ONSAGER CRITICAL SOLUTIONS OF THE FORCED NAVIER-STOKES EQUATIONS

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Abstract. We answer positively to [BDL22, Question 2.4] by building new examples of solutions to the forced $3d$-Navier-Stokes equations with vanishing viscosity, which exhibit anomalous dissipation and which enjoy uniform bounds in the space $L^3_t C_x^{1/3-\varepsilon}$, for any fixed $\varepsilon > 0$. Our construction combines ideas of [BDL22] and [CCS22].

1. Introduction

The forced Navier–Stokes equations on the 3-dimensional torus $\mathbb{T}^3 \simeq \mathbb{R}^3/\mathbb{Z}^3$ are given by

$$\begin{aligned} \partial_t v_\nu + v_\nu \cdot \nabla v_\nu + \nabla p_\nu &= \nu \Delta v_\nu + F_\nu \\ \operatorname{div} v_\nu &= 0, \end{aligned} \tag{NS}$$

where $v_\nu : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ is the velocity field, $p_\nu : [0,T] \times \mathbb{T}^3 \to \mathbb{R}$ is the pressure, $\nu > 0$ is the viscosity parameter and $F_\nu : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ is a (divergence-free) force that may depend on ν. When $\nu = 0$ the Navier–Stokes equations (NS) reduce to the forced Euler equations

$$\begin{aligned} \partial_t v_0 + v_0 \cdot \nabla v_0 + \nabla p_0 &= F_0 \\ \operatorname{div} v_0 &= 0. \end{aligned} \tag{E}$$

We consider both the Navier–Stokes equations (NS) and the Euler equations (E) with a prescribed initial datum v_{in} which is independent of the viscosity parameter ν, namely

$$v_\nu(0, \cdot) = v_{\text{in}} \,. \tag{1.1}$$

Following [BDL22] we study *smooth* solutions of (NS) (namely u_ν and F_ν are both C^∞), which enjoy uniform in ν bounds for v_ν and F_ν in appropriate function spaces X and Y. The purpose is to understand which spaces X and Y allow for u_ν to display *anomalous dissipation*, more precisely whether

$$\limsup_{\nu \downarrow 0} \, \nu \int_0^T \int_{\mathbb{T}^3} |\nabla v_\nu|^2 \, dx \, dt > 0 \,. \tag{1.2}$$

We require that the space Y rules out anomalous dissipation for solutions of the forced linear Stokes equations under the assumption $\sup_\nu \|F_\nu\|_Y < \infty$, namely (1.2) would not hold if we eliminate the nonlinear advective term $v_\nu \cdot \nabla v_\nu$ from (NS) and we have uniform bounds for the body forces in the space Y. As it is noticed in [BDL22, Section 2] the assumption

$$\sup_\nu \|F_\nu\|_{L^{1+\sigma}([0,1]; C^\sigma(\mathbb{T}^3))} < \infty \tag{1.3}$$

for any positive $\sigma > 0$ is in fact sufficient.

In [BDL22] the first and fourth authors give examples of smooth solutions v_ν to (NS) for which:

(i) (1.3) holds (in fact with the stronger bound $\sup_\nu \|F_\nu\|_{L^\infty_t(C^{1-\varepsilon})} < \infty$ for any given positive ε),
(ii) $\sup_\nu \|v_\nu\|_{L^\infty} < \infty$,
(iii) and (1.2) is satisfied.

In [BDL22, Section 2] the authors ask whether this type of behavior is still possible if the uniform L^∞ bound (ii) is replaced by a uniform bound in some space X which is close to be "Onsager critical". The Onsager criticality refers to the famous remark by Onsager [Ons49] that if $\|v\|_{L^\infty(C^{1/3+\varepsilon})} < \infty$ and u solves (E) with $F = 0$, then such solution u is energy conservative. After a first partial result by Eyink in [Eyi94], the latter was rigorously proved by Constantin, E, and Titi in [CET94]. It is straightforward to check that, using the arguments in [CET94], (1.3) and a uniform bound in $\|v_\nu\|_{L^3(C^{1/3+\varepsilon})}$ is in fact enough to rule out (1.2).

Onsager in [Ons49] stated also that the regularity class $L^\infty_t(C^{1/3}_x)$ should in fact be critical, in particular he conjectured the existence of solutions of (E) with $F = 0$ belonging to slightly lower regularity classes of $L^\infty_t(C^{1/3}_x)$ which do not conserve the kinetic energy. After a decade of work in the area which started with [DLS09, DLS13], the Onsager conjecture was proved by Isett in [Ise18] (cf. also [BDLSJV17]) using "convex intgeration methods".

While Onsager's conjecture was motivated by the zero-th law of Kolmogorov's fully developed turbulence, which roughly speaking states that (1.2) should be a "typical" phenomenon, it seems at the moment very hard to show that at least some of the dissipative solutions of the unforced Euler equations found so far in the literature can actually be approximated by a sequence of regular solutions to the unforced Navier-Stokes. For this reason in [BDL22] the authors suggested to consider the forced versions of both equations. The main result of this paper is to show that indeed (1.2) can be achieved for family of solutions $\{v_\nu\}_\nu$ which enjoy a uniform bound in a space which is just below the Onsager-critical $L^3_t(C^{1/3}_x)$, while the corresponding forces F_ν also enjoy a bound like (1.3) which rules out (1.2) for solutions of the linear Stokes equations.

Theorem A (Anomalous dissipation)**.** *Let $T = 1$. For any $\alpha < 1/3$ there exist $\sigma > 0$, a divergence-free initial datum $v_{in} \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ with $\int_{\mathbb{T}^3} v_{\rm in} = 0$, and a family of forces $\{F_\nu\}_{\nu>0} \subset C^\infty([0,1]\times\mathbb{T}^3;\mathbb{R}^3)$ satisfying (1.3) such that:*

- *for each $\nu > 0$ there is a unique solution to (NS) with $v^\nu(0,\cdot) = v_{\rm in}(\cdot)$ which satisfies*
$$\sup_{\nu\in[0,1]} \left(\|v_\nu\|_{L^3([0,1];C^\alpha(\mathbb{T}^3))} + \|v_\nu\|_{L^\infty([0,1]\times\mathbb{T}^3)}\right) < \infty\,, \tag{1.4}$$
- *(1.2) holds.*

Furthermore, we have that $F_\nu \to F_0$ in $L^{1+\sigma}((0,1);C^\sigma(\mathbb{T}^3))$ and $v_\nu \to v_0$ in $L^2((0,1)\times\mathbb{T}^3)$ as $\nu \to 0$, and in particular (v_0, p_0, F_0) is a solution of (E).

Remark 1.1. In our construction all the dissipation occurs at the time $T = 1$, namely (1.2) fails at any $T < 1$. In a forthcoming paper [DRI] De Rosa and Isett point out that this type of "instantaneous loss of energy" cannot occur at a time $T \in (0,1)$ for solutions belonging to $L^p([0,1];C^{1/3-}(\mathbb{T}^3))$ for any $p > 3$. In another forthcoming paper [JS] the authors exhibit a 4-dimensional example for which the loss of energy is "diffused in time". More precisely they prove the existence of unique solutions $\{v_\nu\}_{\nu>0}$ of the 4d forced Navier–Stokes equations with forces $\{F_\nu\}_{\nu>0}$ such that

- $\sup_\nu(\|u_\nu\|_{L^\infty} + \|F_\nu\|_{L^\infty_t(C^\alpha_x)}) < \infty$ for some $\alpha > 0$;
- for a suitable sequence $\nu_k \downarrow 0$ the dissipation $\mu_k(t) := \nu_k \int_{\mathbb{T}^4} |\nabla v_{\nu_k}(t,x)|^2 dx$ converges weakly* to a measure μ which has non-trivial absolutely continuous part.

Remark 1.2. If we only required that the forces $\{F_\nu\}_{\nu>0}$ were uniformly bounded in $L^1((0,1);L^\infty(\mathbb{T}^3))$, then anomalous dissipation would be already possible for solutions of the forced heat equation. Indeed, for any $\nu \in (0,1)$ such that $\nu^{-1/2} \in \mathbb{N}$ we can consider $\vartheta_\nu : [0,1]\times\mathbb{T}^3 \to \mathbb{R}$ defined as
$$\vartheta_\nu(t,x) = (e^{-4\pi^2 t} - 1)\sin(2\pi\nu^{-1/2}x),$$
and observe that it solves
$$\begin{cases} \partial_t\vartheta_\nu - \nu\Delta\vartheta_\nu = -4\pi^2\sin(2\pi\nu^{-1/2}x) =: F_\nu \\ \vartheta_\nu(0,\cdot) \equiv 0. \end{cases} \tag{1.5}$$

It is straightforward to check that $\nu \int_0^1 \int_{\mathbb{T}^3} |\nabla \vartheta_\nu(t,x)|^2 dx dt \geq 1/4$ for every $\nu \in (0,1)$ as above. The latter example can be easily modified to produce an analogous one for the linear Stokes equations.

Note that the crucial point is in the oscillations introduced by the sequence F_ν. In particular, strong convergence in $L^1_t L^2_x$ of F_ν would actually suffice to show that the unique solutions of (1.5) satisfy $\nu \int_0^1 \int_{\mathbb{T}^3} |\nabla \vartheta_\nu(t,x)|^2 dx dt \to 0$.

The following open question was also raised in [BDL22] and at present the methods of this work do not seem strong enough to address it.

Open Question 1. Can Theorem A be shown for Leray solutions but replacing F_ν with a ν-independent force in the space $L^1((0,2); L^\infty(\mathbb{T}^3))$?

In view of Remark 1.2 even producing one such example with force in $L^1((0,2); L^2(\mathbb{T}^3))$ seems interesting and highly nontrivial.

1.1. **Lack of selection principle and non-uniqueness.** As in [CCS22], a byproduct of our techniques is the lack of a selection principle under vanishing viscosity for bounded solutions of the three dimensional forced Euler equations, if the force converges in the vanishing viscosity limit. We say that a weak solution $v \in L^\infty((0,T); L^2(\mathbb{T}^3))$ of the forced Euler equations (E) is admissible if

$$\int_{\mathbb{T}^3} |v(x,t)|^2 dx \leq \int_{\mathbb{T}^3} |v_{\text{in}}(x)|^2 dx + 2 \int_{\mathbb{T}^3} F(x,t) \cdot v(x,t) dx \tag{1.6}$$

for *a.e.* $t \in (0,T)$.

We will show that the problem of uniqueness and vanishing viscosity selection in the class of admissible solutions for (E) is related to having a solution in the space $L^1_t(W^{1,\infty}_x)$ (this is essentially the threshold for classical "weak-strong" uniqueness results, see e.g. [Wie18, DRIS22]). In particular uniqueness and selection both fail for solutions in $L^1((0,T); C^\alpha(\mathbb{T}^3))$ for any $\alpha < 1$.

Remark 1.3. The nonuniqueness of admissible solutions has been already shown in the class $C^\beta((0,T) \times \mathbb{T}^3)$ for $\beta < 1/3$ for the unforced Euler equations using the convex integration technique, cf. the aforementioned papers [DLS09, DLS13, Ise18, BDLSJV17].

Theorem B (Nonuniqueness and lack of selection I)**.** *Let $T = 2$ and let $\alpha' \in [0,1)$ be given. Then there are:*

- *(a) $\sigma > 0$ and a family of smooth body forces F_ν satisfying (1.3),*
- *(b) a limit F_0 such that $F_\nu \to F_0$ in $L^{1+\sigma}((0,2); C^\sigma(\mathbb{T}^3))$,*
- *(c) a divergence-free initial datum $v_{\text{in}} \in C^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^2} v_{\text{in}} = 0$,*
- *(d) and a family $\{v_\nu\}_{\nu>0}$ of (unique) smooth solutions of (NS) and (1.1)*

such that the following holds:

- *(i) $\sup_{\nu \in [0,1]} \|v_\nu\|_{L^\infty((0,2)\times\mathbb{T}^3)} \leq 1$;*
- *(ii) $\{v_\nu\}_{\nu>0}$ has at least two distinct limit points, as $\nu \to 0$, in the L^∞ weak* topology, which are two distinct bounded admissible solutions v_0^{cs} and v_0^{ds} of (E) and (1.1);*
- *(iii) furthermore, $v_0^{cs} \in L^1((0,2); C^{\alpha'}(\mathbb{T}^3)) \cap L^\infty$ satisfies the following energy balance*

$$\|v_0^{cs}(t,\cdot)\|_{L^2}^2 = \|v_{\text{in}}\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{T}^3} F_0 \cdot v_0^{cs} \qquad \text{for a.e. } t \in (0,2), \tag{1.7}$$

while $v_0^{ds} \in L^\infty$ exhibits the strict dissipation

$$\|v_0^{ds}(t,\cdot)\|_{L^2}^2 < \frac{\|v_{\text{in}}\|_{L^2}^2}{2} + 2\int_0^t \int_{\mathbb{T}^3} F_0 \cdot v_0^{ds} \qquad \text{for any } t \in [1,2). \tag{1.8}$$

If we give up the regularity of the conservative solution v_0^{cs} it is possible to show nonuniqueness and lack of selection for much smoother forces.

Theorem C (Nonuniqueness and lack of selection II). *Let $T = 2$ and let $\alpha' \in [0,1)$ be given. Then there are:*

(a) a family $\{F_\nu\}_{\nu>0}$ of smooth forces and a limiting F_0 such that $F_\nu \to F_0$ in $C^{\alpha'}((0,2)\times\mathbb{T}^3)$,
(b) a divergence-free initial datum $v_{\rm in} \in C^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} v_{\rm in} = 0$,
(c) and a family $\{v_\nu\}_{\nu>0}$ of (unique) smooth solutions of (NS) and (1.1),

such that the following holds:

(i) $\sup_{\nu\in[0,1]} \|v_\nu\|_{L^\infty((0,2)\times\mathbb{T}^3)} \leq 1$;
(ii) $\{v_\nu\}$ has at least two distinct limit points, as $\nu \to 0$, in the L^∞ weak topology, which are two distinct bounded admissible solutions v_0^{cs} and v_0^{ds} of (E) and (1.1);*
(iii) v_0^{cs} satisfies (1.7) while v_0^{ds} satisfies (1.8).

Obviously the following are simple corollaries of the previous theorems.

Corollary 1.4 (Non uniqueness for the forced Euler equations I). *Let $\alpha' \in [0,1)$ be given. There exist $\sigma > 0$, a body force $F_0 \in L^{1+\sigma}((0,2);C^\sigma(\mathbb{T}^3))$ and a divergence-free initial datum $v_{\rm in} \in C^\infty(\mathbb{T}^3)$ such that the 3d forced Euler equations (E)-(1.1) admit at least two distinct admissible bounded solutions. Furthermore, one of which belongs $L^1((0,2);C^{\alpha'}(\mathbb{T}^3))$.*

Corollary 1.5 (Non uniqueness for the forced Euler equations II). *Let $\alpha' \in [0,1)$ be given. There exist a body force $F_0 \in C^{\alpha'}((0,2)\times\mathbb{T}^3))$ and a divergence-free initial datum $v_{\rm in} \in C^\infty(\mathbb{T}^3)$ such that the 3d forced Euler equations (E)-(1.1) admit at least two distinct admissible bounded solutions.*

We remark that, with a totally different method, Vishik in [Vis18a, Vis18b] has produced nonuniqueness examples for the incompressible Euler equations in $\mathbb{R}^2$ in vorticity formulation when the solutions have vorticity in $C([0,T], L^\infty \cap L^p)$ for any fixed $p < \infty$, while the curl of the body force belongs to $L^{1+\sigma}([0,T], L^p)$ (cf. the lecture notes [ABC+21]). In particular, using classical Calderon-Zygmund estimates, one can easily see that the velocities of these solutions belong to $C([0,T], W^{1,p}_{\rm loc})$, while the body forces belong to $L^{1+\sigma}([0,T], W^{1,p}_{\rm loc})$. In fact Vishik's techniques have been successfully transposed to even show nonuniqueness of Leray solutions of the forced Navier-Stokes equations *at a fixed positive viscosity* $\nu > 0$, see [ABC22].

While the nature of the nonuniqueness results in [Vis18a, Vis18b, ABC+21, ABC22] is quite different from the constructions of this paper, they also strongly suggest that all the results of this section are likely to hold for body forces $\{F_\nu\}$ enjoying uniform bounds in $L^1([0,T], W^{1,p})$ and solutions of (NS) enjoying uniform bounds in $L^\infty([0,T], W^{1,p})$. They also suggest that the following question has likely a positive answer.

Open Question 2. Can the lack of selection of Theorems B and C be shown with a ν-independent force $F \in L^1((0,2);L^\infty(\mathbb{T}^3))$ replacing the family $\{F_\nu\}_{\nu>0}$ (and $\{v_\nu\}_{\nu>0}$ a family of Leray solutions of (NS)-(1.1))?

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2. Strategy of the proof

We use the same strategy as in [BDL22, JY20a, JY20b] and consider a $2 + 1/2$-dimensional Navier-Stokes solution, for which the evolution decouples into a forced $2d$-Navier-Stokes system and a scalar advection-diffusion equation. The solution v_ν of the forced $2d$-Navier-Stokes system is a suitable regularization of the two-dimensional velocity field $u : [0,2]\times\mathbb{T}^2 \to \mathbb{R}^2$ constructed in [CCS22, Section 4], which is an alternating shear flow, that is, for every $t \in (0,1)$ we have either $u(t,x_1,x_2) = (W(t,x_2),0)$ or $u(t,x_1,x_2) = (0,W(t,x_1))$.

The third component of the $3d$-Navier-Stokes solution solves an advection-diffusion equation and will exhibit anomalous dissipation.

More precisely, we define the solution of the forced $3d$-Navier-Stokes system and the initial condition as

$$v_\nu = \begin{pmatrix} u_\nu \\ \tilde\vartheta_\nu \end{pmatrix}, \qquad v_{\text{in}} = \begin{pmatrix} 0 \\ \vartheta_{\text{in}} \end{pmatrix},$$

where u_ν is a suitable regularization of u (to be defined in Section 4) and $\tilde\vartheta_\nu$ solves the advection-diffusion equation with velocity field u_ν and initial datum ϑ_{in}, i.e.

$$\begin{cases} \partial_t \tilde\vartheta_\nu + u_\nu \cdot \nabla \tilde\vartheta_\nu = \nu \Delta \tilde\vartheta_\nu, \\ \tilde\vartheta_\nu(0,\cdot) = \vartheta_{\text{in}}(\cdot)\,. \end{cases}$$

Since u_ν is also an alternating shear flow (see Section 4, the nonlinear term $u_\nu \cdot \nabla u_\nu$ vanishes identically and therefore the velocity field v_ν solves the forced $3d$-Navier-Stokes system with force

$$F_\nu = \begin{pmatrix} \partial_t u_\nu - \nu \Delta u_\nu \\ 0 \end{pmatrix}.$$

By suitably setting the parameters in the construction of u, we will verify that

$$v_\nu \in L^3((0,1); C^\alpha(\mathbb{T}^3))\,, \quad F_\nu \in L^{1+\sigma}((0,1); C^\sigma(\mathbb{T}^3)) \qquad \text{uniformly in } \nu, \tag{2.1}$$

for some $\sigma > 0$, where $\alpha < 1/3$ is arbitrary. In order to show that v_ν exhibits anomalous dissipation, hence concluding the proof of Theorem A, we employ [CCS22, Theorem A] to get

$$\limsup_{\nu \downarrow 0} 2\,\nu \int_0^1 \int_{\mathbb{T}^3} |\nabla v_\nu(s,x)|^2 \,dx\,ds \geq \limsup_{\nu \downarrow 0} 2\,\nu \int_0^1 \int_{\mathbb{T}^3} |\nabla \tilde\vartheta_\nu(s,x)|^2 \,dx\,ds > 1/2\,. \tag{2.2}$$

To prove that the vanishing viscosity limit does not select a unique solution in the setting of Theorem B and Theorem C we use the corresponding statement in [CCS22, Theorem B] which proves lack of selection for solutions of the advection-diffusion equations with velocity field u. More precisely, we prove that the first two components of v_ν (namely u_ν) strongly converge in $L^2((0,2) \times \mathbb{T}^3)$ to a unique limit whereas the last component of v_ν (namely $\tilde\vartheta_\nu$) for a suitable choice of a sequence of viscosity parameters $\{\tilde\nu_q\}_{q\in\mathbb{N}}$ exhibits anomalous dissipation (2.2) and for another suitable choice of a sequence of viscosity parameters $\{\nu_q\}_{q\in\mathbb{N}}$ converges strongly in $L^2((0,2)\times\mathbb{T}^3)$ to a conservative solution (i.e. the limit satisfies the energy balance (1.7) with the first two components of the velocity field).

3. Construction and main properties of the $2d$ velocity field

In this section we recall the main properties of the velocity field $u : [0,1] \times \mathbb{T}^2 \to \mathbb{R}^2$ constructed in [CCS22] and of the corresponding solution $\vartheta_\nu : [0,1] \times \mathbb{T}^2 \to \mathbb{R}$ of the advection-diffusion equation with velocity field u. This velocity field will be used as a building block for the construction of solutions to the forced $3d$-Navier-Stokes equations in Theorems A, B and C.

3.1. Choice of the parameters.

Let $\alpha \in (0,1)$ and $\beta \in [0, 1/3)$ such that $\alpha + 2\beta < 1$. We consider parameters $\epsilon, \delta \in (0, 1/4)$ sufficiently small such that

$$1 - \frac{2\beta(1+3\epsilon(1+\delta))(1+\delta)}{1-\delta} - \alpha(1+\epsilon\delta)(1+\delta) - \frac{\delta}{8} > 0\,, \tag{3.1a}$$

$$\frac{3\beta(1+3\epsilon(1+\delta))(1+\delta)}{1-\delta} + \frac{\delta}{8} < 1\,, \tag{3.1b}$$

$$\epsilon \leq \frac{\delta^3}{50}\,. \tag{3.1c}$$

Furthermore we introduce the parameter $\gamma > 0$ as

$$\gamma = \frac{3\beta(1+3\epsilon(1+\delta))(1+\delta)}{1-\delta} + \frac{\delta}{8} < 1\,. \tag{3.2}$$

Given $a_0 \in (0,1)$ such that

$$a_0^{\epsilon\delta^2} + a_0^{\epsilon\delta/8} \leq \frac{1}{20}\,, \tag{3.3}$$

we define

$$a_{q+1} = a_q^{1+\delta}, \qquad \lambda_q = \frac{1}{2a_q}. \tag{3.4}$$

3.2. **Construction of the velocity field.** Let us begin by introducing some notation. For any $f : [0,2]\times\mathbb{T}^2 \to \mathbb{R}^2$ we denote by $\mathrm{supp}_T(f)$ the temporal support of the function f, namely the projection on the time interval $[0,2]$ of the support of f. The precise definition is

$$\mathrm{supp}_T(f) := \overline{\{t\in[0,2] :\ \text{there exists } x\in\mathbb{T}^2 \text{ such that } f(t,x)\neq 0\}}\,.$$

Given $\{T_q\}_{q\in\mathbb{N}\cup\{-1\}}$, a decreasing sequence of non-negative numbers such that $T_{-1}=1$ and $T_q \downarrow 0$ as $q\to\infty$, we define the time intervals

$$\mathcal{I}_q = [1-T_q, 1-T_{q+1}]\,, \qquad \mathcal{J}_q = [1+T_{q+1}, 1+T_q]\,, \quad \text{for any } q\in\mathbb{N}\cup\{-1\}\,.$$

The results below are taken from [CCS22].

Proposition 3.1. *Let α, β, γ, ϵ, δ, and $\{a_q\}_{q\in\mathbb{N}}$ as above. Then there exist a decreasing sequence of times $\{T_q\}_{q\in\mathbb{N}\cup\{-1\}}$ satisfying $T_{-1}=1$ and $T_q\downarrow 0$ as $q\to\infty$, an initial datum $\vartheta_{\mathrm{in}}\in C^\infty(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2}\vartheta_{\mathrm{in}}=0$, and a divergence-free velocity field $u\in C^\infty_{\mathrm{loc}}(((0,2)\setminus\{1\})\times\mathbb{T}^2;\mathbb{R}^2)$, such that the following hold:*

(1) (Reflection and shear flow) For any $t\in(0,2)$, $u(t,\cdot)$ coincides either with an horizontal shear flow, or with a vertical one. Moreover $u(t,x)=-u(2-t,x)$ for any $t\in(1,2)$ and $x\in\mathbb{T}^2$.

(2) (Time intervals) For any $q\in\mathbb{N}$ we have $|T_q-T_{q+1}|\leq 4a_q^{\gamma-\gamma\delta}$, and

$$\mathrm{supp}_T(u)\cap(\mathcal{I}_{-1}\cup\mathcal{J}_{-1})=\emptyset\,, \tag{3.5}$$

$$|\mathrm{supp}_T(u)\cap(\mathcal{I}_q\cup\mathcal{J}_q)|\leq 6a_q^{\gamma}. \tag{3.6}$$

Moreover, $u(t,\cdot)\equiv 0$ for any t in a neighborhood of $1-T_q$ and $1+T_q$.

(3) (Regularity of the velocity field) For any $k\in\mathbb{N}$ and $\ell\in\mathbb{N}$ there exists a constant $C>0$ such that

$$\|\partial_t^\ell\nabla^k u\|_{L^\infty((\mathcal{I}_q\cup\mathcal{J}_q)\times\mathbb{T}^2)}\leq C a_q^{1-\gamma}a_{q+1}^{-k(1+\epsilon\delta)}a_q^{-\ell\gamma}\,, \tag{3.7}$$

for any $q\in\mathbb{N}$.

(4) (Regularity of the solution) For any $\nu>0$ there exists a unique bounded solution $\vartheta_\nu:[0,2]\times\mathbb{T}^2\to\mathbb{R}$ of the advection-diffusion equation

$$\partial_t\vartheta_\nu + u\cdot\nabla\vartheta_\nu = \nu\Delta\vartheta_\nu \tag{3.8}$$

with initial datum ϑ_{in}. For $\nu=0$, the advection equation (i.e., (3.8) with $\nu=0$) with velocity field u and initial datum ϑ_{in} has a unique bounded solution with the symmetry $\vartheta_0(t,x)=\vartheta_0(2-t,x)$ for any $t\in(1,2)$ and $x\in\mathbb{T}^2$. The family of solutions $\{\vartheta_\nu\}_{\nu\in[0,1]}$ satisfies

$$\sup_{\nu\in[0,1]}\|\nabla\vartheta_\nu\|_{L^\infty(\mathcal{I}_q\times\mathbb{T}^2)}\leq\|\nabla\vartheta_{\mathrm{in}}\|_{L^\infty}a_{q+1}^{-1-3\epsilon(1+\delta)}\,, \quad \textit{for any } q\in\mathbb{N}.$$

(5) (Anomalous dissipation) For any $q\in\mathbb{N}$ we set

$$\tilde\nu_q = a_q^{2-\frac{\gamma}{1+\delta}+4\epsilon}\,. \tag{3.9}$$

There exists $m\in\mathbb{N}$ such that the sequence $\{\vartheta_{\tilde\nu_q}\}_{q\in\mathbb{N}}$ satisfies

$$2\,\tilde\nu_q\int_0^{1-T_q+\bar t_q}\int_{\mathbb{T}^2}|\nabla\vartheta_{\tilde\nu_q}|^2\,dx\,dt>\frac{1}{2} \qquad \textit{for any } q\in m\mathbb{N}, \tag{3.10}$$

where $\bar t_q\in(T_{q+1},T_q)$ is a suitable intermediate time such that $\mathrm{supp}_T(u)\cap(1-T_q,1-T_q+\bar t_q)=\emptyset$.

Proof. The velocity field with all the above properties is obtained from the one constructed in [CCS22, Section 4] choosing $p = p^\circ = 1/3$. Properties (1) and (2) are a direct consequence of the construction in [CCS22, Section 4]. Property (3) is given in [CCS22, Remark 4.2]. Property (4) has been proved in [CCS22, Section 8]. Property (5) has been proved in [CCS22, Section 7] and it is stated in [CCS22, Theorem A]. □

4. Solution of the forced 3d-Navier-Stokes and Euler equations

Let α, β, γ, ϵ, δ, and $\{a_q\}_{q\in\mathbb{N}}$ be as in Section 3.1. We employ the velocity field u and the initial condition $\vartheta_{\rm in}$ built in Proposition 3.1 to produce (v_ν, p_ν, F_ν) a smooth solution to the forced 3d-Navier-Stokes equations (NS)-(1.1).

For any $q \in \mathbb{N}$, we introduce the closed set $K_q = [0, 1 - T_q] \cup [1 + T_q, 2]$ and define

$$u_q(t,x) = u(t,x)\mathbb{1}_{K_q}(t) \,. \tag{4.1}$$

We observe that u_q is smooth for any $q \in \mathbb{N}$.

We consider the family of viscosity parameters $\tilde\nu_q$ defined in (3.9). For any $\nu \in (0, a_0^2)$ there exists $q \in \mathbb{N}$ such that $\nu \in (\tilde\nu_{q+1}, \tilde\nu_q]$. Let $\tilde\vartheta_\nu : [0,2] \times \mathbb{T}^3 \to \mathbb{R}$ be the unique smooth solution to the advection-diffusion equation (3.8) with diffusion parameter ν, initial datum $\vartheta_{\rm in}$, and velocity field $u_q(t,x)$, i.e.

$$\partial_t \tilde\vartheta_\nu + u_q \cdot \nabla \tilde\vartheta_\nu = \nu \Delta \tilde\vartheta_\nu \,.$$

We define smooth functions $F_\nu, v_\nu : [0,2] \times \mathbb{T}^3 \to \mathbb{R}^3$ and $p_\nu : [0,2] \times \mathbb{T}^3 \to \mathbb{R}$ as

$$\begin{aligned} F_\nu(t,x) &= \begin{pmatrix} \partial_t u_q(t,x) - \nu \Delta u_q(t,x) \\ 0 \end{pmatrix} \\ v_\nu(t,x) &= \begin{pmatrix} u_q(t,x) \\ \tilde\vartheta_\nu(t,x) \end{pmatrix} \\ p_\nu &= 0 \,. \end{aligned}$$

Finally, we set

$$v_{\rm in} = \begin{pmatrix} 0 \\ \vartheta_{\rm in} \end{pmatrix} \,. \tag{4.2}$$

Given Proposition 3.1, the following lemma is immediately checked.

Lemma 4.1. *For any $\nu \in (0, a_0^2)$, given F_ν as above, (v_ν, p_ν) is the unique smooth solution to (NS) with initial datum $v_{\rm in}$. Moreover, at any time $t \in (0,2)$ the velocity field v_ν is an alternating shear flow on the first two components, i.e.*

$$v_\nu(t,x) = \begin{pmatrix} w_\nu^1(t,x_2) \\ 0 \\ * \end{pmatrix} \qquad \text{or} \qquad v_\nu(t,x) = \begin{pmatrix} 0 \\ w_\nu^2(t,x_1) \\ * \end{pmatrix}$$

for suitable one-dimensional functions $w_\nu^i : [0,2] \times \mathbb{T} \to \mathbb{R}$, for $i = 1, 2$.

At least formally, we expect (v_ν, p_ν, F_ν) to converge to a solution of the forced 3d-Euler equations (E)-(1.1) when $\nu \downarrow 0$. We will prove in the next sections that this is the case under suitable assumptions and that

$$F_0(t,x) = \begin{pmatrix} \partial_t u(t,x) \\ 0 \end{pmatrix} \tag{4.3}$$

$$p_0 = 0 \,. \tag{4.4}$$

The following lemma immediately follows from the regularity of u in $(0,1) \times \mathbb{T}^2$ in Proposition 3.1.

Lemma 4.2. *Let u, ϑ_0 be as in Proposition 3.1, and let F_0 be as in (4.3). We have that*

$$F_0 \in C^\infty((0,1)\times\mathbb{T}^3)\,. \tag{4.5}$$

Moreover,

$$v_0(t,x) := \begin{pmatrix} u(t,x) \\ \vartheta_0(t,x) \end{pmatrix}\,, \quad t\in(0,1),\, x\in\mathbb{T}^3\,, \tag{4.6}$$

is the unique smooth solution to (E)-(1.1) in $(0,1)\times\mathbb{T}^3$ with initial datum (4.2).

Remark 4.3. We will see in the next sections that uniqueness for (E)-(1.1) may fail past time $t=1$, where the singularity of F_0 appears.

5. Proof of Theorem A and Theorem B

Let $\alpha\in[0,1/3)$ be fixed as in Theorem A and $\alpha'\in[0,1)$ be fixed as in Theorem B. Without loss of generality and up to increasing α or α', we can assume $\alpha'=3\alpha$. We fix $\beta=\alpha$ and choose the parameters ϵ, δ, and $\{a_q\}_{q\in\mathbb{N}}$ as in Section 3.1. The parameter γ is then determined by (3.2). The viscosity parameter $\tilde\nu_q$ has been chosen in (3.9). Let (v_ν,p_ν,F_ν) be the solution to (NS), with initial datum as in (4.2), built in Section 4.

In order to prove Theorem A and Theorem B we need to show the following facts:

(i) There exists $\sigma>0$ such that

$$\sup_{\nu\in(0,a_0^2)} \|v_\nu\|_{L^3([0,1];C^\alpha(\mathbb{T}^3))} + \|v_\nu\|_{L^\infty([0,2]\times\mathbb{T}^3)} + \|F_\nu\|_{L^{1+\sigma}([0,2];C^\sigma(\mathbb{T}^3))} < \infty\,. \tag{5.1}$$

Moreover, $F_\nu\to F_0$ in $L^{1+\sigma}([0,2];C^\sigma(\mathbb{T}^3))$.

(ii) Let v_0 be as in Lemma 4.2. We have that $v_\nu\to v_0$ in $L^2((0,1)\times\mathbb{T}^3)$ as $\nu\to0$.

(iii) There exist $v_0^{\rm ds}\in L^\infty([0,2]\times\mathbb{T}^3)$ solution to (E) with initial datum (4.2) and a sequence $q_k\to\infty$, such that $v_{\tilde\nu_{q_k}}\to v_0^{\rm ds}$ weakly in $L^2([0,2]\times\mathbb{T}^3)$. Moreover,

$$2\,\tilde\nu_{q_k}\int_0^1\int_{\mathbb{T}^3}|\nabla v_{\tilde\nu_{q_k}}|^2\,dx\,dt > 1/2 \qquad \text{for any } k\in\mathbb{N}. \tag{5.2}$$

In particular $v_0^{\rm ds}$ is an admissible dissipative solution of (E).

(iv) Set $\nu_q=a_q^{2-\gamma+\delta+8\epsilon}$. There exists $v_0^{\rm cs}\in L^1((0,2);C^{\alpha'}(\mathbb{T}^3))$, a conservative (admissible) solution to (E) with initial datum (4.2), such that $v_{\nu_q}\to v_0^{\rm cs}$ strongly in $L^2((0,2)\times\mathbb{T}^3)$ as $q\to\infty$.

Proof of (i). From (3.7) and the maximum principle for the advection-diffusion equation (using that the initial datum is bounded), we deduce that

$$\sup_{\nu\in(0,a_0^2)}\|v_\nu\|_{L^\infty([0,2]\times\mathbb{T}^3)}<\infty\,.$$

Let us now check that

$$\sup_{\nu\in(0,a_0^2)}\|v_\nu\|_{L^3([0,1];C^\alpha(\mathbb{T}^3))}<\infty\,. \tag{5.3}$$

This is a consequence of

$$u\in L^3((0,1);C^\alpha(\mathbb{T}^3)) \qquad \text{and} \qquad \sup_{\nu\in(0,a_0^2)}\|\vartheta_\nu\|_{L^3((0,1);C^\alpha(\mathbb{T}^2))}<\infty \tag{5.4}$$

that we now prove. Indeed, $\tilde\vartheta_\nu\equiv\vartheta_\nu$ in $[0,1-T_q]\times\mathbb{T}^3$ since $u_q\equiv u$ in $[0,1-T_q]\times\mathbb{T}^3$, while $\tilde\vartheta_\nu(t,\cdot)$ solves the heat equation for $t\in[1-T_q,1]$, and the Hölder norm is nonincreasing for solutions of the heat equation.

Let us begin by proving the first property in (5.4). By (3.5) and (3.7) and by interpolation we have

$$\|u\|^3_{L^3((0,1);C^\alpha)} = \sum_{q=0}^\infty\int_{\mathcal{I}_q}\|u(s,\cdot)\|^3_{C^\alpha(\mathbb{T}^2)}ds \le \sum_{q=0}^\infty\int_{\mathcal{I}_q}\|u(s,\cdot)\|^{3(1-\alpha)}_{L^\infty(\mathbb{T}^2)}\|u(s,\cdot)\|^{3\alpha}_{W^{1,\infty}(\mathbb{T}^2)}ds$$

$$\leq C \sum_{q=0}^{\infty} a_q^{\gamma} a_q^{-3(1-\alpha)(\gamma-1)} a_q^{-3\alpha(\gamma-1)} a_{q+1}^{-3\alpha(1+\epsilon\delta)}$$

and the sum is finite if and only if

$$\frac{\gamma}{3} + 1 - \gamma - \alpha(1+\epsilon\delta)(1+\delta) > 0\,,$$

which holds thanks to the choice (3.2) and the condition (3.1a).

Let us show the second property in (5.4). Fix $\nu \in (0, a_0^2)$ and correspondingly let $q \in \mathbb{N}$ such that $\nu \in (\tilde{\nu}_{q+1}, \tilde{\nu}_q]$. Thanks to property (4) of Proposition 3.1 and using $\alpha = \beta$, we get

$$\begin{aligned}\|\vartheta_\nu\|^3_{L^3((0,1);C^\alpha)} &= \sum_{q=0}^{\infty} \int_{\mathcal{I}_q} \|\vartheta_\nu(s,\cdot)\|^3_{C^\beta} ds \leq C \sum_{q=0}^{\infty} a_q^{\gamma-\gamma\delta} a_{q+1}^{-3(\beta+3\beta\epsilon(1+\delta))} \\ &= C \sum_{q=0}^{\infty} a_q^{\gamma-\gamma\delta-3(\beta+3\beta\epsilon(1+\delta))(1+\delta)}\end{aligned}$$

and the sum is finite and independent of ν since as a consequence of (3.2) we have

$$-\gamma(1-\delta) + 3(\beta + 3\beta\epsilon(1+\delta))(1+\delta) < 0\,.$$

We finally prove that

$$F_\nu \in L^{1+\sigma}((0,2); C^\sigma(\mathbb{T}^3))\,, \quad \text{uniformly in } \nu \in (0, a_0^2)\,, \tag{5.5}$$

for some $\sigma > 0$, and $F_\nu \to F_0$ in $L^{1+\sigma}((0,2); C^\sigma(\mathbb{T}^3))$ as $\nu \to 0$. To this aim, it is enough to show that there exists $C > 0$ such that for any $\nu \in (\tilde{\nu}_{q+1}, \tilde{\nu}_q]$ we have

$$\|\partial_t u\|_{L^{1+\sigma}((0,2);C^\sigma(\mathbb{T}^3))} \leq C \qquad \text{and} \qquad \|\nu\Delta u\|_{L^{1+\sigma}((K_q;C^\sigma(\mathbb{T}^3))} \leq C a_q^{\epsilon}\,. \tag{5.6}$$

and that

$$\|\partial_t u\|_{L^{1+\sigma}(K_q^c;C^\sigma(\mathbb{T}^3))} \to 0 \qquad \text{as } q \to \infty. \tag{5.7}$$

For the first property in (5.6), thanks to (3.7) we have

$$\begin{aligned}\|\partial_t u\|^{1+\sigma}_{L^{1+\sigma}((0,2);C^\sigma(\mathbb{T}^3))} &\leq \sum_{j=0}^{\infty} \int_{\mathcal{I}_j \cup \mathcal{J}_j} \left(\|\partial_t u(s,\cdot)\|^{(1-\sigma)}_{L^\infty(\mathbb{T}^3)} \|\partial_t u(s,\cdot)\|^{\sigma}_{W^{1,\infty}(\mathbb{T}^3)} \right)^{1+\sigma} ds \\ &\leq C \sum_{j=0}^{\infty} \int_{\mathcal{I}_j \cup \mathcal{J}_j} \left(a_j^{1-2\gamma} a_{j+1}^{-\sigma(1+\epsilon\delta)} \right)^{1+\sigma} ds \\ &\leq 4C \sum_{j=0}^{\infty} a_j^{\gamma} a_j^{(1+\sigma)(1-2\gamma-\sigma(1+\delta)(1+\epsilon\delta))} < \infty\end{aligned}$$

where we used that $1-\gamma > 0$, and we choose $\sigma > 0$ sufficiently small such that $\gamma + (1+\sigma)(1-2\gamma-\sigma(1+\delta)(1+\epsilon\delta)) > 0$. Property (5.7) follows by noticing that

$$\|\partial_t u\|^{1+\sigma}_{L^{1+\sigma}(K_q^c;C^\sigma(\mathbb{T}^3))} \leq \sum_{j=q}^{\infty} \int_{\mathcal{I}_j \cup \mathcal{J}_j} \left(\|\partial_t u(s,\cdot)\|^{(1-\sigma)}_{L^\infty(\mathbb{T}^3)} \|\partial_t u(s,\cdot)\|^{\sigma}_{C^1(\mathbb{T}^3)} \right)^{1+\sigma} ds \quad \to \quad 0 \qquad \text{as } q \to \infty.$$

For the second property in (5.6), thanks to (3.7), we have

$$\begin{aligned}\|\nu\Delta u\|^{1+\sigma}_{L^{1+\sigma}(K_q;C^\sigma(\mathbb{T}^3))} &\leq \tilde{\nu}_q \sum_{j=0}^{q-1} \int_{\mathcal{I}_j \cup \mathcal{J}_j} \left(\|\Delta u(s,\cdot)\|^{1-\sigma}_{L^\infty(\mathbb{T}^3)} \|\Delta u(s,\cdot)\|^{\sigma}_{W^{1,\infty}(\mathbb{T}^3)} \right)^{1+\sigma} ds \\ &\leq C a_q^{2-\frac{\gamma}{1+\delta}+4\epsilon} \sum_{j=0}^{q-1} \int_{\mathcal{I}_j \cup \mathcal{J}_j} \left(a_j^{1-\gamma} a_{j+1}^{-2(1+\epsilon\delta)(1-\sigma)} a_{j+1}^{-3(1+\epsilon\delta)\sigma} \right)^{1+\sigma} ds\end{aligned}$$

$$\le C a_q^{2-\frac{\gamma}{1+\delta}+2\epsilon} \sum_{j=0}^{q-1} a_j^{\gamma} \left(a_j^{-1-2\delta-\gamma} a_j^{-\sigma(1+\epsilon\delta)(1+\delta)} \right)^{1+\sigma}$$
$$\le C q a_q^{2\epsilon} a_{q-1}^{2+2\delta-\gamma} a_{q-1}^{\gamma} a_{q-1}^{-(1+\sigma)(1+2\delta+\gamma+\sigma(1+\epsilon\delta)(1+\delta))} \le C a_q^{\epsilon},$$

where we used that $q a_q^{\epsilon} \le 1$, $a_{j+1} = a_j^{1+\delta}$, $1-\gamma > 0$ and we choose $\sigma > 0$ sufficiently small to guarantee that $2 + 2\delta - (1+\sigma)(1+2\delta+\gamma+\sigma(1+\epsilon\delta)(1+\delta)) > 0$.

Proof of (ii). Recalling (4.1), it suffices to prove that $\tilde{\vartheta}_\nu \to \vartheta_0$ in $L^2((0,1)\times\mathbb{T}^3)$, as $\nu \to 0$. Fix $\nu \in (0, a_0^2)$, and let $q \in \mathbb{N}$ such that $\nu \in (\tilde{\nu}_{q+1}, \tilde{\nu}_q]$. We employ a standard vanishing viscosity estimate. For any $0 \le t \le 1 - T_q$, we have that $u_q = u$, hence

$$\partial_t(\tilde{\vartheta}_\nu - \vartheta_0) + u \cdot \nabla(\tilde{\vartheta}_\nu - \vartheta_0) = \nu \Delta \tilde{\vartheta}_\nu \qquad \text{for any } 0 \le t \le 1 - T_q .$$

We multiply the above equation by $\tilde{\vartheta}_\nu - \vartheta_0$ and integrate in space-time to get

$$\|\tilde{\vartheta}_\nu(t,\cdot) - \vartheta_0(t,\cdot)\|_{L^2(\mathbb{T}^3)}^2 \le \nu \left| \int_0^t \int_{\mathbb{T}^3} \nabla\tilde{\vartheta}_\nu(s,x) \cdot \nabla\vartheta_0(s,x)\, dx ds \right|$$
$$\le \left(\nu \int_0^t \int_{\mathbb{T}^3} \left|\nabla\tilde{\vartheta}_\nu(s,x)\right|^2 dx ds \right)^{1/2} \left(\nu \int_0^t \int_{\mathbb{T}^3} |\nabla\vartheta_0(s,x)|^2 dx ds \right)^{1/2} .$$

We observe that by the energy equality

$$\nu \int_0^t \int_{\mathbb{T}^3} \left|\nabla\tilde{\vartheta}_\nu(s,x)\right|^2 dx ds \le 1 \qquad \text{for any } t \in [0,1] . \tag{5.8}$$

Let us define $t(\nu) := 1 - T_{k(q)} \le 1 - T_q$, where $k(q)$ is the largest natural number satisfying

$$a_q^{2-\frac{\gamma}{1+\delta}} \exp\left(a_{k(q)}^{2-2\gamma} a_{k(q)+1}^{-2-2\epsilon\delta} \right) \le 1 .$$

We claim that

$$\nu \int_0^{t(\nu)} \int_{\mathbb{T}^3} |\nabla\vartheta_0(s,x)|^2 dx ds \to 0 \qquad \text{and} \qquad t(\nu) \to 1 \qquad \text{as } \nu \to 0 . \tag{5.9}$$

This follows by Grönwall inequality and (3.7), since

$$\nu \int_0^{t(\nu)} \|\nabla\vartheta_0(s,\cdot)\|_{L^\infty(\mathbb{T}^3)}^2 ds \le \nu \|\nabla\vartheta_{\rm in}(\cdot)\|_{L^\infty(\mathbb{T}^3)}^2 \exp\left(\int_0^{t(\nu)} \|\nabla u(s,\cdot)\|_{L^\infty}^2 \right) ds$$
$$\le C a_q^{2-\frac{\gamma}{1+\delta}+4\epsilon} a_0^{-2-2\epsilon\delta} \exp\left(a_{k(q)}^{2-2\gamma} a_{k(q)+1}^{-2-2\epsilon\delta} \right)$$
$$\le C a_q^{4\epsilon} a_0^{-2-2\epsilon\delta} \to 0$$

as $q \to \infty$. Finally, $t(\nu) \to 1$ as $\nu \to 0$ follows by the fact that $k(q) \to \infty$ as $q \to \infty$.

Building upon (5.8), (5.9), and the fact that $\tilde{\vartheta}_\nu$ and ϑ_0 are uniformly bounded by 1, we deduce

$$\|\tilde{\vartheta}_\nu - \vartheta_0\|_{L^2((0,1)\times\mathbb{T}^3)}^2 \le \nu \int_0^{t(\nu)} \|\nabla\vartheta_0(s,\cdot)\|_{L^\infty(\mathbb{T}^3)}^2 ds + C(1 - t(\nu)) \to 0 \tag{5.10}$$

as $q \to \infty$.

Proof of (iii). We observe that the sequence of solutions $\vartheta_{\tilde{\nu}_q}$ of the advection-diffusion equation with diffusion parameter $\tilde{\nu}_q$, velocity field u, and initial datum $\vartheta_{\rm in}$ satisfies

$$2\,\tilde{\nu}_q \int_0^{1-T_q+\bar{t}_q} \|\nabla\vartheta_{\tilde{\nu}_q}(s,\cdot)\|_{L^2}^2\, ds > \frac{1}{2} \qquad \text{for any } q \in m\mathbb{N}, \tag{5.11}$$

as a direct consequence of (5) in Proposition 3.1. Therefore $\tilde{\vartheta}_{\tilde{\nu}_q}$, the third component of $v_{\tilde{\nu}_q}$, satisfies (5.11) as well since $\vartheta_{\tilde{\nu}_q} = \tilde{\vartheta}_{\tilde{\nu}_q}$ in $[0, 1 - T_q + \bar{t}_q]$.

The first two components of v_ν strongly converge to u in $L^\infty((0,2)\times\mathbb{T}^3)$ since $\|u\|_{L^\infty(K_q^c\times\mathbb{T}^3)} \le 2a_q^{1-\gamma} \to 0$ as $q\to\infty$. It is simple to see that $\{\tilde\vartheta_{\tilde\nu_q}\}_{q\in\mathbb{N}}$ admits limit points in the weak topology of $L^2((0,2)\times\mathbb{T}^2)$ and that any such limit point solves the transport equation with velocity field u and initial datum $\vartheta_{\rm in}$. Let us fix a limit point and denote it by $\vartheta^{\rm ds}$. It follows by (5.11) that $\vartheta^{\rm ds}$ is a dissipative solution of the transport equation. We define

$$v_0^{\rm ds}(t,x) := \begin{pmatrix} u(t,x) \\ \vartheta^{\rm ds}(t,x)\end{pmatrix}, \quad t\in(0,2),\ x\in\mathbb{T}^3\,, \tag{5.12}$$

and check that $(v_0^{\rm ds}, p_0, F_0)$ with $p_0=0$ solves (E). Indeed, since the first two components of $v_{\tilde\nu_q}$ strongly converge to u in $L^\infty((0,2)\times\mathbb{T}^3)$ and the last component converges weakly to $\vartheta^{\rm ds}$, the quadratic term $v_{\tilde\nu_q}\cdot\nabla v_{\tilde\nu_q}$ converges in the sense of distributions to $v_0^{\rm ds}\cdot\nabla v_0^{\rm ds}$. It is straightforward to check that all the other terms in the distributional formulation of (E) pass to the limit as $\tilde\nu_q\to 0$. Finally the admissibility condition (1.6) of $v_0^{\rm ds}$ follows from the fact that it is a weak* limit in L^∞ of admissible solutions v_ν with force F_ν and the forces F_ν are strongly converging to F_0 in L^1.

Proof of (iv). Let $\nu_q = a_q^{2-\gamma+\delta+8\epsilon}\in(\tilde\nu_{q+1},\tilde\nu_q]$. As before we have

$$v_{\nu_q}(t,x) = \begin{pmatrix} u(t,x)\mathbb{1}_{K_q}(t) \\ \tilde\vartheta_{\nu_q}(t,x)\end{pmatrix}.$$

Recalling the proof of (iii), we only need to prove that the last component of v_{ν_q} strongly converges in $L^2((0,2)\times\mathbb{T}^3)$ to a velocity field $v_0^{\rm cs}\in L^1((0,2);C^{\alpha'}(\mathbb{T}^3))$ that conserves in time the spatial L^2 norm and the admissibility condition (1.6) will directly follow from the conservative property.

We show that $\tilde\vartheta_{\nu_q}\to\vartheta_0$ in $L^2((0,2)\times\mathbb{T}^3)$, where ϑ_0 is the symmetric solution to the transport equation in Proposition 3.1(4). To this aim, it is enough to show that $\|\tilde\vartheta_{\nu_q}(t,\cdot)-\vartheta_q(t,\cdot)\|_{L^2(\mathbb{T}^2)}\to 0$ as $q\to\infty$ for any $t\in(0,2)$, where ϑ_q is the unique solution of the transport equation with velocity field u_q and initial datum $\vartheta_{\rm in}$. Indeed, this will entail

$$\|\tilde\vartheta_{\nu_q}-\vartheta_0\|_{L^2([0,2]\times\mathbb{T}^2)} \le \|\tilde\vartheta_{\nu_q}-\vartheta_q\|_{L^2([0,2]\times\mathbb{T}^2)} + \|\vartheta_q-\vartheta_0\|_{L^2([0,2]\times\mathbb{T}^2)} \to 0 \qquad \text{as } q\to\infty\,,$$

where the second term $\|\vartheta_q-\vartheta_0\|_{L^2((0,2)\times\mathbb{T}^3)}\to 0$ as $q\to\infty$, thanks to $\vartheta_q(t,\cdot)=\vartheta_0(t,\cdot)$ for any $t\in[1-T_q,1+T_q]^c$, and the L^∞ bound $\|\vartheta_0\|_{L^\infty((0,2)\times\mathbb{T}^2)}+\|\vartheta_q\|_{L^\infty((0,2)\times\mathbb{T}^2)}\le 2$.

For any $t\in(0,2)$, using a standard energy estimate with the regularity bound (4) and the symmetry property (1) from Proposition 3.1, we estimate

$$\begin{aligned}
&\|\tilde\vartheta_{\nu_q}(t,\cdot)-\vartheta_q(t,\cdot)\|^2_{L^2(\mathbb{T}^2)}\\
\le& 2\nu_q\left|\int_0^t\int_{\mathbb{T}^2}\nabla\tilde\vartheta_{\nu_q}(s,x)\cdot\nabla\vartheta_q(s,x)dxds\right|\\
\le& 2\left(\nu_q\int_0^t\int_{\mathbb{T}^2}|\nabla\vartheta_q(s,x)|^2dxds\right)^{1/2}\\
=& 2\left(\nu_q\sum_{j=q}^\infty\int_{\mathcal{I}_j\cup\mathcal{J}_j}\int_{\mathbb{T}^2}|\nabla\vartheta_q(s,x)|^2dxds+\nu_q\sum_{j=0}^{q-1}\int_{\mathcal{I}_j\cup\mathcal{J}_j}\int_{\mathbb{T}^2}|\nabla\vartheta_q(s,x)|^2dxds\right)^{1/2}\\
\le& C\left(a_q^{2-\gamma+\delta+8\epsilon}a_q^\gamma a_q^{-2(1+3\epsilon(1+\delta))}+a_q^{2-\gamma+\delta+8\epsilon}\sum_{j=0}^{q-1}a_j^\gamma a_{j+1}^{-2(1+3\epsilon(1+\delta))}\right)^{1/2}\\
&\le Ca_q^{1-\frac{\gamma}{2}+\frac{\delta}{2}+4\epsilon}a_{q-1}^{\gamma/2}a_q^{-1-3\epsilon(1+\delta)}\le Ca_q^{\frac{\delta}{2}-\frac{\delta\gamma}{2}}\to 0\,,
\end{aligned}$$

as $q\to\infty$, where we used $\gamma<1$, $a_q=a_{q-1}^{1+\delta}$, and $\delta\in(0,1/8)$.

We finally show that $v_0^{\rm cs} \in L^1((0,2); C^{\alpha'}(\mathbb{T}^3))$. Using (3.5) and (3.7) we deduce

$$\|u\|_{L^1((0,2);C^{\alpha'})} = 2\sum_{q=1}^{\infty}\int_{\mathcal{I}_q} \|u(s,\cdot)\|_{C^{\alpha'}(\mathbb{T}^2)} ds \le 2\sum_{q=1}^{\infty}\int_{\mathcal{I}_q} \|u(s,\cdot)\|_{L^\infty(\mathbb{T}^2)}^{1-\alpha'} \|u(s,\cdot)\|_{W^{1,\infty}(\mathbb{T}^2)}^{\alpha'} ds$$
$$\le C\sum_{q=0}^{\infty} a_q^{\gamma} a_q^{1-\gamma} a_{q+1}^{-\alpha'(1+\epsilon\delta)} = C\sum_{q=0}^{\infty} a_q^{1-\alpha'(1+\epsilon\delta)(1+\delta)} < \infty$$

recalling that $\alpha' = 3\alpha$ and $\alpha = \beta$, and the last inequality holds thanks to the condition (3.1a), which implies $1 - 3\alpha(1+\epsilon\delta)(1+\delta) > 0$. For the last component of $v_0^{\rm CS}$, namely ϑ_0, we recall that

$$\vartheta_0(t,x) = \vartheta_0(2-t,x) \qquad \text{for any } x \in \mathbb{T}^3 \text{ and } t \in (1,2]\,,$$

and that it solves the transport equation (namely (3.8) with $\nu = 0$) with velocity field u. Therefore, it is sufficient to estimate ϑ_0 in $[0,1]\times\mathbb{T}^2$. Using (4) in Proposition 3.1 we have

$$\|\vartheta_0\|_{L^1((0,1);C^{\alpha'})} = \sum_{q=0}^{\infty}\int_{\mathcal{I}_q} \|\vartheta_0(s,\cdot)\|_{C^{\alpha'}} ds \le 4\|\nabla\vartheta_{\rm in}\|_{L^\infty} \sum_{q=0}^{\infty} a_q^{\gamma-\gamma\delta} a_{q+1}^{-\alpha'(1+3\epsilon(1+\delta))}$$
$$= 4\|\nabla\vartheta_{\rm in}\|_{L^\infty} \sum_{q=0}^{\infty} a_q^{\gamma-\gamma\delta-\alpha'(1+3\alpha'\epsilon(1+\delta))(1+\delta)} < \infty$$

where the last estimate holds thanks to $\alpha = \alpha'/3$, (3.1a), (3.2), and $\epsilon < \frac{\delta}{16(1+\delta)^2}$ (a consequence of (3.1c)).

6. Proof of Theorem C

Let $\alpha' \in [0,1)$ be as in Theorem C. We fix $\alpha = \alpha'$ and $\beta = 0$ and we choose the parameters δ, ϵ, γ, and $\{a_q\}_{q\in\mathbb{N}}$ as in Section 3.1. The parameters satisfy (3.1), (3.2), (3.3), and the further condition

$$1 - \alpha'(1+\epsilon\delta)(1+\delta) - \frac{\delta}{4} > 0 \tag{6.1}$$

which is compatible with all the other conditions. Let (v_ν, p_ν, F_ν) be the solution to (NS), with initial datum as in (4.2), built in Section 4.

In order to prove Theorem C we need to show the following facts:

(i) There holds
$$\sup_{\nu\in(0,a_0^2)} \|v_\nu\|_{L^\infty([0,2]\times\mathbb{T}^3)} + \|F_\nu\|_{C^{\alpha'}((0,2)\times\mathbb{T}^3)} < \infty\,.$$
Moreover, $F_\nu \to F_0$ in $C^{\alpha'}((0,2)\times\mathbb{T}^3)$.

(ii) There exist $v_0^{\rm ds} \in L^\infty([0,2]\times\mathbb{T}^3)$ solution to (E) with initial datum (4.2) and a sequence $q_k \to \infty$, such that $v_{\tilde\nu_{q_k}} \rightharpoonup v_0^{\rm ds}$ weakly in $L^2([0,2]\times\mathbb{T}^3)$. Moreover,
$$2\,\tilde\nu_{q_k}\int_0^1\int_{\mathbb{T}^3} |\nabla v_{\tilde\nu_{q_k}}|^2\,dx\,dt \ge 1/2 \qquad \text{for any } q_k. \tag{6.2}$$
In particular $v_0^{\rm ds}$ is an admissible dissipative solution of (E).

(iii) Set $\nu_q = a_q^{2+3\epsilon}$. There exists $v_0^{\rm cs} \in L^\infty((0,2)\times\mathbb{T}^3)$, an (admissible) conservative solution to (E) with initiald datum (4.2), such that $v_{\nu_q} \to v_0^{\rm cs}$ in $L^2((0,2)\times\mathbb{T}^3)$.

Proof of (i). Since u is bounded, more precisely $\|u(t,\cdot)\|_{L^\infty((0,2)\times\mathbb{T}^3)} \le 2a_0^{1-\gamma} \le 1$ and $\|\tilde\vartheta_\nu\|_{L^\infty((0,2)\times\mathbb{T}^3)} \le \|\vartheta_{\rm in}\|_{L^\infty(\mathbb{T}^3)} = 1$ by the maximum principle, we have

$$\|v_\nu\|_{L^\infty((0,2)\times\mathbb{T}^3)} \le 1\,.$$

Let us now show the uniform-in-viscosity regularity of F_ν. If suffices to prove that there exists $C>0$ such that for any $\nu \in (\tilde\nu_{q+1}, \tilde\nu_q]$ we have

$$\|\partial_t u\|_{C^{\alpha'}((0,2)\times\mathbb{T}^3)} \le C \qquad \text{and} \qquad \|\nu\Delta u\|_{C^{\alpha}(K_q\times\mathbb{T}^3)} \le C a_q^{\epsilon}\,. \tag{6.3}$$

We estimate the first term. Thanks to (3.7) and the interpolation inequality, we have

$$\begin{aligned}\|\partial_t u\|_{C^{\alpha'}((0,2)\times\mathbb{T}^3)} &\le \sup_{j\in\mathbb{N}} \|\partial_t u\|_{L^\infty(\mathcal{I}_j;C^{\alpha'}(\mathbb{T}^3))} + \sup_{j\in\mathbb{N}} \|\partial_t u\|_{L^\infty(\mathbb{T}^3;W^{1,\infty}(\mathcal{I}_j))}\\ &\le C\left(\sup_{j\in\mathbb{N}} \|\partial_t u\|^{1-\alpha'}_{L^\infty(\mathcal{I}_j;L^\infty(\mathbb{T}^3))}\|\partial_t u\|^{\alpha'}_{L^\infty(\mathcal{I}_j;W^{1,\infty}(\mathbb{T}^3))} + \sup_{j\in\mathbb{N}} a_j^{1-2\gamma}\right)\\ &\le C \sup_{j\in\mathbb{N}} a_j^{1-2\gamma} a_{j+1}^{-\alpha'(1+\epsilon\delta)} + 1 < \infty\end{aligned}$$

where we used (6.1) and $\gamma = \delta/8 < 1/2$. This proves the first property in (6.3). In order to show the second property in (6.3), we exploit (3.7) and tha fact that $\nu \in (\tilde\nu_{q+1}, \tilde\nu_q]$ to obtain

$$\begin{aligned}\|\nu\Delta u\|_{C^{\alpha'}((0,1-T_q)\times\mathbb{T}^3))} &\le \tilde\nu_q \sup_{j\le q-1} \|\Delta u\|_{C^{\alpha'}(\mathcal{I}_j\times\mathbb{T}^3)}\\ &\le C a_q^{2-\frac{\gamma}{1+\delta}+4\epsilon} \sup_{j\le q-1} (a_j^{1-\gamma} a_{j+1}^{-2-2\epsilon\delta} a_{j+1}^{-\alpha'(1+\epsilon\delta)} + a_j^{1-\gamma} a_{j+1}^{-2-2\epsilon\delta} a_j^{-\gamma})\\ &\le C a_q^{\epsilon} a_{q-1}^{1-2\gamma-\alpha'(1+\epsilon\delta)(1+\delta)} \le C a_q^{\epsilon}\,,\end{aligned}$$

where we also used $a_q^{2\epsilon} a_q^{-2\epsilon\delta} \le 1$, (3.1a), and (6.1).

The convergence $F_\nu \to F_0$ in $C^{\alpha'}((0,2)\times\mathbb{T}^3)$ can be shown along the same lines, by observing that $\|\partial_t u\|_{C^{\alpha'}(K_q^c\times\mathbb{T}^2)} \to 0$ as $q\to\infty$.

Proof of (ii). We argue exactly as in the proof of (iii) in Section 5. We first notice that

$$2\,\tilde\nu_q \int_0^{1-T_q+\bar t_q} \|\nabla\tilde\vartheta_{\tilde\nu_q}(s,\cdot)\|^2_{L^2}\,ds = 2\,\tilde\nu_q \int_0^{1-T_q+\bar t_q} \|\nabla\vartheta_{\tilde\nu_q}(s,\cdot)\|^2_{L^2}\,ds > \frac{1}{2} \qquad \text{for any } q\in m\mathbb{N},$$

as a direct consequence of (5) in Proposition 3.1. The first two components of v_ν strongly converge to u in $L^\infty((0,2)\times\mathbb{T}^3)$ while $\{\tilde\vartheta_{\tilde\nu_q}\}_{q\in\mathbb{N}}$ admits a limit point $\vartheta^{\rm ds}$ in the weak topology of $L^2((0,2)\times\mathbb{T}^2)$ which solves the transport equation with velocity field u and initial datum $\vartheta_{\rm in}$. Setting

$$v_0^{\rm ds}(t,x) := \begin{pmatrix} u(t,x)\\ \vartheta^{\rm ds}(t,x)\end{pmatrix}\,, \quad t\in(0,2),\, x\in\mathbb{T}^3\,, \tag{6.4}$$

we can verify that $(v_0^{\rm ds}, p_0, F_0)$ with $p_0 = 0$ solves (E) and $v_0^{\rm ds}$ is an admissible solution by arguing exactly as in the proof of (iii) in Section 5.

Proof of (iii). The first two components of v_{ν_q} strongly converge to u in $L^\infty((0,2)\times\mathbb{T}^3)$. We claim that $\tilde\vartheta_{\nu_q}$, the last component of v_{ν_q}, strongly converges in $L^2((0,2)\times\mathbb{T}^2)$ to ϑ_0 (defined as in (4) of Proposition 3.1). Setting

$$v_0^{\rm cs}(t,x) := \begin{pmatrix} u(t,x)\\ \vartheta_0(t,x)\end{pmatrix}\,, \qquad t\in(0,2),\, x\in\mathbb{T}^3 \tag{6.5}$$

and observing that $\|\vartheta_0(t,\cdot)\|_{L^2} = \|\vartheta_{\rm in}\|_{L^2}$ for any $t\in(0,2)\setminus\{1\}$, $F_{\nu_q}\to F_0$ in $C^{\alpha'}((0,2)\times\mathbb{T}^3)$, and $u_q\to u$ in $L^2((0,2)\times\mathbb{T}^3)$, the claimed convergence suffices to conclude that $v_0^{\rm cs}$ is an (admissible) conservative solution to (E).

We argue as in the proof of (iv) in Section 5. Denoting by $\vartheta_q : (0,2)\times\mathbb{T}^2\to\mathbb{R}$ the unique solution to the transport equation with velocity field u_q and initial datum $\vartheta_{\rm in}$, we have

$$\|\tilde\vartheta_{\nu_q} - \vartheta_0\|_{L^2([0,2]\times\mathbb{T}^2)} \le \|\tilde\vartheta_{\nu_q} - \vartheta_q\|_{L^2([0,2]\times\mathbb{T}^2)} + \|\vartheta_q - \vartheta_0\|_{L^2([0,2]\times\mathbb{T}^2)}\,. \tag{6.6}$$

We notice that $\|\vartheta_q - \vartheta_0\|_{L^2((0,2)\times\mathbb{T}^3)} \to 0$ as $q \to \infty$, thanks to $\vartheta_q(t,x) = \vartheta_0(t,x)$ for any $t \in K_q$ and any $x \in \mathbb{T}^3$ (because of the symmetry of the velocity field u as in property (1) of Proposition 3.1) and to the bound $\|\vartheta_0\|_{L^\infty((0,2)\times\mathbb{T}^2)} + \|\vartheta_q\|_{L^\infty((0,2)\times\mathbb{T}^2)} \le 2$. For any $t \in (0,2)$, we estimate the first term in (6.6) relying on the regularity bound (4) and the symmetry property (1) in Proposition 3.1. We have

$$
\begin{aligned}
&\|\tilde\vartheta_{\nu_q}(t,\cdot) - \vartheta_q(t,\cdot)\|^2_{L^2(\mathbb{T}^2)} \\
\le& 2\nu_q \left| \int_0^t \int_{\mathbb{T}^2} \nabla\tilde\vartheta_{\nu_q}(s,x)\cdot\nabla\vartheta_q(s,x)\,dxds \right| \\
\le& 2\left(\nu_q \int_0^t \int_{\mathbb{T}^2} |\nabla\vartheta_q(s,x)|^2 dxds \right)^{1/2} \\
=& 2\left(\nu_q \sum_{j=q}^{\infty} \int_{\mathcal{I}_j\cup\mathcal{J}_j} \int_{\mathbb{T}^2} |\nabla\vartheta_q(s,x)|^2 dxds + \nu_q \sum_{j=0}^{q-1} \int_{\mathcal{I}_j\cup\mathcal{J}_j} \int_{\mathbb{T}^2} |\nabla\vartheta_q(s,x)|^2 dxds \right)^{1/2} \\
\le& C\left(a_q^{2+3\epsilon} a_q^{\gamma} a_q^{-2(1+3\epsilon(1+\delta))} + a_q^{2+3\epsilon} \sum_{j=0}^{q-1} a_j^{\gamma} a_{j+1}^{-2(1+3\epsilon(1+\delta))} \right)^{1/2} \\
\le& C\left(a_q^{2+3\epsilon} a_{q-1}^{\gamma} a_q^{-2(1+3\epsilon(1+\delta))} \right)^{1/2} \\
\le& Cq a_q^{1+\frac{3\epsilon}{2}} a_{q-1}^{\gamma} a_q^{-1-6\epsilon} \le Cq a_q^{\frac{\delta}{32}-\frac{9\epsilon}{2}} \to 0
\end{aligned}
$$

as $q \to \infty$, where we used $\gamma = \delta/8$, $a_{q+1} = a_q^{1+\delta}$, $\delta \in (0, 1/8)$, $q a_q^{\delta/32} \le 1$, $\sum_{j\ge q} a_j^{\gamma} \le 2a_q^{\gamma}$, and $\delta/8 > 9\epsilon/2$. Therefore, v_0^{CS} satisfies (1.7).

7. Future extensions

In this section we focus on the problem of erasing the dependence of ν on the body force F_ν, which will be replaced just with F_0.

We introduce the shorthand notation

$$\tau_k = 1 - T_k \qquad \mathcal{I}_k = [\tau_k, \tau_{k+1}] \qquad \text{for any } k \in \mathbb{N}$$

Let $u : [0,2]\times\mathbb{T}^2 \to \mathbb{R}^2$ be the the alternated shear flow constructed in [CCS22] which solves 2d Euler equations with body force $F_0 = \partial_t u$ with zero pressure and zero initial datum

$$\partial_t u + u\cdot\nabla u = F_0\,,$$

noticing that $u\cdot\nabla u = 0$, thanks to the alternating shear flow property. Let $f_\nu : [0,2]\times\mathbb{T}^2 \to \mathbb{R}^2$ be the solution to the heat equation with the body force F_0 and zero initial datum, namely

$$\partial_t f_\nu = \nu\Delta f_\nu + F_0\,.$$

We have the Duhamel formula

$$|f_\nu(x,t) - u(x,t)| = |\int_0^t (e^{-\nu(t-s)\Delta}F_0)(x,s) - F_0(x,s)ds|\,.$$

To estimate the previous quantity we introduce $h_{\nu,k} : \mathcal{I}_k \times \mathbb{T}^2 \to \mathbb{R}$

$$\begin{cases} \partial_t h_{\nu,k} = \nu\Delta h_{\nu,k} + F_0\,, & t\in\mathcal{I}_k\,, x\in\mathbb{T}^2 \\ h_{\nu,k}(\tau_k,\cdot) \equiv 0\,. \end{cases}$$

and

$$\begin{cases} \partial_t g_{\nu,k} = \nu\Delta g_{\nu,k}\,, & t\in\mathcal{I}_k\,, x\in\mathbb{T}^2 \\ g_{\nu,k}(\tau_k,\cdot) = f_\nu(\tau_k,\cdot) \end{cases}$$

for any $k \in \mathbb{N}$.

Let us fix $\alpha \in [0,1]$, using the linearity of the equation for f_ν for any $k \in \mathbb{N}$ and for any $t \in \mathcal{I}_k$, we have the following identity

$$\begin{aligned}\|f_\nu(\cdot,t) - u(\cdot,t)\|_{C^\alpha} &\le \|g_{\nu,k}(\cdot,t)\|_{C^\alpha} + h_{\nu,k}(\cdot,t) - \int_{\tau_k}^t F_0(\cdot,t)dt\|_{C^\alpha}\\ &= \|g_{\nu,k}\|_{L^\infty(\mathcal{I}_k;C^\alpha(\mathbb{T}^2))} + \int_{\tau_k}^{\tau_{k+1}} \|e^{-\nu(\tau_{k+1}-s)\Delta}F_0)(\cdot,s) - F_0(\cdot,s)\|_{C^\alpha}\\ &\le \|g_{\nu,k}\|_{L^\infty(\mathcal{I}_k;C^\alpha(\mathbb{T}^2))} + \int_{\tau_k}^{\tau_{k+1}} (\nu(\tau_{k+1}-s))^{1/2}\|F_0(\cdot,s)\|_{C^{1+\alpha}}ds\,.\end{aligned}$$

Therefore, we define the following quantities

$$\begin{cases} A_k = \|f_\nu - u\|_{L^\infty(\mathcal{I}_k;C^\alpha(\mathbb{T}^2)} \\ B_k = \|g_{\nu,k}\|_{L^\infty(\mathcal{I}_k;C^\alpha(\mathbb{T}^2))} \\ C_k = \int_{\tau_k}^{\tau_{k+1}} (\nu(\tau_{k+1}-s))^{1/2}\|F_0(\cdot,s)\|_{C^{1+\alpha}}ds \end{cases}$$

and the previous estimate reads as

$$A_k \le B_k + C_k\,. \tag{7.1}$$

Observing that for any $k \in \mathbb{N}$ we have that $u(\cdot,\tau_k) \equiv 0$ and using that heat equation solutions have C^α norm non-increasing, we deduce that

$$B_k \le \|f_\nu(\tau_k,\cdot)\|_{C^\alpha} = \|f_\nu(\tau_k,\cdot) - u(\tau_k,\cdot)\|_{C^\alpha} \le A_{k-1}$$

therefore, thanks to (7.1), the sequence A_k satisfies

$$A_k \le A_{k-1} + C_k \le A_{k-2} + C_{k-1} + C_k \le \ldots \le \sum_{j=0}^k C_j\,.$$

for any $k \in \mathbb{N}$. Using (3.7) we finally conclude that

$$\begin{aligned}A_{q-1} &\le \sum_{j=0}^{q-1} C_j = \nu^{1/2}\sum_{j=0}^{q-1}\int_{\mathcal{I}_j} (\tau_{j+1}-s)^{1/2}\|F_0(s,\cdot)\|_{C^{1+\alpha}}ds\\ &\le \nu^{1/2}\sum_{j=0}^{q-1} a_j^{3\gamma/2}a_j^{-\gamma}a_j^{1-\gamma}a_{j+1}^{(-1-\alpha)(1+\epsilon\delta)} \le q\nu^{1/2}a_{q-1}^{1-\frac{\gamma}{2}-(1+\alpha)(1+\epsilon\delta)(1+\delta)}\end{aligned}$$

7.1. **Leading order terms discussion.** Redefining $\nu = \nu_q = a_q^{2-\frac{\gamma}{1+\delta}+\kappa}$ (this implies that we should change the length interval where the anomalous dissipation is happening, namely in $[\tau_q, \tau_q + a_q^{\frac{\gamma}{1+\delta}-\kappa}]$), where $0 \le \kappa < \gamma$ is a new variable. We observe that the leading part of the previous estimate is

$$a_q^{1+\frac{\kappa}{2}-\frac{\gamma}{2(1+\delta)}}a_j^{1-\frac{\gamma}{2}-(1+\alpha)}\,.$$

Therefore we observe that, if we have $\kappa > 0$ (considering the leading terms)

$$\|f_\nu - u\|_{L^\infty((0,\tau_q);L^\infty)} \sim a_q^{1+\frac{\kappa}{2}-\frac{\gamma}{2(1+\delta)}}a_q^{-\frac{\gamma}{2}} <<< a_q^{1-\gamma} \sim \|u\|_{L^\infty(\mathcal{I}_j\times\mathbb{T}^2)}$$

and

$$\|f_\nu - u\|_{L^\infty((0,\tau_q);C^1)} \sim a_q^{1+\frac{\kappa}{2}-\frac{\gamma}{2(1+\delta)}}a_q^{-2}a_{q-1}^{1-\frac{\gamma}{2}} \sim a_q^{\frac{\kappa}{2}}a_{q-1}^{-\gamma-\delta} <<< a_{q-1}^{-\gamma-\delta} \sim \|u\|_{L^\infty((0,\tau_q);C^1)}$$

and the last is small and going to zero when $q \to \infty$ and in particular this quantity is much smaller than $\|u\|_{L^1((0,1-T_q);C^1(\mathbb{T}^2))}$ which explodes.

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