

REGULARITY OF AREA MINIMIZING CURRENTS II: CENTER MANIFOLD

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ABSTRACT. This is the second paper of a series of three on the regularity of higher codimension area minimizing integral currents. Here we perform the second main step in the analysis of the singularities, namely the construction of a *center manifold*, i.e. an approximate average of the sheets of an almost flat area minimizing current. Such a center manifold is accompanied by a Lipschitz multivalued map on its normal bundle, which approximates the current with a high degree of accuracy. In the third and final paper these objects are used to conclude the proof of Almgren’s celebrated dimension bound on the singular set.

0. INTRODUCTION

In this second paper on the regularity of area minimizing integer rectifiable currents (we refer to the Foreword of [5] for the precise statement of the final theorem and on overview of its proof) we address one of the main steps in the analysis of the singularities, namely the construction of what Almgren calls the *center manifold*. Unlike the case of hypersurfaces, singularities in higher codimension currents can appear as “higher order” perturbation of smooth minimal submanifolds. In order to illustrate this phenomenon, we can consider the examples of area minimizing currents induced by complex varieties of \mathbb{C}^n , as explained in the Foreword of [5]. Take, for instance, the complex curve:

$$\mathcal{V} := \{(z, w) : (z - w^2)^2 = w^5\} \subset \mathbb{C}^2.$$

The point $0 \in \mathcal{V}$ is clearly a singular point. Nevertheless, in every sufficiently small neighborhood of the origin, \mathcal{V} looks like a small perturbation of the smooth minimal surface $\{z = w^2\}$: roughly speaking, $\mathcal{V} = \{z = w^2 \pm w^{5/2}\}$. One of the main issues of the regularity of area minimizing currents is to understand this phenomenon of “higher order singularities”. Following the pioneering work of Almgren [2], a way to deal with it is to approximate the minimizing current with the graph of a multiple valued function on the normal bundle of a suitable, curved, manifold. Such manifold must be close to the “average of the sheets” of the current (from this the name *center manifold*): the hope is that such a property will guarantee a singular “first order expansion” of the corresponding approximating map.

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A “center manifold” with such an approximation property is clearly very far from being uniquely defined and moreover the relevant estimates are fully justified only by the concluding arguments, which will appear in [7]. In this paper, building upon the works [4, 6, 5], we provide a construction of a center manifold \mathcal{M} and of an associated approximation of the corresponding area minimizing current via a multiple valued function $F : \mathcal{M} \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$.

The corresponding construction of Almgren is given in [2, Chapter 4]. Unfortunately, we do not understand this portion of Almgren’s monograph deeply enough to make a rigorous comparison between the two constructions. Even a comparison between the statements is prohibitive, since the main ones of Almgren (cf. [2, 4.30 & 4.33]) are rather involved and seem to require a thorough understanding of most of the chapter (which by itself has the size of a rather big monograph). At a first sight, our approach seems to be much simpler and to deliver better estimates. In the rest of this introduction we will explain some of the main aspects of our construction.

0.1. Whitney-type decomposition. The center manifold is the graph of a classical function over an m -dimensional plane with respect to which the excess of the minimizing current is sufficiently small. To achieve a suitable accuracy in the approximation of the average of the sheets of the current, it is necessary to define the function at an appropriate scale, which varies locally. Around any given point such scale is morally the first at which the sheets of the current cease to be close. This leads to a Whitney-type decomposition of the reference m -plane, where the refining algorithm is stopped according to three conditions. In each cube of the decomposition the center manifold is then a smoothing of the average of the Lipschitz multiple valued approximation constructed in [5], performed in a suitable orthonormal system of coordinates, which changes from cube to cube.

0.2. $C^{3,\kappa}$ -regularity of \mathcal{M} . The arguments of [7] require that the center manifold is at least C^3 -regular. As it is the case of Almgren’s center manifold, we prove actually $C^{3,\kappa}$ estimates, which are a natural outcome of some Schauder estimates. It is interesting to notice that, if the current has multiplicity one everywhere (i.e., roughly speaking, is made of a single sheet), then the center manifold coincides with it and, hence, we can conclude directly a higher regularity than the one given by the usual De Giorgi-type (or Allard-type) argument. This is already remarked in the introduction of [2] and it has been proved in our paper [3] with a relatively simple and short direct argument. The interested reader might find useful to consult that reference as well, since many of the estimates of this note appear there in a much more elementary form.

0.3. Approximation on \mathcal{M} . Having defined a center manifold, we then give a multivalued map F on its normal bundle which approximates the current. The relevant estimates on this map and its approximation properties are then given locally for each cube of the Whitney decomposition used in the construction of the center manifold. We follow a simple principle: at each scale where the refinement of the Whitney decomposition has stopped, the image of such function coincides (on a large set) with the Lipschitz multiple valued approximation constructed in [5], i.e. the same map whose smoothed average has been used to construct the center manifold. As a result, the graph of F is well centered, i.e. the

average of F is very close (compared to its Dirichlet energy and its L^2 norm) to be the manifold \mathcal{M} itself. As far as we understand Almgren is not following this principle and it seems very difficult to separate his construction of the center manifold from the one of the approximating map.

0.4. Splitting before tilting. The regularity of the center manifold \mathcal{M} and the centering of the approximating map F are not the only properties needed to conclude our proof in [7]. Another ingredient plays a crucial role. Assume that around a certain point, at all scales larger than a given one, say s , the excess decays and the sheets stay very close. If at scale s the excess is not decaying anymore, then the sheets must separate as well. In other words, since the tilting of the current is under control up to scale s , the current must in some sense "split before tilting". We borrow the terminology from a remarkable work of Rivière [10], where this phenomenon was investigated independently of Almgren's monograph in the case of 2-dimensional area-minimizing currents. Rivière's approach relies on a clever "lower epiperimetric inequality", which unfortunately seems limited to the 2-d context.

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1. CONSTRUCTION ALGORITHM AND MAIN EXISTENCE THEOREM

The goal of this section is to specify the algorithm leading to the center manifold. The proofs of the various statements are all deferred to later sections.

1.1. Notation, height and excess. For open balls in \mathbb{R}^{m+n} we use $\mathbf{B}_r(p)$. For any linear subspace $\pi \subset \mathbb{R}^{m+n}$, π^\perp is its orthogonal complement, \mathbf{p}_π the orthogonal projection onto π , $B_r(q, \pi)$ the disk $\mathbf{B}_r(q) \cap (q + \pi)$ and $\mathbf{C}_r(p, \pi)$ the cylinder $\{(x + y) : x \in \mathbf{B}_r(p), y \in \pi^\perp\}$ (in both cases q is omitted if it is the origin and π is omitted if it is clear from the context). We also assume that each π is *oriented* by a k -vector $\vec{\pi} := v_1 \wedge \dots \wedge v_k$ (thereby making a distinction when the same plane is given opposite orientations) and with a slight abuse of notation we write $|\pi_2 - \pi_1|$ for $|\vec{\pi}_2 - \vec{\pi}_1|$ (where $|\cdot|$ stands for the norm associated to the usual inner product of k -vectors).

A primary role will be played by the m -dimensional plane $\mathbb{R}^m \times \{0\}$ with the standard orientation: for this plane we use the symbol π_0 throughout the whole paper.

Definition 1.1 (Excess and height). Given an integer rectifiable m -dimensional current T in \mathbb{R}^{m+n} with finite mass and compact support and m -planes π, π' , we define the *excess*

of T in balls and cylinders as

$$\mathbf{E}(T, \mathbf{B}_r(x), \pi) := (2\omega_m r^m)^{-1} \int_{\mathbf{B}_r(x)} |\vec{T} - \vec{\pi}|^2 d\|T\|, \quad (1.1)$$

$$\mathbf{E}(T, \mathbf{C}_r(x, \pi), \pi') := (2\omega_m r^m)^{-1} \int_{\mathbf{C}_r(x, \pi)} |\vec{T} - \vec{\pi}'|^2 d\|T\|, \quad (1.2)$$

and the *height function* in a set $A \subset \mathbb{R}^{m+m}$ as

$$\mathbf{h}(T, A, \pi) := \sup_{x, y \in \text{spt}(T) \cap A} |\mathbf{p}_{\pi^\perp}(x) - \mathbf{p}_{\pi^\perp}(y)|.$$

In what follows all currents will have compact support and finite mass and will always be considered as currents defined in the entire Euclidean space. As a consequence their restrictions to a set A and their pushforward through a map \mathbf{p} are well-defined as long as A is a Borel set and the map \mathbf{p} is Lipschitz in a neighborhood of their support.

Definition 1.2 (Optimal planes). We say that an m -dimensional plane π *optimizes the excess* of T in a ball $\mathbf{B}_r(x)$ if

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau} \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi). \quad (1.3)$$

Observe that in general the plane optimizing the excess is not unique and $\mathbf{h}(T, \mathbf{B}_r(x), \pi)$ might depend on the optimizer π . Since for notational purposes it is convenient to define a unique ‘‘height’’ $\mathbf{h}(T, \mathbf{B}_r(x))$, we call a plane π as in (1.3) *optimal* if in addition

$$\mathbf{h}(T, \mathbf{B}_r(x), \pi) = \min \{ \mathbf{h}(T, \mathbf{B}_r(x), \tau) : \tau \text{ satisfies (1.3)} \} =: \mathbf{h}(T, \mathbf{B}_r(x)), \quad (1.4)$$

i.e. π optimizes the height among all planes that optimize the excess. However (1.4) does not play any further role apart from simplifying the presentation.

In the case of cylinders, instead, $\mathbf{E}(T, \mathbf{C}_r(x, \pi))$ will denote $\mathbf{E}(T, \mathbf{C}_r(x, \pi), \pi)$ (which coincides with the cylindrical excess used in [5] when $(\mathbf{p}_\pi)_\# T \llcorner \mathbf{C}_r(x, \pi) = Q \llbracket B_r(\mathbf{p}_\pi(x), \pi) \rrbracket$), whereas $\mathbf{h}(T, \mathbf{C}_r(x, \pi))$ will be used for $\mathbf{h}(T, \mathbf{C}_r(x, \pi), \pi)$.

We are now ready to formulate the main assumptions of all the statements in this work.

Assumption 1.3. $\varepsilon_0 \in]0, 1]$ is a fixed constant and $\Sigma \subset \mathbf{B}_{7\sqrt{m}} \subset \mathbb{R}^{m+n}$ is a C^{3, ε_0} $(m + \bar{n})$ -dimensional submanifold with no boundary in $\mathbf{B}_{7\sqrt{m}}$. We moreover assume that, for each $p \in \Sigma$, Σ is the graph of a C^{3, ε_0} map $\Psi_p : T_p \Sigma \cap \mathbf{B}_{7\sqrt{m}} \rightarrow T_p \Sigma^\perp$. We denote by $\mathbf{c}(\Sigma)$ the number $\sup_{p \in \Sigma} \|D\Psi_p\|_{C^{2, \varepsilon_0}}$. T^0 is an m -dimensional integral current of \mathbb{R}^{m+n} with support in $\Sigma \cap \bar{\mathbf{B}}_{6\sqrt{m}}$ and finite mass. It is area-minimizing in Σ (i.e. $\mathbf{M}(T) \leq \mathbf{M}(T + \partial S)$ for any current S with $\text{spt}(S) \subset \Sigma$) and moreover

$$\Theta(0, T^0) = Q \quad \text{and} \quad \partial T^0 \llcorner \mathbf{B}_{6\sqrt{m}} = 0, \quad (1.5)$$

$$\|T^0\|(\mathbf{B}_{6\sqrt{m}\rho}) \leq (\omega_m Q (6\sqrt{m})^m + \varepsilon_2^2) \rho^m \quad \forall \rho \leq 1, \quad (1.6)$$

$$\mathbf{E}(T^0, \mathbf{B}_{6\sqrt{m}}) = \mathbf{E}(T^0, \mathbf{B}_{6\sqrt{m}}, \pi_0), \quad (1.7)$$

$$\mathbf{m}_0 := \max \{ \mathbf{c}(\Sigma)^2, \mathbf{E}(T^0, \mathbf{B}_{6\sqrt{m}}) \} \leq \varepsilon_2^2 \leq 1. \quad (1.8)$$

ε_2 is a positive number whose choice will be specified in each statement.

Constants which depend only upon m, n, \bar{n} and Q will be called geometric and usually denoted by C_0 .

Remark 1.4. Note that (1.8) implies $\mathbf{A} := \|A_\Sigma\|_{C^0(\Sigma)} \leq C_0 \mathbf{m}_0^{1/2}$, where A_Σ denotes the second fundamental form of Σ and C_0 is a geometric constant. Observe further that for $p \in \Sigma$ the oscillation of Ψ_p is controlled in $T_p \Sigma \cap \mathbf{B}_{6\sqrt{m}}$ by $C_0 \mathbf{m}_0^{1/2}$.

In what follows we set $l := n - \bar{n}$. To avoid discussing domains of definitions it is convenient to extend Σ so that it is an entire graph over all $T_p \Sigma$. Moreover we will often need to parametrize Σ as the graph of a map $\Psi : \mathbb{R}^{m+\bar{n}} \rightarrow \mathbb{R}^l$. However we do not assume that $\mathbb{R}^{m+\bar{n}} \times \{0\}$ is tangent to Σ at any p and thus we need the following lemma.

Lemma 1.5. *There are positive constants $C_0(m, \bar{n}, n)$ and $c_0(m, \bar{n}, n)$ such that, provided $\varepsilon_2 < c_0$, the following holds. If Σ is as in Assumption 1.3, then we can (modify it outside $\mathbf{B}_{6\sqrt{m}}$ and) extend it to a complete submanifold of \mathbb{R}^{m+n} which, for every $p \in \Sigma$, is the graph of a global C^{3, ε_0} map $\Psi_p : T_p \Sigma \rightarrow T_p \Sigma^\perp$ with $\|D\Psi_p\|_{C^{2, \varepsilon_0}} \leq C_0 \mathbf{m}_0^{1/2}$. T^0 is still area-minimizing in the extended manifold and in addition we can apply a global affine isometry which leaves $\mathbb{R}^m \times \{0\}$ fixed and maps Σ onto Σ' so that*

$$|\mathbb{R}^{m+\bar{n}} \times \{0\} - T_0 \Sigma'| \leq C_0 \mathbf{m}_0^{1/2} \quad (1.9)$$

and Σ' is the graph a C^{3, ε_0} map $\Psi : \mathbb{R}^{m+\bar{n}} \rightarrow \mathbb{R}^l$ with $\Psi(0) = 0$ and $\|D\Psi\|_{C^{2, \varepsilon_0}} \leq C_0 \mathbf{m}_0^{1/2}$.

From now on we assume, w.l.o.g. that $\Sigma' = \Sigma$. The next lemma is a standard consequence of the theory of area-minimizing currents (we include the proofs of Lemma 1.5 and Lemma 1.6 in Section 4.1 for the reader's convenience).

Lemma 1.6. *There are positive constants $C_0(m, n, \bar{n}, Q)$ and $c_0(m, n, \bar{n}, Q)$ with the following property. If T^0 is as in Assumption 1.3, $\varepsilon_2 < c_0$ and $T := T^0 \llcorner \mathbf{B}_{23\sqrt{m}/4}$, then:*

$$\partial T \llcorner \mathbf{C}_{11\sqrt{m}/2}(0, \pi_0) = 0, \quad (\mathbf{p}_{\pi_0})_\# T \llcorner \mathbf{C}_{11\sqrt{m}/2}(0, \pi_0) = Q \llbracket B_{11\sqrt{m}/2}(0, \pi_0) \rrbracket \quad (1.10)$$

$$\text{and } \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}(0, \pi_0)) \leq C_0 \mathbf{m}_0^{1/2m}. \quad (1.11)$$

In particular for each $x \in B_{11\sqrt{m}/2}(0, \pi_0)$ there is a point $p \in \text{spt}(T)$ with $\mathbf{p}_{\pi_0}(p) = x$.

From now we will always work with the current T of Lemma 1.6. We specify next some notation which will be recurrent in the paper when dealing with cubes of π_0 . For each $j \in \mathbb{N}$, \mathcal{C}^j denotes the family of closed cubes L of π_0 of the form

$$[a_1, a_1 + 2\ell] \times \dots \times [a_m, a_m + 2\ell] \times \{0\} \subset \pi_0, \quad (1.12)$$

where $2\ell = 2^{1-j} =: 2\ell(L)$ is the side-length of the cube, $a_i \in 2^{1-j}\mathbb{Z} \forall i$ and we require in addition $-4 \leq a_i \leq a_i + 2\ell \leq 4$. To avoid cumbersome notation, we will usually drop the factor $\{0\}$ in (1.12) and treat each cube, its subsets and its points as subsets and elements of \mathbb{R}^m . Thus, for the center x_L of L we will use the notation $x_L = (a_1 + \ell, \dots, a_m + \ell)$, although the precise one is $(a_1 + \ell, \dots, a_m + \ell, 0, \dots, 0)$. Next we set $\mathcal{C} := \bigcup_{j \in \mathbb{N}} \mathcal{C}^j$. If H and L are two cubes in \mathcal{C} with $H \subset L$, then we call L an ancestor of H and H a descendant of L . When in addition $\ell(L) = 2\ell(H)$, H is a son of L and L the father of H .

Definition 1.7. A Whitney decomposition of $[-4, 4]^m \subset \pi_0$ consists of a closed set $\Gamma \subset [-4, 4]^m$ and a family $\mathscr{W} \subset \mathscr{C}$ satisfying the following properties:

- (w1) $\Gamma \cup \bigcup_{L \in \mathscr{W}} L = [-4, 4]^m$ and Γ does not intersect any element of \mathscr{W} ;
- (w2) the interiors of any pair of distinct cubes $L_1, L_2 \in \mathscr{W}$ are disjoint;
- (w3) if $L_1, L_2 \in \mathscr{W}$ have nonempty intersection, then $\frac{1}{2}\ell(L_1) \leq \ell(L_2) \leq 2\ell(L_1)$.

Observe that (w1) - (w3) imply

$$\text{sep}(\Gamma, L) := \inf\{|x - y| : x \in L, y \in \Gamma\} \geq 2\ell(L) \quad \text{for every } L \in \mathscr{W}. \quad (1.13)$$

However, we do *not* require any inequality of the form $\text{sep}(\Gamma, L) \leq C\ell(L)$, although this would be customary for what is commonly called a Whitney decomposition in the literature.

1.2. Parameters. The algorithm for the construction of the center manifold involves several parameters which depend in a complicated way upon several quantities and estimates. We introduce these parameters and specify some relations among them in the following

Assumption 1.8. $C_e, C_h, \beta_2, \delta_2, M_0$ are positive real numbers and N_0 a natural number for which we assume always

$$\beta_2 = 4\delta_2 = \min\left\{\frac{1}{2m}, \frac{\gamma_1}{100}\right\}, \quad \text{where } \gamma_1 \text{ is the constant of [5, Theorem 1.4]}, \quad (1.14)$$

$$M_0 \geq C_0(m, n, \bar{n}, Q) \geq 4 \quad \text{and} \quad \sqrt{m}M_02^{7-N_0} \leq 1. \quad (1.15)$$

As we can see, β_2 and δ_2 are fixed. The other parameters are not fixed but are subject to further restrictions in the various statements, respecting the following ‘‘hierarchy’’. As already mentioned, ‘‘geometric constants’’ are assumed to depend only upon m, n, \bar{n} and Q . The dependence of other constants upon the various parameters p_i will be highlighted using the notation $C = C(p_1, p_2, \dots)$.

Assumption 1.9 (Hierarchy of the parameters). In all the statements of the paper

- (a) M_0 is larger than a geometric constant (cf. (1.15)) or larger than a constant $C(\delta_2)$, see Proposition 3.4;
- (b) N_0 is larger than $C(\beta_2, \delta_2, M_0)$ (see for instance (1.15) and Proposition 3.7);
- (c) C_e is larger than $C(\beta_2, \delta_2, M_0, N_0)$ (see the statements of Proposition 1.11, Theorem 1.17 and Proposition 3.4);
- (d) C_h is larger than $C(\beta_2, \delta_2, M_0, N_0, C_e)$ (see Propositions 1.11 and 3.1);
- (e) ε_2 is smaller than $c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$ (which will always be positive).

The functions C and c will vary in the various statements: the hierarchy above guarantees however that there is a choice of the parameters for which *all* the restrictions required in the statements of the next propositions are simultaneously satisfied. In fact it is such a choice which is then made in [7]. To simplify our exposition, for smallness conditions on ε_2 as in (e) we will use the sentence ‘‘ ε_2 is sufficiently small’’.

1.3. The Whitney decomposition. Thanks to Lemma 1.6, for every $L \in \mathcal{C}$, we may choose $y_L \in \pi_L^\perp$ so that $p_L := (x_L, y_L) \in \text{spt}(T)$ (recall that x_L is the center of L). y_L is in general not unique and we fix an arbitrary choice. A more correct notation for p_L would be $x_L + y_L$. This would however become rather cumbersome later, when we deal with various decompositions of the ambient space in triples of orthogonal planes. We thus abuse the notation slightly in using (x, y) instead of $x + y$ and, consistently, $\pi_0 \times \pi_0^\perp$ instead of $\pi_0 + \pi_0^\perp$.

Definition 1.10 (Refining procedure). For $L \in \mathcal{C}$ we set $r_L := M_0 \sqrt{m} \ell(L)$ and $\mathbf{B}_L := \mathbf{B}_{64r_L}(p_L)$. We next define the families of cubes $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{W} = \mathcal{W}_e \cup \mathcal{W}_h \cup \mathcal{W}_n \subset \mathcal{C}$ with the convention that $\mathcal{S}^j = \mathcal{S} \cap \mathcal{C}^j$, $\mathcal{W}^j = \mathcal{W} \cap \mathcal{C}^j$ and $\mathcal{W}_\square^j = \mathcal{W}_\square \cap \mathcal{C}^j$ for $\square = h, n, e$. We define $\mathcal{W}^i = \mathcal{S}^i = \emptyset$ for $i < N_0$. We proceed with $j \geq N_0$ inductively: if no ancestor of $L \in \mathcal{C}^j$ is in \mathcal{W} , then

- (EX) $L \in \mathcal{W}_e^j$ if $\mathbf{E}(T, \mathbf{B}_L) > C_e \mathbf{m}_0 \ell(L)^{2-2\delta_2}$;
- (HT) $L \in \mathcal{W}_h^j$ if $L \notin \mathcal{W}_e^j$ and $\mathbf{h}(T, \mathbf{B}_L) > C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$;
- (NN) $L \in \mathcal{W}_n^j$ if $L \notin \mathcal{W}_e^j \cup \mathcal{W}_h^j$ but it intersects an element of \mathcal{W}^{j-1} ;

if none of the above occurs, then $L \in \mathcal{S}^j$. We finally set

$$\mathbf{\Gamma} := [-4, 4]^m \setminus \bigcup_{L \in \mathcal{W}} L = \bigcap_{j \geq N_0} \bigcup_{L \in \mathcal{S}^j} L. \quad (1.16)$$

Observe that, if $j > N_0$ and $L \in \mathcal{S}^j \cup \mathcal{W}^j$, then necessarily its father belongs to \mathcal{S}^{j-1} .

Proposition 1.11 (Whitney decomposition). *Let Assumptions 1.3 and 1.8 hold and let ε_2 be sufficiently small. Then $(\mathbf{\Gamma}, \mathcal{W})$ is a Whitney decomposition of $[-4, 4]^m \subset \pi_0$. Moreover, for any choice of M_0 and N_0 , there is $C^* := C^*(M_0, N_0)$ such that, if $C_e \geq C^*$ and $C_h \geq C^* C_e$, then*

$$\mathcal{W}^j = \emptyset \quad \text{for all } j \leq N_0 + 6. \quad (1.17)$$

Moreover, the following estimates hold with $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$:

$$\mathbf{E}(T, \mathbf{B}_J) \leq C_e \mathbf{m}_0 \ell(J)^{2-2\delta_2} \quad \text{and} \quad \mathbf{h}(T, \mathbf{B}_J) \leq C_h \mathbf{m}_0^{1/2m} \ell(J)^{1+\beta_2} \quad \forall J \in \mathcal{S}, \quad (1.18)$$

$$\mathbf{E}(T, \mathbf{B}_L) \leq C \mathbf{m}_0 \ell(L)^{2-2\delta_2} \quad \text{and} \quad \mathbf{h}(T, \mathbf{B}_L) \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} \quad \forall L \in \mathcal{W}. \quad (1.19)$$

1.4. Construction algorithm. We fix next two important functions $\vartheta, \varrho : \mathbb{R}^m \rightarrow \mathbb{R}$.

Assumption 1.12. $\varrho \in C_c^\infty(B_1)$ is radial, $\int \varrho = 1$ and $\int |x|^2 \varrho(x) dx = 0$. For $\lambda > 0$ ϱ_λ denotes, as usual, $x \mapsto \lambda^{-m} \varrho(\frac{x}{\lambda})$. $\vartheta \in C_c^\infty([-\frac{17}{16}, \frac{17}{16}]^m, [0, 1])$ is identically 1 on $[-1, 1]^m$.

ϱ will be used as convolution kernel for smoothing maps z defined on m -dimensional planes π of \mathbb{R}^{m+n} . In particular, having fixed an isometry A of π onto \mathbb{R}^m , the smoothing will be given by $[(z \circ A) * \varrho] \circ A^{-1}$. Observe that since ϱ is radial, our map does not depend on the choice of the isometry and we will therefore use the shorthand notation $z * \varrho$.

Definition 1.13 (π -approximations). Let $L \in \mathcal{S} \cup \mathcal{W}$ and π be an m -dimensional plane. If $T \llcorner \mathbf{C}_{32r_L}(p_L, \pi)$ fulfills the assumptions of [5, Theorem 1.4] in the cylinder $\mathbf{C}_{32r_L}(p_L, \pi)$,

then the resulting map $f : B_{8r_L}(p_L, \pi) \rightarrow \mathcal{A}_Q(\pi^\perp)$ given by [5, Theorem 1.4] is a π -approximation of T in $\mathbf{C}_{8r_L}(p_L, \pi)$. The map $\hat{h} : B_{7r_L}(p_L, \pi) \rightarrow \pi^\perp$ given by $\hat{h} := (\boldsymbol{\eta} \circ f) * \varrho_{\ell(L)}$ will be called the *smoothed average of the π -approximation*, where we recall the notation $\boldsymbol{\eta} \circ f(x) := Q^{-1} \sum_{i=1}^Q f_i(x)$ for any Q -valued map $f = \sum_i \llbracket f_i \rrbracket$.

Definition 1.14 (Reference plane π_L). For each $L \in \mathcal{S} \cup \mathcal{W}$ we let $\hat{\pi}_L$ be an optimal plane in \mathbf{B}_L (cf. Definition 1.2) and choose an m -plane $\pi_L \subset T_{p_L} \Sigma$ which minimizes $|\hat{\pi}_L - \pi_L|$.

In what follows we will deal with graphs of multivalued functions f in several system of coordinates. These objects can be naturally seen as currents \mathbf{G}_f (see [6]) and in this respect we will use extensively the notation and results of [6] (therefore Gr will denote the “set-theoretic” graph).

Lemma 1.15. *Let the assumptions of Proposition 1.11 hold and assume $C_e \geq C^*$ and $C_h \geq C^* C_e$ (where C^* is the constant of Proposition 1.11). For any choice of the other parameters, if ε_2 is sufficiently small, then $T \llcorner \mathbf{C}_{32r_L}(p_L, \pi_L)$ satisfies the assumptions of [5, Theorem 1.4] for any $L \in \mathcal{W} \cup \mathcal{S}$. Moreover, if f_L is a π_L -approximation, denote by \hat{h}_L its smoothed average and by \bar{h}_L the map $\mathbf{p}_{T_{p_L} \Sigma}(\hat{h}_L)$, which takes value in the plane $\varkappa_L := T_{p_L} \Sigma \cap \pi_L^\perp$, i.e. the orthogonal complement of π_L in $T_{p_L} \Sigma$. If we let h_L be the map $x \mapsto h_L(x) := (\bar{h}_L(x), \Psi_{p_L}(x, \bar{h}_L(x))) \in \varkappa_L \times T_{p_L} \Sigma^\perp$, then there is a smooth map $g_L : B_{4r_L}(p_L, \pi_0) \rightarrow \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)$.*

Definition 1.16 (Interpolating functions). The maps h_L and g_L in Lemma 1.15 will be called, respectively, the *tilted L -interpolating function* and the *L -interpolating function*. For each j let $\mathcal{P}^j := \mathcal{S}^j \cup \bigcup_{i=N_0}^j \mathcal{W}^i$ and for $L \in \mathcal{P}^j$ define $\vartheta_L(y) := \vartheta(\frac{y-x_L}{\ell(L)})$. Set

$$\hat{\varphi}_j := \frac{\sum_{L \in \mathcal{P}^j} \vartheta_L g_L}{\sum_{L \in \mathcal{P}^j} \vartheta_L} \quad \text{on }]-4, 4[^m, \quad (1.20)$$

let $\bar{\varphi}_j(y)$ be the first \bar{n} components of $\hat{\varphi}_j(y)$ and define $\varphi_j(y) = (\bar{\varphi}_j(y), \Psi(y, \bar{\varphi}_j(y)))$, where Ψ is the map of Lemma 1.5. φ_j will be called the *glued interpolation* at the step j .

Theorem 1.17 (Existence of the center manifold). *Assume that the hypotheses of Lemma 1.15 hold and let $\kappa := \min\{\varepsilon_0/2, \beta_2/4\}$. For any choice of the other parameters, if ε_2 is sufficiently small, then*

- (i) $\|D\varphi_j\|_{C^{2,\kappa}} \leq C \mathbf{m}_0^{1/2}$ and $\|\varphi_j\|_{C^0} \leq C \mathbf{m}_0^{1/2m}$, with $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$.
- (ii) if $L \in \mathcal{W}^i$ and H is a cube concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$, then $\varphi_j = \varphi_k$ on H for any $j, k \geq i + 2$.
- (iii) φ_j converges in C^3 to a map $\boldsymbol{\varphi}$ and $\mathcal{M} := \text{Gr}(\boldsymbol{\varphi}|_{]-4, 4[^m})$ is a $C^{3,\kappa}$ submanifold of Σ .

Definition 1.18 (Whitney regions). The manifold \mathcal{M} in Theorem 1.17 is called a *center manifold of T relative to π_0* and $(\boldsymbol{\Gamma}, \mathcal{W})$ the *Whitney decomposition associated to \mathcal{M}* . Setting $\boldsymbol{\Phi}(y) := (y, \boldsymbol{\varphi}(y))$, we call $\boldsymbol{\Phi}(\boldsymbol{\Gamma})$ the *contact set*. Moreover, to each $L \in \mathcal{W}$ we associate a *Whitney region \mathcal{L}* on \mathcal{M} as follows:

- (WR) $\mathcal{L} := \boldsymbol{\Phi}(H \cap [-\frac{7}{2}, \frac{7}{2}]^m)$, where H is the cube concentric to L with $\ell(H) = \frac{17}{16}\ell(L)$.

2. THE \mathcal{M} -NORMAL APPROXIMATION AND RELATED ESTIMATES

In what follows we assume that the conclusions of Theorem 1.17 apply and denote by \mathcal{M} the corresponding center manifold. For any Borel set $\mathcal{V} \subset \mathcal{M}$ we will denote by $|\mathcal{V}|$ its \mathcal{H}^m -measure and will write $\int_{\mathcal{V}} f$ for the integral of f with respect to \mathcal{H}^m . $\mathcal{B}_r(q)$ denotes the geodesic balls in \mathcal{M} . Moreover, we refer to [6] for all the relevant notation pertaining to the differentiation of (multiple valued) maps defined on \mathcal{M} , induced currents, differential geometric tensors and so on.

Assumption 2.1. We fix the following notation and assumptions.

- (U) $\mathbf{U} := \{x \in \mathbb{R}^{m+n} : \exists! y = \mathbf{p}(x) \in \mathcal{M} \text{ with } |x - y| < 1 \text{ and } (x - y) \perp \mathcal{M}\}$.
- (P) $\mathbf{p} : \mathbf{U} \rightarrow \mathcal{M}$ is the map defined by (U).
- (R) For any choice of the other parameters, we assume ε_2 to be so small that \mathbf{p} extends to $C^{2,\kappa}(\bar{\mathbf{U}})$ and $\mathbf{p}^{-1}(y) = y + \overline{B_1(0, (T_y\mathcal{M})^\perp)}$ for every $y \in \mathcal{M}$.
- (L) We denote by $\partial_l \mathbf{U} := \mathbf{p}^{-1}(\partial\mathcal{M})$ the *lateral boundary* of \mathbf{U} .

The following is then a corollary of Theorem 1.17 and the construction algorithm.

Corollary 2.2. *Under the hypotheses of Theorem 1.17 and of Assumption 2.1 we have:*

- (i) $\text{spt}(\partial(T \llcorner \mathbf{U})) \subset \partial_l \mathbf{U}$, $\text{spt}(T \llcorner [-\frac{7}{2}, \frac{7}{2}]^m \times \mathbb{R}^n) \subset \mathbf{U}$ and $\mathbf{p}_\#(T \llcorner \mathbf{U}) = Q \llbracket \mathcal{M} \rrbracket$;
- (ii) $\text{spt}(\langle T, \mathbf{p}, \Phi(q) \rangle) \subset \{y : |\Phi(q) - y| \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}\}$ for every $q \in L \in \mathscr{W}$, where $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$;
- (iii) $\langle T, \mathbf{p}, p \rangle = Q \llbracket p \rrbracket$ for every $p \in \Phi(\Gamma)$.

The main goal of this paper is to couple the center manifold of Theorem 1.17 with a good approximating map defined on it.

Definition 2.3 (\mathcal{M} -normal approximation). An \mathcal{M} -normal approximation of T is given by a pair (\mathcal{K}, F) such that

- (A1) $F : \mathcal{M} \rightarrow \mathcal{A}_Q(\mathbf{U})$ is Lipschitz (with respect to the geodesic distance on \mathcal{M}) and takes the special form $F(x) = \sum_i \llbracket x + N_i(x) \rrbracket$, with $N_i(x) \perp T_x \mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every x and i .
- (A2) $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\Phi(\Gamma \cap [-\frac{7}{2}, \frac{7}{2}]^m)$ and $\mathbf{T}_{F \llcorner \mathbf{p}^{-1}(\mathcal{K})} = T \llcorner \mathbf{p}^{-1}(\mathcal{K})$.

The map $N = \sum_i \llbracket N_i \rrbracket : \mathcal{M} \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$ is the *normal part* of F .

In the definition above it is not required that the map F approximates efficiently the current outside the set $\Phi(\Gamma \cap [-\frac{7}{2}, \frac{7}{2}]^m)$. However, all the maps constructed in this paper and used in the subsequent note [7] will approximate T with a high degree of accuracy in each Whitney region: such estimates are detailed in the next theorem. In order to simplify the notation, we will use $\|N|_{\mathcal{V}}\|_{C^0}$ (or $\|N|_{\mathcal{V}}\|_0$) to denote the number $\sup_{x \in \mathcal{V}} \mathcal{G}(N(x), Q \llbracket 0 \rrbracket)$.

Theorem 2.4 (Local estimates for the \mathcal{M} -normal approximation). *Let $\gamma_2 := \frac{\gamma_1}{4}$, with γ_1 the constant of [5, Theorem 1.4]. Under the hypotheses of Theorem 1.17 and Assumption 2.1, if ε_2 is suitably small (depending upon all other parameters), then there is an \mathcal{M} -normal approximation (\mathcal{K}, F) such that the following estimates hold on every Whitney*

region \mathcal{L} associated to a cube $L \in \mathscr{W}$, with constants $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$:

$$\text{Lip}(N|_{\mathcal{L}}) \leq C \mathbf{m}_0^{\gamma_2} \ell(L)^{\gamma_2} \quad \text{and} \quad \|N|_{\mathcal{L}}\|_{C^0} \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}, \quad (2.1)$$

$$|\mathcal{L} \setminus \mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C \mathbf{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}, \quad (2.2)$$

$$\int_{\mathcal{L}} |DN|^2 \leq C \mathbf{m}_0 \ell(L)^{m+2-2\delta_2}. \quad (2.3)$$

Moreover, for any $a > 0$ and any Borel $\mathcal{V} \subset \mathcal{L}$, we have (for $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$)

$$\int_{\mathcal{V}} |\boldsymbol{\eta} \circ N| \leq C \mathbf{m}_0 (\ell(L)^{m+3+\beta_2/3} + a \ell(L)^{2+\gamma_2/2} |\mathcal{V}|) + \frac{C}{a} \int_{\mathcal{V}} \mathcal{G}(N, Q[\boldsymbol{\eta} \circ N])^{2+\gamma_2}. \quad (2.4)$$

From (2.1) - (2.3) it is not difficult to infer analogous ‘‘global versions’’ of the estimates.

Corollary 2.5 (Global estimates). *Let \mathcal{M}' be the domain $\Phi([-7/2, 7/2]^m)$ and N the map of Theorem 2.4. Then, (again with $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$)*

$$\text{Lip}(N|_{\mathcal{M}'}) \leq C \mathbf{m}_0^{\gamma_2} \quad \text{and} \quad \|N|_{\mathcal{M}'}\|_{C^0} \leq C \mathbf{m}_0^{1/2m}, \quad (2.5)$$

$$|\mathcal{M}' \setminus \mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{M}')) \leq C \mathbf{m}_0^{1+\gamma_2}, \quad (2.6)$$

$$\int_{\mathcal{M}'} |DN|^2 \leq C \mathbf{m}_0. \quad (2.7)$$

3. ADDITIONAL CONCLUSIONS UPON \mathcal{M} AND THE \mathcal{M} -NORMAL APPROXIMATION

3.1. Height bound and separation. We now analyze more in detail the consequences of the various stopping conditions for the cubes in \mathscr{W} . We first deal with $L \in \mathscr{W}_h$.

Proposition 3.1 (Separation). *There is a constant $C^\sharp(M_0) > 0$ with the following property. Assume the hypotheses of Theorem 2.4 and in addition $C_h^{2m} \geq C^\sharp C_e$. If ε_2 is sufficiently small, then the following conclusions hold for every $L \in \mathscr{W}_h$:*

- (S1) $\Theta(T, p) \leq Q - \frac{1}{2}$ for every $p \in \mathbf{B}_{16r_L}(p_L)$.
- (S2) $L \cap H = \emptyset$ for every $H \in \mathscr{W}_n$ with $\ell(H) \leq \frac{1}{2}\ell(L)$;
- (S3) $\mathcal{G}(N(x), Q[\boldsymbol{\eta} \circ N(x)]) \geq \frac{1}{4} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$ for every $x \in \Phi(B_{2\sqrt{m}\ell(L)}(x_L, \pi_0))$.

A simple corollary of the previous proposition is the following.

Corollary 3.2. *Given any $H \in \mathscr{W}_n$ there is a chain $L = L_0, L_1, \dots, L_j = H$ such that:*

- (a) $L_0 \in \mathscr{W}_e$ and $L_i \in \mathscr{W}_n$ for all $i > 0$;
- (b) $L_i \cap L_{i-1} \neq \emptyset$ and $\ell(L_i) = \frac{1}{2}\ell(L_{i-1})$ for all $i > 0$.

In particular, $H \subset B_{3\sqrt{m}\ell(L)}(x_L, \pi_0)$.

We use this last corollary to partition \mathscr{W}_n .

Definition 3.3 (Domains of influence). We first fix an ordering of the cubes in \mathscr{W}_e as $\{J_i\}_{i \in \mathbb{N}}$ so that their sidelengths do not increase. Then $H \in \mathscr{W}_n$ belongs to $\mathscr{W}_n(J_0)$ (the domain of influence of J_0) if there is a chain as in Corollary 3.2 with $L_0 = J_0$. Inductively, $\mathscr{W}_n(J_r)$ is the set of cubes $H \in \mathscr{W}_n \setminus \cup_{i < r} \mathscr{W}_n(J_i)$ for which there is a chain as in Corollary 3.2 with $L_0 = J_r$.

3.2. Splitting before tilting I. The following proposition contains a “typical” splitting-before-tilting phenomenon: the key assumption of the theorem (i.e. $L \in \mathscr{W}_e$) is that the excess does not decay at some given scale (“tilting”) and the main conclusion (3.2) implies a certain amount of separation between the sheets of the current (“splitting”).

Proposition 3.4. (*Splitting I*) *There are functions $C_1(\delta_2), C_2(M_0, \delta_2)$ such that, if $M_0 \geq C_1(\delta_2)$, $C_e \geq C_2(M_0, \delta_2)$, if the hypotheses of Theorem 2.4 hold and if ε_2 is chosen sufficiently small, then the following holds. If $L \in \mathscr{W}_e$, $q \in \pi_0$ with $\text{dist}(L, q) \leq 4\sqrt{m}\ell(L)$ and $\Omega = \Phi(B_{\ell(L)/4}(q, \pi_0))$, then (with $C, C_3 = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$):*

$$C_e \mathbf{m}_0 \ell(L)^{m+2-2\delta_2} \leq \ell(L)^m \mathbf{E}(T, \mathbf{B}_L) \leq C \int_{\Omega} |DN|^2, \quad (3.1)$$

$$\int_{\mathcal{L}} |DN|^2 \leq C \ell(L)^m \mathbf{E}(T, \mathbf{B}_L) \leq C_3 \ell(L)^{-2} \int_{\Omega} |N|^2. \quad (3.2)$$

3.3. Persistence of Q points. We next state two important properties triggered by the existence of $p \in \text{spt}(T)$ with $\Theta(p, T) = Q$, both related to the splitting before tilting.

Proposition 3.5. (*Splitting II*) *Let the hypotheses of Theorem 1.17 hold and assume ε_2 is sufficiently small. For any $\alpha, \bar{\alpha}, \hat{\alpha} > 0$, there is $\varepsilon_3 = \varepsilon_3(\alpha, \bar{\alpha}, \hat{\alpha}, \beta_2, \delta_2, M_0, N_0, C_e, C_h) > 0$ as follows. If, for some $s \leq 1$*

$$\sup \{ \ell(L) : L \in \mathscr{W}, L \cap B_{3s}(0, \pi_0) \neq \emptyset \} \leq s, \quad (3.3)$$

$$\mathcal{H}_{\infty}^{m-2+\alpha}(\{\Theta(T, \cdot) = Q\} \cap \mathbf{B}_s) \geq \bar{\alpha} s^{m-2+\alpha}, \quad (3.4)$$

and $\min \{s, \mathbf{m}_0\} \leq \varepsilon_3$, then,

$$\sup \{ \ell(L) : L \in \mathscr{W}_e \text{ and } L \cap B_{19s/16}(0, \pi_0) \neq \emptyset \} \leq \hat{\alpha} s.$$

Proposition 3.6. (*Persistence of Q -points*) *Assume the hypotheses of Proposition 3.4 hold. For every $\eta_2 > 0$ there are $\bar{s}, \bar{\ell} > 0$, depending upon $\eta_2, \beta_2, \delta_2, M_0, N_0, C_e$ and C_h , such that, if ε_2 is sufficiently small, then the following property holds. If $L \in \mathscr{W}_e$, $\ell(L) \leq \bar{\ell}$, $\Theta(T, p) = Q$ and $\text{dist}(\mathbf{p}_{\pi_0}(\mathbf{p}(p)), L) \leq 4\sqrt{m}\ell(L)$, then*

$$\int_{B_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q[\boldsymbol{\eta} \circ N])^2 \leq \frac{\eta_2}{\ell(L)^{m-2}} \int_{B_{\ell(L)}(\mathbf{p}(p))} |DN|^2. \quad (3.5)$$

3.4. Comparison between different center manifolds. We list here a final key consequence of the splitting before tilting phenomenon. $\iota_{0,r}$ denotes the map $z \mapsto \frac{z}{r}$.

Proposition 3.7 (Comparing center manifolds). *There is a geometric constant C_0 and a function $\bar{c}_s(\beta_2, \delta_2, M_0, N_0, C_e, C_h) > 0$ with the following property. Assume the hypotheses of Proposition 3.4, $N_0 \geq C_0$, $c_s := \frac{1}{64\sqrt{m}}$ and ε_2 is sufficiently small. If for some $r \in]0, 1[$:*

- (a) $\ell(L) \leq c_s \rho$ for every $\rho > r$ and every $L \in \mathscr{W}$ with $L \cap B_{\rho}(0, \pi_0) \neq \emptyset$;
- (b) $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) < \varepsilon_2$ for every $\rho > r$;
- (c) there is $L \in \mathscr{W}$ such that $\ell(L) \geq c_s r$ and $L \cap \bar{B}_r(0, \pi_0) \neq \emptyset$;

then

- (i) the current $T' := (\iota_{0,r})_{\#} T \llcorner \mathbf{B}_{6\sqrt{m}}$ and the submanifold $\Sigma' := \iota_{0,r}(\Sigma) \cap \mathbf{B}_{7\sqrt{m}}$ satisfy the assumptions of Theorem 2.4 for some plane π in place of π_0 ;
- (ii) for the center manifold \mathcal{M}' of T' relative to π and the \mathcal{M}' -normal approximation N' as in Theorem 2.4, we have

$$\int_{\mathcal{M}' \cap \mathbf{B}_2} |N'|^2 \geq \bar{c}_s \max \{ \mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma')^2 \}. \quad (3.6)$$

4. CENTER MANIFOLD'S CONSTRUCTION

In this section we lay down the technical preliminaries to prove Theorem 1.17, state the related fundamental estimates and show how the theorem follows from them.

4.1. Technical preliminaries and proof of (1.17).

Proof of Lemma 1.6. Recalling that $T := T^0 \llcorner \mathbf{B}_{23\sqrt{m}/4}$, we want to show that (1.10) hold. To this regard, we can argue by contradiction. If for instance the second statement in (1.10) were false, then we would have a sequence of currents T_k^0 in $\mathbf{B}_{6\sqrt{m}}$ and of submanifolds Σ_k satisfying Assumption 1.3 with $\varepsilon_2(k) \downarrow 0$ and $(\mathbf{p}_{\pi_0})_{\#} T_k^0 \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4}) \neq Q \llbracket B_{11\sqrt{m}/2} \rrbracket$. On the other hand, from (1.5), (1.7), (1.8) and the standard monotonicity formula

$$T_k^0 \rightharpoonup T_{\infty} := Q \llbracket B_{6\sqrt{m}} \rrbracket .$$

Also, by standard regularity theory for area minimizing currents, we conclude that $\text{spt}(T_k^0) \cap \mathbf{B}_r$ converges to $\text{spt}(T_{\infty}) \cap \mathbf{B}_r$ in the Hausdorff distance for every $r < 6\sqrt{m}$. Since ∂T_k^0 vanishes in $\mathbf{B}_{6\sqrt{m}}$, $T_k^0 \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4})$ has no boundary in $\mathbf{C}_{11\sqrt{m}/2}$ for k large enough, thereby implying that $(\mathbf{p}_{\pi_0})_{\#} T_k^0 \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4}) = Q_k \llbracket B_{11\sqrt{m}/2} \rrbracket$ for some integer Q_k . Since $T_k^0 \rightharpoonup T_{\infty}$, we deduce that $Q_k = Q$ for k large enough, giving the desired contradiction. Note that the argument actually shows also the first statement in (1.10). The height bound (1.11) now follows from Theorem A.1 because $(\mathbf{p}_{\pi_0})_{\#} T^0 \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4}) = Q \llbracket B_{11\sqrt{m}/2} \rrbracket$ and $\Theta(T^0, 0) = Q$: in particular, the latter assumption and Theorem A.1(iii) imply that there is one single open set \mathbf{S}_1 as in Theorem A.1(i), which in turn must contain the origin.

By the slicing theory of currents (see [12, Section 28] or [8, 4.3.8]) and by (1.10), there is a set $A \subset B_{5\sqrt{m}}$ of full measure such that

$$\langle T, \mathbf{p}_{\pi_0}, x \rangle = \sum_{i=1}^{N(x)} k_i(x) \delta_{(x, y_i(x))} \quad \forall x \in A,$$

where $N(x) \in \mathbb{N}$, $k_i(x) \in \mathbb{Z}$ with $\sum_i k_i = Q$, and $(x, y_i(x)) \in \text{spt}(T)$ with $|y_i(x)| \leq C_0 \mathbf{m}_0^{1/2m}$. By the density of A in $B_{5\sqrt{m}}$, we conclude that $\text{spt}(T) \cap (x + \pi_0^{\perp}) \neq \emptyset$ for every $x \in \bar{B}_{5\sqrt{m}}$. This completes the proof of Lemma 1.6. Observe also that as a consequence, if $L \in \mathcal{C}$, then

$$|y_L| \leq C \mathbf{m}_0^{1/2m} \quad \text{and} \quad |p_L| \leq 4\sqrt{m} + C_0 \mathbf{m}_0^{1/2m} \quad (4.1)$$

(recall that $p_L = (x_L, y_L) \in \pi_0 \times \pi_0^{\perp} \cap \text{spt}(T)$ is the center of \mathbf{B}_L , cf. Definition 1.10). \square

Proof of Lemma 1.5. The first part of the statement, i.e. the extension of the manifold Σ , is a fairly standard fact: it suffices to make the correct extension of the map Ψ_0 to $T_0\Sigma$ and then use the smallness of the norm to show that Σ is globally graphical over every $T_p\Sigma$. The fact that T^0 remains area-minimizing is also fairly simple: any area minimizing current T' in the extended manifold Σ with $T' - T^0 = \partial S$ must be supported in \mathbf{B}_{C_0} for some geometric constant C_0 , by the monotonicity formula. On the other hand, for a sufficiently small ε_2 , $\mathbf{B}_r \cap \Sigma$ is geodesically convex in Σ for every $r \in]0, C_0]$ and thus there is a projection $\mathbf{p} : \mathbf{B}_{C_0} \cap \Sigma \rightarrow \mathbf{B}_{6\sqrt{m}} \cap \Sigma$ which is 1-Lipschitz with respect to the Riemannian metric on Σ . Since $\pi_{\sharp} T'$ cannot have mass smaller than T' , T' must be supported in $\bar{\mathbf{B}}_{6\sqrt{m}}$. But then T' is area-minimizing even in the original (i.e. not extended) Σ and must have the same mass as T^0 .

By Assumption 1.3 and Remark 1.4, $\mathbf{A} \leq C_0 \mathbf{m}_0^{1/2} \leq C_0$. Then, by the monotonicity formula, $\|T^0\|(\mathbf{B}_1) \geq c_0 > 0$ and so there is $p \in \text{spt}(T) \cap \mathbf{B}_1$ such that

$$|\vec{T}(p) - \pi_0|^2 = |\vec{T}^0(p) - \pi_0|^2 \leq C_0 \frac{\mathbf{E}(T^0, \mathbf{B}_1, \pi_0)}{\|T\|(\mathbf{B}_1)} \leq C_0 \mathbf{m}_0.$$

We conclude that, if ε_2 is smaller than a geometric constant, $\mathbf{p}_{T_p\Sigma}(\pi_0)$ is an m -dimensional plane with $|\mathbf{p}_{T_p\Sigma}(\pi_0) - \pi_0| \leq C_0 \mathbf{m}_0^{1/2}$. On the other hand $|\mathbf{p}_{T_0\Sigma} - \mathbf{p}_{T_p\Sigma}| \leq C_0 |T_p\Sigma - T_0\Sigma| \leq C_0 \mathbf{A} \leq C_0 \mathbf{m}_0^{1/2}$ and we conclude $|\mathbf{p}_{T_0\Sigma}(\pi_0) - \pi_0| \leq C_0 \mathbf{m}_0^{1/2}$. Therefore there is an n -dimensional plane \varkappa_0 orthogonal to π_0 such that $|\pi_0 \times \varkappa_0 - T_0\Sigma| \leq C_0 \mathbf{m}_0^{1/2}$. We then find a rotation which fixes π_0 and maps \varkappa_0 onto $\{0\} \times \mathbb{R}^n \times \{0\}$. The remaining statements follows easily from Lemma B.1. \square

Proof of (1.17). Fix $L \in \mathscr{W}^j$ with $N_0 \leq j \leq N_0 + 6$. Since $r_L \leq 2^{-7}$ (cf. Assumption 1.8), (4.1) guarantees $\mathbf{B}_L \subset \mathbf{B}_{5\sqrt{m}}$ if ε_2 is small enough. Moreover

$$\mathbf{E}(T, \mathbf{B}_L, \pi_0) \leq \frac{6^m}{(64M_0 2^{-N_0-6})^m} \mathbf{E}(T^0, \mathbf{B}_{6\sqrt{m}}, \pi_0) \leq \frac{6^m}{(64M_0)^m 2^{-(N_0+6)m}} \mathbf{m}_0.$$

For a suitable $C^*(M_0, N_0)$ the inequality $C_e \geq C^*$ implies

$$\mathbf{E}(T, \mathbf{B}_L) \leq \mathbf{E}(T, \mathbf{B}_L, \pi_0) \leq C_e \mathbf{m}_0 \ell(L)^{2-2\delta_2}.$$

Let now $\hat{\pi}_L$ be an optimal plane in \mathbf{B}_L : since the center p_L belongs to $\text{spt}(T)$, by the monotonicity formula $\|T\|(\mathbf{B}_L) \geq c_0 r_L^m$ (cf. [12, Section 17] or [5, Appendix A]). Thus

$$|\hat{\pi}_L - \pi_0|^2 \leq C_0 (\mathbf{E}(T, \mathbf{B}_L, \pi_0) + \mathbf{E}(T, \mathbf{B}_L, \hat{\pi}_L)) \leq C_0 C_e \mathbf{m}_0 \ell(L)^{2-2\delta_2}, \quad (4.2)$$

where C_0 is a geometric constant. This in turn implies that

$$\begin{aligned} \mathbf{h}(T, \mathbf{B}_L) &\leq C_0 M_0 |\hat{\pi}_L - \pi_0| \ell(L) + \mathbf{h}(T, \mathbf{B}_L, \pi_0) \leq C_0 M_0 C_e^{1/2} \mathbf{m}_0^{1/2} \ell(L)^{2-\delta_2} + \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}) \\ &\stackrel{(1.11)}{\leq} C(M_0, N_0) (C_e^{1/2} + 1) \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \end{aligned}$$

Thus, if $C^*(M_0, N_0)$ is chosen sufficiently large and $C_h \geq C^* C_e \geq (C^*)^2$, neither the condition (EX) nor (HT) apply to L . Therefore, $\mathscr{W}^j = \emptyset$ for every $j \leq N_0 + 6$. \square

4.2. Tilting of planes and proof of Proposition 1.11. Next we compare optimal planes and height functions across different cubes of $\mathcal{W} \cup \mathcal{S}$.

Proposition 4.1 (Tilting of optimal planes). *Assume that the hypotheses of Assumptions 1.3 and 1.8 hold, that $C_e \geq C^*$ and $C_h \geq C^*C_e$, where $C^*(M_0, N_0)$ is the constant of the previous section. If ε_2 is sufficiently small, then*

(i) $\mathbf{B}_H \subset \mathbf{B}_L \subset \mathbf{B}_{5\sqrt{m}}$ for all $H, L \in \mathcal{W} \cup \mathcal{S}$ with $H \subset L$.

Moreover, if $H, L \in \mathcal{W} \cup \mathcal{S}$ and either $H \subset L$ or $H \cap L \neq \emptyset$ and $\frac{\ell(L)}{2} \leq \ell(H) \leq \ell(L)$, then the following holds, for $\bar{C} = \bar{C}(\beta_2, \delta_2, M_0, N_0, C_e)$ and $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$:

(ii) $|\hat{\pi}_H - \pi_H| \leq \bar{C}\mathbf{m}_0^{1/2}\ell(H)^{1-\delta_2}$;

(iii) $|\pi_H - \pi_L| \leq \bar{C}\mathbf{m}_0^{1/2}\ell(L)^{1-\delta_2}$;

(iv) $|\pi_H - \pi_0| \leq \bar{C}\mathbf{m}_0^{1/2}$;

(v) $\mathbf{h}(T, \mathbf{C}_{36r_H}(p_H, \pi_0)) \leq C\mathbf{m}_0^{1/2m}\ell(H)$ and $\text{spt}(T) \cap \mathbf{C}_{36r_H}(p_H, \pi_0) \subset \mathbf{B}_H$;

(vi) For $\pi = \pi_H, \hat{\pi}_H$, $\mathbf{h}(T, \mathbf{C}_{36r_L}(p_L, \pi)) \leq C\mathbf{m}_0^{1/2m}\ell(L)^{1+\beta_2}$ and $\text{spt}(T) \cap \mathbf{C}_{36r_L}(p_L, \pi) \subset \mathbf{B}_L$.

In particular, the conclusions of Proposition 1.11 hold.

Proof. In this proof we will use the following convention: geometric constants will be denoted by C_0 or c_0 , constants depending upon $\beta_2, \delta_2, M_0, N_0, C_e$ will be denoted by \bar{C} or \bar{c} and constants depending upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h will be denoted by C or c .

Proof of (i)–(vi) when $H \subset L$. The proof is by induction over $i = -\log_2(\ell(H))$, where we start with $i = N_0$. For the starting step $i = N_0$ we need to check (i), (ii), (iv), (v) and (vi), all in the special case $H = L$. Observe first that (i) is a consequence of (4.1) and the estimate $64r_L \leq M_0\sqrt{m}2^{-N_0} \leq \sqrt{m}/2$. Since $\mathcal{W}^{N_0} = \emptyset$, for $i = N_0$ we have $H \in \mathcal{S}^{N_0}$, which means that H satisfies neither condition (EX) nor condition (HT). Since by the monotonicity formula $\|T\|(\mathbf{B}_H) \geq c_0 r_H^m$, there exists at least a point $p \in \text{spt}(T) \cap \mathbf{B}_H$ such that

$$|\vec{T}(p) - \hat{\pi}_H|^2 \leq \mathbf{E}(T, \mathbf{B}_H) \frac{C_0 r_H^m}{\|T\|(\mathbf{B}_H)} \leq \bar{C}\mathbf{m}_0 \ell(H)^{2-2\delta_2}. \quad (4.3)$$

Since $\vec{T}(p)$ is an m -vector of $T_p\Sigma$, this implies that $|\mathbf{p}_{T_p\Sigma}(\hat{\pi}_H) - \hat{\pi}_H| \leq \bar{C}\mathbf{m}_0^{1/2}\ell(H)^{1-\delta_2}$. Recalling that $|\mathbf{p}_{T_{p_H}\Sigma} - \mathbf{p}_{T_p\Sigma}| \leq C_0 r_H \mathbf{A} \leq \bar{C}\mathbf{m}_0^{1/2}\ell(H)$, we conclude (ii). (iv) follows simply from (4.2) and (ii). As for (v), observe that the radius of $\mathbf{C}_{36r_H}(p_H, \pi_0)$ is smaller than $\sqrt{m}/2$ and its center $p_H = (x_H, y_H)$ satisfies $|x_H| \leq 4\sqrt{m}$. Thus $\mathbf{C}_{36r_H}(p_H, \pi_0) \subset \mathbf{C}_{5\sqrt{m}}(0, \pi_0) =: \mathbf{C}$ and the first conclusion of (v) is a consequence of (1.11). The second conclusion follows from the first provided $\varepsilon_2 < c$. Finally, with regard to (vi), recall that $H = L$. There are two cases: $\pi = \pi_H$ and $\pi = \hat{\pi}_H$. Since the arguments are entirely analogous, we just give it in the first case. The base point p_H of the cylinder $\mathbf{C}' := \mathbf{C}_{36r_H}(p_H, \pi_H)$ satisfies, by (1.11) $|p_H| \leq 4\sqrt{m} + C_0\mathbf{m}_0^{1/2m}$ and its radius is smaller than $\sqrt{m}/2$. By a simple geometric consideration, $\mathbf{C}' \cap \mathbf{B}_{6\sqrt{m}} \subset \mathbf{C}$ holds provided $|\pi_H - \pi_0|$ and $|p_H| - 4\sqrt{m}$ are smaller than a geometric constant: this requires $\varepsilon_2 \leq \bar{c}$. Under this assumption $\text{spt}(T) \cap \mathbf{C}' \subset \mathbf{C}$ and from (1.11) and (iv) we conclude $\mathbf{h}(T, \mathbf{C}', \pi_H) \leq$

$C_0|\pi_H - \pi_0| + \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}, \pi_0) \leq \bar{C}\mathbf{m}_0^{1/2m}$. it follows then that $\text{spt}(T) \cap \mathbf{C}' \subset \mathbf{B}_H$, provided ε_2 is sufficiently small. Since $H \notin \mathcal{W}$, from (ii) we then conclude

$$\mathbf{h}(T, \mathbf{C}', \pi_H) \leq \mathbf{h}(T, \mathbf{B}_H) + C_0 M_0 \ell(H) |\pi_H - \hat{\pi}_H| \leq C\mathbf{m}_0^{1/2m} \ell(H)^{1+\beta_2} + \bar{C}\mathbf{m}_0^{1/2} \ell(H)^{2-\delta_2}.$$

Now we pass to the inductive step. Thus fix some $H_{i+1} \in \mathcal{S}^{i+1} \cup \mathcal{W}^{i+1}$ and consider a chain $H_{i+1} \subset H_i \subset \dots \subset H_{N_0}$ with $H_l \in \mathcal{S}^l$ for $l \leq i$. We wish to prove all the conclusions (i)–(vi) when $H = H_{i+1}$ and $L = H_j$ for some $j \leq i+1$, recalling that, by inductive assumption, all the statements hold when $H = H_k$ and $L = H_l$ for $l \leq k \leq i$. With regard to (i), it is enough to prove that $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$. By inductive assumption we know (v) holds with $H = H_i$, whereas $|x_{H_i} - x_{H_{i+1}}| \leq \sqrt{m} \ell(H_i)$: so $|p_{H_{i+1}} - p_{H_i}| \leq (\sqrt{m} + C\mathbf{m}_0^{1/2m})2\ell(H)_{i+1}$. In particular, for ε_2 small enough, we conclude $|p_{H_{i+1}} - p_{H_i}| \leq 3\sqrt{m}\ell(H_{i+1})$. Assuming that the geometric constant in the first inequality of (1.15) is large enough, we infer $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$. We show now (ii). By (i),

$$\mathbf{E}(T, \mathbf{B}_{H_{i+1}}) \leq 2^m \mathbf{E}(T, \mathbf{B}_{H_i}) \leq 2^{m+2-2\delta_2} C_e \mathbf{m}_0 \ell(H_{i+1})^{2-2\delta_2}. \quad (4.4)$$

Therefore, we can argue as above in the case $i = N_0$ to achieve (ii). We next come to (iii) and (iv). Fix any $l \in \{N_0 + 1, \dots, i+1\}$. By the inclusion in (i), we can argue similarly to infer

$$|\hat{\pi}_{H_{l-1}} - \hat{\pi}_{H_l}|^2 \leq (\mathbf{E}(T, \mathbf{B}_{H_{l-1}}) + \mathbf{E}(T, \mathbf{B}_{H_l})) \frac{C_0 r_{H_{l-1}}^m}{\|T\|(\mathbf{B}_{H_l})} \leq \bar{C} \mathbf{m}_0 \ell(H_l)^{2-2\delta_2}. \quad (4.5)$$

Using the estimate $\sum_{l=j}^{\infty} \ell(H_l)^{1-\delta_2} \leq C_0 \ell(H_j)^{1-\delta_2}$ and (ii), we conclude (iii) for $H = H_{i+1}$ and $L = H_j$. As for (iv) it follows from (iii) and the case $|\pi_{H_{N_0}} - \pi_0| \leq \bar{C}\mathbf{m}_0^{1/2}$. We next come to (v). (v) holds for H_i and so we conclude $\text{spt}(T) \cap \mathbf{C}_{36r_{H_i}}(p_{H_i}, \pi_0) \subset \mathbf{B}_{H_i}$. Since $|p_{H_{i+1}} - p_{H_i}| \leq 3\sqrt{m}\ell(H_{i+1})$ and $r_{H_{i+1}} = \frac{1}{2}r_{H_i}$, we have $\mathbf{C}_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset \mathbf{C}_{36r_{H_i}}(p_{H_i}, \pi_0)$ provided the geometric constant in the first inequality of (1.15) is large enough. Thus:

$$\begin{aligned} \mathbf{h}(T, \mathbf{C}_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)) &\leq \mathbf{h}(T, \mathbf{B}_{H_i}) + C_0 r_{H_i} |\hat{\pi}_{H_i} - \pi_0| \\ &\stackrel{(iv)}{\leq} C_h \mathbf{m}_0^{1/2m} \ell(H_i)^{1+\beta_2} + \bar{C} \mathbf{m}_0^{1/2} \ell(H_i) \leq C \mathbf{m}_0^{1/2m} \ell(H_i), \end{aligned}$$

where we used $H_i \in \mathcal{S}^i$. Thus (v) follows easily for $H = H_{i+1}$. The inclusion $\text{spt}(T) \cap \mathbf{C}_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset \mathbf{B}_{H_{i+1}}$ is an obvious corollary of the bound and of the fact that the center of the ball $\mathbf{B}_{H_{i+1}}$ (i.e. the point $p_{H_{i+1}}$) belongs to $\text{spt}(T) \cap \mathbf{C}_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)$: we again need to ensure that ε_2 is chosen small enough.

Next we show (vi) for $H = H_{i+1}$ and $L = H_j$ with $j \leq i+1$ (including the case $L = H_{i+1}$). The argument is the same in both cases π_H and $\hat{\pi}_H$ and we show it in the first case. We first prove the second claim of (vi) inductively on j . Observe that for $j = N_0$ we can argue as for the inclusion $\mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}) \cap \mathbf{B}_{6\sqrt{m}} \subset \mathbf{C}_{5\sqrt{m}}(0, \pi_0)$ to infer also $\mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_H) \cap \mathbf{B}_{5\sqrt{m}} \subset \mathbf{C}_{5\sqrt{m}}(0, \pi_0)$: since $|\pi_{H_{N_0}} - \pi_H| \leq \bar{C}\mathbf{m}_0^{1/2} \ell(H_{N_0})^{1-\delta_2}$, such inclusion simply requires a smaller choice of ε_2 . We can then use (1.11) to infer

$$\mathbf{h}(T, \mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_H)) \leq \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}(0, \pi_0), \pi_0) + C_0 r_{H_{N_0}} |\pi_0 - \pi_H| \leq C\mathbf{m}_0^{1/2m} \ell(H_{N_0})^{1+\beta_2}.$$

Again the inclusion $\text{spt}(T) \cap \mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_H) \subset \mathbf{B}_{H_{N_0}}$ follows from assuming ε_2 sufficiently small. Next, assume that the second claim of (vi) holds for H and $L = H_l$. Observe that

$$\mathbf{C}_{36r_{H_{l+1}}}(p_{H_{l+1}}, \pi_H) \subset \mathbf{C}_{36r_{H_l}}(p_{H_l}, \pi_H) :$$

in fact, arguing as above, we have $|p_{H_{l+1}} - p_H| \leq 3\sqrt{m}\ell(H_{l+1})$ and thus such inclusion requires only a sufficiently large geometric constant in the first inequality of (1.15). But then, we know $\mathbf{C}_{36r_{H_{l+1}}}(p_{H_{l+1}}, \pi_H) \subset \mathbf{B}_{H_l}$ and we can therefore conclude

$$\mathbf{h}(T, \mathbf{C}_{36r_{H_{l+1}}}(p_{H_{l+1}}, \pi_H), \pi_H) \leq \mathbf{h}(T, \mathbf{B}_{H_l}) + C_0 r_{H_{l+1}} |\pi_H - \hat{\pi}_{H_l}|.$$

From this we then conclude the second claim of (vi), i.e. $\text{spt}(T) \cap \mathbf{C}_{36r_{H_{l+1}}}(p_{H_{l+1}}, \pi_H) \subset \mathbf{B}_{H_{l+1}}$. Next, the first claim of (vi) is an obvious consequence of the second claim when $L = H_j$ for $j \leq i$ because $L \in \mathcal{S}$: in this case we have, as computed above,

$$\begin{aligned} \mathbf{h}(T, \mathbf{C}_{36r_L}(p_L, \pi_H)) &\leq \mathbf{h}(T, \mathbf{B}_L) + C_0 r_L |\hat{\pi}_L - \pi_H| \\ &\stackrel{\text{(iii)\&(ii)}}{\leq} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} + \bar{C} \mathbf{m}_0^{1/2} \ell(L)^{2-\delta_2} \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \end{aligned}$$

Finally, since $\mathbf{C}_{36r_H}(p_H, \pi_H) \subset \mathbf{C}_{36r_{H_i}}(p_{H_i}, \pi_H) \subset \mathbf{B}_{H_i}$ and the sidelengths $\ell(H)$ and $\ell(H_i)$ differ by a factor 2, we conclude as well that the first claim of (vi) holds for $H = L$.

Proof of Proposition 1.11. Observe that (1.17) has already been shown in the previous subsection and that (1.18) is an obvious consequence of the definition of \mathcal{S} : it only remains to show (1.19). Fix then $L \in \mathcal{W}$ and recall that its father J belongs to \mathcal{S} . However, having proved (i)–(vi) for pairs of cubes in which one is the ancestor of the other, we know that $\mathbf{B}_L \subset \mathbf{B}_J$ and thus we achieve

$$\mathbf{E}(T, \mathbf{B}_L) \leq 2^m \mathbf{E}(T, \mathbf{B}_J) \leq 2^m C_e \mathbf{m}_0 \ell(J)^{2-2\delta_2} \leq 2^{m+2-2\delta_2} C_e \mathbf{m}_0 \ell(L)^{2-2\delta_2} \quad (4.6)$$

$$\mathbf{h}(T, \mathbf{B}_L) \leq \mathbf{h}(T, \mathbf{B}_J) + C_0 r_L |\hat{\pi}_J - \hat{\pi}_L| \stackrel{\text{(ii)\&(iii)}}{\leq} C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \quad (4.7)$$

Proof of (i)–(vi) for neighboring H and L . Observe that in this case we only have to show (iii) and (vi). The argument for (iii) is entirely analogous to the case $H \subset L$. Assume first that $L \notin \mathcal{S}^{N_0}$. Then L has a father J . As already seen we have $|p_L - p_J| \leq 3\sqrt{m}\ell(J)$. On the other hand it is also easy to see that, with the same argument, we conclude $|p_H - p_L| \leq 3\sqrt{m}\ell(L)$ and thus $|p_H - p_J| \leq 5\sqrt{m}\ell(J)$. We therefore easily conclude $\mathbf{B}_H \cup \mathbf{B}_L \subset \mathbf{B}_J$, provided the geometric constant in the first inequality of (1.15) is large enough. Therefore, we can estimate

$$|\hat{\pi}_L - \hat{\pi}_J| \leq C_0 (\mathbf{E}(T, \mathbf{B}_J) + \mathbf{E}(T, \mathbf{B}_L))^{1/2}$$

and use (ii) to conclude. In case $L \in \mathcal{S}^{N_0}$, we can simply replace \mathbf{B}_J with $\mathbf{B}_{5\sqrt{m}}$.

We pass finally to (vi). We can in fact use the very same argument already explained when $H \subset L$: we claim indeed that (vi) holds not only for L but also for all its ancestors J and prove this claim by induction exactly as done above. \square

4.3. Existence of several approximating maps. Next, we prove that the building blocks for the construction of the center manifold are well-defined.

Proposition 4.2 (Existence of interpolating functions). *Assume the conclusions of Proposition 4.1 apply. The following facts are true provided ε_2 is sufficiently small. Let $H, L \in \mathcal{W} \cup \mathcal{S}$ be such that either $H \subset L$ or $H \cap L \neq \emptyset$ and $\frac{\ell(L)}{2} \leq \ell(H) \leq \ell(L)$. Then,*

- (i) *for $\pi = \pi_H, \hat{\pi}_H$, $(\mathbf{p}_\pi)_\# T \lrcorner \mathbf{C}_{32r_L}(p_L, \pi) = Q \llbracket B_{32r_L}(p_L, \pi) \rrbracket$ and T satisfies the assumptions of [5, Theorem 1.4] in the cylinder $\mathbf{C}_{32r_L}(p_L, \pi)$;*
- (ii) *Let f_{HL} be the π_H -approximation of T in $\mathbf{C}_{8r_L}(p_L, \pi_H)$ and $h := (\boldsymbol{\eta} \circ f_{HL}) * \varrho_{\ell(L)}$ be its smoothed average. Set $\varkappa_H := \pi_H^\perp \cap T_{p_H} \Sigma$ and consider the maps*

$$\begin{aligned} x &\mapsto \bar{h}(x) &:= \mathbf{p}_{T_{p_H} \Sigma}(h) &\in \varkappa_H \\ x &\mapsto h_{HL}(x) &:= (\bar{h}(x), \Psi_{p_H}(x, \bar{h}(x))) &\in \varkappa_H \times (T_{p_H}(\Sigma))^\perp. \end{aligned}$$

Then there is a smooth $g_{HL} : B_{4r_L}(p_L, \pi_0) \rightarrow \pi_0^\perp$ s.t. $\mathbf{G}_{g_{HL}} = \mathbf{G}_{h_{HL}} \lrcorner \mathbf{C}_{4r_L}(p_L, \pi_0)$.

Definition 4.3. h_{HL} and g_{HL} will be called, respectively, *tilted (H, L) -interpolating function* and *(H, L) -interpolating function*.

Observe that the tilted (L, L) -interpolating function and the (L, L) -interpolating function correspond to the tilted L -interpolating function and to the L -interpolating function of Definition 1.16. Obviously, Lemma 1.15 is just a particular case of Proposition 4.2.

Proof. We use the convention that C_0 and c_0 denote geometric constants, \bar{C} and \bar{c} denote dependence upon $\beta_2, \delta_2, M_0, N_0$ and C_e , whereas C and c dependence upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h . There are two cases: (i) $\pi = \pi_H$ and (ii) $\pi = \hat{\pi}_H$; since the argument for case (ii) is entirely analogous to that for case (i) we only give it for case (i). First recall that, by Proposition 4.1,

$$\text{spt}(T \lrcorner \mathbf{C}_{32r_L}(p_L, \pi_H)) \subset \mathbf{B}_L \subset \mathbf{B}_{5\sqrt{m}}. \quad (4.8)$$

We thus have $\partial T \lrcorner \mathbf{C}_{32r_L}(p_L, \pi_H) = 0$ and thus, setting $\mathbf{p} := \mathbf{p}_{\pi_H}$, we conclude

$$\mathbf{p}_\# T \lrcorner \mathbf{C}_{32r_L}(p_L, \pi_H) = k \llbracket B_{32r_L}(\mathbf{p}(p_L), \pi_H) \rrbracket \quad (4.9)$$

for some integer k . We will show now that $Q = k$. If J is the father of L , we then have proved in the previous section that $|p_L - p_J| \leq 3\sqrt{m}\ell(L)$. We thus conclude $\mathbf{C}_{32r_L}(p_L, \pi_H) \subset \mathbf{C}_{32r_J}(p_J, \pi_H)$, provided M_0 is larger than a geometric constant. Consider the chain of ancestors $J \subset \dots \subset M$ of L , till $M \in \mathcal{S}^{N_0}$. We then have $\mathbf{C}_{32r_L}(p_L, \pi_H) \subset \mathbf{C}_{32r_M}(p_M, \pi_H)$ and it suffices to show

$$\mathbf{p}_\# T \lrcorner \mathbf{C}_{32r_M}(p_M, \pi_H) = Q \llbracket B_{32r_M}(\mathbf{p}(p_M), \pi_H) \rrbracket \quad (4.10)$$

Observe also that $|\pi_0 - \pi_H| \leq \bar{C} \mathbf{m}_0^{1/2}$, by Proposition 4.1. Join $\pi_H =: \pi(1)$ and $\pi_0 =: \pi(0)$ with a continuous one-parameter family of planes $\pi(t)$ with the property that

$$|\pi(t) - \pi_0| \leq C_0 |\pi_H - \pi_0| \leq \bar{C} \mathbf{m}_0^{1/2}, \quad (4.11)$$

where $C_0 > 0$ is some geometric constant. Since $\bar{C} = \bar{C}(\beta_2, \delta_2, M_0, N_0, C_e)$, it is then clear from (4.11) that, if ε_2 is suitably small, then we have $\mathbf{B}_{6\sqrt{m}} \cap \mathbf{C}_{32r_M}(p_M, \pi_t) \subset \mathbf{C}_{5\sqrt{m}}(0, \pi_0)$ for every $t \in [0, 1]$ (as already argued in the proof of Proposition 4.1). We consider then the

currents $S(t) := (\mathbf{p}_{\pi(t)})_{\#} T \llcorner \mathbf{C}_{32r_M}(p_L, \pi(t))$ and get $S(t) = Q(t) \llbracket B_{34r_M}(\mathbf{p}_{\pi(t)}(p_M), \pi(t)) \rrbracket$, where $Q(t)$ is an integer for every t by the Constancy Theorem. On the other hand $t \mapsto S(t)$ is weakly continuous in the space of currents and thus $Q(t)$ must be constant. Since $Q(0) = Q$ by (1.10), this proves the desired claim.

Observe next that, again from Proposition 4.1,

$$\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_H)) \leq \bar{C} \mathbf{E}(T, \mathbf{B}_L, \pi_H) \leq \bar{C} \mathbf{E}(T, \mathbf{B}_L) + \bar{C} |\pi_H - \hat{\pi}_L|^2 \leq \bar{C} \mathbf{m}_0 \ell(L)^{2-2\delta_2}.$$

If ε_2 is sufficiently small, then $\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_H)) < \varepsilon_1$, where ε_1 is the constant of [5, Theorem 1.4]. Therefore, the current $T \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H)$ and the submanifold Σ satisfy all the assumptions of [5, Theorem 1.4] in the cylinder $\mathbf{C}_{32r_L}(p_L, \pi_H)$: we apply it to construct the π_H -approximation f_{HL} . By [5, Theorem 1.4] and the properties of Ψ_{p_H} , we have

$$\text{Lip}(h_{HL}) \leq C_0 \text{Lip}(\boldsymbol{\eta} \circ f_{HL}) \leq \bar{C} (\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_H)))^{\gamma_1} \leq \bar{C} \mathbf{m}_0^{\gamma_1} \ell(L)^{\gamma_1},$$

and

$$\begin{aligned} \|h_{HL} - \mathbf{p}_{\pi_H^{\pm}}(p_L)\|_{C^0} &\leq C_0 \|\boldsymbol{\eta} \circ f_{HL} - \mathbf{p}_{\pi_H^{\pm}}(p_L)\|_{C^0} \leq C_0 \|\mathcal{G}(f_{HL}, Q \llbracket p_{\pi_H^{\pm}}(p_L) \rrbracket)\|_{C^0} \\ &\leq C_0 \mathbf{h}(T, \mathbf{C}_{32r_L}(p_L, \pi_H)) + (\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_H)))^{1/2} + \mathbf{A} r_L \\ &\leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \end{aligned}$$

Since C does not depend on ε_2 , if the latter is smaller than a suitable positive constant $c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$, we can apply Lemma B.1 to conclude that the interpolating function g_{HL} is well-defined. \square

4.4. Key estimates and proof of Theorem 1.17. We are now ready to state the key construction estimates and show how Theorem 1.17 follows easily from them.

Proposition 4.4 (Construction estimates). *Assume the conclusions of Propositions 4.1 and 4.2 apply and set $\kappa = \min\{\beta_2/4, \varepsilon_0/2\}$. Then, the following holds for any pair of cubes $H, L \in \mathcal{P}^j$ (cf. Definition 1.16), where $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$:*

- (i) $\|g_H\|_{C^0(B)} \leq C \mathbf{m}_0^{1/2m}$ and $\|Dg_H\|_{C^{2,\kappa}(B)} \leq C \mathbf{m}_0^{1/2}$, for $B = B_{4r_H}(x_H, \pi_0)$;
- (ii) if $H \cap L \neq \emptyset$, then $\|g_H - g_L\|_{C^i(B_{r_L}(x_L))} \leq C \mathbf{m}_0^{1/2} \ell(H)^{3+\kappa-i}$ for every $i \in \{0, \dots, 3\}$;
- (iii) $|D^3 g_H(x_H) - D^3 g_L(x_L)| \leq C \mathbf{m}_0^{1/2} |x_H - x_L|^\kappa$;
- (iv) $\|g_H - y_H\|_{C^0} \leq C \mathbf{m}_0^{1/2m} \ell(H)$ and $|\pi_H - T_{(x, g_H(x))} \mathbf{G}_{g_H}| \leq C \mathbf{m}_0^{1/2} \ell(H)^{1-\delta_2} \forall x \in H$;
- (v) if L' is the cube concentric to $L \in \mathcal{W}^j$ with $\ell(L') = \frac{9}{8} \ell(L)$, then

$$\|\varphi_i - g_L\|_{L^1(L')} \leq C \mathbf{m}_0 \ell(L)^{m+3+\beta_2/3} \quad \text{for all } i \geq j.$$

Proof of Theorem 1.17. As in all the proofs so far, we will use C_0 for geometric constants and C for constants which depend upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h . Define $\chi_H := \vartheta_H / (\sum_{L \in \mathcal{P}^j} \vartheta_L)$ for each $H \in \mathcal{P}^j$ and observe that

$$\sum_{H \in \mathcal{P}^j} \chi_H = 1 \quad \text{on } [-4, 4]^m \quad \text{and} \quad \|\chi_H\|_{C^i} \leq C_0 \ell(H)^{-i} \quad \forall i \in \{0, 1, 2, 3, 4\}. \quad (4.12)$$

Set $\mathcal{P}^j(H) := \{L \in \mathcal{P}^j : L \cap H \neq \emptyset\} \setminus \{H\}$ for each $H \in \mathcal{P}^j$. By construction $\frac{1}{2}\ell(L) \leq \ell(H) \leq 2\ell(L)$ for every $L \in \mathcal{P}^j(H)$ and the cardinality of $\mathcal{P}^j(H)$ is bounded by a geometric constant C_0 . The estimate $|\hat{\varphi}_j| \leq C\mathbf{m}_0^{1/2m}$ follows then immediately from Proposition 4.4(i). For $x \in H$ we write

$$\hat{\varphi}_j(x) = \left(g_H \chi_H + \sum_{L \in \mathcal{P}^j(H)} g_L \chi_L \right)(x) = g_H(x) + \sum_{L \in \mathcal{P}^j(H)} (g_L - g_H) \chi_L(x), \quad (4.13)$$

because H does not meet the support of ϑ_L for any $L \in \mathcal{P}^j$ which does not meet H . Using the Leibniz rule, (4.12) and the estimates of Proposition 4.4(i) - (ii), for $i \in \{1, 2, 3\}$ we get

$$\|D^i \hat{\varphi}_j\|_{C^0(H)} \leq \|D^i g_H\|_{C^0} + C_0 \sum_{0 \leq l \leq i} \sum_{L \in \mathcal{P}^j(H)} \|g_L - g_H\|_{C^l(H)} \ell(L)^{l-i} \leq C\mathbf{m}_0^{\frac{1}{2}} (1 + \ell(H)^{3+\kappa-i}),$$

(assuming M_0 is larger than the geometric constant $2\sqrt{m}$, we have $H \subset B_{r_L}(x_L)$ and the estimate of Proposition 4.4(ii) can be applied). Next, using also $\|D^3 g_H - D^3 g_L\|_{C^\kappa(B_{r_L}(x_L))} \leq C\mathbf{m}_0^{1/2}$, we obtain

$$\begin{aligned} [D^3 \hat{\varphi}_j]_{\kappa, H} &\leq C_0 \sum_{0 \leq l \leq 3} \sum_{L \in \mathcal{P}^j(H)} \ell(H)^{l-3} (\ell(H)^{-\kappa} \|D^l(g_L - g_H)\|_{C^0(H)} + [D^l(g_L - g_H)]_{\kappa, H}) \\ &\quad + [D^3 g_H]_{\kappa, H} \leq C\mathbf{m}_0^{1/2}, \end{aligned}$$

where $[a]_{\kappa, D}$ is the usual Hölder seminorm $\sup\{|x - y|^{-\kappa} |a(x) - a(y)| : x \neq y, x, y \in D\}$. Fix now $x, y \in [-4, 4]^m$, let $H, L \in \mathcal{P}^j$ be such that $x \in H$ and $y \in L$. If $H \cap L \neq \emptyset$, then

$$|D^3 \hat{\varphi}_j(x) - D^3 \hat{\varphi}_j(y)| \leq C([D^3 \hat{\varphi}_j]_{\kappa, H} + [D^3 \hat{\varphi}_j]_{\kappa, L}) |x - y|^\kappa. \quad (4.14)$$

If $H \cap L = \emptyset$, we assume w.l.o.g. $\ell(H) \leq \ell(L)$ and observe that

$$\max\{|x - x_H|, |y - x_L|\} \leq \sqrt{m} \ell(L) \leq 2\sqrt{m} |x - y|.$$

Moreover, by construction $\hat{\varphi}_j$ is identically equal to g_H in a neighborhood of its center x_H . Thus, we can estimate

$$\begin{aligned} |D^3 \hat{\varphi}_j(x) - D^3 \hat{\varphi}_j(y)| &\leq |D^3 \hat{\varphi}_j(x) - D^3 \hat{\varphi}_j(x_H)| + |D^3 g_H(x_H) - D^3 g_L(x_L)| + |D^3 \hat{\varphi}_j(x_L) - D^3 \hat{\varphi}_j(y)| \\ &\leq C\mathbf{m}_0^{1/2} (|x - x_H|^\kappa + |x_H - x_L|^\kappa + |y - x_L|^\kappa) \leq C\mathbf{m}_0^{1/2} |x - y|^\kappa, \end{aligned} \quad (4.15)$$

where we used (4.14) and Proposition 4.4(iii). We have thus shown $\|D \hat{\varphi}_j\|_{C^{2, \kappa}} \leq C\mathbf{m}_0^{1/2}$. Since $\varphi_j(x) = (\bar{\varphi}_j(x), \Psi(x, \bar{\varphi}_j(x)))$, where $\bar{\varphi}_j(x)$ denote the first \bar{n} components of $\hat{\varphi}_j(x)$, Theorem 1.17(i) follows easily from the chain rule.

Let $L \in \mathcal{W}^i$ and fix $j \geq i+2$. Observe that, by the inductive procedure defining $\mathcal{S}^j \cup \mathcal{W}^j$, we have $\mathcal{P}^j(L) = \mathcal{P}^{i+2}(L) \subset \mathcal{W}$. Let H be the cube concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$. Then, by Assumption 1.12, $\text{spt}(\vartheta_M) \cap H = \emptyset \forall M \notin \mathcal{P}^j(L)$. Thus, Theorem 1.17(ii) follows.

We now show below that $\|\varphi_j - \varphi_{j+1}\|_{C^0([-4, 4]^m)} \leq C2^{-j}$. This immediately implies the existence of a continuous φ to which φ_j converges uniformly. The bounds of Theorem 1.17(i) immediately implies Theorem 1.17(iii). Fix therefore $x \in [-4, 4]^m$ and assume that

$x \in L \cap H$ with $L \in \mathcal{P}^j$ and $H \in \mathcal{P}^{j+1}$. Without loss of generality, we can make the choice of H and L in such a way that either $H = L$ or H is a son of L . Now, if $\ell(L) \geq 2^{-j+2}$, then by (ii) we have $\varphi_j(x) = \varphi_{j+1}(x)$. Otherwise, from (i) and Proposition 4.4(iv), we can conclude that:

$$\begin{aligned} |\hat{\varphi}_j(x) - \hat{\varphi}_{j+1}(x)| &\leq |\hat{\varphi}_j(x) - \hat{\varphi}_j(x_H)| + |g_H(x_H) - g_L(x_L)| + |\hat{\varphi}_{j+1}(x_L) - \hat{\varphi}_{j+1}(x)| \\ &\leq C(\|\hat{\varphi}_j\|_{C^1} + \|\hat{\varphi}_{j+1}\|_{C^1})2^{-j} + \|g_H - y_H\|_{C^0} + \|g_L - y_L\|_{C^0} + |y_H - y_L| \\ &\leq C\mathbf{m}_0^{1/2m}2^{-j} + |p_H - p_L|. \end{aligned} \quad (4.16)$$

Since $\mathbf{B}_H \subset \mathbf{B}_L$, we conclude $|\hat{\varphi}_j(x) - \hat{\varphi}_{j+1}(x)| \leq C2^{-j}$. Given that Ψ is Lipschitz, we get $\|\varphi_j - \varphi_{j+1}\|_{C^0} \leq C2^{-j}$ and conclude. \square

5. PROOF OF THE THREE KEY CONSTRUCTION ESTIMATES

5.1. Elliptic PDE for the average. This section contains the most important computation, namely the derivation via a first variation argument of a suitable elliptic system for the average of the π -approximations. In order to simplify the notation we introduce the following definition.

Definition 5.1 (Tangential parts). Having fixed $H \in \mathcal{P}^j$ and $\pi := \pi_H \subset T_{p_H}\Sigma$, we let \varkappa be the orthogonal complement of π in $T_{p_H}\Sigma$. For any given point $q \in \mathbb{R}^{m+n}$, any set $\Omega \subset \pi$ and any map $\xi : q + \Omega \rightarrow \pi^\perp$, the map $\mathbf{p}_\varkappa \circ \xi$ will be called the *tangential part* of ξ and usually denoted by $\bar{\xi}$. Analogous notation and terminology will be used for multiple-valued maps.

Proposition 5.2 (Elliptic system). *Assume the conclusions of Proposition 4.1 and 4.2. Let $H \in \mathcal{W}^j \cup \mathcal{S}^j$ and L be either an ancestor or a cube of $\mathcal{W}^j \cap \mathcal{S}^j$ with $H \cap L \neq \emptyset$ (possibly also H itself). Let $f_{HL} : B_{8r_L}(p_L, \pi_H) \rightarrow \mathcal{A}_Q(\pi_H^\perp)$ be the π_H -approximation of T in $\mathbf{C}_{8r_L}(p_L, \pi_H)$, h_{HL} the tilted (H, L) -interpolating functions and \bar{f}_{HL} and \bar{h}_{HL} their tangential parts, according to Definition 5.1. Then, there is a matrix \mathbf{L} , which depends on Σ and H but not on L , such that $|\mathbf{L}| \leq C_0\mathbf{A}^2 \leq C_0\mathbf{m}_0$ for a geometric constant C_0 and (for $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$)*

$$\left| \int (D(\boldsymbol{\eta} \circ \bar{f}_{HL}) : D\zeta + (\mathbf{p}_\pi(x - p_H))^t \cdot \mathbf{L} \cdot \zeta) \right| \leq C\mathbf{m}_0 r_L^{m+1+\beta_2} (r_L \|\zeta\|_{C^1} + \|\zeta\|_{C^0}) \quad (5.1)$$

for every $\zeta \in C_c^\infty(B_{8r_L}(p_L, \pi_L), \varkappa)$. Moreover (for $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$)

$$\|\bar{h}_{HL} - \boldsymbol{\eta} \circ \bar{f}_{HL}\|_{L^1(B_{7r_L}(p_L, \pi_L))} \leq C\mathbf{m}_0 r_L^{m+3+\beta_2}. \quad (5.2)$$

Before coming to the proof we introduce the oscillation of a multivalued function f , which will play an important role also later:

$$\text{osc}(f) := \sup\{|P - P'| : P \in \text{spt}(f(x)), P' \in \text{spt}f(y)\}. \quad (5.3)$$

Observe that the oscillation is comparable to $\sup_{x,y} \mathcal{G}(f(x), f(y))$.

Proof. We use the convention that geometric constants are denoted by C_0 , whereas C denotes constants depending upon the parameters $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h . Set $\pi = \pi_H$. We fix a system of coordinates $(x, y, z) \in \pi \times \mathfrak{z} \times (T_{p_H}\Sigma)^\perp$ so that $p_H = (0, 0, 0)$. Also, in order to simplify the notation, although the domains of the various maps are subsets Ω of $p_L + \pi$, we will from now on consider them as functions of x (i.e. we shift their domains to $\mathbf{p}_\pi(\Omega)$). We also use Ψ_H for the map Ψ_{p_H} of Assumption 1.3. Recall that $\Psi_H(0, 0) = 0$, $D\Psi_H(0, 0) = 0$ and $\|D\Psi_H\|_{C^2, \varepsilon_0} \leq \mathbf{m}_0^{1/2}$. Finally, to simplify the notation we also drop the subscripts HL from the functions f_{HL} , \bar{f}_{HL} and \bar{h}_{HL} (this notation might generate some confusion since h is used in Proposition 4.2 for the smoothed average of f_{HL} ; observe however that the tangential part of such smoothed average does coincide with the tangential part of the tilted (H, L) -approximation).

Given a test function ζ and any point $q = (x, y, z) \in \Sigma$, we consider the vector field $\chi(q) = (0, \zeta(x), D_y\Psi(x, y) \cdot \zeta(x))$. χ is tangent to Σ and therefore $\delta T(\chi) = 0$. Thus,

$$|\delta \mathbf{G}_f(\chi)| = |\delta \mathbf{G}_f(\chi) - \delta T(\chi)| \leq C_0 \int_{\mathbf{C}_{8r_L}(p_L, \pi)} |D\chi| d\|\mathbf{G}_f - T\|. \quad (5.4)$$

Let $r = r_L$ and $B = B_{8r_L}(p_L, \pi)$. Since $\|D\Psi_H\|_0 \leq \mathbf{m}_0^{1/2}$, for ε_2 sufficiently small we achieve $|\chi| \leq 2|\zeta|$ and $|D\chi| \leq 2|\zeta| + 2|D\zeta|$. Set now $E := \mathbf{E}(T, \mathbf{C}_{32r}(p_L, \pi))$ and recall [5, Theorem 1.4] to derive

$$|Df| \leq C_0 E^{\gamma_1} + C_0 r \mathbf{A} \leq C \mathbf{m}_0^{\gamma_1} r^{\gamma_1}, \quad (5.5)$$

$$|f| \leq C_0 \mathbf{h}(T, \mathbf{C}_{32r}(p_L, \pi)) + C_0 (E^{1/2} + r \mathbf{A}) r \leq C \mathbf{m}_0^{1/2m} r^{1+\beta_2}, \quad (5.6)$$

$$\int_B |Df|^2 \leq C_0 r^m E \leq C \mathbf{m}_0 r^{m+2-2\delta_2}, \quad (5.7)$$

and

$$|B \setminus K| \leq C_0 E^{\gamma_1} (E + r^2 \mathbf{A}^2) \leq C \mathbf{m}_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}, \quad (5.8)$$

$$\left| \|T\|(\mathbf{C}_{8r}(p_L, \pi)) - |B| - \frac{1}{2} \int_B |Df|^2 \right| \leq C_0 E^{\gamma_1} (E + r^2 \mathbf{A}^2) \leq C \mathbf{m}_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}, \quad (5.9)$$

where $K \subset B$ is the set

$$B \setminus K = \mathbf{p}_\pi((\text{spt}(T) \Delta \text{Gr}(f)) \cap \mathbf{C}_{8r_L}(p_L, \pi)). \quad (5.10)$$

Concerning (5.6) observe that the statement of [5, Theorem 1.4] bounds indeed $\text{osc}(f)$. However, in our case we have $p_H = (0, 0, 0) \in \text{spt}(T)$ and $\text{spt}(T) \cap \text{Gr}(f) \neq \emptyset$. Thus we conclude $|f| \leq C_0 \text{osc}(f) + C_0 \mathbf{h}(T, \mathbf{C}_{32r}(p_L, \pi))$.

Writing $f = \sum_i \llbracket f_i \rrbracket$ and $\bar{f} = \sum_i \llbracket \bar{f}_i \rrbracket$, $\text{Gr}(f) \subset \Sigma$ implies $f = \sum_i \llbracket (\bar{f}_i, \Psi_H(x, \bar{f}_i)) \rrbracket$. From [6, Theorem 4.1] we can infer that

$$\begin{aligned} \delta \mathbf{G}_f(\chi) &= \int_B \sum_i \left(\underbrace{D_{xy} \Psi_H(x, \bar{f}_i) \cdot \zeta}_{(A)} + \underbrace{(D_{yy} \Psi_H(x, \bar{f}_i) \cdot D_x \bar{f}_i) \cdot \zeta}_{(B)} + \underbrace{D_y \Psi_H(x, \bar{f}_i) \cdot D_x \zeta}_{(C)} \right) \\ &\quad : \left(\underbrace{D_x \Psi_H(x, \bar{f}_i)}_{(D)} + \underbrace{D_y \Psi_H(x, \bar{f}_i) \cdot D_x \bar{f}_i}_{(E)} \right) + \int_B \sum_i D_x \zeta : D_x \bar{f}_i + \text{Err}. \end{aligned} \quad (5.11)$$

To avoid cumbersome notation we use $\|\cdot\|_0$ for $\|\cdot\|_{C^0}$ and $\|\cdot\|_1$ for $\|\cdot\|_{C^1}$. Recalling [6, Theorem 4.1], the error term Err in (5.11) satisfies the inequality

$$|\text{Err}| \leq C \int |D\chi| |Df|^3 \leq \|\zeta\|_1 \int |Df|^3 \leq C \|\zeta\|_1 \mathbf{m}_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}. \quad (5.12)$$

The second integral in (5.11) is obviously $Q \int_B D\zeta : D(\boldsymbol{\eta} \circ \bar{f})$. We therefore expand the product in the first integral and estimate all terms separately. We will greatly profit from the Taylor expansion $D\Psi_H(x, y) = D_x D\Psi_H(0, 0) \cdot x + D_y D\Psi_H(0, 0) \cdot y + O(\mathbf{m}_0^{1/2}(|x|^2 + |y|^2))$. In particular we gather the following estimates:

$$\begin{aligned} |D\Psi_H(x, \bar{f}_i)| &\leq C \mathbf{m}_0^{1/2} r \quad \text{and} \quad D\Psi_H(x, \bar{f}_i) = D_x D\Psi_H(0, 0) \cdot x + O(\mathbf{m}_0^{1/2} r^{1+\beta_2}), \\ |D^2\Psi_H(x, \bar{f}_i)| &\leq \mathbf{m}_0^{1/2} \quad \text{and} \quad D^2\Psi_H(x, \bar{f}_i) = D^2\Psi_H(0, 0) + O(\mathbf{m}_0^{1/2} r). \end{aligned}$$

We are now ready to compute

$$\begin{aligned} \int \sum_i (A) : (D) &= \int \sum_i (D_{xy} \Psi_H(0, 0) \cdot \zeta) : D_x \Psi_H(x, \bar{f}_i) + O(\mathbf{m}_0 r^2 \int |\zeta|) \\ &= \int \sum_i (D_{xy} \Psi_H(0, 0) \cdot \zeta : D_{xx} \Psi_H(0, 0) \cdot x + O(\mathbf{m}_0 r^{1+\beta_2} \int |\zeta|)). \end{aligned} \quad (5.13)$$

Obviously the first integral in (5.13) has the form $\int x^t \cdot \mathbf{L}_{AD} \cdot \zeta$, where the matrix \mathbf{L}_{AD} is a quadratic function of $D^2\Psi_H(0, 0)$. Next, we estimate

$$\int \sum_i (A) : (E) = O(\mathbf{m}_0^{1+\gamma_1} r^{1+\gamma_1} \int |\zeta|), \quad (5.14)$$

$$\int \sum_i (B) : ((D) + (E)) = O(\mathbf{m}_0^{1+\gamma_1} r^{1+\gamma_1} \int |\zeta|), \quad (5.15)$$

$$\int \sum_i (C) : (E) = O(\mathbf{m}_0^{1+\gamma_1} r^{2+\gamma_1} \int |D\zeta|). \quad (5.16)$$

Finally we compute

$$\begin{aligned} \int \sum_i (C) : (D) &= \int \sum_i ((D_{xy}\Psi_H(0,0) \cdot x) \cdot D_x\zeta) : D_x\Psi_H(x, \bar{f}_i) + O\left(\mathbf{m}_0 r^{2+\beta_2} \int |D\zeta|\right) \\ &= \int \sum_i (D_{xy}\Psi_H(0,0) \cdot x) \cdot D_x\zeta : (D_{xx}\Psi_H(0,0) \cdot x) + O\left(\mathbf{m}_0 r^{2+\beta_2} \int |D\zeta|\right). \end{aligned}$$

Integrating by parts the first integral in the last line we reach

$$\int \sum_i (C) : (D) = \int x^t \cdot \mathbf{L}_{CD} \cdot \zeta + O\left(\mathbf{m}_0 r^{2+\beta_2} \int |D\zeta|\right), \quad (5.17)$$

where the matrix \mathbf{L}_{CD} is a quadratic function of $D^2\Psi_H(0,0)$. Set $\mathbf{L} := \mathbf{L}_{AD} + \mathbf{L}_{CD}$. Since $D\Psi_H(0,0) = 0$, \mathbf{L} is in fact a quadratic function of the tensor A_Σ at the point p_H . In order to summarize all our estimates we introduce some simpler notation. We define $\mathbf{f} = \boldsymbol{\eta} \circ \bar{f}$, $\ell := \ell(L)$ and (recalling the set K of (5.10)) the measure μ on B as

$$\mu(E) := |E \setminus K| + \|T\|((E \setminus K) \times \mathbb{R}^n) \quad \text{for every Borel } E \subset B.$$

Since $\|T - \mathbf{G}_f\|(E \times \mathbb{R}^n) \leq C_0\mu(E)$ for every Borel $E \subset B$, we can summarize (5.4) and (5.11) - (5.17) into the following estimate:

$$\begin{aligned} \left| \int (D\mathbf{f} : D\zeta + x^t \cdot \mathbf{L} \cdot \zeta) \right| &\leq C\mathbf{m}_0 r^{1+\beta_2} \int (r|D\zeta(x)| + |\zeta(x)|) dx \\ &\quad + C \int (r|D\zeta(x)| + |\zeta(x)|) (|Df(x)|^3 dx + d\mu(x)). \end{aligned} \quad (5.18)$$

From (5.5) and (5.7) we infer that

$$\int |Df|^3 \leq Cr^m \text{Lip}(f)E \leq C\mathbf{m}_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}. \quad (5.19)$$

Next, observe that

$$\begin{aligned} \mu(B) &= |B \setminus K| + \|T\|((B \setminus K) \times \pi^\perp) \\ &\leq |B \setminus K| + \|T\|(\mathbf{C}_{32r_L}(p_L, \pi)) - \mathbf{M}(\mathbf{G}_f) + \|\mathbf{G}_f\|((B \setminus K) \times \pi^\perp) \\ &\leq C_0|B \setminus K|(1 + \text{Lip}(f)) + \left| \|T\|(\mathbf{C}_{32r_L}(p_L, \pi)) - |B| - \frac{1}{2} \int_B |Df|^2 \right| + C_0 \int_B |Df|^3, \end{aligned}$$

where in the last line we have used the Taylor expansion of the mass of \mathbf{G}_f , cf. [6, Corollary 3.3]. Using next (5.8), (5.9) and (5.19) we conclude

$$\mu(B) \leq C\mathbf{m}_0 r^{m+2-2\delta_2+\gamma_1}. \quad (5.20)$$

Therefore (5.1) follows from (5.18) and our choice of the parameters in Assumption 1.8 (recall, in particular, $\gamma_1 - 2\delta_2 > \beta_2$).

We next come to (5.2). Fix a smooth radial test function $\zeta \in C_c(B_\ell)$ with $\ell = \ell(L)$, and set $\zeta(\cdot) := \zeta(z - \cdot)e_i$, where $e_{m+1}, \dots, e_{m+\bar{n}}$ is an orthonormal base of \mathfrak{x} . Observe

that, if in addition we assume $\int \varsigma = 0$, then $\int x_i \varsigma(z-x) dx = 0$. Under these assumptions, $\int x^t \cdot \mathbf{L} \cdot \varsigma(z-x) dx = 0$ and from (5.18) we get for $z \in B_{7r_L}(p_L, \pi_L)$

$$\begin{aligned} \left| \int_{B_\ell(z)} \langle D\mathbf{f}^i(x), D\varsigma(z-x) \rangle dx \right| &\leq C \int_{B_\ell(z)} |Df|^3(x) (|D\varsigma| + |\varsigma|)(z-x) dx \\ &+ C \int_{B_\ell(z)} (r|D\varsigma| + |\varsigma|)(z-x) d\mu(x) + C\mathbf{m}_0 r^{1+\beta_2} \int_{B_\ell} (r|D\varsigma| + |\varsigma|). \end{aligned} \quad (5.21)$$

Recall the standard estimate on convolutions $\|a * \mu\|_{L^1} \leq \|a\|_{L^1} \mu(B)$, and integrate (5.21) in $z \in B_{7r_L}(p_L, \pi_L)$: by (5.19) and (5.20) (and recalling that $\gamma_1 - 2\delta_2 \geq \beta_2$) we reach

$$\|D\mathbf{f}^i * D\varsigma\|_{L^1(B_{7r_L}(p_L, \pi_L))} \leq C\mathbf{m}_0 r^{m+1+\beta_2} \int_{B_\ell} (r|D\varsigma| + |\varsigma|) \quad \forall \varsigma \in C_c^\infty(B_\ell) \quad \text{with} \quad \int_{B_\ell} \varsigma = 0. \quad (5.22)$$

By a simple density argument, (5.22) holds also when $\varsigma \in W^{1,1}$ is supported in B_ℓ and $\int \varsigma = 0$. Observe next

$$\begin{aligned} \bar{h}(x) - \mathbf{f}(x) &= \int \varrho_\ell(y) (\mathbf{f}(x-y) - \mathbf{f}(x)) dy = \int \varrho_\ell(y) \int_0^1 D\mathbf{f}(x-\sigma y) \cdot (-y) d\sigma dy \\ &= \int \int_0^1 \varrho_\ell\left(\frac{w}{\sigma}\right) D\mathbf{f}(x-w) \cdot \frac{-w}{\sigma^{m+1}} dw = \int D\mathbf{f}(x-w) \cdot \underbrace{(-w) \int_0^1 \varrho_\ell\left(\frac{w}{\sigma}\right) \sigma^{-m-1} d\sigma}_{=: \Upsilon(w)} dw. \end{aligned}$$

Note that Υ is smooth on $\mathbb{R}^m \setminus \{0\}$ and unbounded in a neighborhood of 0. However,

$$\|\Upsilon\|_{L^1} = \int \int_0^1 |w| \left| \varrho\left(\frac{w}{\ell\sigma}\right) \right| \ell^{-m} \sigma^{-m-1} d\sigma dw = \ell \int \int_0^1 |u| |\varrho(u)| d\sigma du \leq Cr. \quad (5.23)$$

Observe also that $\Upsilon(w) = w \psi(|w|)$. Therefore Υ is a gradient. Since $\Upsilon(w)$ vanishes outside a compact set, integrating along rays from ∞ , we can compute a potential for it:

$$\varsigma(w) = \int_{|w|}^\infty \tau \int_0^1 \varrho_\ell\left(\frac{w\tau}{|\tau|\sigma}\right) \sigma^{-m-1} d\sigma d\tau = |w|^2 \int_1^\infty t \int_0^1 \varrho_\ell\left(\frac{wt}{\sigma}\right) \sigma^{-m-1} d\sigma dt. \quad (5.24)$$

Then, ς is a $W^{1,1}$ function, supported in $B_\ell(0)$, $\int \varsigma = 0$ by Assumption 1.12. Summarizing, $\bar{h}^i - \mathbf{f}^i = (D\mathbf{f}^i) * D\varsigma$ for a convolution kernel for which (5.22) holds. Since

$$\begin{aligned} \|\varsigma\|_{L^1} &\leq \int \int_1^\infty \int_0^1 t |w|^2 \left| \varrho\left(\frac{wt}{\sigma}\right) \right| \ell^{-m} \sigma^{-m-1} d\sigma dt dw \\ &= \ell^2 \int_1^\infty \int_0^1 \int |u|^2 |\varrho(u)| du \sigma d\sigma t^{-m-1} dt \leq Cr^2, \end{aligned} \quad (5.25)$$

we then conclude from (5.22) that

$$\int_{B_{7r_L}(p_L, \pi_L)} |\bar{h} - \mathbf{f}| \leq C\mathbf{m}_0 r^{m+1+\beta_2} \int_{B_\ell} (r|D\varsigma| + |\varsigma|) \leq C\mathbf{m}_0 r^{m+3+\beta_2}. \quad \square$$

5.2. C^k estimates for h_{HL} and g_{HL} . Recall the tilted (H, L) -interpolating function h_{HL} and the interpolating function g_{HL} of Definition 4.3.

Lemma 5.3. *Assume that H and L are as in Proposition 5.2 and the hypotheses of Proposition 4.4 hold. Set $B' := B_{5r_H}(p_H, \pi_H)$ and $B := B_{4r_H}(p_H, \pi_0)$. Then, for $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$,*

$$\|h_{HL} - h_H\|_{C^j(B')} + \|g_{HL} - g_H\|_{C^j(B)} \leq C\mathbf{m}_0^{1/2} \ell(L)^{3+2\kappa-j} \quad \forall j \in \{0, \dots, 3\}, \quad (5.26)$$

$$\|h_{HL} - h_H\|_{C^{3,\kappa}(B')} + \|g_{HL} - g_H\|_{C^{3,\kappa}(B)} \leq C\mathbf{m}_0^{1/2} \ell(L)^\kappa. \quad (5.27)$$

As a consequence Proposition 4.4(i) and (iv) hold.

Proof. All the constants C will depend only upon the parameters $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h , unless otherwise specified.

Consider a triple of cubes H, J and L where $H \in \mathcal{S}^j \cup \mathcal{W}^j$ and

- (a) either L is an ancestor of H (possibly H itself) and J is father of L ;
- (b) or J is the father of H , and $L \in \mathcal{S}^j \cup \mathcal{W}^j$ is adjacent to H .

In order to simplify the notation let $\pi := \pi_H$ and $r := r_L$. By Proposition 4.1(i), up to taking the geometric constant in the first inequality of (1.15) sufficiently large, we can assume that $B^b := B_{6r}(p_L, \pi) \subset B^\sharp = B_{13r/2}(p_L, \pi) \subset \bar{B} := B_{7r_J}(p_J, \pi)$. Consider the π -approximations f_{HL} and f_{HJ} , respectively in $\mathbf{C}_{8r}(p_L, \pi)$ and $\mathbf{C}_{8r_J}(p_J, \pi)$, and introduce the corresponding maps

$$\begin{aligned} \bar{\mathbf{f}}_L &:= \mathbf{p}_\pi(\boldsymbol{\eta} \circ f_{HL}) & \text{and} & & \bar{\mathbf{f}}_J &:= \mathbf{p}_\pi(\boldsymbol{\eta} \circ f_{HJ}), \\ \bar{h}_{HL} &:= \bar{\mathbf{f}}_L * \varrho_{\ell(L)} & \text{and} & & \bar{h}_{HJ} &:= \bar{\mathbf{f}}_J * \varrho_{\ell(J)}, \end{aligned}$$

which are the tangential parts of the corresponding maps according to Definition 5.1.

If \mathbf{l} is an affine function on \mathbb{R}^m and ς a radial convolution kernel, then $\varsigma * \mathbf{l} = (\int \varsigma) \mathbf{l}$ because \mathbf{l} is an harmonic function. This means that $\int \langle (\zeta * \varrho), \mathbf{l} \rangle = \int \langle \zeta, \mathbf{l} \rangle$ for any test function ζ and any radial convolution kernel ϱ with integral 1. Similarly $\int \langle (\zeta * \partial^I \varrho), \mathbf{l} \rangle = \int \langle \zeta, \partial^I \mathbf{l} \rangle$ for any partial derivative ∂^I of any order. Consider now a ball \hat{B} concentric to B^b and contained in B^\sharp in such a way that, if $\zeta \in C_c^\infty(\hat{B})$, then $\zeta * \varrho_{\ell(L)}$ and $\zeta * \varrho_{\ell(J)}$ are both supported in B^\sharp . Set $\xi := \bar{h}_{HL} - \bar{h}_{HJ}$ and (assuming $\mathbf{p}_\pi(x_H)$ is the origin of our system of coordinates) compute:

$$\begin{aligned} \int \langle \zeta, \Delta \xi \rangle &= - \int D(\bar{h}_{HL} - \bar{h}_{HJ}) : D\zeta = \int D\bar{\mathbf{f}}_J : D(\zeta * \varrho_{\ell(J)}) - \int D\bar{\mathbf{f}}_L : D(\zeta * \varrho_{\ell(L)}) \\ &= \int (D\bar{\mathbf{f}}_J : D(\zeta * \varrho_{\ell(J)}) + x^t \cdot \mathbf{L} \cdot (\zeta * \varrho_{\ell(J)})) - \int (D\bar{\mathbf{f}}_L : D(\zeta * \varrho_{\ell(L)}) + x^t \cdot \mathbf{L} \cdot (\zeta * \varrho_{\ell(L)})), \end{aligned}$$

where the last line holds for any matrix \mathbf{L} (with constant coefficients) because $x \mapsto x^t \cdot \mathbf{L}$ is a linear function. In particular, we can use the matrix of Proposition 5.2 to achieve

$$\int \langle \zeta, \Delta \xi \rangle \leq C\mathbf{m}_0 r^{m+1+\beta_2} \left(r \|\zeta * \varrho_{\ell(L)}\|_1 + r \|\zeta * \varrho_{\ell(J)}\|_1 + \|\zeta * \varrho_{\ell(J)}\|_0 + \|\zeta * \varrho_{\ell(L)}\|_0 \right),$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ denote the C^0 and C^1 norms respectively. Recalling the inequality $\|\psi * \zeta\|_0 \leq \|\psi\|_\infty \|\zeta\|_{L^1}$ and taking into account that $\ell(L)$ and $\ell(J)$ are both comparable to r (up to a constant depending only on M_0 and m), we achieve $\int \langle \zeta, \Delta \xi \rangle \leq C \mathbf{m}_0 r^{1+\beta_2} \|\zeta\|_{L^1}$. Taking the supremum over all possible test functions with $\|\zeta\|_{L^1} \leq 1$, we obviously conclude $\|\Delta \xi\|_{L^\infty(\hat{B})} \leq C \mathbf{m}_0 r^{1+\beta_2}$. Observe that a similar estimate could be achieved for any partial derivative $D^k \xi$ simply using the identity

$$\int D(D^k(a * \varsigma)) : Db = - \int Da : (Db * D^k \varsigma).$$

Summarizing we conclude

$$\|\Delta D^k(\bar{h}_{HL} - \bar{h}_{HJ})\|_{C^0(\hat{B})} \leq \|\Delta D^k \xi\|_\infty \leq C \mathbf{m}_0 r^{1+\beta_2-k}, \quad (5.28)$$

where the constant C depends upon all the parameters and on $k \in \mathbb{N}$, but not on ε_2 , \mathbf{m}_0 , H , J or L . By [5, Theorem 1.4] (cf. also the proof of Proposition 4.2), we have $\text{osc}(f_{HL}) + \text{osc}(f_{HJ}) \leq C \mathbf{m}_0^{1/2m} r$ and, setting $\mathbf{E} := \mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_H))$ and $\mathbf{E}' := \mathbf{E}(T, \mathbf{C}_{32r_J}(p_J, \pi_H))$,

$$\mathcal{H}^m(\{f_{HL} \neq f_{HJ}\} \cap \hat{B}) \leq C [(\mathbf{E} + \mathbf{A}^2 r^2) \mathbf{E}^{\gamma_1} + (\mathbf{E}' + \mathbf{A}^2 r^2) \mathbf{E}'^{\gamma_1}] r^m \leq C \mathbf{m}_0^{1+\gamma_1} r^{m+2+\gamma_1/2}.$$

Therefore, taking into account (5.2), we conclude $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{L^1(\hat{B})} \leq C \mathbf{m}_0 r^{m+3+\beta_2}$. Thus, we appeal to Lemma C.1 and use the latter estimate together with (5.28) (in the case $k = 0$) to get $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^k(B')}$ $\leq C \mathbf{m}_0 r^{3+\beta_2-k}$ for $k = \{0, 1\}$ and for every concentric smaller ball $B' \subset \hat{B}$ (where the constant depends also on the ratio between the corresponding radii). This implies $\|D(\bar{h}_{HL} - \bar{h}_{HJ})\|_{L^1(B')}$ $\leq C \mathbf{m}_0 r^{m+2+\beta_2}$ and hence we can use again Lemma C.1 (based on the case $k = 1$ of (5.28)) to conclude $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^2(B'')}$ $\leq C \mathbf{m}_0 r^{1+\beta_2}$. Iterating another two times we can then conclude $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^k(B^\sharp)} \leq C \mathbf{m}_0 r^{3+\beta_2-k}$ for $k \in \{0, 1, 2, 3, 4\}$. By interpolation, since $\kappa \leq \beta_2/4$, $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^{3,2\kappa}(B^\sharp)} \leq C \mathbf{m}_0 \ell(L)^{2\kappa}$.

Observe now that, since $h_{HL} = (\bar{h}_{HL}, \Psi(x, \bar{h}_{HL}))$ and $h_{HJ} = (\bar{h}_{HJ}, \Psi_H(x, \bar{h}_{HJ}))$, we deduce the corresponding estimates for h_{HL} and h_{HJ} from the chain rule, namely:

$$\begin{aligned} \|h_{HL} - h_{HJ}\|_{C^j(B^\sharp)} &\leq C \mathbf{m}_0 \ell(L)^{3+2\kappa-j} \quad \forall j \in \{0, \dots, 3\} \\ \|h_{HL} - h_{HJ}\|_{C^{3,2\kappa}(B^\sharp)} &\leq C \mathbf{m}_0 \ell(L)^{2\kappa}. \end{aligned} \quad (5.29)$$

We next want to prove the first estimate of (5.26) and the first estimate of (5.27). We distinguish two cases. In the first L is adjacent to H and has the same side-length. Let then J be the father of H . From the argument above we then know how to bound $h_H - h_{HJ} = h_{HH} - h_{HJ}$ and $h_{HJ} - h_{HL}$. Both estimates follow then from the triangle inequality. In the second case L is an ancestor of H . Let then $H =: L_j \subset L_{j-1} \subset \dots \subset L = L_i$. We then know how to bound $h_{HL_i} - h_{HL_{i-1}}$ on the ball $B^l := B_{13/2r_{L_i}}(p_{L_i}, \pi)$. On the other hand, if the constant in the first inequality of (1.15) is large enough (independently of l), then

$B' \subset B^l$. Summing the corresponding estimates, we get

$$\begin{aligned} \|h_H - h_{HL}\|_{C^{3,2\kappa}(B')} &\leq C \sum_{l=i}^{j-1} \|h_{HL_l} - h_{HL_{l+1}}\|_{C^{3,2\kappa}(B')} \\ &\leq C \mathbf{m}_0 \ell(L)^{2\kappa} \sum_{l=0}^{j-i-1} 2^{-2\kappa l} \leq C \mathbf{m}_0 \ell^{2\kappa}, \end{aligned} \quad (5.30)$$

with a constant C independent of i and j . Obviously a similar estimate holds for $\|h_H - h_{HL}\|_{C^j(B')}$.

We still need to prove the second estimate of (5.26) and the second estimate of (5.27). If H is a fixed cube in the Whitney decomposition and $L_{N_0} \in \mathcal{S}^{N_0}$ its biggest ancestor, we then have $\|h_H - h_{HL_{N_0}}\|_{C^{3,2\kappa}(B')} \leq C \mathbf{m}_0$. On the other hand

$$\|D\mathbf{f}_{HL_{N_0}}\|_{L^2(B^{N_0})}^2 \leq \text{Dir}(f_{HL_{N_0}}) \leq C\mathbf{E}(T, \mathbf{C}_{32r_{L_{N_0}}}(p_{L_{N_0}}, \pi_H)) \leq C\mathbf{m}_0 + C|\pi_H - \pi_0|^2 \leq C\mathbf{m}_0.$$

Thus, by standard convolution estimates, $\|D\bar{h}_{HL_{N_0}}\|_{C^k(B^{N_0})} \leq C\mathbf{m}_0^{1/2}$ (where the constant C depends on $k \in \mathbb{N}$ and on the various parameters). Using (5.30) we then get $\|D\bar{h}_H\|_{C^{2,2\kappa}(B')} \leq \|D\bar{h}_H - D\bar{h}_{HL_{N_0}}\|_{C^{2,2\kappa}(B')} + \|D\bar{h}_{HL_{N_0}}\|_{C^{2,2\kappa}(B^{N_0})} \leq C\mathbf{m}_0^{1/2}$. By the chain rule and the regularity of Ψ we then conclude the general bound $\|Dh_H\|_{C^{3,2\kappa}(B')} \leq C\mathbf{m}_0^{1/2}$. This implies the existence of a constant ξ such that $\|h_H - \xi\|_{C^{3,2\kappa}(B')} \leq C\mathbf{m}_0^{1/2}$. Applying Lemma B.1 we achieve the bound $\|g_H - \zeta\|_{C^{3,2\kappa}(B)} \leq C\mathbf{m}_0^{1/2}$ for some other constant ζ . With a similar argument using the bound $\|\bar{h}_{HL_{N_0}}\|_{C^0(B^{N_0})} \leq C\mathbf{m}_0^{1/2m}$, we achieve $\|\bar{h}_H\|_{C^0(B')} \leq C\mathbf{m}_0^{1/2m}$. Hence again by Lemma B.1 $\|g_H\|_{C^0(B)} \leq C\mathbf{m}_0^{1/2m}$. This shows obviously Proposition 4.4(i).

Next, observe that we have, by the very same arguments, $\|g_{HL} - \zeta\|_{C^{3,2\kappa}(B)} \leq C\mathbf{m}_0^{1/2}$, thus concluding $\|g_{HL} - g_H\|_{C^{3,2\kappa}(B)} \leq C\mathbf{m}_0^{1/2}$. On the other hand, it also follows from the same arguments above that $\|h_{HL} - h_H\|_{L^1(B')} \leq C\mathbf{m}_0 \ell^{m+3+\beta_2} \leq C\mathbf{m}_0 \ell^{m+3+4\kappa}$. Applying Lemma B.1(b) we then conclude $\|g_{HL} - g_H\|_{L^1(B)} \leq C\mathbf{m}_0 \ell^{m+3+4\kappa}$. We can now apply Lemma C.2 to conclude that $\|D^i(g_{HL} - g_H)\|_{C^0(B)} \leq C\mathbf{m}_0^{1/2} \ell^{3-i+4\kappa}$ for every $i \in \{0, 1, 2, 3\}$, reaching the second estimate of (5.26). Interpolating between the latter estimates and $\|g_{HL} - g_H\|_{C^{3,2\kappa}(B)} \leq C\mathbf{m}_0^{1/2}$ we reach as well the second conclusion of (5.27).

Coming to (iv) in Proposition 4.4, the estimate on $g_H - y_H$ is a straightforward consequence of the height bound, [5, Theorem 1.4] and Lemma B.1 (applied to h_H). Next, observe that

$$\|Dh_H\|_{L^2(B')}^2 \leq C(1 + \text{Lip}(\Psi_H)) \|D\bar{h}_H\|_{L^2(B')}^2 + C\|D_x \Psi_H(x, \bar{h})\|_{L^2(B')}^2$$

and

$$\|D\bar{h}_H\|_{L^2(B')}^2 \leq C\|D(\boldsymbol{\eta} \circ f_H)\|_{L^2(B')}^2 \leq C\text{Dir}(f_H, B') \leq C\mathbf{m}_0 \ell(H)^{m+2-2\delta_2}.$$

On the other hand recall that the plane π_H is contained in the plane $T_{p_H}\Sigma$ and thus $D_x\Psi_H(p_H, 0) = 0$. Since $\|D^2\Psi_H\|_0 \leq \mathbf{m}_0^{1/2}$ we obviously conclude that $\|D_x\Psi_H(x, \bar{h})\|_{L^2}^2 \leq C\mathbf{m}_0\ell^{m+2}$. Therefore $\|Dh_H\|_{L^2(B')}^2 \leq C\mathbf{m}_0\ell^{m+2-2\delta_2}$.

Thus, there is at least one point $q \in \text{Gr}(h_H|_{B'})$ such that $|T_q\mathbf{G}_{h_H} - \pi_H| \leq C\mathbf{m}_0^{1/2}\ell(H)^{1-\delta_2}$. Since $\|D^2h_H\|_0 \leq C\mathbf{m}_0^{1/2}$, we then conclude that $|T_{q'}\mathbf{G}_{h_H} - \pi_H| \leq C\mathbf{m}_0^{1/2}\ell(H)^{1-\delta_2}$ holds indeed for any point $q' \in \text{Gr}(h_H|_{B'})$. Since $\text{Gr}(g_H|_B)$ is a subset of $\text{Gr}(h_H|_{B'})$ (with the same orientation!), the second inequality of Proposition 4.4(iv) follows. \square

5.3. Tilted L^1 estimate. In order to achieve Proposition 4.4(ii) and (iii), we need to compare tilted interpolating functions coming from different coordinates. To this aim, we set the following terminology.

Definition 5.4 (Distant relation). Four cubes H, J, L, M make a distant relation between H and L if $J, M \in \mathcal{S}^j \cup \mathcal{W}^j$ have nonempty intersection, H is a descendant of J (or J itself) and L a descendant of M (or M itself).

Lemma 5.5 (Tilted L^1 estimate). *Assume the hypotheses of Proposition 4.4 hold and ε_2 is sufficiently small. Let H, J, L and M be a distant relation between H and L , and let h_{HJ}, h_{LM} be the maps given in Definition 4.3. Then there is a map $\hat{h}_{LM} : B_{4r_J}(p_J, \pi_H) \rightarrow \pi_H^\perp$ such that $\mathbf{G}_{\hat{h}_{LM}} = \mathbf{G}_{h_{LM}} \sqcup \mathbf{C}_{4r_J}(p_J, \pi_H)$ and, for $C = C(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$,*

$$\|h_{HJ} - \hat{h}_{LM}\|_{L^1(B_{2r_J}(p_J, \pi_H))} \leq C\mathbf{m}_0\ell(J)^{m+3+\beta_2/2}. \quad (5.31)$$

Proof. As in the previous proofs we follow the convention that C_0 denotes geometric constants whereas C denotes constants which depend upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h . First observe that Lemma B.1 can be applied because, by Proposition 4.1,

$$|\pi_H - \pi_L| \leq |\pi_H - \pi_J| + |\pi_J - \pi_M| + |\pi_M - \pi_L| \leq C\mathbf{m}_0^{1/2}\ell(J)^{1-\delta_2}.$$

Set $\pi := \pi_H$ and let \varkappa be its orthogonal complement in $T_{p_H}\Sigma$, and similarly $\bar{\pi} = \pi_L$ and $\bar{\varkappa}$ its orthogonal in $T_{p_L}\Sigma$. After a translation we also assume $p_J = 0$, and write $r = r_J = r_M$, $\ell = \ell(J) = \ell(M)$ and $E := \mathbf{E}(T, \mathbf{C}_{32r}(0, \pi))$, $\bar{E} := \mathbf{E}(T, \mathbf{C}_{32r}(p_M, \bar{\pi}))$. Recall that $\max\{E, \bar{E}\} \leq C\mathbf{m}_0\ell^{2-2\delta_2}$. We fix also the maps $\Psi_H : T_{p_H}\Sigma \rightarrow T_{p_H}\Sigma^\perp$ and $\Psi_L : T_{p_L}\Sigma \rightarrow T_{p_L}\Sigma^\perp$ whose graphs coincide with the submanifold Σ . Observe that $|\pi - \bar{\pi}| + |\varkappa - \bar{\varkappa}| \leq C\mathbf{m}_0^{1/2}\ell^{1-\delta_2}$, $\|D\Psi_H\|_{C^{2,\varepsilon_0}} + \|D\Psi_L\|_{C^{2,\varepsilon_0}} \leq C\mathbf{m}_0^{1/2}$ and

$$\ell^{-1} (\|\Psi_H\|_{C^0(B_{8r})} + \|\Psi_L\|_{C^0(B_{8r})}) + \|D\Psi_H\|_{C^0(B_{8r})} + \|D\Psi_L\|_{C^0(B_{8r})} \leq C\mathbf{m}_0^{1/2}\ell.$$

Consider the map $\hat{f}_{LM} : B_{4r}(0, \pi) \rightarrow \mathcal{A}_Q(\pi^\perp)$ such that $\mathbf{G}_{\hat{f}_{LM}} = \mathbf{G}_{f_{LM}} \sqcup \mathbf{C}_{4r}(0, \pi)$, which exists by [6, Proposition 5.2]. Recalling the estimates therein and those of [5, Theorem 1.4],

$$\text{Lip}(f_{HJ}) + \text{Lip}(\hat{f}_{LM}) \leq C\mathbf{m}_0^{\gamma_1}\ell^{\gamma_1} \quad \text{and} \quad |f_{HJ}| + |\hat{f}_{LM}| \leq C\mathbf{m}_0^{1/2m}\ell^{1+\beta_2}, \quad (5.32)$$

$$\text{Dir}(f_{HJ}) + \text{Dir}(\hat{f}_{LM}) \leq C\mathbf{m}_0\ell^{m+2-2\delta_2}. \quad (5.33)$$

Consider next the projections A and \hat{A} onto π of the Borel sets $\text{Gr}(f_{HJ}) \setminus \text{spt}(T)$ and $\text{Gr}(\hat{f}_{LM}) \setminus \text{spt}(T)$. We know from [5, Theorem 1.4] that

$$|A \cup \hat{A}| \leq C \left[\|\mathbf{G}_{f_{HJ}} - T\|(\mathbf{C}_{32}(0, \pi)) + \|\mathbf{G}_{\hat{f}_{LM}} - T\|(\mathbf{C}_{32}(p_M, \bar{\pi})) \right] \leq C \mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1}. \quad (5.34)$$

Recall that

$$\begin{aligned} h_{HJ} &= (\mathbf{p}_\varkappa((\boldsymbol{\eta} \circ f_{HJ}) * \varrho_\ell), \Psi_H(x, \mathbf{p}_\varkappa((\boldsymbol{\eta} \circ f_{HJ}) * \varrho_\ell))) \\ h_{LM} &= (\mathbf{p}_{\bar{\varkappa}}((\boldsymbol{\eta} \circ f_{LM}) * \varrho_\ell), \Psi_L(x, \mathbf{p}_{\bar{\varkappa}}((\boldsymbol{\eta} \circ f_{LM}) * \varrho_\ell))) \end{aligned}$$

and define in addition the maps

$$\begin{aligned} \mathbf{f}_{HJ} &= (\mathbf{p}_\varkappa(\boldsymbol{\eta} \circ f_{HJ}), \Psi_H(x, \mathbf{p}_\varkappa(\boldsymbol{\eta} \circ f_{HJ}))) \\ \mathbf{f}_{LM} &= (\mathbf{p}_{\bar{\varkappa}}(\boldsymbol{\eta} \circ f_{LM}), \Psi_L(x, \mathbf{p}_{\bar{\varkappa}}(\boldsymbol{\eta} \circ f_{LM}))). \end{aligned}$$

Recall that $\hat{h}_{LM} : B_{4r}(0, \pi) \rightarrow \pi^\perp$ satisfies $\mathbf{G}_{\hat{h}_{LM}} = \mathbf{G}_{h_{LM}} \lrcorner \mathbf{C}_{4r}(0, \pi)$ and let $\hat{\mathbf{f}}_{LM}$ be such that $\mathbf{G}_{\hat{\mathbf{f}}_{LM}} = \mathbf{G}_{\mathbf{f}_{LM}} \lrcorner \mathbf{C}_{4r}(0, \pi)$. We use Proposition 5.2, the Lipschitz regularity of Ψ_H and Lemma B.1 to conclude

$$\|\hat{h}_{LM} - \hat{\mathbf{f}}_{LM}\|_{L^1} \leq C \|h_{LM} - \mathbf{f}_{LM}\|_{L^1} \leq C \mathbf{m}_0 r^{m+3+\beta_2}.$$

Likewise $\|h_{HJ} - \mathbf{f}_{HJ}\|_{L^1} \leq C \mathbf{m}_0 r^{m+3+\beta_2}$. We therefore need to estimate $\|\mathbf{f}_{HJ} - \hat{\mathbf{f}}_{LM}\|_{L^1}$. Define next the map $\mathbf{g}_{LM} = (\mathbf{p}_\varkappa(\boldsymbol{\eta} \circ \hat{f}_{LM}), \Psi_H(x, \mathbf{p}_\varkappa(\boldsymbol{\eta} \circ \hat{f}_{LM})))$ and observe that $\|\mathbf{g}_{LM} - \mathbf{f}_{HJ}\|_{L^1} \leq C \|\boldsymbol{\eta} \circ \hat{f}_{LM} - \boldsymbol{\eta} \circ f_{HJ}\|_{L^1}$. On the other hand, since the two maps \hat{f}_{LM} and f_{HJ} differ only on $A \cup \hat{A}$, we can estimate

$$\|\boldsymbol{\eta} \circ \hat{f}_{LM} - \boldsymbol{\eta} \circ f_{HJ}\|_{L^1} \leq C |A \cup \hat{A}| (\|f_{LM}\|_\infty + \|\hat{f}_{HJ}\|_\infty) \leq C \mathbf{m}_0^{1+1/2m} \ell^{3+m+\gamma_1+\beta_2}.$$

It thus suffices to estimate $\|\mathbf{g}_{LM} - \hat{\mathbf{f}}_{LM}\|_{L^1}$. This estimate is independent of the rest and it is an easy consequence of (5.35) in Lemma 5.6 below. \square

Lemma 5.6. *Fix m, n, l and Q . There are geometric constants c_0, C_0 with the following property. Consider two triples of planes (π, \varkappa, ϖ) and $(\bar{\pi}, \bar{\varkappa}, \bar{\varpi})$, where*

- π and $\bar{\pi}$ are m -dimensional;
- \varkappa and $\bar{\varkappa}$ are \bar{n} -dimensional and orthogonal, respectively, to π and $\bar{\pi}$;
- ϖ and $\bar{\varpi}$ l -dimensional and orthogonal, respectively, to $\pi \times \varkappa$ and $\bar{\pi} \times \bar{\varkappa}$.

Assume $\text{An} := |\pi - \bar{\pi}| + |\varkappa - \bar{\varkappa}| \leq c_0$ and let $\Psi : \pi \times \varkappa \rightarrow \varpi$, $\bar{\Psi} : \bar{\pi} \times \bar{\varkappa} \rightarrow \bar{\varpi}$ be two maps whose graphs coincide and such that $|\bar{\Psi}(0)| \leq c_0 r$ and $\|D\bar{\Psi}\|_{C^0} \leq c_0$. Let $u : B_{8r}(0, \bar{\pi}) \rightarrow \mathcal{A}_Q(\bar{\varkappa})$ be a map with $\text{Lip}(u) \leq c_0$ and $\|u\|_{C^0} \leq c_0 r$ and set $f(x) = \sum_i [(u_i(x), \bar{\Psi}(x, u_i(x)))]$ and $\mathbf{f}(x) = (\boldsymbol{\eta} \circ u(x), \bar{\Psi}(x, \boldsymbol{\eta} \circ u(x)))$. Then there are

- a map $\hat{u} : B_{4r}(0, \pi) \rightarrow \mathcal{A}_Q(\varkappa)$ such that the map $\hat{f}(x) := \sum_i [(\hat{u}_i(x), \Psi(x, \hat{u}_i(x)))]$ satisfies $\mathbf{G}_{\hat{f}} = \mathbf{G}_f \lrcorner \mathbf{C}_{4r}(0, \pi)$
- and a map $\hat{\mathbf{f}} : B_{4r}(0, \pi) \rightarrow \varkappa \times \varpi$ such that $\mathbf{G}_{\hat{\mathbf{f}}} = \mathbf{G}_{\mathbf{f}} \lrcorner \mathbf{C}_{4r}(0, \pi)$.

Finally, if $\mathbf{g}(x) := (\boldsymbol{\eta} \circ \hat{u}(x), \Psi(x, \boldsymbol{\eta} \circ \hat{u}(x)))$, then

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1} \leq C_0 (\|f\|_{C^0} + r \text{An}) (\text{Dir}(f) + r^m (\|D\bar{\Psi}\|_{C^0}^2 + \text{An}^2)). \quad (5.35)$$

The proof of the lemma is quite long and we defer it to Appendix D.

5.4. Proof of Proposition 4.4. We are finally ready to complete the proof of Proposition 4.4. Recall that (i) and (iv) have already been shown in Lemma 5.3. In order to show (ii) fix two cubes $H, L \in \mathcal{P}^j$ with nonempty intersection. If $\ell(H) = \ell(L)$, then we can apply Lemma 5.5 to conclude

$$\|h_{HH} - \hat{h}_{LL}\|_{L^1(B_{2r_H}(p_H, \pi_H))} \leq C\mathbf{m}_0 \ell(H)^{m+3+\beta_2/2} \leq C\mathbf{m}_0 \ell(H)^{m+3+2\kappa}. \quad (5.36)$$

If $\ell(H) = \frac{1}{2}\ell(L)$, then let J be the father of H . Obviously, $J \cap L \neq \emptyset$. We can therefore apply Lemma 5.5 above to infer $\|h_{HJ} - \hat{h}_{LL}\|_{L^1(B_{2r_J}(p_J, \pi_H))} \leq C\mathbf{m}_0 \ell(J)^{m+3+\beta_2/2}$. On the other hand, by Lemma 5.3, $\|h_{HH} - h_{HJ}\|_{L^1(B_{2r_H}(p_H, \pi_H))} \leq Cr^m \|h_H - h_{HJ}\|_0 \leq C\mathbf{m}_0 \ell(J)^{m+3+2\kappa}$. Thus we conclude (5.36) as well. Note that $\mathbf{G}_{g_L} \perp \mathbf{C}_{r_H}(x_H, \pi_0) = \mathbf{G}_{\hat{h}_{LL}} \perp \mathbf{C}_{r_H}(x_H, \pi_0)$ and that the same property holds with g_H and h_{HH} . We can thus appeal to Lemma B.1 to conclude

$$\|g_H - g_L\|_{L^1(B_{r_H}(p_H, \pi_0))} \leq C\mathbf{m}_0 \ell(H)^{m+3+2\kappa}. \quad (5.37)$$

However, recall also that $\|D^3(g_H - g_L)\|_{C^\kappa(B_{r_H}(p_H, \pi_0))} \leq C\mathbf{m}_0^{1/2}$. We can then apply Lemma C.2 to conclude (ii).

Now, if $L \in \mathcal{W}^j$ and $i \geq j$, consider the subset $\mathcal{P}^i(L)$ of all cubes in \mathcal{P}^i which intersect L . If L' is the cube concentric to L with $\ell(L') = \frac{9}{8}\ell(L)$, we then have by definition of φ_j :

$$\|\varphi_i - g_L\|_{L^1(L')} \leq C \sum_{H \in \mathcal{P}^i(L)} \|g_H - g_L\|_{L^1(B_{r_L}(p_L, \pi_0))} \leq C\mathbf{m}_0 \ell(H)^{m+3+2\kappa}, \quad (5.38)$$

which is the claim of (v).

As for (iii), observe first that the argument above applies also when L is the father of H . Iterating then the corresponding estimates, it is easy to see that

$$|D^3 g_H(x_H) - D^3 g_J(x_J)| \leq C\mathbf{m}_0^{1/2} \ell(J)^\kappa \quad \text{for any ancestor } J \text{ of } H. \quad (5.39)$$

Fix now any pair $H, L \in \mathcal{P}^j$. Let H_i, L_i be the ‘‘first ancestors’’ of H and L which are adjacent, i.e. among all pairs H', L' of ancestors of H and L with same side-length and nonempty intersection, we assume that the side-length ℓ of H_i, L_i is the smallest possible. We can therefore use the estimates obtained so far to conclude

$$\begin{aligned} |D^3 g_H(x_H) - D^3 g_L(x_L)| &\leq |D^3 g_H(x_H) - D^3 g_{H_i}(x_{H_i})| + |D^3 g_{H_i}(x_{H_i}) - D^3 g_{L_i}(x_{L_i})| \\ &\quad + |D^3 g_{L_i}(x_{L_i}) - D^3 g_L(x_L)| \leq C\mathbf{m}_0^{1/2} \ell^\kappa. \end{aligned}$$

A simple geometric consideration shows that $|x_L - x_H| \geq c_0 \ell$, where c_0 is a dimensional constant, thus completing the proof.

6. EXISTENCE AND ESTIMATES FOR THE \mathcal{M} -NORMAL APPROXIMATION

In this section we continue using the convention that C denotes constants which depend upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h , whereas C_0 denotes geometric constants.

6.1. Proof of Corollary 2.2. The first two statements of (i) follow immediately from Theorem 1.17(i) and Proposition 4.1(v). Coming to the third claim of (i), we extend the function φ to the entire plane π_0 by increasing its $C^{3,\kappa}$ norm by a constant geometric factor. Let $\varphi_t(x) := t\varphi(x)$ for $t \in [0, 1]$, $\mathcal{M}_t := \text{Gr}(\varphi_t|_{]-4,4[^m})$ and set

$$\mathbf{U}_t := \{x + y : x \in \mathcal{M}_t, y \perp T_x \mathcal{M}_t, |y| < 1\}.$$

For ε_2 sufficiently small the orthogonal projection $\mathbf{p}_t : \mathbf{U}_t \rightarrow \mathcal{M}_t$ is a well-defined $C^{2,\kappa}$ map for every $t \in [0, 1]$, which depends smoothly on t . It is also easy to see that $\partial T \lrcorner \mathbf{U}_t = 0$. Thus, $(\mathbf{p}_t)_\#(T \lrcorner \mathbf{U}_t) = Q(t) \llbracket \mathcal{M}_t \rrbracket$ for some integer $Q(t)$. On the other hand these currents depend continuously on t and therefore $Q(t)$ must be a constant. Since $\mathcal{M}_0 =]-4, 4[^m \times \{0\} \subset \pi_0$ and $\mathbf{p}_0 = \mathbf{p}_{\pi_0}$, we conclude $Q(0) = Q$.

With regard to (ii), consider $q \in L \in \mathscr{W}$, set $p := \Phi(q)$ and $\pi := T_p \mathcal{M}$, whereas π_L is as in Definition 1.16. Let J be the cube concentric to L and with side-length $\frac{17}{16}\ell(L)$. By the definition of φ , Theorem 1.17(ii) and Proposition 4.4, we have that, denoting by $\bar{\varphi}$ and \bar{g}_L the first \bar{n} components of the corresponding maps,

$$\|\bar{\varphi} - \bar{g}_L\|_{C^0(J)} \leq \sum_{H \in \mathscr{W}, H \cap L \neq \emptyset} \|g_L - g_H\|_{C^0(J)} \leq C\mathbf{m}_0^{1/2} \ell(L)^{3+\kappa}.$$

So, since $\varphi = (\bar{\varphi}, \Psi(x, \bar{\varphi}))$ and $g_H = (\bar{g}_H, \Psi(x, \bar{g}_H))$, we conclude $\|g_L - \varphi\|_{C^0(J)} \leq C\mathbf{m}_0^{1/2} \ell(L)^{3+\kappa}$. On the other the graph of g_L coincides with the graph of the tilted interpolating function h_L . Consider in $\mathbf{C} := \mathbf{C}_{8r_L}(p_L, \pi_L)$ the π_L -approximation f_L used in the construction algorithm and recall that, by [5, Theorem 1.4].

$$\begin{aligned} \text{osc}(f_L) &\leq C_0 (\mathbf{h}(T, \mathbf{C}_{32r_L}(p_L, \pi_L), \pi_L) + ((\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_L)))^{1/2} + r_L \mathbf{A})r_L) \\ &\leq C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \end{aligned}$$

Recall that $p_L = (z_L, w_L) \in \pi_L \times \pi_L^\perp$ belongs to $\text{spt}(T)$, so we easily conclude that $\|\eta \circ f_L - w_L\|_{C^0} \leq C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$. This implies $\|h_L - w_L\|_{C^0} \leq C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$. Putting all these estimates together, we easily conclude that, for any point p in $\text{spt}(T) \cap \mathbf{C}_{7r_L}(p_L, \pi_L)$ the distance to the graph of h_L is at most $C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$. This shows the claim if we can prove that $\text{spt}(\langle T, \mathbf{p}, p \rangle) \subset \mathbf{B}_{r_L}(p) \subset \mathbf{C}_{7r_L}(p_L, \pi_L)$, for which we argue by contradiction. Assuming the opposite, there is a $p' \in \text{spt}(\langle T, \mathbf{p}, p \rangle)$ and an ancestor J with largest sidelength among those for which $|p' - p| \geq r_J$. Let π be the tangent to \mathcal{M} at p and observe that we have the estimates $|\pi - \pi_J| \leq C\mathbf{m}_0^{1/2}$ and $|\pi - \pi_0| \leq C\mathbf{m}_0^{1/2}$. If J were an element of \mathscr{S}^{N_0} , the height bound (1.11) would imply $|p' - p| \leq C\mathbf{m}_0^{1/2m}$. If $J \notin \mathscr{S}^{N_0}$ and we let H be the father of J , we then conclude that $q \in \mathbf{B}_H$ and thus we have $|p' - p| \leq C\mathbf{h}(T, \mathbf{B}_H) \leq C\mathbf{m}_0^{1/2m} \ell(H)^{1+\beta_2}$. In both cases this would be incompatible with $|p' - p| \geq r_J$, provided $\varepsilon_2 \leq c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$.

Finally, we show (iii). Fix a point $p \in \Gamma$. By construction, there is an infinite chain $L_{N_0} \supset L_{N_0+1} \supset \dots \supset L_j \supset \dots$ of cubes $L_j \in \mathscr{S}^j$ such that $\{p\} = \bigcap_j L_j$. Set $\pi_j := \pi_{L_j}$. From Proposition 4.1 we infer that the planes π_j converge to a plane π with a rate $|\pi_j - \pi| \leq C\mathbf{m}_0^{1/2} 2^{-j(1-\delta_2)}$. Moreover, the rescaled currents $(\iota_{p_{L_j}, 2^{-j}})_\# T$ (where the map

$\iota_{q,r}$ is given by $\iota_{q,r}(z) = \frac{z-q}{r}$ converge to $Q \llbracket \pi \rrbracket$. Since $|\Phi(p) - p_{L_j}| \leq C\sqrt{m}2^{-j}$ for some constant C independent of j , we easily conclude that $\Theta(T, \Phi(p)) = Q$ and $Q \llbracket \pi \rrbracket$ is the unique tangent cone to T at $\Phi(p)$. We next show that $\mathbf{p}^{-1}(\Phi(p)) \cap \text{spt}(T) = \{\Phi(p)\}$. Indeed, assume there were $q \neq \Phi(p)$ which belongs to $\text{spt}(T)$ and such that $\mathbf{p}(q) = \Phi(p)$. Let j be such that $2^{-j-1} \leq |\Phi(p) - q| \leq 2^{-j}$. Provided ε_2 is sufficiently small, Proposition 4.1(v) guarantees that $j \geq N_0$. Consider the cube L_j in the chain above and recall that $\mathbf{h}(T, \mathbf{C}_{32r_{L_j}}(p_{L_j}, \pi_j)) \leq C\mathbf{m}_0^{1/2m}2^{-j(1+\beta_2)}$. Hence,

$$\begin{aligned} 2^{-j-1} &\leq |q - \Phi(p)| = |\mathbf{p}_{\pi^\perp}(q - \Phi(p))| \leq C_0|q - \Phi(p)||\pi - \pi_j| + \mathbf{h}(T, \mathbf{C}_{32r_{L_j}}(p_{L_j}, \pi_j)) \\ &\leq C\mathbf{m}_0^{1/2}2^{-j(1-\delta_2)}2^{-j} + C\mathbf{m}_0^{1/2m}2^{-j(1+\beta_2)} \leq C\varepsilon_2^{1/2m}2^{-j}, \end{aligned}$$

which, for an appropriate choice of ε_2 (depending only on the various other parameters $\beta_2, \delta_2, C_e, C_h, M_0, N_0$) is a contradiction.

6.2. Construction of the \mathcal{M} -normal approximation and first estimates. We set $F(p) = Q \llbracket p \rrbracket$ for $p \in \Phi(\Gamma)$. For every $L \in \mathscr{W}^j$ consider the π_L -approximating function $f_L : \mathbf{C}_{8r_L}(p_L, \pi_L) \rightarrow \mathcal{A}_Q(\pi_L^\perp)$ of Definition 1.13 and $K_L \subset B_{8r_L}(p_L, \pi_L)$ the projection on π_L of $\text{spt}(T) \cap \text{Gr}(f_L)$. In particular we have $\mathbf{G}_{f_L|_{K_L}} = T \llcorner (K_L \times \pi_L^\perp)$. We then denote by $\mathscr{D}(L)$ the portions of the supports of T and $\text{Gr}(f_L)$ which differ:

$$\mathscr{D}(L) := (\text{spt}(T) \cup \text{Gr}(f_L)) \cap [(B_{8r_L}(p_L, \pi_L) \setminus K_L) \times \pi_L^\perp].$$

Observe that, by [5, Theorem 1.4] and Assumption 1.8, we have

$$\mathcal{H}^m(\mathscr{D}(L)) + \|T\|(\mathscr{D}(L)) \leq C_0 E^{\gamma_1} (E + \ell(L)^2 \mathbf{A}^2) \ell(L)^m \leq C\mathbf{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}, \quad (6.1)$$

where $E = \mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_L))$ (cf. with (5.20)). Let \mathcal{L} be the Whitney region in Definition 1.18 and set $\mathcal{L}' := \Phi(J)$ where J is the cube concentric to L with $\ell(J) = \frac{9}{8}\ell(L)$. Observe that our choice of the constants is done in such a way that,

$$L \cap H = \emptyset \iff \mathcal{L}' \cap \mathcal{H}' = \emptyset \quad \forall H, L \in \mathscr{W}, \quad (6.2)$$

$$\Phi(\Gamma) \cap \mathcal{L}' = \emptyset \quad \forall L \in \mathscr{W}. \quad (6.3)$$

We then apply [6, Theorem 5.1] to obtain maps $F_L : \mathcal{L}' \rightarrow \mathcal{A}_Q(\mathbf{U})$, $N_L : \mathcal{L}' \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$ with the following properties:

- $F_L(p) = \sum_i \llbracket p + (N_L)_i(p) \rrbracket$,
- $(N_L)_i(p) \perp T_p \mathcal{M}$ for every $p \in \mathcal{L}'$
- and $\mathbf{G}_{f_L} \llcorner (\mathbf{p}^{-1}(\mathcal{L}')) = \mathbf{T}_{F_L} \llcorner (\mathbf{p}^{-1}(\mathcal{L}'))$.

For each L consider the set $\mathscr{W}(L)$ of elements in \mathscr{W} which have a nonempty intersection with L . We then define the set \mathcal{K} in the following way:

$$\mathcal{K} = \mathcal{M} \setminus \left(\bigcup_{L \in \mathscr{W}} \left(\mathcal{L}' \cap \bigcup_{M \in \mathscr{W}(L)} \mathbf{p}(\mathscr{D}(M)) \right) \right). \quad (6.4)$$

In other words \mathcal{K} is obtained from \mathcal{M} by removing in each \mathcal{L}' those points x for which there is a neighboring cube M such that the slice of \mathbf{T}_{F_M} at x (relative to the projection \mathbf{p}) does not coincide with the slice of T . Observe that, by (6.3), \mathcal{K} contains necessarily $\Phi(\Gamma)$.

Moreover, recall that $\text{Lip}(\mathbf{p}) \leq C$, that the cardinality $\mathscr{W}(L)$ is bounded by a geometric constant and that each element of $\mathscr{W}(L)$ has side-length at most twice that of L . Thus (6.1) implies

$$|\mathcal{L} \setminus \mathcal{K}| \leq |\mathcal{L}' \setminus \mathcal{K}| \leq \sum_{M \in \mathscr{W}(L)} \sum_{H \in \mathscr{W}(M)} \mathbf{p}(\mathcal{D}(H)) \leq C \mathbf{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}. \quad (6.5)$$

On $\Phi(\mathbf{\Gamma})$ we define $F(p) = Q \llbracket p \rrbracket$. By (6.2), if J and L are such that $\mathcal{J}' \cap \mathcal{L}' \neq \emptyset$, then $J \in \mathscr{W}(L)$ and therefore $F_L = F_J$ on $\mathcal{K} \cap (\mathcal{J}' \cap \mathcal{L}')$. We can therefore define a unique map on \mathcal{K} by simply setting $F(p) = F_L(p)$ if $p \in \mathcal{K} \cap \mathcal{L}'$. Our resulting map has the Lipschitz bound of (2.1) in each $\mathcal{L} \cap \mathcal{K}$. Indeed, notice that, by the C^2 estimate on φ and Proposition 4.4(iv), \mathcal{M} is given on $\mathbf{C}_{r_L}(p_L, \pi_L)$ as the graph of a map $\varphi' : B_{r_L}(p_L, \pi_L) \rightarrow \pi_L^\perp$ with $\|D\varphi'\|_{C^0} \leq C \mathbf{m}_0^{1/2} \ell(H)^{1-\delta_1}$ and $\|D^2\varphi'\|_{C^0} \leq C \mathbf{m}_0^{1/2}$. Hence, the Lipschitz constant of N_L can be estimated using [6, Theorem 5.1] as

$$\text{Lip}(N_L) \leq C (\|D^2\varphi'\|_{C^0} \|N\|_{C^0} + \|D\varphi'\|_{C^0} + \text{Lip}(f_L)) \leq C \mathbf{m}_0^{\gamma_2} \ell(L)^{\gamma_2}, \quad (6.6)$$

Moreover, $\mathbf{T}_F = T \perp \mathbf{p}^{-1}(\mathcal{K})$, which implies two facts. First, by Corollary 2.2(ii) we also have that $N(p) := \sum_i \llbracket F_i(p) - p \rrbracket$ enjoys the bound $\|N|_{\mathcal{L} \cap \mathcal{K}}\|_{C^0} \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$. Secondly,

$$\|T\|(\mathbf{p}^{-1}(\mathcal{L} \setminus \mathcal{K})) \leq \sum_{M \in \mathscr{W}(L)} \sum_{H \in \mathscr{W}(M)} \|T\|(\mathcal{D}(H)) \leq C \mathbf{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}. \quad (6.7)$$

Hence, F and N satisfy the bounds (2.1) on \mathcal{K} . We next extend them to the whole center manifold and conclude (2.2) from (6.7) and (6.5). The extension is achieved in three steps:

- we first extend the map F to a map \bar{F} taking values in $\mathcal{A}_Q(\mathbf{U})$;
- we then modify \bar{F} to achieve the form $\hat{F}(x) = \sum_i \llbracket x + \hat{N}_i(x) \rrbracket$ with $\hat{N}_i(x) \perp T_x \mathcal{M}$ for every x ;
- we finally modify \hat{F} to reach the desired extension $F(x) = \sum_i \llbracket x + N_i(x) \rrbracket$, with $N_i(x) \perp T_x \mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every x .

First extension. We use on \mathcal{M} the coordinates induced by its graphical structure, i.e. we work with variables in flat domains. Note that the domain parameterizing the Whitney region for $L \in \mathscr{W}$ is then the cube concentric to L and with side-length $\frac{17}{16} \ell(L)$. The multivalued map N is extended to a multivalued \bar{N} inductively to appropriate neighborhoods of the skeleta of the Whitney decomposition (a similar argument has been used in [4, Section 1.2.2]). The extension of F will obviously be $\bar{F}(x) = \sum_i \llbracket \bar{N}_i(x) + x \rrbracket$. The neighborhoods of the skeleta are defined in this way:

- (1) if p belongs to the 0-skeleton, we let $L \in \mathscr{W}$ be (one of) the smallest cubes containing it and define $U^p := B_{\ell(L)/16}(p)$;
- (2) if $\sigma = [p, q] \subset L$ is the edge of a cube and $L \in \mathscr{W}$ is (one of) the smallest cube intersecting σ , we then define U^σ to be the neighborhood of size $\frac{1}{4} \frac{\ell(L)}{16}$ of σ minus the closure of the unions of the U^r 's, where r runs in the 0-skeleton;

- (3) we proceed inductively till the $m - 1$ -skeleton: given a k -dimensional facet σ and (one of) the smallest cube $L \in \mathscr{W}$ which intersects it, U^σ is its neighborhood of size $4^{-k} \frac{\ell(L)}{16}$ minus the closure of the union of all U^τ 's, where τ runs among all facets of dimension at most $k - 1$.

Denote by \bar{U} the closure of the union of all these neighborhoods and let $\{V_i\}$ be the connected components of the complement. For each V_i there is a $L_i \in \mathscr{W}$ such that $V_i \subset L_i$. Moreover, V_i has distance $c_0 \ell(L)$ from ∂L_i , where c_0 is a geometric constant. It is also clear that if τ and σ are two distinct facets of the same cube L with the same dimension, then the distance between any pair of points x, y with $x \in U^\tau$ and $y \in U^\sigma$ is at least $c_0 \ell(L)$. In Figure 1 the various domains are shown in a piece of a 2-dimensional decomposition.

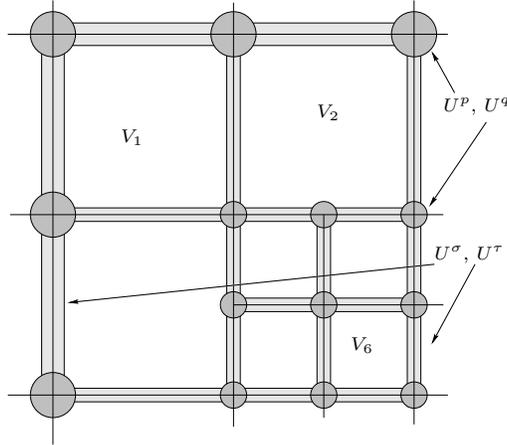


FIGURE 1. The sets U^p , U^σ and V_i .

At a first step we extend N to a new map \bar{N} separately on each U^p , where p are the points in the 0-skeleton. Fix $p \in L$ and let $\text{St}(p)$ be the union of all cubes which contain p . Observe that the Lipschitz constant of $N|_{\mathcal{K} \cap \text{St}(p)}$ is smaller than $C\mathbf{m}_0^{\gamma^2} \ell(L)^{\gamma^2}$ and that $|N| \leq C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$ on $\text{St}(p)$. We can therefore extend the map N to U^p at the price of slightly enlarging this Lipschitz constant and this height bound, using [4, Theorem 1.7]. Being the U^p disjoint, the resulting map, for which we use the symbol \bar{N} , is well-defined.

It is obvious that this map has the desired height bound in each Whitney region. We therefore want to estimate its Lipschitz constant. Consider $L \in \mathscr{W}$ and H concentric to L with side-length $\ell(H) = \frac{17}{16} \ell(L)$. Let $x, y \in H$. If $x, y \in \mathcal{K}$, then there is nothing to check. If $y \in U^p$ for some p and $x \notin \bigcup_q U^q$, then $x \in \text{St}(p)$ and $\mathcal{G}(\bar{N}(x), \bar{N}(y)) \leq C\mathbf{m}_0^{\gamma^2} \ell(L)^{\gamma^2} |x - y|$. The same holds when $x, y \in U^p$. The remaining case is $x \in U^p$ and $y \in U^q$ with $p \neq q$. Observe however that this would imply that p, q are both vertices of L . Given that $L \setminus \mathcal{K}$ has much smaller measure than L there is at least one point $z \in L \cap \mathcal{K}$. It is then obvious that

$$\mathcal{G}(\bar{N}(x), \bar{N}(y)) \leq \mathcal{G}(\bar{N}(x), \bar{N}(z)) + \mathcal{G}(\bar{N}(z), \bar{N}(y)) \leq C\mathbf{m}_0^{\gamma^2} \ell(L)^{\gamma^2} \ell(L),$$

and, since $|x - y| \geq c_0 \ell(L)$, the desired bound readily follows. Observe moreover that, if x is in the closure of some U^q , then we can extend the map continuously to it. By the properties of the Whitney decomposition it follows that the union of the closures of the U^q and of \mathcal{K} is closed and thus, w.l.o.g., we can assume that the domain of this new \bar{N} is in fact closed.

This procedure can now be iterated over all skeleta inductively on the dimension k of the corresponding skeleton, up to $k = m - 1$: in the argument above we simply replace points p with k -dimensional faces σ , defining $\text{St}(\sigma)$ as the union of the cubes which contain σ . In the final step we then extend over the domains V_i 's: this time $\text{St}(V_i)$ will be defined as the union of the cubes which intersect the cube $L_i \supset V_i$. The correct height and Lipschitz bounds follow from the same arguments. Since the algorithm is applied $m + 1$ times, the original constants have been enlarged by a geometric factor.

Second extension: orthogonality. For each $x \in \mathcal{M}$ let $\mathbf{p}^\perp(x, \cdot) : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ be the orthogonal projection on $(T_x \mathcal{M})^\perp$ and set $\hat{N}(x) = \sum_i \llbracket \mathbf{p}^\perp(x, \bar{N}_i(x)) \rrbracket$. Obviously $|\hat{N}(x)| \leq |\bar{N}(x)|$, so the L^∞ bound is trivial. We now want to show the estimate on the Lipschitz constant. To this aim, fix two points p, q in the same Whitney region associated to L and parameterize the corresponding geodesic segment $\sigma \subset \mathcal{M}$ by arc-length $\gamma : [0, d(p, q)] \rightarrow \sigma$, where $d(p, q)$ denotes the geodesic distance on \mathcal{M} . Use [4, Proposition 1.2] to select Q Lipschitz functions $N'_i : \sigma \rightarrow \mathbf{U}$ such that $\bar{N}|_\gamma = \sum \llbracket N'_i \rrbracket$ and $\text{Lip}(N'_i) \leq \text{Lip}(\bar{N})$. Fix a frame ν_1, \dots, ν_n on the normal bundle of \mathcal{M} with the property that $\|D\nu_i\|_{C^0} \leq C\mathbf{m}_0^{1/2m}$ (which is possible since \mathcal{M} is the graph of a $C^{3,\kappa}$ function, cf. [6, Appendix A]). We have $\hat{N}(\gamma(t)) = \sum_i \llbracket \hat{N}_i(t) \rrbracket$, where

$$\hat{N}_i(t) = \sum_j [\nu_j(\gamma(t)) \cdot N'_i(\gamma(t))] \nu_j(t).$$

Hence we can estimate

$$\left| \frac{d\hat{N}_i}{dt} \right| \leq C_0 \text{Lip}(N'_i) + C_0 \sum_j \|D\nu_j\| \|N'_i\|_{C^0} \leq C\mathbf{m}_0^{\gamma_2} \ell(L)^{\gamma_2} + C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} \leq C\mathbf{m}_0^{\gamma_2} \ell(L)^{\gamma_2}.$$

Integrating this inequality we find

$$\mathcal{G}(\hat{N}(p), \hat{N}(q)) \leq C_0 \sum_{i=1}^Q |\hat{N}_i(d(p, q)) - \hat{N}_i(0)| \leq C\mathbf{m}_0^{\gamma_2} \ell(L)^{\gamma_2} d(p, q).$$

Since $d(p, q)$ is comparable to $|p - q|$, we achieve the desired Lipschitz bound.

Third extension and conclusion. For each $x \in \mathcal{M} \subset \Sigma$ consider the orthogonal complement \varkappa_x of $T_x \mathcal{M}$ in $T_x \Sigma$. Let \mathcal{T} be the fiber bundle $\bigcup_{x \in \mathcal{M}} \varkappa_x$ and observe that, by the regularity of both \mathcal{M} and Σ there is a global $C^{2,\kappa}$ trivialization (argue as in [6, Appendix A]). It is then obvious that there is a $C^{2,\kappa}$ map $\Xi : \mathcal{T} \rightarrow \mathbb{R}^{m+n}$ with the following property: for each (x, v) , $q := x + \Xi(x, v)$ is the only point in Σ which is orthogonal to $T_x \mathcal{M}$ and such that $\mathbf{p}_{\varkappa_x}(q - x) = v$. We then set $N(x) = \sum_i \llbracket \Xi(x, \mathbf{p}_{\varkappa_x}(\hat{N}_i(x))) \rrbracket$. Obviously, $N(x) = \hat{N}(x)$ for $x \in \mathcal{K}$, simply because in this case $x + \hat{N}_i(x)$ belongs to Σ .

In order to show the Lipschitz bound, denote by $\Omega(x, q)$ the map $\Xi(x, \mathbf{p}_{x_x}(q))$. Ω is a $C^{2,\kappa}$ map. Thus

$$|\Omega(x, q) - \Omega(x, p)| \leq C_0|q - p|. \quad (6.8)$$

Moreover, since $\Omega(x, 0) = 0$ for every x , we have $D_x\Omega(x, 0) = 0$. We therefore conclude that $|D_x\Omega(x, q)| \leq C_0|q|$ and hence that

$$|\Omega(x, q) - \Omega(y, q)| \leq C_0|q||y - x|. \quad (6.9)$$

Thus, fix two points $x, y \in \mathcal{L}$ and let us assume that $\mathcal{G}(\hat{N}(x), \hat{N}(y))^2 = \sum_i |\hat{N}_i(x) - \hat{N}_i(y)|^2$ (which can be achieved by a simple relabeling). We then conclude

$$\begin{aligned} \mathcal{G}(N(x), N(y))^2 &\leq 2 \sum_i |\Omega(x, \hat{N}_i(x)) - \Omega(x, \hat{N}_i(y))|^2 + 2 \sum_i |\Omega(x, \hat{N}_i(y)) - \Omega(y, \hat{N}_i(y))|^2 \\ &\leq C_0\mathcal{G}(\hat{N}(x), \hat{N}(y))^2 + C \sum_i |\hat{N}_i(y)|^2|x - y|^2 \\ &\leq C\mathbf{m}_0^{2\gamma_2}\ell(L)^{2\gamma_2}|x - y|^2 + C\mathbf{m}_0^{1/2m}\ell(L)^{1+\beta_2}|x - y|^2. \end{aligned} \quad (6.10)$$

This proves the desired Lipschitz bound. Finally, using the fact that $\Omega(x, 0) = 0$, we have $|\Omega(x, v)| \leq C_0|v|$ and the L^∞ bound readily follows.

6.3. Estimates (2.3) and (2.4). First consider the cylinder $\mathbf{C} := \mathbf{C}_{8r_L}(p_L, \pi_L)$. Denote by $\vec{\mathcal{M}}$ the unit m -vector orienting $T\mathcal{M}$ and by $\vec{\tau}$ the one orienting $T\mathbf{G}_{h_L} = T\mathbf{G}_{g_L}$. Recalling that g_L and φ coincide in a neighborhood of x_L , we have

$$\sup_{p \in \mathcal{M} \cap \mathbf{C}} |\vec{\tau}(x_L, g_L(x_L)) - \vec{\mathcal{M}}(p)| \leq C\|D^2\varphi\|_{C^0}\ell(L) \leq C\mathbf{m}_0^{1/2}\ell(L).$$

Since $\|D^2h_L\| \leq C\mathbf{m}_0^{1/2}$ we have $|\vec{\tau}(x_L, g_L(x_L)) - \vec{\tau}(q)| \leq C\mathbf{m}_0^{1/2}\ell(L) \forall q \in \mathcal{M} \cap \mathbf{C}$. Combining the last two inequalities with Proposition 4.4(iv) we infer $\sup_{\mathbf{C} \cap \mathcal{M}} |\vec{\mathcal{M}} - \pi_L| \leq C\mathbf{m}_0^{1/2}\ell(L)^{1-\delta_2}$. Thus, since $\mathbf{p}^{-1}(\mathcal{L}) \cap \text{spt}(T) \subset \mathbf{C}$, we can estimate

$$\begin{aligned} &\int_{\mathbf{p}^{-1}(\mathcal{L})} |\vec{\mathbf{T}}_F(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2 d\|\mathbf{T}_F\|(x) \\ &\leq \int_{\mathbf{p}^{-1}(\mathcal{L})} |\vec{T}(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2 d\|T\|(x) + C\mathbf{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2} \\ &\leq \int_{\mathbf{p}^{-1}(\mathcal{L})} |\vec{T}(x) - \vec{\pi}_L|^2 d\|T\|(x) + C\mathbf{m}_0\ell(L)^{m+2-2\delta_2} \end{aligned} \quad (6.11)$$

In turn the integral in (6.11) is smaller than $C\ell(L)^m \mathbf{E}(T, \mathbf{C}, \pi_L)$. By [6, Proposition 3.4] we then conclude

$$\begin{aligned} \int_{\mathcal{L}} |DN|^2 &\leq C_0 \int_{\mathbf{p}^{-1}(\mathcal{L})} |\vec{\mathbf{T}}_F(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2 d\|\mathbf{T}_F\|(x) + C_0 \|A_{\mathcal{M}}\|_{C^0}^2 \int_{\mathcal{L}} |N|^2 \\ &\quad + C_0 \text{Lip}(N)^2 \int_{\mathcal{L}} |DN|^2 \\ &\leq C \mathbf{m}_0 \ell(L)^{m+2-2\delta_2} + C \mathbf{m}_0 \ell(L)^{m+2+2\beta_2} + C \mathbf{m}_0^{2\gamma_2} \int_{\mathcal{L}} |DN|^2, \end{aligned}$$

where we have used $\|A_{\mathcal{M}}\|_{C^0} \leq C \|D^2\varphi\|_{C^0} \leq C \mathbf{m}_0^{1/2}$. Thus (2.3) follows provided ε_2 is sufficiently small

We finally come to (2.4). First observe that, by (2.1) and (2.2),

$$\int_{\mathcal{L} \setminus \mathcal{K}} |\boldsymbol{\eta} \circ N| \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} |\mathcal{L} \setminus \mathcal{K}| \leq C \mathbf{m}_0^{1+\gamma_2+1/2m} \ell(L)^{m+3+\beta_2+\gamma_2}. \quad (6.12)$$

Fix now $p \in \mathcal{K}$. Recalling that $F_L(x) = \sum_i \llbracket p + (N_L)_i(p) \rrbracket$ is given by [6, Theorem 5.1] applied to the map f_L , we can use [6, Theorem 5.1(5.4)] to conclude

$$\begin{aligned} |\boldsymbol{\eta} \circ N_L(p)| &\leq C |\boldsymbol{\eta} \circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L}^\perp(p)| + C \text{Lip}(N_L|_{\mathcal{L}}) |T_p \mathcal{M} - \pi_L| |N_L(p)| \\ &\leq C |\boldsymbol{\eta} \circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L}^\perp(p)| \\ &\quad + C \mathbf{m}_0^{1/2+\gamma_2} \ell(L)^{1+\gamma_2-\delta_2} (\mathcal{G}(N_L(p), Q \llbracket \boldsymbol{\eta} \circ N_L(p) \rrbracket) + Q |\boldsymbol{\eta} \circ N_L(p)|). \end{aligned}$$

For ε_2 sufficiently small (depending only on $\beta_2, \gamma_2, M_0, N_0, C_e, C_h$), we then conclude that

$$\begin{aligned} |\boldsymbol{\eta} \circ N_L(p)| &\leq C |\boldsymbol{\eta} \circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L}^\perp(p)| + C \mathbf{m}_0^{1/2+\gamma_2} \ell(L)^{1+\gamma_2-\delta_2} \mathcal{G}(N_L(p), Q \llbracket \boldsymbol{\eta} \circ N_L(p) \rrbracket) \\ &\leq C |\boldsymbol{\eta} \circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L}^\perp(p)| + C a \mathbf{m}_0^{1+\gamma_2} \ell(L)^{(1+\gamma_2-\delta_2)\frac{2+\gamma_2}{1+\gamma_2}} \\ &\quad + \frac{C}{a} \mathcal{G}(N_L(p), Q \llbracket \boldsymbol{\eta} \circ N_L(p) \rrbracket)^{2+\gamma_2}. \end{aligned} \quad (6.13)$$

Our choice of δ_2 makes the exponent $(1 + \gamma_2 - \delta_2)\frac{2+\gamma_2}{1+\gamma_2}$ larger than $2 + \gamma_2/2$. Let next $\varphi' : \pi_L \rightarrow \pi_L^\perp$ be such that $\mathbf{G}_{\varphi'} = \mathcal{M}$. Applying Lemma B.1 we conclude that

$$\begin{aligned} \int_{\mathcal{K} \cap \mathcal{V}} |\boldsymbol{\eta} \circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L}^\perp(p)| &\leq C \int_{\mathbf{p}_{\pi_L}(\mathcal{K} \cap \mathcal{V})} |\boldsymbol{\eta} \circ f_L(x) - \varphi'(x)| \\ &\leq C \|g_L(x) - \varphi(x)\|_{C^0(H)} \ell(L)^m, \end{aligned}$$

where H is a cube concentric to L with side-length $\ell(H) = \frac{9}{8}\ell(L)$. From Proposition 4.4(v) we get $\|\varphi - g_L\|_{C^0(H)} \leq C \mathbf{m}_0 \ell(L)^{m+3+\beta_2/3}$ and (2.4) follows integrating (6.13) over $\mathcal{V} \cap \mathcal{K}$ and using (6.12).

6.4. Proof of Corollary 2.5. Observe that $N \equiv 0$ over $\Phi(\Gamma)$ and thus the second inequality in (2.5) follows easily from the second inequality of (2.1), recalling that $\ell(L) \leq 1$ for any cube $L \in \mathscr{W}$. For the same reasons, from (2.3) we conclude

$$\int_{\mathcal{M}'} |DN|^2 \leq C\mathbf{m}_0 \sum_{L \in \mathscr{W}} \ell(L)^{m+2-2\delta_2} \leq C\mathbf{m}_0 \sum_{L \in \mathscr{W}} \ell(L)^m \leq C\mathbf{m}_0.$$

(2.6) follows from (2.2) with similar considerations. Coming to the first inequality in (2.5) fix any two points $p = \Phi(x), q = \Phi(y) \in \mathcal{M}'$. Observe that the length of the geodesic segment joining p and q is comparable, up to constants, to $|x - y|$. If $x, y \in \Gamma$, then $N(p) = N(q) = Q \llbracket 0 \rrbracket$ and so $\mathcal{G}(N(p), N(q)) = 0$. If $x \in \Gamma$ and $y \notin \Gamma$, then y belongs to some $L \in \mathscr{W}$ and, by the properties of the Whitney decomposition, $\ell(L) \leq \frac{1}{2}|x - y|$. Thus, using the second inequality in (2.1) we conclude $\mathcal{G}(N(q), N(p)) = \mathcal{G}(N(q), Q \llbracket 0 \rrbracket) \leq \|N|_{\mathcal{L}}\|_{C^0} \leq C\mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} \leq C\mathbf{m}_0^{1/2m} |x - y|$. Finally, if $x, y \notin \Gamma$ we analyze two cases. If the geodesic segment $[x, y]$ intersects Γ , then we conclude the same inequality as above. Otherwise there are points $x = z_0, z_1, \dots, z_N = y$ in $[x, y]$ such that each segment $[z_{i-1}, z_i]$ is contained in some single $L_i \in \mathscr{W}$ and $\sum_i |z_i - z_{i-1}| = |x - y|$. It then follows from the first bound in (2.1) that

$$\mathcal{G}(N(p), N(q)) \leq \sum_i \mathcal{G}(N(\Phi(z_i)), N(\Phi(z_{i-1}))) \leq C\mathbf{m}_0^{\gamma_2} \sum_i |z_i - z_{i-1}| = C\mathbf{m}_0^{\gamma_2} |x - y|.$$

Recalling that $\gamma_2 \leq \frac{1}{2m}$, all the cases examined prove the first inequality in (2.5).

7. SEPARATION AND SPLITTING BEFORE TILTING

As in the previous sections, C_0 will be used for geometric constants, \bar{C} for constants depending on $\beta_2, \delta_2, M_0, N_0$ and C_e , whereas C will be used for constants depending on all the latter parameters and also C_h .

7.1. Vertical separation. In this section we prove Proposition 3.1 and Corollary 3.2.

Proof of Proposition 3.1. Let J be the father of L . By Proposition 4.1, Theorem A.1 can be applied to the cylinder $\mathbf{C} := \mathbf{C}_{36r_J}(p_J, \pi_J)$. Moreover, $|p_J - p_L| \leq 3\sqrt{m}\ell(J)$. Thus, if M_0 is larger than a geometric constant, we have $\mathbf{B}_L \subset \mathbf{C}_{34r_J}(p_J, \pi_J)$. Denote by $\mathbf{q}_L, \mathbf{q}_J$ the projections $\mathbf{p}_{\hat{\pi}_L^\perp}$ and $\mathbf{p}_{\pi_J^\perp}$ respectively. Since $L \in \mathscr{W}_h$, there are two points $p_1, p_2 \in \text{spt}(T) \cap \mathbf{B}_L$ such that $|\mathbf{q}_L(p_1 - p_2)| \geq C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}$. On the other hand, recalling Proposition 4.1, $|\pi_J - \hat{\pi}_L| \leq \bar{C} \mathbf{m}_0^{1/2} \ell(L)^{1-\delta_2}$, where \bar{C} depends upon $\beta_2, \delta_2, M_0, N_0, C_e$ but not C_h . Thus,

$$|\mathbf{q}_J(p_1 - p_2)| \geq |\mathbf{q}_L(p_1 - p_2)| - C_0 |\hat{\pi}_L - \pi_J| |p_1 - p_2| \geq C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} - \bar{C} \mathbf{m}_0^{1/2} \ell(L)^{2-\delta_2}.$$

Hence, if ε_2 is sufficiently small, we actually conclude

$$|\mathbf{q}_J(p_1 - p_2)| \geq \frac{15}{16} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \quad (7.1)$$

Set $E := \mathbf{E}(T, \mathbf{C})$ and apply Theorem A.1 to \mathbf{C} : the union of the corresponding ‘‘stripes’’ \mathbf{S}_i contains the set $\text{spt}(T) \cap \mathbf{C}_{36r_J(1-C_0E^{1/2m}|\log E|)}(p_J, \pi_J)$. We can therefore assume that they contain $\text{spt}(T) \cap \mathbf{C}_{34r_J}(p_J, \pi_J)$. The width of these stripes is bounded as follows:

$$\sup \{ |\mathbf{q}_J(x - y)| : x, y \in \mathbf{S}_i \} \leq C_0 E^{1/2m} r_J \leq C_0 C_e^{1/2m} M_0 \mathbf{m}_0^{1/2m} \ell(L)^{1+(2-2\delta_2)/2m}.$$

So, if C^\sharp is chosen large enough (depending only upon M_0 , m , n and Q), we actually conclude that p_1 and p_2 must belong to two different stripes, say \mathbf{S}_1 and \mathbf{S}_2 . By Theorem A.1(iii) we conclude that all points in $\mathbf{C}_{34r_J}(p_J, \pi_J)$ have density Θ strictly smaller than $Q - \frac{1}{2}$, thereby implying (S1). Moreover, by choosing C^\sharp appropriately, we achieve that

$$|\mathbf{q}_J(x - y)| \geq \frac{7}{8} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}. \quad \forall x \in \mathbf{S}_1, y \in \mathbf{S}_2. \quad (7.2)$$

Assume next there is $H \in \mathscr{W}$ with $\ell(H) \leq \frac{1}{2}\ell(L)$ and $H \cap L \neq \emptyset$. From our construction it follows that $\ell(H) = \frac{1}{2}\ell(L)$, $\mathbf{B}_H \subset \mathbf{C}_{34r_J}(p_J, \pi_J)$ and $|\pi_H - \pi_J| \leq \bar{C} \mathbf{m}_0^{1/2} \ell(H)^{1-\delta_2}$ (see again Proposition 4.1). Arguing as above (and possibly choosing ε_2 smaller, but only depending upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h) we then conclude

$$|\mathbf{p}_{\pi_H^\perp}(x - y)| \geq \frac{3}{4} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} \geq \frac{3}{2} C_h \mathbf{m}_0^{1/2m} \ell(H)^{1+\beta_2} \quad \forall x \in \mathbf{S}_1, y \in \mathbf{S}_2. \quad (7.3)$$

Now, recalling Proposition 4.1, if ε_2 is sufficiently small, $\mathbf{C}_{32r_H}(p_H, \pi_H) \cap \text{spt}(T) \subset \mathbf{B}_H$. Moreover, by Theorem A.1(ii),

$$(\mathbf{p}_{\pi_J})_\sharp(T \llcorner (\mathbf{S}_i \cap \mathbf{C}_{32r_H}(p_H, \pi_J))) = Q_i \llbracket B_{32r_H}(p_H, \pi_J) \rrbracket \quad \text{for } i = 1, 2, \quad Q_i \geq 1.$$

A simple argument already used several other times allows to conclude that indeed

$$(\mathbf{p}_{\pi_H})_\sharp(T \llcorner (\mathbf{S}_i \cap \mathbf{C}_{32r_H}(p_H, \pi_H))) = Q_i \llbracket B_{32r_H}(p_H, \pi_H) \rrbracket \quad \text{for } i = 1, 2, \quad Q_i \geq 1.$$

Thus, \mathbf{B}_H must necessarily contain two points x, y with $|\mathbf{p}_{\pi_H^\perp}(x - y)| \geq \frac{3}{2} C_h \mathbf{m}_0^{1/2m} \ell(H)^{1+\beta_2}$. Given that $|\hat{\pi}_H - \pi_H| \leq \bar{C} \mathbf{m}_0^{1/2} \ell(H)^{1-\delta_2}$, we conclude (again imposing that ε_2 is sufficiently small) that $|\mathbf{p}_{\pi_H^\perp}(x - y)| \geq \frac{5}{4} C_h \mathbf{m}_0^{1/2m} \ell(H)^{1+\beta_2}$, i.e. the cube H satisfies the stopping condition (HT), which has ‘‘priority over the condition (NN)’’ and thus it cannot belong to \mathscr{W}_n . This shows (S2).

Coming to (S3), set $\Omega := \Phi(B_{2\sqrt{m}\ell(L)}(x_L, \pi_0))$ and observe that $\mathbf{p}_\sharp(T \llcorner (\Omega \cap \mathbf{S}_i)) = Q_i \llbracket \Omega \rrbracket$. Thus, for each $p \in \mathcal{K} \cap \Omega$, the support of $p + N(p)$ must contain at least one point $p + N_1(p) \in \mathbf{S}_1$ and at least one point $p + N_2(p) \in \mathbf{S}_2$. Now, by (7.2)

$$|N_1(p) - N_2(p)| \geq \frac{7}{8} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2} - C_0 \ell(L) |T_p \mathcal{M} - \pi_J|. \quad (7.4)$$

Recalling, however, Proposition 4.4 and that \mathcal{M} and $\text{Gr}(g_J)$ coincide on a nonempty open set, we easily conclude that $|T_p \mathcal{M} - \pi_J| \leq C \mathbf{m}_0^{1/2} \ell(L)^{1-\delta_2}$ and, via (7.4),

$$\mathcal{G}(N(p), Q \llbracket \boldsymbol{\eta} \circ N(p) \rrbracket) \geq \frac{1}{2} |N_1(p) - N_2(p)| \geq \frac{3}{8} C_h \mathbf{m}_0^{1/2m} \ell(L)^{1+\beta_2}.$$

Next observe that, by the property of the Whitney decomposition, any cube touching $B_{2\sqrt{m}\ell(L)}(x_L, \pi)$ has sidelength at most $4\ell(L)$. Thus $|\Omega \setminus \mathcal{K}| \leq C \mathbf{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}$ and

for every point $p \in \Omega$ there exists $q \in \mathcal{K} \cap \Omega$ which has geodesic distance to p at most $C\mathbf{m}_0^{1/m+72/m}\ell(L)^{1+2/m+72/m}$. Given the Lipschitz bound for N and the choice $\beta_2 \leq \frac{1}{2m}$, we then easily conclude (S3):

$$\mathcal{G}(N(q), Q[\boldsymbol{\eta} \circ N(q)]) \geq \frac{3}{8}C_h\mathbf{m}_0^{1/2m}\ell(L)^{1+\beta_2} - C\mathbf{m}_0^{1/m}\ell(L)^{1+2/m} \geq \frac{1}{4}C_h\mathbf{m}_0^{1/2m}\ell(L)^{1+\beta_2},$$

where again we need $\varepsilon_2 < c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$ for a sufficiently small c . \square

Proof of Corollary 3.2. The proof is straightforward. Consider any $H \in \mathcal{W}_n^j$. By definition it has a nonempty intersection with some cube $J \in \mathcal{W}^{j-1}$: this cube cannot belong to \mathcal{W}_h by Proposition 3.1. It is then either an element of \mathcal{W}_e or an element $H_{j-1} \in \mathcal{W}_n^{j-1}$. Proceeding inductively, we then find a chain $H = H_j, H_{j-1}, \dots, H_i =: L$, where $H_{\bar{l}} \cap H_{\bar{l}-1} \neq \emptyset$ for every \bar{l} , $H_{\bar{l}} \in \mathcal{W}_n^{\bar{l}}$ for every $\bar{l} > i$ and $L = H_i \in \mathcal{W}_e^i$. Observe also that

$$|x_H - x_L| \leq \sum_{\bar{l}=i}^{j-1} |x_{H_{\bar{l}}} - x_{H_{\bar{l}+1}}| \leq \sqrt{m}\ell(L) \sum_{\bar{l}=0}^{\infty} 2^{-\bar{l}} \leq 2\sqrt{m}\ell(L).$$

It then follows easily that $H \subset B_{3\sqrt{m}\ell(L)}(L)$. \square

7.2. Unique continuation for Dir-minimizers. Proposition 3.4 is based on a De Giorgi-type decay estimate for Dir-minimizing Q -valued maps which are close to a classical harmonic function with multiplicity Q . The argument involves a unique continuation-type result for Dir-minimizers.

Lemma 7.1 (Unique continuation for Dir-minimizers). *For every $\eta \in (0, 1)$ and $c > 0$, there exists $\gamma > 0$ with the following property. If $w : \mathbb{R}^m \supset B_{2r} \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ is Dir-minimizing, $\text{Dir}(w, B_r) \geq c$ and $\text{Dir}(w, B_{2r}) = 1$, then*

$$\text{Dir}(w, B_s(q)) \geq \gamma \quad \text{for every } B_s(q) \subset B_{2r} \text{ with } s \geq \eta r.$$

Proof. We start showing the following claim:

(UC) if Ω is a connected open set and $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dir-minimizing in any open $\Omega' \subset \subset \Omega$, then either w is constant or $\int_J |Dw|^2 > 0$ on any open $J \subset \Omega$.

We prove (UC) by induction on Q . If $Q = 1$, this is the classical unique continuation for harmonic functions. Assume now it holds for all $Q^* < Q$ and we prove it for Q -valued maps. Assume $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $J \subset \Omega$ is an open set on which $|Dw| \equiv 0$. Without loss of generality, we can assume J connected and $w|_J \equiv T$ for some $T \in \mathcal{A}_Q$. Let J' be the interior of $\{w = T\}$ and $K := \overline{J'} \cap \Omega$. We prove now that K is open, which in turn by connectedness of Ω concludes (UC). We distinguish two cases.

Case (a): the diameter of T is positive. Since w is continuous, for every $x \in K$ there is $B_\rho(x)$ where w separates into $\llbracket w_1 \rrbracket + \llbracket w_2 \rrbracket$ and each w_i is a Q_i -valued Dir-minimizer. Since $J' \cap B_\rho(x) \neq \emptyset$, each w_i is constant in a (nontrivial) open subset of $B_\rho(x)$. By inductive hypothesis each w_i is constant in $B_\rho(x)$ and therefore $w = T$ in $B_\rho(x)$, that is $B_\rho(x) \subset J' \subset K$.

Case (b): $T = Q \llbracket p \rrbracket$ for some p . In this case let J'' be the interior of $\{w = Q \llbracket \boldsymbol{\eta} \circ w \rrbracket\}$. By [4, Definition 0.10], $\partial J'' \cap \Omega$ is contained in the singular set of w . By [4, Theorem 0.11],

$\mathcal{H}^{m-2+\varepsilon}(\Omega \cap \partial J'') = 0$ for every $\varepsilon > 0$. Consider now a point $p \in \partial J'' \cap \Omega$ and a small ball $B_\rho(x) \subset \Omega$. Since $\mathcal{H}^{m-1}(\partial J'' \cap B_\rho(x)) = 0$, by the isoperimetric inequality, either $|B_\rho(x) \setminus J''| = 0$ or $|J''| = 0$. The latter alternative is impossible because J'' is open and has nonempty intersection with $B_\rho(x)$. It then turns out that $|B_\rho(x) \setminus J''| = 0$ and thus the closure of J'' contains $B_\rho(x)$. But then $w = Q \llbracket \boldsymbol{\eta} \circ w \rrbracket$ on $B_\rho(x)$ and thus x cannot belong to $\partial J''$. So $\partial J'' \cap \Omega$ is empty and thus $w = Q \llbracket \boldsymbol{\eta} \circ w \rrbracket$ on Ω . On the other hand $\boldsymbol{\eta} \circ w$ is an harmonic function (cf. [4, Lemma 3.23]). Being $\boldsymbol{\eta} \circ w|_{J'} \equiv p$, by the classical unique continuation $\boldsymbol{\eta} \circ w \equiv p$ on Ω .

We now come to the proof of the lemma. Without loss of generality, we can assume $r = 1$. Arguing by contradiction, there exists sequences $\{w_k\}_{k \in \mathbb{N}} \subset W^{1,2}(B_2, \mathcal{A}_Q(\mathbb{R}^n))$ and $\{B_{s_k}(q_k)\}_{k \in \mathbb{N}}$ with $s_k \geq \eta$ and such that $\text{Dir}(w_k, B_{s_k}(q_k)) \leq \frac{1}{k}$. Without loss of generality, after applying a translation, we can assume that $\boldsymbol{\eta} \circ w_k(0) = 0$. Next, passing to a subsequence, we can either assume that $\sup_k \mathcal{G}(w_k(0), Q \llbracket 0 \rrbracket) < \infty$ or that $\lim_k \mathcal{G}(w_k, Q \llbracket 0 \rrbracket) = \infty$. In the first case, by [4, Proposition 3.20], a subsequence (not re-labeled) converges to $w \in W^{1,2}(B_2, \mathcal{A}_Q(\mathbb{R}^n))$ Dir-minimizing in every open $\Omega' \subset\subset B_2$. Up to subsequences, we can also assume that $q_k \rightarrow q$ and $s_k \rightarrow s \geq \eta > 0$. Thus, $B_s(q) \subset B_2$ and $\text{Dir}(w, B_s(q)) = 0$. By (UC) this implies that w is constant. On the other hand, by [4, Proposition 3.20] $\text{Dir}(w, B_1) = \lim_k \text{Dir}(w_k, B_1) \geq c > 0$ gives the desired contradiction. In the second case, by the Hölder continuity of Dir-minimizers, each w_k splits in $B_{3/2}$ as $w_k = w_k^1 + w_k^2$ where w_k^i is Dir-minimizing and Q_i -valued. After extracting a subsequence we can assume that Q_1 is independent of k and that $\text{Dir}(w_k^1, B_1) \geq \frac{\varepsilon}{2}$. We can then repeat the argument above and either reach a contradiction or split further the sequence in the ball $B_{5/4}$. The splitting procedure must stop after at most Q iterations. \square

Next we show that if the energy of a Dir-minimizer w does not decay appropriately, then w must split. In order to simplify the exposition, in the sequel we fix $\lambda > 0$ such that

$$(1 + \lambda)^{(m+2)} < 2^{\delta_2}. \tag{7.5}$$

Proposition 7.2 (Decay estimate for Dir-minimizers). *For every $\eta > 0$, there is $\gamma > 0$ with the following property. Let $w : \mathbb{R}^m \supset B_{2r} \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ be Dir-minimizing in every $\Omega' \subset\subset B_{2r}$ such that*

$$\int_{B_{(1+\lambda)r}} \mathcal{G}(Dw, Q \llbracket D(\boldsymbol{\eta} \circ w)(0) \rrbracket)^2 \geq 2^{\delta_2 - m - 2} \text{Dir}(w, B_{2r}). \tag{7.6}$$

Then, if we set $\bar{w} = \sum_i \llbracket w_i - \boldsymbol{\eta} \circ w \rrbracket$, the following holds:

$$\gamma \text{Dir}(w, B_{(1+\lambda)r}) \leq \text{Dir}(\bar{w}, B_{(1+\lambda)r}) \leq \frac{1}{\gamma r^2} \int_{B_s(q)} |\bar{w}|^2 \quad \forall B_s(q) \subset B_{2r} \text{ with } s \geq \eta r. \tag{7.7}$$

Before coming to the proof of the Proposition we point out an elementary fact which will be used repeatedly in this section.

Lemma 7.3. *Let $B \subset \mathbb{R}^m$ be a ball centered at 0, $w \in W^{1,2}(B, \mathcal{A}_Q(\mathbb{R}^n))$ Dir-minimizing and $\bar{w} = \sum_i \llbracket w_i - \boldsymbol{\eta} \circ w \rrbracket$. We then have*

$$Q \int_B |D(\boldsymbol{\eta} \circ w) - D(\boldsymbol{\eta} \circ w)(0)|^2 = \int_B \mathcal{G}(Dw, Q \llbracket D(\boldsymbol{\eta} \circ w)(0) \rrbracket)^2 - \text{Dir}(\bar{w}, B). \quad (7.8)$$

Proof. Let $u := \boldsymbol{\eta} \circ w$ and observe that it is harmonic. Thus, using the mean value property of harmonic functions and a straightforward computation we get

$$Q \int_B |Du - Du(0)|^2 = Q \int_B |Du|^2 - Q|B||Du(0)|^2.$$

On the other hand, using again the mean value property of harmonic functions, it is easy to see that

$$\int_B \mathcal{G}(Dw, Q \llbracket Du(0) \rrbracket)^2 + Q|B||Du(0)|^2 = \int_B |Dw|^2 = \int_B |D\bar{w}|^2 + \int_B |Du|^2.$$

Combining the last two inequalities we prove the lemma. \square

Proof of Proposition 7.2. By a simple scaling argument we can assume $r = 1$ and we argue by contradiction. Let w_k be a sequence of local Dir-minimizers which satisfy (7.6), $\text{Dir}(w_k, B_2) = 1$ and

- (a) either $\int_{B_{s_k}(q_k)} |\bar{w}_k|^2 \leq \frac{1}{k}$ for some ball $B_{s_k}(q_k) \subset B_{2r}$ with $s_k \geq \eta$;
- (b) or $\text{Dir}(\bar{w}_k, B_{1+\lambda}) \leq \frac{1}{k}$.

Passing to a subsequence, if necessary, we can assume that $s_k \rightarrow s$ and $q_k \rightarrow q$. Moreover, we can normalize the sequence so that $\int_{B_2} D(\boldsymbol{\eta} \circ w_k) = 0$ and in particular, passing to a subsequence, assume that $\boldsymbol{\eta} \circ w_k$ converges strongly in L^2 . Assume now that (a) holds for an infinite sequence of indices. In that case we can extract a subsequence, not relabeled, which converges locally in $W^{1,2}$ to a Dir-minimizer w : in fact the Hölder bound for Dir-minimizers and (a) imply necessarily that $\sup_k \mathcal{G}(w_k(q_k), Q \llbracket \boldsymbol{\eta} \circ w_k(q_k) \rrbracket) < \infty$ and we can argue as in the proof of Lemma 7.1. We then conclude that $\bar{w} = \sum_i \llbracket w_i - \boldsymbol{\eta} \circ w \rrbracket$ vanishes identically on $B_s(q)$ and we can appeal to Lemma 7.1 to infer that \bar{w} vanishes on B_2 . This means in particular that $\text{Dir}(\bar{w}_k, B_{1+\lambda}) \rightarrow \text{Dir}(\bar{w}, B_{1+\lambda}) = 0$. Summarizing we conclude that $\text{Dir}(\bar{w}_k, B_{1+\lambda})$ converges to 0 in any case.

Let next $u_k := \boldsymbol{\eta} \circ w_k$ and recall that we are assuming that u_k converges to an harmonic function u . Thus from (7.6) and Lemma 7.3 we get

$$\begin{aligned} \int_{B_{1+\lambda}} Q |Du_k - Du_k(0)|^2 &= \int_{B_{1+\lambda}} (\mathcal{G}(Dw_k, Q \llbracket Du_k(0) \rrbracket)^2 - |D\bar{w}_k|^2) \\ &\geq 2^{\delta_2 - m - 2} \int_{B_2} |Dw_k|^2 - \int_{B_{1+\lambda}} |D\bar{w}_k|^2. \end{aligned} \quad (7.9)$$

Letting $k \uparrow \infty$, since $\text{Dir}(w_k, B_2) \leq 1$ and $\text{Dir}(\bar{w}_k, B_{1+\lambda}) \rightarrow 0$, we conclude

$$\int_{B_{1+\lambda}} |Du - Du(0)|^2 \geq 2^{\delta_2 - m - 2} \geq 2^{\delta_2 - m - 2} \int_{B_2} |Du|^2. \quad (7.10)$$

Since $(1 + \lambda)^{m+2} < 2^{\delta_2}$, (7.10) violates the decay estimate for classical harmonic functions:

$$\int_{B_{1+\lambda}} |Du - Du(0)|^2 \leq 2^{-m-2}(1 + \lambda)^{m+2} \int_{B_2} |Du|^2, \quad (7.11)$$

thus concluding the proof. In order to show (7.11) it suffices to decompose Du in series of homogeneous harmonic polynomials $Du(x) = \sum_{i=0}^{\infty} P_i(x)$, where i is the degree. In particular the restriction of this decomposition on any sphere $S := \partial B_\rho$ gives the decomposition of $Du|_S$ in spherical harmonics, see [13, Chapter 5, Section 2]. It turns out, therefore, that the P_i are $L^2(B_\rho)$ orthogonal. Since the constant polynomial P_0 is $Du(0)$ and $\int_{B_{1+\lambda}} |P_i|^2 \leq 2^{-m-2i} \int_{B_2} |P_i|^2$, (7.11) follows at once. \square

7.3. Splitting before tilting I: Proof of Proposition 3.4. As customary we use the convention that constants denoted by C depend upon all the parameters but ε_2 , whereas constants denoted by C_0 depend only upon m, n, \bar{n} and Q .

Given $L \in \mathcal{W}_e^j$, let us consider its ancestors $H \in \mathcal{S}^{j-1}$ and $J \in \mathcal{S}^{j-6}$. Set $\ell = \ell(L), \pi = \hat{\pi}_H$ and $\mathbf{C} := \mathbf{C}_{8r_J}(p_J, \pi)$, and let $f : B_{8r_J}(p_J, \pi) \rightarrow \mathcal{A}_Q(\pi^\perp)$ be the π -approximation of Definition 1.13, which is the result of [5, Theorem 1.4] applied to $\mathbf{C}_{32r_J}(p_J, \pi)$ (recall that Proposition 4.2(i) ensures the applicability of [5, Theorem 1.4] in the latter cylinder). We let $K \subset B_{8r_J}(p_J, \pi)$ denote the set of [5, Theorem 1.4] and recall that $\mathbf{G}_{f|_K} = T \perp K \times \pi^\perp$. Observe that $\mathbf{B}_L \subset \mathbf{B}_H \subset \mathbf{C}$ (this requires, as usual, $\varepsilon_2 \leq c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$). The following are simple consequences of Proposition 4.1:

$$E := \mathbf{E}(T, \mathbf{C}_{32r_J}(p_J, \pi)) \leq C \mathbf{m}_0 \ell^{2-2\delta_2}, \quad (7.12)$$

$$\mathbf{h}(T, \mathbf{C}, \pi) \leq C \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}. \quad (7.13)$$

In particular the positive constant C does not depend on ε_2 . Moreover, since $\mathbf{B}_L \subset \mathbf{C}$, $L \in \mathcal{W}_e$ and $r_L/r_J = 2^{-6}$, we have

$$cC_e \mathbf{m}_0 r_L^{2-2\delta} \leq E, \quad (7.14)$$

where c is only a geometric constant. We divide the proof of Proposition 3.4 in three steps.

Step 1: decay estimate for f . Let $2\rho := 64r_H - C^\sharp \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}$: since $p_H \in \text{spt}(T)$, it follows from (7.13) that, upon choosing C^\sharp appropriately, $\text{spt}(T) \cap \mathbf{C}_{2\rho}(p_H, \pi) \subset \mathbf{B}_H \subset \mathbf{C}$ (observe that C^\sharp depends upon the parameters $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h , but not on ε_2). Setting $B = B_{2\rho}(x, \pi)$ with $x = \mathbf{p}_\pi(p_H)$, using the Taylor expansion in [6, Corollary 3.3] and the estimates in [5, Theorem 1.4], we then get

$$\begin{aligned} \text{Dir}(B, f) &\leq 2|B| \mathbf{E}(T, \mathbf{C}_{2\rho}(x_H, \pi)) + C \mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2} \\ &\leq 2\omega_m (2\rho)^m \mathbf{E}(T, \mathbf{B}_H) + C \mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2}. \end{aligned} \quad (7.15)$$

Consider next the cylinder $\mathbf{C}_{64r_L}(p_L, \pi)$ and set $x' := \mathbf{p}_\pi(p_L)$. Recall that $|x - x'| \leq |p_H - p_L| \leq C\ell(H)$, where C is a geometric constant (cf. Proposition 4.1) and set $\sigma := 64r_L + C\ell(H) = 32r_H + C\ell(H)$. If λ is the constant in (7.5) and M_0 is chosen sufficiently

large (thus fixing a lower bound for M_0 which depends only on δ_2) we reach

$$\sigma \leq \left(\frac{1}{2} + \frac{\lambda}{4}\right) 64 r_H \leq \left(1 + \frac{\lambda}{2}\right) \rho + C^\# \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}.$$

In particular, choosing ε_2 sufficiently small we conclude $\sigma \leq (1 + \lambda)\rho$ and thus also $\mathbf{B}_L \subset \mathbf{C}_{64r_L}(x', \pi) \subset \mathbf{C}_{(1+\lambda)\rho}(x, \pi) =: \mathbf{C}'$. Define $B' := B_{(1+\lambda)\rho}(x, \pi)$, set $A := f_{B'} D(\boldsymbol{\eta} \circ f)$, let $\mathcal{A} : \pi \rightarrow \pi^\perp$ be the linear map $x \mapsto A \cdot x$ and let τ be the plane corresponding to $\mathbf{G}_{\mathcal{A}}$. Using [6, Theorem 3.5], we can estimate

$$\begin{aligned} \frac{1}{2} \int_{B'} \mathcal{G}(Df, Q[[A]])^2 &\geq |B'| \mathbf{E}(T, \mathbf{C}', \tau) - C \mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2} \\ &\geq |B'| \mathbf{E}(T, \mathbf{B}_L, \tau) - C \mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2} \\ &\geq \omega_m ((1 + \lambda)\rho)^m \mathbf{E}(T, \mathbf{B}_L) - C \mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1/2}. \end{aligned} \quad (7.16)$$

Now let ϖ be the $(m + \bar{n})$ -dimensional plane containing $\pi = \hat{\pi}_H$ so that $\pi \times \varkappa$ has the least distance to the plane $T_{p_H} \Sigma$. From the bound $|\pi_H - \hat{\pi}_H| \leq C \mathbf{m}_0^{1/2} \ell^{1-\delta}$ we conclude that $|\varpi - T_{p_H} \Sigma| \leq C \mathbf{m}_0^{1/2} \ell^{1-\delta_2}$. In particular we can apply Lemma B.1 to infer the existence of a C^{3, ε_0} map $\Psi : \varpi \rightarrow \varpi^\perp$ whose graph coincides with Σ and satisfies the bounds $\|D\Psi\|_0 \leq C_0 \|D\Psi_H\|_0 + C \mathbf{m}_0^{1/2} \leq C \mathbf{m}_0^{1/2} \ell^{1-\delta_2} \leq 1$ and $\|D^2\Psi\|_0 \leq C_0 \mathbf{A} \leq C_0 \mathbf{m}_0^{1/2}$ (recall that \mathbf{A} denotes the C^0 norm of the second fundamental form of Σ).

Let \varkappa be the orthogonal complement of π in ϖ and establish the notation $\pi \times \varkappa \ni (y, v) \rightarrow \Psi(y, v)$ and $(v, z) \in \varkappa \times \varpi^\perp$. Since the approximation f takes values in Σ , we infer the existence of a Q -valued map $g = \sum_i [g_i]$ so that $f(y) = \sum_i [g_i(y), \Psi(y, g_i(y))]$. By the chain rule we have $D(\Psi(y, g(y))) = \sum_i [D_y \Psi(y, g_i(y)) + D_v \Psi(y, g_i(y)) \cdot Dg_i(y)]$. Recalling that $\text{osc } f \leq C \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}$ we obtain the same bound for the oscillation of g and thus conclude the existence of a constant vector $\bar{v} \in \varkappa$ such that $|g_i(y) - \bar{v}| \leq C \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}$ for every i and every $y \in B$. We thus achieve

$$\mathcal{G}(D(\Psi(y, g(y))), Q[[D\Psi(y, \bar{v})]]) \leq C \mathbf{m}_0^{1/2+1/2m} \ell^{1+\beta_2} + C \mathbf{m}_0^{1/2} \ell^{1-\delta_2} |Dg|(y) \quad \forall y \in B.$$

Next, $|D\Psi(y, \bar{v}) - D\Psi(x, \bar{v})| \leq C_0 \mathbf{m}_0^{1/2} \rho$, where the latter constant C_0 is indeed independent of $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h . Therefore, if we set $\tilde{A} = f_{B'} \boldsymbol{\eta}(D(\Psi(y, g))) = f_{B'} D(\boldsymbol{\eta} \circ \Psi(y, g))$, we infer

$$\int_{B'} \mathcal{G}(D(\Psi(y, g(y))), Q[[\tilde{A}]])^2 dy \leq C_0 \mathbf{m}_0 \rho^{m+2} + C \mathbf{m}_0 \text{Dir}(B, g) + C \mathbf{m}_0^{1+1/m} \rho^{m+2}.$$

Observe next that $\mathcal{G}(Df, Q[[A]])^2 = \mathcal{G}(Dg, Q[[\bar{A}]])^2 + \mathcal{G}(D(\Psi(y, g)), Q[[\tilde{A}]])^2$, where $\bar{A} = f_{B'} D(\boldsymbol{\eta} \circ g)$. We thus conclude

$$\text{Dir}(B, g) \leq 2\omega_m (2\rho)^m \mathbf{E}(T, \mathbf{B}_H) + C \mathbf{m}_0^{1+\gamma_1} \rho^{m+2}. \quad (7.17)$$

$$\begin{aligned} \int_{B'} \mathcal{G}(Dg, Q[[\bar{A}]])^2 &\geq 2\omega_m ((1 + \lambda)\rho)^m \mathbf{E}(T, \mathbf{B}_L) - C \mathbf{m}_0 \text{Dir}(B, g) \\ &\quad - C_0 \mathbf{m}_0 \rho^{m+2} - C \mathbf{m}_0^{1+\gamma_1} \rho^{m+2}. \end{aligned} \quad (7.18)$$

Step 2: harmonic approximation. From now on, to simplify our notation, we use $B_s(y)$ in place of $B_s(y, \pi)$. Set $p := \mathbf{p}_\pi(p_J)$. From (7.14) we infer that $8r_J \mathbf{A} \leq 8r_J \mathbf{m}_0^{1/2} \leq E^{3/8}$ for ε_2 sufficiently small. Therefore, for every positive $\bar{\eta}$, we can apply [5, Theorem 1.6] to the cylinder \mathbf{C} and achieve a map $w : B_{8r_J}(p, \pi) \rightarrow \mathcal{A}_Q(\pi^\perp)$ of the form $w = (u, \Psi(y, u))$ for a Dir-minimizer u and such that

$$(8r_J)^{-2} \int_{B_{8r_J}(p)} \mathcal{G}(f, w)^2 + \int_{B_{8r_J}(p)} (|Df| - |Dw|)^2 \leq \bar{\eta} E (8r_J)^m, \quad (7.19)$$

$$\int_{B_{8r_J}(p)} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \leq \bar{\eta} E (8r_J)^m. \quad (7.20)$$

Now, since $D(\boldsymbol{\eta} \circ u) = \boldsymbol{\eta} \circ Du$ is harmonic we have $D(\boldsymbol{\eta} \circ u)(x) = f_{B'}(\boldsymbol{\eta} \circ Du)$. So we can combine (7.19) and (7.20) with (7.18) to infer

$$\begin{aligned} \int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}(Du, Q[\boldsymbol{\eta} \circ Du(x)])^2 &\geq 2\omega_m((1+\lambda)\rho)^m \mathbf{E}(T, \mathbf{B}_L) - C\mathbf{m}_0 \text{Dir}(B, u) \\ &\quad - C_0 \mathbf{m}_0 \rho^{m+2} - C\mathbf{m}_0^{1+\gamma_1} \rho^{m+2} - C_0 \bar{\eta}^{1/2} E \rho^m. \end{aligned} \quad (7.21)$$

Now, recall that $\mathbf{E}(T, \mathbf{B}_L) \geq C_e \mathbf{m}_0 \ell(L)^{2-2\delta_2} \geq 2^{2\delta_2-2} \mathbf{E}(T, \mathbf{B}_H)$ and that $E \leq C\mathbf{m}_0 \rho^{2-2\delta_2}$. We can therefore combine (7.21) with (7.12), (7.17) and (7.19) to achieve

$$\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}(Du, Q[D(\boldsymbol{\eta} \circ u)(x)])^2 \geq (2^{2\delta_2-2-m} - \frac{C_0}{C_e} - C\bar{\eta}^{1/2} - C\mathbf{m}_0^{\gamma_1}) \int_{B_{2\rho}(x)} |Du|^2.$$

It is crucial that the constant C , although depending upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h , does not depend on $\bar{\eta}$ and ε_2 , whereas C_0 depends only upon Q, m, \bar{n} and n . So, if C_e is chosen sufficiently large, depending only upon λ (and hence upon δ_2), we can require that $2^{2\delta_2-2-m} - \frac{C_0}{C_e} \geq 2^{3\delta_2/4-2-m}$. We then require $\bar{\eta}$ and ε_2 to be sufficiently small so that $2^{3\delta_2/4-2-m} - C\mathbf{m}_0^{1/2m} - C\bar{\eta}^{1/2} \geq 2^{\delta_2-2-m}$. We can now apply Lemma 7.1 and Proposition 7.2 to u and conclude

$$\hat{C}^{-1} \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 \leq \int_{B_{\ell/8}(q)} \mathcal{G}(Du, Q[D(\boldsymbol{\eta} \circ u)])^2 \leq \hat{C} \ell^{-2} \int_{B_{\ell/8}(q)} \mathcal{G}(u, Q[\boldsymbol{\eta} \circ u])^2,$$

for any ball $B_{\ell/8}(q) = B_{\ell/8}(q, \pi) \subset B_{8r_J}(p, \pi)$, where \hat{C} depends upon δ_2 and M_0 . In particular, being these constants independent of ε_2 and C_e , we can use the previous estimates and reabsorb error terms (possibly choosing ε_2 even smaller and C_e larger) to conclude

$$\begin{aligned} \mathbf{m}_0 \ell^{m+2-2\delta_2} &\leq \tilde{C} \ell^m \mathbf{E}(T, \mathbf{B}_L) \leq \bar{C} \int_{B_{\ell/8}(q)} \mathcal{G}(Df, Q[D(\boldsymbol{\eta} \circ f)])^2 \\ &\leq \check{C} \ell^{-2} \int_{B_{\ell/8}(q)} \mathcal{G}(f, Q[\boldsymbol{\eta} \circ f])^2, \end{aligned} \quad (7.22)$$

where \tilde{C}, \bar{C} and \check{C} are constants which depend upon δ_2, M_0 and C_e , but not on ε_2 .

Step 3: Estimate for the \mathcal{M} -normal approximation. Now, consider any ball $B_{\ell/4}(q, \pi_0)$ with $\text{dist}(L, q) \leq 4\sqrt{m} \ell$ and let $\Omega := \Phi(B_{\ell/4}(q, \pi_0))$. Observe that $\mathbf{p}_\pi(\Omega)$ must

contain a ball $B_{\ell/8}(q', \pi)$, because of the estimates on φ and $|\pi_0 - \hat{\pi}_H|$, and in turn it must be contained in $B_{8r_J}(p, \pi)$. Moreover, $\mathbf{p}^{-1}(\Omega) \cap \text{spt}(T) \supset \mathbf{C}_{\ell/8}(q', \pi) \cap \text{spt}(T)$ and, for an appropriate geometric constant C_0 , Ω cannot intersect a Whitney region \mathcal{L}' corresponding to an L' with $\ell(L') \geq C_0 \ell(L)$. In particular, Theorem 2.4 implies that

$$\|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\Omega)) + \|\mathbf{T}_F - \mathbf{G}_f\|(\mathbf{p}^{-1}(\Omega)) \leq C \mathbf{m}_0^{1+\gamma_2} \ell^{m+2+\gamma_2}. \quad (7.23)$$

Let now F' be the map such that $\mathbf{T}_{F'} \llcorner (\mathbf{p}^{-1}(\Omega)) = \mathbf{G}_f \llcorner (\mathbf{p}^{-1}(\Omega))$ and let N' be the corresponding normal part, i.e. $F'(x) = \sum_i \llbracket x + N'_i(x) \rrbracket$. The region over which F and F' differ is contained in the projection onto Ω of $(\text{Im}(F) \setminus \text{spt}(T)) \cup (\text{Im}(F') \setminus \text{spt}(T))$ and therefore its \mathcal{H}^m measure is bounded as in (7.23). Recalling the height bound on N and f , we easily conclude $|N| + |N'| \leq C \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}$, which in turn implies

$$\int_{\Omega} |N|^2 \geq \int_{\Omega} |N'|^2 - C \mathbf{m}_0^{1+1/m+\gamma_2} \ell^{m+4+2\beta_2+\gamma_2}. \quad (7.24)$$

On the other hand, let $\varphi' : B_{8r_J}(p, \pi) \rightarrow \pi^\perp$ be such that $\mathbf{G}_{\varphi'} = \llbracket \mathcal{M} \rrbracket$ and $\Phi'(z) = (z, \varphi'(z))$; then, applying [6, Theorem 5.1 (5.3)], we conclude

$$|N'(\Phi'(z))| \geq \frac{1}{2\sqrt{Q}} \mathcal{G}(f(z), Q \llbracket \varphi'(z) \rrbracket) \geq \frac{1}{4\sqrt{Q}} \mathcal{G}(f(z), Q \llbracket \eta \circ f(z) \rrbracket),$$

which in turn implies

$$\begin{aligned} \mathbf{m}_0 \ell^{m+2-2\delta_2} &\stackrel{(7.22)}{\leq} C \ell^{-2} \int_{B_{\ell/8}(q', \pi)} \mathcal{G}(f, Q \llbracket \eta \circ f \rrbracket)^2 \leq C \ell^{-2} \int_{\Omega} |N'|^2 \\ &\leq C \ell^{-2} \int_{\Omega} |N|^2 + C \mathbf{m}_0^{1+\gamma_2+1/2m} \ell^{m+2+2\beta_2+\gamma_2}. \end{aligned} \quad (7.25)$$

For ε_2 sufficiently small, this leads to the second inequality of (3.2), while the first one comes from Theorem 2.4 and $\mathbf{E}(T, \mathbf{B}_L) \geq C_e \mathbf{m}_0 \ell^{2-2\delta_2}$.

We next complete the proof showing (3.1). Since $D(\eta \circ f)(z) = \eta \circ Df(z)$ for a.e. z , we obviously have

$$\int_{B_{\ell/8}(q', \pi)} \mathcal{G}(Df, Q \llbracket D(\eta \circ f) \rrbracket)^2 \leq \int_{B_{\ell/8}(q', \pi)} \mathcal{G}(Df, Q \llbracket D\varphi' \rrbracket)^2. \quad (7.26)$$

Let now $\vec{\mathbf{G}}_f$ be the orienting tangent m -vector to \mathbf{G}_f and $\vec{\tau}$ the one to \mathcal{M} . For a.e. z we have the inequality

$$C_0 \sum_i |\vec{\mathbf{G}}_f(f_i(z)) - \vec{\tau}(\varphi'(z))|^2 \geq \mathcal{G}(Df(z), Q \llbracket D\varphi'(z) \rrbracket)^2,$$

for some geometric constant C_0 , because $|\vec{\mathbf{G}}_f(f_i(z)) - \vec{\tau}(\varphi'(z))| \leq C\mathbf{m}_0^{\gamma_2}$ (thus it suffices to have ε_2 sufficiently small). Hence

$$\begin{aligned} \int_{B_{\ell/8}(q',\pi)} \mathcal{G}(Df, Q \llbracket D\varphi' \rrbracket)^2 &\leq C \int_{\mathbf{C}_{\ell/8}(q',\pi)} |\vec{\mathbf{G}}_f(z) - \vec{\tau}(\varphi'(\mathbf{p}_\pi(z)))|^2 d\|\mathbf{G}_f\|(z) \\ &\leq C \int_{\mathbf{C}_{\ell/8}(q',\pi)} |\vec{T}(z) - \vec{\tau}(\varphi'(\mathbf{p}_\pi(z)))|^2 d\|T\|(z) + C\mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\gamma_1}. \end{aligned} \quad (7.27)$$

Now, thanks to the height bound and to the fact that $|\vec{\tau} - \pi| \leq |\vec{\tau} - \pi_H| + |\pi_H - \pi| \leq C\mathbf{m}_0^{1/2} \ell^{1-\delta_2}$ in the cylinder $\hat{\mathbf{C}} = \mathbf{C}_{\ell/8}(q', \pi)$, we have the inequality

$$|\mathbf{p}(z) - \varphi'(\mathbf{p}_\pi(z))| \leq C\mathbf{m}_0^{1/2m+1/2} \ell^{2+\beta_2-\delta_2} \leq C\mathbf{m}_0^{1/2m+1/2} \ell^{2+\beta_2/2} \quad \forall z \in \text{spt}(T) \cap \hat{\mathbf{C}}.$$

Using $\|\varphi'\|_{C^2} \leq C\mathbf{m}_0^{1/2}$ we then easily conclude from (7.27) that

$$\begin{aligned} \int_{B_{\ell/8}(p,\pi)} \mathcal{G}(Df, Q \llbracket D\varphi' \rrbracket)^2 &\leq C_0 \int_{\hat{\mathbf{C}}} |\vec{T}(z) - \vec{\tau}(\mathbf{p}(z))|^2 d\|T\|(z) + C\mathbf{m}_0^{1+\gamma_1} \ell^{m+2+\beta_2/2} \\ &\leq C_0 \int_{\mathbf{p}^{-1}(\Omega)} |\vec{\mathbf{T}}_F(z) - \vec{\tau}(\mathbf{p}(z))|^2 d\|\mathbf{T}_F\|(z) + C\mathbf{m}_0^{1+\gamma_2} \ell^{m+2+\gamma_2}, \end{aligned}$$

where we used (7.23).

Since $|DN| \leq C\mathbf{m}_0^{\gamma_2} \ell^{\gamma_2}$, $|N| \leq C\mathbf{m}_0^{1/2m} \ell^{1+\beta_2}$ on Ω and $\|A_{\mathcal{M}}\|^2 \leq C\mathbf{m}_0$, applying now [6, Proposition 3.4] we conclude

$$\int_{\mathbf{p}^{-1}(\Omega)} |\vec{\mathbf{T}}_F(x) - \tau(\mathbf{p}(x))|^2 d\|\mathbf{T}_F\|(x) \leq (1 + C\mathbf{m}_0^{2\gamma_2} \ell^{2\gamma_2}) \int_{\Omega} |DN|^2 + C\mathbf{m}_0^{1+1/m} \ell^{m+2+2\beta_2}.$$

Thus, putting all these estimates together we achieve

$$\mathbf{m}_0 \ell^{m+2-2\delta_2} \leq C(1 + C\mathbf{m}_0^{2\gamma_2} \ell^{2\gamma_2}) \int_{\Omega} |DN|^2 + C\mathbf{m}_0^{1+\gamma_2} \ell^{m+2+\gamma_2}. \quad (7.28)$$

Since the constant C might depend on the various other parameters but not on ε_2 , we conclude that for a sufficiently small ε_2 we have

$$\mathbf{m}_0 \ell^{m+2-2\delta_2} \leq C \int_{\Omega} |DN|^2. \quad (7.29)$$

But $\mathbf{E}(T, \mathbf{B}_L) \leq C\mathbf{m}_0 \ell^{2-2\delta_2}$ and thus (3.1) follows.

8. PERSISTENCE OF Q -POINTS

8.1. Proof of Proposition 3.5. We argue by contradiction. Assuming the proposition does not hold, there are sequences T_k 's and Σ_k 's satisfying the Assumption 1.3 and radii s_k for which

- (a) either $\mathbf{m}_0(k) := \max\{\mathbf{E}(T_k, \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma_k)^2\} \rightarrow 0$ and $1 \geq \bar{s} = \lim_k s_k > 0$; or $s_k \downarrow 0$;
- (b) the sets $\Lambda_k := \{\Theta(x, T_k) = Q\} \cap \mathbf{B}_{s_k}$ satisfy $\mathcal{H}_{\infty}^{m-2+\alpha}(\Lambda_k) \geq \bar{\alpha} s_k^{m-2+\alpha}$;
- (c) denoting by $\mathcal{W}(k)$ and $\mathcal{S}(k)$ the families of cubes in the Whitney decompositions related to T_k with respect to π_0 , $\sup\{\ell(L) : L \in \mathcal{W}(k), L \cap B_{3s}(0, \pi_0) \neq \emptyset\} \leq s_k$;

(d) there exists $L_k \in \mathcal{W}_e(k)$ with $L_k \cap B_{19s/16}(0, \pi_0) \neq \emptyset$ and $\hat{\alpha}s_k < \ell(L_k) \leq s_k$.

It is not difficult to see that $\mathbf{E}(T_k, \mathbf{B}_{6\sqrt{m}s_k}) \leq C\mathbf{m}_0(k)s_k^{2-2\delta_2}$, where the constant C depends only on $\beta_2, \delta_2, M_0, N_0, C_e, C_h$. Indeed this follows obviously if $s_k \geq c(M_0, N_0) > 0$. Otherwise there is some ancestor H'_k of L_k with $s_k \leq \ell(H'_k) \leq C_0s_k$ for which $\mathbf{B}_{6\sqrt{m}s_k} \subset \mathbf{B}_{H'_k}$.

Consider now the ancestors H_k and J_k of L_k as in Section 7.3, and the corresponding Lipschitz approximations f_k . Consider next the radius $\rho_k := 5/4s_k + 2r_{L_k}$ and observe that [5, Theorem 1.4] can be applied to the cylinder $\hat{\mathbf{C}}_k := \mathbf{C}_{5\rho_k}(0, \hat{\pi}_{H_k})$: again as above, either $s_k \geq c(M_0, N_0)$, and the theorem can be applied using the estimates on the height of T in $\mathbf{C}_{5\sqrt{m}}(0, \pi_0)$ and of its excess in $\mathbf{B}_{6\sqrt{m}}$, or s_k is smaller and then we can use the ancestor H'_k of the argument above. We thus have

$$\mathbf{E}(T_k, \hat{\mathbf{C}}_k, \hat{\pi}_{H_k}) \leq C\mathbf{m}_0(k)s_k^{2-2\delta_2} \quad \text{and} \quad \mathbf{h}(T_k, \hat{\mathbf{C}}_k(0, \hat{\pi}_{H_k}), \hat{\pi}_{H_k}) \leq C\mathbf{m}_0(k)^{1/2m}s_k^{1+\beta_2}. \quad (8.1)$$

We denote by g_k the $\hat{\pi}_{H_k}$ approximation in the cylinder $\mathbf{C}_k := \mathbf{C}_{\rho_k}(0, \hat{\pi}_{H_k})$. Observe that f_k and g_k are defined on the same plane and we also denote by B_k the ball on which f_k is defined. On B_k , which is contained in the domain of definition of g_k , the two maps g_k and f_k coincide outside of a set of measure at most $C\mathbf{m}_0(k)^{1+\gamma_1}s_k^{m+2-2\delta_2+\gamma_1}$ and their oscillation is estimated with $C\mathbf{m}_0^{1/2m}s_k^{1+\beta_2}$. We can therefore conclude that

$$\int_{B_k} \mathcal{G}(f_k, g_k)^2 \leq C\mathbf{m}_0(k)^{1+\gamma_1+1/2m}s_k^{m+4+2\beta_2-2\delta_2+\gamma_1}$$

From Proposition 3.4 (3.1) we easily conclude

$$E_k := \mathbf{E}(T_k, \mathbf{C}_k, \hat{\pi}_{H_k}) \geq c_0\mathbf{E}(T_k, \mathbf{B}_{L_k}) \geq c_0C_e\mathbf{m}_0(k)\ell(L_k)^{2-2\delta_2} \geq c_0(\hat{\alpha})\mathbf{m}_0(k)s_k^{2-2\delta_2}. \quad (8.2)$$

Moreover, applying Proposition 3.4 and arguing as in Step 1 and Step 2 in Section 7.3, we find a ball $B'_k \subset \hat{\pi}_{H_k}$ contained in $B_{5s_k/4}$ and with radius at least $\ell(L_k)/8$ such that

$$\int_{B'_k} \mathcal{G}(f_k, Q[\boldsymbol{\eta} \circ f_k])^2 \geq \bar{c}\mathbf{m}_0(k)\ell(L_k)^{m+4-2\delta_2} \geq c_1(\hat{\alpha})\mathbf{m}_0(k)s_k^{m+4-2\delta_2} \quad (8.3)$$

(cf. (7.22)). Since either $\mathbf{m}_0(k) \downarrow 0$ or $s_k \downarrow 0$, we obviously conclude from (8.1) that

$$\int_{B'_k} \mathcal{G}(g_k, Q[\boldsymbol{\eta} \circ g_k])^2 \geq c(\hat{\alpha})s_k^{m+2}E_k, \quad (8.4)$$

where the constant $c(\hat{\alpha})$ is positive and depends also upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h .

Define next $\mathbf{A}_k^2 := \|A_{\Sigma_k \cap \mathbf{C}_k}\|^2 \leq C_0\mathbf{m}_0(k)$. Note that by (8.2), we have that $\mathbf{A}_k^2 s_k^2 \leq C^* E_k$, for some C^* independent of k . In particular, since either $s_k \downarrow 0$ or $\mathbf{m}_0(k) \downarrow 0$, it turns out that, for k large enough, $\mathbf{A}_k s_k \leq E_k^{3/8}$. For any given $\eta > 0$ we can then apply [5, Theorem 1.6] whenever k is large enough. We thus find a sequence of multivalued maps $w_k = (u_k, \Psi_k(x, u_k))$ on $B_{5s_k/4}(0, \hat{\pi}_{H_k})$ so that each u_k is Dir-minimizing and

$$s_k^{-2} \int_{B_{5s_k/4}(0, \hat{\pi}_{H_k})} \mathcal{G}(g_k, w_k)^2 + \int_{B_{5s_k/4}(0, \hat{\pi}_{H_k})} (|Dg_k| - |Dw_k|)^2 = o(E_k)s_k^m, \quad (8.5)$$

where the domain of Ψ_k is an $m+\bar{n}$ -dimensional plane which includes $\hat{\pi}_{H_k}$ but might change with k , cf. [5, Remark 1.5]; observe also that $\text{Lip}(\Psi_k) \leq CE_k^{1/2}$, again cf. [5, Remark 1.5].

Up to rotations (so to get $\hat{\pi}_{H_k} = \pi_0 = \mathbb{R}^m \times \{0\}$ and $\text{Dom}(\Psi_k) = \mathbb{R}^{m+\bar{n}} \times \{0\}$) and dilations (of a factor s_k) of the system of coordinates, we then end up with a sequence of C^{3,ε_0} $(m + \bar{n})$ -dimensional submanifolds Γ_k of \mathbb{R}^{m+n} , area-minimizing currents S_k in Γ_k , functions h_k and \bar{w}_k with the following properties:

- (1) the excess $E_k := \mathbf{E}(S_k, \mathbf{C}_5(0, \pi_0))$ and the height $\mathbf{h}(S_k, \mathbf{C}_5(0, \pi_0), \pi_0)$ converge to 0 (note that the constant E_k defined here equals the one in (8.2));
- (2) $\mathbf{A}_k^2 := \|A_{\Gamma_k}\|^2 \leq C^* E_k$ and hence it also converges to 0;
- (3) $\text{Lip}(h_k) \leq C E_k^{\gamma_1}$;
- (4) $\|\mathbf{G}_{h_k} - S_k\|(\mathbf{C}_{5/4}(0, \pi_0)) \leq C E_k^{1+\gamma_1}$;
- (5) $\bar{w}_k = (\bar{u}_k, \Psi_k(x, \bar{u}_k))$ for some Dir-minimizing \bar{u}_k in $B_{5/4}(0, \pi_0)$ and

$$\int_{B_{5/4}} (|Dh_k| - |D\bar{w}_k|)^2 + \mathcal{G}(h_k, \bar{w}_k)^2 = o(E_k), \quad (8.6)$$

(where with abuse of notation we keep the symbol Ψ_k for the map whose graph coincides with Γ_k);

- (6) for some positive constant $c(\hat{\alpha})$ (depending also upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h),

$$\int_{B_{5/4}} \mathcal{G}(h_k, \boldsymbol{\eta} \circ h_k)^2 \geq c E_k; \quad (8.7)$$

- (7) $\Xi_k := \{\Theta(S_k, y) = Q\} \cap \mathbf{B}_1$ has the property that $\mathcal{H}_\infty^{m-2+\alpha}(\Xi_k) \geq \bar{\alpha} > 0$ and $0 \in \Xi_k$.

Consider the projections $\bar{\Xi}_k := \mathbf{p}_{\pi_0}(\Xi_k)$. We are therefore in the position of applying [5, Theorem 1.7] to conclude that, for every $\varpi > 0$ there is a $\bar{s}(\varpi) > 0$ (which depends also upon the various parameters $\alpha, \bar{\alpha}, \hat{\alpha}, \beta_2, \delta_2, M_0, N_0, C_e$ and C_h) such that

$$\limsup_{k \rightarrow \infty} \max_{x \in \bar{\Xi}_k} \int_{B_\rho(x)} \mathcal{G}(h_k, Q[\boldsymbol{\eta} \circ h_k])^2 \leq \varpi E_k \quad \forall \rho < \bar{s}(\varpi). \quad (8.8)$$

Up to subsequences we can assume that $\bar{\Xi}_k$ (and hence also Ξ_k) converges, in the Hausdorff sense, to a compact set $\bar{\Xi}$, which is nonempty. Moreover, consider the Dir-minimizing maps $x \mapsto \hat{u}_k(x) = E_k^{-1/2} \sum_i [(\bar{u}_k)_i(x) - \boldsymbol{\eta} \circ \bar{u}_k(x)]$. Note that, by (8.6) and (8.8) we have

$$\limsup_k \int_{B_{\hat{s}}(x_k)} |\hat{u}_k|^2 < \infty$$

for some fixed $\hat{s} = \bar{s}(1) > 0$ and some sequence $\{x_k\} \subset B_1$. In particular, since

$$\limsup_k \int_{B_{5/4}} |D|\hat{u}_k||^2 \leq \limsup_k \text{Dir}(\hat{u}_k, B_{5/4}) < \infty,$$

we easily conclude that $\int_{B_{5/4}} |\hat{u}_k|^2$ is bounded independently of k . Thus, by [4, Proposition 3.20], \hat{u}_k converges, strongly in $L^2(B_{5/4})$ and up to subsequences, to a Dir-minimizing function u with $\boldsymbol{\eta} \circ u = 0$. Observe that

$$\int_{B_{5/4}} \mathcal{G}(\Psi_k(x, \bar{u}_k), Q[\boldsymbol{\eta} \circ \Psi_k(x, \bar{u}_k)])^2 \leq C \text{Lip}(\Psi_k)^2 \int_{B_{5/4}} \mathcal{G}(\bar{u}_k, Q[\boldsymbol{\eta} \circ \bar{u}_k])^2 \leq C E_k^2.$$

Thus (8.6) and (8.7) easily imply that

$$\liminf_k \int_{B_{5/4}} \mathcal{G}(\hat{u}_k, Q \llbracket 0 \rrbracket)^2 \geq \liminf E_k^{-1} \int_{B_{5/4}} \mathcal{G}(\bar{u}_k, \boldsymbol{\eta} \circ \bar{u}_k)^2 \geq c > 0. \quad (8.9)$$

From the strong L^2 convergence of \hat{u}_k we then conclude that u does not vanish identically. On the other hand, by (8.8), (8.6) and the strong convergence of \hat{u}_k we conclude that, for any given $\delta > 0$ there is a $\bar{s} > 0$ such that

$$\int_{B_\rho(x)} \mathcal{G}(u, Q \llbracket 0 \rrbracket)^2 \leq \varpi \quad \forall x \in \Xi \quad \text{and} \quad \forall \rho < \bar{s}(\varpi).$$

Since u is Dir-minimizing and hence continuous, the arbitrariness of ϖ implies $u \equiv Q \llbracket 0 \rrbracket$ on Ξ . On the other hand, $\mathcal{H}_\infty^{m-2+\alpha}(\Xi) \geq \limsup_k \mathcal{H}_\infty^{m-2+\alpha}(\Xi_k) \geq \bar{\alpha} > 0$. Then, by [4, Theorem 0.11] and Lemma 7.1 we conclude $\bar{\Xi} = B_{5/4}$, which contradicts $u \neq 0$.

8.2. Proof of Proposition 3.6. We fix the notation as in Section 7.3 and notice that

$$E := \mathbf{E}(T, \mathbf{C}_{32r_J}(p_J, \hat{\pi}_H)) \leq C \mathbf{m}_0 \ell(L)^{2-2\delta_2} \leq C \mathbf{m}_0 \bar{\ell}^{2-2\delta_2}.$$

By Proposition 3.4 we have

$$\int_{\mathcal{B}_{\ell(L)}(\mathbf{p}(p))} |DN|^2 \geq \bar{c}_1 \mathbf{m}_0 \ell(L)^{m+2-2\delta_2}. \quad (8.10)$$

Next, let $p := (x, y) \in \hat{\pi}_H \times \hat{\pi}_H^\perp$, fix a $\bar{\eta} > 0$, to be chosen later, and note that (7.14) allows us to apply [5, Theorem 1.7]: there exists then $\bar{s} > 0$ such that

$$\int_{\mathcal{B}_{2\bar{s}\ell(L)}(x, \hat{\pi}_H)} \mathcal{G}(f, Q \llbracket \boldsymbol{\eta} \circ f \rrbracket)^2 \leq \bar{\eta} \bar{s}^m \ell(L)^{m+2} E. \quad (8.11)$$

Observe that, no matter how small $\bar{\eta}$ is chosen, such estimate holds when \bar{s} and E are appropriately small: the smallness of E is then achieved choosing $\bar{\ell}$ as small as needed.

Now consider the graph $\text{Gr}(\boldsymbol{\eta} \circ f) \llcorner \mathbf{C}_{2\bar{s}\ell(L)}(x, \hat{\pi}_H)$ and project it down onto \mathcal{M} . Since \mathcal{M} is a graph over $\hat{\pi}_H$ of a function $\hat{\varphi}$ with $\|D\hat{\varphi}\|_{C^{2+\kappa}} \leq C \mathbf{m}_0^{1/2}$ and since the Lipschitz constant of $\boldsymbol{\eta} \circ f$ is controlled by $C \mathbf{m}_0^{\gamma_1}$, provided ε_2 is smaller than a geometric constant we have that $\Omega := \mathbf{p}(\text{Gr}(\boldsymbol{\eta} \circ f) \llcorner \mathbf{C}_{2\bar{s}\ell(L)}(x, \hat{\pi}_H))$ contains a ball $\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))$.

Consider now the map $F'(q) = \sum_i \llbracket q + N'_i(q) \rrbracket$ such that $\mathbf{T}_{F'} \llcorner \mathbf{p}^{-1}(\Omega) = \mathbf{G}_f \llcorner \mathbf{p}^{-1}(\Omega)$ given by [6, Theorem 5.1]. Consider also the map $\xi : \mathcal{B}_{2\bar{s}\ell(L)}(x, \hat{\pi}_H) \ni z \mapsto \mathbf{p}((z, \boldsymbol{\eta} \circ f(z))) \in \Omega$. This map is bilipschitz with controlled constant, again assuming that ε_2 is sufficiently small. Consider now $\hat{n} : \Omega \rightarrow \mathbb{R}^{m+n}$ with the property that $\hat{n}(q) \perp T_q \mathcal{M}$ and $\xi(x) + \hat{n}(\xi(x)) = (x, \boldsymbol{\eta} \circ f(x))$. Applying the estimate of [6, Theorem 5.1 (5.5)] we then get

$$\mathcal{G}(N'(\xi(x)), Q \llbracket \boldsymbol{\eta} \circ N'(\xi(x)) \rrbracket) \leq \mathcal{G}(N'(\xi(x)), Q \llbracket \hat{n}(\xi(x)) \rrbracket) \leq 2\sqrt{Q} \mathcal{G}(f(x), Q \llbracket \boldsymbol{\eta} \circ f(x) \rrbracket).$$

Integrating the latter inequality, changing variable and using $\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p)) \subset \Omega$, we then obtain

$$\int_{\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N', Q \llbracket \boldsymbol{\eta} \circ N' \rrbracket)^2 \leq C \bar{\eta} \bar{s}^m \ell(L)^{m+2} E \leq C \bar{\eta} \mathbf{m}_0 \bar{s}^m \ell(L)^{m+4-2\delta_2}.$$

Next, recalling the height bound and the fact that N and N' coincide outside a set of measure $\mathbf{m}_0^{1+\gamma_1} \ell(L)^{m+2+\gamma_2}$, we infer

$$\int_{\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q[\boldsymbol{\eta} \circ N])^2 \leq C_1 \bar{\eta} \mathbf{m}_0 \bar{s}^m \ell(L)^{m+4-2\delta_2} + C_2 \mathbf{m}_0^{1+\gamma_1} \ell(L)^{m+4+\gamma_2+2\beta_2}. \quad (8.12)$$

Since the constants \bar{c}_1 , C_1 and C_2 in (8.10) and (8.12) are independent of $\ell(L)$ and $\bar{\eta}$, we fix $\bar{\eta}$ (and consequently \bar{s}) so small that $C_1 \bar{\eta} \leq \bar{c}_1 \frac{\eta_2}{2}$. We therefore achieve from (8.12)

$$\int_{\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q[\boldsymbol{\eta} \circ N])^2 \leq \frac{\bar{c}_1}{2} \eta_2 \mathbf{m}_0 \ell(L)^{4-2\delta_2} + C_2 \mathbf{m}_0^{1+\gamma_1} \bar{s}^{-m} \ell(L)^{4+\gamma_2+2\beta_2}. \quad (8.13)$$

Having now fixed \bar{s} we choose $\bar{\ell}$ so small that $C_2 \bar{s}^{-m} \bar{\ell}^{2\delta_2+\gamma_2+2\beta_2} \leq \bar{c}_1 \eta_2 / 2$. For these choices of the parameters, under the assumptions of the proposition we then infer

$$\int_{\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q[\boldsymbol{\eta} \circ N])^2 \leq \eta_2 \bar{c}_1 \mathbf{m}_0 \ell(L)^{4-2\delta_2}. \quad (8.14)$$

The latter estimate combined with (8.10) gives the desired conclusion.

9. COMPARISON BETWEEN DIFFERENT CENTER MANIFOLDS

Proof of Proposition 3.7. We first verify (i). Observe that

$$\mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}) = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \liminf_{\rho \downarrow r} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq \varepsilon_2.$$

Moreover, since Σ' is a rescaling of Σ , $\mathbf{c}(\Sigma') \leq \mathbf{c}(\Sigma) \leq \mathbf{m}_0^{1/2}$. Therefore, (1.8) is fulfilled by Σ' and T' as well; (1.7) follows trivially upon substituting π_0 with an optimal π for T' in $\mathbf{B}_{6\sqrt{m}}$ (which is an optimal plane for T in $\mathbf{B}_{6\sqrt{m}r}$ by a trivial scaling argument); (1.5) is scaling invariant; whereas $\partial T' \llcorner \mathbf{B}_{6\sqrt{m}} = (\iota_{0,r})_{\#}(\partial T \llcorner \mathbf{B}_{6\sqrt{m}r}) = 0$.

We now come to (ii). From now on we assume N_0 to be so large that 2^{-N_0} is much smaller than c_s . In this way we know that r must be much smaller than 1. We have that $\ell(L) = c_s r$, otherwise condition (a) would be violated. Moreover, we can exclude that $L \in \mathcal{W}_n$. Indeed, in this case there must be a cube $J \in \mathcal{W}$ with $\ell(J) = 2\ell(L)$ and nonempty intersection with L . It then follows that, for $\rho := r + 2\sqrt{m}\ell(L) = (1 + 2\sqrt{m}c_s)r$, $B_\rho(0, \pi_0)$ intersects J . Again upon assuming N_0 sufficiently large, such ρ is necessarily smaller than 1. On the other hand, since $2\sqrt{m}c_s < 1$ we then have $c_s \rho < 2c_s r \leq 2\ell(L) = \ell(J)$.

Next observe that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq C \mathbf{m}_0 \rho^{2-2\delta_2}$ for some constant C and for every $\rho \geq r$. Indeed, if ρ is smaller than a threshold r_0 but larger than r , then $\mathbf{B}_{6\sqrt{m}\rho}$ is contained in the ball \mathbf{B}_J for some ancestor J of L with $\ell(J) \leq C\rho$, where the constant C and the threshold r_0 depend upon the various parameters, but not upon ε_2 . Then, $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq C \mathbf{E}(T, \mathbf{B}_J) \leq C \mathbf{m}_0 \rho^{2-2\delta_2}$. If instead $\rho \geq r_0$, we then use simply $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq C(r_0) \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}) \leq C(r_0) \mathbf{m}_0$. This estimate also has the consequence that, if $\pi(\rho)$ is an optimal m -plane in $\mathbf{B}_{6\sqrt{m}\rho}$, then $|\hat{\pi}_L - \pi(\rho)| \leq C \mathbf{m}_0^{1/2} \rho^{1-\delta_2}$.

We next consider the notation introduced in Section 7.3, the corresponding cubes $L \subset H \subset J$ and the $\hat{\pi}_H$ -approximation f introduced there. If $L \in \mathscr{W}_e$, then by (7.22) we get

$$\int_{B_{\ell/8}(x, \hat{\pi}_H)} \mathcal{G}(f, Q[\boldsymbol{\eta} \circ f])^2 \geq c \mathbf{m}_0 \ell^{m+4-2\delta_2} \geq c \mathbf{m}_0 r^{m+4-2\delta_2}, \quad (9.1)$$

where $x = \mathbf{p}_{\hat{\pi}_H}(x_H)$ and $c(\beta_2, \delta_2, M_0, N_0, C_e, C_h) > 0$. On the other hand, if $L \in \mathscr{W}_h$, we can argue as in the proof of Proposition 3.1 and use Theorem A.1 to conclude the existence of at least two stripes \mathbf{S}_1 and \mathbf{S}_2 , at distance $\bar{c} \mathbf{m}_0^{1/2m} \ell^{1+\beta_2}$ with the property that any slice $\langle T, \mathbf{p}_{\hat{\pi}_H}, z \rangle$ with $z \in B_{\ell/8}(x, \hat{\pi}_H)$ must intersect both of them. Since for $x \in K$ such slice coincides with $f(x)$, we then have

$$\begin{aligned} \int_{B_{\ell/8}(x, \hat{\pi}_H)} \mathcal{G}(f, Q[\boldsymbol{\eta} \circ f])^2 &\geq c \mathbf{m}_0^{1/m} \ell^{m+2+2\beta_2} - C \mathbf{m}_0^{1/m} \ell^{2+2\beta_2} |K| \\ &\geq c \mathbf{m}_0^{1/m} \ell^{m+2+2\beta_2} - C \mathbf{m}_0^{1+\gamma_1+1/m} \ell^{(m+2-2\delta_2)(1+\gamma_1)+2\beta_2} \\ &\geq c \mathbf{m}_0 r^{m+4-2\delta_2}. \end{aligned} \quad (9.2)$$

Rescale next through $\iota_{0,r}$ and consider $T' := (\iota_{0,r})_{\#} T$. We also rescale the graph of the corresponding $\hat{\pi}_H$ -approximation f to the graph of a map g , which then has the following properties. If $B \subset \hat{\pi}_H$ is the rescaling of the ball $B_{\ell/8}(x, \hat{\pi}_H)$, then $B \subset B_{3/2}$ and the radius of B is larger than $c_s/8$. On B we have the estimate

$$\int_B \mathcal{G}(g, Q[\boldsymbol{\eta} \circ g])^2 = r^{-m-2} \int_{B_{\ell/8}(x, \hat{\pi}_H)} \mathcal{G}(f, Q[\boldsymbol{\eta} \circ f])^2 \geq \bar{c} \mathbf{m}_0 r^{2-2\delta_2}. \quad (9.3)$$

The Lipschitz constant of g is the same of that of f and hence controlled by $C \mathbf{m}_0^{\gamma_1} r^{\gamma_1}$. On the other hand, we have

$$\hat{\mathbf{m}}_0 := \max \{ \mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma')^2 \} \leq \max \{ C \mathbf{m}_0 r^{2-2\delta_2}, \mathbf{c}(\Sigma)^2 r^2 \} \leq C \mathbf{m}_0 r^{2-2\delta_2}. \quad (9.4)$$

Moreover, denoting by $\hat{\mathbf{C}}$ the rescaling of the cylinder $\mathbf{C}_{8r_j}(p_J, \hat{\pi}_H)$, we have that

$$\| \mathbf{G}_g - T' \|(\hat{\mathbf{C}}) \leq C \mathbf{m}_0^{1+\gamma_1} r^{2+\gamma_1/2}. \quad (9.5)$$

Finally, since $|\pi - \hat{\pi}_H| \leq C \mathbf{m}_0^{1/2} r^{1-\delta_2}$ and because \mathcal{M}' is the graph of a function $\boldsymbol{\varphi}'$ over π with $\|D\boldsymbol{\varphi}'\|_{C^{2,\kappa}} \leq C \hat{\mathbf{m}}_0^{1/2}$ and $\|\boldsymbol{\varphi}'\|_{C^0} \leq C \hat{\mathbf{m}}_0^{1/2m}$, by (9.4) we can actually conclude that \mathcal{M}' is the graph over $\hat{\pi}_H$ of a map $\hat{\boldsymbol{\varphi}} : \hat{\pi}_H \rightarrow \hat{\pi}_H^{\perp}$ with $\|D\hat{\boldsymbol{\varphi}}\|_{C^{2,\alpha}} \leq C \mathbf{m}_0^{1/2} r^{1-\delta_2}$. Similarly, the \mathcal{M}' -approximating map $x \mapsto F'(x) := \sum_i [x + N'_i(x)]$ coincides with T' over a subset $\mathcal{K}' \subset \mathcal{M}'$ with $|\mathcal{M}' \setminus \mathcal{K}'| \leq \hat{\mathbf{m}}_0^{1+\gamma_2} \leq C \mathbf{m}_0^{1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)}$.

Consider now the projection \mathbf{p}' over \mathcal{M}' and hence define the sets

$$\mathcal{H} := \mathbf{p}'(\text{Gr}(g)) \quad (9.6)$$

$$\mathcal{J} := \{ q \in \mathcal{H} : \langle \mathbf{T}_{F'}, \mathbf{p}', q \rangle = \langle \mathbf{G}_g, \mathbf{p}', q \rangle \} \quad (9.7)$$

Since $\mathcal{J} \subset \mathbf{p}'(\text{Im}(F') \setminus \text{spt}(T)) \cup \mathbf{p}'(\text{Gr}(g) \setminus \text{spt}(T))$, we have $|\mathcal{H} \setminus \mathcal{J}| \leq \mathbf{m}_0^{1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)}$. On the other hand, by [6, Theorem 5.1] there is a Lipschitz map G defined on a subset $\text{Dom}(G)$ of $\mathcal{H} \subset \mathcal{M}'$ such that $\text{Im}(G) \supset \text{Gr}(g|_B)$. We then have $G \equiv F'$ on any point of $\mathcal{J} \cap \text{Dom}(G)$, which in turn is contained in $\mathbf{B}_2 \cap \mathcal{M}'$ (at least provided \mathbf{m}_0 is small

enough). Consider next a point $p \in \text{Dom}(G)$ with $p = (x, \hat{\varphi}(x))$ and consider that for this point we have, by [6, Theorem 5.1 (5.3)],

$$\mathcal{G}(g(x), Q \llbracket \eta \circ g(x) \rrbracket) \leq \mathcal{G}(g(x), Q \llbracket \hat{\varphi}(x) \rrbracket) \leq C_0 \mathcal{G}(G(p), Q \llbracket p \rrbracket).$$

Therefore, using (9.3) we can easily estimate

$$\begin{aligned} \int_{\mathbf{B}_2 \cap \mathcal{M}'} |N'|^2 &\geq \int_{\mathcal{J} \cap \text{Dom}(G)} \mathcal{G}(G(p), Q \llbracket p \rrbracket)^2 \geq \int_{\text{Dom}(G)} \mathcal{G}(G(p), Q \llbracket P \rrbracket)^2 - C \hat{\mathbf{m}}_0^{1/m} |\mathcal{H} \setminus \mathcal{J}| \\ &\geq c \mathbf{m}_0 r^{2-2\delta_2} - C \mathbf{m}_0^{1/m+1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)} \geq c \mathbf{m}_0 r^{2-2\delta_2}, \end{aligned} \quad (9.8)$$

where all the constants are independent of ε_2 (but depend upon the other parameters) and as usual ε_2 is assumed to be sufficiently small. Thus finally, by (9.4) we conclude

$$\int_{\mathbf{B}_2 \cap \mathcal{M}'} |N'|^2 \geq \bar{c}_s \hat{\mathbf{m}}_0 = \bar{c}_s \max\{\mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma')^2\}. \quad \square$$

APPENDIX A. HEIGHT BOUND REVISITED

In this section we prove a strengthened version of the so-called ‘‘height bound’’ (see [8, Lemma 5.3.4]), which appeared first in [1]. Our proof follows closely that of [11].

Theorem A.1. *Let Q, m, \bar{n} and n be positive integers. Then there are $\varepsilon(Q, m, \bar{n}, n) > 0$ and $C_0(Q, m, \bar{n}, n)$ with the following property. For $r > 0$ and $\mathbf{C} = \mathbf{C}_r(x_0)$ assume:*

- (h1) $\Sigma \subset \mathbb{R}^{m+n}$ is an $(m + \bar{n})$ -dimensional C^2 submanifold with $\mathbf{A} := \|A_\Sigma\|_0 \leq \varepsilon$;
- (h2) R is an integer rectifiable m -current with $\text{spt}(R) \subset \Sigma$ and area-minimizing in Σ ;
- (h3) $\partial R \llcorner \mathbf{C} = 0$, $(\mathbf{p}_{\pi_0})_\# R \llcorner \mathbf{C} = Q \llbracket B_r(\mathbf{p}_{\pi_0}(x_0), \pi_0) \rrbracket$ and $E := \mathbf{E}(R, \mathbf{C}) < \varepsilon$.

Then there are $k \in \mathbb{N} \setminus \{0\}$, points $\{y_1, \dots, y_k\} \subset \mathbb{R}^n$ and positive integers Q_1, \dots, Q_k such that:

- (i) having set $\sigma := C_0 E^{1/2m}$, the open sets $\mathbf{S}_i := \mathbb{R}^m \times (y_i +] - r\sigma, r\sigma[{}^n)$ are pairwise disjoint and $\text{spt}(R) \cap \mathbf{C}_{r(1-\sigma|\log E|)}(x_0) \subset \cup_i \mathbf{S}_i$;
- (ii) $(\mathbf{p}_{\pi_0})_\# [R \llcorner (\mathbf{C}_{r(1-\sigma|\log E|)}(x_0) \cap \mathbf{S}_i)] = Q_i \llbracket B_{r(1-\sigma|\log E|)}(\mathbf{p}_{\pi_0}(x_0), \pi_0) \rrbracket \forall i \in \{1, \dots, k\}$.
- (iii) for every $p \in \text{spt}(R) \cap \mathbf{C}_{r(1-\sigma|\log E|)}(x_0)$ we have $\Theta(R, p) < \max_i Q_i + \frac{1}{2}$.

Remark A.2. Obviously, $\sum_i Q_i = Q$ and hence $1 \leq k \leq Q$. Most likely the bound on the radius of the inner cylinder could be improved to $1 - \sigma$. However this would not give us any advantage in the rest of the paper and hence we do not pursue the issue here.

Proof. We first observe that, without loss of generality, we can assume $x_0 = 0$ and $r = 1$. Moreover, (iii) follows from (i) and (ii) through the monotonicity formula. Indeed, let $p \in \text{spt}(R)$ be such that $\mathbf{B}_\rho(p) := \mathbf{B}_{E^{1/2m}}(p) \subset \mathbf{C}_{1-\sigma|\log E|}(x_0) =: \mathbf{C}'$. p must be contained in one of the \mathbf{S}_i , say \mathbf{S}_1 . Consider the current $R_1 = R \llcorner (\mathbf{S}_1 \cap \mathbf{C}')$. Observe that R_1 must be area-minimizing in Σ , $\Theta(R_1, p) = \Theta(R, p)$ and that $\mathbf{E}(R_1, \mathbf{C}') \leq E$. On the other hand, if $\|A_\Sigma\|$ is smaller than a geometric constant, the monotonicity formula implies

$$\mathbf{M}(R_1 \llcorner \mathbf{C}_\rho(p)) \geq \mathbf{M}(R_1 \llcorner \mathbf{B}_\rho(p)) \geq \omega_m (\Theta(R, p) - \frac{1}{4}) \rho^m = \omega_m (\Theta(R, p) - \frac{1}{4}) E^{1/2}.$$

On the other hand, $\mathbf{M}(R_1 \llcorner \mathbf{C}_\rho(p)) \leq \omega_m Q_1 \rho^m + E = \omega_m Q_1 E^{1/2} + E$. Thus, if E is smaller than a geometric constant, we ensure $\Theta(R, p) \leq Q_1 + \frac{1}{2}$. This means that, having proved (i) and (ii) for $\sigma = C_0 E^{1/2m}$, (iii) would hold if we redefine σ as $(C_0 + 1)E^{1/2m}$.

The proof of (i) and (ii) is by induction on Q . The starting step $Q = 1$ is Federer's classical statement (cf. with [8, Lemma 5.3.4] and [11, Lemma 2]) and though its proof can be easily concluded from what we describe next, our only concern will be to prove the inductive step. Hence, from now on we assume that the theorem holds for all multiplicities up to $Q - 1 \geq 1$ and we prove it for Q . Indeed, we will show a slightly weaker assertion, i.e. the existence of numbers $a_1, \dots, a_k \in \mathbb{R}$ such that the conclusions (i) and (ii) apply when we replace \mathbf{S}_i with $\Sigma_i = \mathbb{R}^{m+n-1} \times]a_i - \sigma, a_i + \sigma[$. The general statement is obviously a simple corollary. To simplify the notation we use $\bar{\mathbf{p}}$ in place of \mathbf{p}_{π_0} .

Step 1. Let $r \geq \frac{1}{2}$ and $a - b > 2\eta = 2C^b E^{1/2m}$, where C^b is a constant depending only on m and n , which will be determined later. We denote by $W_r(a, b)$ the open set $B_r \times \mathbb{R}^{n-1} \times]a, b[$. In this step we show

$$\|R\|(W_r(a, b)) \leq \frac{2Q-1}{2Q} \omega_m r^m \implies \text{spt}(R) \cap W_{r-\eta}(a + \eta, b - \eta) = \emptyset. \quad (\text{A.1})$$

Without loss of generality, we assume $a = 0$. For each $\tau \in]0, \frac{b}{2}[$ consider the currents $R_\tau := R \llcorner W_r(\tau, b - \tau)$ and $S_\tau := \bar{\mathbf{p}}_\sharp R_\tau$. It follows from the slicing theory that S_τ is a locally integral current for a.e. τ . There are then functions $f_\tau \in BV_{loc}(B_r)$ which take integer values and such that $S_\tau = f_\tau \llbracket B_r \rrbracket$. Since $\|f_\tau\|_1 = \mathbf{M}(S_\tau) \leq \|R\|(W_r(0, b)) \leq \frac{2Q-1}{2Q} \omega_m r^m$, f_τ must vanish on a set of measure at least $\frac{\omega_m}{2Q} r^m$. By the relative Poincaré inequality,

$$\mathbf{M}(S_\tau)^{1-1/m} = \|f_\tau\|_{L^1}^{1-1/m} \leq C \|Df_\tau\|(B_r) = C \|\partial(\bar{\mathbf{p}}_\sharp R_\tau)\|(B_r) \leq C \|\partial R_\tau\|(C_r).$$

We introduce the slice $\langle R, \tau \rangle$ relative to the map $x_{m+n} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}$ which is the projection on the last coordinate. Then the usual slicing theory gives that

$$(\mathbf{M}(S_\tau))^{1-1/m} \leq C \|\partial R_\tau\|(C_r) = C \mathbf{M}(\langle R, \tau \rangle - \langle R, b - \tau \rangle) \quad \text{for a.e. } \tau. \quad (\text{A.2})$$

Let now $\bar{\tau}$ be the supremum of τ 's such that $\mathbf{M}(S_t) \geq \sqrt{E} \forall t < \tau$. If $\mathbf{M}(S_0) < \sqrt{E}$, we then set $\bar{\tau} := 0$. If $\bar{\tau} > 0$, observe that, for a.e. $\tau \in [0, \bar{\tau}[$ we have

$$E^{\frac{m-1}{2m}} \leq C (\mathbf{M}(S_\tau))^{1-1/m} \leq C (\mathbf{M}(\langle R, \tau \rangle - \langle R, b - \tau \rangle)). \quad (\text{A.3})$$

Integrate (A.3) between 0 and $\bar{\tau}$ to conclude

$$\bar{\tau} E^{\frac{m-1}{2m}} \leq C \int_0^{\bar{\tau}} \mathbf{M}(\langle R, \tau \rangle - \langle R, b - \tau \rangle) d\tau = C \int_{W_r(0, \bar{\tau}) \cup W_r(b - \bar{\tau}, b)} |\vec{R} \llcorner dx_{m+n}| d\|R\|. \quad (\text{A.4})$$

We then apply Cauchy-Schwartz and recall

$$\int_{C_1} |\vec{R} \llcorner dx_{m+n}|^2 d\|R\| \leq \mathbf{E}(R, C_1) = E$$

We then conclude $\bar{\tau} E^{\frac{m-1}{2m}} \leq \bar{C} \sqrt{E}$ for some constant \bar{C} depending only on m and n , i.e. $\bar{\tau} \leq \bar{C} E^{1/2m}$. Set $C^b := \bar{C} + 2$ and recall that $\eta = C^b E^{1/2m}$. Observe also that there

must be a sequence of $\tau_k \downarrow \bar{\tau}$ with $\mathbf{M}(S_{\tau_k}) < \sqrt{E}$. Therefore,

$$\|R\|(W_r(\bar{\tau}, b - \bar{\tau})) \leq \liminf_{k \rightarrow \infty} \|R\|(W_r(\tau_k, b - \tau_k)) \leq \liminf_{k \rightarrow \infty} \mathbf{M}(S_{\tau_k}) + E \leq 2\sqrt{E}. \quad (\text{A.5})$$

Assume now the existence of $p \in \text{spt}(T) \cap W_{r-\eta}(\eta, b - \eta)$. By the properties of area-minimizing currents, $\Theta(T, p) \geq 1$. Set $\rho := 2E^{1/2m}$ and $B' := \mathbf{B}_\rho(p) \subset W_r(\bar{\tau}, \ell - \bar{\tau})$. By the monotonicity formula, $\|R\|(B') \geq c 2^m \omega_m \sqrt{E}$, where c depends only on \mathbf{A} (recall that $\rho \leq 1$) and approaches 1 as \mathbf{A} approaches 0. Thus, for ε_2 sufficiently small, this would contradict (A.5). We have therefore proved (A.1).

Step 2. We are now ready to conclude the proof of (i) and (ii). Assume

$$\max \{ \|R\|(W_1(0, \infty)), \|R\|(W_1(-\infty, 0)) \} \leq \frac{1}{2} \mathbf{M}(R). \quad (\text{A.6})$$

Divide the interval $[0, 1[$ into $Q + 1$ intervals $[a_i, a_{i+1}[$ and let $W^i := W_1(a_i, a_{i+1})$. For each i consider $S^i := \bar{\mathbf{p}}_\#(T \llcorner W^i)$. Observe that there must be one i for which $\mathbf{M}(S^i) \leq (1 - \frac{1}{2Q})\omega_m$. Otherwise we would have

$$\omega_m Q + E \geq \mathbf{M}(R) \geq 2 \sum_i \mathbf{M}(S^i) \geq 2(Q + 1)\omega_m \frac{2Q-1}{2Q},$$

which is obviously a contradiction if E is sufficiently small.

It follows from Step 1 that there must be an i so that $\text{spt}(T)$ does not intersect $W_{1-\eta}(a_i + \eta, a_{i+1} - \eta)$. Consider $R_1 := R \llcorner W_{1-\eta}(-\infty, a_i + \eta)$ and $R_2 := R \llcorner W_{1-\eta} \times (a_{i+1} - \eta, \infty)$. By the constancy theorem $\bar{\mathbf{p}}_\# R_i = k_i \llbracket B_{1-\eta} \rrbracket$, where both k_i 's are integers. Indeed, having assumed that E is sufficiently small, each k_i must be nonnegative and their sum is Q . There are now two possibilities.

- (a) Both k_i 's are positive. In this case R_1 and R_2 satisfy again the assumptions of the Theorem with $\mathbf{C}_{1-\eta}(0)$ in place of $\mathbf{C}_1(0)$. After a suitable rescaling we can apply the inductive hypothesis to both currents and hence get the desired conclusion.
- (b) One k_i is zero. In this case $\mathbf{M}(R_i) \leq E$ and it cannot be R_1 , since $\mathbf{M}(R_1) \geq \frac{1}{2} \mathbf{M}(R)$ by (A.6). Thus it is R_2 and, arguing as at the end of Step 1, we conclude $R \llcorner W_{1-2\eta}([a_{i+1} + 2\eta, \infty[) = 0$.

In case (b) we repeat the argument splitting $] - 1, 0]$ into $Q + 1$ intervals. Once again, either we can “separate” the current into two pieces and apply the inductive hypothesis, or we conclude that $\text{spt}(R \llcorner \mathbf{C}_{1-4\eta}) \subset W_{1-4\eta}(-1 - \eta, 1 + \eta) =: W_{1-4\eta}(a_0, b_0)$. If this is the case, we apply once again the argument above and either we “separate” $R^1 := R \llcorner \mathbf{C}_{1-6\eta} \times \mathbb{R}^n$ into two pieces, or we conclude that $\text{spt}(R^1) \subset W_{1-6\eta}(a_1, b_1)$, where

$$b_1 - a_1 \leq (b_0 - a_0) \left(1 - \frac{1}{Q+1} + \eta \right) \leq 2 \left(1 - \frac{1}{Q+2} \right)$$

(provided ε_2 is smaller than a geometric constant). We now iterate this argument a finite number of times, stopping if at any step we “separate” the current and can apply the inductive hypothesis, or if the resulting current is contained in $W_{1-(4+2k)\eta}(a_k, b_k)$ for some a_k, b_k with $b_k - a_k \leq c_1 E^{1/2m}$. The constant c_1 is chosen larger than 1 and in such a way that, if $\ell > c_1 E^{1/2m}$, then $\ell \frac{Q}{Q+1} + \eta \leq \frac{Q+1}{Q+2} \ell$. Observe that, since $\eta = C^b E^{1/2m}$, c_1 depends

only upon Q , m and n . We now want to estimate from above the maximal number of times k that the procedure above gets iterated. Observe that we must have

$$c_1 E^{1/2m} \leq b_{k-1} - a_{k-1} \leq (b_0 - a_0) \left(\frac{Q+1}{Q+2} \right)^{k-1}.$$

Since $b_0 - a_0 = 2 + 2\eta$, we get the estimate

$$-(k-1) \log \left(\frac{Q+1}{Q+2} \right) \leq -\log(2+2\eta) - \log c_1 - \frac{1}{2m} \log E.$$

Since E is assumed to be small we get the bound $k \leq -C \log E$. This completes the proof. \square

APPENDIX B. CHANGING COORDINATES FOR CLASSICAL FUNCTIONS

Lemma B.1. *For any $m, n \in \mathbb{N} \setminus \{0\}$ there are constants $c_0, C_0 > 0$ with the following properties. Assume that*

- (i) $\varkappa, \varkappa_0 \subset \mathbb{R}^{m+n}$ are m -dim. planes with $|\varkappa - \varkappa_0| \leq c_0$ and $0 < r \leq 1$;
- (ii) $p = (q, u) \in \varkappa \times \varkappa^\perp$ and $f, g : B_{7r}^m(q, \varkappa) \rightarrow \varkappa^\perp$ are Lipschitz functions such that

$$\text{Lip}(f), \text{Lip}(g) \leq c_0 \quad \text{and} \quad |f(q) - u| + |g(q) - u| \leq c_0 r.$$

Then there are two maps $f', g' : B_{5r}(p, \varkappa_0) \rightarrow \varkappa_0^\perp$ such that

- (a) $\mathbf{G}_{f'} = \mathbf{G}_f \llcorner \mathbf{C}_{5r}(p, \varkappa_0)$ and $\mathbf{G}_{g'} = \mathbf{G}_g \llcorner \mathbf{C}_{5r}(p, \varkappa_0)$;
- (b) $\|f' - g'\|_{L^1(B_{5r}(p, \varkappa_0))} \leq C_0 \|f - g\|_{L^1(B_{7r}(p, \varkappa))}$;
- (c) if $f \in C^{3, \kappa}(B_{7r}(p, \varkappa))$ then $f' \in C^{3, \kappa}(B_{5r}(p, \varkappa_0))$ with the estimates

$$\|f' - u'\|_{C^0} \leq C \|f - u\|_{C^0} + C |\varkappa - \varkappa_0| r \tag{B.1}$$

$$\|Df'\|_{C^0} \leq C \|Df\|_{C^0} + C |\varkappa - \varkappa_0| \tag{B.2}$$

$$\|D^2 f'\|_{C^{1, \kappa}} \leq \Phi(|\varkappa - \varkappa_0|, \|D^2 f\|_{C^{1, \kappa}}) \tag{B.3}$$

where $(q', u') \in \varkappa_0 \times \varkappa_0^\perp$ coincides with the point $(q, u) \in \varkappa \times \varkappa^\perp$ and Φ is a smooth functions with $\Phi(\cdot, 0) \equiv 0$;

- (d) $\|f' - g'\|_{W^{1,2}(B_{5r}(p, \varkappa_0))} \leq C_0 (1 + \|D^2 f\|_{C^0}) \|f - g\|_{W^{1,2}(B_{7r}(p, \varkappa))}$.

All the conclusions of the Lemma still hold if we replace the exterior radius $7r$ and interior radius $5r$ with ρ and s : the corresponding constants c_0 and C_0 (and functions Φ , Λ and Λ_κ) will then depend also on the ratio $\frac{\rho}{s}$.

Proof. The case of two general radii s and ρ follows easily from that of $\rho = 7r$ and $s = 5r$ and a simple covering argument. In what follows, given a pair of points $x \in \varkappa, y \in \varkappa^\perp$ we use the notation (x, y) for the vector $x + y$. By translation invariance we can assume that $(q, u) = (0, 0)$ (and hence $(q', u') = (0, 0)$). Consider then the maps $F, G : B_{7r}(0, \varkappa) \rightarrow \varkappa_0^\perp$ and $I, J : B_{7r}(0, \varkappa) \rightarrow \varkappa_0$ given by

$$F(x) = \mathbf{p}_{\varkappa_0^\perp}((x, f(x))) \quad \text{and} \quad G(x) = \mathbf{p}_{\varkappa_0^\perp}((x, g(x))),$$

$$I(x) = \mathbf{p}_{\varkappa_0}((x, f(x))) \quad \text{and} \quad J(x) = \mathbf{p}_{\varkappa_0}((x, g(x))).$$

With $c_0 \leq 1$ we can easily estimate

$$|I(x) - I(y)| \geq |x - y|(1 - C_0|\varkappa - \varkappa_0| - C_0\text{Lip}(f)),$$

for some geometric constant C_0 . Thus, if c_0 is small enough I and (for the same reason) J are injective Lipschitz maps. Therefore, the graphs $\text{Gr}_{\varkappa_0}(f)$ and $\text{Gr}_{\varkappa_0}(g)$ of f and g in the “original” coordinates system $\varkappa_0 \times \varkappa_0$ coincide, in the new coordinate system $\varkappa \times \varkappa$, with the graphs $\text{Gr}_{\varkappa}(f')$ and $\text{Gr}_{\varkappa}(g')$ of the functions $f' = F \circ I^{-1}$ and $g' = G \circ J^{-1}$ defined respectively in $D := I(B_{7r}(0, \varkappa))$ and $\bar{D} := J(B_{7r}(0, \varkappa))$. If c_0 is chosen sufficiently small, then we also conclude

$$\text{Lip}(I), \text{Lip}(J), \text{Lip}(I^{-1}), \text{Lip}(J^{-1}) \leq 1 + C c_0, \quad (\text{B.4})$$

and

$$|I(0)|, |J(0)| \leq C c_0 r, \quad (\text{B.5})$$

where the constant C is only geometric. Clearly, (B.4) and (B.5) easily imply that $B_{5r}(0, \varkappa_0) \subset D \cap \bar{D}$ when c_0 is smaller than a geometric constant, thereby implying (a) if we restrict the domain of definition of f' and g' to $B_{5r}(0, \varkappa_0)$. We claim next that, for a sufficiently small c_0 ,

$$|f'(x') - g'(x')| \leq 2|f(I^{-1}(x')) - g(I^{-1}(x'))| \quad \forall x' \in B_{5r}(0, \varkappa_0), \quad (\text{B.6})$$

from which, using the change of variables formula for bilipschitz homeomorphisms and (B.4), (b) follows.

In order to prove (B.6), consider any $x' \in B_r(q')$, set $x := I^{-1}(x')$ and

$$p_1 := (x, f(x)) \in \varkappa \times \varkappa^\perp, \quad p_2 := (x, g(x)) \in \varkappa \times \varkappa^\perp \quad \text{and} \quad p_3 := (x', g'(x')) \in \varkappa_0 \times \varkappa_0^\perp.$$

Obviously $|f'(x') - g'(x')| = |p_1 - p_3|$ and $|f(x) - g(x)| = |p_1 - p_2|$. Note that, $g(x) = f(x)$ if and only if $g'(x') = f'(x')$, and in this case (B.6) follows trivially. If this is not the case, the triangle with vertices p_1, p_2 and p_3 is non-degenerate. Let θ_i be the angle at p_i . Note that, $\text{Lip}(g) \leq c_0$ implies $|\frac{\pi}{2} - \theta_2| \leq C c_0$ and $|\varkappa - \varkappa_0| \leq c_0$ implies $|\theta_1| \leq C c_0$, for some dimensional constant C . Since $\theta_3 = \pi - \theta_1 - \theta_2$, we conclude as well $|\frac{\pi}{2} - \theta_3| \leq C c_0$. Therefore, if c_0 is small enough, we have $1 \leq 2 \sin \theta_3$, so that, by the Sinus Theorem,

$$|f'(x') - g'(x')| = |p_1 - p_3| = \frac{\sin \theta_2}{\sin \theta_3} |p_1 - p_2| \leq 2 |p_1 - p_2| = 2 |f(x) - g(x)|,$$

thus concluding the claim.

As for (c), observe that I, F and J, G are obviously as regular as f and g . So, when the latter are C^1 , I and J are also C^1 . In the latter case, if we put suitable coordinates on both \varkappa and \varkappa_0 (identifying them with \mathbb{R}^m) we can easily estimate $|dI - \text{Id}| \leq C_0(\|Df\|_0 + |\varkappa - \varkappa_0|)$, where C_0 is a geometric constant, dI the differential of I and Id the identity. Thus for c_0 sufficiently small we can apply the inverse function theorem: so I^{-1} is as regular as I and hence as f . Since $f' = F \circ I^{-1}$, also f' is as regular as f . Recall next that we are assuming $q = 0$ and $u = 0$. Define the map $\tilde{I}(x) = I^{-1}(x) - x$. Since $f' = F \circ I^{-1}$, the bounds claimed in (c) follows easily if we can prove the very same bounds for the map $\tilde{I}(x)$. If we set $\bar{I}(x) = I(x) - x$, the inverse function theorem gives $\|\tilde{I}\|_{C^1} \leq 2\|\bar{I}\|_{C^1}$ provided c_0

is sufficiently small. The bounds on the higher derivatives can then be easily concluded differentiating the identity $dI^{-1}(x) = [dI]^{-1}(I^{-1}(x))$.

We finally come to (d). The estimate $\|f' - g'\|_{L^2} \leq C\|f - g\|_{L^2}$ is an obvious consequence of (B.6). Given next a point p in the graph of f , resp. in the graph of g , we denote by $\sigma(p)$, resp. $\tau(p)$, the oriented tangent plane to the corresponding graphs. Observe that the points are described by the pairs $(x', f(x'))$ and $(x', g(x'))$, in the coordinates $\kappa \times \kappa^\perp$, and by $(I^{-1}(x'), f(I^{-1}(x')))$ and $(J^{-1}(x'), g(J^{-1}(x')))$, in the coordinates $\varkappa_0 \times \varkappa_0^\perp$. Thus

$$\begin{aligned} & |\nabla f'(x') - \nabla g'(x')| \leq C|\sigma(p) - \tau(q)| \leq C|\nabla f(I^{-1}(x')) - \nabla g(J^{-1}(x'))| \\ & \leq C|\nabla f(I^{-1}(x')) - \nabla f(J^{-1}(x'))| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))| \\ & \leq C\|D^2 f\|_{C^0}|I^{-1}(x') - J^{-1}(x')| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))| \\ & \leq C\|D^2 f\|_{C^0}|f'(x') - g'(x')| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))|. \end{aligned} \quad (\text{B.7})$$

Integrating this last inequality in x' and changing variables we then conclude

$$\|\nabla f' - \nabla g'\|^2 \leq C\|\nabla f - \nabla g\|_{L^2} + C\|D^2 f\|_{C^0}\|f' - g'\|_{L^2},$$

which, together with the L^2 estimate, gives (d). \square

APPENDIX C. TWO INTERPOLATION INEQUALITIES

Lemma C.1. *Let $A > 0$ and $\psi \in C^2(B_\rho, \mathbb{R}^n)$ satisfy $\|\psi\|_{L^1} \leq A\rho^{m+1}$ and $\|\Delta\psi\|_{L^\infty} \leq \rho^{-1}A$. Then, for every $r < \rho$ there is a constant $C > 0$ (depending only on m and $\frac{\rho}{r}$) such that*

$$\rho^{-1}\|\psi\|_{L^\infty(B_r)} + \|D\psi\|_{L^\infty(B_r)} \leq CA. \quad (\text{C.1})$$

Proof. By a simple covering argument we can, w.l.o.g., assume $\rho = 3r$. Moreover, if we apply the scaling $\psi_r(x) := r^{-1}\psi(rx)$ we see that $\|\psi_r\|_{L^1(B_3)} = (\rho/3)^{-m-1}\|\psi\|_{L^1(B_\rho)}$, $\|\psi_r\|_\infty = (\rho/3)^{-1}\|\psi\|_\infty$, $\|D\psi_r\|_\infty = \|D\psi\|_\infty$ and $\|\Delta\psi_r\|_\infty = (\rho/3)\|\Delta\psi\|_\infty$. We can therefore assume $r = 1$. Consider the harmonic function $\zeta : B_2 \rightarrow \mathbb{R}$ with boundary data $\psi|_{\partial B_2}$,

$$\begin{cases} \Delta\zeta = 0 & \text{in } B_2, \\ \zeta = \psi & \text{on } \partial B_2. \end{cases}$$

Set $u := \psi - \zeta$ and note that $u = 0$ on ∂B_2 , $\|\Delta u\|_{C^0(B_2)} \leq A$. Hence, using the Poincaré inequality, we can estimate the L^1 -norm of u in the following way:

$$\|u\|_{L^1} \leq \|u\|_{L^2} \leq C\|Du\|_{L^2} \leq C\left(\int_{B_2} |\Delta u u|\right)^{1/2} \leq C\|\Delta u\|_{C^0}^{1/2}\|u\|_{L^1}^{1/2} \leq CA.$$

Choose now $a \in]0, 1[$ and $s \in]1, \infty[$ such that $\frac{1}{m} + a\left(\frac{1}{s} - \frac{2}{m}\right) + 1 - a < 0$ (which exist because for $s \rightarrow \infty$ and $a \rightarrow 1$ the expression converges to $-\frac{1}{m}$). By a classical interpolation inequality, (see [9])

$$\|Du\|_{L^\infty} \leq C\|D^2 u\|_{L^s}^a \|u\|_{L^1}^{1-a} + C\|u\|_{L^1}.$$

Using the L^s -estimate for the Laplacian, we deduce

$$\|Du\|_{L^\infty} \leq C\|\Delta u\|_{L^s}^a \|u\|_{L^1}^{1-a} + C\|u\|_{L^1} \leq C\|\Delta u\|_\infty^a \|u\|_{L^1}^{1-a} + \|u\|_{L^1} \leq CA. \quad (\text{C.2})$$

From (C.2) and $u|_{\partial B_2} = 0$ it follows trivially $\|u\|_{L^\infty} \leq A$. To infer (C.1), we observe that, by $\|\zeta\|_{L^1(B_2)} \leq \|u\|_{L^1(B_2)} + \|\psi\|_{L^1(B_2)} \leq C A$ and the harmonicity of ζ ,

$$\|\zeta\|_{L^\infty(B_1)} + \|D\zeta\|_{L^\infty(B_1)} \leq C\|\zeta\|_{L^1(B_2)} \leq C A. \quad \square$$

Lemma C.2. *For every $m, r < s$ and κ there is a positive constant C (depending on m, κ and $\frac{s}{r}$) with the following property. Let f be a $C^{3,\kappa}$ function in the ball $B_s \subset \mathbb{R}^m$. Then*

$$\|D^j f\|_{C^0(B_r)} \leq C r^{-m-j} \|f\|_{L^1(B_s)} + C r^{3+\kappa-j} [D^3 f]_{\kappa, B_s} \quad \forall j \in \{0, 1, 2, 3\}. \quad (\text{C.3})$$

Proof. A simple covering argument reduces the lemma to the case $s = 2r$. Moreover, define $f_r(x) := f(rx)$ to see that we can assume $r = 1$. So our goal is to show

$$\sum_{j=0}^3 |D^j f(y)| \leq C \|f - g\|_{L^1} + C [D^3 f]_{\kappa} \quad \forall y \in B_1, \forall f \in C^{3,\kappa}(B_2). \quad (\text{C.4})$$

By translating it suffices then to prove the estimate

$$\sum_{j=0}^3 |D^j f(0)| \leq C \|f\|_{L^1(B_1)} + C [D^3 f]_{\kappa, B_1} \quad \forall f \in C^{3,\kappa}(B_1). \quad (\text{C.5})$$

Consider now the space of polynomials R in m variables of degree at most 3, which we write as $R = \sum_{j=0}^3 A_j x^j$. This is a finite dimensional vector space, on which we can define the norms $|R| := \sum_{j=0}^3 |A_j|$ and $\|R\| := \int_{B_1} |R(x)| dx$. These two norms must then be equivalent, so there is a constant C (depending only on m), such that $|R| \leq C \|R\|$ for any such polynomial. In particular, if P is the Taylor polynomial of third order for f at the point 0, we conclude

$$\begin{aligned} \sum_{j=0}^3 |D^j f(0)| &= |P| \leq C \|P\| = C \int_{B_1} |P(x)| dx \leq C \|f\|_{L^1(B_1)} + C \|f - P\|_{L^1(B_1)} \\ &\leq C \|f\|_{L^1} + C [D^3 f]_{\kappa}. \end{aligned} \quad \square$$

APPENDIX D. PROOF OF LEMMA 5.6

D.1. Reduction to special triples of planes. We first observe that, by a simple scaling, we can assume $r = 1$. The rescaling which we apply to any map φ is the usual $x \mapsto r^{-1}\varphi(rx) =: \varphi_r$. It is easy to see that (5.35) is then scaling invariant.

We next fix the following terminology: we say that $R \in SO(m + \bar{n} + l)$ is a *2d-rotation* if there are two orthonormal vectors e_1, e_2 and an angle θ such that $R(e_1) = \cos \theta e_1 + \sin \theta e_2$, $R(e_2) = \cos \theta e_2 - \sin \theta e_1$ and $R(v) = v$ for every $v \perp \text{span}(e_1, e_2)$. Given a triple $(\bar{\pi}, \bar{\varkappa}, \bar{\omega})$ we then say that:

- R is of type A with respect to $(\bar{\pi}, \bar{\varkappa}, \bar{\omega})$ if $e_1 \in \bar{\varkappa}$ and $e_2 \in \bar{\omega}$;
- R is of type B with respect to $(\bar{\pi}, \bar{\varkappa}, \bar{\omega})$ if $e_1 \in \bar{\pi}$ and $e_2 \in \bar{\varkappa}$;
- R is of type C with respect to $(\bar{\pi}, \bar{\varkappa}, \bar{\omega})$ if $e_1 \in \bar{\pi}$ and $e_2 \in \bar{\omega}$.

The following lemma will then allow us to reduce the general case of Lemma 5.6 to the particular ones in which $(\bar{\pi}, \bar{\varkappa}, \bar{\varpi})$ is obtained from (π, \varkappa, ϖ) through a (small) rotation of type A, B or C.

Lemma D.1. *There are constants $C_0(m, \bar{n}, l)$ and $\bar{N}(m, \bar{n}, l)$ with the following property. If c_0 in Lemma 5.6 is sufficiently small, then there are $N \leq \bar{N}$ triples $(\pi_j, \varkappa_j, \varpi_j)$ “joining” $(\pi, \varkappa, \varpi) = (\pi_N, \varkappa_N, \varpi_N)$ with $(\bar{\pi}, \bar{\varkappa}, \bar{\varpi}) = (\pi_0, \varkappa_0, \varpi_0)$ such that each $(\pi_j, \varkappa_j, \varpi_j)$ is the image of $(\pi_{j-1}, \varkappa_{j-1}, \varpi_{j-1})$ under a $2d$ -rotation of type A, B or C and angle θ_j with $|\theta_j| \leq C_0(|\pi - \bar{\pi}| + |\varkappa - \bar{\varkappa}|)$.*

Proof. We first show that, if $\varpi = \bar{\varpi}$, or $\varkappa = \bar{\varkappa}$ or $\pi = \bar{\pi}$, then the claim can be achieved with small $2d$ -rotations all of the same type, namely of type B, C and A, respectively. Assume for instance that $\varpi = \bar{\varpi}$. Let ω be the intersection of π and $\bar{\pi}$ and ω' be the intersection of \varkappa and $\bar{\varkappa}$. Pick a vector $e \in \pi$ which is not contained in $\bar{\pi}$ and is orthogonal to ω . Let $\bar{e} := \frac{\mathbf{p}_{\bar{\pi}}(e)}{|\mathbf{p}_{\bar{\pi}}(e)|}$. Then, \bar{e} is necessarily orthogonal to ω and the angle between \bar{e} and e is controlled by $|\pi - \bar{\pi}|$. There is therefore a $2d$ -rotation R such that $R(e) = \bar{e}$ and obviously its angle is controlled by $|\pi - \bar{\pi}|$. It turns out that R keeps ϖ and ω fixed. So the new triple $(R(\pi), R(\varkappa), R(\varpi))$ has the property that $R(\varpi) = \varpi = \bar{\varpi}$ and the dimension of $R(\pi) \cap \bar{\pi}$ is larger than that of $\pi \cap \bar{\pi}$. This procedure can be repeated and after $N \leq m$ times it leads to a triple of planes $(\pi_N, \varkappa_N, \varpi_N)$ with $\varpi_N = \bar{\varpi}$ and $\pi_N = \bar{\pi}$. This however implies necessarily $\bar{\varkappa} = \varkappa_N$.

Assume therefore that ϖ and $\bar{\varpi}$ do not coincide. Let $\omega := (\varkappa \times \pi) \cap (\bar{\varkappa} \times \bar{\pi})$. There is then a unit vector $\bar{e} \in \bar{\varkappa}$ or a unit vector $\bar{e} \in \bar{\pi}$ which does not belong to $\pi \times \varkappa$ and which is orthogonal to ω . Assume for the moment that we are in the first case, and consider the vector $e := \frac{\mathbf{p}_{\pi \times \varkappa}(\bar{e})}{|\mathbf{p}_{\pi \times \varkappa}(\bar{e})|}$. The vector e forms an angle with the plane \varkappa bounded by $C_0|\varkappa - \bar{\varkappa}|$. Therefore there is a rotation R with angle smaller than $C_0|\varkappa - \bar{\varkappa}|$ of the plane $\pi \times \varkappa$ with the property that $R(\varkappa)$ contains e and fixes ω , which is orthogonal to e . By the previous step, R can be written as composition $R_{N'} \circ \dots \circ R_1$ of small $2d$ -rotations of type B keeping ϖ fixed. Since $e \perp R(\pi)$, we can then find a small $2d$ -rotation S of type A with respect to $(R(\pi), R(\varkappa), \varpi)$ acting on the plane spanned by e and \bar{e} and such that $S(R(\varkappa)) \ni \bar{e}$. S keeps then ω fixed. An analogous argument works if the vector $e \in \bar{\pi}$. We therefore conclude that, after applying a finite number of rotations $R_1, \dots, R_{N'}, R_{N'+1}$ of the three types above, the dimension of $R_{N'+1} \circ R_{N'} \circ \dots \circ R_1(\pi \times \varkappa) \cap \bar{\pi} \times \bar{\varkappa}$ is larger than that $\pi \times \varkappa \cap \bar{\pi} \times \bar{\varkappa}$ (where the number N' is smaller than a geometric constant depending only on m and \bar{n}). Obviously, after at most $m + \bar{n}$ iterations of this argument, we are reduced to the situation $\pi \times \varkappa = \bar{\pi} \times \bar{\varkappa}$. \square

Assume now to have proved Lemma 5.6 for some constants c_0 and C_0 and for all $2d$ rotations which are of type A, B or C with respect to one of the two triples of planes. We next claim that, at the price of possibly enlarging the constants, the Lemma holds for any pair of triples. To this purpose we now fix two triples as in the statement of the lemma and choose a chain $(\pi_j, \varkappa_j, \varpi_j)$ as in Lemma D.1. As already observed it suffices to prove the statement when $r = 1$, but we assume of having proved it for *any* radius in the case of small $2d$ rotations of type A, B or C. Lemma D.1 implies that $|\pi_j - \pi_i| + |\varkappa_j - \varkappa_i| \leq \bar{C}_0 \Delta_n$ for

some geometric constant \bar{C}_0 . For each i we therefore have Lipschitz maps $\Psi^i : \pi_i \times \varkappa_i \rightarrow \varpi_i$ and Lipschitz maps $f^i : B_4(0, \pi) \rightarrow \mathcal{A}_Q(\varkappa \times \varpi)$ whose graph coincides with the ones of $\bar{\Psi}$ and f (the latter restricted to $\mathbf{C}_4(0, \pi)$): their existence is ensured by [6, Proposition 5.2] which also implies

$$\|D\Psi_i\|_{C^0} \leq \bar{C}_0(\|D\Psi\|_{C^0} + \text{An}) \quad (\text{D.1})$$

$$|\Psi_i(0)| \leq \bar{C}_0(|\Psi(0)| + \|D\Psi\|_{C^0} + \text{An}) \quad (\text{D.2})$$

$$\text{Lip}(f^i) \leq \bar{C}_0(\text{Lip}(f) + \text{An}) \quad (\text{D.3})$$

$$\|f^i\|_{C^0} \leq \bar{C}_0(\|f\|_{C^0} + \text{An}) \quad (\text{D.4})$$

Set now $r_i := 2^{2-i}$. By assuming the constant c_0 sufficiently small we can therefore assume that the Lemma can be applied to the pairs $(\pi_{i-1}, \varkappa_{i-1}, \varpi_{i-1})$ and $(\pi_i, \varkappa_i, \varpi_i)$, to the maps $\Psi_{i-1}, \Psi_i, f^{i-1}, f^i$ and to the radius $r_i/4$. In order to streamline the argument, for $j > i$ we use the notation $f^j = \mathbf{R}_{ij}f^i$ to underline that the graph of f^j coincides, in the cylinder $\mathbf{C}_{r_i}(0, \pi_i)$, with the graph of f^i . Likewise, if u^i is the multivalued map into $\mathcal{A}_Q(\varkappa_i)$ such that $f^i(x) = \sum_l \llbracket (u_l^i(x), \Psi_i(u_l^i(x))) \rrbracket$, we then denote by $\text{Av}(f^i)$ the map $(\boldsymbol{\eta} \circ u^i, \Psi_i(\boldsymbol{\eta} \circ u^i))$. With this notation we observe that $\hat{\mathbf{f}} = \mathbf{R}_{N0}(\text{Av}(f^0)) = \mathbf{R}_{N0}(\text{Av}(f))$ and $\mathbf{g} = \text{Av}(\mathbf{R}_{N0}(f^0)) = \text{Av}(\mathbf{R}_{N0}(f))$. We can then estimate

$$\begin{aligned} \|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1(B_{r_N}(\pi, 0))} &\leq \underbrace{\|\mathbf{R}_{N(N-1)}(\mathbf{R}_{(N-1)0}(\text{Av}(f))) - \mathbf{R}_{N(N-1)}(\text{Av}(\mathbf{R}_{(N-1)0}(f)))\|_{L^1(B_{r_N}(\pi_N, 0))}}_{(I)} \\ &+ \underbrace{\|\mathbf{R}_{N(N-1)}(\text{Av}(\mathbf{R}_{(N-1)0}(f))) - \text{Av}(\text{Rot}_{N(N-1)}(\mathbf{R}_{(N-1)0}(f)))\|_{L^1(B_{r_N}(\pi_N, 0))}}_{(II)}. \end{aligned}$$

Now, to the first summand we apply Lemma B.1(b) and we bound it with

$$(I) \leq \bar{C}_0 \|\mathbf{R}_{(N-1)0}(\text{Av}(f)) - \text{Av}(\mathbf{R}_{(N-1)0}(f))\|_{L^1(B_{r_{N-1}}(0, \pi_{N-1}))}.$$

As for the second summand, observing that $\mathbf{R}_{(N-1)0}(f) = f^{N-1}$, we can apply Lemma 5.6 for the special case of a 2d rotation of type A, B or C and conclude

$$\begin{aligned} (II) &\leq \bar{C}_0(\|f^{N-1}\|_{C^0} + |\pi_N - \pi_{N-1}| + |\varkappa_N - \varkappa_{N-1}|)(\text{Dir}(f^{N-1}) \\ &\quad + \|D\Psi_{N-1}\|_{C^0}^2 + (|\pi_N - \pi_{N-1}| + |\varkappa_N - \varkappa_{N-1}|)^2) \\ &\leq \bar{C}_0(\|f^{N-1}\|_{C^0} + \text{An})(\text{Dir}(f^{N-1}) + \|D\bar{\Psi}\|_{C^0}^2 + \text{An}^2). \end{aligned}$$

On the other hand, by the Taylor expansion of the mass in [6, Corollary 3.3],

$$\begin{aligned} \text{Dir}(f^{N-1}) &\leq 4\mathbf{E}(\mathbf{G}_{f^{N-1}}, \mathbf{C}_{r_{N-1}}(0, \pi_{N-1})) \\ &\leq 4\mathbf{E}(\mathbf{G}_{f^{N-1}}, \mathbf{C}_{r_{N-1}}(0, \pi_{N-1}), \pi_N) + \bar{C}_0|\pi_{N-1} - \pi_N|^2 \\ &\leq 4\mathbf{E}(\mathbf{G}_f, \mathbf{C}_8(0, \bar{\pi})) + \bar{C}_0\text{An}^2 \leq 8\text{Dir}(f) + \bar{C}_0\text{An}^2. \end{aligned} \quad (\text{D.5})$$

Putting all these estimates together we then conclude

$$\begin{aligned} \|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1(B_{r_N}(\pi,0))} &\leq \bar{C}_0 \|\mathbf{R}_{(N-1)0}(\text{Av}(f)) - \text{Av}(\mathbf{R}_{(N-1)0}(f))\|_{L^1(B_{r_{N-1}}(0,\pi_{N-1}))} \\ &\quad + \bar{C}_0 (\|f\|_{C^0} + \text{An}) (\text{Dir}(f) + \|D\Psi\|_{C^0}^2 + \text{An}^2). \end{aligned}$$

We can now iterate $N - 1$ more times this argument to finally achieve

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1(B_{r_N}(\pi,0))} \leq \bar{C}_0 (\|f\|_{C^0} + \text{An}) (\text{Dir}(f) + \|D\Psi\|_{C^0}^2 + \text{An}^2).$$

Of course this is not yet the estimate claimed in Lemma 5.6 since the inner radius r_N equals 2^{2-N} rather than 4. However a simple covering argument allows to conclude the proof. In the remaining sections we focus our attention on 2D rotations of coordinates of type A, B and C.

D.2. Type A. As already observed it suffices to show the lemma in the case $r = 1$. We use the notation $(z, w) \in \varkappa \times \varpi$ and $(\bar{z}, \bar{w}) \in \bar{\varkappa} \times \bar{\varpi}$ for the same point. In what follows we will drop the \cdot when writing the usual products between matrices. We then have $\bar{z} = Uz + Vw$ and $\bar{w} = Wz + Zw$, where the orthogonal matrix

$$L := \begin{pmatrix} U & V \\ W & Z \end{pmatrix}$$

has the property that $|L - \text{Id}| \leq C_0 \text{An}$. Clearly, Ψ and $\bar{\Psi}$ are related by the identity

$$Wz + Z\Psi(x, z) = \bar{\Psi}(x, Uz + V\Psi(x, z)). \quad (\text{D.6})$$

Fix x and $\hat{f}(x) = \sum_i \llbracket (\hat{u}_i(x), \Psi(x, \hat{u}_i(x))) \rrbracket =: \sum_i \llbracket (z_i, \Psi(x, z_i)) \rrbracket$. We then have

$$\mathbf{g}(x) = (a, b) := \left(\frac{1}{Q} \sum z_i, \Psi\left(x, \frac{1}{Q} \sum z_i\right) \right) \quad \text{in } \varkappa \times \varpi,$$

and

$$\hat{\mathbf{f}}(x) = L^{-1} \left(U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i), \bar{\Psi}\left(x, U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i)\right) \right) =: L^{-1}(c, d).$$

Since L is orthogonal, we have

$$\begin{aligned} |\hat{\mathbf{f}}(x) - \mathbf{g}(x)| &= |L(a, b) - (c, d)| \\ &= \left| \left(V \left(\Psi\left(x, \frac{1}{Q} \sum z_i\right) - \frac{1}{Q} \sum_i \Psi(x, z_i) \right), W \frac{1}{Q} \sum_i z_i + Z \Psi\left(x, \frac{1}{Q} \sum_i z_i\right) \right. \right. \\ &\quad \left. \left. - \bar{\Psi}\left(x, U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i)\right) \right) \right| \\ &\stackrel{(\text{D.6})}{=} \left| \left(V \left(\Psi\left(x, \frac{1}{Q} \sum z_i\right) - \frac{1}{Q} \sum_i \Psi(x, z_i) \right), \right. \right. \\ &\quad \left. \left. \bar{\Psi}\left(x, U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i)\right) - \bar{\Psi}\left(x, U \frac{1}{Q} \sum z_i + V \frac{1}{Q} \sum \Psi(x, z_i)\right) \right) \right|. \end{aligned}$$

Thus,

$$|\hat{\mathbf{f}}(x) - \mathbf{g}(x)| \leq (1 + \text{Lip}(\bar{\Psi})) |V| \left| \frac{1}{Q} \sum \Psi(x, z_i) - \Psi\left(x, \frac{1}{Q} \sum z_i\right) \right|.$$

Observe that $|V| \leq |L - \text{Id}| \leq C|\varkappa - \bar{\varkappa}|$. Moreover, with a simple Taylor expansion around the point $(x, \frac{1}{Q} \sum z_i)$ we achieve

$$\left| \frac{1}{Q} \sum \Psi(x, z_i) - \Psi\left(x, \frac{1}{Q} \sum z_i\right) \right| \leq C_0 \|D\Psi\|_0 \sum_i \left| z_i - \frac{1}{Q} \sum z_i \right| \leq C_0 \|D\Psi\|_0 \|\hat{u}\|_{C^0}.$$

Since we have $\|D\Psi\|_0 \leq C_0 \|D\bar{\Psi}\|_0 + C_0 \text{An}$ and $\|\hat{u}\|_{C^0} \leq \|\hat{f}\|_{C^0} \leq C\|f\|_{C^0} + C_0 \text{An}$ we conclude the pointwise estimate

$$|\hat{\mathbf{f}}(x) - \mathbf{g}(x)| \leq C_0 \text{An} (\|D\bar{\Psi}\|_0 + \text{An}) (\|f\|_0 + \text{An}),$$

which obviously implies (5.35).

D.3. Type B. In this case $\Psi = \bar{\Psi}$ and thus

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1} \leq C_0 (1 + \|D\bar{\Psi}\|_0) \|\boldsymbol{\eta} \circ \hat{u} - \mathbf{p}_\varkappa(\hat{\mathbf{f}})\|_{L^1}. \quad (\text{D.7})$$

Fix next an orthonormal base $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+\bar{n}}$, where the first m vectors span π and the remaining span \varkappa . We also assume that the rotation R acts on the plane spanned by $\{e_m, e_{m+1}\}$ and set $v = R(e_m) = a e_m + b e_{m+1}$ and $v_{m+1} = R(e_{m+1})$. We then define two systems of coordinates: given $q \in \mathbb{R}^m \times R^{\bar{n}}$, we write

$$\begin{aligned} q &= \sum_{1 \leq i \leq m-1} z_i(q) e_i + t(q) e_m + \tau(q) e_{m+1} + \sum_{2 \leq j \leq \bar{n}} y^j(q) e_{j+m} \\ &= \sum_i z_i(q) e_i + s(q) v_m + \sigma(q) v_{m+1} + \sum_j y^j(q) e_{j+m}. \end{aligned}$$

The first will be called (t, τ) -coordinates and the second (s, σ) -coordinates.

We fix for the moment $x \in \mathbb{R}^{m-1}$ with $|x| \leq 4$ and focus our attention on the interval $I_x = \{s : |(x, s)| \leq 6\}$. We restrict the map u to this interval and, by [4, Proposition 1.2] we know that there is a Lipschitz selection such that $u(x, s) = \sum_i \llbracket \theta_i(s) \rrbracket$ in the (s, σ) -coordinates: $\text{Gr}(\theta_i) = \{(x, s, \theta_i^1(s), \dots, \theta_i^{\bar{n}}(s)) : s \in I_x\}$. In the (t, τ) coordinates we can choose functions ϑ_i , also defined on an appropriate interval J_x , whose graphs coincide with the ones of the θ_i . We then obviously must have $\hat{u}(x, t) = \sum_i \llbracket \vartheta_i(t) \rrbracket$ on the domain of definition of \hat{f} . The coordinate functions θ_i^j and ϑ_i^j are linked by the following relations

$$\begin{cases} \Phi_i(t) = a t + b \vartheta_i^1(t), \\ \theta_i^1(\Phi_i(t)) = -b t + a \vartheta_i^1(t), \\ \theta_i^l(\Phi_i(t)) = \vartheta_i^l(t), \end{cases} \quad \text{for } l = 2, \dots, \bar{n}. \quad (\text{D.8})$$

Observe that $\text{Lip}(\Phi_i) \leq (1 + C_0 |\pi - \bar{\pi}|) \leq 2$. Likewise we can assume that $\text{Lip}(\Phi_i^{-1}) \leq 2$. Consider now $v(s) = \boldsymbol{\eta} \circ u(x, s) = \frac{1}{Q} \sum_i \theta_i(s)$ and the corresponding $t \mapsto \hat{v}(t) = \mathbf{p}_\varkappa \circ \hat{\mathbf{f}}(x, t)$, linked to $v = \boldsymbol{\eta} \circ u(x, \cdot)$ through a relation as in (D.8) with a corresponding map Φ :

$$\begin{cases} \Phi(t) = a t + b \hat{v}^1(t), \\ \frac{1}{Q} \sum_i \theta_i^1(\Phi(t)) = v^1(\Phi(t)) = -b t + a \hat{v}^1(t), \\ \frac{1}{Q} \sum_i \theta_i^l(\Phi(t)) = v^l(\Phi(t)) = \hat{v}^l(t), \end{cases} \quad \text{for } l = 2, \dots, \bar{n}. \quad (\text{D.9})$$

Moreover, write $\tilde{v}(t) = \frac{1}{Q} \sum_i \vartheta_i(t) = \boldsymbol{\eta} \circ \hat{u}(x, t)$. We can then compute

$$\begin{aligned} \boldsymbol{\eta} \circ \hat{u}(x, t) - \mathbf{p}_\varkappa(\hat{\mathbf{f}}(x, t)) &= \tilde{v}(t) - \hat{v}(t) = Q^{-1} \sum_i (\vartheta_i(t) - \hat{v}(t)) \\ &= Q^{-1} \sum_i \left(\underbrace{a^{-1}\theta_i^1(\Phi_i(t)) - a^{-1}\theta_i^1(\Phi(t))}_{1^{\text{st}} \text{ component}}, \dots, \underbrace{\theta_i^l(\Phi_i(t)) - \theta_i^l(\Phi(t))}_{l^{\text{th}} \text{ component}}, \dots \right). \end{aligned} \quad (\text{D.10})$$

This implies that

$$|\boldsymbol{\eta} \circ \hat{u}(x, t) - \mathbf{p}_\varkappa(\hat{\mathbf{f}}(x, t))| = |\tilde{v}(t) - \hat{v}(t)| \leq C_0 \sum_i \left| \int_{\Phi(t)}^{\Phi_i(t)} D\theta(\tau) d\tau \right|. \quad (\text{D.11})$$

Next we compute

$$\Phi_i(t) - \Phi(t) = b(\vartheta_i^1(t) - \hat{v}^1(t)) = b(\vartheta_i^1(t) - \tilde{v}^1(t)) + b(\tilde{v}^1(t) - \hat{v}^1(t)). \quad (\text{D.12})$$

Since $|b| \leq C\text{An}$, the terms in (D.12) can be estimated respectively as follows:

$$\begin{aligned} |b|\|\vartheta_i^1(t) - \tilde{v}^1(t)\| &= |b|\|\hat{u}_i^1(x, t) - (\boldsymbol{\eta} \circ \hat{u})^1(t)\| \leq C_0\text{An} \|\hat{u}\|_{C^0}, \\ |\tilde{v}^1(t) - \hat{v}^1(t)| &\stackrel{(\text{D.11})}{\leq} \|D\theta\|_{L^\infty} \sum_{i=1}^Q |\Phi_i(t) - \Phi(t)| \leq C_0 \text{Lip}(u) \sum_{i=1}^Q |\Phi_i(t) - \Phi(t)|. \end{aligned}$$

Recall that $\text{Lip}(u) \leq \text{Lip}(f) \leq C_0\text{Lip}(\hat{f}) + \text{An}$. Combining the last two inequalities with (D.12), we therefore conclude, when c_0 is sufficiently small,

$$\sum_{i=1}^Q |\Phi_i(t) - \Phi(t)| \leq C_0\text{An}\|\hat{f}\|_{C^0} =: \rho. \quad (\text{D.13})$$

With this estimate at our disposal we can integrate (D.11) in t to conclude

$$\int_{J_x} |\tilde{v}(t) - \hat{v}(t)| \leq C_0 \int_{J_x} \int_{\Phi(t)-\rho}^{\Phi(t)+\rho} |D\theta(\tau)| d\tau dt \leq C_0 \int_{I_x} \int_{s-\rho}^{s+\rho} |Du|(x, \tau) d\tau ds,$$

where in the latter inequality we have used the change of variables $s = \Phi(t)$ and the fact that both the Lipschitz constants of Φ and its inverse are under control. Integrating over x and recalling that $\tilde{v}(t) - \hat{v}(t) = \boldsymbol{\eta} \circ \hat{u}(x, t) - \mathbf{p}_\varkappa(\hat{\mathbf{f}}(x, t))$ we achieve

$$\begin{aligned} \int_{B_4} |\boldsymbol{\eta} \circ \hat{u} - \mathbf{p}_\varkappa \circ \hat{\mathbf{f}}| &\leq C_0 \int_{B_4} \int_{-\sqrt{36-|x|^2}}^{\sqrt{36-|x|^2}} \int_{s-\rho}^{s+\rho} |Du|(x, \tau) d\tau ds dx \leq C_0\rho \int_{B_{6+12\rho}} |Du| \\ &\leq C_0\text{An}\|\hat{f}\|_{C^0} \left(\int_{B_8} |Du|^2 \right)^{1/2} \leq C_0\text{An}(\|f\|_{C^0} + \text{An})\text{Dir}(f)^{1/2}. \end{aligned} \quad (\text{D.14})$$

Clearly (D.14) and (D.7) imply the desired estimate.

D.4. **Type C.** Consider $\boldsymbol{\eta} \circ f$ and the $\xi : B_4(0, \pi) \rightarrow \pi^\perp$ such that $\mathbf{G}_\xi = \mathbf{G}_{\boldsymbol{\eta} \circ f} \perp \mathbf{C}_4(0, \pi)$. We can then apply the argument of the estimate for type B to conclude

$$\|\boldsymbol{\eta} \circ \hat{u} - \mathbf{p}_\varkappa(\xi)\|_{L^1(B_4)} \leq \|\boldsymbol{\eta} \circ \hat{f} - \xi\|_{L^1(B_4)} \leq C \text{An}(\|f\|_{C^0} + \text{An}) \text{Dir}(f)^{1/2}. \quad (\text{D.15})$$

We need only to estimate $\|\mathbf{p}_\varkappa(\xi) - \mathbf{p}_\varkappa(\hat{\mathbf{f}})\|_{L^1}$: since $\mathbf{g}(x) = (\boldsymbol{\eta} \circ \hat{u}(x), \Psi(x, \boldsymbol{\eta} \circ \hat{u}(x)))$ and $\hat{\mathbf{f}}(x) = (\mathbf{p}_\varkappa(\hat{\mathbf{f}}(x)), \Psi(x, \mathbf{p}_\varkappa(\hat{\mathbf{f}}(x))))$, we can then estimate

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1} \leq C_0(1 + \|D\Psi\|_0) \left(\|\mathbf{p}_\varkappa(\hat{\mathbf{f}}) - \mathbf{p}_\varkappa(\xi)\|_{L^1} + \|\mathbf{p}_\varkappa(\xi) - \boldsymbol{\eta} \circ \hat{u}\|_{L^1} \right). \quad (\text{D.16})$$

Define the maps v, w and w' as follows:

$$\begin{aligned} \mathbf{f}(\bar{x}) &= (\boldsymbol{\eta} \circ u(\bar{x}), \bar{\Psi}(\boldsymbol{\eta} \circ u(\bar{x}))) =: (v(\bar{x}), w(\bar{x})), \\ \boldsymbol{\eta} \circ f(\bar{x}) &= (\boldsymbol{\eta} \circ u(\bar{x}), \frac{1}{Q} \sum_i \bar{\Psi}(\bar{x}, u_i(\bar{x}))) =: (v(\bar{x}), w'(\bar{x})). \end{aligned}$$

Using the Lipschitz bound for $\bar{\Psi}$ we conclude

$$\|\mathbf{f} - \boldsymbol{\eta} \circ f\|_{C^0} = \|w - w'\|_0 \leq C \|D\bar{\Psi}\|_{C^0} \sum_i |u_i - \boldsymbol{\eta} \circ u| \leq C \|D\bar{\Psi}\|_{C^0} \|f\|_{C^0}. \quad (\text{D.17})$$

Consider an orthogonal transformation

$$L = \begin{pmatrix} U & V \\ W & Z \end{pmatrix}$$

with the properties that $(\bar{x}, \bar{z}) \in \bar{\pi} \times \bar{\varpi}$ corresponds to $(U\bar{x} + V\bar{z}, W\bar{x} + Z\bar{z}) \in \pi \times \varpi$ and $|L - \text{Id}| \leq C_0 \text{An}$. We then have the following relations: $\mathbf{p}_\varkappa(\hat{\mathbf{f}}(x)) = v(\Phi^{-1}(x))$ and $\mathbf{p}_\varkappa(\xi(x)) = v((\Phi')^{-1}(x))$, where Φ^{-1} and $(\Phi')^{-1}$ are the inverse, respectively, of the maps $\Phi(\bar{x}) = U\bar{x} + Vw(\bar{x})$ and $\Phi'(\bar{x}) = U\bar{x} + Vw'(\bar{x})$. Recalling that $|V| \leq |L - \text{Id}| \leq C_0 \text{An}$, we conclude that

$$|\Phi'(\bar{x}) - \Phi(\bar{x})| \leq |V| |w(\bar{x}) - w'(\bar{x})| \leq C_0 \|D\bar{\Psi}\|_{C^0} \|f\|_{C^0} \text{An} \quad \text{for every } \bar{x}.$$

On the other hand we also know that Φ^{-1} has Lipschitz constant at most 2 and so we achieve $|\Phi^{-1}(\Phi'(\bar{x})) - \bar{x}| \leq C_0 \|D\bar{\Psi}\|_{C^0} \|f\|_{C^0} \text{An}$. Being valid for any \bar{x} we can apply it to $\bar{x} = (\Phi')^{-1}(x)$ to conclude $|\Phi^{-1}(x) - (\Phi')^{-1}(x)| \leq C_0 \|D\bar{\Psi}\|_{C^0} \|f\|_{C^0} \text{An}$. Using then $\text{Lip}(v) \leq \text{Lip}(u) \leq c_0$, we conclude the pointwise bound

$$|\mathbf{p}_\varkappa(\hat{\mathbf{f}}(x)) - \mathbf{p}_\varkappa(\xi(x))| = |v(\Phi^{-1}(x)) - v((\Phi')^{-1}(x))| \leq C_0 \|D\bar{\Psi}\|_{C^0} \|f\|_{C^0} \text{An}.$$

After integrating in x , the latter bound combined with (D.15) and (D.16) gives the desired estimate.

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