On the chain rule for the divergence of BV like vector fields: applications, partial results, open problems

Luigi Ambrosio, Camillo De Lellis, Jan Malý

Scuola Normale Superiore, Piazza dei Cavalieri, 56126 Pisa, Italy
Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75, Praha 8, Czech Republic

1 Introduction

In this paper we study the distributional divergence of vector fields U in $\mathbf{R}^d$ of the form $U = wB$, where w is scalar function and B is a weakly differentiable vector field (or more in general the divergence of tensor fields of the form $w \otimes B$). In particular we are interested in a kind of *chain rule* property, relating the divergence of $h(w)B$ to the divergence of wB. In some sense, if one replaces "divergence" by "derivative" this problem is reminiscent to the problem of writing a chain rule for weakly differentiable functions, a theme that has been investigated in several papers (we mention for instance Vol'pert's paper [36] and [3] in the BV setting). However, the "divergence" problem seems to be much harder than the "derivative" problem, due to much stronger cancellation effects. For instance it may happen that $U \in L^\infty_{\rm loc}$ has distributional divergence $f \in L^\infty_{\rm loc}$, but $f \neq 0$ $\mathscr{L}^d$-a.e. on $\{U = 0\}$. This cannot happen for distributional derivatives, see (16).

The problem of writing a chain rule for vector fields $U = wB$ arises in a natural way when one studies the well-posedness of the PDE $D\cdot(wB) = c$, for instance when B has a space-time structure. Indeed one can use $h(s) = s^\pm$ to establish uniqueness and comparison principles, very much like in Kruzhkov's theory of scalar conservation laws (see [29]). When B belongs to a Sobolev space $W^{1,p}_{\rm loc}$ and $w \in L^q_{\rm loc}$, with p, q dual exponents, the chain rule has been established in [25], obtaining

$$D\cdot(h(w)B) = (h(w) - wh'(w))\, D\cdot B + h'(w) D\cdot(wB), \qquad (1)$$

provided $D\cdot(wB)$ is absolutely continuous with respect to $\mathscr{L}^d$. This result has been extended in [6] to the case when $q = \infty$, $B \in BV_{\rm loc}$ and both $D\cdot B$, $D\cdot(wB)$ are absolutely continuous with respect to $\mathscr{L}^d$.

Here we are interested in extending the validity of (1) to the case when these divergences are not necessarily absolutely continuous. As we will see the solution of this problem would have important applications already in the case when wB is divergence-free, and even in this case we still do not have a complete solution.

Looking at (1), it is clear that this extension seems to require the existence of a "good representative" of w, defined not only $\mathscr{L}^d$-a.e., but also up to $|D\cdot B|$–negligible sets when $|D\cdot B|$ is not absolutely continuous with respect to $\mathscr{L}^d$. Our analysis of this problem takes advantage of the techniques introduced in [6] and of the approximate continuity properties for solutions of transport equations with BV coefficients recently proved in [11].

However, our results are not conclusive and they can be summarized as follows. First, in Section 3 we prove that $D\cdot(h(w)B)$ is a measure (even in a vector-valued setting) and we show a chain rule for the absolutely continuous parts of the divergences

$$D^a\cdot(h(w)B) = (h(w) - wh'(w))\, D^a\cdot B + h'(w)D^a\cdot(wB). \tag{2}$$

Second, in Section 4 we characterize the jump part $D^j\cdot(h(w)B)$ (i.e. the one concentrated on $(d-1)$-dimensional sets, see Section 2 for a precise definition):

$$\begin{aligned} D^j\cdot(h(w)B) = \Biggl[&\mathrm{Tr}^+(B,\Sigma)h\left(\frac{\mathrm{Tr}^+(wB,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)}\right) \\ &-\mathrm{Tr}^-(B,\Sigma)h\left(\frac{\mathrm{Tr}^-(wB,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)}\right)\Biggr]\mathscr{H}^{d-1}\llcorner\Sigma,\,. \end{aligned} \tag{3}$$

Here Σ is any countably rectifiable set on which $D^j\cdot B$ and $D^j\cdot(wB)$ are concentrated, whereas $\mathrm{Tr}^+(U,\Sigma)$ and $\mathrm{Tr}^-(U,\Sigma)$ are the normal traces of U on Σ, according to [12], [20], [11]. These one–sided traces coincide for divergence-free vector fields (in general they coincide when the divergence has no jump part). So, a consequence of (3) is that $D^j\cdot(h(w)B) = 0$ when both B and wB are divergence-free.

It remains to characterize the remaining part of the divergence, the so-called Cantor part $D^c(h(w)B)$, and it is this part of the problem that has not been completely settled by now.

In Section 5 we show a new representation of the commutators (20), see Lemma 2. These commutators play a key role in all proofs of the chain rule property known so far. In Section 6 we use this new representation to show that

$$\begin{aligned} &D^c(h(w)B) \;= \\ &(h(\tilde w) - \tilde w h'(\tilde w))\, D^c\cdot B\llcorner\Omega\setminus S_w \;+\; h'(\tilde w)D^c\cdot(wB)\llcorner\Omega\setminus S_w \;+\; \sigma, \end{aligned} \tag{4}$$

where the "error" measure σ is absolutely continuous with respect to $|D^c\cdot B| + |D^c\cdot(wB)|$ and concentrated on S_w, the L^1–approximate discontinuity set of w (see Theorem 7 for a more general result). For the definition of the Lebesgue limit $\tilde{w}(x)$ we refer to Subsection 2.4.

It remains to understand how large S_w can be. Let us first introduce some terminology.

Definition 1 (Tangential set of B). *Let $B \in BV_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$, let $|DB|$ denote the total variation of its distributional derivative and denote by $\tilde{E}$ the Borel set of points $x \in \Omega$ s.t.*

- *The following limit exists and is finite:*

$$M(x) := \lim_{r \downarrow 0} \frac{DB(B_r(x))}{|DB|(B_r(x))}.$$

- *The Lebesgue limit $\tilde{B}(x)$ exists.*

We call tangential set of B *the Borel set*

$$E := \{x \in \tilde{E} \text{ such that } M(x) \cdot \tilde{B}(x) = 0\}.$$

The following result has been proved by a blow-up argument in [11] (see Theorem 6.5 and (6.8) therein).

Proposition 1. *Let $B \in BV_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$ and let $w \in L^\infty_{\mathrm{loc}}(\Omega)$ be such that $D \cdot (Bw)$ is a locally finite Radon measure in Ω. Then the inclusion $S_w \subset E$ holds up to $|D^s B|$-negligible sets.*

Arguing on the single components of w, the proposition above obviously extends to vector-valued functions w. Since the error measure σ in (4) is concentrated on E and absolutely continuous with respect to $|D \cdot B| + |D \cdot (wB)|$, we were thus led to the following question concerning BV vector fields:

(Q) Let $B \in BV_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$. Does the Cantor part of the divergence $|D^c \cdot B|$ vanish on the tangential set?

If this were the case, then the theorems of this paper (see (2), (3), (4)) would give a solution to the chain rule problem whenever the measure $D \cdot (wB)$ is absolutely continuous. Unfortunately the answer to Question (Q) is negative, as it is proved in Section 8:

Proposition 2. *There exists $B \in BV(\mathbf{R}^2, \mathbf{R}^2)$ such that $|D^c \cdot B|(E) > 0$, where E denotes the tangential set of B.*

Still we can pose the following

Question 1 (Divergence problem). Let $B \in BV_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$. Under which conditions the Cantor part of the divergence $|D^c \cdot B|$ vanishes on the tangential set?

The following more concrete version of Question 1 would still give useful partial answers to the chain rule problem (see also Remark 1 below):

Question 2. Let $B \in BV_{\rm loc} \cap L^\infty_{\rm loc}(\Omega, \mathbf{R}^d)$ and let $\rho \in L^\infty(\Omega)$ be such that $\rho \geq C > 0$ and $D \cdot (\rho B) = 0$. Is it true that $|D^c \cdot B|$ vanishes on the tangential set of B ?

In particular, as we explain in Section 7, an affirmative answer to Question 2, combined with some elementary computations and with some remarks of [10], would give an extension of the DiPerna–Lions theory of renormalized solutions to transport equations

$$\begin{cases} \partial_t u + b \cdot \nabla_x u \;=\; 0 \\ \\ u(0, \cdot) \;=\; u_0 \end{cases}$$

when the coefficients b are BV and nearly incompressible. Here by nearly incompressible we mean that there exists a positive function ρ with $\log \rho \in L^\infty$ satisfying

$$\partial_t \rho + D \cdot (\rho b) \;=\; 0 \qquad \text{in the sense of distributions on } \mathbf{R}^+_t \times \mathbf{R}^n \ . \tag{5}$$

Then, as remarked in [10], we could use this extension of DiPerna–Lions theory to prove the following conjecture of Bressan on compactness of ODEs (which indeed was our initial main motivation for investigating the chain rule):

Conjecture 1 (Bressan's compactness conjecture). Let $b_n : \mathbf{R}_t \times \mathbf{R}^d_x \to \mathbf{R}^d$ be smooth maps and denote by Φ_n the solution of the ODEs:

$$\begin{cases} \dfrac{d}{dt}\Phi_n(t,x) \;=\; b_n(t, \Phi_n(t,x)) \\ \\ \Phi_n(0,x) \;=\; x\,. \end{cases} \tag{6}$$

Assume that the fluxes Φ_n are nearly incompressible, i.e. that for some constant C we have

$$C^{-1} \;\leq\; \det(\nabla_x \Phi_n(t,x)) \;\leq\; C\,, \tag{7}$$

and that $\|b_n\|_\infty + \|\nabla b_n\|_{L^1}$ is uniformly bounded. Then the sequence $\{\Phi_n\}$ is strongly precompact in $L^1_{\rm loc}$.

We refer to Section 7 for the details.

Remark 1. We close this introduction by pointing out some natural conditions under which one could investigate the Divergence Problem:

- $B = \nabla \alpha \in BV_{\rm loc}(\Omega)$ for some $\alpha \in W^{1,\infty}_{loc}$ (in this case $D \cdot B = \Delta \alpha$);
- B is a (semi)-monote operator, that is

$$\langle B(y) - B(x), y - x \rangle \;\geq\; \lambda |x - y|^2 \qquad \forall x, y \in \Omega\,. \tag{8}$$

- B is both curl–free and (semi)-monotone.

2 Main notation and preliminary results

2.1 Decomposition of measures

We denote by $\mathscr{L}^d$ the Lebesgue measure in $\mathbf{R}^d$ and by $\mathscr{H}^k(E)$ the Hausdorff k-dimensional measure of a set $E \subset \mathbf{R}^d$. In the sequel we denote by Ω a generic open set in $\mathbf{R}^d$. If μ is a nonnegative Borel measure in Ω we say that μ is concentrated on a Borel set F if $\mu(\Omega \setminus F) = 0$. For a Borel set $F \subset \Omega$, the restriction $\mu \llcorner F$ is defined by

$$\mu \llcorner F(E) := \mu(F \cap E) \qquad \text{for any Borel set } E \subset \Omega.$$

The same operation can be defined for vector valued measures μ with finite total variation in Ω. Unless otherwise stated, weak* convergence of measures is understood in the duality with continuous and compactly supported functions.

We now recall the following elementary results in Measure Theory (see for instance Proposition 1.62(b) of [5]):

Proposition 3. *Let $\{\mu_h\}$ be a sequence of Radon measures on $\Omega \subset \mathbf{R}^d$, which converge weakly* to μ and assume that $|\mu_h|$ converge weakly* to λ. Then $\lambda \geq |\mu|$ and if E is a compact set such that $\lambda(\partial E) = 0$, then $\mu_h(E) \to \mu(E)$.*

Proposition 4. *Let μ be a Radon measure on Ω. We fix a standard kernel $\rho \in C_c^\infty(\mathbf{R}^d)$ supported in the unit ball, we take the standard family of mollifiers $\{\rho_\delta\}$ and on every $\tilde{\Omega} \subset\subset \Omega$ we consider $\mu * \rho_\delta$ for $\delta < \mathrm{dist}\,(\tilde{\Omega}, \partial\Omega)$. Then $\mu * \rho_\delta$ converges weakly* to μ in $\tilde{\Omega}$ and $|\mu * \rho_\delta|$ converges weakly* to $|\mu|$ in $\tilde{\Omega}$.*

Let μ be a Radon vector valued measure on Ω. By the Lebesgue decomposition theorem, μ has a unique decomposition into *absolutely continuous part* μ^a and *singular part* μ^s with respect to Lebesgue measure $\mathscr{L}^d$. Further, by the Radon-Nikodym theorem there exists a unique $f \in L^1_{\mathrm{loc}}(\Omega, \mathbf{R}^k)$ such that $\mu_a = f\mathscr{L}^d$.

One can further decompose μ^s as follows:

Proposition 5 (Decomposition of the singular part). *If $|\mu^s|$ vanishes on any $\mathscr{H}^{d-1}$-negligible set, then μ^s can be uniquely written as a sum $\mu^c + \mu^j$ of two measures such that*

(a) $\mu^c(A) = 0$ for every A such that $\mathscr{H}^{d-1}(A) < +\infty$;
(b) $\mu^j = f\mathscr{H}^{d-1} \llcorner J_\mu$ for some Borel set J_μ σ-finite with respect to $\mathscr{H}^{d-1}$.

The proof of this Proposition is analogous to the proof of decomposition of derivatives of BV functions (and indeed in this case the decompositions coincide), see Proposition 3.92 of [5]. In this proof, the Borel set J_μ is defined as

$$J_\mu := \left\{ x \in \Omega \;\middle|\; \limsup_{r \downarrow 0} \frac{|\mu|(B_r(x))}{r^{d-1}} > 0 \right\}. \tag{9}$$

These measures will be called, respectively, *jump part* and *Cantor part* of the measure μ.

If μ is given by a distributional divergence $D\cdot U$, then μ^a, μ^j, and μ^c will be denoted respectively by $D^a\cdot U$, $D^j\cdot U$, and $D^c\cdot U$.

2.2 Normal traces of divergence–measure fields

In this section we recall some basic facts about the trace properties of vector fields whose divergence is a measure (see [12], the unpublished work [14], [20], and finally [11]).

Thus, let $U\in L^\infty_{\rm loc}(\Omega,\mathbf{R}^d)$ be such that its distributional divergence $D\cdot U$ is a measure with locally finite variation in Ω. The starting point is to define for every C^1 open set $\Omega'\subset\Omega$ the distribution $\mathrm{Tr}(U,\partial\Omega')$ as

$$\langle \mathrm{Tr}(U,\partial\Omega'),\varphi\rangle \;:=\; \int_{\Omega'}\nabla\varphi\cdot U+\int_{\Omega'}\varphi d\,[D\cdot U] \qquad \forall\varphi\in C^\infty_c(\Omega). \tag{10}$$

It was proved in [12] that there exists a unique $g\in L^\infty_{\rm loc}(\Omega\cap\partial\Omega')$ such that

$$\langle \mathrm{Tr}(U,\partial\Omega'),\varphi\rangle \;=\; \int_{\partial\Omega'} g\varphi\, d\mathscr{H}^{d-1}\,.$$

By a slight abuse of notation, we denote the function g by $\mathrm{Tr}(U,\partial\Omega')$ as well.

Given an oriented C^1 hypersurface Σ, we can locally view it as the boundary of an open set Ω_1 having ν_Σ as unit exterior normal. In this way the trace $\mathrm{Tr}^+(U,\Sigma)$ is well defined, and the trace $\mathrm{Tr}^-(U,\Sigma)$ is defined analogously.

In order to extend the notion of trace to countably $\mathscr{H}^{d-1}$-rectifiable sets, defined below, we need a stronger locality property: in [12] it was proved (see also the recent proof in [11]) that the trace operator is local in a strong sense, i.e. if $\Omega_1,\Omega_2\subset\subset\Omega$ are two C^1 open sets, then

$$\mathrm{Tr}(U,\partial\Omega_1)\;=\;\mathrm{Tr}(U,\partial\Omega_2)\qquad \mathscr{H}^{d-1}\text{–a.e. on } \partial\Omega_1\cap\partial\Omega_2, \tag{11}$$

if the exterior unit normals coincide on $\partial\Omega_1\cap\partial\Omega_2$.

Definition 2 (Countably $\mathscr{H}^{d-1}$-rectifiable sets). *We say that $\Sigma\subset\mathbf{R}^d$ is countably $\mathscr{H}^{d-1}$-rectifiable if there exist (at most) countably many C^1 embedded hypersurfaces $\Gamma_i\subset\mathbf{R}^d$ such that*

$$\mathscr{H}^{d-1}\left(\Sigma\setminus\bigcup_i\Gamma_i\right)=0.$$

Using the decomposition of a rectifiable set Σ in pieces of C^1 hypersurfaces we can define an orientation of Σ and the normal traces of U on Σ as follows: by the rectifiability property we can find countably many oriented C^1 hypersurfaces Σ_i and pairwise disjoint Borel sets $E_i\subset\Sigma_i\cap\Sigma$ such that

$\mathscr{H}^{d-1}(\Sigma \setminus \cup_i E_i) = 0$; then we define $\nu_\Sigma(x)$ equal to the classical normal to Σ_i for any $x \in E_i$. Analogously, we define

$$\mathrm{Tr}^+(U,\Sigma) := \mathrm{Tr}^+(U,\Sigma_i), \quad \mathrm{Tr}^-(U,\Sigma) := \mathrm{Tr}^-(U,\Sigma_i) \qquad \mathscr{H}^{d-1}\text{-a.e. on } E_i.$$

The locality property ensures that this definition depends on the orientation ν_Σ, as in the case of oriented C^1 hypersurfaces, but it does not depend on the choice of Σ_i and E_i, up to $\mathscr{H}^{d-1}$-negligible sets.

We end this subsection by stating two useful propositions, which correspond to Propositions 3.4 and 3.6 of [11] (see also [20]).

Proposition 6 (Jump part of $D\cdot U$). *Let the divergence of $C \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$ be a measure with locally finite variation in Ω. Then:*

(a) $|D\cdot U|(E) = 0$ for any $\mathscr{H}^{d-1}$-negligible set $E \subset \Omega$.
(b) If $\Sigma \subset \Omega$ is a C^1 hypersurface then

$$D\cdot U \llcorner \Sigma \;=\; (\mathrm{Tr}^+(U,\Sigma) - \mathrm{Tr}^-(U,\Sigma))\mathscr{H}^{d-1}\llcorner \Sigma. \tag{12}$$

Thanks to Proposition 6(a) it turns out that for any $U \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$ whose divergence is a locally finite measure in Ω there exist a Borel function f and a set $J = J_{D\cdot U}$ such that

$$D^j \cdot U \;=\; f\mathscr{H}^{d-1}\llcorner J_{D\cdot U}\,. \tag{13}$$

Proposition 7 (Fubini's Theorem for traces). *Let U be as above and let $F \in C^1(\Omega)$. Then*

$$\mathrm{Tr}(U, \partial\{F > t\}) = U\cdot\nu \qquad \mathscr{H}^{d-1}\text{–a.e. on } \Omega \cap \partial\{F > t\}$$

for $\mathscr{L}^1$-a.e. $t \in \mathbf{R}$, where ν denotes the exterior unit normal to $\{F > t\}$.

Notice that the coarea formula gives that $\{F = t\} \cap \{|\nabla F| = 0\}$ is $\mathscr{H}^{d-1}$-negligible for $\mathscr{L}^1$-a.e. $t \in \mathbf{R}$, therefore the theory of traces applies to the sets $\Sigma = \{F = t\}$ for $\mathscr{L}^1$-a.e. $t \in \mathbf{R}$.

2.3 BV and BD functions

For $B \in L^1_{\mathrm{loc}}(\Omega;\mathbf{R}^m)$ we denote by $DB = (D_i B^l)$ the derivative in the sense of distributions of B, i.e. the $\mathbf{R}^{m\times d}$-valued distribution defined by

$$D_i B^l(\varphi) := -\int_\Omega B^l \frac{\partial\varphi}{\partial x_i}\,dx \quad \forall\,\varphi \in C^\infty_c(\Omega), \quad 1 \le i \le d, 1 \le l \le m.$$

In the case when $m = d$ we denote by EB the symmetric part of the distributional derivative of B, i.e.,

$$EB := (E_{il}B), \qquad E_{il}B := \frac{1}{2}(D_i B^l + D_l B^i) \quad 1 \le i,\, l \le d.$$

Definition 3 (*BV* and *BD* functions). *We say that $B \in L^1(\Omega;\mathbf{R}^m)$ has bounded variation in Ω, and we write $B \in BV(\Omega;\mathbf{R}^m)$, if DB is representable by a $\mathbf{R}^{m\times d}$-valued measure, still denoted with DB, with finite total variation in Ω.*
We say that $B \in L^1(\Omega;\mathbf{R}^d)$ has bounded deformation in Ω, and we write $B \in BD(\Omega)$, if $E_{ij}B$ is a Radon measure with finite total variation in Ω for any i, $j = 1,\dots,d$.

It is a well known fact that for $B \in BV$ one has $DB << \mathscr{H}^{d-1}$. The same property holds for EB when $B \in BD$ (see for instance Remark 3.3 of [4]). Therefore we can apply the decomposition of Subsection 2.1 to the measures DB and EB and we will use the notation D^aB (E^aB), D^cB (E^cB), and D^jB (E^jB), respectively for the absolutely continuous part, Cantor part, and jump part of DB (EB). The distributional divergence $D\cdot B := \sum_i D_iB^i = \sum_i E_{ii}B$ is a well defined measure with finite total variation in Ω when $B \in BD(\Omega)$.

2.4 Fine properties of *BV* functions

In this subsection we recall the fine properties of $\mathbf{R}^m$-valued BV functions defined in an open set $\Omega \subset \mathbf{R}^d$.

The L^1*–approximate discontinuity set* $S_B \subset \Omega$ of a locally summable $B : \Omega \to \mathbf{R}^m$ and the *Lebesgue limit* are defined as follows: $x \notin S_B$ if and only if there exists $z \in \mathbf{R}^m$ satisfying

$$\lim_{r\downarrow 0} r^{-d}\int_{B_r(x)} |B(y)-z|\,dy = 0.$$

The vector z, if it exists, is unique and denoted by $\tilde{B}(x)$, the Lebesgue limit of B at x. It is easy to check that the set S_B is Borel and that $\tilde{B}$ is a Borel function in its domain (see §3.6 of [5] for details). By Lebesgue differentiation theorem the set S_B is Lebesgue negligible and $\tilde{B} = B$ $\mathscr{L}^d$-a.e. in $\Omega \setminus S_B$.

In a similar way one can define the L^1*–approximate jump set* $J_B \subset S_B$, by requiring the existence of a, $b \in \mathbf{R}^m$ with $a \neq b$ and of a unit vector ν such that

$$\lim_{r\downarrow 0} r^{-d}\int_{B_r^+(x,\nu)} |B(y)-a|\,dy = 0, \qquad \lim_{r\downarrow 0} r^{-d}\int_{B_r^-(x,\nu)} |B(y)-b|\,dy = 0,$$

where

$$\begin{aligned} B_r^+(x,\nu) &:= \{y \in B_r(x) :\ \langle y-x,\nu\rangle > 0\}\,, \\ B_r^-(x,\nu) &:= \{y \in B_r(x) :\ \langle y-x,\nu\rangle < 0\}\,. \end{aligned} \tag{14}$$

The triplet (a,b,ν), if exists, is unique up to a permutation of a and b and a change of sign of ν, and denoted by $(B^+(x),B^-(x),\nu(x))$, where $B^\pm(x)$ are called *Lebesgue one-sided limits* of B at x. It is easy to check that the set

J_B is Borel and that $B^{\pm}$ and ν can be chosen to be Borel functions in their domain (see again §3.6 of [5] for details).

Denoting by $\eta \otimes \xi$ the linear map from $\mathbf{R}^d$ to $\mathbf{R}^m$ defined by $v \mapsto \eta\langle \xi, v\rangle$, the following structure theorem holds (see for instance Theorem 3.77 and Proposition 3.92 of [5]):

Proposition 8 (BV structure theorem). *If $B \in BV_{\rm loc}(\Omega, \mathbf{R}^m)$, then $\mathscr{H}^{d-1}(S_B \setminus J_B) = 0$ and J_B is a countably $\mathscr{H}^{d-1}$–rectifiable set. Moreover*

$$D^j B = (B^+ - B^-) \otimes \nu \mathscr{H}^{d-1} \llcorner J_B\,, \tag{15}$$

$$\begin{gathered} |D^a B|(u^{-1}(N)) + |D^c B|(\tilde{u}^{-1}(N)) = 0 \\ \textit{for any } \mathscr{L}^1\textit{-negligible Borel set } N \subset \mathbf{R}. \end{gathered} \tag{16}$$

As a corollary, since $D^a B$ and $D^c B$ are both concentrated on $\Omega \setminus S_B$, we conclude that $|D^a B| + |D^c B|$–a.e. x is a Lebesgue point for B, with value $\tilde{B}(x)$. The space of functions of special bounded variation (denoted by SBV) is defined as follows:

Definition 4 (SBV). *Let $\Omega \subset \mathbf{R}^d$ be an open set. The space $SBV(\Omega, \mathbf{R}^m)$ is the set of all $u \in BV(\Omega, \mathbf{R}^m)$ such that $D^c u = 0$.*

2.5 Fine properties of BD functions

As in the case of BV functions, also for BD functions B the set J_B is countably $\mathscr{H}^{d-1}$–rectifiable (see [4]). Though the question whether $\mathscr{H}^{d-1}(S_B \setminus J_B) = 0$ is still open, in [4] it was proved that:

Proposition 9. *If $B \in BD(\Omega)$, then*

$$E^j B = (B^+ - B^-) \odot \nu\, \mathscr{H}^{d-1} \llcorner J_B\,,$$

where $2a \odot b := a \otimes b + b \otimes a$.

Similarly, we can define:

Definition 5 (SBD). *$SBD(\Omega)$ is the set of all $B \in BD(\Omega)$ such that $E^c B = 0$.*

2.6 Vol'pert chain rule and Alberti's rank one Theorem

We end this section by recalling the classical chain–rule formula for BV functions of Vol'pert (see [36] and Theorem 3.96 of [5]) and a deep result of Alberti concerning the structure of $D^s B$ (see [1]).

Theorem 1 (Vol'pert chain rule). *Let $v \in BV_{\rm loc}(\Omega, \mathbf{R}^m)$ and let $\Phi \in C^1(\mathbf{R}^m, \mathbf{R}^h)$ be a map with a bounded gradient. Then $\Phi \circ v \in BV_{\rm loc}(\Omega, \mathbf{R}^h)$ and the measure $D(\Phi \circ v)$ can be explicitly computed as*

$$D(\Phi \circ v) = \nabla\Phi(v) \cdot D^a v + \nabla\Phi(\tilde{v}) \cdot D^c v + \big(\Phi(v^+) - \Phi(v^-)\big) \otimes \nu\, \mathscr{H}^{d-1} \llcorner J_v\,.$$

Theorem 2 (Alberti's rank one theorem). *Let $B \in BV_{\mathrm{loc}}(\Omega, \mathbf{R}^m)$. Then there exist Borel functions $\xi : \Omega \to \mathbf{S}^{d-1}, \eta : \Omega \to \mathbf{S}^{m-1}$ such that*

$$D^s B \;=\; \eta \otimes \xi |D^s B| \,. \tag{17}$$

3 Chain rule: The absolutely continuous part

In this and in the next three sections we will study the problem of computing the divergence $D\cdot(h(w)B)$ when B is a BV function and w is an L^∞ function such that $D \cdot (wB)$ is a measure.

To simplify the statements, in this and in the next two sections we will always assume that the divergence of wB is a measure with locally finite total variation.

Remark 2. Note that, when $U \in L^\infty(\mathbf{R}^d, \mathbf{R}^d)$ and $D \cdot U$ is a measure with locally finite total variation, one has the estimate

$$|D \cdot U(B_r(x))| \;\le\; \alpha_{d-1} \|U\|_\infty r^{d-1} \,,$$

where α_{d-1} denotes the $d-1$–dimensional volume of the unit sphere. By standard arguments, this implies that $|D \cdot U|(E) = 0$ for every Borel set E such that $\mathscr{H}^{d-1}(E) = 0$.

Therefore we can decompose $D \cdot U$ into its absolutely continuous, Cantor, and jump part, which will be denoted respectively by $D^a \cdot U$, $D^c \cdot U$, and $D^j \cdot U$.

Definition 6 (BV measures). *We say that a positive locally finite measure σ in Ω is a BV measure if there exists an at most countable Borel partition $\{\Omega_l\}_{l\in I}$ of Ω and functions $f_l \in BV_{\mathrm{loc}}(\Omega)$ such that $\sigma \llcorner \Omega_l \ll |Df_l|$ for any $l \in I$.*

Notice that it is not restrictive to assume that the functions f_l are bounded and nonnegative, by a truncation argument. Also, it is immediate to check using the uniqueness of decomposition in Cantor and jump part that $\sigma \llcorner \Omega_l \ll |Df_l|$ implies $\sigma^j \llcorner \Omega_l \ll |D^j f_l|$ and $\sigma^c \llcorner \Omega_l \ll |D^c f_l|$. As a consequence, since $|D^j f|$ is concentrated on a countably $\mathscr{H}^{d-1}$-rectifiable set for any $f \in BV_{\mathrm{loc}}$ (precisely the L^1–approximate jump set of f), the same is true for the jump part σ^j of a BV measure σ.

Theorem 3 (Absolutely continuous part). *Let $B \in BD_{\mathrm{loc}}(\Omega)$, $w \in L^\infty_{\mathrm{loc}}(\Omega; \mathbf{R}^k)$, and $h \in C^1(\mathbf{R}^k)$, and assume that $D \cdot (wB)$ is a measure with locally finite total variation. Then*

(a) $D \cdot (h(w)B)$ *is a measure with locally finite variation in Ω and*

$$D^a \cdot (h(w)B) \;=\; \left[h(w) - \sum_{i=1}^k w_i \frac{\partial h}{\partial z_i}(w) \right] D^a \cdot B + \sum_{i=1}^k \frac{\partial h}{\partial z_i}(w) D^a \cdot (w_i B) \,. \tag{18}$$

(b) *If $B \in BV_{\text{loc}}$, then for any open set $A \subset\subset \Omega$ we have*

$$|D^s \cdot (h(w)B)| \llcorner A \leq L_1 |D^s \cdot B| + L_2 |D^s \cdot (wB)|\,, \tag{19}$$

where the constants L_1 and L_2 depend only on $L := \|w\|_{L^\infty(A)}$ and $\|h\|_{C^1(B_L(0))}$.

(c) *If $B \in BV_{\text{loc}}$ and $D \cdot (w_i B)$ are BV measures, then $|D \cdot (h(w)B)|$ is a BV measure as well.*

Before going into the proof of the previous theorem, we need some definitions and preliminary lemmas. First of all, ρ_δ will denote a standard family of mollifiers in $\mathbf{R}^d$, that is $\rho \in C^\infty_c(B_1(0))$ is even with $\rho \geq 0$, $\int \rho = 1$ and $\rho_\delta(x) = \delta^{-d}\rho\left(\frac{x}{\delta}\right)$. We set

$$I(\rho) \;:=\; \int_{\mathbf{R}^d} |z||\nabla\rho(z)|\, dz\,.$$

Moreover, we define the commutators

$$T_\delta \;:=\; (D \cdot (Bw)) * \rho_\delta - D \cdot (B(w * \rho_\delta))\,, \tag{20}$$

and we denote by T^i_δ the component $(D \cdot (Bw_i)) * \rho_\delta - D \cdot (B(w_i * \rho_\delta))$. The next Proposition is Theorem 2.6 of [11].

Proposition 10 (BD commutators estimate). *Let $B \in BD_{\text{loc}}(\Omega)$ and let $w \in L^\infty_{\text{loc}}(\Omega)$. Let ρ be a radial convolution kernel. Then:*

(a) *The distributions defined by (20) are induced by measures with locally uniformly bounded variation in Ω as $\delta \downarrow 0$.*

(b) *Any weak* limit σ of a subsequence of $\{|T_\delta|\}_{\delta\downarrow 0}$ as $\delta \downarrow 0$ is a singular measure which satisfies the bound*

$$\sigma \llcorner A \;\leq\; \|w\|_{L^\infty(A)}\big(d + I(\rho)\big)|E^s B| \quad \textit{for any open set } A \subset\subset \Omega. \tag{21}$$

In the case where $B \in BV_{\text{loc}}$ we can consider more general convolution kernels and give more refined estimates. In this we follow [6] and define, for every convolution kernel ρ and any $d \times d$ matrix M, the quantity:

$$\Lambda(M, \rho) \;:=\; \int_{\mathbf{R}^d} \big|\langle M \cdot z, \nabla\rho(z)\rangle\big|\, dz\,. \tag{22}$$

In the following proposition we write the matrix valued measure $D^s B$ as $M|D^s B|$, where $M : \Omega \to \mathbf{R}^{d\times d}$ is a Borel function with $|M| = 1$ $|D^s B|$-a.e. in Ω.

Proposition 11 (BV commutators estimate). *Let $B \in BV_{\text{loc}}(\Omega; \mathbf{R}^d)$ and let ρ be an even convolution kernel. Then, any measure σ which is the weak* limit in Ω of a subsequence of $\{|T_\delta|\}$ satisfies the estimate*

$$\sigma \llcorner A \;\leq\; \|w\|_{L^\infty(A)}\, [\Lambda(M(\cdot), \rho)|D^s B| + |D^s \cdot B|] \tag{23}$$

for all open sets $A \subset\subset \Omega$.

This proposition is the analog of Theorem 3.2 of [6], with the only difference that the commutators considered here in (20) are more general than those considered in [6]. Indeed the commutators considered in [6] can be written only under the assumption that the divergence of B is absolutely continuous. In Appendix A we show the minor modifications needed to adapt the proof of Theorem 3.2 of [6].

The final ingredient for the proof of Theorem 3 is the following elementary lemma

Lemma 1. *Let*

$$K \ := \ \left\{ \rho \in C^\infty_c(B_1(0)) \text{ such that } \rho \geq 0 \text{ is even, and } \int_{B_1(0)} \rho = 1 \right\} . \tag{24}$$

If $D \subset K$ is dense with respect to the strong $W^{1,1}$ topology, then for every ξ, $\eta \in \mathbf{R}^d$ we have

$$\inf_{\rho \in D} \Lambda(\eta \otimes \xi, \rho) \ = \ |\langle \xi, \eta \rangle| \ = \ \left| \mathrm{tr}\,(\eta \otimes \xi) \right| . \tag{25}$$

The proof of this lemma is equal to the the proof of Lemma 3.3 of [6] (see also Remark 3.8(1) in the same paper), but since the statement of Lemma 3.3 of [6] is slightly weaker, for the reader's convenience we include the proof of Lemma 1 in Appendix B. We now come to

Proof (of Theorem 3). **(a)** Let us fix a radial convolution kernel ρ and define T_δ as in (20). Then, we compute

$$\begin{aligned}
& D \cdot (h(w * \rho_\delta) B) \\
&= \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta) \nabla (w_i * \rho_\delta) \cdot B + h(w * \rho_\delta) D \cdot B \\
&= \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta) D \cdot [(w_i * \rho_\delta) B] \\
&\quad + \left[h(w * \rho_\delta) - \sum_i (w_i * \rho_\delta) \frac{\partial h}{\partial z_i}(w * \rho_\delta) \right] D \cdot B \\
&= \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta) [(D \cdot (B w_i)) * \rho_\delta] - \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta) T^i_\delta \qquad (26) \\
&\quad + \left[h(w * \rho_\delta) - \sum_i (w_i * \rho_\delta) \frac{\partial h}{\partial z_i}(w * \rho_\delta) \right] D \cdot B \, . \qquad (27)
\end{aligned}$$

As $\delta \downarrow 0$, $D \cdot (h(w * \rho_\delta) B)$ converges to $D \cdot (h(w) B)$ in the distribution sense. On any open set $A \subset\subset \Omega$ the measures $[(D \cdot (Bw)) * \rho_\delta]$ enjoy uniform bounds on their total variations. In view of Proposition 10, the same holds for T_δ. Since $w \in L^\infty_{\mathrm{loc}}$ we conclude easily that the sum of (26) and (27) converges, up to subsequences, to a Radon measure $\mu = D \cdot (h(w) B)$ on Ω.

Define $S_\delta := \frac{\partial h}{\partial z_i}(w * \rho_\delta) T^i_\delta$. Note that $|S_\delta| \le C|T^i_\delta|$, where C locally depends only on $\|w\|_\infty$ and h. Hence from Proposition 10 we conclude that any weak limit of a subsequence of $|S_\delta|$ is singular and from Proposition 3 we conclude that any limit point of S_δ as $\delta \downarrow 0$ is a singular measure.

We use the decomposition $[(D \cdot (Bw)) * \rho_\delta] = [(D^a \cdot (Bw)) * \rho_\delta] + [(D^s \cdot (Bw)) * \rho_\delta]$. By Proposition 3 again, the measures

$$\mu_2^\delta := \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta)[(D^s \cdot (Bw_i)) * \rho_\delta]$$

converge (up to subsequences) to singular measures. Moreover, if we write $D^a \cdot (Bw) = f \mathscr{L}^d$, we get that $[(D^a \cdot (Bw)) * \rho_\delta] = f * \rho_\delta \mathscr{L}^d$. Since $\frac{\partial h}{\partial z_i}(w * \rho_\delta)$ converges to $\frac{\partial h}{\partial z_i}(w)$ pointwise almost everywhere and $f * \rho_\delta$ converges to f strongly in L^1_{loc}, we conclude that

$$\mu_1^\delta := \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta)[(D^a \cdot (Bw_i)) * \rho_\delta]$$

converges to

$$\left[\sum_i \frac{\partial h}{\partial z_i}(w) f_i\right] \mathscr{L}^d = \sum_i \frac{\partial h}{\partial z_i}(w)\, D^a \cdot (Bw_i)\,.$$

In a similar way we can treat the last term in (27) and we conclude that the sum of the expressions (26) and (27) converges (up to subsequences) to $\mu = \mu_1 + \mu_2$, where

- μ_2 is singular with respect to $\mathscr{L}^d$ and is the limit of

$$\mu_2^\delta + \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta) T^i_\delta + \left[h(w * \rho_\delta) - \sum_i (w_i * \rho_\delta) \frac{\partial h}{\partial z_i}(w * \rho_\delta)\right] D^s \cdot B\,;$$

- μ_1 is absolutely continuous and

$$\begin{aligned} \mu_1 &= \lim_{\delta \downarrow 0} \left\{ \mu_1^\delta + \left[h(w * \rho_\delta) - \sum_i (w_i * \rho_\delta) \frac{\partial h}{\partial z_i}(w * \rho_\delta)\right] D^a \cdot B \right\} \\ &= \left[h(w) - \sum_{i=1}^k w_i \frac{\partial h}{\partial z_i}(w)\right] D^a \cdot B + \sum_{i=1}^k \frac{\partial h}{\partial z_i}(w) D^a \cdot (w_i B)\,. \end{aligned}$$

From this we easily get (18).

(b) From the argument of the previous step, we conclude that $D^s \cdot (h(w)B)$ is the limit (in the sense of distributions) of the sums of the following expressions:

$$\left\{\sum_i \frac{\partial h}{\partial z_i}(w*\rho_\delta)\big[(D^s\cdot(Bw_i))*\rho_\delta\big]\right.$$
$$\left. + \left[h(w*\rho_\delta)-\sum_i(w_i*\rho_\delta)\frac{\partial h}{\partial z_i}(w*\rho_\delta)\right]D^s\cdot B\right\}, \tag{28}$$

$$\sum_i \frac{\partial h}{\partial z_i}(w*\rho_\delta)T^i_\delta\,. \tag{29}$$

Clearly, any limit point of the sum in (28) is a measure which is bounded on any open set $A\subset\subset\Omega$ by

$$L_1|D^s\cdot(Bw)|+L_2|D^s\cdot B|\,,$$

where L_1 and L_2 only depend on $L:=\|w\|_{L^\infty(A)}$ and $\|h\|_{C^1(B_L(0))}$.

Now, fix an open set $A\subset\subset\Omega$ and let ν be any limit point of (29). According to Proposition 11 we have

$$|\nu|\llcorner A\ \le\ \|w\|_{L^\infty(A)}\Lambda(M(\cdot),\rho)|D^sB|+\|w\|_{L^\infty(A)}|D^s\cdot B|$$

Thus, choosing subsequences for which both terms in (28) and (29) are converging, we find

$$|D^s\cdot(h(w)B))|\llcorner A\ \le\ L_1|D^s\cdot(Bw)|+L_2|D^s\cdot B|+L_3\Lambda(M(\cdot),\rho)|D^sB| \tag{30}$$

for some constants L_i independent of Q and of ρ and depending only on A. Now, let τ be the positive part of the measure $|D^s(h(w)B)|-L_1|D^s\cdot(Bw)|$ and let g be its Radon–Nikodym derivative with respect to $|D^sB|$. Then from (30) it follows that, for any even convolution kernel ρ, the inequality

$$g(x)\ \le\ L_2|\mathrm{tr}(M(x))|+L_3\Lambda(M(x),\rho)$$

holds for $|D^sB|$–a.e. $x\in A$. Let D be a countable set of mollifiers which is dense in the $W^{1,1}$ strong topology in the set K of (24). Then,

$$g(x)\ \le\ L_2\big|\mathrm{tr}\,(M(x))\big|+L_3\inf_{\rho\in D}\Lambda(M(x),\rho)\qquad\text{for } |D^sB|\text{–a.e. } x\in A.$$

Recall Alberti's Theorem: $M(x)=\eta(x)\otimes\xi(x)$. Thus, from Lemma 1 it follows that

$$g(x)\ \le\ L_2\big|\mathrm{tr}\,(M(x))\big|+L_3\big|\mathrm{tr}\,(M(x))\big|\,,$$

so that $\tau\llcorner A\le(L_2+L_3)|D^s\cdot B|$. Hence, setting $L_4=L_2+L_3$, we conclude that

$$\begin{aligned}|D^s\cdot(h(w)B))| &\le L_1|D^s\cdot(Bw)|+L_4|\mathrm{tr}(M)||D^sB|\\ &= L_1|D^s\cdot(Bw)|+L_4|D^s\cdot B|\,.\end{aligned} \tag{31}$$

(c) It is an immediate consequence of (19). □

4 Chain rule: The jump part

In this section we prove the following

Theorem 4 (Jump part). *Let $h \in C^1(\mathbf{R}^k)$, $B \in BD_{\rm loc}$ and let $\Sigma \subset \Omega$ be any oriented countably $\mathscr{H}^{d-1}$-rectifiable set. Then*

$$D \cdot (h(w)B) \llcorner \Sigma = \left[\mathrm{Tr}^+(B,\Sigma) h\left(\frac{\mathrm{Tr}^+(wB,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)}\right) \right. \\ \left. - \mathrm{Tr}^-(B,\Sigma) h\left(\frac{\mathrm{Tr}^-(wB,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)}\right)\right] \mathscr{H}^{d-1} \llcorner \Sigma,$$

where the ratio $\frac{\mathrm{Tr}^+(wB,\Sigma)}{\mathrm{Tr}^+(B,\Sigma)}$ (resp. $\frac{\mathrm{Tr}^-(wB,\Sigma)}{\mathrm{Tr}^-(B,\Sigma)}$) is arbitrarily defined at points where the trace $\mathrm{Tr}^+(B,\Sigma)$ (resp. $\mathrm{Tr}^-(B,\Sigma)$) vanishes.

Moreover, if $D^j \cdot (w_i B)$ are concentrated on a countably $\mathscr{H}^{d-1}$-rectifiable set Σ, then $D^j(h(w)B)$ is concentrated on Σ.

The key for proving Theorem 4 is the following theorem. The scalar case is proved in [11]. In the vector-valued case the proof is analogous, but we give a detailed one for the reader's convenience.

Theorem 5 (Change of variables for traces). *Let $\Omega' \subset\subset \Omega$ be an open domain with a C^1 boundary and let $h \in C^1(\mathbf{R}^k)$. Then*

$$\mathrm{Tr}(h(w)B, \partial\Omega') = h\left(\frac{\mathrm{Tr}(wB,\partial\Omega')}{\mathrm{Tr}(B,\partial\Omega')}\right)\mathrm{Tr}(B,\partial\Omega') \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial\Omega'.$$

Proof. It is not restrictive to assume that the larger open set Ω is bounded and it has a C^1 boundary.
STEP 1. Let $\Omega'' = \Omega \setminus \overline{\Omega'}$. In this step we prove that

$$\mathrm{Tr}(h(w)B, \partial\Omega'') = h\left(\frac{\mathrm{Tr}(wB,\partial\Omega'')}{\mathrm{Tr}(B,\partial\Omega'')}\right)\mathrm{Tr}(B,\partial\Omega'') \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial\Omega'',$$

under the assumption that the components of B and w are bounded and belong to the Sobolev space $W^{1,1}(\Omega'')$. Indeed, the identity is trivial if both w and B are continuous up to the boundary, and the proof of the general case can be immediately achieved by a density argument based on the strong continuity of the trace operator from $W^{1,1}(\Omega'')$ to $L^1(\partial\Omega'', \mathscr{H}^{d-1} \llcorner \partial\Omega'')$ (see for instance Theorem 3.88 of [5]).
STEP 2. In this step we prove the general case. Let us apply Gagliardo's theorem on the surjectivity of the trace operator from $W^{1,1}$ into L^1 to obtain a bounded vector field $B_1 \in W^{1,1}(\Omega''; \mathbf{R}^d)$ whose trace on $\partial\Omega' \subset \partial\Omega''$ is equal to the trace of B, seen as a function in $BD(\Omega')$. In particular $\mathrm{Tr}(B,\partial\Omega') = -\mathrm{Tr}(B_1, \partial\Omega'')$. Defining

$$\tilde{B}(x) := \begin{cases} B(x) & \text{if } x \in \Omega' \\ B_1(x) & \text{if } x \in \Omega'', \end{cases}$$

it turns out that $\tilde{B} \in BD_{\mathrm{loc}}(\Omega)$ and that

$$|E\tilde{B}|(\partial\Omega') = 0. \tag{32}$$

Let us consider the function $\theta := \mathrm{Tr}(wB, \partial\Omega')/\mathrm{Tr}(B, \partial\Omega')$ (set equal to 0 wherever the denominator is 0) and let us prove that $\|\theta\|_{L^\infty(\partial\Omega')}$ is less than $\|w\|_{L^\infty(\Omega')}$. Indeed, writing $\partial\Omega'$ as the 0-level set of a C^1 function F with $|\nabla F| > 0$ on $\partial\Omega'$ and $\{F = t\} \subset \Omega'$ for $t > 0$ sufficiently small, by Proposition 7 we have

$$\begin{aligned} -\|w\|_{L^\infty(\Omega')}\mathrm{Tr}(B, \partial\{F > t\}) &\le \mathrm{Tr}(wB, \partial\{F > t\}) \\ &\le \|w\|_{L^\infty(\Omega')}\mathrm{Tr}(B, \partial\{F > t\}) \end{aligned}$$

$\mathscr{H}^{d-1}$-a.e. on $\{F = t\}$ for $\mathscr{L}^1$-a.e. $t > 0$ sufficiently small. Passing to the limit as $t \downarrow 0$ and using the w^*-continuity of the trace operator (see [20], [11]) we recover the same inequality on $\{F = 0\}$, proving the boundedness of θ.

Now, still using Gagliardo's theorem, we can find a bounded function $w_1 \in W^{1,1}(\Omega''; \mathbf{R}^k)$ whose trace on $\partial\Omega'$ is given by θ, so that the normal trace of $w_{1i}B_1$ on $\partial\Omega''$ is equal to $\mathrm{Tr}(w_iB, \partial\Omega')$ on the whole of $\partial\Omega'$. Defining

$$\tilde{w}(x) := \begin{cases} w(x) & \text{if } x \in \Omega' \\ w_1(x) & \text{if } x \in \Omega'', \end{cases}$$

by Proposition 6 we obtain

$$|D \cdot (\tilde{w}_i\tilde{B})|(\partial\Omega') = 0 \qquad i = 1, \ldots, k. \tag{33}$$

Let us apply now (19) in Theorem 3 and (32), (33), to obtain that the divergence of the vector field $h(\tilde{w})\tilde{B}$ is a measure with finite total variation in Ω, whose restriction to $\partial\Omega'$ vanishes. As a consequence, Proposition 6 gives

$$\mathrm{Tr}^+(h(\tilde{w})\tilde{B}, \partial\Omega') = \mathrm{Tr}^-(h(\tilde{w})\tilde{B}, \partial\Omega') \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial\Omega'. \tag{34}$$

By applying (34), Step 1, and finally our choice of B_1 and w_1 the following chain of equalities holds $\mathscr{H}^{d-1}$-a.e. on $\partial\Omega'$:

$$\begin{aligned} \mathrm{Tr}(h(w)B, \partial\Omega') &= \mathrm{Tr}^+(h(\tilde{w})\tilde{B}, \partial\Omega') = \mathrm{Tr}^-(h(\tilde{w})\tilde{B}, \partial\Omega') \\ &= \mathrm{Tr}(h(w_1)B_1, \partial\Omega'') = h\left(\frac{\mathrm{Tr}(w_1B_1, \partial\Omega'')}{\mathrm{Tr}(B_1, \partial\Omega'')}\right)\mathrm{Tr}(B_1, \partial\Omega'') \\ &= h\left(\frac{\mathrm{Tr}(wB, \partial\Omega')}{\mathrm{Tr}(B, \partial\Omega')}\right)\mathrm{Tr}(B, \partial\Omega'). \end{aligned}$$

□

Proof (of Theorem 4). If Σ is a compact set contained in a C^1 hypersurface $\partial\Omega'$ the statement is a direct consequence of Proposition 6 and of Theorem 5. The general case follows by the rectifiability of Σ, recalling the way in which traces on rectifiable sets have been defined. Finally, the last statement is a direct consequence of (19). □

5 Concentration of commutators on S_w

In this section we improve the results of Section 3, showing that the commutators

$$\mathcal{T}^i_\delta := \frac{\partial h}{\partial z_i}(w*\rho_\delta)\,T^i_\delta = \frac{\partial h}{\partial z_i}(w*\rho_\delta)\left[(D\cdot(Bw_i))*\rho_\delta - D\cdot((w_i*\rho_\delta)B)\right] \quad (35)$$

concentrate on the L^1–approximate discontinuity set S_w. The main ingredient in the proof of this fact is the following lemma, which gives a representation of the product ΦT_δ, where Φ is a test function, as the integral with respect to DB of a kind of anisotropic convolution product between w and Φ (it is indeed a standard 1-d convolution along the direction y, weighted by $\nabla\rho(y)$).

It is easy to check (see for instance [11]) that T^i_δ can be written as $r^i_\delta \mathscr{L}^d - (w_i*\rho_\delta)D\cdot B$, where

$$r^i_\delta(x) := \int_{\mathbf{R}^d} w_i(x')\left[(B(x)-B(x'))\cdot\nabla\rho_\delta(x'-x)\right]dx'\,. \quad (36)$$

Lemma 2 (Double averages lemma). *Let $\Phi \in L^\infty(\Omega)$ and assume that its support is a compact subset of Ω. Then, for δ sufficiently small, we have*

$$\int_{\mathbf{R}^d} \Phi(x) r^i_\delta(x)\,dx = \sum_{j,k}\int_{\mathbf{R}^d} A^{ijk}_\delta(\xi)\,d[D_k B^j](\xi)\,, \quad (37)$$

where the functions A^{ijk}_δ are given by the double average

$$A^{ijk}_\delta(\xi) := -\frac{1}{\delta}\int_0^\delta\int_{\mathbf{R}^d} y_k \frac{\partial\rho}{\partial z_j}(y)\Phi(\xi-\tau y)w_i(\xi+(\delta-\tau)y)\,dy\,d\tau\,. \quad (38)$$

Proof. Fix $\Phi\in L^\infty(\Omega)$ and with compact support contained in Ω and assume without loss of generality that w is globally bounded. Then, if δ is sufficiently small, A^{ijk}_δ has compact support contained in Ω. We now prove that A^{ijk}_δ is a continuous function. Taking into account that Φ and w are bounded, it suffices to show that

$$R_\varepsilon(\xi) := \int_\varepsilon^{\delta-\varepsilon}\int_{\mathbf{R}^d} y_k\frac{\partial\rho}{\partial z_j}(y)\Phi(\xi-\tau y)w_i(\xi+(\delta-\tau)y)\,dy\,d\tau$$

is continuous for any $\varepsilon\in(0,\delta/2)$. This claim can be proved as follows. First of all, without loss of generality, we can assume that both w and Φ are compactly supported. Next we take sequences $\{w^l\}$ and $\{\Phi^l\}$ of continuous compactly supported functions such that $\|w-w^l\|_{L^2}+\|\Phi-\Phi^l\|_{L^2}\downarrow 0$. If we set

$$R^l_\varepsilon(\xi) := \int_\varepsilon^{\delta-\varepsilon}\int_{\mathbf{R}^d} y_k\frac{\partial\rho}{\partial z_j}(y)\Phi^l(\xi-\tau y)w^l_i(\xi+(\delta-\tau)y)\,dy\,d\tau\,,$$

then each R^l_ε is continuous. Moreover one can easily check that

$$|R^l_\varepsilon(\xi) - R_\varepsilon(\xi)| \;\le\; C\delta\varepsilon^{-n}\big(\|\Phi\|_{L^2}\|w-w^l\|_{L^2} + \|w^l\|_{L^2}\|\Phi^l-\Phi\|_{L^2}\big)\,.$$

Therefore $R^l_\varepsilon \to R_\varepsilon$ uniformly, and we conclude that R_ε is continuous.

Now, fix B and δ as in the statement of the lemma. We approximate B in $L^1_{\rm loc}$ with a sequence of smooth functions B_n, in such a way that $D_kB^j_n$ converge weakly* to D_kB^j on Ω. Hence, we have that

$$R^i_n(x) \;:=\; \int_{\mathbf{R}^d} w_i(x')\big[(B_n(x)-B_n(x'))\cdot\nabla\rho_\delta(x'-x)\big]\,dx'$$

converge strongly in $L^1_{\rm loc}$ to r^i_δ. Moreover, since A^{ijk}_δ is a continuous and compactly supported function, we have

$$\lim_{n\to\infty}\int A^{ijk}_\delta(\xi)d[D_kB^j_n](\xi) \;=\; \int A^{ijk}_\delta(\xi)d[D_kB^j](\xi)\,.$$

Hence it is enough to prove the statement of the lemma for B_n, which are smooth functions.

Thus, we fix a smooth function B and compute

$$\begin{aligned}
&-\int r^i_\delta(x)\Phi(x)\,dx\\
&= -\int_{\mathbf{R}^d}\Phi(x)\int_{\mathbf{R}^d} w_i(x')\big[(B(x)-B(x'))\cdot\nabla\rho_\delta(x'-x)\big]\,dx'\,dx\\
&= -\int_{\mathbf{R}^d\times\mathbf{R}^d}\Phi(x)w_i(x+\delta y)\frac{B(x)-B(x+\delta y)}{\delta}\cdot\nabla\rho(y)\,dy\,dx\\
&= \int_{\mathbf{R}^d\times\mathbf{R}^d}\Phi(x)w_i(x+\delta y)\frac{1}{\delta}\int_0^\delta\sum_{k,j} y_k\frac{\partial B^j}{\partial z_k}(x+\tau y)\frac{\partial\rho}{\partial z_j}(y)\,d\tau\,dy\,dx\\
&= \sum_{k,j}\int_{\mathbf{R}^d}\left[\frac{1}{\delta}\int_0^\delta\int_{\mathbf{R}^d} y_k\frac{\partial\rho}{\partial z_j}(y)\Phi(\xi-\tau y)w_i(\xi+(\delta-\tau)y)\,dy\,d\tau\right]\frac{\partial B^j}{\partial z_k}(\xi)\,d\xi\,.
\end{aligned}$$

Since the measure $\frac{\partial B^j}{\partial z_k}\mathscr{L}^d$ is equal to D_kB^j, the claim of the lemma follows. □

Theorem 6 (Concentration of commutators on S_w). *Assume that $B\in BV_{\rm loc}(\Omega;\mathbf{R}^d)$ and $w\in L^\infty_{\rm loc}(\Omega;\mathbf{R}^k)$. Then any limit point as $\delta\downarrow 0$ of $\mathcal{T}^i_\delta$ is a measure concentrated on S_w.*

Proof. We rewrite $\mathcal{T}^i_\delta$ as

$$\mathcal{T}^i_\delta \;=\; \frac{\partial h}{\partial z_i}(w*\rho_\delta)\,r^i_\delta\,\mathscr{L}^d - \frac{\partial h}{\partial z_i}(w*\rho_\delta)\,(w_i*\rho_\delta)\,D\cdot B\,. \tag{39}$$

We define the matrix–valued measures

$$\alpha := DB \llcorner (\Omega \setminus S_w)$$
$$\beta := DB \llcorner S_w$$

and the measures

$$\gamma := [D \cdot B] \llcorner (\Omega \setminus S_w)$$
$$\lambda := [D \cdot B] \llcorner S_w \,.$$

Then we introduce the measures S^i_δ and R^i_δ given by the following linear functionals on $\varphi \in C_c(\Omega)$:

$$\begin{aligned} \langle S^i_\delta, \varphi \rangle &:= \sum_{j,k} \int_{\mathbf{R}^d} g^{ijk}_\delta(\xi)\, d[\alpha_{kj}](\xi) \\ &\quad - \int_{\mathbf{R}^d} \varphi(x) \frac{\partial h}{\partial z_i}(w * \rho_\delta(x))\, w_i * \rho_\delta(x)\, d\gamma(x) \qquad (40) \\ \langle R^i_\delta, \varphi \rangle &:= \sum_{j,k} \int_{\mathbf{R}^d} g^{ijk}_\delta(\xi)\, d[\beta_{kj}](\xi) \\ &\quad - \int_{\mathbf{R}^d} \varphi(x) \frac{\partial h}{\partial z_i}(w * \rho_\delta(x))\, w_i * \rho_\delta(x)\, d\lambda(x)\,, \qquad (41) \end{aligned}$$

where

$$\begin{aligned} g^{ijk}_\delta(\xi) := -\frac{1}{\delta} \int_0^\delta \int_{\mathbf{R}^d} y_k \frac{\partial \rho}{\partial z_j}(y) \varphi(\xi - \tau y) \\ \cdot \frac{\partial h}{\partial z_i}(w * \rho_\delta(\xi - \tau y)) w_i(\xi + (\delta - \tau) y)\, dy\, d\tau \,. \qquad (42) \end{aligned}$$

This formula for g^{ijk}_δ comes from the formulas for A^{ijk}_δ of Lemma 2, where we choose as Φ the function

$$\Phi \;:=\; \varphi \frac{\partial h}{\partial z_i}(w * \rho_\delta)\,.$$

Hence, comparing (42) with (39) and (38), from Lemma 2 we conclude that $\mathcal{T}^i_\delta = S^i_\delta + R^i_\delta$.

Let R^i_0 be any weak limit of a subsequence $\{R^i_{\delta_n}\}_{\delta_n \downarrow 0}$ and let S^i_0 be any weak limit of a subsequence (not relabeled) of $\{S^i_{\delta_n}\}$. In what follows we will prove that

(i) $R^i_0 \ll |\lambda| + |\beta|$
(ii) $S^i_0 = 0$.

Since $|\lambda|$ and $|\beta|$ are concentrated on S_w, (i) and (ii) prove the Theorem.

Proof of (i) Let us fix an open set $\tilde{\Omega} \subset\subset \Omega$ and a smooth function φ with $|\varphi| \leq 1$ and with support $K \subset \tilde{\Omega}$. If we define g^{ijk}_δ as in (42), from the fact that w is locally bounded we conclude that there exists a constant C, depending only on $\tilde{\Omega}$, w and h, such that $\|g^{ijk}_\delta\|_\infty \leq C$. Hence, it follows that

$$\left|\int \varphi\, dR^i_\delta\right| \;\leq\; C\|w\|_\infty \left\{|\beta|\left(\bigcup_{j,k} \mathrm{supp}\,(g^{ijk}_\delta)\right) + |\lambda|(K)\right\}. \tag{43}$$

Moreover, it is easy to check that, if K_ε denotes the ε–neighborhood of K, then $\mathrm{supp}\,(g^{ijk}_\delta) \subset K_{2\delta}$. Hence, passing into the limit in (43), we conclude that

$$\left|\int \varphi dR^i_0\right| \;\leq\; C\|w\|_\infty \Big(|\lambda|(K) + |\beta|(K)\Big).$$

From the arbitrariness of $\varphi \in C^\infty_c(\tilde{\Omega})$ it follows easily that $R^i_0 \llcorner \tilde{\Omega} \leq C(|\beta| + |\lambda|)$.

Proof of (ii) By definition of S_w, w has Lebesge limit $\tilde{w}(x)$ at every $x \in \Omega \setminus S_w$. Hence it follows that

$$\lim_{\delta\downarrow 0} w * \rho_\delta(x) \;=\; \tilde{w}(x) \tag{44}$$

Fix φ and define g^{ijk}_δ as in (42). We will show that for every $\xi \in \Omega \setminus S_w$ we have that

$$\lim_{\delta\downarrow 0} g^{ijk}_\delta(\xi) \;=\; g^{ijk}(\xi) \tag{45}$$

where

$$g^{ijk}(\xi) \;:=\; -\varphi(\xi)\frac{\partial h}{\partial z_i}(\tilde{w}(\xi))\tilde{w}_i(\xi)\int_{\mathbf{R}^d} y_k \frac{\partial \rho}{\partial z_j}(y)\, dy\,.$$

Integrating by parts we get

$$g^{ikk}(\xi) = \varphi(\xi)\frac{\partial h}{\partial z_i}(\tilde{w}(\xi))\tilde{w}_i(\xi) \tag{46}$$

$$g^{ijk}(\xi) = 0 \qquad\qquad \text{for } j \neq k. \tag{47}$$

Recall that g^{ijk}_δ, φ, $w*\rho_\delta$, $h(w*\rho_\delta)$, and $\nabla h(w*\rho_\delta)$ are all uniformly bounded. Hence, letting $\delta \downarrow 0$ in (40), from (44), (45), (46), (47), and the dominated convergence theorem we conclude that

$$\begin{aligned}\langle S^i_0, \varphi\rangle = &\sum_k \int_{\mathbf{R}^d} \frac{\partial h}{\partial z_i}(\tilde{w}(\xi))\, \tilde{w}_i(\xi)\, \varphi(\xi)\, d[\alpha_{kk}](\xi) \\ &- \int_{\mathbf{R}^d} \frac{\partial h}{\partial z_i}(\tilde{w}(x))\, \tilde{w}_i(x)\, \varphi(x)\, d\gamma(x)\,.\end{aligned}$$

Recalling that $\sum_k \alpha_{kk} = \sum_k D^c_k B^k \llcorner(\Omega \setminus S_w) = D^c \cdot B \llcorner(\Omega \setminus S_w)$ and $\gamma = D^c \cdot B \llcorner(\Omega \setminus S_w)$, we conclude that $\langle S^i_0, \varphi\rangle = 0$. The arbitrariness of φ gives (ii).

Hence, to finish the proof, it suffices to show (45). Recalling the smoothness of φ and the fact that ρ is supported in the ball $B_1(0)$ we conclude that it suffices to show that

$$I_\delta := \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| \frac{\partial h}{\partial z_j}(w * \rho_\delta(\xi - \tau y))\, w_i(\xi + (\delta - \tau)y) - \frac{\partial h}{\partial z_j}(\tilde{w}(\xi))\, \tilde{w}_i(\xi) \right| dy\, d\tau \tag{48}$$

converges to 0. Then, we write

$$\begin{aligned}
I_\delta \le\; & \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| \frac{\partial h}{\partial z_j}(w * \rho_\delta(\xi - \tau y)) - \frac{\partial h}{\partial z_j}(\tilde{w}(\xi)) \right| |w_i(\xi + (\delta - \tau)y)|\, dy\, d\tau \\
& + \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| \frac{\partial h}{\partial z_j}(\tilde{w}(\xi)) \right| \left| w_i(\xi + (\delta - \tau)y) - \tilde{w}_i(\xi) \right| dy\, d\tau \\
\le\; & \frac{C_1}{\delta}\int_0^\delta \int_{B_1(0)} \left| w * \rho_\delta(\xi - \tau y) - \tilde{w}(\xi) \right| dy\, d\tau \\
& + \frac{C_2}{\delta}\int_0^\delta \int_{B_1(0)} \left| w(\xi + (\delta - \tau)y) - \tilde{w}(\xi) \right| d\xi\, d\tau \\
=:\; & C_1 J_\delta^1 + C_2 J_\delta^2
\end{aligned}$$

where the constants C_1 and C_2 depend only on ξ, w, and h. Note that

$$\begin{aligned}
J_\delta^1 &= \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| w(\xi + \tau y) - \tilde{w}(\xi) \right| dy\, d\tau \\
&= \frac{1}{\delta}\int_0^\delta \left[\frac{1}{\tau^d} \int_{B_\tau(\xi)} \left| w(z) - \tilde{w}(\xi) \right| dz \right] d\tau\,,
\end{aligned}$$

and

$$\begin{aligned}
J_\delta^2 &= \frac{1}{\delta}\int_0^\delta \int_{B_1(0)} \left| w * \rho_\delta(\xi + \tau y) - \tilde{w}(\xi) \right| dy\, d\tau \\
&= \frac{1}{\delta}\int_0^\delta \left[\frac{1}{\tau^d} \int_{B_\tau(\xi)} \left| w * \rho_\delta(z) - \tilde{w}(\xi) \right| dz \right] d\tau\,.
\end{aligned}$$

Hence, since $\tilde{w}(\xi)$ is the Lebesgue limit of w at ξ, we conclude that $J_\delta^1 + J_\delta^2 \to 0$. This completes the proof. □

6 Chain rule: The Cantor part

The following theorem provides together with Theorem 3 and Theorem 5 a full chain rule for the distributional divergence out of S_w.

Theorem 7 (Cantor part). *Assume that* $B \in BV_{\rm loc}(\Omega, \mathbf{R}^d)$ *and* $w \in L^\infty_{\rm loc}(\Omega, \mathbf{R}^k)$*. Then, for every* $h \in C^1(\mathbf{R}^k)$ *we have*

$$[D^c \cdot (h(w)B)] \llcorner (\Omega \setminus S_w) = \left[h(\tilde w) - \sum_{i=1}^k \tilde w_i \frac{\partial h}{\partial z_i}(\tilde w) \right] D^c \cdot B \llcorner (\Omega \setminus S_w) + \sum_{i=1}^k \frac{\partial h}{\partial z_i}(\tilde w) D^c \cdot (w_i B) \llcorner (\Omega \setminus S_w)\,. \qquad (49)$$

As a consequence (4) holds, with σ *concentrated on* S_w *and absolutely continuous with respect to* $|D^c \cdot B| + |D^c \cdot (wB)|$.

Proof. Let us fix a convolution kernel ρ. First of all, we follow the same computations as in the first part of the proof of Theorem 3 to conclude that

$$D \cdot (h(w * \rho_\delta)B) = \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta)[(D \cdot (Bw_i)) * \rho_\delta] + \sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta) T^i_\delta \qquad (50)$$

$$+ \left[h(w * \rho_\delta) - \sum_i (w_i * \rho_\delta) \frac{\partial h}{\partial z_i}(w * \rho_\delta) \right] D \cdot B\,. \qquad (51)$$

Next, let us consider the decompositions:

$$[(D \cdot (Bw_i)) * \rho_\delta] = (D^a \cdot (Bw_i)) * \rho_\delta + (D^c \cdot (Bw_i)) * \rho_\delta + (D^j \cdot (Bw_i)) * \rho_\delta \qquad (52)$$

$$D \cdot B = D^a \cdot B + D^c \cdot B + D^j \cdot B\,. \qquad (53)$$

The proof of Theorem 3 yields that:

(i) The measures

$$\sum_i \frac{\partial h}{\partial z_i}(w * \rho_\delta)\big[(D^a \cdot (Bw_i)) * \rho_\delta\big] + \left[h(w * \rho_\delta) - \sum_i w_i * \rho_\delta \frac{\partial h}{\partial z_i}(w * \rho_\delta) \right] D^a \cdot B$$

converge to $D^a \cdot (h(w)B)$.

Moreover, from Proposition 3 and the fact that w is locally uniformly bounded it follows that:

(ii) Any weak limit μ of a subsequence as $\delta \downarrow 0$ of the measures

$$\sum_{i=1}^k \frac{\partial h}{\partial z_i}(w * \rho_\delta)\big[(D^j \cdot (Bw_i)) * \rho_\delta\big] + \left[h(w * \rho_\delta) - \sum_{i=1}^k (w_i * \rho_\delta) \frac{\partial h}{\partial z_i}(w * \rho_\delta) \right] D^j \cdot B \qquad (54)$$

satisfies $|\mu| \ll |D^j \cdot (Bw)| + |D^j \cdot B|$.

We further split:

$$(D^c\cdot(Bw_i))*\rho_\delta \;=\; \big[D^c\cdot(Bw_i))\llcorner(\Omega\setminus S_w)\big]*\rho_\delta+\big[D^c\cdot(Bw_i))\llcorner S_w\big]*\rho_\delta \quad (55)$$

$$D^c\cdot B \;=\; [D^c\cdot B]\llcorner(\Omega\setminus S_w)+[D^c\cdot B]\llcorner S_w\,. \quad (56)$$

From Propositions 3 and 4 it follows that

(iii) If μ is the weak limit of a subsequence of

$$\sum_i \frac{\partial h}{\partial z_i}(w*\rho_\delta)\Big\{\big[D^c\cdot(Bw_i))\llcorner S_w\big]*\rho_\delta\Big\} + \left[h(w*\rho_\delta)-\sum_i (w_i*\rho_\delta)\frac{\partial h}{\partial z_i}(w*\rho_\delta)\right][D^c\cdot B]\llcorner S_w\,, \quad (57)$$

as $\delta\downarrow 0$, then $|\mu|\ll|D^c\cdot(Bw)|\llcorner S_w+|D^c\cdot B|\llcorner S_w$. Hence $|\mu|(\Omega\setminus S_w)=0$.

In what follows, we will also prove that

(iv) The measures

$$\sum_i \frac{\partial h}{\partial z_i}(w*\rho_\delta)\Big\{\big[D^c\cdot(Bw_i))\llcorner(\Omega\setminus S_w)\big]*\rho_\delta\Big\} + \left[h(w*\rho_\delta)-\sum_i (w_i*\rho_\delta)\frac{\partial h}{\partial z_i}(w*\rho_\delta)\right][D^c\cdot B]\llcorner(\Omega\setminus S_w)\,. \quad (58)$$

converge to

$$\sum_i \frac{\partial h}{\partial z_i}(\tilde w)\,\big[D^c\cdot(Bw_i)\big]\llcorner(\Omega\setminus S_w) + \left[h(\tilde w)-\sum_i \tilde w_i\frac{\partial h}{\partial z_i}(\tilde w)\right]\,D^c\cdot B\llcorner(\Omega\setminus S_w)\,.$$

In order to prove this claim, we notice that the functions

$$f_\delta \;:=\; \left[h(w*\rho_\delta)-\sum_i (w_i*\rho_\delta)\frac{\partial h}{\partial z_i}(w*\rho_\delta)\right]$$

converge to

$$f \;:=\; \left[h(\tilde w)-\sum_i \tilde w_i\frac{\partial h}{\partial z_i}(\tilde w)\right]$$

pointwise on $\Omega\setminus S_w$. Since f_δ are locally uniformly bounded, by the dominated convergence theorem we conclude that f_δ converges to f in $L^1_{\rm loc}(|D^c\cdot B|\llcorner(\Omega\setminus S_w))$. Thus, it is sufficient to prove that for every continuous function ψ the measures

$$\nu_\delta \ := \ \psi(w*\rho_\delta)\Big\{\big[(D^c\cdot(Bw_i))\llcorner(\Omega\setminus S_w)\big]*\rho_\delta\Big\}$$

converge to

$$\psi(\tilde w)\,\big[D^c\cdot(Bw_i)\big]\llcorner(\Omega\setminus S_w)\,. \tag{59}$$

Let us denote by ν the measure $\big[D^c\cdot(Bw_i)\big]\llcorner(\Omega\setminus S_w)$ and by f_δ the functions $\psi(w*\rho_\delta)$. Then, if $\varphi\in C^\infty_c(\Omega)$ is any test function, we have

$$\int\varphi f_\delta d(\nu*\rho_\delta)=\int(\varphi f_\delta)*\rho_\delta d\nu=\int_{\Omega\setminus S_w}(\varphi f_\delta)*\rho_\delta d[D^c\cdot(Bw_i)]\,.$$

We claim that

$$\lim_{\delta\downarrow 0}\big[(\varphi f_\delta)*\rho_\delta\big](x)=\lim_{\delta\downarrow 0}\big[(\varphi\psi(w*\rho_\delta))*\rho_\delta\big](x)=\varphi(x)\psi(\tilde w(x))$$

for any $x\in\Omega\setminus S_w$. Indeed, since φ and f are regular and w is uniformly bounded on a neighborhood of the support of φ, we can write

$$\begin{aligned}\mathrm{osc}_\delta &:= \sup_{y\in B_\delta(x)}|\varphi(y)f_\delta(y)-\varphi(x)\psi(\tilde w(x))|\\ &\le\ C_1\delta+C_2\sup_{y\in B_\delta(x)}|w*\rho_\delta(y)-\tilde w(x)|\\ &=\ C_1\delta+C_2\sup_{y\in B_\delta(x)}\frac{1}{\delta^d}\left|\int_{B_\delta(y)}\big[w(z)-\tilde w(x)\big]\rho\left(\frac{z-y}{\delta}\right)dz\right|\\ &\le\ C_1\delta+\frac{C_3}{\delta^d}\int_{B_{2\delta}(x)}|w(z)-\tilde w(x)|\,dz\,.\end{aligned} \tag{60}$$

Since w has Lebesgue limit $\tilde w(x)$ at x, it follows that the right hand side of (60) tends to 0 as $\delta\downarrow 0$. Thus, we can conclude

$$\begin{aligned}&\lim_{\delta\downarrow 0}\left|\big[(\varphi f_\delta)*\rho_\delta\big](x)-\varphi(x)\psi(\tilde w(x))\right|\\ &=\lim_{\delta\downarrow 0}\frac{1}{\delta^n}\left|\int_{B_\delta(x)}(\varphi(y)f_\delta(y)-\varphi(x)\psi(\tilde w(x)))\rho\left(\frac{y-x}{\delta}\right)dy\right|\\ &\le C\lim_{\delta\downarrow 0}\mathrm{osc}_\delta\ =\ 0\,.\end{aligned}$$

The pointwise convergence of $[(\varphi f_\delta)*\rho_\delta](x)$ just proved gives

$$\lim_{\delta\downarrow 0}\int_{\Omega\setminus S_w}(\varphi f_\delta)*\rho_\delta d[D^c\cdot(Bw_i)]\ =\ \int_{\Omega\setminus S_w}\varphi\psi(\tilde w)d[D^c\cdot(Bw_i)]\,.$$

This implies that the measures ν_δ converge weakly to (59), concluding the proof of claim (iv).

(v) Any limit point of the measures

$$\sum_i\frac{\partial h}{\partial z_i}(w*\rho_\delta)T^i_\delta$$

is concentrated on S_w. This is precisely the statement of Proposition 6.

The proof can now achieved noticing that the decompositions above yield that $D\cdot(h(w)B)$ is the sum of absolutely continuous measures (the one considered in item (i)), jump measures (the ones considered in item (ii)), measures concentrated on S_w (the ones considered in items (iii) and (v)) and finally the measures in item (iv). Restricting the divergence to the $\mathscr{L}^d$-negligible set S_w, the Cantor part of all these contributions, with the exception of the one considered in (iv), disappear. Finally, we obtain (4) from (49) with $\sigma := D^c \cdot (h(w)B) \llcorner S_w$, and Theorem 3 gives that this measure is absolutely continuous with respect to $|D^c \cdot B| + |D^c \cdot (wB)|$. □

7 Bressan's compactness conjecture and its variants

In the following section we use the theorems proved so far to study transport equations and Bressan's compactness conjecture. In Subsection 7.1 we show that the results of the previous sections provide a DiPerna–Lions theory for nearly incompressible BV fields which satisfy a certain technical assumption. In Subsection 7.2 we show how this implies certain cases of Conjecture 1. In both sections we also explain why a positive answer to Question 2 would remove the technical assumption, giving a DiPerna–Lions theory for *all* nearly incompressible BV fields and a full positive answer to Conjecture 1. Finally in Section 7.3 we remark that Theorem 3 and Theorem 4 yields a DiPerna–Lions theory for nearly incompressible SBD fields, and hence allows to prove a variant of Conjecture 1.

7.1 DiPerna–Lions theory and continuity equation for nearly incompressible fields

We first introduce the following notion

Definition 7 (Near incompressibility). *A vector field* $b \in L^\infty(\mathbf{R}_t \times \mathbf{R}^n_x, \mathbf{R}^n_x)$ *is called* nearly incompressible *if there exists a positive function* ρ *with* $\log\rho \in L^\infty$ *such that*

$$\partial_t \rho + D_x \cdot (\rho b) \;=\; 0 \qquad \textit{in the sense of distributions on } \mathbf{R}^+_t \times \mathbf{R}^n \;. \tag{61}$$

Next we introduce a concept of weak solution for transport equations with nearly incompressible BV coefficients. The problem in defining a solution of (62) under the assumptions above is that the distribution $b \cdot \nabla_x w$ cannot be defined as $\mathrm{div}\,(bw) - w\mathrm{div}\, b$, since w is an L^∞ function, defined up to sets of 0 Lebesgue measure, and $\mathrm{div}\, b$ can have nontrivial singular part. This problem is overcome by using the existence of the function ρ in the definition of near incompressibility.

Definition 8 (Weak solutions). *Fix a nearly incompressible vector field $b \in L^\infty \cap BV$ (or $b \in L^\infty \cap BD$) and a function ρ as in Definition 7. For any $c \in L^\infty$ we say that $u \in L^\infty$ is a ρ–*weak solution *of*

$$\begin{cases} \partial_t u + b \cdot \nabla_x u \ = \ c\mathscr{L}^d \\ \\ u(0,\cdot) \ = \ u_0 \end{cases} \tag{62}$$

if u solves the following Cauchy problem

$$\begin{cases} \partial_t(\rho u) + D_x \cdot (b\rho u) \ = \ \rho c\mathscr{L}^d \\ \\ u(0,\cdot) \ = \ u_0 \end{cases} \tag{63}$$

in the following (distributional) sense:

$$\int_{\mathbf{R}^+\times\mathbf{R}^n} \rho(t,x)\Big\{u(t,x)\big[\partial_t\varphi(t,x) + b(t,x)\cdot\nabla_x\varphi(t,x)\big] + c(t,x)\varphi(t,x)\Big\}\,dx\,dt$$
$$= \ -\int_{\mathbf{R}^n} \rho(0,x)u_0(x)\varphi(0,x)\,dx \tag{64}$$

for every test function $\varphi \in C^\infty(\mathbf{R}\times\mathbf{R}^n)$.

Remark 3. The attainment of initial conditions as in (64) is justified by the following remarks. Set $B = (1,b)$. From (61) we get that $D\cdot(\rho B) = 0$. We orient the hyperplane $I := \{t = 0\} \subset \mathbf{R}_t \times \mathbf{R}^n_x$ with the vector $(1,0,\ldots,0)$. Thus, the vector field ρB has a well defined normal trace $\mathrm{Tr}^+(\rho B, I)$. Since the normal trace $\mathrm{Tr}^+(B,I)$ is identically equal to 1, the trace of ρ on I can be uniquely defined as $\rho(0,\cdot) := \mathrm{Tr}^+(\rho B, I)$.

Then, in this subsection we will prove:

Theorem 8 (Uniqueness of weak solutions). *Let $b \in BV_{\rm loc} \cap L^\infty(\mathbf{R}^+\times\mathbf{R}^n,\mathbf{R}^n)$ be nearly incompressible. Consider $B := (1,b)$ and assume that $D^c\cdot B$ vanishes on its tangential set E. Then:*

(a) If ρ and ζ satisfy (61), then any ρ–weak solution of (62) is a ζ–weak solution.
(b) If u is a ρ–weak solution of (62) and γ is a C^1 function, then $\gamma(u)$ is a ρ–weak solution of the Cauchy problem (62) with initial data $\gamma(u_0)$ and right hand side $\gamma'(u)c\mathscr{L}^d$;
(c) For any $u_0 \in L^\infty$ there exists a unique ρ–weak solution of (62).

Thus it makes sense to call the function u of Definition 8 the *weak solution of (62).*

Remark 4. Clearly, a positive answer to Question 2 would give that the assumption $|D^c\cdot B|(E) = 0$ satisfied by any nearly incompressible b. Hence Theorem 8 would give a DiPerna–Lions theory for every nearly incompressible $BV_{\rm loc}\cap L^\infty$ field.

The proof of Theorem 8 is based on the following "renormalization" lemma:

Lemma 3 (Renormalization Lemma). *Let $\Omega \subset \mathbf{R}^d$ be open, $B \in BV_{\rm loc} \cap L^\infty(\Omega, \mathbf{R}^d)$, $\rho \in L^\infty(\Omega)$ and u, $s \in L^\infty(\Omega, \mathbf{R}^l)$. Assume that*

$$\begin{aligned} D \cdot (\rho B) \ &= \ 0 \qquad \textit{and} \qquad \rho \ \geq \ C \ > \ 0 & (65)\\ D \cdot (\rho u_i B) \ &= \ s_i \mathscr{L}^d \qquad \textit{for } i = 1, \ldots, l. & (66)\end{aligned}$$

Then, if E denotes the tangential set of B, we have

$$D \cdot (\rho\beta(u)B) - \sum_{i=1}^{l} \frac{\partial \beta}{\partial y_i}(u) s_i \mathscr{L}^d \ \ll \ |D^c \cdot B| \llcorner E \qquad \forall \beta \in C^1(\mathbf{R}^l)\,. \quad (67)$$

Proof. Set $k := l + 1$ and define $w \in L^\infty(\Omega, \mathbf{R}^k)$ as $w_1 = \rho$, $w_i = \rho u_{i-1}$ for $i = 2, \ldots, l$. Moreover define $H : \mathbf{R}^k \to \mathbf{R}$ as

$$H(z_1, \ldots, z_k) \ = \ z_1\,\beta\left(\frac{z_2}{z_1}, \ldots, \frac{z_k}{z_1}\right)\,. \quad (68)$$

Clearly, H is C^1 on the set $\mathbf{R}^k \setminus \{z_1 = 0\}$. Define $m := \max\{\|\rho\|_\infty, \|w\|_\infty\}$ and let $h \in C^1(\mathbf{R}^k)$ be such that $h = H$ on the set

$$D \ := \ \left\{ z_1 \ \geq \ \frac{C}{2} \quad \text{and} \quad |z| \ \leq \ m+1 \right\}\,. \quad (69)$$

Then we have $D \cdot (w_1 B) = 0$, $D \cdot (w_i B_i) = s_i$ for $i = 2, \ldots, k$, and $D \cdot (\rho\beta(u)B) = D \cdot (h(w)B)$.

Absolutely continuous part. We use formula (18) and we compute

$$D^a \cdot (h(w)B) \ := \ \left[h(w) - \sum_{i=1}^{k} \frac{\partial h}{\partial z_i}(w) w_i \right] D^a \cdot B + \sum_{i=2}^{k} \frac{\partial h}{\partial z_i}(w) s_i\,.$$

Note that h is a 1–homogeneous function on D. Thus

$$h(z) - \sum_{i=1}^{k} \frac{\partial h}{\partial z_i}(z) z_i \ = \ 0 \qquad \text{for every } z \in D.$$

Since the essential range of w is contained in D, we get that

$$h(w) - \sum_{i=1}^{k} \frac{\partial h}{\partial z_i} w_i \ = \ 0 \qquad \mathscr{L}^d\text{-almost everywhere}\,,$$

$$\sum_{i=2}^{k} \frac{\partial h}{\partial z_i}(w) s_i \ = \ \sum_{i=1}^{l} \frac{\partial \beta}{\partial y_i}(u) s_i \qquad \mathscr{L}^d\text{-almost everywhere.}$$

Thus we conclude that

$$D^c \cdot (h(w)B) \;=\; \sum_{i=1}^{l} \frac{\partial \beta}{\partial y_i}(u) s_i \mathscr{L}^d \,. \tag{70}$$

Cantor part According to (49) we have

$$\big[D^c \cdot (h(w)B)\big] \llcorner (\Omega \setminus E) \;=\; \left[h(\tilde{w}) - \sum_{i=1}^{k} \frac{\partial h}{\partial z_i}(\tilde{w})\tilde{w}_i\right] D^c \cdot B \llcorner (\Omega \setminus E)\,,$$

where $\tilde{w}(x)$ is the Lebesgue limit of w at x, which on $\Omega \setminus E$ exists $|D^c \cdot B|$–a.e.. From the very definition of Lebesgue limit, it follows that whenever it exists, it belongs to the closure of the essential range of w, that is still contained in D. Hence, arguing as above, we conclude that

$$\big[D^c \cdot (h(w)B)\big] \llcorner (\Omega \setminus E) \;=\; 0\,. \tag{71}$$

Jump part Let J_B be the jump set, let ν be an orienting unit normal to J_B, and let B^+ and B^- be respectively the right and left traces. Then it follows from Theorem 4 that there exist Borel functions w^+ and w^-, characterized as quotients of traces of wB and B, such that

$$D^j \cdot (h(w)B) \;=\; \big[h(w^+)B^+ \cdot \nu - h(w^-)B^- \cdot \nu\big]\, \mathscr{H}^{d-1} \llcorner J_B\,. \tag{72}$$

If we define the functions $h_i : \mathbf{R}^k \to \mathbf{R}$ as

$$h_i(z_1, \ldots, z_k) \;=\; z_i$$

we can apply the same theorem in order to get

$$D^j \cdot (h_i(w)B) \;=\; \big[w_i^+ B^+ \cdot \nu - w_i^- B^- \cdot \nu\big]\, \mathscr{H}^{d-1} \llcorner J_B\,. \tag{73}$$

But since $D \cdot (h_i(w)B) = D \cdot (w_i B) = 0$, we conclude that

$$w_i^+ B^+ \cdot \nu \;=\; w_i^- B^- \cdot \nu \qquad \mathscr{H}^{d-1}\text{–a.e. on } J_B. \tag{74}$$

We fix $x \in J_B$ such that $w_i^+(x)B^+(x) \cdot \nu(x) = w_i^-(x)B^- \cdot \nu(x)$ and we distinguish two cases:

Case 1 $B^+(x) \cdot \nu(x) \neq 0 \neq B^-(x) \cdot \nu(x)$.

Then $w^+(x)$ and $w^-(x)$ are the right and left Lebesgue limits of w at x. This means that they both belong to the essential range of w and hence to D. To simplify the notation in the following formulas we drop the (x) dependence.

The formulas (74) give that

$$w_i^+ \;=\; w_i^- \left[\frac{B^- \cdot \nu}{B^+ \cdot \nu}\right].$$

Recall that $D \subset \{z_1 \neq 0\}$ and hence we conclude

$$\frac{w_i^+}{w_1^+} = \frac{w_i^-}{w_1^-} \qquad \text{for } i = 1, \dots, k. \tag{75}$$

Since $h = H$ on D, plugging (75) into (68) and using (74) we conclude

$$\begin{aligned} & h(w^+)B^+ \cdot \nu - h(w^-)B^- \cdot \nu \\ &= w_1^+ \beta\left(\frac{w_2^+}{w_1^+}, \dots, \frac{w_k^+}{w_1^+}\right) B^+ \cdot \nu - w_1^- \beta\left(\frac{w_2^-}{w_1^-}, \dots, \frac{w_k^-}{w_1^-}\right) B^- \cdot \nu \\ &= \beta\left(\frac{w_2^+}{w_1^+}, \dots, \frac{w_k^+}{w_1^+}\right) \left[w_1^+ B^+ \cdot \nu - w_1^- B^- \cdot \nu\right] = 0\,. \end{aligned}$$

Case 2 Remaining cases.
If both $B^+(x) \cdot \nu(x)$, $B^-(x) \cdot \nu(x)$ vanish, then clearly

$$h(w^+(x))B^+(x) \cdot \nu(x) - h(w^-(x))B^-(x) \cdot \nu(x) = 0\,.$$

Assume that one of them vanishes but the other not. Without loosing our generality we assume that $B^+(x) \cdot \nu(x) = 0 \neq B^-(x) \cdot \nu(x)$. Then, $w^-(x)$ is in D. This means that $w_1^-(x) \neq 0$. But then we would have

$$w_1^+(x)B^+(x) \cdot \nu(x) = 0 \neq w_1^-(x)B^-(x) \cdot \nu(x)\,,$$

which contradicts (74).

From the analysis of these two cases we conclude that

$$D^j \cdot (h(w)B) = 0\,. \tag{76}$$

Conclusion From (70), (71), and (76) we conclude that

$$D \cdot (h(w)B) = D^c \cdot (h(w)B) \llcorner (\Omega \setminus E)\,. \tag{77}$$

From (19) we know that

$$D^s \cdot (h(w)B) \ll |D^s \cdot B|\,. \tag{78}$$

Thus, (77) and (78) give (67). $\square$

Proof (of Theorem 8). (a) Assume that u is a ρ–weak solution and that ζ is another weak solution of (61). Let $u_1 := u$, $u_2 := \zeta/\rho$, $B = (1, b)$, $s_1 = c\rho$ and $s_2 = 0$. Apply the renormalization Lemma 3 to $\beta(u_1, u_2) = u_1 u_2$ to conclude that

$$\begin{aligned} & \partial_t(\beta(u_1, u_2)\rho) + D_x \cdot (b\beta(u_1, u_2)\rho) \\ &= \left(s_1 \frac{\partial \beta}{\partial y_1}(u_1, u_2) + s_2 \frac{\partial \beta}{\partial y_2}(u_1, u_2)\right) \mathscr{L}^{d+1}\,. \end{aligned}$$

Note that $s_2 = 0$ and that $\beta_{y_1}(u_1, u_2) = u_2$, hence

$$\partial_t(u\zeta) + D_x \cdot (u\zeta) \;=\; c\zeta\mathscr{L}^{d+1}\,. \tag{79}$$

This means that u and ζ solve (79) in the sense of distributions in $\mathbf{R}^+ \times \mathbf{R}^n$. To take care of the initial condition in the sense of Remark 3, it is sufficient to apply Theorem 5.

(b) Applying Lemma 3 we conclude that $\partial_t(\rho\gamma(u)) + D_x \cdot (b\rho\gamma(u)) = \rho c\gamma'(u)\mathscr{L}^d$ in the sense of distributions in $\mathbf{R}^+ \times \mathbf{R}^n$. As above, we apply Theorem 5 in order to take care of the initial condition in the sense of Remark 3.

(c) Let u_1 and u_2 be two ρ–solutions of the same Cauchy problem. Define $u := u_1 - u_2$ and note that u solves in the sense of distributions the Cauchy problem

$$\begin{cases} \partial_t(u\rho) + D_x \cdot (u\rho b) \;=\; 0 \\ \\ u(0, \cdot) \;=\; 0\,. \end{cases}$$

From (b) we conclude that u^2 solves the same Cauchy problem. This means that for every $\varphi \in C_c^\infty(\mathbf{R} \times \mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^+ \times \mathbf{R}^n} \rho(t,x)u^2(t,x)\big[\partial_t\varphi(t,x) + b(t,x) \cdot \nabla_x\varphi(t,x)\big]\,dx\,dt \;=\; 0\,.$$

Thus, we can apply Lemma 2.11 of [10] to conclude that $u^2 \equiv 0$. □

7.2 Some cases of Conjecture 1

We now apply the results of the previous section to prove the following:

Proposition 12 (Partial answer to Bressan's Conjecture). *Let b_n be as in Conjecture 1 and assume that $b_n \to b$ strongly in $L^1_{\rm loc}$. If $D^c \cdot B$ vanishes on the tangential set of B, then the conclusion of Conjecture 1 holds and the fluxes $\{\Phi_n\}$ have a unique limit.*

Remark 5. If Question 2 has a positive answer, then any limiting B satisfies the assumption of Proposition 12, and therefore Conjecture 1 would have a full positive answer.

Proof. The arguments are the same as those given in Section 4 of [10] and they consist in a standard modification of the usual arguments used in DiPerna–Lions theory of renormalized solutions to pass from a uniqueness theorem on transport equations to compactness properties of solutions to ordinary differential equations (see [25] and also [6], [7] for a different approach).

Since (Φ_n) is locally uniformly bounded it suffices to prove, thanks to the dominated convergence theorem, that for every $T \in \mathbf{R}$ there exists a function

$\Phi(T,\cdot) \in L^1_{\rm loc}(\mathbf{R}^d,\mathbf{R}^d)$ such that $\Phi_n(T,\cdot)$ converge strongly to $\Phi(T,\cdot)$ in $L^1_{\rm loc}$. We will prove this property for $T<0$, the case $T>0$ being analogous.

Step 1. For each t denote by $\Psi_n(t,\cdot)$ the inverse of $\Phi_n(t,\cdot)$. Let us consider the ODE

$$\begin{cases} \dfrac{d}{dt}\Lambda_n(t,x) \;=\; b_n(t,\Lambda_n(t,x)) \\ \\ \Lambda_n(T,x) \;=\; x\,, \end{cases}$$

and note that

$$\Lambda_n(t,x) \;=\; \Phi_n(t,\Psi_n(T,x))\,. \tag{80}$$

Thus, if we denote by $J_n(t,\cdot)$ the Jacobian of $\Lambda_n(t,\cdot)$, we get that $C^{-2} \le J_n(t,\cdot) \le C^2$. Denote by $\Gamma_n(t,\cdot)$ the inverse of $\Lambda_n(t,\cdot)$ and set

$$\rho_n(t,x) := \frac{1}{J_n(t,\Gamma_n(t,x))}\,.$$

Since, by the area formula, $\rho_n(t,\cdot)$ is the density of the of the image of $\mathscr{L}^d$ by the flow map $\Lambda_n(t,\cdot)$, the maps ρ_n solve the continuity equation

$$\begin{cases} \partial_t\rho_n + \mathrm{div}_x\,(b_n\rho_n) \;=\; 0 \\ \\ \rho_n(T,\cdot) \;=\; 1 \end{cases} \tag{81}$$

in the sense of distributions.

Any weak* limit point ρ of a subsequence of (ρ_n) will still solve

$$\begin{cases} \partial_t\rho + \mathrm{div}_x\,(b\rho) \;=\; 0 \\ \\ \rho(T,\cdot) \;=\; 1 \end{cases} \tag{82}$$

in the sense of distributions. If ζ is any other distributional solution of (82), then $w := \zeta/\rho$ is a weak solution (in the sense of Definition 8, up to a time shift) of

$$\begin{cases} \partial_t w + b\cdot\nabla_x w \;=\; 0 \\ \\ w(T,\cdot) \;=\; 1\,. \end{cases} \tag{83}$$

Therefore from Theorem 8 (again up to a time shift) we conclude $w=1$, that is $\zeta=\rho$. Therefore we conclude that the *whole sequence* (ρ_n) converges to ρ.

Step 2. Now fix $\overline{w} \in L^\infty(\mathbf{R}^d)$ and set $w_n(t,x) := \overline{w}(\Gamma_n(t,x))$. These functions are weak solutions to the transport equations

$$\begin{cases} \partial_t w_n + b_n\cdot\nabla_x w_n \;=\; 0 \\ \\ w_n(T,\cdot) \;=\; \overline{w}(\cdot)\,. \end{cases} \tag{84}$$

Passing to a subsequence we can assume that $\rho_{n(k)}w_{n(k)}$ converge to a function v weakly* in L^∞. If we define $w := v/\rho$, then w is the *unique* weak solution of

$$\begin{cases} \partial_t w + b\cdot\nabla_x w \;=\; 0 \\ w(T,\cdot) \;=\; \overline{w}(\cdot)\,. \end{cases} \tag{85}$$

Therefore the *whole sequence* $(\rho_n w_n)$ converges to ρw. From Theorem 8(b) it follows that, for every $\beta\in C^1$, the functions $\tilde w_n := \beta(w_n)$ and $\tilde w := \beta(w)$ are the unique weak solutions of

$$\begin{cases} \partial_t \tilde w_n + b_n\cdot\nabla_x \tilde w_n \;=\; 0 \\ \tilde w_n(T,\cdot) \;=\; \beta\big(\overline{w}(\cdot)\big)\,, \end{cases} \qquad \begin{cases} \partial_t \tilde w + b\cdot\nabla_x \tilde w \;=\; 0 \\ \tilde w(T,\cdot) \;=\; \beta\big(\overline{w}(\cdot)\big)\,. \end{cases} \tag{86}$$

Therefore arguing as above we conclude that

$$\rho_n\beta(w_n) \text{ weak}^*\text{-converge in } L^\infty \text{ to } \rho\beta(w) \text{ for every } \beta\in C^1. \tag{87}$$

Step 3. For every given smooth function $\varphi\in C_c(\mathbf{R}^+\times\mathbf{R}^d)$ we write

$$\int\varphi\rho_n(w_n-w)^2 \;=\; \int\varphi\rho_n w_n^2 - 2\int\rho_n w_n w + \int\rho_n w^2\,.$$

Then, from (87) with $\beta(s)=s^2$ we get:

$$\lim_{n\to\infty}\int\varphi\rho_n w_n^2 \;=\; \int\varphi\rho w^2\,,$$

$$\lim_{n\to\infty}\int\varphi\rho_n w_n w \;=\; \int\varphi\rho w^2\,,$$

$$\lim_{n\to\infty}\int\varphi\rho_n w^2 \;=\; \int\varphi\rho w^2\,,$$

hence

$$\lim_{n\to\infty}\int\varphi\rho_n(w_n-w)^2 \;=\; 0\,.$$

Since $\rho_n\geq C^{-2}>0$ and φ is arbitrary, we conclude that (w_n) converges to w strongly in $L^2_{\rm loc}$. Since $w_n(t,x) \;=\; \overline{w}(\Gamma_n(t,x))$ we conclude that for every $\overline{w}\in L^\infty(\mathbf{R}^d)$ the functions $\overline{w}(\Gamma_n(t,x))$ converge strongly in $L^1_{\rm loc}$ to a unique function. Therefore we conclude that Γ_n converge strongly in $L^1_{\rm loc}$ to a function Γ on $[T,0]\times\mathbf{R}^d$.

Step 4. Fix $R>0$ and note that for each x the curves $\Gamma_n(\cdot,x)$ have a uniformly bounded Lipschitz constant, and so possibly modifying Γ in a $\mathscr{L}^{d+1}$-negligible set we can assume that the same is true for Γ. Since the mean value theorem provides us an infinitesimal sequence $(\epsilon_n)\subset(T,0)$ such that

$$\lim_{n\to\infty} \int_{B_R} |\Gamma_n(\epsilon_n, x) - \Gamma(\epsilon_n, x)|\, dx = 0$$

we obtain as a consequence that $\Gamma_n(0,\cdot) \to \Gamma(0,\cdot)$ strongly in $L^1(B_R)$. Recalling the identity (80) we have $\Lambda_n(0,x) = \Psi_n(T,x)$. Therefore $\Gamma_n(0,\cdot)$ is the inverse of $\Psi_n(T,\cdot)$, which means $\Gamma_n(0,\cdot) = \Phi_n(T,\cdot)$. This allows us to conclude that $\Phi_n(T,\cdot)$ converge strongly in $L^1(B_R)$ to $\Gamma(0,\cdot) =: \Phi(T,\cdot)$. Since R is arbitrary the proof of the convergence of (Φ_n) is achieved. □

7.3 SBD–variant of Conjecture 1

The following are corollaries of Theorem 3 and Theorem 4:

Theorem 9 (Uniqueness of weak solutions for SBD coefficients). *Let $b \in SBD_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R}^n)$ be nearly incompressible. Then:*

(a) If ρ and ζ satisfy (I), then any ρ–weak solution of (62) is a ζ–weak solution;
(b) If u is a ρ–weak solution of (62) and γ is a C^1 function, then $\gamma(u)$ is a ρ–weak solution of the Cauchy problem (62) with initial data $\gamma(u_0)$ and with right hand side $\gamma'(u)c\mathscr{L}^d$;
(c) For any $u_0 \in L^\infty$ there exists a unique ρ–weak solution of (62).

Thus it makes sense to call the u of Definition 8 the *weak solution of (62).*

Proposition 13 (SBD variant of Bressan's compactness conjecture). *Let b_n be a sequence of smooth maps $b_n : \mathbf{R}_t \times \mathbf{R}^d_x \to \mathbf{R}^d_x$, uniformly bounded. Let Φ_n be as in (6) and assume that they satisfy condition (7). If $b_n \to b$ and $b \in SBD_{\rm loc}$, then $\{\Phi_n\}$ has a unique strong limit in $L^1_{\rm loc}$.*

Since the proof are essentially analogous to the proofs of Theorem 8 and Proposition 12 we do not give the details here.

8 Proof of Proposition 2

We set $\Omega := \{(x,y) \in \mathbf{R}^2 \;:\; 1 < x < 2\,, 0 < y < x\}$. We construct a scalar function $u \in L^\infty \cap BV(\Omega)$ with the following properties:

(a) $D^c_y u \neq 0$;
(b) $D_x u + D_y(u^2/2)$ is a pure jump measure, i.e. it is concentrated on the jump set J_u.

Given such a function u, the field $B = (1,u)\mathbf{1}_\Omega$ meets the requirements of the proposition. Indeed, let $\tilde{B} = (1,\tilde{u})\mathbf{1}_\Omega$ be the precise representative of B. Due to (b) the Cantor part of $D_x u + D_y(u^2/2)$ vanishes. Hence using the chain rule of Vol'pert we get

$$D^c_x u + \tilde{u} D^c_y u \;=\; 0\,. \tag{88}$$

Denote by $M(x)$ the Radon–Nikodym derivative $DB/|DB|$. Then we have

$$M\cdot\tilde B|D^cB| = D^cB\cdot\tilde B = \begin{pmatrix}0 & 0\\ D^c_x u & D^c_y u\end{pmatrix}\cdot\begin{pmatrix}1\\ \tilde u\end{pmatrix} = \begin{pmatrix}0\\ D^c_x u+\tilde u D^c_y u\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Hence we conclude that $M(x)\cdot\tilde B(x)=0$ for $|D^cB|$–a.e. x, that is $|D^cB|$ is concentrated on the tangential set E of B. Therefore $|D^c\cdot B|(\Omega\setminus E)=0$. On the other hand, from (a) we have $D^c\cdot B = D^c_y u\neq 0$. Hence we conclude $|D^c\cdot B|(E)>0$.

We now come to the construction of the desired u. This is achieved as the limit of a suitable sequence of functions u_k.

Step 1. Construction of u_k.
Consider the auxiliary 1-periodic function $\sigma:\mathbf{R}\to\mathbf{R}$ defined by

$$\sigma(p+x) \;=\; 1-x\,,\qquad 0<x\le 1\,,\quad p\in\mathbf{Z}\,.$$

We let $\gamma_k:[0,1]\to[0,1]$ be the usual piecewise linear approximation of the Cantor ternary function, that is $\gamma_0(z)=z$ and, for $k\ge 1$,

$$\gamma_k(z) \;=\; \begin{cases}\frac12\gamma_{k-1}(3z), & 0<z\le\frac13\,,\\[1ex] \frac12, & \frac13<z\le\frac23\,,\\[1ex] \frac12\big(1+\gamma_{k-1}(3z-2)\big), & \frac23<z\le 1\,.\end{cases}$$

Notice that

$$\gamma_k'(z)\in\big\{0,\ \big(\frac32\big)^k\big\} \tag{89}$$

and

$$|\gamma_k(z)-\gamma_{k-1}(z)|\le\frac13\cdot 2^{-k}\,. \tag{90}$$

We set $G:=]1,2[\times]0,1[$ and we define $\varphi_k:G\to\mathbf{R}$ by

$$\varphi_k(x,z) \;=\; xz \;+\; \sum_{j=1}^{k} 4^{1-j}\sigma(4^{j-1}x)\big(\gamma_{j-1}(z)-\gamma_j(z)\big).$$

Note that φ_k is bounded. To describe more precisely the behavior of this function we introduce the following sets: The strips

$$S^k_i \;:=\; \big]1+(i-1)4^{1-k},1+i4^{1-k}\big[\;\times\;\mathbf{R}\qquad i=1,\dots,4^{k-1},$$

and the vertical lines

$$V^k_i \;:=\; \{i4^{1-k}\}\;\times\;\mathbf{R}\qquad i=1,\dots,4^{k-1}-1\,.$$

Then φ_k is Lipschitz on each rectangle $S_i^k \cap G$ and it has jump discontinuities on the segments $V_i^k \cap G$. Therefore φ_k is a BV function and satisfies the identities $D_x\varphi_k = D_x^j\varphi_k + D_x^a\varphi_k$ and $D_y\varphi_k = D_y^a\varphi_k$. Moreover, denoting by $(\partial_x\varphi_k, \partial_y\varphi_k)$ the density of the absolutely continuous part of the derivative, we get

$$\begin{aligned}\partial_x\varphi_k(x,z) &= z + (\gamma_1(z) - z) + (\gamma_2(z) - \gamma_1(z)) + \cdots + (\gamma_k(z) - \gamma_{k-1}(z)) \\ &= \gamma_k(z)\,. \end{aligned} \tag{91}$$

Clearly

$$0 \;\le\; 4^{1-j}\sigma(4^{j-1}x) - 4^{-j}\sigma(4^j x) \;\le\; 3\cdot 4^{-j}\,.$$

Therefore, using also (89), on each rectangle $S_i^k \cap G$ we can estimate

$$\begin{aligned}\partial_z\varphi_k(x,z) &= x + \sigma(x) - \big(\sigma(x) - 4^{-1}\sigma(4x)\big)\ \gamma_1'(z) \\ &\qquad - \big(4^{-1}\sigma(4x) - 4^{-2}\sigma(4^2x)\big)\ \gamma_2'(z)\ -\ \cdots \\ &\qquad - \big(4^{2-k}\sigma(4^{k-1}x) - 4^{1-k}\sigma(4^{k-1}x)\big)\ \gamma_{k-1}'(z) \\ &\qquad - 4^{1-k}\sigma(4^{k-1}x)\ \gamma_k'(z) \\ &\ge 2 - 3\left(4^{-1}\gamma_1'(z) + \cdots + 4^{1-k}\gamma_k'(z)\right) - 4^{1-k}\gamma_k'(z) \\ &\ge 2 - 3\left(\frac{3}{8} + \cdots + \left(\frac{3}{8}\right)^{k-1}\right) - 4\left(\frac{3}{8}\right)^k .\end{aligned}$$

Since

$$4\left(\frac{3}{8}\right)^k \le 3\left(\left(\frac{3}{8}\right)^k + \left(\frac{3}{8}\right)^{k+1} + \cdots\right),$$

we obtain

$$\partial_z\varphi_k \;\ge\; 2 - 3\left(\frac{3}{8} + \left(\frac{3}{8}\right)^2 + \cdots\right) = \frac{1}{5}. \tag{92}$$

Hence, since $\varphi_k(x,\cdot)$ maps $[0,1]$ onto $[0,x]$, the function

$$\Phi_k(x,y) \;:=\; (x, \varphi_k(x,y))$$

maps each rectangle $S_i^k \cap G$ onto $S_i^k \cap \Omega$, and it is bi-Lipschitz on each such rectangle. This allows to define u_k by the implicit equation

$$u_k\big(x, \varphi_k(x,z)\big) \;=\; \gamma_k(z)\,, \tag{93}$$

and to conclude that $0 \le u_k \le 1$ and that u_k is Lipschitz on each $S_i^k \cap \Omega$. Therefore $u_k \in L^\infty \cap BV(\Omega)$, $D_x u_k = D_x^a u_k + D_x^j u_k$ and $D_y u_k = D_y^a u_k$.

Step 2. BV bounds.

We prove in this step that $|Du_k|(\Omega)$ is uniformly bounded. This claim and the bound $\|u_k\|_\infty \le 1$ allow to apply the BV compactness theorem to get a subsequence which converges to a bounded BV function u, strongly in L^p for every $p < \infty$. In Steps 3 and 4 we will then complete the proof by showing that u satisfies both the requirements (a) and (b).

By differentiating (93) and using (91) we get the following identity for $\mathscr{L}^2$–a.e. $(x,z) \in S_i^k \cap G$:

$$\begin{aligned} 0 &= \frac{\partial u_k(x,\varphi_k(x,z))}{\partial x} + \frac{\partial u_k(x,\varphi_k(x,z))}{\partial y} \frac{\partial \varphi_k(x,z)}{\partial x} \\ &= \frac{\partial u_k(x,\varphi_k(x,z))}{\partial x} + \frac{\partial u_k(x,\varphi_k(x,z))}{\partial y} \gamma_k(x) \\ &= \frac{\partial u_k(x,\varphi_k(x,z))}{\partial x} + \frac{\partial u_k(x,\varphi_k(x,z))}{\partial y} u_k(x,\varphi_k(x,z)). \end{aligned}$$

Since Φ_k is bi-Lipschitz we get

$$\partial_x u_k(x,y) + u_k \partial_y u_k(x,y) \;=\; 0 \qquad \text{for } \mathscr{L}^2\text{–a.e. } (x,y) \in S_i^k \cap \Omega\,. \tag{94}$$

If $4^{k-1}x \notin \mathbf{N}$ the function $u_k(x,\cdot)$ is non decreasing. Therefore

$$|D_y u_k|(\Omega) \;=\; D_y u_k(\Omega) \;=\; \int_1^2 \big(u_k(x,x) - u_k(x,0)\big)\, dx \;=\; 1\,. \tag{95}$$

From (94) we get

$$|D_x^a u_k|(\Omega) \;\le\; |D_y^a u_k|(\Omega) \;=\; 1\,. \tag{96}$$

Therefore it remains to bound $|D_x^j u_k|(\Omega)$. This consists of

$$|D_x^j u_k|(\Omega) \;=\; \sum_{i=1}^{4^{k-1}-1} \int_{V_i^k} |u_k^+ - u_k^-|\, d\mathscr{H}^1. \tag{97}$$

For each x of type $1 + i4^{1-k}$ we compute

$$\begin{aligned} \int_{V_i^k} |u_k^+ - u_k^-|\, d\mathscr{H}^1 \;&=\; \int_0^x |u_k(x^+,y) - u_k(x^-,y)|\, dy \\ &=\; \int_0^1 |\{y : u_k(x^-,y) < t < u_k(x^+,y)\}|\, dt \\ &\quad + \int_0^1 |\{y : u_k(x^+,y) < t < u_k(x^-,y)\}|\, dt \\ &=\; \int_0^1 |\{y : u_k(x^-,y) < \gamma_k(z) < u_k(x^+,y)\}|\, \gamma_k'(z)\, dz \\ &\quad + \int_0^1 |\{y : u_k(x^+,y) < \gamma_k(z) < u_k(x^-,y)\}|\, \gamma_k'(z)\, dz \\ &=\; \int_0^1 |\varphi_k(x^+,z) - \varphi_k(x^-,z)|\, \gamma_k'(z)\, dz \\ &\le\; \sup_{z\in]0,1[} |\varphi_k(x^+,z) - \varphi_k(x^-,z)| \end{aligned}$$

$$\stackrel{(90)}{\leq} \frac{4}{3}\sum_{j=1}^{k} 8^{-j}\big(\sigma(4^{j-1}x^+) - \sigma(4^{j-1}x^-)\big)\,. \tag{98}$$

Combining (97) and (98) we get

$$\begin{aligned} |D^j_x u_k|(\Omega) &\leq \frac{4}{3}\sum_{i=1}^{4^{k-1}-1}\sum_{j=1}^{k} 8^{-j}\big(\sigma(4^{j-1}4^{1-k}i^+) - \sigma(4^{j-1}4^{1-k}i^-)\big) \\ &= \frac{4}{3}\sum_{j=1}^{k} 8^{-j}\sum_{i=1}^{4^{k-1}-1}\big(\sigma(4^{j-k}i^+) - \sigma(4^{j-k}i^-)\big) \\ &= \frac{4}{3}\sum_{j=1}^{k} 8^{-j}4^{j-1} \;\leq\; \frac{1}{3}\,. \end{aligned} \tag{99}$$

Step 3. Proof of (a).

We now fix a bounded BV function u and a subsequence of u_k, not relabeled, which converges to u strongly in L^1. We claim that (a) holds. More precisely we will show that:

(Cl) For $\mathscr{L}^1$–a.e. x the function $u(x,\cdot)$ is a nonconstant BV function of one variable which has no absolutely continuous part and no jump part.

(Cl) gives (a) by the slicing theory of BV functions, see Theorem 3.108 of [5].

In order to prove (Cl) we proceed as follows. By possibly extracting another subsequence we assume that u_k converges to u $\mathscr{L}^2$–a.e. in Ω. We then show (Cl) for every x such that:

- $4^k x \notin \mathbf{N}$ for every k;
- $u_k(x,y)$ converges to $u(x,y)$ for $\mathscr{L}^1$–a.e. y.

Clearly $\mathscr{L}^1$–a.e. x meets these requirements.

Fix any such x. Note that x is never on the boundary of any strip S^k_i. Therefore we can denote by g^x_k the inverse of $\varphi_k(x,\cdot)$ and we can use (93) to write

$$u_k(x,y) \;=\; \gamma_k(g^x_k(y))\,. \tag{100}$$

Thanks to (92), the Lipschitz constant of g_k is uniformly bounded. Therefore, after possibly extracting a subsequence, we can assume that g_k uniformly converge to a Lipschitz function g. Since γ_k uniformly converge to the Cantor ternary function γ, we can pass into the limit in (100) to conclude

$$u(x,y) \;=\; \gamma(g(y))\,. \tag{101}$$

Therefore $u(x,\cdot)$ is continuous, nondecreasing, nonconstant, and locally constant outside a closed set of zero Lebesgue measure ($g^{-1}(C)$, where C is the Cantor set). This proves (Cl).

Step 4. Proof of (b).
Let u be as in Step 3. From the construction of u_k it follows that

$$D_x u_k + D_y(u_k^2/2) \;=\; D_x^j u_k\,. \tag{102}$$

After possibly extracting a subsequence we can assume that $D_x^j u_k$ converges weakly* to a measure μ. This gives

$$D_x u + D_y(u^2/2) \;=\; \mu\,. \tag{103}$$

Therefore it suffices to prove that μ is concentrated on a set of σ–finite 1–dimensional Hausdorff measure. Indeed μ is concentrated on the union of the countable family of segments $\{V^k\}_{k,i}$. In order to prove this claim it suffices to show the following tightness property: for every $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$|D_x^j u_k|\left(\bigcup_{l\geq N}\bigcup_{i=1}^{4^{l-1}-1} V_i^l\right) \;\leq\; \varepsilon \qquad \text{for every } k. \tag{104}$$

Note that

$$|D_x^j u_k|\left(\bigcup_{l\geq N}\bigcup_{i=1}^{4^{l-1}-1} V_i^l\right) \;\leq\; \sum_{l\geq N}\sum_{i=1}^{4^{l-1}-1}\int_{V_i^l} |u_k^+ - u_k^+|\,.$$

Then the same computations leading to (98) and (99) give

$$|D_x^j u_k|\left(\bigcup_{l\geq N}\bigcup_{j=1}^{4^{l-1}-1} V_j^l\right) \;\leq\; \frac{4}{3}\sum_{l=N}^{k} 8^{-l}4^{l-1} \;\leq\; \frac{1}{3\cdot 2^{N-1}}\,. \tag{105}$$

This concludes the proof. □

Remark 6. The function u constructed in Proposition 2 solves Burgers' equation with a measure source

$$D_t u + D_x(u^2/2) \;=\; \mu\,, \tag{106}$$

and has nonvanishing Cantor part. On the other hand in [9] it has been proved that *entropy* solutions to Burgers' equation without source are SBV, i.e. the Cantor part of their derivative is trivial. It would be interesting to understand whether this gain of regularity is due to the entropy condition, or instead BV distributional solutions of (106) with $\mu = 0$ are always SBV.

A Proof of Proposition 11

In order to prove Proposition 11 we need the following Theorem on the decomposition of difference quotients of BV functions, which is part of Theorem 2.4 of [6] (we refer to [6] for the proof). In what follows, for $B \in BV_{\rm loc}$, we denote by ∇B the $L^1_{\rm loc}$ (matrix valued) function such that $D^a B = \nabla B\,\mathscr{L}^d$ and by $\operatorname{div} B$ the $L^1_{\rm loc}$ function such that $D^a\cdot B = \operatorname{div} B\,\mathscr{L}^d$.

Theorem 10 (Difference quotients). *Let $B \in BV_{\rm loc}(\mathbf{R}^d, \mathbf{R}^m)$ and let $z \in \mathbf{R}^d$. Then the difference quotients*

$$\frac{B(x+\delta z) - B(x)}{\delta}$$

can be canonically written as $B^1_\delta(z)(x) + B^2_\delta(z)(x)$, where

(a) $B^1_\delta(z)$ converges strongly in $L^1_{\rm loc}$ to $z \cdot \nabla B$ as $\delta \downarrow 0$.
(b) For any compact set $K \subset \mathbf{R}^d$ we have

$$\limsup_{\delta\downarrow 0} \int_K \left|B^2_\delta(z)(x)\right| dx \ \le \ |z||D^s B|(K)\,. \tag{107}$$

(c) For every compact set $K \subset \mathbf{R}^d$ we have

$$\sup_{\delta\in]0,\varepsilon[} \int_K \left|B^1_\delta(z)(x)\right| + \left|B^2_\delta(z)(x)\right| dx \ \le \ |DB|(K_\varepsilon) \tag{108}$$

where $K_\varepsilon := \{x : {\rm dist}\,(x, K) \le \varepsilon\}$.

Proof (of Theorem 11). As in (36) we write

$$T_\delta \ := \ r_\delta \mathscr{L}^d - (w * \rho_\delta) D \cdot B,$$

where

$$\begin{aligned} r_\delta(x) &:= \int_{\mathbf{R}^d} w(y)\big[(B(x) - B(y)) \cdot \nabla\rho_\delta(y-x)\big]\, dy \\ &= -\int_{\mathbf{R}^d} w(x+\delta y) \left[\frac{B(x+\delta y) - B(x)}{\delta} \cdot \nabla\rho(y)\right] dy\,. \end{aligned}$$

Recalling the notation $D^a \cdot B = {\rm div}\, B\, \mathscr{L}^d$, we get

$$|T_\delta| \ := \ \left|r_\delta - (w * \rho_\delta){\rm div}\, B\right| \mathscr{L}^d + |w * \rho_\delta||D^s \cdot B|\,.$$

Next, using Theorem 10 we write r_δ as $r^1_\delta + r^2_\delta$, where

$$\begin{aligned} r^1_\delta(x) &:= \int_{\mathbf{R}^d} w(x+\delta y) B^1_\delta(y)(x) \cdot \nabla\rho(y)\, dx \\ r^2_\delta(x) &:= \int_{\mathbf{R}^d} w(x+\delta y) B^2_\delta(y)(x) \cdot \nabla\rho(y)\, dx \end{aligned}$$

Let σ be the weak* limit of a subsequence of $|T_\delta|$, and fix a nonnegative $\varphi \in C_c(A)$. Then we get

$$\int_{\mathbf{R}^d} \varphi\, d\sigma \le \limsup_{\delta\downarrow 0} \left\{ \int_{\mathbf{R}^d} \varphi(x) \left| r^1_\delta(x) - w * \rho_\delta(x) \operatorname{div} B(x) \right| dx \right.$$
$$+ \int_{\mathbf{R}^d} \varphi(x) \left| r^2_\delta(x) \right| dx$$
$$\left. + \int_{\mathbf{R}^d} \varphi(x) |w * \rho_\delta(x)|\, d|D^s \cdot B|(x) \right\} . \quad (109)$$

We now analyze the behavior of the three integrals above.

First Integral From Theorem 10(a) and (c), and from the strong L^1_{loc} convergence of $w * \rho_\delta$ to w, it follows that

$$\lim_{\delta\downarrow 0} \int_{\mathbf{R}^d} \varphi(x) \left| r^1_\delta(x) - w * \rho_\delta(x) \operatorname{div} B(x) \right| dx$$
$$= \int_{\mathbf{R}^d} \varphi(x) \left| - \int_{\mathbf{R}^d} w(x) \left\langle \nabla B(x)(y), \nabla \rho(y) \right\rangle dy - w(x) \operatorname{div} B(x) \right| dx \quad (110)$$

Integrating by parts we get that

$$\int_{\mathbf{R}^d} w(x) \left\langle \nabla B(x)(y), \nabla \rho(y) \right\rangle dy \ = \ -w(x) \operatorname{div} B(x) \qquad \forall x \in \mathbf{R}^d$$

and therefore (110) vanishes.

Second Integral Let us write $D^s B = M|D^s B|$. Then from Theorem 10(b) and (c), and using the definition of Λ, we conclude that

$$\lim_{\delta\downarrow 0} \int_{\mathbf{R}^d} \varphi(x) \left| r^2_\delta(x) \right| dx$$
$$\le \|w\|_{L^\infty(A)} \int_{\mathbf{R}^d} \varphi(x) \int_{\mathbf{R}^d} |\langle M(x)(y), \nabla\rho(y)\rangle|\ dy\, d|D^s B|(x)$$
$$= \|w\|_{L^\infty(A)} \int_{\mathbf{R}^d} \varphi(x) \Lambda(M(x), \rho)\, d|D^s B|(x) . \quad (111)$$

Third Integral Finally, we have

$$\lim_{\delta\downarrow 0} \int_{\mathbf{R}^d} \varphi(x) |w * \rho_\delta(x)|\, d|D^s \cdot B|(x) \ \le \ \|w\|_{L^\infty(A)} \int_{\mathbf{R}^d} \varphi(x)\, d|D^s \cdot B|(x) . \quad (112)$$

Conclusion From (109), (110), (111), and (112) we get

$$\int_{\mathbf{R}^d} \varphi\, d\sigma \le \|w\|_{L^\infty(A)} \int_{\mathbf{R}^d} \varphi(x) \Lambda(M(x), \rho)\, d|D^s B|(x)$$
$$+ \|w\|_{L^\infty(A)} \int_{\mathbf{R}^d} \varphi(x)\, d|D^s \cdot B|(x) \quad (113)$$

for every nonnegative $\varphi \in C_c(A)$. □

B Proof of Lemma 1

Proof. Note that since the map $\rho \in C_c^\infty(B_1(0)) \mapsto \Lambda(M,\rho)$ is continuous with respect to the strong $W^{1,1}$ topology, it is sufficient to prove that

$$\inf_{\rho\in K} \Lambda(\eta\otimes\xi,\rho) \;=\; |\mathrm{tr}\, M|\,, \tag{114}$$

where K is the set in (24).

If $d=2$ we can fix an orthonormal basis of coordinates z_1, z_2 in such a way that $\xi=(a,b)$ and $\eta=(0,c)$. Consider the rectangle $R_\varepsilon := [-\varepsilon/2,\varepsilon/2]\times[-1/2,1/2]$ and consider the kernel $\rho_\varepsilon := \frac{1}{\varepsilon}\mathbf{1}_{R_\varepsilon}$. Let $\zeta\in K$ and denote by ζ_δ the family of mollifiers generated by ζ. Clearly $\rho_\varepsilon * \zeta_\delta \in K$ for $\varepsilon+\delta$ small enough.

Denote by $\nu=(\nu_1,\nu_2)$ the unit normal to ∂R_ε and recall that

$$\lim_{\delta\downarrow 0}\left|\frac{\partial(\rho_\varepsilon*\zeta_\delta)}{\partial z_i}\right| \to \frac{|\nu_i|}{\varepsilon}\mathscr{H}^1 \llcorner \partial R_\varepsilon\,. \tag{115}$$

in the sense of measures.

If $M=\eta\otimes\xi$ we can compute

$$\begin{aligned}\limsup_{\delta\downarrow 0}\Lambda(M,\rho_\varepsilon*\zeta_\delta) &\le \limsup_{\delta\downarrow 0}\int_{\mathbf{R}^2}\big(|az_1|+|bz_2|\big)|c|\left|\frac{\partial(\rho_\varepsilon*\zeta_\delta)}{\partial z_2}\right| dz_1dz_2\\ &= \frac{2|c|}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon/2}\left(|az_1|+\frac{|b|}{2}\right)dz_1 \;=\; |ac|\frac{\varepsilon}{2}+|bc|\,.\end{aligned}$$

Note that $bc=\mathrm{tr}\,M$. Thus, if we define the convolution kernels $\lambda_{\varepsilon,\delta} := \rho_\varepsilon*\zeta_\delta$ we get:

$$\limsup_{\varepsilon\downarrow 0}\limsup_{\delta\downarrow 0}\Lambda(M,\rho_\varepsilon*\zeta_\delta) \;\le\; |\mathrm{tr}\,M|\,. \tag{116}$$

For $d\ge 2$ we consider a system of coordinates $x_1,x_2,\dots,x_d$ such that $\eta=(a,b,0,\dots,0)$, $\xi=(0,c,0,\dots,0)$ and we define the convolution kernels

$$\lambda_{\varepsilon,\delta}(x) \;:=\; [\rho_\varepsilon*\zeta_\delta](x_1,x_2)\cdot\zeta(x_3)\cdot\ldots\cdot\zeta(x_d)\,.$$

Then (116) holds as well and we conclude that for any d we have

$$\inf_{\rho\in K}\Lambda(\eta\otimes\xi,\rho) \;\le\; |\mathrm{tr}\,M|\,.$$

On the other hand, for every $\rho\in K$ and every $d\times d$ matrix M, we have

$$\begin{aligned}\Lambda(M,\rho) \ge \left|\int_{B_1(0)}\langle M\cdot y,\nabla\rho(y)\rangle\right| &= \left|\sum_{k,j}M_{jk}\int_{B_1(0)} y_j\frac{\partial\rho}{z_k}(y)\right| dy\\ &= \left|-\sum_{k,j}M_{jk}\int_{B_1(0)}\delta_{jk}\rho(y)\,dy\right| \;=\; |\mathrm{tr}\,M|\,.\end{aligned}$$

This concludes the proof. □

Acknowledgments The research of Luigi Ambrosio has been partially supported by a MIUR PRIN 2004 grant. The research of Camillo De Lellis has been partially supported by the Swiss National Foundation. The research of Jan Malý has been supported by the research project MSM 0021620839 of the Czech Ministry MŠMT and by the grant GAČR 201/03/0931.

References

1. G.Alberti: *Rank-one properties for derivatives of functions with bounded variation.* Proc. Roy. Soc. Edinburgh Sect. A, **123** (1993), 239–274.
2. G.Alberti & L.Ambrosio: *A geometrical approach to monotone functions in* $\mathbf{R}^n$. Math. Z., **230** (1999), 259–316.
3. L.Ambrosio & G.Dal Maso: *A general chain rule for distributional derivatives.* Proc. Amer. Math. Soc., **108** (1990), 691–702.
4. L.Ambrosio, A.Coscia & G.Dal Maso: *Fine properties of functions in BD.* Arch. Rat. Mech. Anal., **139** (1997), 201–238.
5. L.Ambrosio, N.Fusco & D.Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, 2000.
6. L.Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Inventiones Mathematicae, **158** (2004), 227–260.
7. L.Ambrosio: *Lecture notes on transport equation and Cauchy problem for BV vector fields and applications.* To be published in the proceedings of the School on Geometric Measure Theory, Luminy, October 2003, available at http://cvgmt.sns.it .
8. L.Ambrosio & C.De Lellis: *Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions.* IMRN, **41** (2003), 2205–2220.
9. L.Ambrosio & C.De Lellis: *A note on admissible solutions of 1d scalar conservation laws and 2d Hamilton–Jacobi equations* Journal of Hyperbolic Differential Equations, **1** (4) (2004), 813–826.
10. L.Ambrosio, F.Bouchut & C.De Lellis: *Well–posedness for a class of hyperbolic systems of conservation laws in several space dimensions.* Comm. Partial Differential Equations, **29** (2004), 1635–1651.
11. L.Ambrosio, G.Crippa & S.Maniglia: *Traces and fine properties of a BD class of vector fields and applications.* Preprint, 2004.
12. G.Anzellotti: *Pairings between measures and bounded functions and compensated compactness.* Ann. Mat. Pura App., **135** (1983), 293–318.
13. G.Anzellotti: *The Euler equation for functionals with linear growth.* Trans. Amer. Mat. Soc., **290** (1985), 483–501.
14. G.Anzellotti: *Traces of bounded vectorfields and the divergence theorem.* Unpublished preprint, 1983.
15. F.Bouchut & F.James: *One dimensional transport equation with discontinuous coefficients.* Nonlinear Analysis, **32** (1998), 891–933.
16. F.Bouchut: *Renormalized solutions to the Vlasov equation with coefficients of bounded variation.* Arch. Rational Mech. Anal., **157** (2001), 75–90.
17. F.Bouchut, F.James & S.Mancini: *Uniqueness and weak stability for multidimensional transport equations with one-sided Lipschitz coefficients.* Preprint, 2004.

18. A.Bressan: *An ill posed Cauchy problem for a hyperbolic system in two space dimensions.* Rend. Sem. Mat. Univ. Padova, **110** (2003), 103–117.
19. I.Capuzzo Dolcetta & B.Perthame: *On some analogy between different approaches to first order PDE's with nonsmooth coefficients.* Adv. Math. Sci Appl., **6** (1996), 689–703.
20. G.-Q.Chen & H.Frid: *Divergence-measure fields and conservation laws.* Arch. Rational Mech. Anal., **147** (1999), 89–118.
21. G.-Q.Chen & H.Frid: *Extended divergence-measure fields and the Euler equation of gas dynamics.* Comm. Math. Phys., **236** (2003), 251–280.
22. F.Colombini & N.Lerner: *Uniqueness of continuous solutions for BV vector fields.* Duke Math. J., **111** (2002), 357–384.
23. F.Colombini & N.Lerner: *Uniqueness of L^∞ solutions for a class of conormal BV vector fields.* Preprint, 2003.
24. C.M.Dafermos: *Hyperbolic conservation laws in continuum physics.* Springer-Verlag, Berlin, 1999.
25. R.J.Di Perna & P.L.Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511–547.
26. L.C.Evans & R.F.Gariepy: *Lecture notes on measure theory and fine properties of functions,* CRC Press, 1992.
27. H.Federer: *Geometric measure theory*, Springer, 1969.
28. B.L.Keyfitz & H.C.Kranzer: *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory.* Arch. Rational Mech. Anal., **72** (1980), 219–241.
29. S.N. Kruzhkov *First order quasilinear equations in several independent variables.* Math USSR–Sbornik, **10** (1970), 217–243.
30. P.L.Lions: *Generalized Solutions of Hamilton-Jacobi Equations*, volume 69 of *Research Notes in Mathematics.* Pitman Advanced Publishing Program, London, 1982.
31. P.L.Lions: *Sur les équations différentielles ordinaires et les équations de transport.* C. R. Acad. Sci. Paris Sér. I, **326** (1998), 833–838.
32. G.Petrova & B.Popov: *Linear transport equation with discontinuous coefficients.* Comm. PDE, **24** (1999), 1849–1873.
33. F.Popaud & M.Rascle: *Measure solutions to the liner multidimensional transport equation with non-smooth coefficients.* Comm. Partial Differential Equations, **22** (1997), 337–358.
34. R.Temam: *Problémes mathématiques en plasticité.* Gauthier-Villars, Paris, 1983.
35. A.Vasseur: *Strong traces for solutions of multidimensional scalar conservation laws.* Arch. Ration. Mech. Anal., **160** (2001), 181–193.
36. A.I.Vol'pert: *Spaces* BV *and quasilinear equations.* Mat. Sb. (N.S.) **73 (115)** (1967), 255–302.