As Marco Inversi and Luigi De Rosa pointed out to me, the identity (10) is not rigorously derived in the paper: the problem is that $\nabla(\varphi v)$ is not continuous (it is indeed not even bounded), while the $\lambda_t(dx)$ is a measure which is possibly singular. In particular φv is not an admissible test in the definition of measure-valued solution. In fact if we want to be literal we need $\varphi v \in C^\infty_c$ according to Definition 1 and while it is a simple exercise to show that we can also admit tests which are C^1_c, we are not allowed to use tests which just L^p, or L^∞. The way out is to in fact to test with a suitable regularization of v, in particular if we mollify in space and time with a standard kernel and denote by v_ε the mollified vector field, the map φv_ε becomes an admissible test. Observe that we can then write

$$-\iint \chi' \varphi\, \bar\nu \cdot v\, dxdt = \lim_{\varepsilon\downarrow 0} - \iint \chi'\varphi\, \bar\nu \cdot v_\varepsilon\, dxdt\,.$$

We can next write

$$\begin{aligned}
-\iint \chi'\varphi\,\bar\nu\cdot v_\varepsilon\, dxdt &= \iint -\partial_t(\chi\varphi v_\varepsilon)\cdot\bar\nu - \chi\varphi\bar\nu\cdot\big(\text{div }(v\otimes v)_\varepsilon + \nabla p_\varepsilon\big)\, dxdt\\
&= \iint \chi\nabla(\varphi v_\varepsilon) : \langle \nu, \xi\otimes\xi\rangle - \chi\varphi\bar\nu\cdot \text{div }(v\otimes v)_\varepsilon\, dxdt\\
&+ \iint \chi\nabla(\varphi v_\varepsilon) : \langle \nu^\infty, \theta\otimes\theta\rangle\, \lambda_t(dx)dt\\
&- \iint \chi\varphi\bar\nu\cdot\nabla p_\varepsilon\, dxdt.
\end{aligned} \tag{1}$$

We next symmetrize all the terms in $\nabla(\varphi v)$ (which leaves the identity invariant because they appear in Hilbert-Schmidt products with symmetric matrices) and achieve

$$\begin{aligned}
&-2\iint \chi'\varphi\,\bar\nu\cdot v_\varepsilon\, dxdt\\
&\quad= \iint \chi[\nabla(\varphi v_\varepsilon) + \nabla(\varphi v_\varepsilon)^T] : \langle \nu, \xi\otimes\xi\rangle - \chi\varphi\bar\nu\cdot\text{div }(v\otimes v)_\varepsilon\, dxdt\\
&\qquad+ \iint \chi[\nabla(\varphi v_\varepsilon) + \nabla(\varphi v_\varepsilon)^T] : \langle \nu^\infty, \theta\otimes\theta\rangle\, \lambda_t(dx)dt\\
&\qquad- \iint \chi\varphi\bar\nu\cdot\nabla p_\varepsilon\, dxdt.
\end{aligned} \tag{2}$$

We now let $\varepsilon\downarrow 0$ to conclude

$$\begin{aligned}
&-2\iint \chi'\varphi\,\bar\nu\cdot v\, dxdt\\
&\quad= \iint \chi[\nabla(\varphi v) + \nabla(\varphi v)^T] : \langle \nu, \xi\otimes\xi\rangle - \chi\varphi\bar\nu\cdot\text{div }(v\otimes v)\, dxdt\\
&\qquad+ \lim_{\varepsilon\downarrow 0}\iint \chi[\nabla(\varphi v_\varepsilon) + \nabla(\varphi v_\varepsilon)^T] : \langle \nu^\infty, \theta\otimes\theta\rangle\, \lambda_t(dx)dt\\
&\qquad- \iint \chi\varphi\bar\nu\cdot\nabla p\, dxdt.
\end{aligned} \tag{3}$$

We now follow the remaining computations of the paper, namely we let $\varphi = \varphi_k$ where $0\leq\varphi_k\leq 1$, $\varphi_k\equiv 1$ on $B_k(0)$. We just require additionally that $\|\varphi_k\|_{C^0}\leq Ck^{-1}$ and that the support of φ_k is contained in B_{2k}, which can be easily achieved by making the support of the cut-off sufficiently large. In place of equation (13) of the paper we then achieve

$$\int_0^T \chi'(t)F(t)\,dt = \int_0^T \chi'(t)E(t)\,dt + \frac{1}{2}\int_0^T \chi' \int_{\mathbb{R}^n} |v|^2 dx\,dt$$
$$+\frac{1}{2}\iint \chi(\nabla v + \nabla v^T) : \langle \nu, (\xi - v)\otimes(\xi - v)\rangle\, dxdt$$
$$+ \lim_{k\uparrow\infty}\lim_{\varepsilon\downarrow 0}\frac{1}{2}\underbrace{\iint \chi(\nabla(\varphi_k v_\varepsilon + \nabla(\varphi_k v_\varepsilon)^T) : \langle \nu^\infty, \theta\otimes\theta\rangle\, \lambda_t(dx)dt}_{=:(I)_{k,\varepsilon}}\,. \tag{4}$$

We next estimate

$$(I)_{k,\varepsilon} \le \int_0^T \chi(t)\|\nabla(\varphi_k v_\varepsilon(t)) + \nabla(\varphi_k v_\varepsilon(t))^T\|_\infty F(t)\,dt\,.$$

Observe that

$$\|\nabla(\varphi_k v_\varepsilon) + \nabla(\varphi_k v_\varepsilon)^T\|_{L^1_t L^\infty_x} \le \|\nabla v_\varepsilon + \nabla v_\varepsilon(t)\|_{L^1_t L^\infty_x} + \frac{2}{k}\|v_\varepsilon(t)\|_{L1_t L^\infty_x}$$
$$\le \|\nabla v(t) + \nabla v(t)^T\|_{L^1_t L^\infty_x} + \frac{2}{k}\|v(t)\|_{L^1_t L^\infty_x}$$

Next, as already observed in the paper, the bounds on v imply that $\|v(t)\|_\infty \le C\|\nabla v(t) + \nabla v(t)^T\|_\infty + C\|v(t)\|_{L^2}$, from we conclude that $\|v(t)\|_\infty$ is in fact an L^1 function. In particular we can estimate

$$\lim_{k\uparrow\infty}\lim_{\varepsilon\downarrow 0}\frac{1}{2}(I)_{k,\varepsilon} \le \int_0^T \chi(t)\|\nabla v(t) + \nabla v(t)^T\|_\infty F(t)\,dt\,.$$

We have thus justified inequality (14) of the paper, which then yields the desired conclusion.