0,\cdot) \to \rho(0,\cdot)$ *and* $\overline{u}_n \to \overline{u}$ *strongly in* L^1_{loc}*.*

If u_n*,* u *solve* (56) *and* (57)*, then* $\rho_n u_n \to \rho u$ *strongly in* L^1_{loc}*.*

Proof. From the Proof of Corollary 3.20 we conclude that $\rho_n(u_n-u)^2 \to 0$ strongly in L^1_{loc}. Since $\|\rho_n\|_\infty$ is uniformly bounded, we get that $(\rho_n u_n - \rho_n u)^2 \to 0$, and hence $|\rho_n u_n - \rho_n u| \to 0$ strongly in L^1_{loc}. But $|u\rho_n - \rho u| \leq \|u\|_\infty |\rho_n - \rho| \to 0$ strongly in L^1_{loc}, and thus we finally get $|\rho_n u_n - \rho u| \to 0$, which is the desired conclusion. □

3.5. **Existence, uniqueness, and stability of regular Lagrangian flows.** We will now show existence, uniqueness, and stability of the regular Lagrangian flows using the stability results for transport and continuity equations proved in the previous sections.

Theorem 3.22. *Let* b *a bounded nearly incompressible vector field with the renormalization property. Then there exists a unique regular Lagrangian flow* Φ *for* b*. Moreover, let* b_n *be a sequence of bounded nearly incompressible vector fields with the renormalization property such that*

- $\|b_n\|_\infty$ *is uniformly bounded and* $b_n \to b$ *strongly in* L^1_{loc}*;*
- *The densities* ρ_n *generated by* b_n *satisfy* $\limsup_n (\|\rho_n\|_\infty + \|\rho_n^{-1}\|_\infty) < \infty$.

Then the regular Lagrangian flows Φ_n *generated by* b_n *converge in* L^1_{loc} *to* Φ*.*

Proof. **Uniqueness** Let Φ and Ψ be two regular Lagrangian flows associated to the same nearly incompressible vector field. For any $\overline{\zeta} \in L^\infty(\mathbb{R}^n)$ consider the bounded functions ζ and $\hat{\zeta}$ given by

$$\int \varphi(t,x)\zeta(t,x)\,dt\,dx \;=\; \int \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx$$

$$\int \varphi(t,x)\hat{\zeta}(t,x)\,dt\,dx \;=\; \int \varphi(t,\Psi(t,x))\overline{\zeta}(x)\,dt\,dx\,.$$

According to Proposition 3.5, ζ and $\hat{\zeta}$ solve both the same equation

$$\begin{cases} \partial_t \zeta + D_x \cdot (\zeta b) \;=\; 0 \\ \\ \zeta(0,\cdot) \;=\; \overline{\zeta}\,. \end{cases}$$

When b has the renormalization property we can apply Proposition 3.13 to conclude that $\zeta = \hat{\zeta}$. Therefore, when b has the renormalization property we conclude that, for any compactly supported $\varphi \in L^\infty(\mathbb{R}\times\mathbb{R}^m)$ and $\overline{\zeta} \in L^\infty(\mathbb{R}^m)$, we have

$$\int \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx \;=\; \int \varphi(t,\Psi(t,x))\overline{\zeta}(x)\,dt\,dx\,.$$

This easily implies that $\Psi = \Phi$ $\mathscr{L}^{m+1}$–a.e..

Stability Next consider a sequence of $b_n \to b$ as in the statement of the Proposition. Let Φ and Φ_n be regular Lagrangian flows generated by b and b_n. Fix again any $\overline{\zeta} \in L^\infty$ and define ζ as in the previous step and ζ_n by

$$\int \varphi(t,x)\zeta_n(t,x) \;=\; \int \varphi(t,\Phi_n(t,x))\overline{\zeta}(x)\,dt\,dx\,.$$

Applying the comparison principle we get that $\|\zeta_n\|_\infty$ is uniformly bounded, and from Corollary 3.19 we conclude that $\zeta_n \rightharpoonup^* \zeta$. Therefore we get that

$$\int \varphi(t,\Phi_n(t,x))\overline{\zeta}(x)\,dt\,dx \;\to\; \int \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx \tag{61}$$

for every bounded $\overline{\zeta}$ and every φ which is bounded and has bounded support.

Note that, since $\|b_n\|_\infty$ is uniformly bounded, for every $R > 0$, $\|\Phi_n\|_{L^\infty([0,R]\times B_R(0))}$ is uniformly bounded. Therefore, if $\overline{\zeta}$ has bounded support, then (61) holds for every bounded φ which has support bounded in time. Thus, we can apply (61) with $\overline{\zeta} = \mathbf{1}_{B_R(0)}$ and $\varphi(t,x) = \mathbf{1}_{[0,R]}(t)|x|^2$ in order to get

$$\int_{[0,R]\times B_R(0)} |\Phi_n(t,x)|^2\,dt\,dx \;\to\; \int_{[0,R]\times B_R(0)} |\Phi(t,x)|^2\,dt\,dx \tag{62}$$

Next, apply (61) with $\varphi(t,x) = \mathbf{1}_{[0,R]}(t)\gamma(t)x\cdot v$ and $\overline{\zeta} = \beta\mathbf{1}_{B_R(0)}$. Then we conclude that

$$\int_{[0,R]\times B_R(0)} \Phi_n(t,x)\cdot v\,\gamma(t)\beta(x)\,dt\,dx \;\to\; \int_{[0,R]\times B_R(0)} \Phi(t,x)\cdot v\,\gamma(t)\beta(x)\,dt\,dx\,.$$

By linearity, we conclude that

$$\int_{[0,R]\times B_R(0)} \sum_{i=1}^N \Phi_n(t,x)\cdot v_i\,\gamma_i(t)\beta_i(x)\,dt\,dx \;\to\; \int_{[0,R]\times B_R(0)} \sum_{i=1}^N \Phi(t,x)\cdot v_i\,\gamma_i(t)\beta_i(x)\,dt\,dx$$

for any choice of the bounded functions γ_i, β_i, and v_i. However, by a standard argument, we can approximate Φ strongly in $L^1([0,R]\times B_R(0))$ with functions of type $\sum_{i=1}^N v_i\gamma_i(t)\beta_i(x)$.

This gives

$$\int_{[0,R]\times B_R(0)} \Phi_n(t,x)\cdot\Phi(t,x)\,dt\,dx \;\to\; \int_{[0,R]\times B_R(0)} |\Phi(t,x)|^2\,dt\,dx\,. \tag{63}$$

Therefore, from (62) and (63) we get

$$\lim_{n\uparrow\infty}\int_{[0,R]\times B_R(0)} |\Phi_n(t,x)-\Phi(t,x)|^2\,dt\,dx \;=\; 0\,.$$

From the arbitrariness of R we conclude that $\Phi_n \to \Phi$ in L^1_{loc}.

Existence. Step 1: Regular Approximation We finally address the existence of a regular Lagrangian flow. Fix two kernels $\chi \in C^\infty_c(]0,\infty[)$ and $\psi \in C^\infty(\mathbb{R}^m)$, let $\{\chi_\varepsilon\}_\varepsilon$ and $\{\psi_\varepsilon\}_\varepsilon$ be the two standard families of mollifiers generated by χ and η, and set $\varphi_\varepsilon(t,x) := \chi_\varepsilon(t)\eta_\varepsilon(x)$.

Let ρ be the density generated by b and set $\rho_\varepsilon := \rho * \varphi_\varepsilon$, $b_\varepsilon := b*\varphi_\varepsilon/\rho_\varepsilon$. Note that

- $\|b_\varepsilon\|_\infty + \|\rho_\varepsilon\|_\infty + \|\rho_\varepsilon^{-1}\|_\infty$ is uniformly bounded;
- $b_\varepsilon \to b$ and $\rho_\varepsilon \to \rho$ in L^1_{loc};
- $\rho_\varepsilon(t,\cdot) \rightharpoonup^* \rho(t,\cdot)$ in $L^\infty(\mathbb{R}^m)$ for every $t\geq 0$.

For each ε, b_ε is globally Lipschitz, and therefore we can apply the classical Cauchy Lipschitz Theorem to get the unique regular Lagrangian flow Φ_ε generated by b_ε.

Note that $\|\Phi_\varepsilon\|_{L^\infty(K)}$ is uniformly bounded for every compact set K. Thus we can extract a sequence $\{\Phi_n\} = \{\Phi_{\varepsilon_n}\}$ which locally converges weakly* to a map Φ. We will show that Φ_n converges strongly in L^1_{loc}. From this we easily conclude that Φ is a regular Lagrangian flow for b. From now on, in order to simplify the notation we will use b_n, ρ_n for b_{ε_n} and ρ_{ε_n}.

Existence. Step 2: Strong convergence Note that each $\Phi_n(t,\cdot)$ is a diffemorphism of $\mathbb{R}^m$. Therefore we can define $\Psi_n(t,\cdot) := [\Phi_n(t,\cdot)]^{-1}$. Fix $T>0$ and solve the following ODE backward in time:

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\Lambda_n(t,x) &=& b_n(t,\Lambda_n(t,x))\\[2ex] \Lambda_n(T,x) &=& x\,. \end{array}\right.$$

Note that $\Lambda_n(t,\cdot) = \Phi_n(t,\Psi_n(T,\cdot))$. Thus, if we denote by $J_n(t,\cdot)$ the Jacobian of $\Lambda_n(t,\cdot)$, we get that $0\leq C^{-2}\leq J_n(t,\cdot)\leq C^2$. Denote by $\Gamma_n(t,\cdot)$ the inverse of $\Lambda_n(t,\cdot)$ and set $\zeta_n(t,x) := J_n(t,\Gamma_n(t,x))$. Moreover, for every $\overline{w}\in L^\infty(\mathbb{R}^m,\mathbb{R}^m)$ define the function $w_n(t,x) := \overline{w}(\Gamma_n(t,x))$. Clearly we have

$$\left\{\begin{array}{rcll} \partial_t(\zeta_n w_n) + D_x\cdot(\zeta_n w_n\otimes b_n) &=& 0 & \quad\text{on } [0,T]\times\mathbb{R}^m\\[2ex] \zeta_n w_n(T,x) &=& \overline{w}(x)\,, & \end{array}\right.$$

(the first line is just a shorthand notation for the equations $\partial_t(\zeta_n w_n^i) + D_x\cdot(\zeta_n w^i_n b_n) = 0$ for $i\in\{1,\ldots,m\}$). We claim that the ζ_n's have a unique weak* limit. Indeed, assume that ζ and $\hat\zeta$ are weak* limits of two convergent subsequences of ζ_n's. Then $\partial_t\zeta + D_x\cdot(b\zeta) = 0$ and

$\partial_t \hat{\zeta} + D_x \cdot (b\hat{\zeta}) = 0$. Moreover, both ζ and $\hat{\zeta}$ have weak trace equal to 1 at $t = T$. Thus by the backward uniqueness of Remark 3.18, we conclude that ζ and $\hat{\rho}$ coincide with the unique solution of

$$\begin{cases} \partial_t \beta + D_x \cdot (\beta b) \;=\; 0 \qquad \text{on } [0,T] \times \mathbb{R}^n \\ \\ \beta(T,\cdot) \;=\; 1\,. \end{cases}$$

Note that there exists a constant C such that $|\Gamma_n(t,x) - x| \leq C(T-t)$ for every t, x and j. Fix $r > 0$ and choose $R > 0$ so large that $R - CT > r$. Let $\overline{w}$ be the vector valued map $x \to x\,\mathbf{1}_{B_R(0)}(x)$. Thus, for every $t < T$ and every $|x| < r$, $w_n(t,x)$ is equal to the vector $\Gamma_n(t,x)$. Thanks to Remark 3.18, w_n converges strongly in L^1_{loc} the unique w solving

$$\begin{cases} \partial_t(\beta w) + D_x \cdot (\beta w \otimes b) \;=\; 0 \qquad \text{on } [0,T] \times \mathbb{R}^m \\ \\ [\beta w]\,(0,\cdot) \;=\; \overline{w}\,. \end{cases}$$

Hence, by the arbitrariness of r we conclude that Γ_n converges to a unique Γ strongly in L^1_{loc}.

For each x, $\Gamma_n(\cdot,x)$ is a Lipschitz curve, with Lipschitz constant uniformly bounded. Thus we infer that, for a.e. x, $\Gamma_n(\cdot,x)$ converges uniformly to the curve $\Gamma(\cdot,x)$ on $[0,T]$. Hence, we conclude that, after possibly changing Γ on a set of measure 0, for every $t \geq 0$ the maps $\Gamma_n(t,\cdot)$ converge to $\Gamma(t,\cdot)$ in $L^1_{loc}(\mathbb{R}^m)$.

Since $\Gamma_n(0,\cdot) = \Phi_n(T,\cdot)$ we conclude that for every T there exists a $\Phi(T,\cdot)$ such that $\Phi_n(T,\cdot)$ converges to $\Phi(T,\cdot)$ in $L^1_{loc}(\mathbb{R}^m)$. Since Φ_n is locally uniformly bounded, we conclude that Φ_n converges to Φ strongly in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^m)$.

Existence. Step 3: Near incompressibility Note that, by our construction, there exists a constant C such that, for every t and every n,

$$C^{-1}\mathscr{L}^m \;\leq\; \Phi_n(t,\cdot)_\# \mathscr{L}^m \;\leq\; C\mathscr{L}^m\,. \tag{64}$$

Let $\varphi \in C_c([0,\infty[\times\mathbb{R}^m)$ be given. Then

$$\int |\varphi(t,\Phi_n(t,x))|\,dx\,dt \;\leq\; C\int |\varphi(t,y)|\,dy\,dt \;<\; \infty\,.$$

Up to extracting another subsequence, not relabeled, we can assume that $\Phi_n(t,x) \to \Phi(t,x)$ for $\mathscr{L}^{m+1}$–a.e. (t,x). Thus, by the dominated convergence theorem:

$$\begin{aligned} \lim_{n\uparrow\infty} \int \varphi\, d\mu_{\Phi_n} \;&=\; \lim_{n\uparrow\infty} \int \varphi(t,\Phi_n(t,x))\,dx\,dt \\ &=\; \int \varphi(t,\Phi(t,x))\,dt\,dx \;=\; \int \varphi\,d\mu_\Phi\,. \end{aligned}$$

Therefore, from (64) we get $C^{-1}\mathscr{L}^{m+1} \leq \mu_\Phi \leq C\mathscr{L}^{m+1}$. Therefore Φ satisfies condition (a) of Definition 3.1.

Existence. Step 4: Final ODE Next, we show that $b_n(t,\Phi_n(t,x)) \to b(t,\Phi(t,x))$ strongly in L^1_{loc}, from which (b) of Definition 3.1 follows. Let R be any given positive number.

Since $\|b_n\|_\infty \le C$, we have $\|\Phi_n\|_{L^\infty([0,R[\times B_R(0))} \le (C+1)R$. Thus, set $b'_n := b_n \mathbf{1}_{[0,R]\times B_{(C+1)R}(0)}$ and $b' := b\mathbf{1}_{[0,R]\times B_{(C+1)R}(0)}$. Using Egorov's and Lusin's Theorems, for any given $\varepsilon > 0$ choose $\hat{b}_n, \hat{b} \in C_c([0,\infty[\times\mathbb{R}^m)$ such that

- $\|\hat{b}_n - b'_n\|_{L^1} + \|\hat{b} - b'\|_{L^1} < \varepsilon$;
- $\hat{b}_n \to \hat{b}$ uniformly.

Then, $\hat{b}_n(t, \Phi_n(t,x)) \to \hat{b}(t, \Phi(t,x))$ for $\mathscr{L}^{m+1}$–a.e. (t,x). Thus,

$$
\begin{aligned}
& \limsup_{n\uparrow\infty} \|b_n(\cdot, \Phi_n(\cdot)) - b(\cdot, \Phi(\cdot))\|_{L^1([0,R]\times B_R(0)}\\
= \;& \limsup_{n\uparrow\infty} \|b'_n(\cdot, \Phi_n(\cdot)) - b'(\cdot, \Phi(\cdot))\|_{L^1([0,R]\times B_R(0)}\\
\le \;& \limsup_{n\uparrow\infty} \|\hat{b}_n(\cdot, \Phi_n(\cdot)) - \hat{b}(\cdot, \Phi(\cdot))\|_{L^1([0,R]\times B_R(0)}\\
& + \limsup_{n\uparrow\infty} \Big(\|(\hat{b}_n - b'_n)(\cdot, \Phi_n(\cdot))\|_{L^1} + \|(\hat{b} - b')(\cdot, \Phi(\cdot))\|_{L^1}\Big)\\
= \;& \limsup_{n\uparrow\infty} \Big(\|(\hat{b}_n - b'_n)(\cdot, \Phi_n(\cdot))\|_{L^1} + \|(\hat{b} - b')(\cdot, \Phi(\cdot))\|_{L^1}\Big)\\
\stackrel{(64)}{\le} \;& C \limsup_{n\uparrow\infty} \big(\|\hat{b}_n - b'_n\|_{L^1} + \|\hat{b} - b'\|_{L^1}\big) \;\le\; C\varepsilon\,.
\end{aligned}
$$

By the arbitrariness of R and ε, we get the desired convergence. This completes the proof. □

4. Commutator estimates and Ambrosio's Renormalization Theorem

In this section we study the following problem. Let $\Omega \subset \mathbb{R}^d$ be an open set and $B : \Omega \to \mathbb{R}^d$ a bounded BV vector field. Assume $w^1, \ldots, w^k$ are L^∞ functions which satisfy

$$D \cdot (w^i B) \;=\; 0 \qquad \text{distributionally in } \Omega \text{ for every } i\,,$$

(that is $D \cdot (w \otimes B) = 0$) and let $H \in C^1(\mathbb{R}^k)$. What are the properties of the distribution $D \cdot (H(w)B)$?

In particular, our final goal is to show the following theorem, which has been proved in [10] by slightly adapting the ideas of [2]:

Theorem 4.1. *Let B, Ω, w and H be as above. Then, $D \cdot (H(w)B)$ is a Radon measure and*

$$\left| D \cdot (H(w)B) - \left(H(w) - \sum_{i=1}^{d} \frac{\partial H}{\partial v_i}(w) w^i \right) D^a \cdot B \right| \;\le\; C|D^s \cdot B|\,, \tag{65}$$

where the constant C depends only on $R := \|w\|_\infty$ and $\|H\|_{C^1(B_R(0))}$.

Our approach to this problem is to consider appropriate "commutators" and get estimates for them. More precisely, fix a standard kernel η in $\mathbb{R}^d$ supported in the ball $B_r(0)$ and let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be the standard family of mollifiers generated by η. Thus, for any distribution T in Ω the convolution $T * \rho_\varepsilon$ is a well defined distribution in the open set $\Omega_\varepsilon \;:=\; \{x \in \Omega :$

$\mathrm{dist}\,(x, \partial\Omega) > \varepsilon r\}$ in the usual way. Since $w^i * \eta_\varepsilon \to w^i$ converges strongly in $L^1(K)$ to w^i for any $K \subset\subset \Omega$, we conclude $D \cdot (H(w * \eta_\delta)B)$ converges in the sense of distributions to $D \cdot (H(w)B)$ in every open set $\Omega' \subset\subset \Omega$. Since $w * \eta_\delta$ is smooth, the usual chain rule applies and we can compute

$$\begin{aligned} D \cdot (H(w * \eta_\delta)B) \;&=\; \sum_{i=1}^d \frac{\partial H}{\partial v_i}(w * \eta_\delta) D \cdot (w^i * \eta_\delta B) \\ &\quad + \left(H(w * \eta_\delta)) - \sum_{i=1}^d \frac{\partial H}{\partial v_i}(w * \eta_\delta) w^i * \eta_\delta \right) D \cdot B\,. \end{aligned}$$

Moreover notice that $(D \cdot (w^i B)) * \eta_\delta = 0$. Thus we can write

$$\begin{aligned} D \cdot (H(w * \eta_\delta)B) \;&=\; \sum_{i=1}^d \frac{\partial H}{\partial v_i}(w * \eta_\delta) \big[D \cdot (w^i * \eta_\delta B) - (D \cdot (w^i B)) * \eta_\delta \big] \\ &\quad + \left(H(w * \eta_\delta) - \sum_{i=1}^d \frac{\partial H}{\partial v_i}(w * \eta_\delta) w^i * \eta_\delta \right) D \cdot B\,. \end{aligned} \tag{66}$$

Motivated by these computations we introduce the following terminology and notation.

Definition 4.2. *For every fixed kernel η, we denote by $T^i_{\delta,\eta}$ the commutators*

$$T^i_{\delta,\eta} \;:=\; (D \cdot (Bw^i)) * \eta_\delta - D \cdot (Bw^i * \eta_\delta)\,. \tag{67}$$

Moreover, the vector–valued distribution $(T^1_{\delta,\eta}, \ldots, T^k_{\delta,\eta})$ will be denoted by $T_{\delta,\eta}$. When no confusion can arise, we drop the η from $T^i_{\delta,\eta}$ and $T_{\delta,\eta}$.

Clearly, in our case the commutators $T_\delta = D \cdot (w \otimes B) * \eta_\delta - D \cdot ((w * \eta_\delta) \otimes B)$ are equal to $-D \cdot ((w * \eta_\delta) \otimes B)$. Since $w * \eta_\delta$ is smooth and B is a BV vector field, $(w * \eta_\delta) \otimes B$ is a BV matrix–valued function. Thus T_δ is a vector valued measure. However this turns out to hold even when we do not assume $D \cdot (w \otimes B) = 0$: The commutators T_δ are always measures, for every BV vector field B and every L^∞ map w (see Proposition 4.6(a)).

Next, write $D \cdot B = D^a \cdot B + D^s \cdot B$, and from (66) get the inequality

$$\begin{aligned} &\left| D \cdot (H(w * \eta_\delta)B) - \left(H(w * \eta_\delta) - \sum_{i=1}^d \frac{\partial H}{\partial v_i}(w * \eta_\delta) w^i * \eta_\delta \right) D^a \cdot B \right| \\ &\leq\; C\big(|T_{\delta,\eta}| + C|D^s \cdot B| \big), \end{aligned} \tag{68}$$

where the constant C depend on H and $\|w\|_\infty$.

Comparing (65) and (68), it is clear that we might try to prove Theorem 4.1 by careful analyzing the behavior of the commutators $|T_{\delta,\eta}|$. This is done in Proposition 4.6, with the help of a technical Proposition 4.3 concerning difference quotients of BV functions, which is proved in Subsection 4.1. The key commutator estimate of Proposition 4.6 is stated and proved in 4.2. In Subsection 4.3 we state two lemmas. The first one is due to Bouchut and it was used in the first proof of the results of [2], in combination with the Rank–one Theorem (see Theorem 2.13). The second lemma is a generalization of Bouchut's one, suggested by Alberti. This new lemma can replace the one by Bouchut and the Rank–one Theorem in the proof of Theorem 4.1, yielding a much more transparent and self–contained argument. In Subsection 4.4 we give both these proofs of Theorem 4.1.

4.1. **Difference quotients of BV functions.** In what follows, for BV vector fields B, we denote, as usual, by DB their distributional derivative, which is a Radon measure. If $DB = M\mathscr{L}^d + D^sB$ is the Radon–Nykodim decomposition of DB with respect to $\mathscr{L}^d$, then we denote M by ∇B.

Proposition 4.3. *Let $B \in BV_{\rm loc}(\mathbb{R}^d, \mathbb{R}^m)$ and let $z \in \mathbb{R}^d$. Then the difference quotients*

$$\frac{B(x+\delta z) - B(x)}{\delta}$$

can be canonically written as $B_{1,\delta}(z)(x) + B_{2,\delta}(z)(x)$, where

(a) *$B_{1,\delta}(z)$ converges strongly in $L^1_{\rm loc}$ to $\nabla B \cdot z$ as $\delta \downarrow 0$.*
(b) *For any compact set $K \subset \mathbb{R}^d$ we have*

$$\limsup_{\delta\downarrow 0} \int_K \big|B_{2,\delta}(z)(x)\big|\, dx \;\leq\; |D^sB\cdot z|(K)\,. \tag{69}$$

(c) *For every compact set $K \subset \mathbb{R}^d$ we have*

$$\sup_{\delta\in]0,\varepsilon[} \int_K \big|B_{1,\delta}(z)(x)\big| + \big|B_{2,\delta}(z)(x)\big|\, dx \;\leq\; |z||DB|(K_\varepsilon) \tag{70}$$

where $K_\varepsilon := \{x : {\rm dist}\,(x, K) \leq \varepsilon\}$.

Remark 4.4. *The decomposition of the proof is canonical in the sense that we give an explicit way of constructing $B_{1,\delta}$ and $B_{2,\delta}$ from the measures $D^aB\cdot z$ and $D^sB\cdot z$. One important consequence of this explicit construction is the following linearity property: If $B^1, B^2 \in BV_{loc}(\mathbb{R}^d, \mathbb{R}^m)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $z \in \mathbb{R}^d$, then*

$$(\lambda_1 B^1 + \lambda_2 B^2)_{i,\delta}(z)(x) \;=\; \lambda_1 B^1_{i,\delta}(z)(x) + \lambda_2 B^2_{i,\delta}(z)(x)\,. \tag{71}$$

Proof. Let $e_1, \ldots, e_d$ be orthonormal vectors in $\mathbb{R}^d$. In the corresponding system of coordinates we use the notation $x = (x_1, \ldots, x_{d-1}, x_d) = (x', x_d)$. Without loss of generality we can assume that $z = e_d$. Recall the following elementary fact: If μ is a Radon measure on $\mathbb{R}$, then the functions

$$\hat{\mu}_\delta(t) \;:=\; \frac{\mu([t, t+\delta])}{\delta} \;=\; \mu * \frac{\mathbf{1}_{[-\delta,0]}}{\delta}(t) \qquad t\in\mathbb{R}$$

satisfy

$$\int_K |\hat{\mu}_\delta|\, dt \;\leq\; \mu(K_\delta) \tag{72}$$

for every compact set $K \subset \mathbb{R}$, where K_δ denotes the δ–neighborhood of K.

Consider the measure $D_{e_d}B = DB \cdot e_d$, and the vector–valued function $\nabla B \cdot e_d$. Clearly this function is the Radon–Nykodim derivative of $D_{e_d}B$ with respect to $\mathscr{L}^d$ and we denote by $D^s_{e_d}B$ the singular measure $D^s B \cdot e_d = D_{e_d}B - \nabla B \cdot e_d \mathscr{L}^d$.

We define

$$B_{1,\delta}(x', x_d) \;=\; \frac{1}{\delta}\int_{x_d}^{x_d+\delta} \nabla B \cdot e_d(x', s)\, ds\,.$$

By Fubini's Theorem and standard arguments on convolutions, we get that $B_{1,\delta} \to \nabla B \cdot e_d$ strongly in L^1_{loc}.

Next set

$$B_{2,\delta}(x', x_2) \;:=\; \frac{B(x', x_d+\delta) - B(x', x_d)}{\delta} - B_{1,\delta}(x', x_d)\,,$$

and, for $\mathscr{L}^{d-1}$–a.e. $y \in \mathbb{R}^{d-1}$, define $B_y : \mathbb{R} \to \mathbb{R}$ by $B_y(s) = B(y, s)$.

We recall the following slicing properties of BV functions (see Theorem 3.103, Theorem 3.107, and Theorem 3.108 of [11]):

(a) $B_y \in BV_{loc}(\mathbb{R}, \mathbb{R}^m)$ for $\mathscr{L}^{d-1}$–a.e. y;

(b) If we let $D^s B_y + B'_y \mathscr{L}^1$ be the Radon–Nykodim decomposition of DB_y, then we have

$$\nabla B(y, s) \cdot e_d \;=\; B'_y(s) \qquad \text{for } \mathscr{L}^d\text{–a.e. } (y, s)$$

and

$$|D^s_{e_d}|(A) \;=\; \int_{\mathbb{R}^{d-1}} |D^s B_y|(A \cap \{(y, s) : s \in \mathbb{R}\})\, dy\,;$$

(c) $B_y(s+\delta) - B_y(s) = DB_y([s, s+\delta])$.

Therefore, for any $\delta > 0$ and for $\mathscr{L}^{d-1}$–a.e. y we have

$$\begin{aligned}
\frac{B(y, x_d+\delta) - B(y, x_d)}{\delta} \;&=\; \frac{B_y(x_d+\delta) - B_y(x_d)}{\delta} \;=\; \frac{DB_y([x_d, x_d+\delta])}{\delta} \\
&=\; (\widehat{B'_x \mathscr{L}^1})_\delta(x_d) + (\widehat{D^s B_y})_\delta(x_d) \\
&=\; B_{1,\delta}(y, x_d) + (\widehat{D^s B_y})_\delta(x_d) \qquad \text{for } \mathscr{L}^1\text{–a.e. } x_d.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_K |B_{2,\delta}| \;&\le\; \int_{\mathbb{R}^{d-1}} \int_{\{x_d : (y, x_d) \in K\}} \left| (\widehat{D^s B_y})_\delta(x_d) \right| dx_d\, dy \\
&\le\; \int_{\mathbb{R}^{d-1}} |D^s B_y| \left(\{x_d : (y, x_d) \in K_\delta\} \right) dy \\
&=\; |D^s B \cdot e_d|(K_\delta) \;\le\; |D^s B|(K_\delta)\,. \qquad (73)
\end{aligned}$$

Letting $\delta \downarrow 0$, this gives (69).

Note moreover that

$$\begin{aligned}\int_K |B_{1,\delta}| &\leq \int_{\mathbb{R}^{d-1}} \int_{\{x_d:(y,x_d)\in K\}} \left|\big(\widehat{B'_y\mathscr{L}^1}\big)_\delta(x_d)\right| dx_d\, dy\\ &\leq \int_{K_\delta} |\nabla B\cdot e_d|(y,x_d)\, dy\, dx_d\\ &\leq \int_{K_\delta} |\nabla B|(y,x_d)\, dy\, dx_d\,. \qquad (74)\end{aligned}$$

Adding the bounds (73) and (74) we get (70) □

4.2. **Commutator estimate.** In this subsection we use the technical proposition proved above in order to show the key commutator estimate which, together with Lemma 4.8 will give Theorem 4.1. In order to state it we introduce the following notation.

Definition 4.5. *For any $\eta\in C^\infty_c(\mathbb{R}^d)$ and any matrix M we define*

$$\Lambda(M,\eta) := \int_{\mathbb{R}^d} |\nabla\eta(z)\cdot M\cdot z|\, dz\,. \qquad (75)$$

Proposition 4.6 (Commutators estimate)**.** *Let $B\in BV\cap L^\infty(\Omega,\mathbb{R}^d)$ and $w\in L^\infty(\Omega,\mathbb{R}^k)$. Assume η is an even convolution kernel and denote by M the Borel matrix–valued measure given by the Radon–Nykodim decomposition $DB = M|DB|$. Then:*

(a) *The commutators (67) are induced by measures and the total variation of these measures is uniformly bounded on any compact subset of Ω;*
(b) *Any weak* limit σ of a subsequence of $\{|T_\delta|\}_{\delta\downarrow 0}$ as $\delta\downarrow 0$ is a singular measure which satisfies the bound*

$$\sigma\llcorner A \;\leq\; \|w\|_{L^\infty(A)}\big(|D^s\cdot B| + \Lambda(M,\eta)|D^sB|\big) \quad \textit{for any open set } A\subset\subset\Omega. \qquad (76)$$

Proof. Let $\delta>0$ be fixed and choose $\lambda>0$ such that the support of η is contained in $B_\lambda(0)$. Next, let A be any open set such that $\delta\lambda<\mathrm{dist}\,(A,\partial\Omega)$. First of all, note that, in A, we have

$$T_\delta \;=\; r_\delta\mathscr{L}^d - w*\eta_\delta\, D\cdot B\,, \qquad (77)$$

where r_δ is an L^1 function which will be computed below. Note that the formula $w*\eta_\delta\, D\cdot B$ makes sense, because $D\cdot B$ is a measure and $w*\eta_\delta$ is a continuous function.

Indeed, fix a test function $\varphi \in C_c^\infty(A)$ and notice that

$$\begin{aligned}
\langle T_\delta^i, \varphi\rangle &= \langle D\cdot((w^i B)*\eta_\delta), \varphi\rangle - \langle D\cdot(w^i*\eta_\delta B),\varphi\rangle \\
&= \int_{\mathbb{R}^d} D_x \cdot \left(\int_{\mathbb{R}^d} w(y)B(y)\eta_\delta(x-y)\,dy\right)\varphi(x)\,dx + \int_{\mathbb{R}^d} w^i*\eta_\delta B\cdot\nabla\varphi \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(y)B(y)\cdot\nabla_x\eta_\delta(x-y)\,dy\,\varphi(x)\,dx \\
&\quad -\int_{\mathbb{R}^d}\nabla(w^i*\eta_\delta)\cdot B\,\varphi - \int_{\mathbb{R}^d} w^i*\eta_\delta\varphi\,d[D\cdot B] \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} w^i(y)B(y)\cdot\nabla_x\eta_\delta(x-y)\,dy\right)\varphi(x)\,dx \\
&\quad + \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} w^i(y)\nabla_y\eta_\delta(x-y)\,dy\right)\cdot B(x)\,\varphi(x)\,dx - \int_{\mathbb{R}^d} w^i*\eta_\delta\varphi\,d[D\cdot B] \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} w^i(y)(B(x)-B(y))\cdot\nabla_y\eta_\delta(x-y)\,dy\right)\varphi(x)\,dx - \int_{\mathbb{R}^d} w^i*\eta_\delta\varphi\,d[D\cdot B]\,.
\end{aligned}$$

This proves (77) with

$$\begin{aligned}
r_\delta(x) &= \int_{\mathbb{R}^d} w(y)(B(x)-B(y))\cdot\nabla_y\eta_\delta(x-y)\,dy \\
&= -\int_{\mathbb{R}^d} w(x+\delta y)\left[\frac{B(x+\delta y)-B(x)}{\delta}\cdot\nabla\eta(y)\right]\,dy\,. \qquad (78)
\end{aligned}$$

We denote by $\nabla\cdot B$ the Radon–Nykodim derivative of the measure $D\cdot B$ with respect to $\mathscr{L}^d$, that is $D\cdot B = D^s\cdot B + \nabla\cdot B\mathscr{L}^d$. Thus, we have $T_\delta = (r_\delta - w*\eta_\delta\nabla\cdot B)\mathscr{L}^d - w*\eta_\delta D^s\cdot B$, and

$$|T_\delta| = |r_\delta - w*\eta_\delta\nabla\cdot B|\mathscr{L}^d + |w*\eta_\delta||D^s\cdot B|\,. \qquad (79)$$

Using Proposition 4.3 we write r_δ as $r_{1,\delta}+r_{2,\delta}$, where

$$\begin{aligned}
r_{1,\delta}(x) &:= -\int_{\mathbb{R}^d} w(x+\delta y)B_{1,\delta}(y)(x)\cdot\nabla\eta(y)\,dy \\
r_{2,\delta}(x) &:= -\int_{\mathbb{R}^d} w(x+\delta y)B_{2,\delta}(y)(x)\cdot\nabla\eta(y)\,dy
\end{aligned}$$

Let σ be the weak* limit of a subsequence of $|T_\delta|$, and fix a nonnegative $\varphi \in C_c(A)$. Then we get

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi\, d\sigma \quad \leq \quad \limsup_{\delta\downarrow 0} \Bigg\{ & \int_{\mathbb{R}^d} \varphi(x)\big|r_{1,\delta}(x) - w * \eta_\delta(x)\, \nabla\cdot B(x)\big|\, dx \\ & + \int_{\mathbb{R}^d} \varphi(x)\big|r_{2,\delta}(x)\big|\, dx \\ & + \int_{\mathbb{R}^d} \varphi(x)|w*\eta_\delta(x)|\, d|D^s\cdot B|(x) \Bigg\}\,. \end{aligned} \tag{80}$$

We now analyze the behavior of the three integrals above.

First Integral From Proposition 4.3(a) and (c), and from the strong $L^1_{\rm loc}$ convergence of $w*\eta_\delta$ to w, it follows that

$$\begin{aligned} & \lim_{\delta\downarrow 0} \int_{\mathbb{R}^d} \varphi(x)\big|r_{1,\delta}(x) - w*\eta_\delta(x)\,\nabla\cdot B(x)\big|\, dx \\ = & \int_{\mathbb{R}^d} \varphi(x) \left| - \int_{\mathbb{R}^d} w(x)\,[\nabla\eta(y)\cdot \nabla B(x)\cdot y]\, dy - w(x)\,\nabla\cdot B(x)\right|\, dx\,. \end{aligned} \tag{81}$$

Let $\mathcal{B}_{ij}(x)$ be the components of $\nabla B(x)$. For every $x\in\mathbb{R}^d$ we then compute

$$\begin{aligned} \int_{\mathbb{R}^d} w(x)\,[\nabla\eta(y)\cdot\nabla B(x)\cdot y]\; dy \;&=\; w(x)\sum_{i,j}\mathcal{B}_{ij}(x)\int_{\mathbb{R}^d}\partial_{y_i}\eta(y)\, y_j\, dy \\ &=\; -w(x)\sum_i \mathcal{B}_{ii}(x)\int_{\mathbb{R}^d}\eta(y)\, dy \;=\; -w(x)\,\nabla\cdot B(x)\,, \end{aligned}$$

and therefore (81) vanishes.

Second Integral From now on, δ is assumed to be so small that if $\operatorname{supp}\varphi + \operatorname{supp}\eta_\delta \subset A$. Let us write $D^sB = M|D^sB|$, set $K_t := \{\varphi \geq t\}$ and write

$$\int_{\mathbb{R}^d} \varphi(x)|r_{2,\delta}|(x)\, dx \;=\; \int_0^\infty \int_{K_t} |r_{2,\delta}(x)|\, dx\, dt\,. \tag{82}$$

Note that $K_t = \emptyset$ for $t > \|\varphi\|_{C^0} =: T$ and $K_t \subset \operatorname{supp}(\varphi) =: \Gamma$ for $t > 0$. On the other hand $\int_\Gamma |r_{2,\delta}(x)|dx$ is bounded by a constant C independent of δ by Proposition 4.3(c). This means that the functions $t \mapsto \int_{K_t}|r_{2,\delta}(x)|dx$ are bounded by the L^1 function $t\mapsto C\,\mathbf{1}_{]0,T]}(t)$. Hence, by the Dominated Convergence Theorem

$$\limsup_{\delta\downarrow 0}\int_{\mathbb{R}^d}\varphi(x)|r_{2,\delta}|(x)\, dx \;\leq\; \int_0^\infty \left\{\limsup_{\delta\downarrow 0}\int_{K_t}|r_{2,\delta}(x)|\, dx\right\} dt\,. \tag{83}$$

Next, fix any compact set K, and consider

$$\int_K |r_{2,\delta}(x)|\, dx \;\leq\; \|w\|_{L^\infty(A)} \int_{\operatorname{supp}(\eta)}\int_K |B_{2,\delta}(y)(x)\cdot\nabla\eta(y)|\, dx\, dy\,. \tag{84}$$

By the bound (c) in Proposition 4.3, the function

$$y \;\mapsto\; \int_K |B_{2,\delta}(y)(x)\cdot\nabla\eta(y)|\,dx \tag{85}$$

is uniformly bounded for $y \in \operatorname{supp}(\eta)$. Hence, again by the Dominated Convergence Theorem:

$$\limsup_{\delta\downarrow 0}\int_K |r_{2,\delta}(x)|\,dx \;\leq\; \|w\|_{L^\infty(A)}\int_{\mathbb{R}^d}\left\{\limsup_{\delta\downarrow 0}\int_K |B_{2,\delta}(y)(x)\cdot\nabla\eta(y)|\,dx\right\}dy\,. \tag{86}$$

For any fixed y, use Remark 4.4 to get $B_{2,\delta}(y)(x)\cdot\nabla\eta(y) = [B\cdot\nabla\eta(y)]_{2,\delta}(y)(x)$. By Proposition 4.3(b) we then conclude

$$\limsup_{\delta\downarrow 0}\int_K |r_{2,\delta}(x)|\,dx \;\leq\; \|w\|_{L^\infty(A)}\int_{\mathbb{R}^d}|D^s(B\cdot\nabla\eta(y))\cdot y|(K)\,dy\,. \tag{87}$$

On the other hand

$$|D^s(B\cdot\nabla\eta(y))\cdot y|(K) \;=\; \int_K |\nabla\eta(y)\cdot M(x)\cdot y|\,d|D^sB|(x)\,. \tag{88}$$

Using (86), (87), and (88), and exchanging the order of integration, we get

$$\limsup_{\delta\downarrow 0}\int_K |r_{2,\delta}(x)|\,dx \;\leq\; \|w\|_{L^\infty(A)}\int_K\left[\int_{\mathbb{R}^d}|\nabla\eta(y)\cdot M(x)\cdot y|\,dy\right]d|D^sB|(x)\,. \tag{89}$$

Plugging (89) into (83), and recalling the definition of $\Lambda(M,\eta)$, we get

$$\begin{aligned}\limsup_{\delta\downarrow 0}\int_{\mathbb{R}^d}\varphi(x)|r_{2,\delta}|(x)\,dx \;&\leq\; \|w\|_{L^\infty(A)}\int_0^\infty\int_{K_t}\varphi(x)\Lambda(M(x),\eta)\,d|D^sB|(x)\,dt\\ &=\; \|w\|_{L^\infty(A)}\int\varphi(x)\Lambda(M(x),\eta)\,d|D^sB|(x).\end{aligned} \tag{90}$$

Third Integral Finally, we have

$$\lim_{\delta\downarrow 0}\int_{\mathbb{R}^d}\varphi(x)|w*\eta_\delta(x)|\,d|D^s\cdot B|(x) \;\leq\; \|w\|_{L^\infty(A)}\int_{\mathbb{R}^d}\varphi(x)\,d|D^s\cdot B|(x)\,. \tag{91}$$

Conclusion From (80), (81), (90), and (91) we get

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi\,d\sigma \;&\leq\; \|w\|_{L^\infty(A)}\int_{\mathbb{R}^d}\varphi(x)\Lambda(M(x),\eta)\,d|D^sB|(x)\\ &\quad+\|w\|_{L^\infty(A)}\int_{\mathbb{R}^d}\varphi(x)\,d|D^s\cdot B|(x)\end{aligned} \tag{92}$$

for every nonnegative $\varphi \in C_c(A)$, which implies the desired estimate

$$\sigma\llcorner A \;\leq\; \|w\|_{L^\infty(A)}\Lambda(M,\eta)|D^sB| + \|w\|_{L^\infty(A)}|D^s\cdot B|\,.$$

□

4.3. **Bouchut's Lemma and Alberti's Lemma.** The following lemma was first proved by Bouchut in [15] and it was the starting point of Ambrosio's original proof of his commutator estimate (see [2]).

Lemma 4.7 (Bouchut)**.** *Let*

$$K \;:=\; \left\{\eta \in C^\infty_c(B_1(0)) \text{ such that } \eta \geq 0 \text{ is even, and } \textstyle\int_{B_1(0)} \eta = 1\right\}. \tag{93}$$

If $D \subset K$ is dense with respect to the strong $W^{1,1}$ topology, then for every ξ, $\chi \in \mathbb{R}^d$ we have

$$\inf_{\eta\in D} \Lambda(\chi\otimes\xi, \eta) \;=\; |\langle \xi, \chi\rangle| \;=\; \big|\mathrm{tr}\,(\chi\otimes\xi)\big|\,. \tag{94}$$

However, Ambrosio's original proof made use of the difficult Rank–one Theorem. Recently, Alberti has proposed an elementary proof of the following generalization of Bouchut's Lemma

Lemma 4.8 (Alberti)**.** *Let K be as in Lemma 4.7 and let M be a $d\times d$ matrix. Then*

$$\inf_{\eta\in D} \Lambda(M, \eta) \;=\; \big|\mathrm{tr}\, M\big|\,. \tag{95}$$

Proof of Lemma 4.7. Set $M := \chi\otimes\xi$. Note that, since the map $\eta \in C^\infty_c(B_1(0)) \mapsto \Lambda(M,\eta)$ is continuous with respect to the strong $W^{1,1}$ topology, it is sufficient to prove that

$$\inf_{\eta\in K} \Lambda(M, \eta) \;=\; |\mathrm{tr}\, M|\,, \tag{96}$$

where K is the set in (93).

If $d = 2$ we can fix an orthonormal basis of coordinates z_1, z_2 in such a way that $\xi = (a, b)$ and $\chi = (0, c)$. Consider the rectangle $R_\varepsilon := [-\varepsilon/2, \varepsilon/2] \times [-1/2, 1/2]$ and consider the kernel $\eta_\varepsilon := \frac{1}{\varepsilon}\mathbf{1}_{R_\varepsilon}$. Let $\zeta \in K$ and denote by ζ_δ the family of mollifiers generated by ζ. Clearly $\eta_\varepsilon * \zeta_\delta \in K$ for $\varepsilon + \delta$ small enough.

Denote by $\nu = (\nu_1, \nu_2)$ the unit normal to ∂R_ε and recall that

$$\lim_{\delta\downarrow 0} \left|\frac{\partial(\eta_\varepsilon * \zeta_\delta)}{\partial z_i}\right| \rightharpoonup^* \frac{|\nu_i|}{\varepsilon}\mathscr{H}^1 \llcorner \partial R_\varepsilon\,. \tag{97}$$

in the sense of measures.

Thus, we can compute

$$\begin{aligned}
\limsup_{\delta\downarrow 0} \Lambda(M, \eta_\varepsilon * \zeta_\delta) \;&\leq\; \limsup_{\delta\downarrow 0} \int_{\mathbb{R}^2} \big(|az_1| + |bz_2|\big)|c| \left|\frac{\partial(\eta_\varepsilon * \zeta_\delta)}{\partial z_2}\right| dz_1 dz_2\\
&=\; \frac{2|c|}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \left(|az_1| + \frac{|b|}{2}\right) dz_1 \;=\; |ac|\frac{\varepsilon}{2} + |bc|\,.
\end{aligned}$$

Note that $bc = \mathrm{tr}\, M$. Thus, if we define the convolution kernels $\lambda_{\varepsilon,\delta} := \eta_\varepsilon * \zeta_\delta$ we get:

$$\limsup_{\varepsilon\downarrow 0}\limsup_{\delta\downarrow 0} \Lambda(M, \eta_\varepsilon * \zeta_\delta) \;\leq\; |\mathrm{tr}\, M|\,. \tag{98}$$

For $d \geq 2$ we consider a system of coordinates $x_1, x_2, \dots, x_d$ such that $\eta = (a, b, 0, \dots, 0)$, $\xi = (0, c, 0, \dots, 0)$ and we define the convolution kernels

$$\lambda_{\varepsilon,\delta}(x) \;:=\; [\eta_\varepsilon * \zeta_\delta](x_1, x_2) \cdot \zeta(x_3) \cdot \ldots \cdot \zeta(x_d)\,.$$

Then (98) holds as well and we conclude that for any d we have

$$\inf_{\eta \in K} \Lambda(M, \eta) \;\leq\; |\mathrm{tr}\, M|\,.$$

On the other hand, for every $\eta \in K$ and every $d \times d$ matrix M, we have

$$\begin{aligned}\Lambda(M,\eta) \;&\geq\; \left|\int_{B_1(0)} \langle M \cdot y, \nabla \eta(y) \rangle\right| \;=\; \left|\sum_{k,j} M_{jk} \int_{B_1(0)} y_j \frac{\partial \eta}{z_k}(y)\right| dy \\ &=\; \left|-\sum_{k,j} M_{jk} \int_{B_1(0)} \delta_{jk} \eta(y)\, dy\right| \;=\; |\mathrm{tr}\, M|\,. \qquad (99)\end{aligned}$$

This concludes the proof. □

The proof of the second Lemma follows mainly [3].

Proof of Lemma 4.8. As in the first proof, we note that it is sufficient to prove that

$$\inf_{\eta \in K} \Lambda(M, \eta) \;=\; |\mathrm{tr}\, M|\,, \qquad (100)$$

and that the lower bound $\inf_{\eta\in K} \Lambda(M, \eta) \geq |\mathrm{tr}\, M|$ follows immediately from (99) (the argument leading to (99) does need the assumption $M = \chi \otimes \xi$). Therefore it remains to show the upper bound. Again by the identity $\langle M \cdot z, \nabla \eta(z)\rangle = \mathrm{div}\,(M \cdot z\eta(z)) - \mathrm{tr}\, M\eta(z)$, it suffices to show that for every $T > 0$ there exists $\eta \in K$ such that

$$\int_{\mathbb{R}^n} |\mathrm{div}(M \cdot z\eta(z))|\, dz \;\leq\; \frac{2}{T}\,. \qquad (101)$$

Given a smooth nonnegative convolution kernel θ with compact support, we claim that the function

$$\eta(z) \;=\; \frac{1}{T} \int_0^T \theta(e^{-tM} \cdot z)\, e^{-t\mathrm{tr}\, M}\, dt$$

has the required properties. Here e^{tM} is the matrix

$$\sum_{i=0}^{\infty} \frac{t^i M^i}{i!}\,.$$

Thus $e^{tM} \cdot z$ is just the solution of the ODE $\dot\gamma = M \cdot \gamma$ with initial condition $\gamma(0) = z$, and $e^{-t\mathrm{tr}\, M}$ is the determinant of e^{-tM}. The usual change of variables yields

$$\begin{aligned}\int \eta(z)\varphi(z)\, dz \;&=\; \frac{1}{T}\int_0^T \int \varphi(z)\theta(e^{-tM} \cdot z) e^{-t\mathrm{tr}\, M}\, dz\, dt \\ &=\; \frac{1}{T}\int_0^T \int \varphi(e^{tM} \cdot \zeta)\theta(\zeta)\, d\zeta\, dt\,, \qquad (102)\end{aligned}$$

for any integrable bounded φ. Hence $\eta\mathscr{L}^d$ is the time average of the pushforward of the measure $\theta\mathscr{L}^d$ along the trajectories of $\dot\gamma = M\cdot\gamma$. This is the point of view taken in [3] to prove (101), for which we argue with the direct computations shown below.

Note that

$$\operatorname{div}(M\cdot z\eta(z)) \;=\; \frac{1}{T}\int_0^T \operatorname{div}(M\cdot z\theta(e^{-tM}\cdot z))e^{-t\operatorname{tr} M}\,dt\,.$$

We compute

$$\begin{aligned}
&\operatorname{div}(M\cdot z\theta(e^{-tM}\cdot z))e^{-t\operatorname{tr} M}\\
=\;& \operatorname{tr} M\,\theta(e^{-tM}\cdot z)e^{-t\operatorname{tr} M} + \langle M\cdot z, e^{-tM}\cdot\nabla\theta(e^{-tM}\cdot z)\rangle e^{-t\operatorname{tr} M}\\
=\;& -\frac{d}{dt}\left(e^{-t\operatorname{tr} M}\right)\theta(e^{-tM}\cdot z) + \langle e^{-tM}\cdot M\cdot z, \nabla\theta(e^{-tM}\cdot z)\rangle e^{-t\operatorname{tr} M}\\
=\;& -\frac{d}{dt}\left(e^{-t\operatorname{tr} M}\right)\theta(e^{-tM}\cdot z) - \left\langle \frac{d}{dt}\left(e^{-tM}\cdot z\right), \nabla\theta(e^{-tM}\cdot z)\right\rangle e^{-t\operatorname{tr} M}\\
=\;& -\frac{d}{dt}\left(e^{-t\operatorname{tr} M}\right)\theta(e^{-tM}\cdot z) - \frac{d}{dt}\left(\theta(e^{-tM}\cdot z)\right)e^{-t\operatorname{tr} M} \;=\; -\frac{d}{dt}\left(\theta(e^{-tM}\cdot z)e^{-t\operatorname{tr} M}\right)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^d} |\operatorname{div}(M\cdot z\eta(z))|\,dz \;&=\; \int_{\mathbb{R}^d}\frac{1}{T}\left|\int_0^T \operatorname{div}(M\cdot z\theta(e^{-tM}\cdot z))e^{-t\operatorname{tr} M}\,dt\right|\,dz\\
&=\; \int_{\mathbb{R}^d}\frac{1}{T}\left|\int_0^T \frac{d}{dt}\left(\theta(e^{-tM}\cdot z)e^{-t\operatorname{tr} M}\right)\,dt\right|\,dz\\
&=\; \int_{\mathbb{R}^d}\frac{1}{T}\left|\theta(e^{-TM}\cdot z)e^{-T\operatorname{tr} M} - \theta(z)\right|\,dz\\
&\le\; \frac{1}{T}\left(\int_{\mathbb{R}^d}\theta(e^{-TM}\cdot z)e^{-T\operatorname{tr} M}\,dz + \int_{\mathbb{R}^d}\theta(z)\,dz\right)\\
&=\; \frac{1}{T}\left(\int_{\mathbb{R}^d}\theta(\zeta)\,d\zeta + \int_{\mathbb{R}^d}\theta(z)\,dz\right) \;=\; \frac{2}{T}\,,
\end{aligned}$$

where in the last line we changed variables as in (102). This shows (101) and concludes the proof. □

4.4. **Proof of Theorem 4.1.** We finally come to the Proof of Theorem 4.1

Proof of Theorem 4.1. Let η be any smooth even convolution kernel. Set $\sigma_\delta := |T^i_\delta|$. From Proposition 4.6 we know that the total variation of these measures is uniformly bounded. Thus, recalling the computation of Section 4, and in particular (66), we conclude that $D\cdot(H(w)B)$ is a measure. Next, set

$$\alpha \;:=\; D\cdot(H(w)B) - \left(H(w) - \sum_{i=1}^d \frac{\partial H}{\partial v_i}(w)w^i\right)D^a\cdot B$$

and let σ be the weak* limit of any subsequence of the measures $\{\sigma_\delta\}$. Then, from (68) we get

$$|\alpha| \;\le\; C\sigma + C|D^s\cdot B|\,. \tag{103}$$

According to Proposition 4.6(b), this gives $|\alpha| << |D^sB|$, and thus we have $|\alpha| = g|D^sB|$ for some nonnegative Borel function g. Denote by M the Radon–Nykodim derivative of D^sB with respect to $|D^sB|$. Then $|D^s\cdot B| = \mathrm{tr}\,M|D^sB|$. Thus, from (68) and (76) we conclude

$$g(x) \;\le\; C\big(|\mathrm{tr}\,M(x)| + \Lambda(M(x),\eta)\big) \qquad \text{for } |D^sB|\text{–a.e. } x. \tag{104}$$

Note that (104) holds for any even convolution kernel η. Let K be as in Lemma 4.8 and choose a countable set $D\subset K$ which is dense in the $W^{1,1}$ topology. Then

$$g(x) \;\le\; C\big(|\mathrm{tr}\,M(x)| + \inf_{\eta\in D}\Lambda(M(x),\eta)\big) \qquad \text{for } |D^sB|\text{–a.e. } x. \tag{105}$$

Therefore, from Lemma 4.8 we conclude

$$g(x) \;\le\; C|\mathrm{tr}\,M(x)|\,,$$

which implies $|\alpha|\le C|D^s\cdot B|$. Following the argument, one can readily check that C depends only on $R := \|w\|_\infty$ and $\|H\|_{C^1(B_R(0))}$. □

Remark 4.9. *In this last step, the original proof of Ambrosio in* [2] *used Bouchut's Lemma and Alberti's Rank–one Theorem 2.13. Indeed, by Theorem 2.13 there exists two Borel vector valued maps χ,ξ such that $M(x) = \chi(x)\otimes\xi(x)$ for $|D^sB|$–a.e. x. Therefore, using this information one might rewrite* (104) *and* (105) *with*

$$g(x) \;\le\; C\big(|\mathrm{tr}\,M(x)| + \Lambda(\chi(x)\otimes\xi(x),\eta)\big) \qquad \textit{for } |D^sB|\textit{–a.e. } x. \tag{106}$$

and

$$g(x) \;\le\; C\big(|\mathrm{tr}\,M(x)| + \inf_{\eta\in D}\Lambda(\chi(x)\otimes\xi(x),\eta)\big) \qquad \textit{for } |D^sB|\textit{–a.e. } x. \tag{107}$$

From (107) *it suffices to apply Lemma 4.7 to get*

$$g(x) \;\le\; C|\mathrm{tr}\,M(x)|\,.$$

5. Existence, uniqueness, and stability for the Keyfitz and Kranzer system

In this section we consider the Cauchy problem for the Keyfitz and Kranzer system

$$\begin{cases} \partial_t u^i + \sum_{\alpha=1}^m \partial_{x_\alpha}(g^\alpha(|u|)u^i) \;=\; 0 \\[2ex] u^i(0,\cdot) \;=\; \overline{u}^i(\cdot) \end{cases} \tag{108}$$

Before stating the main theorem, we recall the notion of entropy solution of a scalar conservation law and the classical theorem of Kruzhkov, which provides existence, stability and uniqueness of entropy solutions to the Cauchy problem for scalar laws.

Definition 5.1. *Let $g \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^m)$. A pair (h,q) of functions $h \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R})$, $q \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^m)$ is called an* entropy–entropy flux pair *relative to g if*

$$q' \;=\; h'g' \qquad \mathscr{L}^1\text{–almost everywhere on } \mathbb{R}. \tag{109}$$

If, in addition, h is a convex function, then we say that (h,q) is a convex entropy–entropy flux *pair. A weak solution $\rho \in L^\infty(\mathbb{R}^+_t \times \mathbb{R}^m_x)$ of*

$$\begin{cases} \partial_t \rho + D_x \cdot [g(\rho)] \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; \overline{\rho}(\cdot) \end{cases} \tag{110}$$

is called an entropy solution if $\partial_t[h(\rho)] + D_x \cdot [q(\rho)] \le 0$ in the sense of distributions for every convex entropy–entropy flux pair (h,q).

In what follows, we say that $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^m)$ *has a strong trace $\overline{\rho}$ at* 0 if for every bounded $\Omega \subset \mathbb{R}^n$ we have

$$\lim_{T\downarrow 0} \frac{1}{T} \int_{[0,T]\times\Omega} |\rho(t,x) - \overline{\rho}(x)|\, dx\, dt \;=\; 0\,.$$

Theorem 5.2 ([36] Kruzhkov)**.** *Let $g \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^m)$ and $\overline{\rho} \in L^\infty$. Then there exists a* unique *entropy solution ρ of* (110) *with a strong trace at $t=0$. If in addition $\overline{\rho} \in BV_{loc}(\mathbb{R}^m)$, then, for every open set $A \subset\subset \mathbb{R}^m$ and for every $T \in]0,\infty[$, there exists an open set $A' \subset\subset \mathbb{R}^m$ (whose diameter depends only on A, T, g and $\|\overline{\rho}\|_\infty$) such that*

$$\|\rho\|_{BV(]0,T[\times A)} \;\le\; \|\overline{\rho}\|_{BV(A')}\,. \tag{111}$$

Often, in what follows we will use the terminology *Kruzhkov solution* for entropy solutions of (110) with a strong trace at $t=0$.

Remark 5.3. In many cases the requirement that ρ has strong trace at 0 is not necessary. Indeed, when g is sufficiently regular and satisfies suitable assumptions of genuine nonlinearity, Vasseur proved in [39] that *any* entropy solution has a strong trace at 0.

We are now ready to introduce the particular class of weak solutions of (108) for which we are able to prove existence, uniqueness, and continuous dependence with respect to the initial data.

Definition 5.4. *A weak solution u of* (108) *is called a* renormalized entropy solution *if $|u|$ is an Kruzhkov solution of the scalar law*

$$\begin{cases} \partial_t \rho + \sum\limits_{\alpha=1}^m \partial_{x_\alpha}\big(g^\alpha(\rho)\rho\big) \;=\; 0 \\ \\ \rho(0,\cdot) \;=\; \overline{\rho}(\cdot)\,. \end{cases} \tag{112}$$

In the class of renormalized entropy solutions we have the following well–posedness theorem for bounded initial data $\overline{u}$ such that $|\overline{u}| \in BV_{loc}$

Theorem 5.5. *Let* $g \in W^{1,\infty}_{loc}(\mathbb{R}, \mathbb{R}^k)$ *and* $|\overline{u}| \in L^\infty \cap BV_{loc}$. *Then there exists a unique renormalized entropy solution* u *of* (108). *If* $\overline{u}^j$ *is a sequence of initial data such that*

(a) $|\overline{u}^j| \leq C$ *for some constant* C,
(b) *for every bounded open set* Ω, *there is a constant* $C(\Omega)$ *such that* $\big\| |\overline{u}^j| \big\|_{BV(\Omega)} \leq C(\Omega)$,
(c) $\overline{u}^j \to \overline{u}$ *strongly in* L^1_{loc},

then the corresponding renormalized entropy solutions converge strongly in L^1_{loc} *to* u.

The suggestion of using the terminology "renormalized entropy solutions" has been taken from [32]. This terminology is more appropriate than the one of "entropy solutions" used in [8], because the usual notion of *entropy* (or *admissible*) solution of a hyperbolic system of conservation laws does not coincide with the one of renormalized entropy solutions. Let us recall the usual notion of entropy solution for systems (cp. Section 4.3 of [22]):

Definition 5.6. *Let* $F^\alpha : \mathbb{R}^k \to \mathbb{R}^k$, $\alpha = 1, \ldots, n$, *be Lipschitz and consider the system*

$$\partial_t u + \sum_{\alpha=1}^{m} \partial_{x_\alpha}[F^\alpha(u)] \;=\; 0 \qquad u : \Omega \subset \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^k \,. \tag{113}$$

A pair (H, Q) *of functions* $H \in W^{1,\infty}_{loc}(\mathbb{R}^k, \mathbb{R})$, $Q \in W^{1,\infty}_{loc}(\mathbb{R}^k, \mathbb{R}^m)$ *is called a* convex entropy–entropy flux pair *for the system* (113) *if* H *is convex and if* $DQ^\alpha = DH \cdot DF^\alpha$, *for every* $\alpha \in \{1, \ldots, m\}$.

A distributional solution u *of* (113) *supplemented by the initial condition*

$$u(0, \cdot) \;=\; \overline{u}(\cdot)$$

is called an entropy *solution if for every convex entropy–entropy flux pair* (H, Q) *and for every smooth test function* $\psi \geq 0$,

$$\int_{t>0}\int_{\mathbb{R}^m} [\partial_t\psi(t,z)\, H(u(t,z)) + \nabla_z\psi(t,z) \cdot Q(u(t,z))]\, dt\, dz + \int_{\mathbb{R}^m} \psi(0,z)\eta(\overline{u}(z))\, dz \;\geq\; 0\,. \tag{114}$$

The (nonpositive) entropy production *measure*

$$\partial_t[H(u)] + D_x \cdot [Q(u)]$$

will be denoted by μ_H.

The system of Keyfitz and Kranzer corresponds to the particular case $F(u) = u \otimes g(|u|)$. We will later show that, under suitable assumptions on g, for every convex entropy H for (108) there exists a convex function $h : \mathbb{R}^k \to \mathbb{R}$ and a Lipschitz function $\hat{H} : \mathbf{S}^{k-1} \to \mathbb{R}$ such that

$$H(v) \;=\; h(|v|) + |v|\hat{H}(v/|v|) \qquad \text{for every } v \neq 0$$

(see Lemma 5.11 and compare with Lemma 1.1 of [32]).

Using this lemma we will show that if u is a renormalized entropy solution, then u is an entropy solution in the sense of Definition 5.6.

Proposition 5.7. *Assume $g \in C^1$ and $\mathscr{L}^1(\{s > 0 : g'(s) = 0\}) = 0$. Then every renormalized entropy solution of* (108) *is an entropy solution.*

Actually we expect this statement to be true even if we drop the assumption $\mathscr{L}^1(\{s > 0 : g'(s) = 0\}) = 0$. However Lemma 5.11 does not hold in general and therefore a more refined approach is required.

Clearly, another natural question is whether the opposite inclusion

$$\big\{\text{entropy solutions}\big\} \;\subset\; \big\{\text{renormalized entropy solutions}\big\}$$

holds. It can be shown that, already in one space dimension, there exist entropy solutions of (108) which are not renormalized entropy solutions (see for instance [22]). This is essentially caused by the degeneration at the origin of the the hyperbolicity of the Keyfitz and Kranzer system. However under appropriate assumptions on the initial data, it is reasonable to expect that any entropy solution coincides with the unique renormalized entropy solution. In particular we propose the following

Conjecture 5.8. *Let u be a bounded entropy solution of* (108) *and denote by C the closure of the convex hull of its essential image. If $0 \notin C$ or if it is an extremal point of C, then u is a renormalized entropy solution.*

A partial answer to this Conjecture is given by the following

Proposition 5.9. *Let $f \in W^{1,\infty}_{loc}$ and $\overline{u} \in L^\infty(\mathbb{R}^m, \mathbb{R}^k)$. Denote by C be the closure of the convex hull of the essential image of u and assume that*

(a) *Either $0 \notin C$ or it is an extremal point of C;*
(b) *u is a bounded entropy solution of* (108)*;*
(c) *$u \in BV(]0,T[\times\Omega)$ for some $T > 0$ and for some bounded open $\Omega \subset \mathbb{R}^m$.*

Then u is a renormalized entropy solution of (108) *on $]0,T[\times\Omega$.*

5.1. **Proof of Theorem 5.5.** The proof of Theorem 5.5 follows from the theory of transport equations for nearly incompressible fields via Ambrosio's renormalization Theorem. More precisely, the key point is the following

Lemma 5.10. *Let $\rho \in L^\infty([0,\infty[\times\mathbb{R}^m), b \in L^\infty([0,\infty[\times\mathbb{R}^m, \mathbb{R}^m)$ be such that*

- *$b, \rho \in BV([0,T[\times K)$ for every compact set K;*
- (29) *holds, that is $\partial_t \rho + D_x \cdot (\rho b) = 0$;*
- *$\rho(0,\cdot) \in BV_{loc}$.*

Then the pair (b,ρ) has the renormalization property.

Proof. Recall that, from the trace properties of BV functions we have

$$\lim_{T\downarrow 0} \frac{1}{T} \int_0^T \int_K |\rho(t,x) - \rho(0,x)| + |b(t,x) - b(0,x)|\, dx\, dt \;=\; 0$$

for every compact set $K \subset \mathbb{R}^m$. We define $\hat{\rho} \in BV_{loc}(\mathbb{R}^{m+1}), \hat{b} \in BV_{loc}(\mathbb{R}^{m+1})$ by setting

$$\hat{\rho}(t,x) \;=\; \begin{cases} \rho(0,x) & \text{if } t \leq 0 \\ \rho(t,x) & \text{if } t > 0 \end{cases} \qquad \text{and} \qquad \hat{b}(t,x) \;=\; \begin{cases} 0 & \text{if } t \leq 0 \\ b(t,x) & \text{if } t > 0\,. \end{cases}$$

Now, let $u \in L^\infty([0,\infty[\times\mathbb{R}^m)$ and $\overline{u} \in L^\infty(\mathbb{R}^m)$ be such that

$$\begin{cases} \partial_t(\rho u) + D_x \cdot (b\rho u) \;=\; 0 \\ \\ [\rho u]\,(0,\cdot) \;=\; \rho(0,\cdot)\overline{u} \end{cases} \tag{115}$$

and define

$$\hat{u}(t,x) \;=\; \begin{cases} \overline{u}(x) & \text{if } t < 0 \\ u(t,x) & \text{if } t \geq 0. \end{cases}$$

Then $\partial_t(\hat{\rho}\hat{u}) + D_x \cdot (\hat{\rho}\hat{u}\hat{b}) = 0$ distributionally on $\mathbb{R}^{m+1}$. Thus, if we apply Theorem 4.1 to $B = (\hat{\rho}, \hat{\rho}\hat{b})$, $\hat{u}$ and $H(v) = v^2$, since $D \cdot B = 0$, we conclude that

$$\partial_t(\hat{u}^2\hat{\rho}) + D_x \cdot (\hat{u}^2\hat{\rho}\hat{b}) \;=\; 0\,.$$

From Lemma 3.7 we have that, up to change $\hat{\rho}\hat{u}^2$ on a set of measure zero, the map $t \mapsto \hat{\rho}(t,\cdot)\hat{u}^2(t,\cdot)$ is weakly continuous. Since for $t < 0$ we have $\hat{\rho}(t,\cdot)\hat{u}^2(t,\cdot) = \rho(0,\cdot)\overline{u}^2(\cdot)$ and for $t > 0$ we have $\hat{\rho}(t,\cdot)\hat{u}^2(t,\cdot) = \rho(t,\cdot)u^2(t,\cdot)$ we conclude that $\rho(0,\cdot)\overline{u}^2(\cdot)$ is the trace at $t = 0$ of the function ρu^2. Thus we get

$$\begin{cases} \partial_t(\rho u^2) + D_x \cdot (b\rho u^2) \;=\; 0 \\ \\ [\rho u^2]\,(0,\cdot) \;=\; \rho(0,\cdot)\overline{u}^2\,. \end{cases}$$

With an analogous argument one shows that if

$$\begin{cases} \partial_t(\rho u) + D_x \cdot (b\rho u) \;=\; 0 \\ \\ [\rho u]\,(0,\cdot) \;=\; \rho(0,\cdot)\overline{u} \\ \\ [\rho u]\,(T,\cdot) \;=\; \rho(T,\cdot)\hat{u}\,, \end{cases} \tag{116}$$

then $v = u^2$ solves

$$\begin{cases} \partial_t(\rho v) + D_x \cdot (b\rho v) \;=\; 0 \\ \\ [\rho v]\,(0,\cdot) \;=\; \rho(0,\cdot)\overline{u}^2 \\ \\ [\rho v]\,(T,\cdot) \;=\; \rho(T,\cdot)\hat{u}^2\,. \end{cases}$$

□

Proof of Theorem 5.5. **Existence** Let g and $\overline{u}$ be as in the statement. First of all, let ρ be the Kruzhkov solution of

$$\left\{\begin{array}{rcl} \partial_t \rho + D_x \cdot (\rho g(\rho)) & = & 0 \\ \\ \rho(0,\cdot) & = & |\overline{u}|(\cdot)\,. \end{array}\right. \tag{117}$$

Then, Kruzhkov's theory gives $\|\rho\|_\infty \leq \|\overline{u}\|_\infty$ and $\rho \in BV([0,T[\times K)$ for every compact set. Since g is locally Lipschitz, $g(\rho) \in BV([0,T[\times K)$. Therefore, by Lemma 5.10, the pair $(b,\rho) := (g(\rho),\rho)$ has the renormalization property.

Next let $\overline{\theta} \in L^\infty(\mathbb{R}^n, \mathbf{S}^{k-1})$ be any function such that $\overline{u} = |\overline{u}|\overline{\theta}$ and apply Proposition 3.13 to get a bounded solution θ of

$$\left\{\begin{array}{rcl} \partial_t(\rho\theta) + D_x \cdot (\theta \otimes (\rho g(\rho))) & = & 0 \\ \\ \left[\rho\theta\right](0,\cdot) & = & \overline{\rho}(0,\cdot)\overline{\theta}(\cdot)\,. \end{array}\right. \tag{118}$$

Consider the continuous function $H : \mathbb{R}^k \to [0,\infty[$ given by $H(v) := |v|$. Applying Lemma 5.10 and Proposition 3.10 we conclude that

$$\left\{\begin{array}{rcl} \partial_t(\rho|\theta|) + D_x \cdot (\rho|\theta| g(\rho)) & = & 0 \\ \\ \left[\rho|\theta|\right](0,\cdot) & = & \overline{\rho}(0,\cdot)|\overline{\theta}(\cdot)| \;\; = \;\; \overline{\rho}(0,\cdot)\,. \end{array}\right.$$

Thus, from Proposition 3.13, it follows $\rho|\theta| = \rho$. Therefore, if we define $u := \rho\theta$, we have $|u| = \rho$ and hence

- $|u|$ is a Kruzkov solution of (117);
- u solves

$$\left\{\begin{array}{rcl} \partial_t u + D_x \cdot (u \otimes g(|u|)) & = & 0 \\ \\ u(0,\cdot) & = & \overline{u}\,. \end{array}\right.$$

Uniqueness The uniqueness follows easily from the uniqueness of Kruzhkov solutions for the Cauchy problem of scalar conservation laws and from Proposition 3.13.

Stability The stability follows directly from the stability of Kruzhkov solutions for scalar conservation laws and from Corollary 3.21. □

5.2. **Renormalized entropy solutions are entropy solutions.** In this subsection we prove Proposition 5.7. The key remark is the following lemma (see [32]):

Lemma 5.11. *Assume $g \in C^1([0,\infty[,\mathbb{R}^k)$ and $\mathscr{L}^1(\{s > 0 : g'(s) = 0\}) = 0$. Consider the map $F^\alpha \in W^{1,\infty}_{loc}(\mathbb{R}^k,\mathbb{R}^k)$ given by $F^\alpha(u) = g^\alpha(|u|)u$. If (H,Q) is a convex entropy–entropy flux pair in the sense of Definition 5.6, then there exist a convex $h \in W^{1,\infty}_{loc}([0,\infty[)$ and an $\hat{H} \in W^{1,\infty}(\mathbf{S}^{k-1})$ such that*

$$H(u) \;=\; h(|u|) + |u|\hat{H}(u/|u|) \qquad \text{for any } u \neq 0.$$

In order to simplify the notation, in what follows, if $\hat{H} : \mathbf{S}^{k-1} \to \mathbb{R}$ is a bounded function, we extend the function

$$\mathbb{R}^k \setminus \{0\} \ni u \;\to\; |u|\hat{H}(u/|u|) \in \mathbb{R}$$

by defining as 0 its value at 0. Clearly this extension is Lipschitz whenever $\hat{H}$ is Lipschitz.

Remark 5.12. *Note that at least the assumption that $\{g' = 0\}$ has empty interior is needed in order to conclude Lemma 5.11. Indeed, assume $]a, b[\subset \{g' = 0\}$. Then g is constantly equal to some vector γ on that interval. Consider any convex function $H \in C^2(\mathbb{R}^k)$ with the following properties*

- $H = 0$ *on* $\{0 \le |v| \le (a + b)/2]\}$,
- $H(v) = |v|$ *on* $\{v \in \mathbb{R}^k : |v| \ge b\}$,

and let Q be given by

- $Q(v) = H(v)\gamma$ *for* $0 \le |v| \le b$;
- $Q(v) = |v|f(|v|)$ *for* $|v| \ge b$.

Then (H, Q) is a convex entropy–entropy flux pair, but H is not necessarily of the form $h(|u|) + |u|\hat{H}(u/|u|)$.

Nonetheless we expect that the conclusion of Proposition 5.7 holds in general. Indeed, if $g' = 0$ on $[a, b]$ and u is a solution of (108) *such that $a \le |u| \le b$, then u solves k decoupled transport equations with constant coefficients. Thus u is trivially an entropy solution. However, a more refined analysis would be needed if the range of $|u|$ contains both intervals where g' vanishes and intervals where $g' \neq 0$.*

Lemma 5.11 easily implies Proposition 5.7.

Proof of Proposition 5.7. Let g be as in the proposition, let u be any renormalized entropy solution and let H, Q be an entropy–entropy flux pair. We apply Lemma 5.11 to get $H(u) = h(|u|) + |u|\hat{H}(u/|u|)$, where h is convex and $\hat{H}$ is Lipschitz. Let $q \in W^{1,\infty}(\mathbb{R})$ be such that $q(0) = Q(0)$ and $q'(r) = h'(r)g'(r)r + h'(r)g(r)$. Then it follows easily that $Q(u) = q(|u|) + |u|g(|u|)\hat{H}(u/|u|)$. Let $\psi \in C^\infty_c(]-\infty, \infty[\times\mathbb{R}^m)$ be any test function. Since $|u|$ is a Kruzkov solution of

$$\begin{cases} \partial_t\rho + D_x \cdot (g(\rho)\rho) \;=\; 0 \\[2ex] \rho(0, \cdot) \;=\; |u(0, \cdot)|\,, \end{cases}$$

we have

$$\int_{t>0}\int_{\mathbb{R}^m} [\partial_t\psi(t, z)\, h(|u(t, z)|) + \nabla_z\psi(t, z) \cdot q(|u(t, z)|)]\, dt\, dz + \int_{\mathbb{R}^m} \psi(0, z)h(|\overline{u}(z)|)\, dz \;\ge\; 0\,. \tag{119}$$

Moreover, from the renormalization property applied to θ we must have

$$\int_{t>0}\int_{\mathbb{R}^m} |u|(t,x)\hat{H}\left(\frac{u(t,x)}{|u|(t,x)}\right)[\partial_t\psi(t,z)\,+\nabla_z\psi(t,z)\cdot g(|u(t,z)|)]\,dt\,dz \tag{120}$$

$$+\int_{\mathbb{R}^m}\psi(0,z)\,|\overline{u}(z)|\hat{H}\left(\frac{\overline{u}(z)}{|\overline{u}(z)|}\right)dz \;=\; 0\,. \tag{121}$$

Summing (119) and (121) we conclude (114). This completes the proof. □

Proof of Lemma 5.11. If g, H, and Q satisfy the assumptions of the Lemma, then Q is a Lipschitz function and the identity

$$\nabla Q^\alpha(v) = \nabla H(v)\cdot\nabla(g^\alpha(|v|)|v|) \tag{122}$$

is valid for $\mathscr{L}^k$–a.e. $v\in\mathbb{R}^k\setminus\{0\}$.

Now consider a smooth system of coordinates $\omega_1,\ldots,\omega_{k-1}$ on $\mathbf{S}^{k-1}$ and let $\omega_1,\ldots,\omega_{k-1},r$ be polar coordinates on $\mathbb{R}^k\setminus\{0\}$. It is not difficult to see that (122) becomes

$$\left\{\begin{array}{rcl}\partial_{\omega_i}Q^\alpha(r,\omega) &=& g^\alpha(r)\partial_{\omega_i}H(r,\omega)\\[2ex] \partial_r Q^\alpha(r,\omega) &=& ((g^\alpha)'(r)r+g^\alpha(r))\partial_r H(r,\omega)\end{array}\right. \tag{123}$$

(in other words, $\omega_1,\ldots,\omega_{k-1},r$ is a coordinate system of Riemann invariants for the Keyfitz and Kranzer system).

These identities hold pointwise a.e. and hence (since Q and H are Lipschitz) in the sense of distributions. Therefore, from $\partial^2_{r\omega_i}Q^\alpha=\partial^2_{\omega_i r}Q^\alpha$ we conclude

$$\partial_r(g^\alpha(r)\partial_{\omega_i}H(r,\omega)) \;=\; \partial_{\omega_i}\big\{((g^\alpha)'(r)r+g^\alpha(r))\partial_r H(r,\omega))\big\}\,. \tag{124}$$

Recall that H is convex, and hence its second derivatives are measures. Thus

$$\partial_r(g^\alpha(r)\partial_{\omega_i}H(r,\omega)) \;=\; (g^\alpha)'(r)\partial_{\omega_i}H(r,\omega)+g^\alpha(r)\partial^2_{r\omega_i}H\,, \tag{125}$$

where the product $g^\alpha(r)\partial^2_{r\omega_i}H$ makes sense because $g^\alpha(r)$ is continuous.

For the same reason, since $\partial^2_{r\omega_i}H$ is a measure and $(g^\alpha)'(r)$ is continuous, a standard smoothing argument justifies

$$\partial_{\omega_i}\big\{((g^\alpha)'(r)r+g^\alpha(r))\partial_r H(r,\omega))\big\} \;=\; (g^\alpha(r)+(g^\alpha)'(r)r)\partial^2_{r\omega_i}H \tag{126}$$

Comparing (124) with (125) and (126), we get

$$(g^\alpha)'(r)\partial_{\omega_i}H(r,\omega)+g^\alpha(r)\partial^2_{r\omega_i}H \;=\; (g^\alpha(r)+(g^\alpha)'(r)r)\partial^2_{r\omega_i}H$$

and hence

$$(g^\alpha)'(r)(r\partial^2_{r\omega_i}H-\partial_{\omega_i}H) \;=\; 0\,. \tag{127}$$

If we set $p(r):=\sum_\alpha|(g_\alpha)'(r)|$, we obtain

$$p(r)(r\partial^2_{r\omega_i}H-\partial_{\omega_i}H) \;=\; 0\,. \tag{128}$$

We claim that, since $\mathscr{L}^1(\{r:p(r)=0\})=0$, we have

$$r\partial^2_{r\omega_i}H-\partial_{\omega_i}H \;=\; 0 \qquad \text{distributionally on } \mathbb{R}^k\setminus\{0\}. \tag{129}$$

Indeed, consider the measures $\mu := r\partial_{r,\omega_i}H$ and $\alpha := \mu - \partial_{\omega_i}H$ and let $\Omega \subset \mathbb{R}^2 \setminus \{0\}$ be the open set $\{x \in \mathbb{R}^k \setminus \{0\} : |p(|x|)| = 0\}$. Then $\alpha \equiv 0$ on Ω. Hence it suffices to show $|\alpha|(\mathbb{R}^2 \setminus \Omega) = 0$. Since $\mathscr{L}^k(\mathbb{R}^k \setminus (\{0\} \cup \Omega)) = 0$ and $\partial_{\omega_i} << \mathscr{L}^k$, it suffices to show

$$|\mu|(\mathbb{R}^2 \setminus (\{0\} \cup \Omega)) \;=\; 0\,.$$

In order to prove this identity, recall that $\mu = \partial_{\omega_i}(\partial_r H)$ and that $\partial_r H$ is a BV function, because H is convex. Consider for every $\tau > 0$ the function $\sigma_\tau(\omega) := \partial_r H(\tau, \omega)$. From the slicing theory of BV functions, it follows that $\sigma_\tau \in BV(\mathbf{S}^{k-1})$ for $\mathscr{L}^1$–a.e. $\tau > 0$ and that

$$|\mu| \;=\; \int_0^\infty |\partial_{\omega_i}\sigma_\tau|\, d\tau\,.$$

Thus, since $\mathscr{L}^1(\{\tau : p(\tau) = 0\}) = 0$, we have $|\mu|(\mathbb{R}^k \setminus (\{0\} \cup \Omega)) = 0$, which concludes the proof of (129).

Note that (129) can be rewritten as

$$r^2\partial_r\left(\frac{\partial_{\omega_i}H}{r}\right) \;=\; 0$$

and hence we get that

$$\partial_{\omega_i}H(r,\omega) \;=\; r\psi_i(\omega)$$

for some locally bounded function ψ_i. Let N be the north pole of $\mathbf{S}^{k-1}$, i.e. the point corresponding to $(1, 0, \ldots, 0)$ for some orthonormal system of coordinates on $\mathbb{R}^k \supset \mathbf{S}^{k-1}$. Consider the restriction $H|_{\mathbf{S}^{k-1}}$ of H on $\mathbf{S}^{k-1}$ and let $\hat{H} \in C(\mathbf{S}^{k-1})$ be given by $\hat{H}(\omega) = H|_{\mathbf{S}^{k-1}}(\omega) - H(N)$. Then $\partial_{\omega_i}(r\hat{H}(\omega)) = r\psi_i(\omega)$. Therefore

$$\partial_{\omega_i}\big(H(r,\omega) - r\hat{H}(\omega)\big) \;=\; 0$$

and hence $H(r,\omega) - r\hat{H}(\omega) = h(r)$ for some function h. Moreover, we have

$$h(r) \;=\; H(r,N) - r\hat{H}(N) \;=\; H(r,N)\,.$$

That is, h is given by the restriction of H to the half–line $\{(\tau, 0, \ldots, 0) : \tau \geq 0\}$. Therefore h is necessarily convex. □

5.3. **Proof of Proposition 5.9.**

Proof of Proposition 5.9. Let u and Ω be as in the statement. Define $\rho := |u|$ and $\overline{\rho} := |\overline{u}|$. The goal is to show that ρ is an entropy solution of the scalar law

$$\begin{cases} \partial_t\rho + D_x\cdot[g(\rho)\rho] \;=\; 0 \\[2ex] \rho(0,\cdot) \;=\; |\overline{u}| \end{cases} \tag{130}$$

in $]0,T[\times\Omega$.

Actually it is sufficient to show that ρ is a *weak* solution of (130) in $]0,T[\times\Omega$. Indeed, note that for every $h : \mathbb{R}^+ \to \mathbb{R}$ which is convex and increasing, $h(|u|)$ is a convex entropy for the system (108) (the entropy flux is of the form $q(|u|)$ for q such that $q' = h'g'$). Thus we have

$$\int_{t>0}\int_{\mathbb{R}^m} [\partial_t\psi(t,z)\, h(\rho(t,z)) + \nabla_x\psi(t,z)\cdot q(\rho(t,z))]\, dt\, dz + \int_{\mathbb{R}^m} \psi(0,z)h(\overline{\rho}(z))\, dz \;\geq\; 0\,, \quad (131)$$

for every nonnegative smooth test function ψ. Moreover, if ρ is a weak solution of (130) in $]0,T[\times\Omega$, L a linear function $L:\mathbb{R}\to\mathbb{R}$ and $Q:\mathbb{R}\to\mathbb{R}^m$ the map given by $Q = (L(g^1),\ldots,L(g^m))$, then

$$\int_{t>0}\int_{\mathbb{R}^m} [\partial_t\psi(t,z)\, L(\rho(t,z)) + \nabla_x\psi(t,z)\cdot Q(\rho(t,z))]\, dt\, dz + \int_{\mathbb{R}^m} \psi(0,z)L(\overline{\rho}(z))\, dz \;=\; 0\,, \quad (132)$$

for every test function $\psi \in C^\infty_c(]-T,T[\times\Omega)$. Given any convex function ξ we can write it as $L+h$, where L is an appropriate linear function and h is increasing on the half–line $\mathbb{R}^+$. Thus, summing (131) and (132), we conclude that ρ satisfies the entropy inequality for ξ and for every nonnegative $\psi\in C^\infty_c(]-T,T[\times\Omega)$, and hence that ρ is an entropy solution of (130) in $]0,T[\times\Omega$.

We now come to the proof that ρ is a weak solution of (130), which we split in several steps.

Step 1

Recall that ρ is a weak solution of (130) in $]0,T[\times\Omega$ if it satisfies the identity

$$\int_{t>0}\int_{\mathbb{R}^m} \rho(t,z)\left[\partial_t\psi(t,z) + g(\rho(t,z))\cdot\nabla_x\psi(t,z)\right] dt\, dz + \int_{\mathbb{R}^m} \psi(0,z)\overline{\rho}(z)\, dz \;=\; 0\,, \quad (133)$$

for every $\psi\in C^\infty_c(]-T,T[\times\Omega)$.

Recall that $\|u\|_{BV(\Omega\times]0,T[)}$ is finite. Hence, we claim that thanks to the trace properties of BV functions, in order to prove (133) it suffices to check that

$$\text{the Radon measure} \qquad \mu \;=\; \partial_t\rho + D_x\cdot(\rho g(\rho)) \qquad \text{vanishes on }]0,T[\times\Omega. \quad (134)$$

Indeed, by a standard approximation argument we get the following estimate for every $t<T$:

$$\int_0^t\int_\Omega |u(\tau,z)-\overline{u}(z)|\, dz\, d\tau \;\leq\; \int_0^t |\partial_t u|(]0,\tau[\times\Omega)\, d\tau \;\leq\; t|\partial_t u|\big(]0,t[\times\Omega\big)\,.$$

From this we conclude

$$\int_0^t\int_\Omega |\rho(\tau,z)-\overline{\rho}(z)|\, dz\, d\tau \;\leq\; t|\partial_t u|\big(]0,t[\times\Omega\big)\,. \quad (135)$$

Fix $\psi\in C^\infty_c(]-T,T[\times\Omega)$ and let $\{\chi_i\}\subset C^\infty([0,T])$ be such that

- $\chi_i = 1$ for $t\geq 2/i$;
- $\chi_i = 0$ for $t \leq 1/i$;
- $0 \;\leq\; \chi_i' \;\leq\; 4i$.

Then, $\psi\chi_i$ is compactly supported in $]0,T[\times\Omega$ and from (134) we get

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^m} \chi_i(\tau)\rho(\tau,z)\Big[\partial_t\psi(\tau,z) + g(\rho(\tau,z))\cdot\nabla_x\psi(\tau,z)\Big]\,dz\,d\tau \\ &+ \int_0^{2/k}\int_{\mathbb{R}^m} \chi_i'(\tau)\rho(\tau,z)\psi(\tau,z)\,dz\,d\tau \;=\; 0\,. \end{aligned} \tag{136}$$

As $i\uparrow\infty$, the first integral in (136) converges to

$$\int_0^T \int_{\mathbb{R}^m} \rho(\tau,z)\Big[\partial_t\psi(\tau,z) + g(\rho(\tau,z))\cdot\nabla_x\psi(\tau,z)\Big]\,dz\,d\tau\,.$$

Concerning the second integral, we recall that $\int_0^{2/i}\chi_i' = 1$ and we write:

$$\begin{aligned} &\left|\int_0^{2/i}\int_{\mathbb{R}^m} \chi_i'(\tau)\rho(\tau,z)\psi(\tau,z)\,dz\,d\tau - \int_{\mathbb{R}^m}\overline{\rho}(z)\psi(0,z)\,dz\right| \\ =\;& \left|\int_0^{2/i}\int_{\mathbb{R}^m} \chi_i'(\tau)\Big[\rho(\tau,z)\psi(\tau,z) - \overline{\rho}(z)\psi(0,z)\Big]\,dz\,d\tau\right| \\ \leq\;& 4i\int_0^{2/i}\int_{\mathbb{R}^m}\big|\rho(\tau,z)\psi(\tau,z) - \overline{\rho}(z)\psi(0,z)\big|\,d\tau\,dz \\ \leq\;& 4i\|\rho\|_\infty\int_0^{2/i}\int_{\mathbb{R}^m}\big|\psi(\tau,z)-\psi(0,z)\big|\,d\tau\,dz + 4i\|\psi\|_\infty\int_0^{2/i}\int_{\mathbb{R}^m}\big|\rho(\tau,z)-\rho(0,z)\big|\,d\tau\,dz\,. \end{aligned}$$

Note that, for $i\uparrow\infty$, the first term tends to 0 because ψ is smooth. Thanks to (135) the second term is bounded by

$$C|\partial_t u|\big(]0,2/i[\times\Omega\big) \tag{137}$$

where C is a constant independent of t, and Ω is a bounded set. Since $|\partial_t u|$ is Radon measure, we conclude that the expression (137) tends to 0 for $i\uparrow\infty$. Thus we conclude that

$$\lim_{i\uparrow\infty}\int_0^{2/i}\int_{\mathbb{R}^m}\chi_i'(\tau)\rho(\tau,z)\psi(\tau,z)\,dz\,d\tau \;=\; \int_{\mathbb{R}^m}\overline{\rho}(z)\psi(0,z)\,dz\,.$$

Hence, passing into the limit in (136) we get (133). Therefore, we are left with the task of proving (134).

Step 2

We wish to use the entropy inequalities and to apply Theorem 2.11 to conclude that μ is supported on the jump set (or shock set) J_u. However this is not possible since the function $|u|$ is not C^1 in the origin (compare with Remark 2.12). We approximate this function uniformly with smooth C^1 convex functions of the form $h_n(|u|)$. Clearly, also these functions are entropies for the system of Keyfitz and Kranzer and their entropy fluxes are of the form $q_n(|u|)$ for some functions $q_n(t)$ which converge uniformly to $tf(t)$.

Let $\nu : J_u \to \mathbb{R}^m$ be a Borel vector field and $\zeta : J_u \to \mathbb{R}$ be a nonnegative Borel function such that $(\zeta, \nu)/\sqrt{\zeta^2 + |\nu|^2}$ is normal to J_u $\mathscr{H}^m$–a.e.. Then, the chain rule of Vol'pert gives that

$$\begin{aligned}&\partial_t[h_n(\rho)] + D_x \cdot [q_n(\rho)]\\ = \ & (\zeta^2 + |\nu|^2)^{-1/2}\Big[\big(h_n(|u^+|) - h_n(|u^-|)\big)\zeta + \big(q_n(|u^+|) - q_n(|u^-|)\big)\cdot \nu\Big]\,\mathscr{H}^m \llcorner J_u\,.\end{aligned}$$

Passing to the limit in n we get:

$$\mu \;=\; (\zeta^2 + |\nu|^2)^{-1/2}\Big[\big(|u^+| - |u^-|\big)\zeta + \big(|u^+|g(|u^+|) - |u^-|g(|u^-|)\big)\cdot\nu\Big]\,\mathscr{H}^m \llcorner J_u\,. \qquad (138)$$

Thus, we must prove that

$$\big(\zeta + g(|u^+|)\cdot\nu\big)|u^+| \;=\; \big(\zeta + g(|u^-|)\cdot\nu\big)|u^-| \qquad \mathscr{H}^m\text{–a.e. on } J_u. \qquad (139)$$

In what follows, for the sake of simplicity, we will drop the "$\mathscr{H}^m$–a.e.".

Since u is a weak solution of (108), when $F(v) := g(|v|)\otimes v$ is C^1 we can apply Theorem 2.11 to get

$$\big(g(|u^+|)\cdot\nu + \zeta\big)u^+ \;=\; \big(g(|u^-|)\cdot\nu + \zeta\big)u^-\,. \qquad (140)$$

In order to derive (140) when 0 is a singularity for DF we approximate F with $F_n := g(h_n(u))\otimes u$. Then we get

$$\begin{aligned}\partial_t u + D_x\cdot(F_n(u)) \;&=\; D^d u + DF_n(\tilde u)\cdot D^d u\\ &\quad + \Big[\big(u^+ - u^-\big)\zeta + \big(F(u^+) - F(u^-)\big)\cdot\nu\Big]\,\mathscr{H}^m \llcorner J_u\,. \qquad (141)\end{aligned}$$

Clearly the left hand side converges to $0 = \partial_t u + D_x\cdot(F(u))$. Moreover the second term of the right hand side converges to

$$\Big[\big(g(|u^+|)\cdot\nu + \zeta\big)u^+ - \big(g(|u^-|)\cdot\nu + \zeta\big)u^-\Big]\,\mathscr{H}^m \llcorner J_u$$

in the sense of measures.

Note that the approximations F_n can be chosen in such a way that DF_n are locally uniformly bounded. In this case, let σ be any weak* limit of any subsequence of $DF_n(\tilde u)\cdot D^d u$. Since $|DF_n\cdot D^d u| \le C|D^d u|$, this weak* limit satisfies $\sigma << |D^d u|$. On the other hand, passing into the limit in (141) we get

$$0 \;=\; \sigma + \Big[\big(g(|u^+|)\cdot\nu + \zeta\big)u^+ - \big(g(|u^-|)\cdot\nu + \zeta\big)u^-\Big]\,\mathscr{H}^m \llcorner J_u\,.$$

Since $|D^d u|(J_u) = 0$, we conclude that (140) holds $\mathscr{H}^m$–a.e. on J_u.

From (140) we get

$$\big|g(|u^+|)\cdot\nu + \zeta\big||u^+| \;=\; \big|g(|u^-|)\cdot\nu + \zeta\big||u^-|\,. \qquad (142)$$

If $|u^+|$ (or $|u^-|$) vanishes, (139) follows trivially. Hence, after setting $\rho^\pm := |u^\pm|$ we restrict our attention to the subset of J_u given by $G := \{\rho^+ \neq 0 \neq \rho^-\}$. On this set we define $\theta^\pm := u^\pm/\rho^\pm$ and we note that (140) becomes

$$\big[\big(g(\rho^+)\cdot\nu + \zeta\big)\big]\rho^+\theta^+ \;=\; \big[\big(g(\rho^-)\cdot\nu + \zeta\big)\big]\rho^-\theta^- \qquad (143)$$

Since $\theta^{\pm} \in \mathbf{S}^{k-1}$ we conclude that, either $\theta^+ = \theta^-$, or $\theta^+ = -\theta^-$. In the next step we will prove that, if D is the closure of the convex hull of the essential image of $u|_{]0,T[\times\mathbb{R}^m}$, then either $0 \notin D$, or 0 is an extremal point of D. This rules out the alternative $\theta^+ = -\theta^-$. Therefore we conclude that $\theta^+ = \theta^-$ on G, from which (139) easily follows.

Step 3

In order to complete the proof it remains to show that, if D denotes the closure of the convex hull of the essential image of $u|_{]0,T[\times\mathbb{R}^m}$, then either the origin is not contained in D, or it is an extremal point of D. Recalling (a), this property is true for the closure C of the convex hull of the essential image of $\overline{u}$. Choose $\xi_1, \ldots, \xi_k$ unit vectors of $\mathbb{R}^k$ such that

$$C \subset \left\{x|\, x \cdot \xi_i \leq 0 \quad \text{for every } i\right\}$$

and 0 is an extremal point of $\left\{x|\, x \cdot \xi_i \leq 0 \,\text{for every } i\right\}$. We will show that the essential image of u is contained in $\{x|\, x \cdot \xi_i \leq 0\}$ for every i.

Fix i and denote by $H : \mathbb{R}^k \to \mathbb{R}$, $Q : \mathbb{R}^k \to \mathbb{R}^m$ the functions

$$H(v) := \begin{cases} 0 & \text{if } \xi_i \cdot v \leq 0 \\ \xi_i \cdot v & \text{otherwise.} \end{cases} \qquad Q(v) := f(|v|)H(v)\,.$$

Note that (H, Q) is a convex entropy–entropy flux pair. Clearly $H(\overline{u}) = 0$ and thus the boundary term in the entropy inequality (114) disappears. Thus, if we set $w := H(u)$ and $b := Q(u)$ we get that

$$\begin{cases} \partial_t w + D_x \cdot b \leq 0 \\ \\ w(0, \cdot) = 0\,. \end{cases}$$

Note that there exists a constant C such that $|b| \leq Cw$. Therefore we can apply Lemma 3.17 to conclude $w \equiv 0$. This completes the proof. □

6. Blow–up of the BV norm for the Keyfitz and Kranzer system

In one space dimension, the fundamental result of Glimm (see [22]) gives the existence of BV entropy solutions for (108) if one starts with initial data which have sufficiently small total variation. Moreover, from Proposition 5.9 we get that, when the convex hull of the essential image of the initial data $\overline{u}$ does not contain the origin (or the origin is an extremal point of it), such solution is the unique renormalized entropy solution.

Hence it is natural to ask whether renormalized entropy solutions u of (108) enjoy BV regularity when the whole initial datum $\overline{u}$ (and not only its modulus) belongs to BV. In analogy with the one–dimensional case, one could ask if such regularity holds at least for small times and when $\overline{u}$ is close to a constant different from 0, in both the L^∞ and the BV norms. We will show that this is not the case. More precisely we will show that

Theorem 6.1. *Let $k \geq 2$, $m \geq 3$, $g \in C^3_{loc}$ and let $c \in \mathbb{R}^k \setminus \{0\}$ such that $g'(|c|) \neq 0$. Then there exists a sequence of initial data $\overline{u}_n : \mathbb{R}^m \to \mathbb{R}^k$ such that*

- $\|\overline{u}_n - c\|_{BV(\mathbb{R}^m)} + \|\overline{u}_n - c\|_\infty \to 0$ *for $n \uparrow \infty$;*
- $\overline{u}_n = c$ *on $\mathbb{R}^m \setminus B_R(0)$ for some $R > 0$ independent of n;*

- *If* u_n *is* any *bounded entropy solution of* (108) *with initial data* $\overline{u}_n$*, then there exists* $r>0$ *(independent of* n*) such that* $\|u_n\|_{BV(]0,T[\times B_r(0))}=\infty$ *for every positive* T*.*

When $m=2$ *the same statement holds if in addition we assume that* $g''(|c|)$ *is parallel to* $g'(|c|)$ *(or vanishes).*

We remark that the system of Keyfitz and Kranzer, in contrast to general hyperbolic systems of conservation laws, has remarkably many features. Indeed consider the system of conservation laws

$$\partial_t u + D_x \cdot [F(u)] \;=\; 0 \qquad u : \Omega \subset \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^k\,, \tag{144}$$

where $F:\mathbb{R}^k\to\mathbb{R}^{k\times m}$ is a C^1 function. In what follows we will use the notation $F=(F^1,\ldots,F^m)$, where each F^i is a map from $\mathbb{R}^k$ to $\mathbb{R}^k$. The Keyfitz and Kranzer system corresponds to the choice $F(v)=v\otimes g(|v|)$, where $g\in C^1(\mathbb{R},\mathbb{R}^m)$. (Note that in this case the requirement $F\in C^1$ implies $g'(0)=0$. However, in the rest of the forthcoming sections we will not impose this condition, since it is not needed in any of the proofs.) Therefore the Keyfitz and Kranzer system falls into the category of symmetric systems of conservation laws, i.e. the systems (144) for which $DF^i(v)$ is a symmetric matrix for every i and for every $v\in\mathbb{R}^k$.

It is known, by a result of Rauch based on a previous paper of Brenner for linear hyperbolic systems (see [16] and [37]), that certain type of BV–estimates (and L^p estimates for $p\neq 2$) fail for all the systems (144) which do not satisfy the commutator conditions

$$DF^i(v)\cdot DF^j(v) \;=\; DF^j(v)\cdot DF^i(v) \qquad \text{for every } v\in\mathbb{R}^k. \tag{145}$$

When $m=2$, it was proved in [23] that (145) is also sufficient to get L^p estimates for every $p\leq 2$ and, under additional conditions, also for $p=\infty$.

Note that the Keyfitz and Kranzer system does satisfy Rauch's commutator condition (145). Moreover we remark that when (145) does not hold, Rauch's result implies that estimates of a certain kind are not available, but it does not exclude BV regularity.

6.1. **Preliminary lemmas.** In this section we collect some facts which will be used in the proof of Theorem 6.1.

Riemann Problem for scalar laws Let us consider the Cauchy problem

$$\left\{\begin{array}{lcl}\partial_t\rho + D_x\cdot[h(\rho)] &=& 0\\ \\ \rho(0,\cdot) &=& \overline{\rho}\end{array}\right. \qquad \rho:\mathbb{R}^+\times\mathbb{R}^m \;\to\; \mathbb{R}\,, \tag{146}$$

where $h:\mathbb{R}\to\mathbb{R}^m$ is of class C^3. Fix $\beta,\gamma,\alpha\in\mathbb{R}$, set $\varepsilon:=\max\{|\alpha-\beta|,|\alpha-\gamma|\}$, and choose

$$\overline{\rho}(x_1,\ldots,x_m) \;=\; \left\{\begin{array}{ll}\beta & \text{for } x_m<0\\ \gamma & \text{for } x_m>0.\end{array}\right.$$

Consider the entropy solution ρ of (146). It is easy to see that ρ depends only on t and x_m. For each $T > 0$ define:

$$\xi := \max\left\{x_m | \rho(T, \cdot, x_m) = \beta\right\} \tag{147}$$
$$\zeta := \min\left\{x_m | \rho(T, \cdot, x_m) = \gamma\right\}. \tag{148}$$

Then the following lemma has an elementary proof:

Lemma 6.2. *Let $T > 0$ and $\alpha \in \mathbb{R}$ be given. For any real α and β, set ε, ξ, and ζ as above. If we denote by $(h^m)'$ and $(h^m)''$ the m–th components of the vector–valued functions h' and h'', then there exist constants C and δ (depending only on h) such that*

$$\max\left\{|\xi - T(h^m)'(\alpha)|, |\zeta - T(h^m)'(\alpha)|\right\} \leq 2|(h^m)''(\alpha)|\varepsilon + C\varepsilon^2 \qquad \text{for } \varepsilon \leq \delta. \tag{149}$$

Regular lagrangian flows Let u be a renormalized entropy solution of (108). Assume that the initial data $\overline{u}$ is bounded away from the origin, i.e. that $|\overline{u}| \geq c > 0$. Then, from the maximum principle for scalar conservation laws, it turns out that the renormalized entropy solution u is bounded away from zero as well, i.e. that $|u| \geq c > 0$. Hence the angular parts $\overline{\theta} := \overline{u}/|\overline{u}|$, $\theta := u/|u|$ are well defined and solve the transport equation (118).

Let Φ be the unique regular lagrangian flow given by Theorem 3.22:

$$\begin{cases} \frac{d}{dt}\Phi(s,x) = g(\rho(s, \Phi(s,x))) \\ \\ \Phi(0,x) = x \end{cases} \tag{150}$$

Then the following holds

Proposition 6.3. *There exists a locally bounded map $\Psi : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ such that $\Phi(s, \Psi(s,x)) = \Psi(s, \Phi(s,x)) = x$ for $\mathscr{L}^{m+1}$–a.e. (s,x). Moreover $\theta(t,x) = \overline{\theta}(\Psi(t,x))$.*

Proof. Let $\{f_n\} \subset C^\infty$ be a uniformly bounded sequence such that $f_n \to g(\rho)$ in L^1_{loc} and $\{\rho_n\} \subset C^\infty$ a sequence of positive functions such that

- $\|\rho_n^{-1}\|_\infty + \|\rho_n\|_\infty$ is uniformly bounded;
- $\rho_n \to \rho$ and $\rho_n(0,\cdot) \to \rho(0,\cdot)$ in L^1_{loc};
- $\partial_t \rho_n + D_x \cdot (\rho_n f_n) = 0$.

These approximating sequences can be constructed as in the in the proof of the existence part of Theorem 3.22 (in particular see Step 1). Let Φ_n be the solutions of the ODEs

$$\begin{cases} \frac{d}{dt}\Phi_n(s,x) = f_n(s, \Phi_n(s,x)) \\ \\ \Phi_n(0,x) = x. \end{cases} \tag{151}$$

Then for some constant C we have $C^{-1} \leq \det \nabla_x \Phi_n \leq C$. Thus, if we let $\Psi_n : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ be such that $\Psi(t, \Phi(t,x)) = (t,x)$, then $\left\{\|\Psi_n\|_{L^\infty([0,T]\times K)}\right\}$ for every $T > 0$ and every compact set $K \subset \mathbb{R}^m$.

From Theorem 3.22, Φ_n converges to Φ strongly in L^1_{loc}. Moreover, from the proof of the stability property of Theorem 3.22, it follows easily that $\Psi_n \to \Psi$ strongly in L^1_{loc} to some bounded map Ψ. From these convergence and from the bounds

$$C^{-1} \;\leq\; \det \nabla_x \Phi_n \;\leq\; C \qquad C^{-1} \;\leq\; \det \nabla_x \Psi_n \;\leq\; C\,,$$

it is easy to conclude that $\Psi(t,\Phi(t,x)) = \Phi(t,\Psi(t,x)) = x$ for $\mathscr{L}^{m+1}$–a.e. (t,x).

Set $\tilde\theta(t,x) := \overline{\theta}(\Psi(t,x))$, then, for $\mathscr{L}^m$–a.e. x, the function $\tilde\theta(\cdot,\Phi(\cdot,c))$ is constant. Therefore, by Proposition 3.5, we get that $\tilde\theta$ solves (118). From Corollary 3.14 we conclude that $\tilde\theta = \theta$. □

Proposition 6.4. *For $\mathscr{L}^m$–a.e. x we have that:*

(a) $\Phi(\cdot,x)$ *is Lipschitz (and hence it is differentiable in t for $\mathscr{L}^1$–a.e. t);*
(b) $(t,\Phi(t,x))$ *is a point of approximate continuity of ρ for $\mathscr{L}^1$–a.e. t;*
(c) $\frac{d}{dt}\Phi(t,x) = g(\rho(t,\Phi(t,x))$ *for $\mathscr{L}^1$–a.e. t.*

Proof. **Step 1** Consider again two sequences of smooth maps $\{f_n\}, \{\rho_n\}$ as in the proof of the previous proposition. Denote by Φ_n the solutions of (151) and set $J_n := \det(\nabla_x \Phi_n)$. From Liouville's Theorem it follows that $\partial_t J_n + \mathrm{div}\,(f_n J_n) = 0$. Since $J_n(0,\cdot) = 1$, the maximum principle of Proposition 3.13 applied to the continuity equation $\partial_t w + \mathrm{div}\,(f_n w) = 0$ yields that $C^{-1}\rho_n \leq J_n \leq C\rho_n$, and hence $C^{-2} \leq J_n \leq C^2$.

Recall that $\Phi_n \to \Phi$ strongly in L^1_{loc}. Since for every x the curves $\Phi_n(\cdot,x)$ are uniformly Lipschitz, we conclude that $\Phi(\cdot,x)$ is a Lipschitz curve for $\mathscr{L}^m$–a.e. x. This gives (a).

Step 2 Next, fix a t and a subsequence (not relabeled) of $\Phi_n(t,\cdot)$ which converges to $\Phi(t,\cdot)$ in $L^1_{loc}(\mathbb{R}^m)$ (such a subsequence exists for $\mathscr{L}^1$–a.e. t). Let $E \subset \mathbb{R}^m$ be an open set. It is not difficult to show that

$$\mathscr{L}^m(\Phi(t,\cdot)^{-1}(E)) \;\leq\; \limsup_{n\uparrow\infty} \mathscr{L}^m(\Phi_n(t,\cdot)^{-1}(E)) \;\leq\; C^2 \mathscr{L}^m(E)\,. \tag{152}$$

Hence, for $\mathscr{L}^1$–a.e. t, this bound holds for every open set E. This property gives that for $\mathscr{L}^1$–a.e. t, $\Phi(t,\cdot)^{-1}$ maps sets of measure zero into sets of measure zero. Thus (b) follows from the fact that ρ is almost everywhere approximately continuous.

Step 3 The strong convergence of Φ_n implies that, if $h_n \in C(\mathbb{R}\times\mathbb{R}^m)$ converges locally uniformly to $h \in C(\mathbb{R}\times\mathbb{R}^m)$, then $h_n(\cdot,\Phi_n)$ converges to $h(\cdot,\Phi)$ strongly in L^1_{loc}. If $h_n \to h$ strongly in L^1_{loc} and it is uniformly bounded, applying Egorov's theorem we find a closed set E such that h_n converges locally uniformly to h on E and $\mathscr{L}^{m+1}(\mathbb{R}\times\mathbb{R}^m \setminus E)$ is as small as desired. Recall that Φ_n is locally uniformly bounded. From Step 2 it follows that $h_n(\cdot,\Phi_n)$ converges strongly to $h(\cdot,\Phi)$.

Step 4 Since Φ_n solves (151) we have

$$\Phi_n(t,x) \;=\; x + \int_0^t f_n(\tau,\Phi_n(\tau,x))\,d\tau\,. \tag{153}$$

Applying Step 3 to $h_n = f_n$ and $h = g(\rho)$ we get a subsequence (not relabeled) of $\{\Phi_n\}$ such that $f_n(\cdot, \Phi_n)$ converges to $g(\rho(\cdot, \Phi))$ pointwise a.e. on $\mathbb{R} \times \mathbb{R}^m$. From the dominated convergence theorem we get

$$\Phi(t,x) \;=\; x + \int_0^t g(\rho(\tau, \Phi(\tau,x)))\, d\tau \qquad\qquad \text{for } \mathscr{L}^{m+1}\text{–a.e. } (t,x).$$

From this identity we easily conclude (c). □

6.2. **Proof of Theorem 6.1.** Theorem 6.1 is a corollary of Proposition 5.9 and of the following

Proposition 6.5. *Let $k \geq 2$, $m \geq 3$, and $g \in C^3_{loc}$. Then, for every $c \in \mathbb{R}^k \setminus \{0\}$ such that $g'(|c|) \neq 0$, there exists a sequence of initial data $\overline{u}_n : \mathbb{R}^m \to \mathbb{R}^k$ such that*

- $\|\overline{u}_n - c\|_{BV(\mathbb{R}^m)} + \|\overline{u}_n - c\|_\infty \to 0$ *for $n \uparrow \infty$;*
- $\overline{u}_n = c$ *on $\mathbb{R}^m \setminus B_R(0)$ for some $R > 0$ independent of n;*
- *If u_n denotes the unique renormalized entropy solution of (108) with $u_n(0,\cdot) = \overline{u}_n$, then there exists $r > 0$ such that $u_n(t,\cdot) \notin BV(B_r(0))$ for every n and for every $t \in]0,1[$.*

When $m = 2$ the same statement holds if in addition $g''(|c|)$ is parallel to $g'(|c|)$ or $g''(|c|) = 0$.

Proof of Theorem 6.1. Let $\overline{u}_n$ be the initial data of Proposition 6.5 and let $r > 0$ be such that the corresponding renormalized entropy solutions $u_n(t,\cdot)$ are not in $BV(B_r(0))$ for any $t \in]0,1[$. Let $\hat{u}_n$ be any other entropy solution of (108) with the same initial data. For any any $c > \|\overline{u}_n\|_\infty$, we apply the argument of Step 3 of the proof of Proposition 5.9 to the entropy $h(|u|) := (|u| - c)\mathbf{1}_{|u|\geq c}$. It turns out that $h(|u|) = 0$, from which we conclude $\|\hat{u}_n\|_\infty \leq \|\overline{u}_n\|_\infty$. Hence $\hat{u}_n$ is uniformly bounded.

Fix $T \in]0,1[$ and let $\gamma \geq 0$ be the supremum of the nonnegative R's such that $\hat{u}_n \in BV(]0,T[\times B_R(0))$. We want to bound γ with a constant times r. From Proposition 5.9 we get that $\hat{u}_n$ is a renormalized entropy solution on $]0,T[\times B_\gamma(0)$. Therefore $\hat{\rho}_n := |\hat{u}_n|$ is a Kruzkov solution of

$$\begin{cases} \partial_t \hat{\rho}_n + D_x \cdot (\hat{\rho}_n g(\hat{\rho}_n)) \;=\; 0 & \text{on }]0,T[\times B_\gamma(0) \\ \\ \hat{\rho}_n(0,\cdot) \;=\; \overline{\rho}_n\,. \end{cases}$$

From the finite speed of propagation of scalar conservation laws, it follows that there exists positive constants T_1 and γ_1 such that $\rho_n = \hat{\rho}_n$ on $]0,T_1[\times B_{\gamma_1}(0)$. Moreover, we can choose

$$\gamma_1 \;\geq\; c\gamma \qquad T_1 \;\geq\; cT \tag{154}$$

where the constant $c > 0$ depends only on $\|\overline{u}_n\|_\infty$ on g.

Set $\hat{\theta}_n = \hat{u}_n/\hat{\rho}_n$ and $\theta_n = u_n/\rho_n$, with the convention that $\hat{\theta}_n = 0$ where $\hat{\rho}_n = 0$ and $\theta_n = 0$ where $\rho_n = 0$. Then $\hat{\theta}_n$ and θ_n solve both the transport equation

$$\begin{cases} \partial_t(\rho_n\omega) + D_x \cdot (\rho_n g(\rho_n)\omega) \;=\; 0 & \text{in }]0,T_1[\times B_{\gamma_1}(0) \\ \\ [\rho_n\omega]\,(0,\cdot) \;=\; \overline{u}_n\,. \end{cases}$$

Thus, by the renormalization property, we get that $w = |\theta_n - \hat{\theta}_n|$ solves

$$\begin{cases} \partial_t(\rho_n w) + D_x \cdot (\rho_n g(\rho_n) w) \;=\; 0 \qquad \text{in }]0, T_1[\times B_{\gamma_1}(0) \\ \\ [\rho_n w]\,(0, \cdot) \;=\; 0\,. \end{cases}$$

From Lemma 3.17, we conclude that there exists two positive constants $\gamma_2 < \gamma_1$ and $T_2 < T_1$ such that $w = 0$ on $]0, T_2[\times B_{\gamma_2}(0)$, and that we can choose

$$\gamma_2 \;\geq\; c'\gamma_1 \qquad T_2 \;\geq\; c' T_1\,, \tag{155}$$

where c' depends only on $\|\rho_n\|_\infty \leq \|\overline{u}_n\|_\infty$ and g.

Since $\|\overline{u}_n\|_\infty$ is uniformly bounded, the constants c and c' in (154) and (155) can be chosen independently of n. Recall that $u_n \notin BV(]0, T_2[\times B_r(0))$. This implies the desired bound $\gamma < cc'r$. Indeed, if such a bound did not hold, then we would have $\gamma_2 \geq r$ and hence $u_n = \hat{u}_n$ on $]0, T_2[\times B_r(0)$. This would imply $u_n \in BV]0, T_2[\times B_r(0)$m, which is a contradiction. □

In the next section we will give a proof of Proposition 6.5. But first we consider the special case of system (108) when $g = (f, 0, \ldots, 0)$, that is

$$\begin{cases} \partial_t u + \partial_{x_1}[f(|u|)u] \;=\; 0 \\ \\ u(0, \cdot) \;=\; u_0\,. \end{cases} \tag{156}$$

The following is a corollary of Proposition 6.5

Proposition 6.6. *Let $k \geq 2$, $m \geq 2$ and $c \in \mathbb{R}^k \setminus \{0\}$ be such that $f'(|c|) \neq 0$. Then there exists a sequence of initial data $\overline{u}_n : \mathbb{R}^m \to \mathbb{R}^k$ such that*

- *$\|\overline{u}_n - c\|_{BV(\mathbb{R}^m)} + \|\overline{u}_m - c\|_\infty \to 0$ for $n \uparrow \infty$;*
- *$\overline{u}_n = c$ on $\mathbb{R}^m \setminus B_R(0)$ for some $R > 0$ independent of n;*
- *If u_n denotes the unique renormalized entropy solution of (156) with $u_n(0, \cdot) = \overline{u}_n$, then there exists $r > 0$ such that $u_n(t, \cdot) \notin BV_{loc}(B_r(0))$ for every n and for every $t \in]0, 1[$.*

Roughly speaking, the proof of Proposition 6.5 is based on the following remark: When $m = 3$ we can choose initial data, close to a constant, in such a way that the behavior of the renormalized entropy solutions of (108) is close to the behavior of solutions of (156). This seems to be no longer true for $m = 2$, unless $g''(|c|)$ is parallel to $g'(|c|)$ (or $g''(|c|) = 0$). Due to this remark, we choose to give a quick self–contained proof of Proposition 6.6.

Remark 6.7. *Concerning the behavior of u_n for large times, in the case of Proposition 6.6 one can construct initial data $\overline{u}_n$ such that $u_n(t, \cdot) \notin BV_{loc}$ for any positive time $t > 0$. In the case of Proposition 6.5 it is difficult to track what happens for large times, since in order to carry on our proof we need that the rarefaction waves generated by $|u_n|$ do not interact.*

Proof of Proposition 6.6. In the following, for any real number α, we denote by $[\alpha]$ the largest integer which is less than or equal to α.

For the sake of simplicity we prove the proposition when $m = 2$, $f'(|c|) = 1$, and $f(|c|) = 0$. Only minor adjustments are needed to handle the general case. To simplify the notation, on $\mathbb{R}^2$ we will use the coordinates (x, y) in place of (x_1, x_2).

Let $\{m_i\}$ be a sequence of positive even numbers such that

$$\sum_i m_i 2^{-i} \; < \; \infty \,. \tag{157}$$

Let $\delta > 0$ be so small that:

- f is injective on $[|c| - 2\delta, |c| + 2\delta]$;
- $\big[-\delta, \delta\big] \subset f\big([|c| - 2\delta, |c| + 2\delta]\big)$.

Then, for i sufficiently large, we define r_i as the unique number in $[-2\delta, 2\delta]$ such that $f(|c| + r_i) = 2^{-i}$. Notice that for i sufficiently large we have $r_i \le 2^{-i+1}$. Set $\alpha = c/|c|$ and for every i choose an $\alpha_i \in \mathbf{S}^{k-1}$ such that $|\alpha_i - \alpha| = i^{-2}$.

Let I_i be the interval $[2^{-i}, 2^{-i+1}[$ and subdivide it in m_i equal subintervals

$$I_i^j \; := \; \left[2^{-i} + \frac{(j-1)2^{-i}}{m_i}\,,\; 2^{-i} + \frac{j 2^{-i}}{m_i} \right[\qquad j \in \{1, \ldots, m_i\}\,.$$

Next define the functions $\psi_i : \mathbb{R}^2 \to \mathbf{S}^{k-1}$ as

$$\psi_i(x, y) \; := \; \begin{cases} \alpha_i & \text{if } y \in I_i \text{ and } [x 2^i] \text{ is odd} \\ \alpha & \text{otherwise} \end{cases}$$

and the functions $\chi_i : \mathbb{R}^2 \to \mathbb{R}$ as

$$\chi_i(x, y) \; := \; \begin{cases} r_i & \text{if } y \in I_i^j \text{ for } j \text{ even and } x \in [-M, M] \\ r_{i+1} & \text{if } y \in I_i^j \text{ for } j \text{ odd and } x \in [-M, M] \\ 0 & \text{otherwise}\,. \end{cases}$$

Here M is a positive real number which will be chosen later. Finally we define

$$\begin{aligned}
\overline{\rho}_n \quad & := \quad |c| + \textstyle\sum_{i=n}^{\infty} \chi_i \,, \\
\overline{\theta}_n(x, y) \quad & := \quad \begin{cases} \psi_i(x, y) & \text{if } y \in I_i \text{ for some } i \ge n \text{ and } x \in [-M, M] \\ \alpha & \text{otherwise} \quad , \end{cases} \\
\overline{u}_n \quad & := \quad \overline{\rho}_n \overline{\theta}_n \,.
\end{aligned}$$

Figure 1 gives a picture of the partition of $\mathbb{R}^2$ on which we based the definition of $\overline{u}_n$.

Clearly $\|\overline{u}_n - c\|_\infty \le |c||\alpha_n - \alpha| + r_n$. Hence, as $n \uparrow \infty$ we have $\|\overline{u}_n - c\|_\infty \to 0$. Moreover notice that $\overline{u}_n - c$ is supported on $[-M, M] \times [0, 1]$. From now on we assume that M will be chosen large than 1.

In order to show that

$$\|\overline{u}_n - c\|_{BV(\mathbb{R}^2)} \; \to \; 0$$

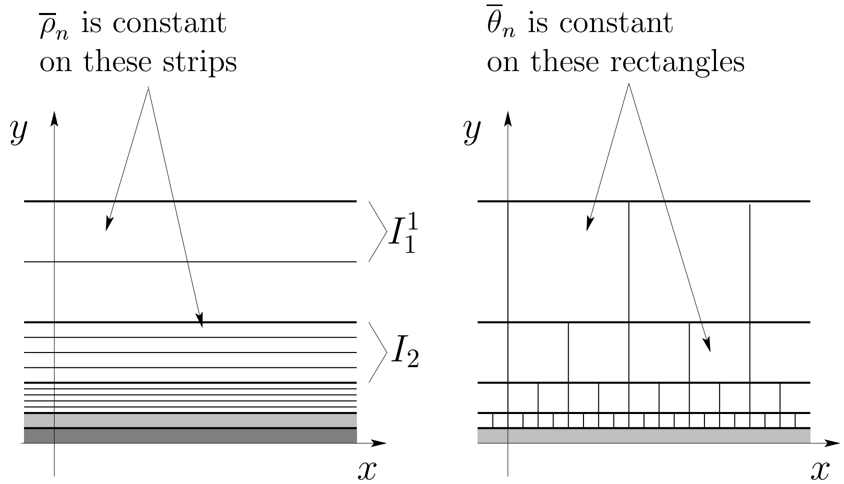

FIGURE 1. Decomposition of the plane in open sets where $\overline{\rho}_n$ (resp. $\overline{\theta}_n$) is constant.

it is sufficient to show

$$\|\overline{\rho}_n - |c|\|_{BV([-2M,2M]^2)} \to 0 \tag{158}$$
$$\|\overline{\theta}_n - \alpha\|_{BV([-2M,2M]^2)} \to 0 \tag{159}$$

Note that

$$\begin{aligned}\|\overline{\rho}_n - |c|\|_{BV([-2M,2M]^2)} &\leq 4\|\overline{u}_n - c\|_\infty M^2 + 2M\sum_{i\geq n} m_i r_i + (4M+2)r_n\\ &\leq 4\|\overline{u}_n - c\|_\infty M^2 + 4M\sum_{i\geq n} m_i 2^{-i} + (4M+2)r_n\end{aligned}$$

and since $\sum 2^{-i}m_i$ is summable, we get (158). Moreover,

$$\begin{aligned}\|\overline{\theta}_n - \alpha\|_{BV([-2M,2M]^2)} &\leq 4\|\overline{\theta}_n - \alpha\|_\infty M^2\\ &+ 2M\sum_{i\geq n} 2^{-i}i^{-2}2^i + 2M\sum_{i\geq n}\left[i^{-2} + (i+1)^{-2}\right] + (4M+2)n^{-2}\end{aligned}$$

and the summability of $\sum i^{-2}$ gives (159).

Now we let u_n be the unique renormalized solution of (156). Recall that $\rho_n := |u_n|$ is the unique entropy solution of (117) with initial data $\overline{\rho}_n$, which in our case is given by

$$\left\{\begin{aligned} \partial_t\rho_n + \partial_x(f(\rho_n)\rho_n) &= 0\\ \rho_n(0,\cdot) &= \overline{\rho}_n\,. \end{aligned}\right.$$

Hence, if $\overline{\rho}_n$ did not depend on x, we would have $\rho_n(t,y,x) = \overline{\rho}_n(x,y)$. Since $\overline{\rho}_n$ is "truncated", this is not true. However, $\overline{\rho}_n(\cdot,y)$ is constant on $[-M,M]$ and by the finite speed of propagation of scalar laws it follows that $\rho_n(t,x,y) = \overline{\rho}_n(x,y)$ if (t,x,y) belongs to the cone

$$\left\{ \sqrt{y^2+x^2} \ \le \ c(M-t) \right\} ,$$

where c is a constant which depends only on $\|\overline{\rho}_n\|_\infty$. Thus, for every $\lambda > 1$, we can choose M large enough (but independent of n) so that

$$\rho_n(t,x,y) \ = \ \overline{\rho}_n(x,y) \qquad \text{for } t \in [0,1] \text{ and } (x,y) \in [-\lambda,\lambda]\times[0,1] \ .$$

To find the angular part $\theta_n(t,x,y) := u_n/|u_n|(t,x,y)$ we use the fact that θ_n is constant on the curves $\Phi_n(\cdot,x)$, where Φ_n solves the ODEs

$$\begin{cases} \frac{d}{dt}\Phi_n(s,x,y) \ = \ g(\rho_n(s,\Phi_n(s,x,y))) \\ \\ \Phi_n(0,x,y) \ = \ (x,y) \end{cases} \tag{160}$$

in the sense of Propositions 6.3 and 6.4. Hence it follows that, for $\mathscr{L}^3$–a.e. (τ,x_1,y_1) there is $(x_0,y_0)\in\mathbb{R}^2$ such that:

- The curve $\Phi(\cdot,x_0,y_0)$ is Lipschitz;
- $\Phi(\tau,x_0,y_0) = (x_1,y_1)$;
- $\Phi(\cdot,x_0,y_0)$ solves (160) in the sense of Proposition 6.4.

Therefore every connected component of the intersection of the curve $\Phi(\cdot,x_0,y_0)$ with $[0,1]\times[-\lambda,\lambda]\times[0,1]$ is a straight segment lying on a plane $\{y = \mathit{const}\}$. If $(\tau,x_1,y_1)\in[0,1]^3\subset[0,1]\times[-\lambda,\lambda]\times[0,1]$, one of these segments contains (τ,x_1,y_1) and hence its slope is given by $f(\rho_n(\tau,x_1,y_1))$. If we choose λ large enough, the curve $\Phi(\cdot,x_0,y_0)$ remains "trapped" on the plane $\{y=y_1\}$ for the whole time interval $]0,\tau[$. Note that this choice of λ depends only on f and on the L^∞ norm of ρ_n, which is uniformly bounded.

From now on, we assume that λ (and hence M) have been chosen so to satisfy the requirement above. Recall that for $\mathscr{L}^3$–a.e. $(t,x,y)\in[0,1]^3$, we have $\rho_n(t,x,y) = |c| + r_i$ for some i, and hence $f(\rho_n(t,x,y))) = 2^{-i}$. From the previous discussion we conclude the following formulas, valid for $\mathscr{L}^3$–a.e. $(t,x,y)\in[0,1]^3$:

- If $\overline{\rho}_n(x,y) = |c|$, then $\theta_n(t,x,y) = \overline{\theta}_n(x,y)$;
- If $\overline{\rho}_n(x,y) = |c| + r_i$, then $\theta_n(t,x,y) = \overline{\theta}_n(x - t2^{-i}, y)$.

Hence, for $j\in\{1, m_i - 1\}$, $i \ge n$, and $l \in \{1,\dots,2^i-1\}$, the function $\theta_n(t,\cdot)$ jumps on the segments

$$S_{j,i,l} \ := \ \left\{ y \ = \ 2^{-i} + \frac{j2^{-i}}{m_i} \quad x \in [l2^{-i},(l+t)2^{-i}] \right\} .$$

See Figure 2.

The total amount of this jump is given by

$$J_i \ := \ \int_{S_{j,i,l}} \left| \big(\theta_n\big)^+(t,x,y) - \big(\theta_n\big)^-(t,x,y) \right| d\mathscr{H}^1(x) \ = \ t2^{-i}|\alpha_i - \alpha| \ = \ t2^{-i}i^{-2} .$$

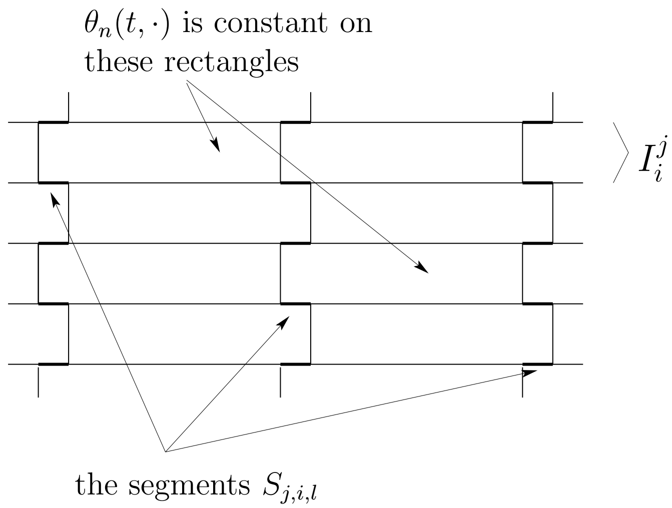

FIGURE 2. The function $\theta_n(t,\cdot)$ and the segments $S_{j,i,l}$.

Thus

$$\|\theta_n(t,\cdot)\|_{BV([0,1]^2)} \;\geq\; \sum_{i\geq n}\sum_{j=1}^{m_i-1}\sum_{l=1}^{2^i-1} J_i \;=\; \sum_{i\geq n}\big(2^i-1\big)(m_i-1)J_i \;\geq\; \frac{t}{2}\sum_{i\geq n}(m_i-1)i^{-2}\,. \quad (161)$$

Clearly, since $|u_n|(t,\cdot)\in BV\cap L^\infty$ for every t and it is bounded away from zero, it is sufficient to show that $\theta_n(t,\cdot)\notin BV([0,1]^2)$ for any $t\in]0,1[$.

Recall that the bound (157) is the only condition required on the sequence of even numbers $\{m_i\}$. If we set $m_i=2i^2$, then (157) is clearly satisfied, whereas (161) is infinite. □

6.3. **Proof of Proposition 6.5.**

Proof of Proposition 6.5. As in the proof of Proposition 6.6, for $\beta\in\mathbb{R}$ we denote by $[\beta]$ the largest integer which is less than or equal to β.

The idea is to mimic the construction of Proposition 6.6. Hence we want to start with piecewise constant initial moduli $\overline{\rho}_n$ which are constant along $m-1$ orthogonal directions $e_1,\ldots,e_{m-1}$ and oscillate along the direction ω orthogonal to each e_i. The solution ρ_n of the scalar law (117) will then be constant along the directions $e_1,\ldots,e_{m-1}$. Moreover, for small times, this solution will consist of shocks and rarefaction waves which do not interact. We will impose two requirements on this construction:

- We choose ω and the sizes and heights of the oscillations in such a way that the distinct shocks and rarefaction waves do not interact for times less than 1. Hence, in this range of times, between each couple of nearby shock and rarefaction wave, there will be a space–time strip on which ρ is constant (see Figure 3).
- We choose ω in such a way that the trajectories of solutions of (150) are “trapped” in the strips for a sufficiently long time.

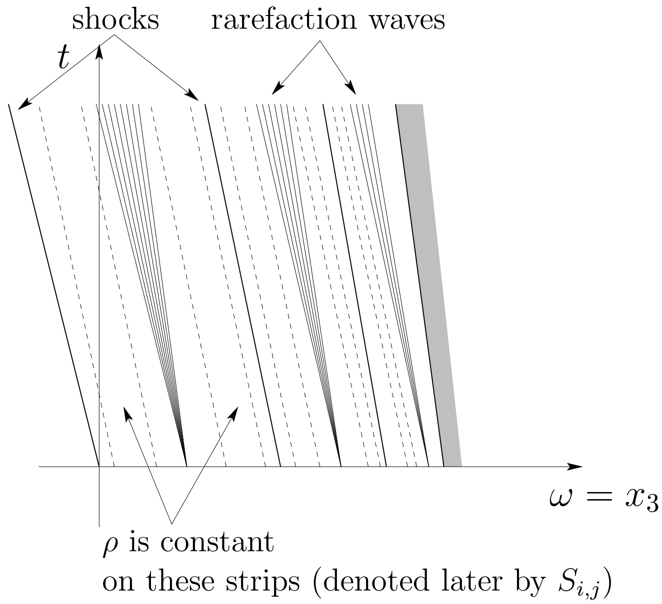

FIGURE 3. A (t, ω)–slice of the evolution of ρ_n.

Finally we choose initial data $\overline{\theta}_n$ which oscillate along a direction perpendicular to ω, in such a way that in the strip mentioned above θ_n reproduce the behavior of the construction of Proposition 6.6.

These requirements translate into geometric conditions on ω and into analytical ones on the various parameters which govern the oscillations. When $m \geq 3$ and g is not constant we can always satisfy these conditions. When $m = 2$, we are able to do it only in some cases.

Since the construction is the same, we only present the proof when $m \geq 3$ and, without loosing our generality, we assume $m = 3$. We denote by h the function given by $h(\rho) = \rho g(\rho)$ and by β the positive real number $|c|$. Clearly there exists a unit vector $\omega \in \mathbb{R}^3$ such that

$$\omega \cdot g(\beta) = \omega \cdot h'(\beta) \tag{162}$$

$$\omega \cdot g'(\beta) = 0 \tag{163}$$

$$\omega \cdot h''(\beta) = 0\,. \tag{164}$$

Indeed, since $h'(\beta) = g(\beta) + \beta g'(\beta)$, (162) reduces to (163). Thus, the conditions above reduce to find a unit vector $\omega \in \mathbb{R}^3$ which is perpendicular to both the vectors $g'(\beta)$ and $h''(\beta)$. We fix an orthonormal system of coordinates in $\mathbb{R}^3$ in such a way that $\omega = (0, 0, 1)$.

Step 1 Construction of the modulus

Let $\{\sigma_l\}$ be a sequence of vanishing positive real numbers such that $\sum \sigma_l < \infty$ and let $I_l \subset \mathbb{R}$ be the intervals

$$I_1 := [0, \sigma_1[\qquad\qquad I_l := \left[\sum_{i\leq l-1} \sigma_i, \sum_{i\leq l} \sigma_i \right] .$$

Let m_l be a strictly increasing sequence of even integers and divide every I_l in m_l equal subintervals I_l^j for $j \in \{1, \ldots m_l\}$. Finally, let $\{a_l\}$ be a vanishing sequence of real numbers and set

$$\rho^{in}(x_1, x_2, x_3) := \begin{cases} \beta + a_l & \text{if } x_3 \in I_l^j \text{ for some even } j \\ \beta & \text{otherwise.} \end{cases}$$

Then, let ρ be the entropy solution of the Cauchy problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x [h(\rho)] = 0 \\ \\ \rho(0, \cdot) = \rho^{in} \end{cases} \tag{165}$$

Clearly ρ is a function of t and x_3 only. Moreover, recalling that $(h^3)''(\beta) = 0$, we can apply Lemma 6.2 in order to get the following property.

(T) For every $C_1 > 0$, there exists a $C_2 > 0$ such that if

$$\frac{\sigma_l}{m_l} \geq C_2 a_l^2 , \tag{166}$$

then every I_l^j contains a subinterval J_l^j such that
- The length of J_l^j is bigger than $C_1 a_l^2$;
- For every $(t, \xi_1, \xi_2, \xi_3) \in [0,1] \times \mathbb{R}^2 \times J_l^j$ we have

$$\rho\big(t, \xi_1, \xi_2, \xi_3 + t(h^3)'(\beta)\big) = \rho\big(0, \xi_1, \xi_2, \xi_3\big) . \tag{167}$$

For each couple j, l we let $S_{l,j}$ be the strip

$$S_{l,j} := \left\{ (t, x_1, x_2, x_3) \middle| 0 \leq t \leq 1 \quad \text{and} \quad (x_3 - t h_3'(\beta)) \in J_l^j \right\}$$

Step 2 The flux generated by ρ.

Denote by $B_R \subset \mathbb{R}^3$ the ball of radius R centered at the origin. It is easy to check that there exists a constant C_3 such that:

$$\|\rho^{in}\|_{BV(B_R)} \leq C_3 R^3 + C_3 R^2 \left(\sum_l (m_l + 1)|a_l| \right) . \tag{168}$$

Hence, to insure that $\rho^{in} \in BV_{loc}$ it is sufficient to assume

$$\sum_l (m_l + 1)|a_l| < \infty . \tag{169}$$

Assuming that this condition is fulfilled, from the classical result of Kruzhkov we get the existence of a constant M such that $\|\rho\|_{BV(]0,1[\times B_R)} \leq M\|\rho^{in}\|_{BV(B_{R+Mt})}$. Thus we can consider the regular Lagrangian flow Φ for the ODE

$$\begin{cases} \frac{d}{dt}\Phi(s,\cdot) &= g(\rho(s,\Phi(s,\cdot))) \\ \Phi(0,x) &= x \end{cases}$$

(see Propositions 6.3 and 6.4). Fix any strip $S_{l,j}$ as defined in Step 1. Clearly, for a.e. x, every connected component of the intersection of the trajectory curve $\gamma_x := \{\Phi(t,x)|t \in \mathbb{R}\}$ with the strip $S_{l,j}$ is a straight segment. If j is even, then this segment is parallel to $(1, g(\beta))$, otherwise it is parallel to $(1, g(\beta + a_l))$. Thus, if j is even and $(t,x) \in S_{l,j}$, then the portion of trajectory

$$T_{t,x} := \big\{\Phi(s,\xi) \quad \text{for } \xi \text{ such that } \Phi(t,\xi) = x \text{ and for } s \in [0,t]\big\}$$

is a straight segment contained in $S_{l,j}$.

Let us now turn to the case where j is odd. Note that

$$g(\beta + a_l) = g(\beta) + g'(\beta)a_l + O(a_l^2)\,. \tag{170}$$

Thanks to the properties of $\omega = (0,0,1)$, we have that the segments of the form

$$\big\{(t, \xi + t(g(\beta) + a_l g'(\beta)))\big|\, 0 \leq t \leq 1 \quad \text{and} \quad (0,\xi) \in S_{l,j}\big\} \tag{171}$$

are subsets of $S_{l,j}$. Recall (T) of Step 1. From (170) and (171) it follows that, for C_1 in (T) sufficiently large, there exists a subinterval $K_{l,j}$ such that:

- The length of $K_{l,j}$ is bigger than a_l^2;
- If $t \in [0,1]$ and $x_3 - tg'(\beta) \in K_{l,j}$, then the set

  $$T_{t,x} = \big\{\Phi(s,\xi)\big|\, s \in [0,t] \quad \text{and} \quad \Phi(t,\xi) = x\big\}$$

  is a straight segment contained in $S_{l,j}$.

From now on we fix a C_1 (and hence C_2) in such a way to ensure the existence of the segments $K_{l,j}$.

Step 3 Construction of the angular part.

We recall that $g_3'(\beta) = g'(\beta)\cdot\omega = 0$ and that $g_3'(\beta) \neq 0$. Since the construction of the previous step is independent of the choice of the coordinates x_1 and x_2, we can choose them so that $g'(\beta) = (0, C_4, 0)$, with $C_4 > 0$. Choose the a_l's in such a way that

$$g_2(\beta + a_l) - g_2(\beta) = 2^{-l}\,.$$

Then, clearly, there exists a constant C_5 such that

$$\frac{2^{-l}}{C_5} \leq a_l \leq C_5 2^{-l}\,. \tag{172}$$

Set $\eta = \frac{c}{|c|}$ and let $\eta_l \in \mathbf{S}^{k-1}$ be such that $|\eta_l - \eta| = l^{-2}$. Then define

$$\theta^{in}(x_1,x_2,x_3) \;:=\; \begin{cases} \eta_l & \text{if } x_3 \in I_l \text{ and } [2^l x_2] \text{ is even} \\ \eta & \text{otherwise}. \end{cases}$$

Set $u^{in} := \rho^{in}\theta^{in}$. Let u be the renormalized entropy solution of

$$\begin{cases} \partial_t u + \operatorname{div}_z[g(|u|)u] \;=\; 0 \\ \\ u(0,\cdot) \;=\; u^{in}. \end{cases} \tag{173}$$

We denote by θ the angular part $u/|u|$. According to Propositions 6.3 and 6.4, θ is given by the formula

$$\theta(t,x) \;=\; \theta^{in}(\Psi(t,x)),$$

where Ψ is a map such that $\Phi(t,\Psi(t,x)) = \Psi(t,\Phi(t,x)) = x$ for $\mathscr{L}^4$–a.e. (t,x). In what follows we denote by Φ_t^{-1} the map $\Psi(t,\cdot)$.

Step 4 Choice of parameters.

We will prove that, for an appropriate choice of the various parameters, $u^{in} \in BV_{loc}$, whereas $u(t,\cdot)$ is not in BV_{loc} for any $t \in]0,1]$. Recall that $\rho^{in} = |u^{in}|$ and $\rho(t,\cdot) = |u|(t,\cdot)$ are both in BV_{loc} and that $C_6^{-1} \le \rho \le C_6$ for some positive constant C_6. Thus our goal is to choose the parameters σ_l and m_l in such a way that $\theta^{in} \in BV_{loc}$ and $\theta(t,\cdot) \notin BV_{loc}$ for every $t \in]0,1]$. Note that, for some constant C_7,

$$\|\theta^{in}\|_{BV(B_R)} \;\le\; C_7 R^3 + C_7 R^2 \left(\sum_l \frac{2^l}{l^2}\sigma_l + \sum_l l^{-2}\right). \tag{174}$$

Hence, choosing $\sigma_l = 2^{-l}$ we conclude that $\theta^{in} \in BV(B_R)$ for every $R > 0$.

Now, we choose $m_l = 2l^2$, and since from (172) we have $a_l \le C_5^2 2^{-l}$, we clearly fulfill the condition (169), which is the only one we required on the sequence $\{m_l\}$. Thus we get

$$\frac{\sigma_l}{m_l} \;=\; l^{-2} 2^{-l+1}.$$

Since from (172) we have $a_l^2 \le C_5 2^{-2l}$, clearly (166) is fulfilled for any constant C_2, provided l is large enough. Thus, we get the existence of a constant C_8 such that the segments $K_{l,j}$ of Step 2 exist for any $l \ge C_8$.

Fix $t \in]0,1]$ and $l \ge C_8$. Recalling that $\theta(t,x) = \theta^{in}(\Phi_t^{-1}(x))$ and taking into account the properties of Φ proved in the Step 2, we conclude what follows

- If $j \in [1,m_l]$ is even and $\xi_{l,j}$ belongs to the segment $J_{l,j}$, then
$$\theta(t,x_1,x_2,\xi_{l,j} + tg^3(\beta)) \;=\; \begin{cases} \eta_l & \text{if } \left[2^l(x_2 - tg^2(\beta))\right] \text{ is even} \\ \eta & \text{otherwise.} \end{cases}$$
- If $j \in [1,m_l]$ is odd and $\xi_{l,j}$ belongs to the segment $K_{l,j}$, then
$$\theta(t,x_1,x_2,\xi_{l,j} + tg^3(\beta)) \;=\; \begin{cases} \eta_l & \text{if } \left[2^l(x_2 - tg^2(\beta + a_l))\right] \text{ is even} \\ \eta & \text{otherwise.} \end{cases}$$

Recall that $g_2(\beta + a_l) - g_2(\beta) = 2^{-l}$. Thus, for any $j \in [1, m_l - 1]$, we have

$$\begin{aligned} A_{l,j} &:= \int_{[0,1]^2} \left|\theta(t, x_1, x_2, \xi_{l,j} + tg^3(\beta)) - \theta(t, x_1, x_2, \xi_{l,j+1} + tg^3(\beta))\right| \, dx_1 \, dx_2 \\ &= t\left|\eta_l - \eta\right| \;=\; tl^{-2} \,. \end{aligned}$$

Thus,

$$\sum_{l \geq C_8} \sum_{1 \leq j \leq m_l - 1} A_{l,j} \;=\; t \sum_{l \geq C_8} \frac{m_l - 1}{l^2} \;=\; t \sum_{l \geq C_8} \frac{2l^2 - 1}{l^2} \;=\; \infty \,. \tag{175}$$

Note that if $\theta(t, \cdot)$ were locally in BV, then $\partial_{x_3}\theta(t, \cdot)$ would be a Radon measure. Denote by μ the total variation measure of $\partial_{x_3}\theta(t, \cdot)$ and by $\mathcal{S}_{l,j}$ the stripes

$$\mathcal{S}_{l,j} \;:=\; \left\{(x_1, x_2, x_3) \middle| \, (x_1, x_2) \in [0,1]^2 \quad \text{and} \quad (x_3 - tg^3(\beta)) \in [\xi_{l,j}, \xi_{l,j+1}]\right\} .$$

Then $A_{l,j} \leq \mu(\mathcal{S}_{l,j})$. The $\mathcal{S}_{l,j}$ are pairwise disjoint and for R' sufficiently large, they are all contained in the ball $B_{R'}$. Thus, we would get

$$\sum_{l \geq C_8} \sum_{1 \leq j \leq m_l - 1} A_{l,j} \;\leq\; \sum_{l \geq C_8} \sum_{1 \leq j \leq m_l - 1} \mu(\mathcal{S}_{l,j}) \;\leq\; \mu(B_{R'}) \;<\; \infty \,,$$

which contradicts (175). Hence, we conclude that $\theta(t, \cdot)$ is not in $BV(B_{R'})$ for any $t \in]0, 1]$.

Step 5 Truncation of the construction and conclusion.
Next, define $\hat{u}^{in}_n : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$\hat{u}^{in}_n(x_1, x_2, x_3) \;:=\; \begin{cases} u^{in}(x_1, x_2, x_3) & \text{if } x_3 \in I_l \text{ for some } l \geq n \\ c & \text{otherwise.} \end{cases}$$

Clearly $\|\hat{u}^{in}_n - c\|_\infty + \|\hat{u}^{in}_n - c\|_{BV(\Omega)} \to 0$ for every bounded open set $\Omega \subset \mathbb{R}^3$. Moreover, if we denote by $\hat{u}_n$ the renormalized entropy solution of

$$\begin{cases} \partial_t u + \operatorname{div}_z[g(|u|)u] \;=\; 0 \\ \\ u(0, \cdot) \;=\; \hat{u}^{in}_n \,, \end{cases} \tag{176}$$

then $\hat{u}_n(t, \cdot) \notin BV(B_{R'})$ for any $t \in]0, 1]$. Finally, let $M > 0$ and define

$$\overline{u}_n(x_1, x_2, x_3) \;:=\; \begin{cases} \hat{u}^{in}_n(x_1, x_2, x_3) & \text{if } x_1^2 + x_2^2 + x_3^2 \leq M \\ c & \text{otherwise.} \end{cases}$$

Let u_n be the renormalized entropy solution of

$$\begin{cases} \partial_t u + \operatorname{div}_x[g(|u|)u] \;=\; 0 \\ \\ u(0, \cdot) \;=\; \overline{u}_n \,. \end{cases} \tag{177}$$

For any $M' > 0$, by the finite speed of propagation for scalar laws, if we choose M sufficiently large, then $|u_n| = |\hat{u}_n|$ on $[0, 1] \times B_{M'}(0)$. Using Lemma 3.17 and arguing as in the proof of Theorem 6.1, we conclude that $u_n = \hat{u}_n$ on $[0, 1] \times B_{R'}(0)$, provided M' is chosen sufficiently large. $\square$

7. Partial regularity and trace properties of solutions to transport equations

In this chapter we will show two regularity properties of solutions to transport equations proved in [6]. The first one is a trace property. Namely, if

- B is a bounded BV vector field and μ a Radon measure,
- w is a bounded solution of the equation

$$D\cdot(wB) \;=\; \mu\,, \tag{178}$$

- and Σ is a non–characteristic hypersurface for (178),

then w has a strong L^1 trace on Σ.

More precisely

Theorem 7.1. *Let B be a bounded BV vector field in $\Omega\subset\mathbb{R}^d$ and w an L^∞ function such that $D\cdot(wB)$ is a Radon measure. Let Σ be an oriented C^1 hypersurface with normal ν such that $\nu\cdot B^+\neq 0$ $\mathscr{H}^{d-1}$–a.e. on Σ. Then for $\mathscr{H}^{d-1}$–a.e. $x\in\Sigma$ there exists $w^+(x)\in\mathbb{R}$ such that*

$$\lim_{r\downarrow 0}\frac{1}{r^d}\int_{B^+(x,\nu)}|w(y)-w^+(x)|\,dy \;=\; 0\,. \tag{179}$$

Remark 7.2. *In* [6] *the authors proved this result for the larger class of vector fields B of bounded deformation. The proof of this stronger result is not substantially different but it needs some adjustments, which go beyond the aims of these notes.*

The second property concerns Lebesgue points of w. Before stating it let us introduce the tangential set of a BV vector field.

Definition 7.3 (Tangential set of B)**.** *Let $B\in BV_{\rm loc}(\Omega,\mathbb{R}^d)$, let $|DB|$ denote the total variation of its distributional derivative and denote by $\tilde{E}$ the Borel set of points $x\in\Omega$ s.t.*

- *The following limit exists and is finite:*

$$M(x) \;:=\; \lim_{r\downarrow 0}\frac{DB(B_r(x))}{|DB|(B_r(x))}\,.$$

- *The Lebesgue limit $\tilde{B}(x)$ exists.*

We call tangential set of B *the Borel set*

$$E \;:=\; \{x\in\tilde{E} \text{ such that } M(x)\cdot\tilde{B}(x)=0\}\,.$$

Theorem 7.4. *Let $B\in BV_{\rm loc}(\Omega,\mathbb{R}^d)$ and let $w\in L^\infty_{\rm loc}(\Omega)$ be such that $D\cdot(Bw)$ is a locally finite Radon measure in Ω. Then $|D^cB|$–a.e. point $x\notin E$ is a Lebesgue point for w, and hence for any such x there exists $\tilde{w}(x)$ such that*

$$\lim_{r\downarrow 0}\frac{1}{r^n}\int_{B_r(x)}|w(y)-\tilde{w}(x)|\,dy \;=\; 0\,. \tag{180}$$

The proof of this Theorem relies on Theorem 7.1, on the Alberti's Rank one Theorem 2.13 and on the coarea formula.

7.1. **Anzellotti's weak trace for measure–divergence bounded vector fields.** In this section we recall some basic facts about the trace properties of vector fields whose divergence is a measure (see [12], the unpublished work [14], [20], and finally [6]).

Thus, let $U \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^d)$ be such that its distributional divergence $D\cdot U$ is a measure with locally finite variation in Ω. The starting point is to define for every C^1 open set $\Omega' \subset \Omega$ the distribution $\mathrm{Tr}(U, \partial\Omega')$ as

$$\langle \mathrm{Tr}(U,\partial\Omega'), \varphi\rangle \;:=\; \int_{\Omega'} \nabla\varphi\cdot U + \int_{\Omega'} \varphi d\,[D\cdot U] \qquad \forall \varphi \in C^\infty_c(\Omega). \tag{181}$$

It was proved in [12] that

Proposition 7.5. *There exists a unique $g \in L^\infty_{\rm loc}(\Omega\cap\partial\Omega')$ such that*

$$\langle \mathrm{Tr}(U,\partial\Omega'), \varphi\rangle \;=\; \int_{\partial\Omega'} g\varphi\, d\mathscr{H}^{d-1}\,.$$

Proof. Clearly, the support of the distribution $\mathrm{Tr}(U, \partial\Omega')$ is contained in $\partial\Omega'$.

Next we claim that for any $\varphi \in C^\infty_c(\Omega)$ and any $\varepsilon > 0$ there exists $\hat\varphi_\varepsilon \in C^\infty_c(\Omega)$ such that

(i) $\hat\varphi_\varepsilon - \varphi_\varepsilon$ vanishes in a neighborhood of $\partial\Omega'$;
(ii) $\|\hat\varphi_\varepsilon\|_\infty \le \|\varphi\|_\infty$;
(iii) $\hat\varphi_\varepsilon = 0$ on $\Omega'_\varepsilon := \{x \in \Omega' : \mathrm{dist}\,(x, \partial\Omega') > \varepsilon\}$;
(iv) $\int_{\Omega'} |\nabla\hat\varphi_\varepsilon| \;\le\; \varepsilon + \int_{\partial\Omega} |\varphi|$.

Having such a $\hat\varphi_\varepsilon$ we can easily estimate

$$\begin{aligned} |\langle \mathrm{Tr}(U,\partial\Omega'),\varphi\rangle| \;&=\; |\langle \mathrm{Tr}(U,\partial\Omega'),\hat\varphi_\varepsilon\rangle| \;\le\; \left|\int_{\Omega'} \hat\varphi_\varepsilon\, d[D\cdot U]\right| + \|U\|_{L^\infty(\Omega')}\int_{\Omega'}|\nabla\hat\varphi_\varepsilon| \\ &\le\; \int_{\Omega'\setminus\Omega'_\varepsilon} |\hat\varphi_\varepsilon| d|D\cdot U| + \|U\|_{L^\infty(\Omega')}\left(\int_{\partial\Omega'}|\varphi| + \varepsilon\right) \\ &\le\; \|\varphi\|_\infty |D\cdot U|(\Omega'\setminus\Omega'_\varepsilon) + \|U\|_{L^\infty(\Omega')}\left(\int_{\partial\Omega'}|\varphi| + \varepsilon\right)\,. \end{aligned}$$

Letting $\varepsilon \downarrow 0$ we get $|\langle \mathrm{Tr}(U,\partial\Omega'),\varphi\rangle| \;\le\; \|U\|_\infty \|\varphi\|_{L^1(\partial\Omega')}$. This estimate is valid for any $\varphi \in C^\infty_c(\Omega)$ and therefore implies the claim of the Proposition.

It remains to prove the existence of the function $\hat\varphi_\varepsilon$. Using the fact that $\partial\Omega'$ is locally the graph of a C^1 function, we can find a family of open sets $\{\Omega_h\}_{h\in\mathbb{N}}$ such that $\Omega_h \subset\subset \Omega$, $\Omega_h \uparrow \Omega'$ and

$$\limsup_{h\uparrow\infty} |D\mathbf{1}_{\Omega_h}|(\mathbb{R}^d) \;\le\; |D\mathbf{1}_{\Omega'}|(\mathbb{R}^d)\,.$$

Let $\varphi \in C^\infty_c(\Omega)$ and $\varepsilon > 0$ be given and consider h so large that $\overline{\Omega'_\varepsilon} \subset \Omega_h$. Let $\{\eta_\delta\}_{\delta>0}$ be a standard family of mollifiers and choose $\delta = \delta(h) < \mathrm{dist}\,(\partial\Omega', \partial\Omega_h)$ so small that $\Omega'_\varepsilon \subset \{\mathbf{1}_{\Omega_h} * \eta_{\delta(h)} = 1\}$. Set $\zeta_h := \mathbf{1}_{\Omega_h} * \eta_{\delta(h)}$ and $\hat\varphi_\varepsilon := \varphi(1-\zeta_h)$. Clearly $\hat\varphi_\varepsilon$ satisfies (i), (ii), and (iii). Therefore it remains to check that (iv) holds for h sufficiently large. Indeed, note that

$$\limsup_{h\uparrow\infty} \int |\nabla\zeta_h| \;\leq\; |D\mathbf{1}_{\Omega'}|(\mathbb{R}^d)\,.$$

Since $\zeta_h \to \mathbf{1}_{\Omega'}$ in L^1, for every open set A we get

$$\liminf_{h\uparrow\infty} \int_A |\nabla\zeta_h| \;\geq\; |D\mathbf{1}_{\Omega'}|(A)$$

for every open set A. Therefore we conclude that the measures $|\nabla\zeta|\mathscr{L}^d$ converges weakly* to $\mathscr{H}^{d-1}\llcorner\partial\Omega'$. Hence we have

$$\int_{\Omega'} |\nabla\hat{\varphi}_\varepsilon| \;\leq\; \int_{\Omega'} (1-\zeta_h)|\nabla\varphi| + \int_{\Omega'} |\varphi||\nabla\zeta_h| \;\to\; \int_{\partial\Omega'} |\varphi|\, d\mathscr{H}^{d-1}\,.$$

This shows that (iv) holds for h sufficiently large, and thus completes the proof of the Proposition. □

By a slight abuse of notation, we denote the function g by $\mathrm{Tr}(U,\partial\Omega')$ as well.

Remark 7.6. *Clearly the notion of trace is local, that is, if $A \subset \partial\Omega_1\cap\partial\Omega_2$ is relatively open and the outer normals of $\partial\Omega_1$ and $\partial\Omega_2$ coincide on Σ, then $\mathrm{Tr}(U,\partial\Omega_1) = \mathrm{Tr}(U,\partial\Omega_2)$ on Σ.*

Given an oriented C^1 hypersurface Σ, we can always view it locally as the boundary of an open set Ω_1 having ν_Σ as unit exterior normal. In this way, we can define the positive trace $\mathrm{Tr}^+(U,\Sigma)$ as $\mathrm{Tr}(U,\partial\Omega_1)$ and the negative trace $\mathrm{Tr}^-(U,\Sigma)$ as $-\mathrm{Tr}(U,\Omega_2\setminus\Omega_1)$ where Ω_2 is any open set such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. The locality property of Remark 7.6 gives that both $\mathrm{Tr}^-(U,\Sigma)$ and $\mathrm{Tr}^+(U,\Sigma)$ are well defined.

In order to extend the notion of trace to countably $\mathscr{H}^{d-1}$-rectifiable sets, we need a stronger locality property: In [12] it was proved that

Proposition 7.7. *If $\Omega_1, \Omega_2 \subset\subset \Omega$ are two C^1 open sets, then*

$$\mathrm{Tr}(U,\partial\Omega_1) \;=\; \mathrm{Tr}(U,\partial\Omega_2) \qquad \mathscr{H}^{d-1}\text{–a.e. on } \partial\Omega_1\cap\partial\Omega_2, \tag{182}$$

if the exterior unit normals coincide on $\partial\Omega_1\cap\partial\Omega_2$.

Here we follow the recent proof of [6].

Proof. Set $\mu := |D\cdot U|\llcorner\Omega_1\cup\Omega_2$ and $E := \partial\Omega_1\cap\Omega_2$, and denote by T_i the $L^\infty(\partial\Omega_i)$ function which gives the trace $\mathrm{Tr}(U,\partial\Omega_i)$. Note thatfrom our assumptions it follows that $\mu(E) = 0$. This implies that

(i) $\mu(B_r(x)) = o(r^{d-1})$ for $\mathscr{H}^{d-1}$–a.e. $x \in E$ (see for instance Theorem 2.53 of [11]);
(ii) $\mathscr{H}^{d-1}$–a.e. $x \in E$ is a Lebesgue point for T_1 and T_2.

It suffices to show $T_1(x) = T_2(x)$ for any x satisfying both (i) and (ii).

Thus, let x be any such point and fix a test function $\chi \in C^\infty_c(B_1(0))$ with $0 \le \chi \le 1$. Set $\chi_r(y) := \chi((y-x)/r)$ for every positive r. When r is small enough, we get $\mathrm{supp}\,(\chi_r) \subset \Omega$ and thus

$$\int_{\partial\Omega_i} T_i\chi_r = \int_{\Omega_i} \nabla\chi_r \cdot U + \int_{\Omega_i} \chi_r d[D\cdot U]\,.$$

Hence,

$$\left|\int_{\partial\Omega_1} T_1\chi_r - \int_{\partial\Omega_2} T_2\chi_r\right| \;\le\; \left|\int_{\Omega_1} \nabla\chi_r \cdot U - \int_{\Omega_2} \nabla\chi_r \cdot U\right| + \left|\int_{\Omega_1} \chi_r d[D\cdot U] - \int_{\Omega_2} \chi_r d[D\cdot U]\right|\,.$$

Note that, since x is a Lebesgue point for both T_i's, for some constant C_χ (depending only on χ) we have

$$\lim_{\rho\downarrow 0} \frac{1}{r^{d-1}} \left|\int_{\partial\Omega_1} T_1\chi_r - \int_{\partial\Omega_2} T_2\chi_r\right| \;=\; C_\chi |T_1(x) - T_2(x)|\,. \tag{183}$$

Moreover C_χ is positive if, for instance, $\chi = 1$ on $B_{1/2}(0)$. Therefore it suffices to show that

$$\lim_{\rho\downarrow 0} \frac{1}{r^{d-1}} \left|\int_{\Omega_1} \nabla\chi_r \cdot U - \int_{\Omega_2} \nabla\chi_r \cdot U\right| \;=\; 0 \tag{184}$$

and

$$\lim_{\rho\downarrow 0} \frac{1}{r^{d-1}} \left|\int_{\Omega_1} \chi_r d[D\cdot U] - \int_{\Omega_2} \chi_r d[D\cdot U]\right| \;=\; 0 \tag{185}$$

to conclude that the RHS of (183) vanishes and $T_1(x) = T_2(x)$.

Since $|\nabla\chi_r| \ge C/r$, we have

$$\left|\int_{\Omega_1} U\cdot\nabla\chi_r - \int_{\Omega_2} U\cdot\nabla\chi_r\right| \;\le\; \frac{C}{r}\mathscr{L}^d((\Omega_1\setminus\Omega_2 \cup \Omega_2\setminus\Omega_1)\cap B_r(x)) \;=\; o(r^{d-1})\,,$$

which shows (184).

On the other hand

$$\begin{aligned}\left|\int_{\Omega_1} \chi_r d[D\cdot U] - \int_{\Omega_2} \chi_r d[D\cdot U]\right| \;&\le\; \|\chi_r\|_\infty |D\cdot U|((\Omega_1\setminus\Omega_2 \cup \Omega_2\setminus\Omega_1)\cap B_r(x))\\ &\le\; \mu(B_r(x)) \;=\; o(r^{d-1})\,,\end{aligned}$$

which implies (185). □

Using the decomposition of a rectifiable set Σ in pieces of C^1 hypersurfaces we can define an orientation of Σ and the normal traces of U on Σ as follows:

Definition 7.8. *By the rectifiability property we can find countably many oriented C^1 hypersurfaces Σ_i and pairwise disjoint Borel sets $E_i \subset \Sigma_i \cap \Sigma$ such that $\mathscr{H}^{d-1}(\Sigma \setminus \cup_i E_i) = 0$; then we define $\nu_\Sigma(x)$ equal to the classical normal to Σ_i for any $x \in E_i$. Analogously, we define*

$$\mathrm{Tr}^+(U,\Sigma) := \mathrm{Tr}^+(U,\Sigma_i), \quad \mathrm{Tr}^-(U,\Sigma) := \mathrm{Tr}^-(U,\Sigma_i) \qquad \mathscr{H}^{d-1}\text{-a.e. on } E_i.$$

The locality property of Proposition 7.7 ensures that this definition depends on the orientation ν_Σ, as in the case of oriented C^1 hypersurfaces, but, up to $\mathscr{H}^{d-1}$-negligible sets, it does not depend on the choice of Σ_i and E_i.

7.2. **Further properties of Anzellotti's weak trace.** In this section we follow [6] and collect three important properties of the trace of bounded vector fields with measure divergence.

Proposition 7.9 (Jump part of $D \cdot U$). *Let the divergence of $U \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^d)$ be a measure with locally finite variation in Ω. Then:*

(a) $|D \cdot U|(E) = 0$ *for any $\mathscr{H}^{d-1}$-negligible set $E \subset \Omega$.*
(b) *If $\Sigma \subset \Omega$ is a C^1 hypersurface then*

$$D \cdot U \llcorner \Sigma \;=\; (\mathrm{Tr}^+(U,\Sigma) - \mathrm{Tr}^-(U,\Sigma))\mathscr{H}^{d-1} \llcorner \Sigma. \tag{186}$$

Thanks to Proposition 7.9(a) it turns out that for any $U \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^d)$ whose divergence is a locally finite measure in Ω there exist a Borel function f and a set $J = J_{D \cdot U}$ such that

$$D^j \cdot U \;=\; f \mathscr{H}^{d-1} \llcorner J_{D\cdot U}\,. \tag{187}$$

Proposition 7.10 (Fubini's Theorem for traces). *Let U be as above and let $F \in C^1(\Omega)$. Then*

$$\mathrm{Tr}(U, \partial\{F > t\}) = U \cdot \nu \qquad \mathscr{H}^{d-1}\text{–a.e. on } \Omega \cap \partial\{F > t\}$$

for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$, where ν denotes the exterior unit normal to $\{F > t\}$.

Notice that the coarea formula gives $\mathscr{H}^{d-1}\big(\{F = t\} \cap \{|\nabla F| = 0\}\big) = 0$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$. Therefore the theory of traces applies to the sets $\Sigma_t = \{F = t\}$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$.

Theorem 7.11 (Weak continuity of traces). *Let $U \in L^\infty(\Omega, \mathbb{R}^d)$ be such that $D \cdot U$ is a Radon measure and let $f \in C^1(\mathbb{R}^{d-1})$. For $t \in \mathbb{R}$ consider the surfaces*

$$\Sigma_t \;:=\; \big\{x \;:\; x_d = t + f(x_1, \ldots, x_{d-1})\big\} \cap \Omega$$

and set

$$\alpha_t(x_1, \ldots, x_{d-1}) \;:=\; \mathrm{Tr}(U, \Sigma_t)\big(x_1, \ldots, x_{d-1}, f(x_1, \ldots, x_{d-1}) + t\big)\,.$$

If $D \subset \mathbb{R}^{d-1}$ is an open set and $I \subset \mathbb{R}$ an interval such that $\Omega' := \{(x', f(x') + t) : (x', t) \in D \times I\} \subset \Omega$, then for every $t_0 \in I$ we have $\alpha_t \rightharpoonup^* \alpha_{t_0}$ in $L^\infty(D)$ as $t \to t_0$.

Proof of Proposition 7.9. Claim (a) has been proved in Lemma 2.4. Concerning claim (b), by the locality of the statement it suffices to prove that, if $A \subset\subset \Omega$ and $F \in C^1(A)$ are such that $\Sigma \cap A = \{F = 0\}$ and $\nabla F \neq 0$ on A, then

$$\int_\Sigma \varphi d[D \cdot U] \;=\; \int_\Sigma \varphi \left[\mathrm{Tr}(U, \partial\{F > 0\}) + \mathrm{Tr}(U, \partial\{F < 0\})\right] \qquad \text{for every } \varphi \in C^\infty_c(A).$$

Note that

$$\begin{aligned}\int_\Sigma \varphi d[D\cdot U] &= \int_A \varphi d[D\cdot U] - \int_{\{F>0\}} \varphi d[D\cdot U] - \int_{\{F<0\}} \varphi d[D\cdot U]\\ &= -\int_A \nabla\varphi\cdot U + \int_{\{F>0\}} \nabla\varphi\cdot U + \int_\Sigma \varphi \mathrm{Tr}(U,\partial\{F>0\})\\ &\quad + \int_{\{F<0\}} \nabla\varphi\cdot U + \int_\Sigma \varphi\mathrm{Tr}(U,\partial\{F<0\})\\ &= \int_\Sigma \varphi[\mathrm{Tr}(U,\partial\{F>0\}) + \mathrm{Tr}(U,\partial\{F<0\})]\varphi\,.\end{aligned}$$

□

Proof of Proposition 7.10. The statement of the Proposition is trivial if U is smooth. In the general case we will prove it by approximation.

Indeed let U be a field as in the statement of the Proposition, choose a standard family of mollifiers $\{\eta_\varepsilon\}_{\varepsilon>0}$ and set $U_\varepsilon := U*\eta_\varepsilon$. Recall that $|D\cdot U_\varepsilon|\rightharpoonup^*|D\cdot U|$ in the sense of measures. Note that the set $S:=\{t:|D\cdot U|(\Sigma_t)=0\}$ is at most countable. A For $t\not\in S$ we have

- $\mathrm{Tr}^+(U,\Sigma_t)=\mathrm{Tr}^-(U,\Sigma_t)$. by Proposition 7.9;
- $(D\cdot U_\varepsilon)\llcorner\{F>t\}\rightharpoonup^*(D\cdot U)\llcorner\{F>t\}$ and $(D\cdot U_\varepsilon)\llcorner\{F<t\}\rightharpoonup^*(D\cdot U)\llcorner\{F<t\}$ by Proposition 2.1.

Therefore, from the Definition of trace it follows that

$$\mathrm{Tr}(U_\varepsilon,\partial\{F>t\}) \rightharpoonup \mathrm{Tr}(U,\partial\{F>t\})$$

in the sense of distributions for every $t\not\in S$.

Since U_ε is smooth, $\mathrm{Tr}(U_\varepsilon,\partial\{F>t\})=U_\varepsilon\cdot\nu_t$ and therefore it suffices to prove that

- There exists a vanishing sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}\subset\mathbb{R}^+$ such that

  $$U_{\varepsilon_h}\cdot\nu\to U\cdot\nu \qquad \text{in } L^1(\Sigma_t)$$

  for $\mathscr{L}^1$–a.e. t.

Such a property holds for every "fast" converging subsequence $\{U_{\varepsilon_h}\}$, i.e. such that

$$\sum_{h=1}^\infty \|U_{\varepsilon_h}-U\|_{L^1(\Omega)} \;<\; \infty\,.$$

Indeed for such a subsequence we can use the coarea formula to estimate

$$\begin{aligned}\int_\mathbb{R}\sum_h \|U_{\varepsilon_h}-U\|_{L^1(\Sigma_t)}\,dt &\le \sum_h\int_\mathbb{R}\|U_{\varepsilon_h}-U\|_{L^1(\Sigma_t)}\,dt\\ &\le \sum_h\int_\Omega|\nabla F||U_{\varepsilon_h}-U| \;\le\; \|F\|_{C^1}\sum_h\|U_{\varepsilon_h}-U\|_{L^1(\Omega)}<\infty\,.\end{aligned}$$

Thus, for $\mathscr{L}^1$–a.e. t the series $\sum_h\|U_{\varepsilon_h}-U\|_{L^1(\Sigma_t)}$ must be finite, and this implies that for any such t, $U_{\varepsilon_h}\to U$ strongly in $L^1(\Sigma_t)$. □

Proof of Theorem 7.11. Let $\varphi \in C^\infty_c(D)$ be given and consider the function $\psi \in C^1(\Omega')$ given by $\psi(x', x_d) = \varphi(x')$. It is not difficult to see that ψ can be extended to a function in $C^1_c(\Omega)$. Next, set $\sigma(x') := \sqrt{1 + |\nabla f(x')|^2}$ and for every $t > t_0$ define the open set

$$\Omega_t \;:=\; \left\{(x', f(x') + \tau) \;:\; x' \in D,\, \tau \in]t_0, t[\right\}.$$

In analogous way we define Ω_t for $t < t_0$. Then, using the definition of trace, we easily get

$$\begin{aligned}
&\left|\int_D \varphi(x')\sigma(x')\big(\alpha_t(x') - \alpha_{t_0}(x')\big)\, dx'\right| \\
=\;& \left|\int_{\partial\Omega_t} \big(\mathrm{Tr}^+(U, \Sigma_t)(x)\psi(x) - \mathrm{Tr}^-(U, \Sigma_{t_0})(x)\psi(x)\big)\; d\mathscr{H}^{d-1}(x)\right| \\
=\;& \left|-\int_{\Omega_t} \nabla\psi \cdot U - \int_{\Omega_t} \psi\, d[D\cdot U]\right| \;\leq\; \|\nabla\psi\|_{L^\infty(\Omega_t)}\|U\|_\infty|\Omega_t| + \|\Phi\|_\infty |D\cdot U|(\Omega_t)\,.
\end{aligned}$$

Since the last expressions converge to 0 as $t \to t_0$, we get that

$$\int_D \varphi(x')\sigma(x')\big(\alpha_t(x') - \alpha_{t_0}(x')\big)\, dx' \;\to\; 0$$

for every $\varphi \in C^\infty_c(D)$. Since $\|\alpha_t\|_\infty$ is bounded by $\|U\|_\infty$, we conclude that $\alpha_t\sigma$ converges weakly* in $L^\infty(D)$ to $\alpha_{t_0}\sigma$. Note that $\sigma \geq 1$, and hence $\alpha_t \rightharpoonup^* \alpha_{t_0}$, which is the desired conclusion. □

7.3. **Change of variables for traces.** This section is devoted to prove the core result of [6], namely the following "chain rule" for traces:

Theorem 7.12 (Change of variables for traces)**.** *Let $B \in BV \cap L^\infty(\Omega, \mathbb{R}^d)$ and $w \in L^\infty(\Omega)$ be such that $D\cdot(wB)$ is a Radon measure. If $\Omega' \subset\subset \Omega$ is an open domain with a C^1 boundary and $h \in C^1(\mathbb{R}^k)$, then*

$$\mathrm{Tr}(h(w)B, \partial\Omega') = h\left(\frac{\mathrm{Tr}(wB, \partial\Omega')}{\mathrm{Tr}(B, \partial\Omega')}\right)\mathrm{Tr}(B, \partial\Omega') \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial\Omega'.$$

Here we use the convention that when $\mathrm{Tr}(B, \partial\Omega')(x) = 0$, the expression

$$h\left(\frac{\mathrm{Tr}(wB, \partial\Omega')(x)}{\mathrm{Tr}(B, \partial\Omega')(x)}\right)\mathrm{Tr}(B, \partial\Omega')(x)$$

is zero as well.

Remark 7.13. *In* [6] *the authors proved the previous Theorem for the class of vector fields B of bounded deformation (compare with Remark 7.2).*

In order to prove the Theorem, we need the following renormalization lemma

Lemma 7.14. *Let B, w, and h be as above. Then $D \cdot (h(w)B)$ is a Radon measure and, if $R := \|w\|_\infty$, then*

$$
\begin{aligned}
|D \cdot (h(w)B))| \quad &\leq \quad \|\nabla h\|_{L^\infty(B_R(0))} \big(|D \cdot (wB)| + 2R|D^s \cdot B \big) \\
&\qquad + \left(\sup_{v \in B_R(0)} \left| h(v) - \sum v^i \frac{\partial h}{\partial v^i}(v) \right| \right) |D \cdot B| \, .
\end{aligned}
$$

Proof. Let $\{\eta_\delta\}$ be a family of standard mollifiers and set $w_\delta := w * \eta_\delta$, $T_\delta := (D \cdot (wB) * \eta_\delta - D \cdot (w_\delta B)$. Then we compute

$$
\begin{aligned}
D \cdot h(w_\delta) \quad &= \quad \sum_i \frac{\partial h}{\partial v^i}(w_\delta) \big(D \cdot (B w^i) \big) * \eta_\delta + \sum_i \frac{\partial h}{\partial v^i}(w_\delta) T^i_\delta \\
&\qquad + \left(h(w_\delta) - \sum_i \frac{\partial h}{\partial v^i}(w_\delta) w^i_\delta \right) D \cdot B \, .
\end{aligned}
$$

Using the commutator estimate of Proposition 4.6 and Lemma 4.8 we easily conclude (compare with the proof of Theorem 4.1). □

Proof of Theorem 7.12. It is not restrictive to assume that the larger open set Ω is bounded and it has a C^1 boundary.

Step 1 Let $\Omega'' = \Omega \setminus \overline{\Omega'}$. In this step we prove that

$$
\mathrm{Tr}(h(w)B, \partial\Omega'') = h \left(\frac{\mathrm{Tr}(wB, \partial\Omega'')}{\mathrm{Tr}(B, \partial\Omega'')} \right) \mathrm{Tr}(B, \partial\Omega'') \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial\Omega'',
$$

under the assumption that the components of B and w are bounded and belong to the Sobolev space $W^{1,1}(\Omega'')$. Indeed, the identity is trivial if both w and B are continuous up to the boundary, and the proof of the general case can be immediately achieved by a density argument based on the strong continuity of the trace operator from $W^{1,1}(\Omega'')$ to $L^1(\partial\Omega'', \mathscr{H}^{d-1} \llcorner \partial\Omega'')$ (see for instance Theorem 3.88 of [11]).

Step 2 In this step we prove the general case. Let us apply Gagliardo's Theorem (see [33]) on the surjectivity of the trace operator from $W^{1,1}$ into L^1 to obtain a bounded vector field $B_1 \in W^{1,1}(\Omega''; \mathbb{R}^d)$ whose trace on $\partial\Omega' \subset \partial\Omega''$ is equal to the trace of B, seen as a function in $BV(\Omega')$. In particular $\mathrm{Tr}(B, \partial\Omega') = -\mathrm{Tr}(B_1, \partial\Omega'')$. Defining

$$
\hat{B}(x) := \begin{cases} B(x) & \text{if } x \in \Omega' \\ B_1(x) & \text{if } x \in \Omega'', \end{cases}
$$

it turns out that $\hat{B} \in BV(\Omega)$ and that

$$
|D\hat{B}|(\partial\Omega') = 0. \tag{188}
$$

Let us consider the function $\theta := \mathrm{Tr}(wB, \partial\Omega') / \mathrm{Tr}(B, \partial\Omega')$ (set equal to 0 wherever the denominator is 0) and let us prove that $\|\theta\|_{L^\infty(\partial\Omega')}$ is less than $\|w\|_{L^\infty(\Omega')}$. Indeed, writing

$\partial\Omega'$ as the 0-level set of a C^1 function F with $|\nabla F| > 0$ on $\partial\Omega'$ and $\{F = t\} \subset \Omega'$ for $t > 0$ sufficiently small, by Proposition 7.10 we have

$$\begin{aligned} -\|w\|_{L^\infty(\Omega')}\mathrm{Tr}(B, \partial\{F > t\}) &\le \mathrm{Tr}(wB, \partial\{F > t\}) \\ &\le \|w\|_{L^\infty(\Omega')}\mathrm{Tr}(B, \partial\{F > t\}) \end{aligned}$$

$\mathscr{H}^{d-1}$-a.e. on $\{F = t\}$ for $\mathscr{L}^1$-a.e. $t > 0$ sufficiently small. Passing to the limit as $t \downarrow 0$ and using Theorem 7.11 we recover the same inequality on $\{F = 0\}$, proving the boundedness of θ.

Now, still using Gagliardo's theorem, we can find a bounded function $w_1 \in W^{1,1}(\Omega''; \mathbb{R}^k)$ whose trace on $\partial\Omega'$ is given by θ, so that the normal trace of $w_{1i}B_1$ on $\partial\Omega''$ is equal to $\mathrm{Tr}(w_i B, \partial\Omega')$ on the whole of $\partial\Omega'$. Defining

$$\hat{w}(x) := \begin{cases} w(x) & \text{if } x \in \Omega' \\ w_1(x) & \text{if } x \in \Omega'', \end{cases}$$

by Proposition 7.9 we obtain

$$|D\cdot(\hat{w}^i\hat{B})|(\partial\Omega') = 0 \qquad i = 1, \ldots, k. \tag{189}$$

Let us apply now Lemma 7.14 and (188), (189), to obtain that the divergence of the vector field $h(\hat{w})\hat{B}$ is a measure with finite total variation in Ω, whose restriction to $\partial\Omega'$ vanishes. As a consequence, Proposition 7.9 gives

$$\mathrm{Tr}^+(h(\hat{w})\hat{B}, \partial\Omega') = \mathrm{Tr}^-(h(\hat{w})\hat{B}, \partial\Omega') \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial\Omega' \tag{190}$$

(here, by a slight abuse of notation, we consider $\partial\Omega'$ as a C^1 oriented surface whose orienting normal coincides with the outer normal to $\partial\Omega'$).

By applying (190), Step 1, and finally our choice of B_1 and w_1 the following chain of equalities holds $\mathscr{H}^{d-1}$-a.e. on $\partial\Omega'$:

$$\begin{aligned} \mathrm{Tr}(h(w)B, \partial\Omega') &= \mathrm{Tr}^+(h(\tilde{w})\tilde{B}, \partial\Omega') = \mathrm{Tr}^-(h(\tilde{w})\tilde{B}, \partial\Omega') \\ &= \mathrm{Tr}(h(w_1)B_1, \partial\Omega'') = h\left(\frac{\mathrm{Tr}(w_1B_1, \partial\Omega'')}{\mathrm{Tr}(B_1, \partial\Omega'')}\right)\mathrm{Tr}(B_1, \partial\Omega'') \\ &= h\left(\frac{\mathrm{Tr}(wB, \partial\Omega')}{\mathrm{Tr}(B, \partial\Omega')}\right)\mathrm{Tr}(B, \partial\Omega'). \end{aligned}$$

□

7.4. **Proof of Theorem 7.1.** In this section we combine the change of variables for traces with a blow–up argument in order to prove Theorem 7.1.

Proof. Let Σ be as in the statement. Without loss of generality we can assume that Σ is the boundary of some open set $\Omega' \subset\subset \Omega$, and that the normal ν to Σ is the outer normal of Ω'. Arguing as in the proof of Theorem 7.12, we can build a vector field $\hat{B} \in BV \cap L^\infty(\Omega)$ and a bounded function $\hat{w}$ such that

- $\hat{w} = w$ and $\hat{B} = B$ on $\Omega \setminus \Omega'$;
- $|D\cdot(wB)|(\partial\Omega') = |DB|(\partial\Omega') = 0$.

Given any $x \in \partial\Omega'$, note that

$$\lim_{r\downarrow 0} \frac{|\Omega' \cap B_r^+(x,\nu)|}{r^d} \;=\; 0$$

and thus it suffices to prove the claim for $\hat{w}$ and $\hat{B}$. In order to simplify the notation, from now on we will write w and B instead of $\hat{w}$ and $\hat{B}$. Moreover, note that the change of variables for traces implies that $|D\cdot(w^2B)|(\partial\Omega') = 0$.

Next, fix any $x \in \partial\Omega$ such that $\mathrm{Tr}(B,\partial\Omega')(x)\cdot\nu \neq 0$ and choose a system of coordinates $(x_1,\ldots,x_{d-1},x_d) = (x',x_d)$ in such a way that $\nu = (0,\ldots,0,1)$.

From now on we simply write $B(x)$ for $\mathrm{Tr}(B,\partial\Omega')(x)$ and for any $r > 0$ consider the $d-1$ dimensional cube

$$C_r \;:=\; \big\{x + (y_1,\ldots,y_{d-1},0) \;:\; |y_i| < r\big\}\,,$$

the d–dimensional parallelogram

$$Q_r \;:=\; \big\{y + \rho B(x) \;:\; y \in C_r, |\rho| < r\big\}$$

and the open set $Q_r^+ := Q_r \setminus \overline{\Omega'}$. We denote by 2α the volume of Q_1 (that is, $\alpha = |B(x)\cdot\nu|$).

Clearly, there exists constant C such that $|B_r^+(x,\nu)\setminus Q_r^+| = o(r^d)$, and therefore it suffices to prove that

$$\lim_{r\downarrow 0} \frac{1}{r^d}\int_{Q_r^+(x)} \left| w(y) - \frac{\mathrm{Tr}(wB,\partial\Omega')(x)}{B(x)}\right| dy \;=\; 0\,. \tag{191}$$

We will prove that this holds for any point x which satisfy the following requirements:

(a) x is a Lebesgue point for $\mathrm{Tr}(wB,\partial\Omega')$ and $\mathrm{Tr}(w^2B,\partial\Omega')$, that is

$$\lim_{r\downarrow 0}\frac{1}{r^{d-1}}\int_{\partial\Omega'\cap B_r(x)} \Big[\big|\mathrm{Tr}(wB,\partial\Omega')(y) - \mathrm{Tr}(wB,\partial\Omega')(x)\big| + \big|\mathrm{Tr}(w^2B,\partial\Omega')(y) - \mathrm{Tr}(w^2B,\partial\Omega')(x)\big|\Big]\,dy \;=\; 0\,,$$

and it is a Lebesgue point for B, that is

$$\lim_{r\downarrow 0}\frac{1}{r^d}\int_{B_r(x)} |B(y)-B(x)|\,dy \;=\; 0\,;$$

(b) $B(x)\cdot\nu\,\mathrm{Tr}(w^2B,\partial\Omega') = [\mathrm{Tr}(wB,\partial\Omega')]^2$;

(c) $|D\cdot(wB)|(B_r(x)) + |D\cdot(w^2B)|(B_r(x)) = o(r^{d-1})$.

Since these conditions are satisfied $\mathscr{H}^{d-1}$–a.e. on the set $\partial\Omega'\setminus\{\mathrm{Tr}(B,\partial\Omega') = 0\}$, this claim will prove the Theorem.

Step 1 Let x be any point which satisfies the conditions (a), (b), and (c). In order to simplify the notation, from now on we assume that $x = 0$. Let $r > 0$. Note that using a simple Fubini–type argument we get the existence of an $s(r) \in]r, 2r[$ such that

$$\int_{\partial Q_{s(r)}} |B(y)-B(0)|\,dy \;\leq\; Cr^{-1}\int_{Q_{2r}} |B(y)-B(0)|\,dy \tag{192}$$

where C is a constant. Moreover, by Proposition 7.10, we can also assume that, if ζ denotes the outer unit normal to $\partial Q^+_{s(r)}$, then

$$\mathrm{Tr}(B,\partial Q^+_{s(r)}) ;= \; B\cdot\zeta \qquad \text{and} \qquad \mathrm{Tr}(wB,\partial Q^+_{s(r)}) \;=\; wB\cdot\zeta \qquad \mathscr{H}^{d-1}\text{–a.e. on } \partial Q^+_{s(r)}. \tag{193}$$

Denote by B^d the component in direction $(0,\ldots,0,1)=\nu$ of B and, without loss of generality, assume that $B^d(0)>0$. Moreover, note that $\alpha = |B(0)\cdot\nu| = B^d(0)$. We will show that

$$\lim_{s\downarrow 0} s(r)^{-d}\int_{Q^+_{s(r)}} w(y)B^d(y)\,dy \;=\; \alpha\mathrm{Tr}(wB,\partial\Omega')(0) \tag{194}$$

and

$$\lim_{r\downarrow 0} s(r)^{-d}\int_{Q^+_{s(r)}} w^2(y)B^d(y)\,dy \;=\; \alpha(\mathrm{Tr}(wB,\partial\Omega')(0))^2\,. \tag{195}$$

This will complete the proof, because

$$\begin{aligned}
&\lim_{r\downarrow 0} s(r)^{-d}\int_{Q^+_{s(r)}}\left|w(y)-\frac{\mathrm{Tr}(wB,\partial\Omega')(0)}{B(0)}\right|dy\\
\le\;& \lim_{r\downarrow 0} s(r)^{-d}\int_{Q^+_r}\left|w(y)B^d(0)-\mathrm{Tr}(wB,\partial\Omega')(0)\right|\\
=\;& \lim_{r\downarrow 0} s(r)^{-d}\int_{Q^+_{s(r)}}\Big[w^2(y)(B^d(y))^2-\mathrm{Tr}(wB,\partial\Omega')(0)w(y)B(y)\Big]\,dy+[\mathrm{Tr}(wB,\partial\Omega')(0)]^2\alpha\\
=\;& \lim_{r\downarrow 0} s(r)^{-d}\int_{Q^+_{s(r)}} w^2(y)B^d(y)B^d(0)\,dy-\alpha[\mathrm{Tr}(wB,\partial\Omega')(0)]^2\\
=\;& \alpha B^d(0)\mathrm{Tr}(w^2B,\partial\Omega')(0)-\alpha[\mathrm{Tr}(wB,\partial\Omega')(0)]^2 \;=\; 0\,.
\end{aligned}$$

Step 2 In this step we show (194). The proof of (195) is completely analogous and therefore we omit it.

Denote by $D_{s(r)}$ the top face of $\partial Q^+_{s(r)}$, that is

$$D_{s(r)} \;=\; \big\{(y_1,\ldots,y_{d-1},0)+s(r)B : |y_i|\le s(r)\big\}\,.$$

Then consider the test function $\varphi_r(y) := s(r)B^d(0)-y_d$ and apply the definition of weak trace to get

$$-\int_{Q^+_{s(r)}} w(y)B^d(y)\,dy \;=\; -\int_{Q^+_{s(r)}}\varphi_r\,d[D\cdot(wB)]+\int_{\partial Q^+_{s(r)}\setminus D_{s(r)}}\varphi_r\mathrm{Tr}(wB,\partial Q^+_{s(r)})\,d\mathscr{H}^{d-1}\,.$$

Recall that for some constant C we have $B_{C^{-1}r}(0)\subset Q_r\subset B_{Cr}(0)$. Therefore the first integral in the right hand side is $o(s(r)^d)$ by (c). Next, we split the surface $\partial Q^+_{s(r)}\setminus D_{s(r)}$ into

$\partial\Omega' \cap Q_{s(r)}$ and $L := \partial Q^+_{s(r)} \setminus (D_{s(r)} \cup \partial\Omega')$. Thus

$$\lim_{r\downarrow 0} \frac{1}{(s(r))^d} \int_{Q^+_{s(r)}} w(y)B^d(y)\,dy \;=\; \lim_{r\downarrow 0} \frac{1}{(s(r))^d} \int_{\partial\Omega'\cap Q_{s(r)}} \varphi_r \mathrm{Tr}(wB, \partial Q^+_{s(r)}) + \lim_{r\downarrow 0} \frac{1}{(s(r))^d}\frac{1}{(s(r))^d} \int_L \varphi_r \mathrm{Tr}(wB, \partial Q^+_{s(r)})\,. \quad (196)$$

Note that $\varphi_r = B^d(0)s(r) + o(s(r)) = \alpha s(r) + o(s(r))$ on $Q_{s(r)} \cap \partial\Omega'$. Moreover, note that $\mathscr{H}^{d-1}(\partial\Omega' \cap Q_{s(r)}) = s(r)^{d-1} + o(s(r)^{d-1})$. Thus, from (a) we conclude that

$$\begin{aligned}\lim_{r\downarrow 0} \frac{1}{(s(r))^d} \int_{\partial\Omega'\cap Q_{s(r)}} \varphi_r \mathrm{Tr}(wB, \partial Q^+_{s(r)}) \;&=\; \lim_{r\downarrow 0} \frac{\alpha}{(s(r))^{d-1}} \int_{\partial\Omega'\cap Q_{s(r)}} \mathrm{Tr}(wB, \partial Q^+_{s(r)}) \\ &=\; \alpha \mathrm{Tr}(wB, \partial\Omega')(0)\,. \end{aligned} \quad (197)$$

Recall that our goal is to show (194). Thus, taking into account (196) and (197), it remains to show that

$$\lim_{r\downarrow 0} \frac{1}{(s(r))^d} \int_L \varphi \mathrm{Tr}(wB, \partial Q^+_{s(r)}) \;=\; 0\,. \quad (198)$$

Note that

$$\left| \int_L \varphi \mathrm{Tr}(wB, \partial Q^+_{s(r)}) \right| \;\leq\; Cs(r) \int_L |\mathrm{Tr}(wB, \partial Q^+_{s(r)})|\,. \quad (199)$$

Denote by ζ the normal to ∂L and note that $B(0)\cdot\zeta = 0$. Thus,

$$\int_L |\mathrm{Tr}(B, \partial Q^+_{s(r)})| \;\overset{(193)}{=}\; \int_L |B\cdot\zeta| \;\leq\; \int_L |B(y) - B(0)| \;\overset{(192)}{=}\; o(s(r)^{d-1})\,. \quad (200)$$

On the other hand, by (193), $|\mathrm{Tr}(wB, \partial Q^+_{s(r)})| \leq \|w\|_\infty |\mathrm{Tr}(B, \partial Q^+_{s(r)})|$, and hence (200) and (199) give (198). □

7.5. **Proof of Theorem 7.4.** Given $B \in BV$, the coarea formula and the Alberti's Rank–one Theorem induce a natural fibration of $|D^cB|$ into codimension one rectifiable sets In this section we use this property to show Theorem 7.4 from Theorem 7.1.

Proof. Let $B^1, \dots B^d$ be the components of B. Moreover, recall that $\tilde{B}(x)$ denote the approximate limit of B at x whenever it exists.

Note that $|D^cB| \leq \sum_i |D^cB^i|$. Therefore it suffices to prove (180) for $|D^cB^i|$–a.e. $x \notin E$. According to Alberti's rank–one theorem, there exist Borel functions $\xi : \mathbb{R}^d \to \mathbb{R}^d$ and $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ such that $D^cB = \xi\otimes\zeta|D^cB|$. So it suffices to prove (180) for $|D^cB^i|$–a.e. $x \in F$, where F is the set of points x where the approximate limit of B exists and $\zeta(x)\cdot\tilde{B}(x) = 0$.

Recall that for $\mathscr{L}^1$–a.e. t, the set $\Omega_t := \{B^i > t\}$ is a Caccioppoli set and therefore $D\mathbf{1}_{\Omega_t} = \nu_t \mathscr{H}^{d-1} \llcorner \partial^*\Omega_t$, where $\partial^*\Omega_t$ is a rectifiable set an ν_t the approximate exterior unit normal. From the coarea formula for BV functions (see Theorem 2.10), we have

$$\int_\Omega \varphi\, d|DB^i| \;=\; \int_{\mathbb{R}} \int_{\partial^*\Omega_t} \varphi\, d\mathscr{H}^{d-1}\, dt\,.$$

Therefore, it suffices to prove (180) for points x in the set

$$F' \;:=\; \bigcup_{\{t\,:\,\Omega_t \text{ is a Caccioppoli set}\}} \partial^*\Omega_t \cap F\,.$$

Moreover, recall that

$$\int_\Omega \Phi\cdot dDB^i \;=\; \int_{\mathbb{R}}\int_{\partial^*\Omega_t} \Phi\cdot \nu_t d\mathscr{H}^{d-1}\,dt\,.$$

Thus, for $\mathscr{L}^1$–a.e. t, we have

(a) $\zeta|_{\partial^*_t\Omega}(x) = \nu_t(x)$ for $\mathscr{H}^{d-1}$–a.e. x.

Moreover, note that, for $\mathscr{L}^1$–a.e. t, we have

(b) $\tilde{B}|_{\partial^*\Omega_t\cap F}(x) = \mathrm{Tr}(B,\Omega_t)|_{\partial^*\Omega_t\cap F}(x)$ for $\mathscr{H}^{d-1}$–a.e. x.

Therefore, it suffices to prove the claim for every $x\in F'$ which satisfies (a) and (b).

Next, note that if for $s<t$ Ω_s and Ω_t are both Caccioppoli sets and $x\in\partial^*\Omega_s\cap\partial^*\Omega_t$, then B^1 cannot have approximate limit at x. Therefore, the sets $\partial^*\Omega_t\cap F'$ are all disjoint, and hence the set E of t's such that $|D\cdot(wB)|(\partial^*\Omega_t\cap F')>0$ is at most countable. By the coearea formula, we conclude that

$$|D_c B^i|\left(\bigcup_{t\in E}\partial^*\Omega_t\right) \;=\; 0\,.$$

We finally define the set $F''\subset F'$ of points $x\in\partial^*\Omega_t$ with t and x such that:

- The approximate limit $\tilde{B}(x)$ of B at x exists and $\zeta(x)\cdot\tilde{B}(x)=0$;
- Ω_t is a Caccioppoli set and $|D\cdot(wB)|(\partial^*\Omega_t\cap F')=0$;
- $\nu_t(x)=\zeta(x)$, and hence $\nu_t(x)\cdot\tilde{B}(x)=0$;
- $\tilde{B}(x)=\mathrm{Tr}^+(B,\partial^*\Omega_t)(x)$ (where we take ν_t as orienting normal for $\partial^*\Omega_t$.

Summarizing what discussed so far, it suffices to prove (180) for $\mathscr{H}^{d-1}$–a.e. $x\in\partial^*\Omega_t\cap F''$.

So fix a t such that $\partial^*\Omega_t\cap F''\neq\emptyset$ and let $\{\Sigma_j\}_j$ be a countable family of C^1 surfaces which cover $\mathscr{H}^{d-1}$–a.e. $\partial^*\Omega_t$. If we denote by ν_j the unit normals to Σ_j we have $\nu_j=\nu_t$ $\mathscr{H}^{d-1}$–a.e. on $\Sigma_j\cap\partial^*\Omega_t$. Thus it suffices to show (180) for $\mathscr{H}^{d-1}$–a.e. $x\in\Sigma_j\cap\partial^*\Omega_t$ such that $\nu_j(x)\cdot\mathrm{Tr}(B,\Sigma_j)(x)\neq 0$. From Theorem 7.1, for $\mathscr{H}^{d-1}$–a.e. such x we have

$$\lim_{r\downarrow 0}\frac{1}{r^d}\int_{B^+_r(x,\nu)}\left|w(y)-\frac{\mathrm{Tr}^+(wB,\Sigma_j)(x)}{\mathrm{Tr}^+(B,\Sigma_j)}(x)\right|\,dy \;=\; 0 \tag{201}$$

and

$$\lim_{r\downarrow 0}\frac{1}{r^d}\int_{B^+_r(x,\nu)}\left|w(y)-\frac{\mathrm{Tr}^-(wB,\Sigma_i)(x)}{\mathrm{Tr}^-(B,\Sigma_j)}(x)\right|\,dy \;=\; 0\,. \tag{202}$$

From the definition of F'', $\mathrm{Tr}^+(B,\Sigma_j)(x)=\mathrm{Tr}^-(B,\Sigma_j)(x)=\mathrm{Tr}^+(B,\partial^*\Omega_t)(x)=\tilde{B}(x)$ for $\mathscr{H}^{d-1}$–a.e. $x\in\Sigma_j\cap\partial^*\Omega_t$. Moreover, since $|D\cdot(wB)|(\Sigma_i\cap\partial^*\Omega_t)=0$, from Proposition 7.9 we conclude $\mathrm{Tr}^+(wB,\Sigma_j)(x)=\mathrm{Tr}^-(wB,\Sigma_j)(x)$ for $\mathscr{H}^{d-1}$–a.e. $x\in\partial^*\Omega_t\cap\Sigma_j$. Therefore (201) and (202) give the desired claim. □

8. Bressan's compactness conjecture and the Renormalization conjecture for nearly incompressible BV vector fields

In [17] Bressan proposed the following

Conjecture 8.1 (Bressan's compactness conjecture)**.** *Let $b_n : \mathbb{R}_t \times \mathbb{R}^m_x \to \mathbb{R}^m$ be smooth maps and denote by Φ_n the solution of the ODEs:*

$$\begin{cases} \dfrac{d}{dt}\Phi_n(t,x) \;=\; b_n(t,\Phi_n(t,x)) \\ \\ \Phi_n(0,x) \;=\; x\,. \end{cases} \tag{203}$$

Assume that the fluxes Φ_n are nearly incompressible, i.e. that for some constant C we have

$$C^{-1} \;\leq\; \det(\nabla_x \Phi_n(t,x)) \;\leq\; C\,, \tag{204}$$

and that $\|b_n\|_\infty + \|\nabla b_n\|_{L^1}$ is uniformly bounded. Then the sequence $\{\Phi_n\}$ is strongly precompact in $L^1_{\rm loc}$.

This conjecture was advanced in connection with the Keyfitz and Kranzer system, in particular to provide the existence of suitable weak solutions. Though, as shown in Section 5, one can prove well–posedness for this system bypassing it, Conjecture 8.1 is an interesting and challenging question. In this section we will show some recent partial results on it, contained in [10].

First of all, we note that Bressan's compactness Conjecture would follow from the following

Conjecture 8.2 (Renormalization Conjecture)**.** *Any nearly incompressible bounded BV vector field has the renormalization property of Definition 3.12.*

Conjecture 8.2 $\Longrightarrow$ Conjecture 8.1. Let $\rho_n := ({\rm id}\,,\Phi_n)_\# \mathscr{L}^{m+1}$ be the density generated by the flows Φ_n. From (204) it follows that $C_1 \geq \rho_n \geq C_1^{-1} > 0$ for some constant $C_1 > 0$.

From the BV compactness Theorem and the weak* compactness of L^∞, it suffices to prove Conjecture 8.1 under the additional assumptions that $b_n \to b$ strongly in L^1_{loc} for some BV vector field b and that $\rho_n \rightharpoonup^* \rho$ in L^∞ for some bounded ρ. Note that

- $\partial_t \rho_n + D_x \cdot (\rho_n b_n)$ converge to $\partial_t \rho + D_x \cdot (\rho b)$ in the sense of distributions, and thus $\partial_t \rho + D_x \cdot (\rho b) = 0$;
- $\rho \geq C_1^{-1}$;
- $\|b\|_\infty < \infty$.

Hence, b is a bounded nearly incompressible vector field, and if Conjecture 8.2 has an affirmative answer, then b has the renormalization property. In this case we can apply Theorem 3.22 to conclude that Φ_n converges strongly in L^1_{loc} to the unique regular lagrangian flow generated by b. □

The main result of [10] is the following

Theorem 8.3. *Let $b \in BV \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^m, \mathbb{R}^m)$ be a nearly incompressible vector field. Consider the vector field $B \in BV(\mathbb{R}^+ \times \mathbb{R}^m, \mathbb{R} \times \mathbb{R}^m)$ given by $B := (1, b)$ and denote by*

E its tangential set (see Definition 7.3). If $|D^c_{t,x}\cdot B|(E)=|D^c_x\cdot b|(E)=0$, then b has the renormalization property.

More precisely, we will show

Proposition 8.4. *Let $\Omega\subset\mathbb{R}^d$, $B\in BV\cap L^\infty(\Omega,\mathbb{R}^d)$ and $\rho,w\in L^\infty(\Omega)$ be such that $D\cdot(\rho B)=D\cdot(w\rho B)=0$ and $\rho\geq c>0$. Denote by L the set of Lebesgue points of (ρ,w). Then for every $h\in C^1(\mathbb{R})$, the measure $D\cdot(\rho h(w)B)$ satisfies the bound $|D\cdot(\rho h(w)B)|\leq C|D^c\cdot B|\llcorner(\Omega\setminus L)$ for some constant C.*

Using the same arguments as in the proof of Lemma 5.10, Theorem 8.3 follows from Proposition 8.4 and Theorem 7.4.

These results naturally raise the following problem:

Question 8.5 (Divergence problem)**.** *Let $B\in BV_{\rm loc}\cap L^\infty_{\rm loc}(\Omega,\mathbb{R}^d)$. Under which conditions the Cantor part of the divergence $|D^c\cdot B|$ vanishes on the tangential set of B?*

In Section 9 we will prove that indeed some condition is needed, namely we show a planar BV vector field B such that $|D^c\cdot B|$ does not vanish on the tangential set of Bsa. However we do not know the answer to the following question. Note that in view of Theorem 8.3 a positive answer would imply the Renormalization Conjecture:

Question 8.6. *Let $B\in BV_{\rm loc}\cap L^\infty_{\rm loc}(\Omega,\mathbb{R}^d)$ and let $\rho\in L^\infty(\Omega)$ be such that $\rho\geq C>0$ and $D\cdot(\rho B)=0$. Is it true that $|D^c\cdot B|$ vanishes on the tangential set of B ?*

8.1. **Absolutely continuous and jump parts of the measure $D\cdot(\rho h(w)B)$.** Let B, ρ, and w be as in Proposition 8.4. Let c be such that $\rho\leq c$ and define $H:[c,\infty[\times\mathbb{R}$ by $H(r,u):=rh(u/r)$. Clearly H is C^1 and we can extend it to a C^1 function of $\mathbb{R}^2$. Next set $v:=\rho w$. Then we have

$$D\cdot(\rho B)=0\qquad D\cdot(vB)=0\qquad D\cdot(\rho h(w)b)=D\cdot(H(\rho,v)B)\,.$$

and we can apply Theorem 4.1 in order to get

$$\left|D\cdot(H(\rho,v)B)-\left(H(\rho,v)-\frac{\partial H}{\partial r}(\rho,v)\rho-\frac{\partial H}{\partial u}(\rho,v)v\right)D^a\cdot B\right|\;\leq\;C|D^s\cdot B|\,.$$

On the other hand, since the essential range of (ρ,v) is in $[c,\infty[\times\mathbb{R}$, one immediately sees that

$$H(\rho,v)-\frac{\partial H}{\partial r}(\rho,v)\rho-\frac{\partial H}{\partial u}(\rho,v)v\;=\;0\,.$$

Hence, we have concluded

Corollary 8.7. *Let B, ρ, w, and h be as in Proposition 8.4. Then $D\cdot(\rho h(w)B)$ is a Radon measure and there exists a constant C such that $|D\cdot(\rho h(w)B)|\leq C|D^s\cdot B|$.*

We will next use the trace properties of divergence measure fields in order to show the following

Proposition 8.8. *Let B, ρ, w, and h be as in Proposition 8.4. Then there exists a constant C such that* $|D\cdot(\rho h(w)B)| \le C|D^c\cdot B|$.

Proof. Consider the jump set J_B of B, its approximate unit normal $\nu$and the approximate left and right traces of B on J_B. Then $|D^j\cdot B| = |(B^+ - B^-)\cdot\nu|\mathscr{H}^{d-1}\llcorner J_B$ and, by Corollary 8.7,

$$|D\cdot(\rho h(w)B)| \;\le\; C|(B^+ - B^-)\cdot\nu|\mathscr{H}^{d-1}\llcorner J_B + C|D^c\cdot B|\,. \tag{205}$$

Now, let $\{\Sigma_i\}_i$ be a countable family of hypersurfaces such that $B\subset\cup_i\Sigma_i$. In order to complete the proof it suffices to show that $D\cdot(\rho h(w)B)\llcorner\Sigma_i = 0$ for every i. Next, fix any $\varepsilon>0$ such that $\varepsilon\le\rho$ a.e. and consider the function $F_\varepsilon : \big(-]\infty,-\varepsilon]\cup[\varepsilon,\infty[\big)\times\mathbb{R}\to\mathbb{R}$ defined by $F_\varepsilon(r,u) := h(u/r)$. Extend it to a C^1 function defined on all $\mathbb{R}^2$. Next set $H_\varepsilon(r,u) := rh(u/r)$. Then, recalling that $D\cdot(\rho B) = 0$ and $D\cdot((\rho w)B) = 0$, we can use Proposition 7.9 and Theorem 7.12 to get

$$\begin{aligned}
[D\cdot(h(w)\rho B)]\llcorner\Sigma &= [D\cdot(H_\varepsilon(w\rho,\rho)B]\llcorner\Sigma\\
&= \Bigg[H_\varepsilon\left(\frac{\mathrm{Tr}^+(w\rho B,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)},\frac{\mathrm{Tr}^+(\rho B,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)}\right)\mathrm{Tr}^+(B,\Sigma) -\\
&\qquad H_\varepsilon\left(\frac{\mathrm{Tr}^-(w\rho B,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)},\frac{\mathrm{Tr}^-(\rho B,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)}\right)\mathrm{Tr}^-(B,\Sigma)\Bigg]\,\mathscr{H}^{d-1}\llcorner\Sigma\,.
\end{aligned}\tag{206}$$

Now consider the set

$$\Sigma' := \Big\{x\in\Sigma\;:\;\mathrm{Tr}^+(B,\Sigma)(x)=0\quad\text{or}\quad\mathrm{Tr}^-(B,\Sigma)(x)=0\Big\}\,.$$

Applying Theorem 7.12 to $H\equiv 1$, we conclude that, up to $\mathscr{H}^{d-1}$–negligible sets,

$$\Sigma'\;\subset\;\Sigma_0\;:=\;\big\{x\in\Sigma\;:\;\mathrm{Tr}^-(\rho B,\Sigma)(x)=0\quad\text{or}\quad\mathrm{Tr}^+(\rho B,\Sigma)(x)=0\big\}\,.$$

Next note that, by Proposition 7.9,

$$0\;=\;D\cdot(\rho B)\llcorner\Sigma\;=\;\big[\mathrm{Tr}^+(\rho B,\Sigma)-\mathrm{Tr}^-(\rho B,\Sigma)\big]\,\mathscr{H}^{d-1}\llcorner\Sigma\,.$$

and

$$0\;=\;D\cdot(\rho wB)\llcorner\Sigma\;=\;\big[\mathrm{Tr}^+(\rho wB,\Sigma)-\mathrm{Tr}^-(\rho wB,\Sigma)\big]\,\mathscr{H}^{d-1}\llcorner\Sigma\,.$$

Thus, we conclude that $\mathrm{Tr}^-(\rho B,\Sigma) = \mathrm{Tr}^+(\rho B,\Sigma)$ and $trplus\rho wB\Sigma = \mathrm{Tr}^-(\rho wB,\Sigma)$ a.e. on Σ. Recall the definition of H_ε. Then:

- The expression

$$\begin{aligned}
E \;&:=\; H_\varepsilon\left(\frac{\mathrm{Tr}^+(w\rho B,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)},\frac{\mathrm{Tr}^+(\rho B,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)}\right)\mathrm{Tr}^+(B,\Sigma)\\
&\quad -H_\varepsilon\left(\frac{\mathrm{Tr}^-(w\rho B,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)},\frac{\mathrm{Tr}^-(\rho B,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)}\right)\mathrm{Tr}^-(B,\Sigma)
\end{aligned}$$

  vanishes $\mathscr{H}^{d-1}$–a.e. on Σ_0.

- $\mathscr{H}^{d-1}$–a.e. on $\Sigma_\varepsilon := \{|\mathrm{Tr}^+(\rho B, \Sigma)| \geq \varepsilon\}$ we have $|\mathrm{Tr}^-(\rho B, \Sigma)| \geq \varepsilon$ and $\mathrm{Tr}^+(B, \Sigma) \neq 0 \neq \mathrm{Tr}^-(B, \Sigma)$. Thus we can compute
$$E \;=\; h\left(\frac{\mathrm{Tr}^+(\rho w B, \Sigma)}{\mathrm{Tr}^-(\rho B, \Sigma)}\right)\mathrm{Tr}^+(\rho B, \Sigma) - h\left(\frac{\mathrm{Tr}^-(\rho w B, \Sigma)}{\mathrm{Tr}^-(\rho B, \Sigma)}\right)\mathrm{Tr}^-(\rho B, \Sigma)\,.$$
Recalling that $\mathrm{Tr}^+(\rho B, \Sigma) = \mathrm{Tr}^-(\rho B, \Sigma)$ and $\mathrm{Tr}^+(\rho w B, \Sigma) = \mathrm{Tr}^-(\rho w B, \Sigma)$, we conclude that E vanishes $\mathscr{H}^{d-1}$–a.e. on Σ_ε.

Therefore, by (206) we have
$$0 \;=\; [D\cdot(\rho h(w)B)] \llcorner \left\{x \in \Sigma : 0 < |\mathrm{Tr}^+(\rho B, \Sigma)(x)| < \varepsilon\right\}.$$
Letting $\varepsilon \downarrow 0$ we get $D\cdot(\rho h(w)B) \llcorner \Sigma = 0$, which is the desired conclusion. □

8.2. **Proof of Proposition 8.4 and concentration of commutators.** In the previous section we proved that, under the assumptions of Proposition 8.4, $|D\cdot(\rho h(w)B)| \leq C|D^c\cdot B|$. Here we will state a new commutator estimate and with the help of it we will complete the proof of Proposition 8.4.

As in the previous section:

- We fix w, ρ, b and h as in Proposition 8.4;
- We let $c > 0$ be such that $c < \rho$ a.e. and we define $H : [c, \infty[\times\mathbb{R} \to \mathbb{R}$ setting $H(r, u) := rh(u/r)$;
- We extend H to a C^1 function on $\mathbb{R}^2$.

Next we fix a nonnegative kernel $\eta \in C^\infty_c(\mathbb{R}^d)$ and consider the standard family of mollifiers $\{\eta_\varepsilon\}_{\varepsilon>0}$. If we set $v := \rho w$, then $D\cdot(\rho h(w)B) = D\cdot(H(\rho, v)B)$ is the weak limit of
$$\begin{aligned}
& D\cdot(H(\rho*\eta_\varepsilon, v*\eta_\varepsilon)B)\\
=\;& \left[\frac{\partial H}{\partial r}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)D(\rho*\eta_\varepsilon)\cdot B + \frac{\partial H}{\partial u}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)D(v*\eta_\varepsilon)\cdot B\right]\\
& + H(\rho*\eta_\varepsilon, v*\eta_\varepsilon)D\cdot B\\
=\;& \left[\frac{\partial H}{\partial r}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)D\cdot(\rho*\eta_\varepsilon B) + \frac{\partial H}{\partial u}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)D\cdot(v*\eta_\varepsilon B)\right]\\
& + \left[H(\rho*\eta_\varepsilon, v*\eta_\varepsilon) - \frac{\partial H}{\partial r}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)\rho*\eta_\varepsilon + \frac{\partial H}{\partial u}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)v*\eta_\varepsilon\right] D\cdot B\,.
\end{aligned}$$
Next, note that the range of $\rho*\eta_\varepsilon$ is contained in $[c, \infty[$. Thus, from the definition of H it follows that it is a 1–homogeneous function on the range of $(\rho*\eta_\varepsilon, v*\varepsilon)$. This implies that
$$\frac{\partial H}{\partial r}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)\rho*\eta_\varepsilon + \frac{\partial H}{\partial u}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)v*\eta_\varepsilon - H(\rho*\eta_\varepsilon, v*\eta_\varepsilon) \;=\; 0\,.$$
Recalling that $D\cdot((\rho B)*\eta_\varepsilon) = D\cdot((vB)*\eta_\varepsilon) = 0$ we conclude that $D\cdot(H(\rho, v)B)$ is the limit, in the distributional sense, of the expressions
$$\frac{\partial H}{\partial r}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)[D\cdot(\rho*\eta_\varepsilon B) - D\cdot(\rho B)*\eta_\varepsilon] + \frac{\partial H}{\partial u}(\rho*\eta_\varepsilon, v*\eta_\varepsilon)[D\cdot(v*\eta_\varepsilon B) - D\cdot(vB)*\eta_\varepsilon]\,. \tag{207}$$

This discussion justifies the introduction of the following notation and terminology.

Definition 8.9. *Let $\Omega \subset \mathbb{R}^d$, $B \in BV(\Omega, \mathbb{R}^d)$, $z \in L^\infty(\Omega, \mathbb{R}^k)$, and $H \in C^1(\mathbb{R}^k)$. If $\{\eta_\varepsilon\}_{\varepsilon>0}$ is a standard family of mollifiers, then we define the commutators*

$$T^i_\delta \;:=\; (D\cdot(z^i B)) * \eta_\varepsilon - D\cdot(z^i * \eta_\varepsilon B)$$

$$\mathcal{T}^i_\delta \;:=\; \frac{\partial H}{\partial u_i}(z * \eta_\delta) T^i_\delta\,.$$

Note that the commutators T^i_δ coincide with the commutators T^i_δ of Definition 4.2. Recalling Proposition 4.6, we conclude that the distributions $\mathcal{T}^i_\delta$ are measures with uniformly bounded total variations. Then Proposition 8.4 follows from the following theorem, which will be proved in the next section.

Theorem 8.10. *Let $\mathcal{T}^i_\delta$ be as in Definition 8.9 and consider the set L_z of Lebesgue points of z. Then any weak* limit of $\mathcal{T}^i_\delta$ is a measure ν such that $|\nu|(\Omega \setminus L_z) = 0$.*

8.3. **Proof of Theorem 8.10.** Recalling the proof of Proposition 4.6, T^i_δ can be written as $r^i_\delta \mathscr{L}^d - (z^i * \eta_\delta) D\cdot B$, where

$$r^i_\delta(x) \;:=\; \int_{\mathbb{R}^d} z^i(x')\big[(B(x) - B(x'))\cdot \nabla\eta_\delta(x'-x)\big]\,dx'\,. \tag{208}$$

An important step towards the proof of Theorem 8.10 is the following representation lemma.

Lemma 8.11 (Double averages lemma)**.** *Let $\Phi \in L^\infty(\Omega)$ and assume that its support is a compact subset of Ω. Then, for δ sufficiently small, we have*

$$\int_{\mathbb{R}^d} \Phi(x) r^i_\delta(x)\,dx \;=\; \sum_{j,l}\int_{\mathbb{R}^d} A^{ijl}_\delta(\xi)\,d[D_l B^j](\xi)\,, \tag{209}$$

where the functions A^{ijl}_δ are given by the double average

$$A^{ijl}_\delta(\xi) \;:=\; -\frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}^d} y_l \frac{\partial\eta}{\partial x_j}(y)\Phi(\xi - \tau y) z^i(\xi + (\delta-\tau)y)\,dy\,d\tau\,. \tag{210}$$

Proof. Fix $\Phi \in L^\infty(\Omega)$ and with compact support contained in Ω. Then, if δ is sufficiently small, A^{ijl}_δ has compact support contained in Ω. We now prove that A^{ijl}_δ is a continuous function. Taking into account that Φ and z are bounded, it suffices to show that

$$R_\varepsilon(\xi) \;:=\; \int_\varepsilon^{\delta-\varepsilon}\int_{\mathbb{R}^d} y_l \frac{\partial\eta}{\partial x_j}(y)\Phi(\xi-\tau y) z^i(\xi + (\delta-\tau)y)\,dy\,d\tau$$

is continuous for any $\varepsilon \in]0, \delta/2[$. This claim can be proved as follows. First of all, without loss of generality, we can assume that both z and Φ are compactly supported. Next we take sequences $\{z_n\}$ and $\{\Phi_n\}$ of continuous compactly supported functions such that $\|z - z_n\|_{L^2} + \|\Phi - \Phi_n\|_{L^2} \downarrow 0$. If we set

$$R_{n\varepsilon}(\xi) \ := \ \int_\varepsilon^{\delta-\varepsilon} \int_{\mathbb{R}^d} y_l \frac{\partial \eta}{\partial x_j}(y) \Phi^l(\xi - \tau y) z^i_n(\xi + (\delta - \tau)y)\, dy\, d\tau\,,$$

then each $R_{n,\varepsilon}$ is continuous. Moreover one can easily check that

$$|R_{n,\varepsilon}(\xi) - R_\varepsilon(\xi)| \ \leq \ C\delta\varepsilon^{-d}\big(\|\Phi\|_{L^2}\|z - z_n\|_{L^2} + \|z_n\|_{L^2}\|\Phi_n - \Phi\|_{L^2}\big)\,.$$

Therefore $R_{n,\varepsilon} \to R_\varepsilon$ uniformly, and we conclude that R_ε is continuous.

Now, fix B and δ as in the statement of the lemma. We approximate B in $L^1_{\rm loc}$ with a sequence of smooth functions B_n, in such a way that $D_k B^j_n$ converge weakly* to $D_k B^j$ on Ω. Hence, we have that

$$R^i_n(x) \ := \ \int_{\mathbb{R}^d} z^i(x') \big[(B_n(x) - B_n(x')) \cdot \nabla \eta_\delta(x' - x)\big]\, dx'$$

converge strongly in $L^1_{\rm loc}$ to r^i_δ. Moreover, since A^{ijl}_δ is a continuous and compactly supported function, we have

$$\lim_{n\to\infty} \int A^{ijl}_\delta(\xi) d[D_l B^j_n](\xi) \ = \ \int A^{ijl}_\delta(\xi) d[D_l B^j](\xi)\,.$$

Hence it is enough to prove the statement of the lemma for B_n, which are smooth functions.

Thus, we fix a smooth function B and compute

$$\begin{aligned}
& -\int r^i_\delta(x)\Phi(x)\, dx \\
= & \ -\int_{\mathbb{R}^d} \Phi(x) \int_{\mathbb{R}^d} z^i(x')\big[(B(x) - B(x')) \cdot \nabla \eta_\delta(x' - x)\big]\, dx'\, dx \\
= & \ -\int_{\mathbb{R}^d\times\mathbb{R}^d} \Phi(x) z^i(x + \delta y) \frac{B(x) - B(x + \delta y)}{\delta} \cdot \nabla \eta(y)\, dy\, dx \\
= & \ \int_{\mathbb{R}^d\times\mathbb{R}^d} \Phi(x) z^i(x + \delta y) \frac{1}{\delta} \int_0^\delta \sum_{l,j} y_l \frac{\partial B^j}{\partial x_l}(x + \tau y) \frac{\partial \eta}{\partial x_j}(y)\, d\tau\, dy\, dx \\
= & \ \sum_{k,l} \int_{\mathbb{R}^d} \left[\frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}^d} y_l \frac{\partial \eta}{\partial x_j}(y) \Phi(\xi - \tau y) z^i(\xi + (\delta - \tau)y)\, dy\, d\tau\right] \frac{\partial B^j}{\partial x_l}(\xi)\, d\xi\,.
\end{aligned}$$

Since the measure $\frac{\partial B^j}{\partial z_l}\mathscr{L}^d$ is equal to $D_l B^j$, the claim of the lemma follows. □

Proof of Theorem 8.10. We rewrite $\mathcal{T}^i_\delta$ as

$$\mathcal{T}^i_\delta \ = \ \frac{\partial H}{\partial u_i}(z * \eta_\delta)\, r^i_\delta\, \mathscr{L}^d - \frac{\partial H}{\partial u_i}(z * \eta_\delta)\, (z^i * \eta_\delta)\, D \cdot B\,. \tag{211}$$

We define the matrix–valued measures

$$\begin{aligned} \alpha &:= DB\llcorner L_z \\ \beta &:= DB\llcorner(\Omega \setminus L_z) \end{aligned}$$

and the measures

$$\begin{aligned} \gamma &:= [D \cdot B]\llcorner L_z \\ \lambda &:= [D \cdot B]\llcorner(\Omega \setminus L_z)\,. \end{aligned}$$

Then we introduce the measures S^i_δ and R^i_δ given by the following linear functionals on $\varphi \in C_c(\Omega)$:

$$\langle S^i_\delta, \varphi \rangle := \sum_{j,l} \int_{\mathbb{R}^d} g^{ijl}_\delta(\xi)\, d[\alpha_{lj}](\xi) - \int_{\mathbb{R}^d} \varphi(x) \frac{\partial H}{\partial u_i}(z * \eta_\delta(x))\, z^i * \eta_\delta(x)\, d\gamma(x) \tag{212}$$

$$\langle R^i_\delta, \varphi \rangle := \sum_{j,l} \int_{\mathbb{R}^d} g^{ijl}_\delta(\xi)\, d[\beta_{lj}](\xi) - \int_{\mathbb{R}^d} \varphi(x) \frac{\partial H}{\partial u_i}(z * \eta_\delta(x))\, z^i * \eta_\delta(x)\, d\lambda(x)\,, \tag{213}$$

where

$$g^{ijl}_\delta(\xi) := -\frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}^d} y_l \frac{\partial \eta}{\partial x_j}(y) \varphi(\xi - \tau y) \cdot \frac{\partial h}{\partial u_i}(z * \eta_\delta(\xi - \tau y)) z^i(\xi + (\delta - \tau) y)\, dy\, d\tau\,. \tag{214}$$

This formula for g^{ijl}_δ comes from the formulas for A^{ijl}_δ of Lemma 8.11, where we choose as Φ the function

$$\Phi := \varphi \frac{\partial H}{\partial u_i}(z * \eta_\delta)\,.$$

Hence, comparing (214) with (211) and (210), from Lemma 8.11 we conclude that $\mathcal{T}^i_\delta = S^i_\delta + R^i_\delta$.

Let R^i_0 be any weak limit of a subsequence $\{R^i_{\delta_n}\}_{\delta_n \downarrow 0}$ and let S^i_0 be any weak limit of a subsequence (not relabeled) of $\{S^i_{\delta_n}\}$. In what follows we will prove that

(i) $R^i_0 \ll |\lambda| + |\beta|$
(ii) $S^i_0 = 0$.

Since $|\lambda|$ and $|\beta|$ are concentrated on $\Omega \setminus L_z$, (i) and (ii) prove the Theorem.

Proof of (i) Let us fix a smooth function φ with $|\varphi| \leq 1$ and with support $K \subset\subset \Omega$. If we define g^{ijl}_δ as in (214), there exists a constant C, depending only on w and H, such that

$\|g_\delta^{ijl}\|_\infty \leq C$. Hence, it follows that

$$\left|\int \varphi\, dR^i_\delta\right| \;\leq\; C\|z\|_\infty \left\{|\beta|\left(\bigcup_{j,l} \mathrm{supp}\,(g_\delta^{ijl})\right) + |\lambda|(K)\right\}. \tag{215}$$

Moreover, it is easy to check that, if K_ε denotes the ε–neighborhood of K, then $\mathrm{supp}\,(g_\delta^{ijl}) \subset K_{2\delta}$. Hence, passing into the limit in (215), we conclude that

$$\left|\int \varphi dR^i_0\right| \;\leq\; C\|z\|_\infty \Big(|\lambda|(K) + |\beta|(K)\Big).$$

From the arbitrariness of $\varphi \in C^\infty_c(\tilde\Omega)$ it follows easily that $R^i_0 \llcorner \tilde\Omega \leq C(|\beta| + |\lambda|)$.

Proof of (ii) By definition of L_z, z has Lebesgue limit $\tilde z(x)$ at every $x \in L_z$. Hence it follows that

$$\lim_{\delta\downarrow 0} z * \eta_\delta(x) \;=\; \tilde z(x) \tag{216}$$

Fix φ and define g_δ^{ijl} as in (214). We will show that for every $\xi \in L_z$ we have that

$$\lim_{\delta\downarrow 0} g_\delta^{ijl}(\xi) \;=\; g^{ijl}(\xi) \tag{217}$$

where

$$g^{ijl}(\xi) \;:=\; -\varphi(\xi)\frac{\partial H}{\partial u_i}(\tilde z(\xi))\tilde z^i(\xi)\int_{\mathbb{R}^d} y_l \frac{\partial \eta}{\partial x_j}(y)\, dy\,.$$

Integrating by parts we get

$$g^{ill}(\xi) \;=\; \varphi(\xi)\frac{\partial H}{\partial u_i}(\tilde z(\xi))\tilde z^i(\xi) \tag{218}$$

$$g^{ijl}(\xi) \;=\; 0 \qquad\qquad \text{for } j \neq l. \tag{219}$$

Recall that g_δ^{ijl}, φ, $z*\eta_\delta$, $H(z*\eta_\delta)$, and $\nabla H(z*\eta_\delta)$ are all uniformly bounded. Hence, letting $\delta \downarrow 0$ in (212), from (216), (217), (218), (219), and the dominated convergence theorem we conclude that

$$\begin{aligned}\langle S^i_0, \varphi\rangle \;&=\; \sum_l \int_{\mathbb{R}^d} \frac{\partial H}{\partial u_i}(\tilde z(\xi))\, \tilde z^i(\xi)\, \varphi(\xi)\, d[\alpha_{ll}](\xi) \\ &\quad - \int_{\mathbb{R}^d} \frac{\partial H}{\partial u_i}(\tilde z(x))\, \tilde z^i(x)\, \varphi(x)\, d\gamma(x)\,.\end{aligned}$$

Recalling that $\sum_l \alpha_{ll} = \sum_l D^c_l B^l \llcorner L_z = D^c \cdot B \llcorner L_z$ and $\gamma = D^c \cdot B \llcorner L_z$, we conclude that $\langle S^i_0, \varphi\rangle = 0$. The arbitrariness of φ gives (ii).

Hence, to finish the proof, it suffices to show (217). Recalling the smoothness of φ and the fact that η is supported in the ball $B_1(0)$ we conclude that it suffices to show that

$$I_\delta := \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| \frac{\partial H}{\partial u_j}(z * \eta_\delta(\xi - \tau y))\, z^i(\xi + (\delta - \tau)y) - \frac{\partial H}{\partial u_j}(\tilde{z}(\xi))\, \tilde{z}^i(\xi) \right| dy\, d\tau \tag{220}$$

converges to 0. Then, we write

$$\begin{aligned} I_\delta &\leq \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| \frac{\partial H}{\partial u_j}(z * \eta_\delta(\xi - \tau y)) - \frac{\partial h}{\partial u_j}(\tilde{z}(\xi)) \right| |z^i(\xi + (\delta - \tau)y)|\, dy\, d\tau \\ &\quad + \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| \frac{\partial H}{\partial u_j}(\tilde{z}(\xi)) \right| \left| z^i(\xi + (\delta - \tau)y) - \tilde{z}^i(\xi) \right| dy\, d\tau \\ &\leq \frac{C_1}{\delta}\int_0^\delta \int_{B_1(0)} \left| z * \eta_\delta(\xi - \tau y) - \tilde{z}(\xi) \right| dy\, d\tau \\ &\quad + \frac{C_2}{\delta}\int_0^\delta \int_{B_1(0)} \left| z(\xi + (\delta - \tau)y) - \tilde{z}(\xi) \right| d\xi\, d\tau \\ &=: C_1 J^1_\delta + C_2 J^2_\delta \end{aligned}$$

where the constants C_1 and C_2 depend only on ξ, z, and H. Note that

$$\begin{aligned} J^1_\delta &= \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| z(\xi + \tau y) - \tilde{z}(\xi) \right| dy\, d\tau \\ &= \frac{1}{\delta}\int_0^\delta \left[\frac{1}{\tau^d} \int_{B_\tau(\xi)} \left| z(y') - \tilde{w}(\xi) \right| dy' \right] d\tau\,, \end{aligned}$$

and

$$\begin{aligned} J^2_\delta &= \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| z * \eta_\delta(\xi + \tau y) - \tilde{z}(\xi) \right| dy\, d\tau \\ &= \frac{1}{\delta}\int_0^\delta \left[\frac{1}{\tau^d} \int_{B_\tau(\xi)} \left| z * \eta_\delta(y') - \tilde{z}(\xi) \right| dy' \right] d\tau\,. \end{aligned}$$

Hence, since $\tilde{z}(\xi)$ is the Lebesgue limit of z at ξ, we conclude that $J^1_\delta + J^2_\delta \to 0$. This completes the proof. □

9. Tangential sets of BV vector fields

In this section we will show the following

Proposition 9.1. *There exists $B \in BV \cap L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that $|D^c \cdot B|(E) > 0$, where E denotes the tangential set of B.*

As already explained in Section 8, this Proposition motivates Question 8.5 and in particular Question 8.6. There are other natural conditions under which it would be interesting to investigate the validity of $|D^c \cdot B|(E) = 0$, such as

- $B = \nabla\alpha \in BV_{\rm loc}(\Omega)$ for some $\alpha \in W^{1,\infty}_{loc}$ (in this case $D \cdot B = \Delta\alpha$);
- B is a (semi)-monotone operator, that is

$$\langle B(y) - B(x), y - x\rangle \;\geq\; \lambda|x-y|^2 \qquad \forall x, y \in \Omega\,. \tag{221}$$

- B is both curl–free and (semi)-monotone.

Proof of Proposition 9.1. We set $\Omega := \big\{(x,y) \in \mathbb{R}^2 \;:\; 1 < x < 2\,, 0 < y < x\big\}$. We construct a scalar function $u \in L^\infty \cap BV(\Omega)$ with the following properties:

(a) $D^c_y u \neq 0$;
(b) $D_x u + D_y(u^2/2)$ is a pure jump measure, i.e. it is concentrated on the jump set J_u.

Given such a function u, the field $B = (1,u)\mathbf{1}_\Omega$ meets the requirements of the proposition. Indeed, let $\tilde{B} = (1,\tilde{u})\mathbf{1}_\Omega$ be the precise representative of B. Due to (b) the Cantor part of $D_x u + D_y(u^2/2)$ vanishes. Hence using the chain rule of Vol'pert we get

$$D^c_x u + \tilde{u} D^c_y u \;=\; 0\,. \tag{222}$$

Denote by $M(x)$ the Radon–Nikodym derivative $DB/|DB|$. Then we have

$$\begin{aligned} M \cdot \tilde{B}|D^c B| &= D^c B \cdot \tilde{B} \\ &= \begin{pmatrix} 0 & 0 \\ D^c_x u & D^c_y u \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 0 \\ D^c_x u + \tilde{u} D^c_y u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence we conclude that $M(x) \cdot \tilde{B}(x) = 0$ for $|D^c B|$–a.e. x, that is $|D^c B|$ is concentrated on the tangential set E of B. Therefore $|D^c \cdot B|(\Omega \setminus E) = 0$. On the other hand, from (a) we have $D^c \cdot B = D^c_y u \neq 0$. Hence we conclude $|D^c \cdot B|(E) > 0$.

We now come to the construction of the desired u. This is achieved as the limit of a suitable sequence of functions u_k.

Step 1 Construction of u_k.
Consider the auxiliary 1-periodic function $\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(p + x) \;=\; 1 - x\,, \qquad 0 < x \leq 1\,, \quad p \in \mathbb{Z}\,.$$

We let $\gamma_k : [0,1] \to [0,1]$ be the usual piecewise linear approximation of the Cantor ternary function, that is $\gamma_0(z) = z$ and, for $k \geq 1$,

$$\gamma_k(z) \;=\; \begin{cases} \frac{1}{2}\gamma_{k-1}(3z), & 0 < z \leq \frac{1}{3}\,, \\ \frac{1}{2}, & \frac{1}{3} < z \leq \frac{2}{3}\,, \\ \frac{1}{2}\big(1 + \gamma_{k-1}(3z-2)\big), & \frac{2}{3} < z \leq 1\,. \end{cases}$$

Notice that

$$\gamma_k'(z) \in \left\{0,\ \left(\frac{3}{2}\right)^k\right\} \tag{223}$$

and

$$|\gamma_k(z) - \gamma_{k-1}(z)| \le \frac{1}{3} \cdot 2^{-k}\,. \tag{224}$$

We set $G :=]1,2[\times]0,1[$ and we define $\varphi_k : G \to \mathbb{R}$ by

$$\varphi_k(x,z) \;=\; xz \;+\; \sum_{j=1}^{k} 4^{1-j}\sigma(4^{j-1}x)\big(\gamma_{j-1}(z) - \gamma_j(z)\big).$$

Note that φ_k is bounded. To describe more precisely the behavior of this function we introduce the following sets: The strips

$$S_i^k \;:=\; \big]1 + (i-1)4^{1-k}, 1 + i4^{1-k}\big[\;\times\;\mathbb{R} \qquad i = 1,\dots,4^{k-1},$$

and the vertical lines

$$V_i^k \;:=\; \{i4^{1-k}\}\;\times\;\mathbb{R} \qquad i = 1,\dots,4^{k-1}-1\,.$$

Then φ_k is Lipschitz on each rectangle $S_i^k \cap G$ and it has jump discontinuities on the segments $V_i^k \cap G$. Therefore φ_k is a BV function and satisfies the identities $D_x\varphi_k = D_x^j\varphi_k + D_x^a\varphi_k$ and $D_y\varphi_k = D_y^a\varphi_k$. Moreover, denoting by $(\partial_x\varphi_k, \partial_y\varphi_k)$ the density of the absolutely continuous part of the derivative, we get

$$\begin{aligned}\partial_x\varphi_k(x,z) &= z + (\gamma_1(z) - z) + (\gamma_2(z) - \gamma_1(z)) + \cdots + (\gamma_k(z) - \gamma_{k-1}(z))\\ &= \gamma_k(z)\,.\end{aligned} \tag{225}$$

Clearly

$$0 \;\le\; 4^{1-j}\sigma(4^{j-1}x) - 4^{-j}\sigma(4^j x) \;\le\; 3\cdot 4^{-j}\,.$$

Therefore, using also (223), on each rectangle $S_i^k \cap G$ we can estimate

$$\begin{aligned}\partial_z\varphi_k(x,z) &= x + \sigma(x) - \big(\sigma(x) - 4^{-1}\sigma(4x)\big)\;\gamma_1'(z)\\ &\qquad - \big(4^{-1}\sigma(4x) - 4^{-2}\sigma(4^2x)\big)\;\gamma_2'(z) \;-\; \cdots\\ &\qquad - \big(4^{2-k}\sigma(4^{k-1}x) - 4^{1-k}\sigma(4^{k-1}x)\big)\;\gamma_{k-1}'(z)\\ &\qquad - 4^{1-k}\sigma(4^{k-1}x)\;\gamma_k'(z)\\ &\ge\; 2 - 3\left(4^{-1}\gamma_1'(z) + \cdots + 4^{1-k}\gamma_k'(z)\right) - 4^{1-k}\gamma_k'(z)\\ &\ge\; 2 - 3\left(\frac{3}{8} + \cdots + \left(\frac{3}{8}\right)^{k-1}\right) - 4\left(\frac{3}{8}\right)^k.\end{aligned}$$

Since

$$4\left(\frac{3}{8}\right)^k \le 3\left(\left(\frac{3}{8}\right)^k + \left(\frac{3}{8}\right)^{k+1} + \cdots\right),$$

we obtain

$$\partial_z \varphi_k \;\geq\; 2 - 3\left(\frac{3}{8} + \left(\frac{3}{8}\right)^2 + \cdots\right) = \frac{1}{5}. \tag{226}$$

Hence, since $\varphi_k(x,\cdot)$ maps $[0,1]$ onto $[0,x]$, the function

$$\Phi_k(x,y) \;:=\; (x, \varphi_k(x,y))$$

maps each rectangle $S_i^k \cap G$ onto $S_i^k \cap \Omega$, and it is bi-Lipschitz on each such rectangle. This allows to define u_k by the implicit equation

$$u_k\big(x, \varphi_k(x,z)\big) \;=\; \gamma_k(z)\,, \tag{227}$$

and to conclude that $0 \leq u_k \leq 1$ and that u_k is Lipschitz on each $S_i^k \cap \Omega$. Therefore $u_k \in L^\infty \cap BV(\Omega)$, $D_x u_k = D_x^a u_k + D_x^j u_k$ and $D_y u_k = D_y^a u_k$.

Step 2 *BV* bounds.

We prove in this step that $|Du_k|(\Omega)$ is uniformly bounded. This claim and the bound $\|u_k\|_\infty \leq 1$ allow to apply the BV compactness theorem to get a subsequence which converges to a bounded BV function u, strongly in L^p for every $p < \infty$. In Steps 3 and 4 we will then complete the proof by showing that u satisfies both the requirements (a) and (b).

By differentiating (227) and using (225) we get the following identity for $\mathscr{L}^2$–a.e. $(x,z) \in S_i^k \cap G$:

$$\begin{aligned}
0 \;&=\; \frac{\partial u_k(x,\varphi_k(x,z))}{\partial x} + \frac{\partial u_k(x,\varphi_k(x,z))}{\partial y}\,\frac{\partial \varphi_k(x,z)}{\partial x}\\
&=\; \frac{\partial u_k(x,\varphi_k(x,z))}{\partial x} + \frac{\partial u_k(x,\varphi_k(x,z))}{\partial y}\,\gamma_k(x)\\
&=\; \frac{\partial u_k(x,\varphi_k(x,z))}{\partial x} + \frac{\partial u_k(x,\varphi_k(x,z))}{\partial y}\,u_k(x,\varphi_k(x,z)).
\end{aligned}$$

Since Φ_k is bi-Lipschitz we get

$$\partial_x u_k(x,y) + u_k \partial_y u_k(x,y) \;=\; 0 \qquad \text{for } \mathscr{L}^2\text{–a.e. } (x,y) \in S_i^k \cap \Omega\,. \tag{228}$$

If $4^{k-1}x \notin \mathbb{N}$ the function $u_k(x,\cdot)$ is non decreasing. Therefore

$$|D_y u_k|(\Omega) \;=\; D_y u_k(\Omega) \;=\; \int_1^2 \big(u_k(x,x) - u_k(x,0)\big)\,dx \;=\; 1\,. \tag{229}$$

From (228) we get

$$|D_x^a u_k|(\Omega) \;\leq\; |D_y^a u_k|(\Omega) \;=\; 1\,. \tag{230}$$

Therefore it remains to bound $|D_x^j u_k|(\Omega)$. This consists of

$$|D_x^j u_k|(\Omega) \;=\; \sum_{i=1}^{4^{k-1}-1} \int_{V_i^k} |u_k^+ - u_k^-|\,d\mathscr{H}^1. \tag{231}$$

For each x of type $1+i4^{1-k}$ we compute

$$
\begin{aligned}
\int_{V_i^k} |u_k^+ - u_k^-|\, d\mathscr{H}^1 &= \int_0^x |u_k(x^+,y) - u_k(x^-,y)|\, dy \\
&= \int_0^1 |\{y : u_k(x^-,y) < t < u_k(x^+,y)\}|\, dt \\
&\quad + \int_0^1 |\{y : u_k(x^+,y) < t < u_k(x^-,y)\}|\, dt \\
&= \int_0^1 |\{y : u_k(x^-,y) < \gamma_k(z) < u_k(x^+,y)\}|\, \gamma_k'(z)\, dz \\
&\quad + \int_0^1 |\{y : u_k(x^+,y) < \gamma_k(z) < u_k(x^-,y)\}|\, \gamma_k'(z)\, dz \\
&= \int_0^1 |\varphi_k(x^+,z) - \varphi_k(x^-,z)|\, \gamma_k'(z)\, dz \\
&\leq \sup_{z\in]0,1[} |\varphi_k(x^+,z) - \varphi_k(x^-,z)| \\
&\overset{(224)}{\leq} \frac{4}{3}\sum_{j=1}^{k} 8^{-j}\big(\sigma(4^{j-1}x^+) - \sigma(4^{j-1}x^-)\big)\,. \qquad (232)
\end{aligned}
$$

Combining (231) and (232) we get

$$
\begin{aligned}
|D_x^j u_k|(\Omega) &\leq \frac{4}{3}\sum_{i=1}^{4^{k-1}-1}\sum_{j=1}^{k} 8^{-j}\big(\sigma(4^{j-1}4^{1-k}i^+) - \sigma(4^{j-1}4^{1-k}i^-)\big) \\
&= \frac{4}{3}\sum_{j=1}^{k} 8^{-j}\sum_{i=1}^{4^{k-1}-1}\big(\sigma(4^{j-k}i^+) - \sigma(4^{j-k}i^-)\big) \\
&= \frac{4}{3}\sum_{j=1}^{k} 8^{-j}4^{j-1} \;\leq\; \frac{1}{3}\,. \qquad (233)
\end{aligned}
$$

Step 3 Proof of (a).

We now fix a bounded BV function u and a subsequence of u_k, not relabeled, which converges to u strongly in L^1. We claim that (a) holds. More precisely we will show that:

(Cl) For $\mathscr{L}^1$–a.e. x the function $u(x,\cdot)$ is a nonconstant BV function of one variable which has no absolutely continuous part and no jump part.

(Cl) gives (a) by the slicing theory of BV functions, see Theorem 3.108 of [11].

In order to prove (Cl) we proceed as follows. By possibly extracting another subsequence we assume that u_k converges to u $\mathscr{L}^2$–a.e. in Ω. We then show (Cl) for every x such that:

- $4^k x \notin \mathbb{N}$ for every k;

- $u_k(x,y)$ converges to $u(x,y)$ for $\mathscr{L}^1$–a.e. y.

Clearly $\mathscr{L}^1$–a.e. x meets these requirements.

Fix any such x. Note that x is never on the boundary of any strip S^k_i. Therefore we can denote by g^x_k the inverse of $\varphi_k(x,\cdot)$ and we can use (227) to write

$$u_k(x,y) \;=\; \gamma_k(g^x_k(y))\,. \tag{234}$$

Thanks to (226), the Lipschitz constant of g_k is uniformly bounded. Therefore, after possibly extracting a subsequence, we can assume that g_k uniformly converge to a Lipschitz function g. Since γ_k uniformly converge to the Cantor ternary function γ, we can pass into the limit in (234) to conclude

$$u(x,y) \;=\; \gamma(g(y))\,. \tag{235}$$

Therefore $u(x,\cdot)$ is continuous, nondecreasing, nonconstant, and locally constant outside a closed set of zero Lebesgue measure ($g^{-1}(C)$, where C is the Cantor set). This proves (Cl).

Step 4 Proof of (b).

Let u be as in Step 3. From the construction of u_k it follows that

$$D_x u_k + D_y(u_k^2/2) \;=\; D^j_x u_k\,. \tag{236}$$

After possibly extracting a subsequence we can assume that $D^j_x u_k$ converges weakly* to a measure μ. This gives

$$D_x u + D_y(u^2/2) \;=\; \mu\,. \tag{237}$$

Therefore it suffices to prove that μ is concentrated on a set of σ–finite 1–dimensional Hausdorff measure. Indeed μ is concentrated on the union of the countable family of segments $\{V^k\}_{k,i}$. In order to prove this claim it suffices to show the following tightness property: for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that

$$|D^j_x u_k|\left(\bigcup_{l\geq N}\bigcup_{i=1}^{4^{l-1}-1} V^l_i\right) \;\leq\; \varepsilon \qquad \text{for every } k. \tag{238}$$

Note that

$$|D^j_x u_k|\left(\bigcup_{l\geq N}\bigcup_{i=1}^{4^{l-1}-1} V^l_i\right) \;\leq\; \sum_{l\geq N}\sum_{i=1}^{4^{l-1}-1}\int_{V^l_i}|u^+_k - u^+_k|\,.$$

Then the same computations leading to (232) and (233) give

$$|D^j_x u_k|\left(\bigcup_{l\geq N}\bigcup_{j=1}^{4^{l-1}-1} V^l_j\right) \;\leq\; \frac{4}{3}\sum_{l=N}^{k} 8^{-l}4^{l-1} \;\leq\; \frac{1}{3\cdot 2^{N-1}}\,. \tag{239}$$

This concludes the proof. □

Remark 9.2. *The function u constructed in Proposition 9.1 solves Burgers' equation with a measure source*

$$D_t u + D_x(u^2/2) \;=\; \mu\,, \tag{240}$$

and has nonvanishing Cantor part. On the other hand in [9] *it has been proved that* entropy *solutions to Burgers' equation without source are SBV, i.e. the Cantor part of their derivative is trivial. It would be interesting to understand whether this gain of regularity is due to the entropy condition, or instead BV distributional solutions of* (240) *with $\mu = 0$ are always SBV.*

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