DIRICHLET ENERGY-MINIMIZERS WITH ANALYTIC BOUNDARY

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Abstract. In this paper, we consider multi-valued graphs with a prescribed real analytic interface that minimize the Dirichlet energy. Such objects arise as a linearized model of area minimizing currents with real analytic boundaries and our main result is that their singular set is discrete in 2 dimensions. This confirms (and provides a first step to) a conjecture by B. White [21] that area minimizing 2-dimensional currents with real analytic boundaries have a finite number of singularities. We also show that, in any dimension, Dirichlet energy-minimizers with a C^1 boundary interface are Hölder continuous at the interface.

1. Introduction and main result

Consider a smooth closed curve Γ in $\mathbb{R}^{2+n}$. The existence of oriented surfaces which bound Γ and minimize the area can be approached in two different ways. Following the classical work of Douglas and Rado we can fix an abstract connected smooth surface Σ_g of genus g whose boundary $\partial\Sigma_g$ consists of a single connected component and look at smooth maps $\Phi : \Sigma_g \to \mathbb{R}^{2+n}$ with the property that the restriction of Φ to $\partial\Sigma_g$ is an homeomorphism onto Γ. We then consider the infimum $A_g(\Gamma)$ over all such maps Φ and all smooth Riemannian metrics h on Σ of the energy

$$\int_{\Sigma_g} |\nabla\Phi|^2 \mathrm{dvol}_h\,.$$

If $A_g(\Gamma) < A_{g-1}(\Gamma)$, then there is a minimizer, cf. [16, 4], whose image is an immersed surface of genus g, with possible branch points. A different, more intrisic, approach was pioneered by De Giorgi, cf. [5], in the codimension 1 case, and by Federer and Fleming in higher codimension, cf. [17]. The latter looks at integral currents T (a suitable measure-theoretic generalization of classical oriented submanifolds with boundary) whose boundary is given by $[\![\Gamma]\!]$ and minimizes their mass, a suitable measure-theoretic generalization of the volume of classical submanifolds. The minimizer then always exists via the direct methods of the calculus of variations.

There is a very natural question relating the two approaches: is every minimizer T found by the Federer-Fleming theory a classical minimal surface with finite topology, namely a parametrized surface of some genus g? Note that if this were the case, then the sequence $\{A_g(\Gamma)\}_{g\in\mathbb{N}}$ would become constant for sufficiently large g. When the codimension n equals 1 and Γ is of class $C^{2,\alpha}$ for some $\alpha > 0$, the interior regularity theorem of De Giorgi in [6] and the boundary regularity theorem of Hardt and Simon in [19] imply that every minimizer T is in fact a $C^{2,\alpha}$ embedded surface up to the boundary. Thus T has finite genus g_0 and any conformal parametrization Φ

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over an abstract Riemann surface Σ_{g_0} gives a minimizer in the sense of Douglas and Rado. On the other hand, Fleming in [18]. showed a closed embedded curve Γ in $\mathbb{R}^3$ of finite length for which $\{A_g(\Gamma)\}_{g\in\mathbb{N}}$ is not asymptotically constant.

The question is much more subtle in higher codimension, because singularities might arise, both at the interior and at the boundary. In the work [21] White asks whether the topology of the minimizer T is finite when Γ is real analytic. If this conjecture were true, then T would have finitely many singularities by the main theorem of [21]. The aim of this paper is to start a sort of reverse program to White's: under the assumption of real analyticity for the boundary Γ we wish to show first that the set of boundary and interior singular points of T is finite and hence to analyze the singularities and conclude that the topology of the minimizer is finite.

It has been shown by Chang in [3] that T is smooth in $\mathbb{R}^n \setminus \Gamma$ up to a discrete set of singular branch points and in sufficiently small neighborhoods of such singular points the resulting branched surface is topologically a disk. We in fact refer to [14, 11, 13, 12] for a complete proof, as Chang needs a suitable modification of the techniques of Almgren's monumental monograph [2] to start his argument, and the former has been given in full details in [13]. In order to attack White's conjecture it suffices therefore to deal with boundary regularity. In fact, even for Γ of class $C^{2,\alpha}$, under the assumption that Γ lies in the boundary of a uniformly convex set, the boundary regularity theorem of Allard [1] implies that any minimizer T is smooth at Γ. The general problem is however very subtle. So far the best available result is given in [8] and shows that the set of boundary regular points is dense in Γ when Γ is of class $C^{3,\alpha}$ for $\alpha > 0$. The work [8] gives also an example of a smooth curve in $\mathbb{R}^4$ for which there is a unique Federer-Fleming minimizer with a sequence of singularities accumulating to a boundary branch point. This example has been modified in [7] to produce C^∞ embedded curves in complete C^∞ Riemannian 4-dimensional manifolds for which there is a unique Federer-Fleming minimizer with infinite topology. In particular there is a strong contrast to the codimension 1 case: the real analyticity assumption in White's conjecture is, in a certain sense, needed*.

1.1. **Linearized model.**

The analysis of interior singularities of area minimizing currents was pioneered by Almgren's monumental work in [2] in the early eighties and recently revisited from a modern perspective by the first author and Emanuele Spadaro in [15]. The work [8] gives an Almgren type theory at the boundary, whereas the works [14, 11, 13, 12, 20, 9, 10] extend the interior theory to other objects (almost calibrated currents and area minimizing currents modulo p). The starting point of all these papers, an essential discovery of Almgren, is to analyze the singularities for a suitable "linearized model". The main purpose of the present paper is to state and prove the appropriate linearized counterpart of White's conjecture.

First of all we recall the notation $\mathcal{A}_Q(\mathbb{R}^n)$ for the set of unordered Q-tuples of $\mathbb{R}^n$, which we will regard as nonnegative atomic measures with integer coefficients and total mass Q, cf. [15, Introduction] for the formal definition and for the standard complete metric $\mathcal{G}$ which we will use

*The examples of [7] are curves in smooth almost Kähler manifolds $(\mathbb{R}^4, g)$, whose smooth metrics can be taken arbitrarily close to the euclidean one. However it is currently not known whether such examples exist in the Euclidean space.

on it. For atoms we will use the notation $[\![P]\!]$ and thus elements in $\mathcal{A}_Q(\mathbb{R}^n)$ will be denoted by $\sum_i [\![P_i]\!]$. In what follows we will often write $\mathcal{A}_Q$ instead of $\mathcal{A}_Q(\mathbb{R}^n)$. We recall that for Sobolev functions $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ (cf. again [15, Introduction]) we set

$$|Df|^2 := \sum_{j=1}^{m} |\partial_j f|^2,$$

where

$$|\partial_j f| = \sup_{i\in\mathbb{N}} |\partial_j \mathcal{G}(f, T_i)| \quad \text{almost everywhere in } \Omega, \tag{1.1}$$

and $\{T_i\}_{i\in\mathbb{N}}$ is a countable dense subset of $\mathcal{A}_Q$. While such abstract definition is very direct and useful to work with, the Dirichlet energy turns out to be the sum of the Dirichlet energies of the different sheets in all cases where the multifunction f can be "nicely decomposed". In an appropriate sense this can be justified also for any Sobolev functions, the reader is again referred to [15] for the relevant details.

We now recall the notion of interior regular points.

Definition 1.2 (Interior regular point, Definition 0.10 of [15])**.** A function $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i : B \to \mathbb{R}^n$ such that

$$f(y) = \sum_i [\![f_i(y)]\!] \quad \text{for almost every } y \in B,$$

and either $f_i(y) \neq f_j(y)$ for every $y \in B$, or $f_i \equiv f_j$. The complement of interior regular points is called the set of interior singular points.

The following theorem on the interior regularity of Dir-minimizers was proven in [15], refining a previous fundamental result by Almgren in [2]:

Theorem 1.3 (Theorem 0.12 in [15])**.** *Let $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ be Dir-minimizing and $m = 2$. Then the interior singular set of f consists of isolated points.*

We now come to the boundary counterpart, following the approach of [8]. Suppose a hypersurface γ divides a connected open set $\Omega \subset \mathbb{R}^m$ into two connected components Ω^+ and Ω^-. For any set $K \subset \Omega$ we will use the notation $K^\pm$ for $K \cap \Omega^\pm$. Moreover, in order to avoid confusion, in the rest of the paper we will use the double integral symbol to indicate integration over subsets of $\mathbb{R}^m$ with respect to the Lebesgue measure, and the single integral symbol to indicate integration over subsets of the hypersurface γ with respect to the usual Hausdorff $(m-1)$-dimensional measure.

Definition 1.4. We say that the pair $f = (f^+, f^-)$ is a $\left(Q - \frac{1}{2}\right)$-map with interface (γ, φ) of class $W^{1,2}$ if there is some (classical) function $\varphi \in H^{1/2}(\gamma, \mathbb{R}^n)$ such that

(i) $f^+ \in W^{1,2}(\Omega^+, \mathcal{A}_Q)$ and $f^- \in W^{1,2}(\Omega^-, \mathcal{A}_{Q-1})$;
(ii) $f^+|_\gamma = f^-|_\gamma + [\![\varphi]\!]$.

We refer to [15, 8] for the trace theorems which allow to make sense of (ii) under our assumptions. For the corresponding set of pairs we will use the shorthand notation $W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ and for each $f = (f^+, f^-) \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ we define its Dirichlet energy as

$$\mathrm{Dir}(f, \Omega) := \mathrm{Dir}(f^+, \Omega^+) + \mathrm{Dir}(f^-, \Omega^-) = \iint_{\Omega^+} |Df^+|^2 + \iint_{\Omega^-} |Df^+|^2 .$$

Finall, we say that $f = (f^+, f^-) \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ is *Dir-minimizing with interface* (γ, φ), if $\mathrm{Dir}(g, \Omega) \geq \mathrm{Dir}(f, \Omega)$ for any other function $g \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ with interface (γ, φ) which agrees with f outside of a compact set $K \subset \Omega$.

The goal of the paper is to show that when the interface (γ, φ) is real analytic and the domain is 2-dimensional, Dir-minimizers enjoy a regularity theorem which is analogous to Theorem 1.3. First of all a point $p \in \Omega \setminus \gamma$, namely belonging to either Ω^+ or Ω^-, will be called regular if it is a regular point for, respectively,f^+ or f^- (cf. Definition 1.2). Its complement in $\Omega \setminus \gamma$ is the set of interior singular points, denoted by Σ_f^i. It remains to define regular points at the interface γ.

Definition 1.5 (Boundary regular point, Definition 2.6 of [8])**.** Let $f = (f^+, f^-)$ be a map in $W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ with interface (γ, φ). A point $p \in \gamma$ is regular if there are a ball $B_r(p)$, $(Q-1)$-functions $u_1, \cdots, u_{Q-1} : B_r(p) \to \mathbb{R}^n$ and a function $u_Q : B_r^+(p) \to \mathbb{R}^n$ such that

- $f^+ = \sum_{i=1}^{Q} [\![u_i]\!]$ on $B_r^+(p)$ and $f^- = \sum_{i=1}^{Q-1} [\![u_i]\!]$ on $B_r^-(p)$;
- For any pair $i, j \in \{1, \cdots, Q-1\}$ either the graphs of u_i and u_j are disjoint or they completely coincide;
- For any $i \in \{1, \cdots, Q-1\}$ either the graphs of u_i and u_Q are disjoint in $B_r^+(p)$ or the graph of u_Q is contained in that of u_i.

The complement in γ of the set of regular points is called the set of boundary singular points, denoted by Σ_f^b.

We can now state our main theorem:

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^2$ and (γ, φ) be an interface for which both γ and φ are real analytic. If $f \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ is Dir-minimizing with interface (γ, φ), then the singular set $\Sigma_f = \Sigma_f^i \cup \Sigma_f^b$ is discrete.*

In passing, we need a suitable estimate on the Hölder continuity of minimizers at the interface γ. The latter result is however not confined to the special dimension $m = 2$ nor to real analytic interfaces (γ, φ) and, although it is not immediately relevant for our main purposes, we state it in a more general case in the following

Theorem 1.7. *Let $m \in \mathbb{N} \setminus \{0, 1\}$ and suppose (f^+, f^-) is a Dir-minimizing $(Q - \frac{1}{2})$-map in $\Omega \subset \mathbb{R}^m$ with interface (γ, φ) of class C^1. Then (f^+, f^-) is Hölder regular.*

In fact it is possible to give a precise estimate on a suitable Hölder seminorm of $f^{\pm}$ in terms of the regularity of the interface (γ, φ) and the Dirichlet energy of the minimizer. For the precise statement we refer to Theorem 3.1.

1.2. **Plan of the paper.** The remaining sections are organized as follows. First of all in Section 2 we make some preliminary elementary considerations on planar minimziers which will be particularly useful in the planar case of Theorem 1.7 and in Theorem 1.6. In Section 3 we address the general Hölder regularity result and prove therefore Theorem 1.7. In the subsequent Section 4 we give the fundamental computations leading to the monotonicity of the frequency function, a celebrated result of Almgren away from interface, extended at general interfaces in [8]: in our case the computations are simpler than in [8] because we can "straighten the boudary" using complex analysis. In Section 5 we use the frequency function estimate and the Hölder regularity to prove the existence of suitable blow-ups, or tangent functions, at singular points. A suitable modification of the argument given in [15] (which in turn borrowed from key ideas in [3]) shows then the uniqueness of such objects. In Section 6 we give a list of necessary conditions that tangent functions must satisfy, which in turn leads to a suitable decomposition of them in simpler pieces (which we call irreducible maps). Such decomposition is combined together with the rate of convergence proven in Section 5 in order to decompose general Dir-minimizers at boundary singular points: the latter fact is then used in the final Section 7 to conclude the proof of Theorem 1.6.

2. Reduction and preliminaries for the planar case

2.1. **Reduction of Theorem 1.6.** In this section we use elementary considerations in complex analysis to reduce Theorem 1.6 to a much simpler case. In order to state our theorem, we recall the definition of the map $\boldsymbol{\eta} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^n$ which gives the barycenter of the atomic measure T:

$$\boldsymbol{\eta}\left(\sum_{i=1}^{Q} [\![P_i]\!]\right) = \frac{1}{Q}\sum_{i=1}^{Q} P_i\,.$$

In particular, if $(f^+, f^-) \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ we can define two maps (η^+, η^-) which are, respectively the center of mass of the maps f^+ and f^-. In particular $\eta^\pm := \boldsymbol{\eta} \circ f^\pm$, where we make a slight abuse of notation because we keep the same symbol $\boldsymbol{\eta}$ for two different maps, one defined on $\mathcal{A}_Q$ and the other on $\mathcal{A}_{Q-1}$. Specifically:

$$\eta^+(x) = \frac{1}{Q}\sum_{i=1}^{Q} f_i^+(x) \qquad \text{and} \qquad \eta^-(x) = \frac{1}{Q-1}\sum_{i=1}^{Q-1} f_i^-(x)\,.$$

Theorem 1.6 can then be reduced to the following particular case:

Theorem 2.1. *Let $m = 2$ and assume (f^+, f^-) is Dir-minimizing in the unit disk $\mathbb{D}$ with interface $(\gamma, 0)$, where γ is the coordinate axis $\{(x_1, 0) : x_1 \in \mathbb{R}\}$. Assume further that $Q\eta^+ = (Q-1)\eta^-$. Then the singular set Σ_f is discrete.*

From now on, we introduce the convention that, if $\gamma = \{(x_1, 0) : x_1 \in \mathbb{R}\}$, then the interface (γ, φ) is denoted by $(\mathbb{R}, \varphi)$. This is motivated by the fact that we will often identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, via $(x_1, x_2) \mapsto x_1 + ix_2$. The set $\{x_2 = 0\}$ is then the real axis of $\mathbb{C}$ after such identification. The above theorem will be proved at the end of the paper. In the next paragraph we show how the general case of Theorem 1.6 follows from it.

Assume (f^+, f^-) is as in Theorem 1.6. First of all observe that, if Σ_f is not discrete, then by Theorem 1.3 Σ_f must have an accumulation point $p \in \gamma$. Modulo translation we may assume p is the origin. Since γ is analytic, we may choose a coordinate system so that the tangent to γ satisfies $T_0\gamma = \{x_2 = 0\} = \mathbb{R}$. In particular γ must be (locally) the graph $\{(t, \zeta(t))\}$ of a function $\zeta(t)$ whose Taylor series at the origin is $\sum_{k\geq 2} \alpha_k t^k$ (where $\alpha_k = \frac{\zeta^{(k)}(0)}{k!} \in \mathbb{R}$). Identify $\mathbb{R}^2$ with the complex plane and consider, in a neighborhood of the origin, the holomorphic map Φ given by $\Phi(z) = z + \sum_{k\geq 2} i\alpha_k z^k$. By the inverse function theorem the latter map is invertible in a sufficiently small neighborhood U of the origin (which can be assumed to be a disk) and its inverse over $\Phi(U)$ is also holomorphic. Since Φ is conformal, $(f^+ \circ \Phi^{-1}, f^- \circ \Phi^{-1})$ is clearly a minimizer in $V := \Phi(U)$ and the interface is $(T_0\gamma, \varphi \circ \Phi^{-1})$. Moreover Φ maps the segment $\{\operatorname{Im} z = 0\} \cap U$ onto γ. We can thus assume, without loss of generality, that $\gamma = \mathbb{R}$.

Next, since φ is real analytic, by the Cauchy-Kowalevski Theorem φ has a harmonic extension in a neighborhood of the origin, still denoted by φ. We then replace $f = (f^+, f^-)$ with

$$f^+(x) \mapsto g^+(x) := \sum_{i=1}^{Q} [\![f_i^+(x) - \varphi(x)]\!], \quad f^-(x) \mapsto g^-(x) := \sum_{i=1}^{Q-1} [\![f_i^-(x) - \varphi(x)]\!].$$

Indeed, given a map $(\bar g^+, \bar g^-)$ with interface $(\mathbb{R}, 0)$ and same trace on $\partial\mathbb{D}$ as $(\bar g^+, \bar g^-)$, consider the corresponding map (h^+, h^-) where we add φ on each side. The latter has interface $(\mathbb{R}, \varphi)$ and coincides with (f^+, f^-) on $\partial\mathbb{D}$. Moreover we compute

$$\iint_{\mathbb{D}^+} |Dh^+|^2 = \iint_{\mathbb{D}^+} |D\bar g^+|^2 + Q \iint_{\mathbb{D}^+} |D\varphi|^2 + 2\, \underbrace{Q \iint_{\mathbb{D}^+} D\boldsymbol{\eta} \circ \bar g^+ : D\varphi}_{=:I^+}$$

$$\iint_{\mathbb{D}^-} |Dh^-|^2 = \iint_{\mathbb{D}^-} |D\bar g^-|^2 + (Q-1) \iint_{\mathbb{D}^-} |D\varphi|^2 + 2\, \underbrace{(Q-1) \iint_{\mathbb{D}^-} D\boldsymbol{\eta} \circ \bar g^- : D\varphi}_{=:I^-}$$

Using that the function φ is harmonic we compute

$$I^+ = \underbrace{Q \int_{(\partial\mathbb{D})^+} \boldsymbol{\eta} \circ \bar g^+ \cdot \frac{\partial\varphi}{\partial\nu}}_{=:J^+} - \underbrace{Q \int_{\mathbb{R}\cap\mathbb{D}} \boldsymbol{\eta} \circ \bar g^+ \cdot \frac{\partial\varphi}{\partial x_2}}_{=:K^+}$$

$$I^- = \underbrace{(Q-1) \int_{(\partial D)^-} \boldsymbol{\eta} \circ \bar g^- \cdot \frac{\partial\varphi}{\partial\nu}}_{=:J^-} + \underbrace{(Q-1) \int_{\mathbb{R}\cap\mathbb{D}} \boldsymbol{\eta} \circ \bar g^- \cdot \frac{\partial\varphi}{\partial x_2}}_{=:K^-}$$

Observe that J^+ and J^- are both independent of the choice of $(\bar g^+, \bar g^-)$, because the traces of the respective maps on $(\partial\mathbb{D})^\pm$ equals those of (g^+, g^-). On the other hand $Q\boldsymbol{\eta} \circ \bar g^+ - (Q-1)\boldsymbol{\eta} \circ \bar g^- = 0$ on $\mathbb{R} \cap \mathbb{D}$. Therefore $K^- - K^+ = 0$. This implies that the difference

$$\iint_{\mathbb{D}^+} |Dh^+|^2 + \iint_{\mathbb{D}^-} |Dh^-|^2 - \iint_{\mathbb{D}^+} |D\bar g^+|^2 - \iint_{\mathbb{D}^-} |D\bar g^-|^2$$

is actually a constant. In particular, if we could find a competitor for (g^+, g^-) with lower energy, then we could transform it into a competitor for (f^+, f^-) with lower energy: we conclude that (g^+, g^-) must be a Dir minimizer with interface $(\mathbb{R}, 0)$.

Observe next that $\eta^+ = \boldsymbol{\eta} \circ f^+$ and $\eta^- = \boldsymbol{\eta} \circ f^-$ are harmonic functions in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively. For any $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $\bar{x} = (x_1, -x_2)$ the reflection point of x across $\mathbb{R}$. We define a function $\phi : \mathbb{D} \to \mathbb{R}^n$ as

$$\phi(x) = \begin{cases} \dfrac{Q\eta^+(x) - (Q-1)\eta^-(\bar{x})}{2Q-1}, & x_2 \geq 0, \\ \dfrac{(Q-1)\eta^-(x) - Q\eta^+(\bar{x})}{2Q-1}, & x_2 \leq 0. \end{cases} \tag{2.2}$$

Clearly ϕ is harmonic in $\mathbb{D} \setminus \mathbb{R}$. By the boundary condition $f^+|_\gamma = f^-|_\gamma + [\![0]\!]$, we know

$$Q\eta^+ = (Q-1)\eta^- \quad \text{ on } \mathbb{R}.$$

Hence ϕ is continuous and odd in the variable x_2. In particular ϕ is harmonic on all of $\mathbb{D}$. Therefore by modifying (f^+, f^-) as follows

$$f^+(x) \mapsto \widetilde{f}^+(x) := \sum_{i=1}^{Q} [\![f_i^+(x) - \phi(x)]\!], \quad x \in \mathbb{R}^m_+,$$

$$f^-(x) \mapsto \widetilde{f}^-(x) := \sum_{i=1}^{Q-1} [\![f_i^-(x) - \phi(x)]\!], \quad x \in \mathbb{R}^m_-$$

and repeating the same computations as above we conclude that the new function $(\widetilde{f}^+, \widetilde{f}^-)$ is still a Dir-minimizer with the same interface $(\mathbb{R}, 0)$. Notice also that

$$\sum_{i=1}^{Q} \widetilde{f}_i^+(x) = Q\eta^+(x) - Q\phi(x) = \frac{Q(Q-1)}{2Q-1}\left(\eta^+(x) + \eta^-(\bar{x})\right),$$

$$\sum_{i=1}^{Q-1} \widetilde{f}_i^-(x) = (Q-1)\eta^-(x) - (Q-1)\phi(x) = \frac{Q(Q-1)}{2Q-1}\left(\eta^-(x) + \eta^+(\bar{x})\right),$$

and thus

$$\sum_{j=1}^{Q} \widetilde{f}_j^+(x) = \sum_{j=1}^{Q-1} \widetilde{f}_j^-(\bar{x}).$$

For simplicity we still denote the new function as (f^+, f^-), except that their center of mass now enjoy an additional symmetry:

$$Q\eta^+(x) = (Q-1)\eta^-(\bar{x}). \tag{2.3}$$

This symmetry is invariant under translation, scaling and uniform limit.

2.2. **Decomposition into irreducible maps.** In this section we extend a suitable decomposition of Q-valued maps on the circle to the case of $(Q-\frac{1}{2})$-valued maps. Recall that map $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q)$ is called *irreducible* if there is no decomposition of g into two simpler $W^{1,p}$ functions (cf. [15]), namely if there are no integers $Q_1, Q_2 > 0$ and maps $g_1 \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q_1})$, $g_2 \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q_2})$ such that $g = g_1 + g_2$ (in particular $Q_1 + Q_2 = Q$).

Definition 2.4 (Irreducible $(Q-\frac{1}{2})$-maps on $\mathbb{S}^1$)**.** A map $g = (g^+, g^-) \in W^{1,p}(\mathbb{S}^1, \mathcal{A}^{\pm}_Q)$ with interface $(\mathbb{R}, \varphi)$ is called *irreducible* if there is no decomposition of g into the "sum" of a map $g_1 \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q_1})$ and a map $g_2 \in W^{1,p}(\mathbb{S}^1, \mathcal{A}^{\pm}_{Q_2})$ with the same interface (γ, φ), where the positive integers Q_1, Q_2 satisfy $Q_1 + Q_2 = Q$. The "sum" is understood in the following sense:

$$
\begin{aligned}
g^+ &= g_1 + g_2^+ \qquad \text{on } (\mathbb{S}^1)^+ = \{z \in \mathbb{C} : |z| = 1, \operatorname{Re} z > 0\} \\
g^- &= g_1 + g_2^- \qquad \text{on } (\mathbb{S}^1)^- = \{z \in \mathbb{C} : |z| = 1, \operatorname{Re} z < 0\}.
\end{aligned}
$$

Remark 2.5. By the above definition, clearly any function $g \in W^{1,p}([0,\pi], \mathbb{R}^n)$ satisfying $g(0) = g(\pi) = 0$ is irreducible with $Q = 1$ (the interface being $(\mathbb{R}, 0)$).

The decomposition of $W^{1,p}$ $(Q-\frac{1}{2})$-valued map on the circle is then a corollary of the following proposition for Q-valued maps, where, for any interval $I = [a,b] \subset \mathbb{R}$, we denote by $\mathrm{AC}(I, \mathcal{A}_Q)$ the space of absolutely continuous functions taking values in the metric space $(\mathcal{A}_Q, \mathcal{G})$.

Proposition 2.6 (Porposition 1.2 of [15])**.** *Let $g \in W^{1,p}(I, \mathcal{A}_Q)$. Then*

(a) $g \in \mathrm{AC}(I, \mathcal{A}_Q)$ and moreover, $g \in C^{0,1-\frac{1}{p}}(I, \mathcal{A}_Q)$ for $p > 1$;
(b) There are $g_1, \cdots, g_Q \in W^{1,p}(I, \mathbb{R}^n)$ s.t. $f = \sum_i [\![g_i]\!]$ and $|Dg_i| \le |Dg|$ a.e.

Proposition 2.7 (Decomposition of $W^{1,p}(\mathbb{S}^1, \mathcal{A}^{\pm}_Q)$ into irreducible maps)**.** *A map $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}^{\pm}_Q)$ with interface $(\mathbb{R}, \varphi)$ is either irreducible, or it can be decomposed as $g = g_0 + \sum_{j=1}^J g_j$, where $g_0 \in W^{1,p}(\mathbb{S}^1, \mathcal{A}^{\pm}_{Q_0})$ is irreducible with interface $(\mathbb{R}, \varphi)$, and each $g_j \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q_j})$ is irreducible. Moreover, a map $g \in W^{1,2}(\mathbb{S}^1, \mathcal{A}^{\pm}_Q)$ with interface $(\mathbb{R}, \varphi)$ is irreducible if and only if the following two conditions are satisfied:*

(i) $\mathrm{card}(g^+(\theta)) = Q$ *for every* $\theta \in [0,\pi]$, *and* $\mathrm{card}(g^-(\theta)) = Q - 1$ *for every* $\theta \in [\pi, 2\pi]$.
(ii) There exists a $W^{1,p}$ map $\zeta : \mathbb{S}^1 \to \mathbb{R}^n$ with $\zeta(0) = \varphi(1)$ and $\zeta(2\pi) = \varphi(-1)$ such that g unwinds to ζ, in the following sense: $g^+ = \sum_{j=1}^{Q} [\![g_j^+]\!]$ and $g^- = \sum_{j=1}^{Q-1} [\![g_j^-]\!]$ with

$$
g_j^+(\theta) = \zeta\left(\frac{2\theta}{2Q-1} + \frac{4\pi}{2Q-1}(j-1)\right), \quad \theta \in [0,\pi],\ j = 1, \cdots, Q, \tag{2.8}
$$

$$
g_j^-(\theta) = \zeta\left(\frac{2\theta}{2Q-1} + \frac{4\pi}{2Q-1}(j-1)\right), \quad \theta \in [\pi, 2\pi],\ j = 1, \cdots, Q-1. \tag{2.9}
$$

Proof. The existence of an irreducible decomposition in the above sense is an obvious consequence of the definition of irreducible maps. It remains to show the characterization of irreducible maps.

By Proposition 2.6 a map satisfying (i) and (ii) is clearly irreducible with interface (γ, φ). Suppose $g \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_Q^{\pm})$ with interface (γ, φ) is irreducible. Without loss of generality (i.e. after possible subtracting to all sheets an extension of φ) we can assume $\varphi \equiv 0$. Namely

$$g^+|_\gamma = g^-|_\gamma + [\![0]\!]. \tag{2.10}$$

Recall Proposition 2.6, we consider $g_1^+, \cdots, g_Q^+ \in W^{1,p}([0,\pi], \mathbb{R}^n)$ a selection of $g^+ \in W^{1,p}([0,\pi], \mathcal{A}_Q)$, and $g_1^-, \cdots, g_{Q-1}^- \in W^{1,p}([\pi, 2\pi], \mathbb{R}^n)$ a selection of $g^- \in W^{1,p}([\pi, 2\pi], \mathcal{A}_{Q-1})$. We assume without loss of generality that $g_1^+(0) = [\![0]\!]$. By the boundary condition (2.10), there exists an integer Q_0, $1 \leq Q_0 \leq Q$, such that after reordering the selections $g_i^+(\pi) = g_i^-(\pi) \neq 0$ and $g_i^-(2\pi) = g_{i+1}^+(0)$ for all $i = 1, \cdots, Q_0 - 1$, $g_{Q_0}^+(\pi) = 0$. Suppose $Q_0 < Q$, then we define

$$f_1^+ = \sum_{i=1}^{Q_0} [\![g_i^+]\!], \quad f_1^- = \sum_{i=1}^{Q_0-1} [\![g_i^-]\!],$$

and

$$f_2 = \begin{cases} \sum\limits_{i=Q_0+1}^{Q} [\![g_i^+]\!], & \theta \in [0, \pi] \\ \sum\limits_{i=Q_0}^{Q-1} [\![g_i^-]\!], & \theta \in [\pi, 2\pi] \end{cases}$$

By (2.10), the map $f_1 := (f_1^+, f_1^-)$ lies in $W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q_0}^{\pm})$ with interface (γ, φ); the map f_2 is well-defined on γ, i.e. $f_2(\pi-) = f_2(\pi+)$ and $f_2(2\pi) = f_2(0)$, and moreover $f_2 \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q-Q_0})$. In other words, this gives a nontrivial decomposition of the irreducible map g, contradiction. Hence $Q_0 = Q$, and we define the function ζ by following g_i^+, g_i^-, g_{i+1}^+ in order.

Suppose $\mathrm{card}(g^+) \neq Q$, that is, there exist $\theta_0 \in [0, \pi]$ and $i_1 < i_2$ such that $g_{i_1}^+(\theta_0) = g_{i_2}^+(\theta_0)$. Let

$$\widetilde{g}^+ = \begin{cases} g_{i_1}^+, & \theta \in [0, \theta_0] \\ g_{i_2}^+, & \theta \in [\theta_0, \pi]. \end{cases}$$

Then the following map gives a decomposition of g:

$$f_1^+ = \sum_{i=1}^{i_1-1} [\![g_i^+]\!] + [\![\widetilde{g}]\!] + \sum_{i=i_2+1}^{Q} [\![g_i^+]\!], \quad f_1^- = \sum_{i=1}^{i_1-1} [\![g_i^-]\!] + \sum_{i=i_2}^{Q-1} [\![g_i^-]\!].$$

Since $(f_1^+, f_1^-) \in W^{1,p}(\mathbb{S}^1, \mathcal{A}_{Q+i_1-i_2}^{\pm})$ with interface (γ, φ), and $Q + i_1 - i_2 < Q$, this is a nontrivial decomposition of the irreducible map g, contradiction. Hence $\mathrm{card}(g^+) = Q$. Similarly $\mathrm{card}(g^-) = Q - 1$. $\square$

2.3. **Rolling and unrolling.** The decomposition of the previous section can be used to construct efficient competitors to Dirichlet minimizers in the planar case. Again the situation is similar to that of Q-valued maps. Keeping our identification $\mathbb{R}^2 = \mathbb{C}$ we will denote by $[0,1]$ the "slit" $\{(x_1, 0) : 0 \leq x_1 \leq 1\}$ and on the domain $\mathbb{D} \setminus [0,1]$ we will consider polar coordinates $(r, \theta) \in]0,1[\times]0, 2\pi[$, via the usual parametrization $(r, \theta) \mapsto re^{i\theta}$. Given a map $\zeta \in W^{1,2}(\mathbb{D} \setminus [0,1], \mathbb{R}^n)$ we can define two maps $\zeta^u, \zeta^l \in H^{1/2}([0,1], \mathbb{R}^n)$ which are, respectively, the "upper" and "lower" traces of ζ on the slit $[0,1]$. In particular in polar coordinates we can naturally extend ζ to

$]0,1[\times[0,2\pi]$ setting $\zeta(r,0)=\zeta^u(r)$ and to $\zeta(r,2\pi)=\zeta^l(r)$. In the next lemma and its applications we will follow the latter convention.

Lemma 2.11 (Unrolling, analogue of Lemma 3.12 [15])**.** *Suppose $\zeta\in W^{1,2}(\mathbb{D}\setminus[0,1],\mathbb{R}^n)$ and consider the $\left(Q-\frac{1}{2}\right)$-valued function $f=(f^+,f^-)$ defined as follows:*

$$f_j^+(r,\theta)=\zeta\left(r^{\frac{2}{2Q-1}},\frac{2\theta}{2Q-1}+\frac{4\pi}{2Q-1}(j-1)\right),\quad \theta\in[0,\pi],\ j=1,\cdots,Q,\tag{2.12}$$

$$f_j^-(r,\theta)=\zeta\left(r^{\frac{2}{2Q-1}},\frac{2\theta}{2Q-1}+\frac{4\pi}{2Q-1}(j-1)\right),\quad \theta\in[\pi,2\pi],\ j=1,\cdots,Q-1.\tag{2.13}$$

(For $Q=1$ we just ignore f^-.) Then $f\in W^{1,2}(\mathbb{D},\mathcal{A}_Q^\pm)$ and

$$\mathrm{Dir}(f,\mathbb{D})=\iint_{\mathbb{D}}|D\zeta|^2.\tag{2.14}$$

Moreover, if $\zeta|_{\mathbb{S}^1}\in W^{1,2}(\mathbb{S}^1,\mathbb{R}^n)$, then $f|_{\mathbb{S}^1}\in W^{1,2}(\mathbb{S}^1,\mathcal{A}_Q^\pm)$ and

$$\mathrm{Dir}(f|_{\mathbb{S}^1},\mathbb{S}^1)=\frac{2}{2Q-1}\int_{\mathbb{S}^1}|\partial_\tau\zeta|^2,\tag{2.15}$$

where ∂_τ denotes the tangential derivative on $\mathbb{S}^1$.

Proof. We define the following subsets of the unit disk,

$$C=\left\{re^{i\theta}:0<r<1,\theta\neq 0\right\},$$

$$C^+=\left\{re^{i\theta}:0<r<1,0<\theta<\pi\right\},\quad C^-=\left\{re^{i\theta}:0<r<1,\pi<\theta<2\pi\right\};$$

$$\mathcal{D}_j=\left\{re^{i\theta}:0<r<1,\ \frac{2\pi}{2Q-1}2(j-1)<\theta<\frac{2\pi}{2Q-1}2j\right\},\quad j=1,\cdots,Q-1,$$

$$\mathcal{D}_j^+=\left\{re^{i\theta}:0<r<1,\ \frac{2\pi}{2Q-1}2(j-1)<\theta<\frac{2\pi}{2Q-1}(2j-1)\right\},\quad j=1,\cdots,Q,$$

$$\mathcal{D}_j^-=\left\{re^{i\theta}:0<r<1,\ \frac{2\pi}{2Q-1}(2j-1)<\theta<\frac{2\pi}{2Q-1}2j\right\},\quad j=1,\cdots,Q-1.$$

For $j=1,\cdots,Q-1$, we define $\varphi_j:C\to\mathcal{D}_j$ as

$$\varphi_j(re^{i\theta})=r^{\frac{2}{2Q-1}}e^{i\left(\frac{2\theta}{2Q-1}+\frac{4\pi}{2Q-1}(j-1)\right)};$$

and we define $\varphi_Q:C^+\to\mathcal{D}_Q^+$ as

$$\varphi_Q(re^{i\theta})=r^{\frac{2}{2Q-1}}e^{i\left(\frac{2\theta}{2Q-1}+\frac{4\pi}{2Q-1}(Q-1)\right)}.$$

Then

$$f^+=\sum_{j=1}^{Q}[\![\zeta\circ\varphi|_{C^+}]\!]\text{ and }f^-=\sum_{j=1}^{Q-1}[\![\zeta\circ\varphi|_{C^-}]\!].$$

Since $re^{i\theta} \mapsto r^{\frac{2}{2Q-1}} e^{i\frac{2\theta}{2Q-1}}$ is a conformal map, each φ_j is conformal. So by the invariance of the Dirichlet energy under conformal mappings, we deduce that $f^+ \in W^{1,2}(C^+, \mathcal{A}_Q)$, $f^- \in W^{1,2}(C^-, \mathcal{A}_{Q-1})$ and

$$\begin{aligned}\mathrm{Dir}(f, C) = \mathrm{Dir}(f^+, C^+) + \mathrm{Dir}(f^-, C^-) &= \sum_{j=1}^{Q} \mathrm{Dir}(\zeta \circ \varphi_j, C^+) + \sum_{j=1}^{Q-1} \mathrm{Dir}(\zeta \circ \varphi_j, C^-) \\ &= \sum_{j=1}^{Q} \mathrm{Dir}(\zeta, \mathcal{D}_j^+) + \sum_{j=1}^{Q-1} \mathrm{Dir}(\zeta, \mathcal{D}_j^-) = \mathrm{Dir}\left(\zeta, \cup_{j=1}^{Q-1} \mathcal{D}_j \cup \mathcal{D}_Q^+\right) = \iint_{\mathbb{D}} |D\zeta|^2.\end{aligned}$$

On the other hand, since

$$\partial_\tau \left(\zeta \circ \varphi_j\right) = \partial_\theta \left(\zeta \circ \varphi_j\right) = \frac{2}{2Q-1} \partial_\tau \zeta \circ \varphi_j,$$

we have

$$\mathrm{Dir}(\zeta \circ \varphi_j|_{\mathbb{S}^1}, (\mathbb{S}^1)^+) = \int_{(\mathbb{S}^1)^+} \left(\frac{2}{2Q-1}\right)^2 |\partial_\tau \zeta \circ \varphi_j|^2 = \frac{2}{2Q-1} \int_{\frac{2\pi}{2Q-1}2(j-1)}^{\frac{2\pi}{2Q-1}(2j-1)} |\partial_\tau \zeta|^2.$$

An entirely analogous computations on $(\mathbb{S}^1)^-$ makes it straightforward to show that $f|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q^\pm)$ and

$$\mathrm{Dir}(f|_{\mathbb{S}^1}, \mathbb{S}^1) = \frac{2}{2Q-1} \int_{\mathbb{S}^1} |\partial_\tau \zeta|^2. \qquad \square$$

3. Hölder continuity at the interface

In this section we prove the Hölder regularity Theorem 1.7, whose conclusion we make more quantitative in the following statement.

Theorem 3.1 (Boundary Hölder regularity of Dir-minimizer, analogue of Theorem 3.9 [15])**.** *For every $0 < \delta < \frac{1}{2}$, there exist constant $\alpha = \alpha(m, Q) \in (0,1)$ and $C = C(m, n, Q, \delta)$ with the following property. Assume that γ is a C^1 graph of a function ζ over $\mathbb{R}$ passing through the origin with $\|\zeta\|_{C^1} \leq 1$ and that $\varphi \in C^1(\gamma)$. If $f \in W^{1,2}(B_1, \mathcal{A}_Q^\pm)$ is Dir-minimizing with interface (γ, φ), then*

$$\begin{aligned}[f]_{C^{0,\alpha}(\overline{B}_\delta)} &:= \max\left\{ \sup_{x,y \in \overline{B_\delta^+}} \frac{\mathcal{G}(f^+(x), f^+(y))}{|x-y|^\alpha}, \sup_{x,y \in \overline{B_\delta^-}} \frac{\mathcal{G}(f^-(x), f^-(y))}{|x-y|^\alpha} \right\} \\ &\leq C\, \mathrm{Dir}(f, B_1)^{\frac{1}{2}} + C\|D\varphi\|_{C^0}. \end{aligned} \tag{3.2}$$

The proof consists of two main steps. A comparison argument is used to prove a suitable decay of the Dirichlet energy on balls with vanishing radius. the decay is then combined with a Campanato-Morrey estimate to show Hölder regularity.

3.1. **Campanato-Morey estimate.** We first record the following extension of a classical result by Morrey. In the case of Q-valued maps we refer to [15]. In our case we need a suitable additional argument to treat the case of $\left(Q-\frac{1}{2}\right)$-valued functions.

Lemma 3.3 (Campanato-Morrey estimate)**.** *Suppose $(f^+, f^-) \in W^{1,2}(B_1, \mathcal{A}_Q^{\pm})$ is a map with interface (γ, φ) as in Theorem 3.1. If there exists $\beta \in (0,1]$ and $A \geq 0$ such that*

$$\iint_{B_r(y)} |Df|^2 \leq A r^{m-2+2\beta} \quad \text{for every } y \in B_1 \text{ and almost every } r \in (0, 1-|y|), \tag{3.4}$$

then for every $0 < \delta < 1$, there is a constant $C = C(m, \beta, \delta, \gamma)$ such that

$$[f]_{C^{0,\beta}(\overline{B}_\delta)} \leq C\sqrt{A} + C\delta^{1-\beta}\|D\varphi\|_{C^0}\,.$$

Proof. We first extend (f^+, f^-) to a function $g : B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ as follows. We use the C^1 regularity of γ and φ to extend φ to a C^1 function ϕ over B_1 satisfying the estimate $\|D\phi\|_{C^0(B_1)} \leq C\|D\varphi\|_{C^0(\gamma)}$, where C depends on m and the C^1-norm of γ.

$$g(x) := \begin{cases} f^+(x), & x \in B_1^+ \cup \gamma, \\ f^-(x) + [\![\phi(x)]\!], & x \in B_1^-. \end{cases} \tag{3.5}$$

Since $f^+|_\gamma = f^-|_\gamma + [\![\varphi]\!]$, the function g belongs to $W^{1,2}(B_1, \mathcal{A}_Q)$, by the trace theory of [15]. Moreover, the theory in [15] can be easily used to prove that

$$\iint_{B_r} |Dg|^2 = \iint_{B_r} |Df|^2 + \iint_{B_r^-} |D\phi|^2 \leq r^{m-2+2\beta}\left(A + r^{2-2\beta}\|D\phi\|^2_{C^0}\right).$$

By the Camapanato-Morrey estimate for Q-valued functions (see Proposition 2.14 in [15]), we conclude that

$$\sup_{x,y\in\overline{B_\delta}} \frac{\mathcal{G}(g(x), g(y))}{|x-y|^\beta} \leq C\left(\iint_{B_1} |Dg|^2\right)^{\frac{1}{2}}.$$

Since clearly $\mathcal{G}(g(x), g(y)) = \mathcal{G}(f^+(x), f^+(y))$ for every $x, y \in B_1^+$, we conclude the desired esimate on the Hölder continuity of f^+. The one for f^- is slightly more subtle. Consider indeed two points $x, y \in B_1^-$. It then turns out that there are $i, j \in \{1, Q-1\}$ and an invertible map $\sigma : \{1, \ldots, Q-1\} \setminus \{j\} \to \{1, \ldots, Q-1\} \setminus \{i\}$ with the property that

$$\mathcal{G}(g(x), g(y))^2 = |\phi(x) - f_i^-(y)|^2 + |f_j^-(x) - \phi(y)|^2 + \sum_{k\in\{1,\ldots,Q-1\}\setminus\{j\}} |f_k^-(x) - f_{\sigma(k)}^-(y)|^2\,.$$

Observe therefore that, by the triangle inequality

$$|f_j^-(x) - f_i^-(y)| \leq |\phi(x) - \phi(y)| + 2\mathcal{G}(g(x), g(y))\,.$$

In particular, using the obervation

$$\mathcal{G}(f^-(x), f^-(y))^2 \leq |f_j^-(x) - f_i^-(y)|^2 + \sum_{k\in\{1,\ldots,Q-1\}\setminus\{j\}} |f_k^-(x) - f_{\sigma(k)}^-(y)|^2\,,$$

we achieve

$$\mathcal{G}(f^-(x), f^-(y))^2 \leq 2|\phi(x) - \phi(y)|^2 + 5\mathcal{G}(g(x), g(y))^2 \leq 2\|D\phi\|^2_{C^0}|x-y|^2 + 5\mathcal{G}(g(x), g(y))^2\,.$$

Combinining the latter inequality with the estimate for $[g]_{C^{0,\beta}}$ we conclude the desired estimate for the Hölder seminorm of f^-. □

3.2. **Almgren's retractions and maximum principle.** An important tool in proving the decay of the Dirichlet energy for Q-valued minimizers is a family of retraction maps which can be used, for instance, to prove suitable generalizations of the classical maximum principle for harmonic functions. These maps were introduced by Almgren in his pioneering work and we refer to [15] for an elementary account of them. In order to deal with $\left(Q - \frac{1}{2}\right)$-maps we need an additional property of such retractions, which is not recorded in [15] (nor in [2]) We start by recalling the following notation:

Definition 3.6 (Diameter and separation)**.** Let $T = \sum_i [\![P_i]\!] \in \mathcal{A}_Q$. The *diameter* and *separation* of T are defined, respectively, as

$$d(T) := \max_{i,j} |P_i - P_j| \text{ and } s(T) := \min\{|P_i - P_j| : P_i \neq P_j\},$$

with the convention that $s(T) = +\infty$ if $T = Q[\![P]\!]$.

For $Y = \sum_i [\![P_i]\!]$ we denote by $\mathrm{spt}\,(T)$ the set of points $\{P_1, \ldots, P_Q\} \subset \mathbb{R}^n$. Clearly

$$\mathrm{dist}(\mathrm{spt}(T), q) = \min_i |P_i - q|. \tag{3.7}$$

We have a triangle inequality

$$\mathrm{dist}(\mathrm{spt}(T), q) \leq \mathrm{dist}(\mathrm{spt}(S), q) + \mathcal{G}(T, S), \qquad \text{for every } T, S \in \mathcal{A}_Q. \tag{3.8}$$

Lemma 3.9. *Let $T \in \mathcal{A}_Q$ and $r < s(T)/4$. Then there exists a retraction $\vartheta : \mathcal{A}_Q \to \overline{B_r(T)}$ such that*

(i) $\mathcal{G}(\vartheta(S_1), \vartheta(S_2)) < \mathcal{G}(S_1, S_2)$ if $S_1 \notin \overline{B_r(T)}$,
(ii) $\vartheta(S) = S$ for every $S \in \overline{B_r(T)}$,
(iii) If a point q belongs to $\mathrm{spt}(T)$ and to $\mathrm{spt}(S)$, then it belongs to $\mathrm{spt}(\vartheta(S))$ too.

Proof. We define ϑ in the same way as Lemma 3.7 of [15]. The properties (i) and (ii) are proved in [15, Lemma 3.7] whereas (iii) is an obvious definition of the explicit formula given in there. □

Proposition 3.10 (Maximum principle)**.** *Let $f \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ be a Dir-minimizer with interface $(\gamma, 0)$. Suppose $T \in \mathcal{A}_Q$, $0 \in \mathrm{spt}\,(T)$ and $0 < r < s(T)/4$. If*

$$\mathcal{G}(f(x), T) \leq r \qquad \textit{for } \mathcal{H}^{m-1}\textit{-a.e. } x \in (\partial\Omega)^+ \textit{ and} \tag{3.11}$$

$$\mathcal{G}(f(x) + [\![0]\!], T) \leq r \qquad \textit{for } \mathcal{H}^{m-1}\textit{-a.e. } x \in (\partial\Omega)^-, , \tag{3.12}$$

then

$$\mathcal{G}(f, T) \leq r \qquad \textit{a.e. in } \Omega^+ \textit{ and} \tag{3.13}$$

$$\mathcal{G}(f + [\![0]\!], T) \leq r \qquad \textit{a.e. in } \Omega^-\,. \tag{3.14}$$

Proof. We argue by contradiction. Suppose $f \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ is a Dir-minimizer with interface $(\gamma, 0)$ satisfying (3.11) and (3.12) and assume in addition that there exists a set of positive measure $E \subset \Omega$, such that $f(x) \notin \overline{B_r(T)}$ for every $x \in E \cap \Omega^+$ and $f(x) + \{0\} \notin \overline{B_r(T)}$ for every $x \in E \cap \Omega^-$.

In particular there exist $\delta > 0$ and a set $E' \subset E$ with positive measure such that $f(x) \notin \overline{B_{r+\delta}(T)}$ for every $x \in E' \cap \Omega^+$ and $f(x) + [\![0]\!] \notin \overline{B_{r+\delta}(T)}$ for every $x \in E' \cap \Omega^-$. As in the proof of Lemma 3.3 we consider the Q-valued function on Ω which coincides with f^+ on Ω^+ and with $f + [\![0]\!]$ in Ω^-. Let $\vartheta : \mathcal{A}_Q \to \overline{B_r(T)}$ be the retraction operator in Lemma 3.9. By (iii) $\mathrm{spt}(\vartheta \circ g(x))$ contains the origin for every $x \in \Omega^-$. We can thus consider the $(Q-1)$-valued function on Ω^- given by $\vartheta \circ g - [\![0]\!]$. If we set $h^+ = \vartheta \circ g$ on Ω^+ we then get a $(Q - \frac{1}{2})$-valued map (h^+, h^-) with interface $(\gamma, 0)$. By Lemma 3.9(ii) we also know that $h^{\pm} = f^{\pm}$ on $(\partial\Omega)^{\pm}$. Therefore $h = (h^+, h^-)$ is a suitable competitor for $f = (f^{\cdot} f^-)$. On the other hand, by Lemma 3.9 (i) we know $|D(\vartheta \circ f)| \leq |Df|$ a.e. on Ω and moreover, recalling the definition of ϑ by linear interpolation and that $\mathcal{G}(f(x), T) > r + \delta$, we get that

$$|D(\vartheta \circ f)| \leq t_0 |Df| < |Df| \quad \text{a.e. on } E', \tag{3.15}$$

where $t_0 \leq \frac{r-\delta}{r+\delta} < 1$. Here we compute the partial derivatives by the first order approximation, see the definition and discussions in Definition 1.9 Corollary 2.7 and Proposition 2.17 of [15]. We conclude that $\mathrm{Dir}(h, \Omega) < \mathrm{Dir}(f, \Omega)$, contradicting the minimality of f. □

3.3. **Decomposition.** The maximum principle of the previous section triggers a decomposition lemma for Dir-minimizers with $(\gamma, 0)$ interface.

Proposition 3.16 (Decomposition of $(Q - \frac{1}{2})$-valued Dir-minimizers)**.** *There exists a positive constant $\alpha(Q) > 0$ with the following property. Assume that $f \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ is a Dir-minimizer with interface $(\gamma, 0)$, and that there exists $T \in \mathcal{A}_Q$ with $0 \in \mathrm{spt}\,(T)$ such that (3.11) and (3.12) hold with $r = \alpha(Q) f(T)$. Then there exists a decomposition $f = (f^+, f^-) = (g^+ + h, g^- + h)$, where h is a Q_1-valued Dir-minimizer, (g^+, g^-) a $(Q_2 - \frac{1}{2})$ Dir-minimizer with interface $(\gamma, 0)$, $Q_1 + Q_2 = Q$ and $1 \leq Q_1 \leq Q - 1$.*

Proof. When $d(T) = 0$, our assumption implies $\mathcal{G}(f(x), T) = 0$, namely $f \equiv T$, and there is nothing to prove. So we assume $d(T) > 0$. If $\alpha(Q) d(T) < s(T)/4$ (for a fixed value of $\alpha(Q)$), the proposition follows directly by the maximum principle and the definition of $s(T)$. Suppose therefore $4\alpha(Q) d(T) \geq s(T)$. We fix a positive real number ϵ so that

$$\left(\sqrt{Q} + 2\right) \frac{\epsilon}{1 - \epsilon} = \frac{1}{8}.$$

Recalling Lemma 3.8 of [15], we may *collapse* some points in the support T and find an element $S = \sum_{j=1}^{J} k_j [\![S_j]\!] \in \mathcal{A}_Q$ (with $J \geq 2$) satisfying

$$\beta(\epsilon, Q) d(T) \leq s(S) < +\infty, \tag{3.17}$$

$$\mathcal{G}(S, T) \leq \epsilon s(S). \tag{3.18}$$

We set $\alpha(Q) = \epsilon \beta(\epsilon, Q)$, so that

$$\mathcal{G}(f(x), T) \leq \alpha(Q) d(T) \leq \epsilon s(S) \quad \text{for } \mathcal{H}^{m-1}\text{-a.e. } x \in \partial\Omega. \tag{3.19}$$

Since $0 \in \mathrm{spt}(T)$, we have, by the triangle inequality (3.8),

$$\mathrm{dist}(\mathrm{spt}\,(S), 0) \le \min \mathrm{dist}(\mathrm{spt}\,(T), 0) + \mathcal{G}(S, T) \le \epsilon s(S).$$

Without loss of generality, we assume $|S_1| = \mathrm{dist}(\mathrm{spt}(S), 0)$. Let $\widetilde{S} = k_1[\![0]\!] + \sum_{j=2}^{J} k_j[\![S_j]\!]$. Clearly

$$\mathcal{G}(S, \widetilde{S}) = \sqrt{k_1|S_1|^2} \le \sqrt{Q}\min S \le \epsilon\sqrt{Q}\,s(S). \tag{3.20}$$

On the other hand $s(\widetilde{S}) \ge (1-\epsilon)s(S)$. In fact, either $s(\widetilde{S}) = |S_i - S_j|$ for some $i, j \neq 1$, in which case $s(\widetilde{S}) \ge s(S)$; or $s(\widetilde{S}) = |S_i|$ for some $i \neq 1$, and then

$$s(\widetilde{S}) = |S_i| \ge |S_i - S_1| - |S_1| \ge s(S) - \epsilon s(S). \tag{3.21}$$

Combining (3.19), (3.18), (3.20), (3.21) and the choice of ϵ, we conclude

$$\begin{aligned}\mathcal{G}(f(x), \widetilde{S}) &\le \mathcal{G}(f(x), T) + \mathcal{G}(S, T) + \mathcal{G}(S, \widetilde{S}) \le \epsilon s(S) + \epsilon s(S) + \epsilon\sqrt{Q}\,s(S)\\ &\le \left(\sqrt{Q} + 2\right)\frac{\epsilon}{1-\epsilon}\,s(\widetilde{S}) = \frac{1}{8}s(\widetilde{S}),\end{aligned}$$

for $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$. Again it follows by the maximum principle that $\mathcal{G}(f, \widetilde{S}) \le s(\widetilde{S})/8$ almost everywhere on Ω. We thus have a decomposition of f into simpler multiple-valued functions. □

3.4. **Interpolation that preserves the interface value.** In this subsection, we construct interpolations between pairs of $(Q - \frac{1}{2})$ maps with a common interface $(\gamma, 0)$ defined on concentric spheres and estimate its Dirichlet energy. Later we will use the interpolation to construct competitors for Dir-minimizing maps, so it is crucial that the interpolation has the same interface $(\gamma, 0)$. This is also the major difference from the interior case, proved in Lemma 2.15 in [15]. For our current purpose, namely the proof of the decay of the Dirichlet energy for minimizers, we actually need the existence of the interpolation only in the case $m \ge 3$. However later on Lemma 3.31 will be used on planar maps to show the compactness of minimizers, a crucial point in the proof of Theorem 1.6. We therefore state and proof also the 2-dimensional case (separately).

Lemma 3.22 (Interpolation when $m = 2$)**.** *Let f, g be maps in $W^{1,2}(\partial B_1, \mathcal{A}^{\pm}_Q(\mathbb{R}^n))$ satisfying*

$$f^+|_\gamma = f^-|_\gamma + [\![0]\!], \quad g^+|_\gamma = g^-|_\gamma + [\![0]\!], \tag{3.23}$$

and $\sup_{x\in\partial B_1} \mathcal{G}(f(x), g(x)) < +\infty$. Let $\delta = \frac{1}{N}$ for some $N \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. Then there exists $h \in W^{1,2}(B_1 \setminus B_{1-\delta}, \mathcal{A}^{\pm}_Q(\mathbb{R}^n))$ satisfying $h^+|_\gamma = h^-|_\gamma + [\![0]\!]$ and

$$h(x) = f(x) \text{ for } x \in \partial B_1, \quad h(x) = g\left(\frac{1}{1-\delta}x\right) \text{ for } x \in \partial B_{1-\delta}.$$

Moreover

$$\mathrm{Dir}(h, B_1 \setminus B_{1-\delta}) \le C\delta\,\mathrm{Dir}(f, \partial B_1) + C\delta\,\mathrm{Dir}(g, \partial B_1) + \frac{C}{\delta}\sup_{x\in\partial B_1} \mathcal{G}(f(x), g(x)). \tag{3.24}$$

Proof. By applying a diffeomorphism, we can assume that $\gamma = \mathbb{R}$. We first interpolate f^+ and g^+ in the upper half annulus $B_1^+ \setminus B_{1-\delta}^+$. After parametrizing a biLipschitz diffeomorphism $\phi : [0, 1] \to \partial B_1^+$ to the functions f^+ and g^+, we may assume f^+, g^+ are $W^{1,2}$ maps defined on $[0, 1]$. We will interpolate f^+ and g^+ and get a $W^{1,2}$ map on $[0, 1] \times [0, \delta]$.

We define a cubical decomposition $D_i = [i\delta, (i+1)\delta] \times [0, \delta]$ with $i = 0, 1, \cdots, N-1$, and vertical lines $\ell_i = \{i\delta\} \times [0, \delta]$ with $i = 0, 1, \cdots, N$. For $i = 1, 2, \cdots, N-1$, we define

$$h(x,t) = \boldsymbol{\xi}^{-1} \circ \boldsymbol{\rho}\left(\left(1 - \frac{t}{\delta}\right)\boldsymbol{\xi} \circ g^+(x) + \frac{t}{\delta}\boldsymbol{\xi} \circ f^+(x)\right), \quad (x,t) \in \ell_i,$$

where $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ is the embedding of Q-valued metric space, and $\boldsymbol{\rho} : \mathbb{R}^N \to \boldsymbol{\xi}(\mathcal{A}_Q)$ is the retraction, see Theorem 2.1 [15]. It is clear that

$$|Dh(x,t)| \leq \frac{C}{\delta}\mathcal{G}(g^+(x), f^+(x)), \tag{3.25}$$

where the constant depends on the Lipschitz constants of $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. For $i = 0$ or N, $x = i\delta$, recall (3.32) we denote

$$g^+(x) = \sum_{j=1}^{Q} [\![a_j]\!] = [\![0]\!] + g^-(x), \quad f^+(x) = \sum_{j=1}^{Q} [\![b_j]\!] = [\![0]\!] + f^-(x)$$

and $f^+(x) = \sum_{j=1}^{Q} [\![b_j]\!]$. Here we assume $a_1 = b_1 = 0$ without loss of generality. Suppose τ is a permutation of $\{2, \cdots, Q\}$ such that

$$\mathcal{G}(g^-(x), f^-(x)) = \sqrt{\sum_{j=2}^{Q} |a_j - b_{\tau(j)}|^2}.$$

We define

$$h(x,t) = [\![0]\!] + \sum_{j=2}^{Q} \left[\!\!\left[\left(1 - \frac{t}{\delta}\right)a_j + \frac{t}{\delta}b_{\tau(j)}\right]\!\!\right]. \tag{3.26}$$

(3.32) implies that

$$\mathcal{G}(g^+(x), f^+(x)) \leq \mathcal{G}(g^-(x), f^-(x)) \leq \sqrt{2}\mathcal{G}(g^+(x), f^+(x)). \tag{3.27}$$

Hence

$$|Dh(x,t)| = \frac{1}{\delta}\sqrt{\sum_{j=2}^{Q} |a_j - b_{\tau(j)}|^2} = \frac{1}{\delta}\mathcal{G}(g^-(x), f^-(x)) \leq \frac{\sqrt{2}}{\delta}\mathcal{G}(g^+(x), f^+(x)). \tag{3.28}$$

In this way h is well-defined for each ∂D_i. We now wish to use (3.39) (and a biLipschitz homeomorphism of squares to disks) and claim the existence of an extension h on D_i satisfying

$$\mathrm{Dir}(h, D_i) \leq C\delta\,\mathrm{Dir}(h, \partial D_i). \tag{3.29}$$

Note that this can be done because the proof of (3.39) given later in the planar case is *not* using the current proposition (it uses interpolation, however, if the domain is at least 3-dimensional). Summing up we get

$$\mathrm{Dir}(h, [0,1] \times [0,\delta]) = \sum_{i=0}^{N-1} \mathrm{Dir}(h, D_i)$$

$$\leq C\delta\left(\mathrm{Dir}(h,[0,1]\times\{0\})+\mathrm{Dir}(h,[0,1]\times\{\delta\})+\sum_{i=0}^{N}\mathrm{Dir}(h,\ell_i)\right)$$
$$\leq C\delta\,\mathrm{Dir}(g,[0,1])+C\delta\,\mathrm{Dir}(f,[0,1])+C\sum_{i=0}^{N}\mathcal{G}(g^+(i\delta),f^+(i\delta))$$
$$\leq C\delta\,\mathrm{Dir}(g,[0,1])+C\delta\,\mathrm{Dir}(f,[0,1])+\frac{C}{\delta}\sup_{x\in[0,1]}\mathcal{G}(g^+(x),f^+(x)).$$

Applying the biLipschitz homeomorphism $\phi : [0,1] \to \partial B_1^+$, we get an interpolation $h^+ \in W^{1,2}(B_1^+ \setminus B_{1-\delta}^+, \mathcal{A}_Q)$.

Similarly, we define an interpolation $h^- \in W^{1,2}(B_1^- \setminus B_{1-\delta}^+, \mathcal{A}_{Q-1})$ between g^- and f^-. By (3.32) and the construction (3.26), we know $h^+|_\gamma = h^-|_\gamma + [\![0]\!]$, $h \in W^{1,2}(B_1 \setminus B_{1-\delta}, \mathcal{A}_Q^\pm)$ and moreover

$$\mathrm{Dir}(h,B_1\setminus B_{1-\delta})\leq C\delta\,\mathrm{Dir}(g,B_1)+C\delta\,\mathrm{Dir}(f,B_1)+\frac{C}{\delta}\sup_{x\in\partial B_1}\mathcal{G}(g(x),f(x)). \tag{3.30}$$

□

Lemma 3.31 (Interpolation when $m \geq 3$)**.** *Let f, g be maps in $W^{1,2}(\partial B_1, \mathcal{A}_Q^\pm(\mathbb{R}^n))$ satisfying*

$$f^+|_\gamma = f^-|_\gamma + [\![0]\!], \quad g^+|_\gamma = g^-|_\gamma + [\![0]\!], \tag{3.32}$$

and $\int_{\partial B_1}\mathcal{G}(f,g) < +\infty$. Let $\delta = \frac{1}{N}$ for some $N \in \mathbb{N}\setminus\{0,1,2,3\}$. Then there exists $h \in W^{1,2}(B_1 \setminus B_{1-\delta}, \mathcal{A}_Q^\pm(\mathbb{R}^n))$ satisfying $h^+|_\gamma = h^-|_\gamma + [\![0]\!]$ and

$$h(x) = f(x) \text{ for } x\in\partial B_1, \quad h(x) = g\left(\frac{1}{1-\delta}x\right) \text{ for } x \in \partial B_{1-\delta}.$$

Moreover

$$\mathrm{Dir}(h,B_1\setminus B_{1-\delta})\leq C\delta\,\mathrm{Dir}(f,\partial B_1)+C\delta\,\mathrm{Dir}(g,\partial B_1)+\frac{C}{\delta}\int_{\partial B_1}\mathcal{G}(f,g). \tag{3.33}$$

Proof. By applying a diffeomorphism, we can assume that $\gamma = \{x_m = 0\}$. Let C be the boundary of the cube $[-1,1]^m$. Notice that C is tangent to the sphere ∂B_1. We define the functions $\hat{f}$ and $\hat{g}$ on C by radial projection:

$$\hat{f}(z) := f\left(\frac{z}{|z|}\right), \quad \hat{g}(z) := g\left(\frac{z}{|z|}\right), \quad \text{for every } z \in C.$$

After the radial projection, the tangential derivative on C at z is just a multiple of the tangential derivative on ∂B_1 at $z/|z|$, where the factor is uniformly bounded above and below by dimensional constants. In particular $\hat{f}, \hat{g} \in W^{1,2}(C, \mathcal{A}_Q^\pm)$, that is,

$$\hat{f}^+, \hat{g}^+ \in W^{1,2}(C^+, \mathcal{A}_Q), \quad \hat{f}^-, \hat{g}^- \in W^{1,2}(C^-, \mathcal{A}_{Q-1}),$$

and

$$\hat{f}^+|_\gamma = \hat{f}^-|_\gamma + [\![0]\!], \quad \hat{g}^+|_\gamma = \hat{g}^-|_\gamma + [\![0]\!],$$

where $C^+ = C \cap \{x_m > 0\}$, $C^- = C \cap \{x_m < 0\}$ and $\gamma = \{x_m = 0\}$. We want to construct a function $\hat{h} : C \times [0, \delta] \to \mathcal{A}_Q^{\pm}$ which satisfies $\hat{h}(\cdot, 0) = \hat{g}$, $\hat{h}(\cdot, \delta) = \hat{f}$, $\hat{h} \in W^{1,2}$ and

$$\hat{h}^+|_{\gamma\times[0,\delta]} = \hat{h}^-|_{\gamma\times[0,\delta]} + [\![0]\!];$$

and in turn, we define a function $h : B_1 \setminus B_{1-\delta} \to \mathcal{A}_Q^{\pm}$ by

$$h\left(t\frac{z}{|z|}\right) := \hat{h}\left(z, t-(1-\delta)\right), \qquad \text{for each } z \in C \text{ and } 1-\delta < t < 1,$$

such that $h \in W^{1,2}(B_1 \setminus B_{1-\delta}, \mathcal{A}_Q^{\pm})$ with the desired boundary data.

Let F be any of the $2m$ faces of C, then it is an $(m-1)$-dimensional solid cube (i.e. including the interior) with side length 2. Take for example

$$F = \left\{(-1, x_2, \cdots, x_m) : -1 \le x_j \le 1 \text{ for every } j = 2, \cdots, m\right\}.$$

We will first define $\hat{h}$ on $F \times [0, \delta]$ using the similar construction as in the interior case, see Step 1 of Lemma 4.12 in [15] and the erratum therein. To that end we first need to extend $\hat{f}$ and $\hat{g}$ to a fatter region

$$F_\delta := \left\{(-1, x_2, \cdots, x_m) : -1-\delta \le x_j \le 1+\delta \text{ for every } j = 2, \cdots, m\right\},$$

by using their respective values on neighboring faces of F and scaling appropriately on the corners. For example, for any $x_2 \in [-1-\delta, -1)$ fixed (the other possibility being $x_2 \in (1, 1+\delta]$), we consider the slice

$$S_{x_2} := \left\{(-1, x_2, x_3, \cdots, x_m) : -|x_2| \le x_j \le |x_2| \text{ for every } j = 3, \cdots, m\right\} \subset F_\delta,$$

and define $\hat{f}, \hat{g}$ by their values on a neighboring face of F:

$$F' := \left\{(x_1, -1, x_3, \cdots, x_m) : -1 \le x_j \le 1 \text{ for every } j = 1, 3, \cdots, m\right\}. \tag{3.34}$$

To be precise on S_{x_2} we define

$$\hat{f}(-1, x_2, x_3, \cdots, x_m) := \hat{f}(|x_2| - 2,\ -1,\ \varphi_\delta(x_3), \cdots, \varphi_\delta(x_m)) \tag{3.35}$$

where $\varphi_\delta : [-|x_2|, |x_2|] \to [-1, 1]$ is a piecewise linear function as follows

$$\varphi_\delta(t) = \begin{cases} -1 + \dfrac{\delta}{-1+\delta+|x_2|}(t + |x_2|), & -|x_2| \le t \le -1+\delta \\ t, & -1+\delta \le t \le 1-\delta \\ 1 + \dfrac{\delta}{-1+\delta+|x_2|}(t - |x_2|), & 1-\delta \le t \le |x_2|. \end{cases} \tag{3.36}$$

That is, in the inner region of S_{x_2}, $\hat{f}$ (and $\hat{g}$) take the value on F' faithfully; in the outer region $\hat{f}$ (and $\hat{g}$) is a scaled version of its value on F', with a scaling factor at most 2. The former is to guarantee that the construction of $\hat{h}$ remains faithful to $\hat{f}, \hat{g}$ near the boundary $\gamma \times [0, \delta]$.

For any vector $v \in [-1-\delta, -1]^{m-1}$, consider the cubical decomposition of F_δ induced by the lattice points $\{-1\} \times \left(v + \delta\mathbb{Z}^{m-1}\right)$. For $k \in \{0, \cdots, m-1\}$ we define accordingly the k-dimensional skeleta contained in F_δ, which are the families $\mathcal{S}^k(v)$ of all closed k-dimensional faces of the cubes. By Fubini, for almost every v and face $E \in \mathcal{S}^k(v)$, we have that $\hat{f}|_E, \hat{g}|_E \in W^{1,2}$, and moreover

$$\int_{v\in[-1-\delta,-1]^{m-1}} \left(\sum_{E\in\mathcal{S}^k(v)} \int_E \left(|D\hat{f}|^2 + |D\hat{g}|^2 + \mathcal{G}(\hat{f},\hat{g})^2\right) \right) dv \le C(k,m)\delta^k \int_{F_\delta} \left(|D\hat{f}|^2 + |D\hat{g}|^2 + \mathcal{G}(\hat{f},\hat{g})^2\right).$$

By standard arguments we can choose a vector v such that

- For every $k \ge 1$, for each $E \in \mathcal{S}^k(v)$ and each $G \in \mathcal{S}^{k-1}(v)$ with $G \subset E$, the restrictions $\hat{f}|_E, \hat{f}|_G, \hat{g}|_E, \hat{g}|_G$ are all $W^{1,2}$ and moreover the traces of $\hat{f}|_E$ and $\hat{g}|_E$ on G are precisely $\hat{f}|_G$ and $\hat{g}|_G$;
- For every $k \ge 1$,
$$\sum_{E\in\mathcal{S}^k(v)} \int_E \left(|D\hat{f}|^2 + |D\hat{g}|^2\right) \le C\delta^{k-(m-1)} \int_{F_\delta} \left(|D\hat{f}|^2 + |D\hat{g}|^2\right);$$
- For $k = 0$,
$$\sum_{p\in\mathcal{S}^0(v)} \mathcal{G}(\hat{f}(p),\hat{g}(p))^2 \le C\delta^{-(m-1)} \int_{F_\delta} \mathcal{G}(\hat{f},\hat{g})^2;$$
- Whenever $E \in \mathcal{S}^k(v)$ intersects γ, the center of E, denoted by x_E, lies in C^+, tin other words x_E lies above the boundary γ.

For any $k = 0, \cdots, m-1$ and any $E \in \mathcal{S}^k(v)$ not intersecting γ, we follows the same construction as in the interior case (for Q-valued or $(Q-1)$-valued functions) and define $\hat{h}$ on $E \times [0,\delta]$ by interpolation $\hat{f}^+$ and $\hat{g}^+$, or $\hat{f}^-$ and $\hat{g}^-$ respectively. Across the boundary γ, we temporarily extend the functions trivially by zero, that is, we set

$$\hat{f}_0 = \begin{cases} \hat{f}^+, & \text{on } C^+ \\ \hat{f}^- + [\![0]\!], & \text{on } C^-, \end{cases} \qquad \hat{g}_0 = \begin{cases} \hat{g}^+, & \text{on } C^+ \\ \hat{g}^- + [\![0]\!], & \text{on } C^-, \end{cases}$$

so that $\hat{f}_0, \hat{g}_0$ are Q-valued functions. Notice that the values of $|D\hat{f}|$, $|D\hat{g}|$, $\mathcal{G}(\hat{f}^+,\hat{g}^+)$ stay the same, and on C^-

$$\frac{1}{\sqrt{2}}\mathcal{G}(\hat{f}^-,\hat{g}^-) \le \mathcal{G}(\hat{f}_0,\hat{g}_0) \le \mathcal{G}(\hat{f}^-,\hat{g}^-),$$

see (3.27). Recall that for any $p \in \mathcal{S}^0(v)$ contained in C^-, we define $\hat{h}$ on $p \times [0,\delta]$ as a linear interpolation between $\hat{f}$ and $\hat{g}$, and that

$$\mathrm{Dir}(\hat{h}, p \times [0,\delta]) \le \frac{C}{\delta}\mathcal{G}(\hat{f}^-(p),\hat{g}^-(p))^2 \le \frac{C'}{\delta}\mathcal{G}(\hat{f}_0(p),\hat{g}_0(p))^2. \tag{3.37}$$

Now we construct $\hat{h}$ by an induction on the dimension k. Suppose $E \in \mathcal{S}^k(v)$ intersects γ, where $k = 1, \cdots, m-1$. Either by the inductive hypothesis or by the base case $k = 0$ (see (3.37) and assume $\hat{h}_0(p) = \hat{h}(p) + [\![0]\!]$), we assume that for all lower skeleta $G \in \mathcal{S}^{k-1}(v)$ with $G \subset E$, we have defined a Q-valued function $\hat{h}_0$ on $G \times [0, \delta]$ with the desired properties. Since

$$\partial (E \times [0, \delta]) = \bigcup_{\substack{G \in \mathcal{S}^{k-1}(v) \\ G \subset E}} G \times [0, \delta] \bigcup E \times \{0\} \bigcup E \times \{\delta\},$$

we can define $\hat{h}_0$ on $E \times [0, \delta]$ as the 0-homogeneous extension of $\hat{h}_0|_{\partial(E \times [0,\delta])}$. Simple computations show that

$$\mathrm{Dir}(\hat{h}_0, E \times [0, \delta]) \le C\delta \, \mathrm{Dir}\left(\hat{h}_0, \partial (E \times [0, \delta])\right).$$

More importantly, notice that every point on $(E \cap C^-) \times (0, \delta)$ lies in a line segment between the center $x_E \times \{\delta/2\}$ and some point in $(E \cap C^-) \times \{0\}$, $(E \cap C^-) \times \{\delta\}$ or $(G \cap C^-) \times [0, \delta]$ for some $G \in \mathcal{S}^{k-1}(v)$ and $G \subset E$, this construction guarantees that on $C^- \times [0, \delta]$, the Q-valued function $\hat{h}_0$ always has an element $[\![0]\!]$; in particular we may define $\hat{h} \in \mathcal{A}_Q^{\pm}$ accordingly and it satisfies the desired boundary condition. To sum up, we construct a function $\hat{h}_F$ defined on $\widetilde{F}_\delta \times [0, \delta]$, where $F \subset \widetilde{F}_\delta \subset F_\delta$, and it satisfies

$$\hat{h}_F(\cdot, 0) = \hat{g}, \quad \hat{h}_F(\cdot, \delta) = \hat{f} \text{ on } F;$$

$$\hat{h}_F^+(\cdot, t)\Big|_\gamma = \hat{h}_F^-(\cdot, t)\Big|_\gamma + [\![0]\!] \qquad \text{for every } t \in [0, \delta];$$

$$\mathrm{Dir}(\hat{h}_F, F \times [0, \delta]) \le C\delta \, \mathrm{Dir}(\hat{f}, F_\delta) + C\delta \, \mathrm{Dir}(\hat{g}, F_\delta) + \frac{C}{\delta} \int_{F_\delta} \mathcal{G}(\hat{f}, \hat{g})^2.$$

We would like to repeat the same argument for any neighboring face of F, take for example F' as in (3.34); but we need to be careful and make sure the new function $\hat{h}_{F'}$ is consistent with $\hat{h}_F$ on their domains of overlap, since $\hat{h}_F$ is defined on a small neighborhood near $F \cap F'$ by projecting the fatterned region $\widetilde{F}_\delta$ onto F':

$$\mathrm{Ng}(F) := F' \cap \{-1 \le x_1 \le -1 + \delta'\},$$

where $\delta' \in [0, \delta)$ is determined by the choice of v.

We sketch the necessary technical modifications below. As before, we consider a fattened region F'_δ of F'; and we then choose a cubical decomposition of F'_δ to satisfy, in addition to the requirements stated above, that all skeleta (orthogonal to x_1-axis) ought to be at least $\delta/2$-distance away from $\mathrm{Ng}(F)$. On the interior region

$$\mathrm{Ng}^i(F) := \mathrm{Ng}(F) \cap \left\{-1 + \delta \le x_j \le 1 - \delta \text{ for every } j = 3, \cdots, m\right\},$$

we use $\hat{h}_F$ as the boundary condition to construct $\hat{h}_{F'}$ to make sure they agree; outside, on each $(m-1)$-dimensional δ-cube E contained in $\mathrm{Ng}(F) \setminus \mathrm{Ng}^i(F)$, we replace and reconstruct $\hat{h}_F$ on $E \times [0, \delta]$ as above. This way $\hat{h}_F = \hat{h}_{F'}$ on their domains of overlap $\mathrm{Ng}(F)$; moreover, since we do not redefine $\hat{h}_F$ near the boundary $\gamma \times [0, \delta]$, it still satisfies the desired boundary condition. $\square$

3.5. **Decay estimate.** The key point in the proof of Theorem 3.1 is a suitable decay estimate for the Dirichlet energy, which is essentially the content of the following proposition.

Proposition 3.38. *Suppose f is a $(Q-\frac{1}{2})$ Dir-minimizing map on B_1 with interface (γ,φ) and assume that γ is the graph of a function ζ with $\|\zeta\|_{C^1} \leq 1$. Let $0 < r < 1$ and assume that $f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q^{\pm})$. Then we have*

$$\mathrm{Dir}(f, B_r) \leq C(m) r \,\mathrm{Dir}(f, \partial B_r) + C r^m \|D\varphi\|^2_{C^0}\,, \tag{3.39}$$

where $C(m) < (m-2)^{-1}$.

Remark 3.40. By translation, the same estimate holds for any ball $\overline{B_r(y)} \subset B_1$ with $y \in \gamma$. If $\overline{B_r(y)} \cap \gamma = \emptyset$, the analogous interior estimate was proven in Proposition 3.10 of [15].

Proof. We will prove (3.39) for $r = 1$, because the general case follows from a scaling argument. Moreover we will assume, without loss of generality, that $\varphi \equiv 0$. Indeed, for a general φ, we let ϕ be an extension to B_1 with the property that $\|D\phi\|_{C^0(B_1)} \leq C\|D\varphi\|_{C^0(\gamma)}$, since the interface γ is given by the graph of ζ satisfying $\|\zeta\|_{C^1} \leq 1$. Define then (h^+, h^-) as

$$h^{\pm}(x) = \sum_i [\![f^{\pm}(x) - \phi(x)]\!] \,.$$

Moreover, let $k^{\pm}$ be a Dir-minimizer with boundary values $h^{\pm}$ and interface $(\gamma, 0)$ and construct a corresponding competitor for f by setting

$$\bar{g}^{\pm}(x) = \sum_i [\![k^{\pm} + \phi(x)]\!] \,.$$

Observe that for every ε there is a constant $C(\varepsilon)$ such that

$$\begin{aligned}|D_\tau h^{\pm}(x)|^2 &\leq (1+\varepsilon)|D_\tau f^{\pm}(x)|^2 + C(\varepsilon)|D_\tau \phi(x)|^2,\\ |D\bar{g}^{\pm}(x)|^2 &\leq (1+\varepsilon)|Dk^{\pm}(x)|^2 + C(\varepsilon)|D\phi(x)|^2.\end{aligned}$$

Here D_τ denotes the tangential derivative on the boundary ∂B_1. After proving the Proposition for interfaces $(\gamma, 0)$ we will know that there is a constant $C'(m) < \frac{1}{m-2}$ such that

$$\mathrm{Dir}(k, B_1) \leq C'(m)\,\mathrm{Dir}(k, \partial B_1) = C'(m)\,\mathrm{Dir}(h, \partial B_1) \leq C'(m)(1+\varepsilon)\,\mathrm{Dir}(f, \partial B_1) + C(m,\varepsilon)\|D\phi\|^2_{C^0}$$

Hence we could estimate

$$\begin{aligned}\mathrm{Dir}(f, B_1) \leq \mathrm{Dir}(\bar{g}, B_1) &\leq (1+\varepsilon)\,\mathrm{Dir}(k, B_1) + C(\varepsilon)\|D\phi\|^2_{C^0}\\ &\leq C'(m)(1+\varepsilon)^2\,\mathrm{Dir}(f, \partial B_1) + C'(m,\varepsilon)\|D\phi\|^2_{C^0}\,.\end{aligned}$$

Since $C'(m) < \frac{1}{m-2}$ it suffices to choose ε so that $C(m) := C'(m)(1+\varepsilon)^2 < \frac{1}{m-2}$. From now on we restrict therefore our attention to the case $\varphi \equiv 0$.

The planar case. Set $g := f|_{\partial B_1}$ and let $g = g_0 + \sum_{j=1}^J g_j$ be a decomposition into irreducible maps as in Proposition 2.7. Suppose g_0 unwinds to $\zeta_0 : \mathbb{S}^1 \to \mathbb{R}^n$ as in Proposition 2.7 (ii); and each g_j unwinds to a $W^{1,2}$ function $\zeta_j : \mathbb{S}^1 \to \mathbb{R}^n$ as in Proposition 1.5 (ii) of [15]:

$$g_j(x) = \sum_{z^{Q_j}=x} [\![\zeta_j(z)]\!].$$

Now we construct an admissible competitor for f as follows. Recall that $\zeta_0(0) = \zeta_0(2\pi) = 0$, we consider its Fourier expansion

$$\zeta_0(\theta) = \sum_{l=1}^{\infty} c_l \sin\left(\frac{l\theta}{2}\right).$$

We then extend ζ_0 to be a $W^{1,2}$ function defined on all of B_1 as:

$$\overline{\zeta}_0(r,\theta) = \sum_{l=1}^{\infty} r^{\frac{l}{2}} c_l \sin\left(\frac{l\theta}{2}\right).$$

Note that $\overline{\zeta}_0$ is not harmonic, but it vanishes on all of the positive real axis. We also consider the harmonic extension of each ζ_j, denoted by $\overline{\zeta}_j$. Simple computations show that

$$\iint_{\mathbb{D}} |D\overline{\zeta}_0|^2 \le 2\int_{\mathbb{S}^1} |\partial_\tau \zeta_0|^2, \qquad \iint_{\mathbb{D}} |D\overline{\zeta}_j|^2 \le \frac{1}{2}\int_{\mathbb{S}^1} |\partial_\tau \zeta_j|^2. \tag{3.41}$$

We then unroll $\overline{\zeta}_0$ to a $(Q_0 - \frac{1}{2})$-valued function $h_0 = (h_0^+, h_0^-)$ as in Lemma 2.11. By definition, it follows that h_0 satisfies the boundary condition

$$h_0^+|_\gamma = h_0^-|_\gamma + [\![0]\!].$$

We also unroll each $\overline{\zeta}_j$ to Q_j-valued function h_j by

$$h_j(x) = \sum_{z^{Q_j}=x} [\![\overline{\zeta}_j(z)]\!].$$

The function $h = (h_0^+, h_0^-) + \sum_{j=1}^J h_j$ has interface (γ, φ), agrees with f on $\mathbb{S}^1$, and thus is an admissible competitor for f in B_1. Therefore by Lemma 2.11, Lemma 3.12 of [15] and (3.41), we get

$$\begin{aligned}
\mathrm{Dir}(f, B_1) \le \mathrm{Dir}(h, B_1) &= \sum_{j=0}^{J} \mathrm{Dir}(h_j, B_1) = \sum_{j=0}^{J} \iint_{\mathbb{D}} |D\overline{\zeta}_j|^2 \le 2\sum_{j=0}^{J} \int_{\mathbb{S}^1} |\partial_\tau \zeta_j|^2 \\
&= (2Q_0 - 1)\,\mathrm{Dir}(g_0, \mathbb{S}^1) + \sum_{j=1}^{J} 2Q_j\,\mathrm{Dir}(g_j, \mathbb{S}^1) \le 2Q\,\mathrm{Dir}(g, \partial B_1).
\end{aligned}$$

In particular, the above inequality says that the constant in (3.39) satisfies $C(2) = 2Q(1+\epsilon)^2$, and we may assume that $C(2) = 3Q$ for example.

The non-planar case. We define Q-valued functions $\tilde{g}$ and $\tilde{f}$ by adding a "0 sheet" to g^- and f^-, as in (3.5). Observe that $|D\tilde{g}(x)| = |Dg^{\pm}(x)|$ and $|D_\tau \tilde{f}(x)| = |D_\tau f(x)|$. So, rather than exhibiting a competitor for g we wish to exhibit a competitor, say h, for $\tilde{g}$: we just have to respect the property that spt $h(x) \ni 0$ for every $x \in B_1^-$. With a slight abuse of notation we thus keep the notation g and f for $\tilde{g}$ and $\tilde{f}$.

Step 1. Radial competitors. Let $\bar{g} = \sum_i [\![\bar{g}_i]\!] \in \mathcal{A}_Q$ be a mean for g so that the Poincaré inequality of Proposition 2.12 in [15] holds, i.e.

$$\left(\int_{\partial B_1} \mathcal{G}(g,\bar{g})^p\right)^{1/p} \le C(p)\left(\int_{\partial B_1} |Dg|^2\right)^{1/2}, \tag{3.42}$$

where the exponent p can be taken to be any finite real $p \ge 1$ if $m = 3$ and any real $1 \le p \le 2^*$ (with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{m-1}$) when $m \ge 4$. Assume the diameter of $\bar{g}$ is smaller than a constant $M > 0$ (whose value is to be determined later),

$$d(\bar{g}) \le M.$$

Recall next (3.7) and define the function $m(x) := \mathrm{dist}(\mathrm{spt}\; g(x), 0)$. Observe that $T \mapsto \mathrm{dist}(\mathrm{spt}\,(T), 0)$ is a Lipschitz map with constant less or equal than 1 by (3.8): thus $|Dm| \le |Dg|$ and $|D_\tau m| \le D_\tau g|$. Moreover m obviously vanishes on ∂B_1^- (whose surface measure is larger than a geometric constant). By the relative Poincaré inequality, we know

$$\int_{\partial B_1} m(x)^2 \le C\int_{\partial B_1} |Dm(x)|^2 \le C\int_{\partial B_1} |Dg(x)|^2. \tag{3.43}$$

Hence

$$\bar{m}^2 := \mathrm{dist}(\mathrm{spt}(\bar{g}), 0)^2 = \fint_{\partial B_1} m^2 \lesssim \int_{\partial B_1} m^2 + \int_{\partial B_1} \mathcal{G}(g(x),\bar{g})^2 \le C\int_{\partial B_1} |Dg(x)|^2. \tag{3.44}$$

Combined with the assumption $d(\bar{g}) \le M$, it follows that

$$|\bar{g}|^2 = \sum_i |\bar{g}_i|^2 \le Q(C + M^2).$$

Thus

$$\int_{\partial B_1} |g|^2 \le 2\int_{\partial B_1} \mathcal{G}(g,\bar{g})^2 + 2\int_{\partial B_1} |\bar{g}|^2 \le C_{Q,M},$$

where $C_{Q,M}$ is a constant depending on Q and M with positive correlation. Let φ be a real-valued function in $W^{1,2}([0,1])$ with $\varphi(1) = 1$. Then

$$\hat{f}(x) := \varphi(|x|)g\left(\frac{x}{|x|}\right)$$

is a suitable competitor for f. Simple computation shows that

$$\begin{aligned}\iint_{B_1} |D\hat{f}|^2 &= \left(\int_{\partial B_1} |g|^2\right)\int_0^1 \varphi'(r)^2 r^{m-1}dr + \left(\int_{\partial B_1} |Dg|^2\right)\int_0^1 \varphi(r)^2 r^{m-3}dr \\ &\le \int_0^1 \left(\varphi(r)^2 r^{m-3} + C_{Q,M}\varphi'(r)^2 r^{m-1}\right)dr =: I(\varphi).\end{aligned}$$

By minimality we deduce that

$$\mathrm{Dir}(f, B_1) \le \inf_{\substack{\varphi\in W^{1,2}([0,1])\\ \varphi(1)=1}} I(\varphi).$$

We notice that $I(1) = \frac{1}{m-2}$ ($\varphi \equiv 1$ corresponds to the trivial radial competitor for f). On the other hand $\varphi \equiv 1$ can not be a minimum for I because it does not satisfy the corresponding Euler-Lagrange equation. So there exists a constant $\gamma = \gamma(Q, M) > 0$ such that

$$\mathrm{Dir}(f, B_1) \le \inf_{\substack{\varphi\in W^{1,2}([0,1])\\ \varphi(1)=1}} I(\varphi) = \frac{1}{m-2} - \gamma.$$

In particular, when $Q = 1$, the diameter $d(\bar g) = 0$ and we are done. We will prove the proposition by an induction on Q.

Step 2. Splitting procedure: the inductive step. Let $Q \ge 2$ be fixed and assume that the proposition holds for every $Q^* < Q$. Assume moreover that $d(\bar g) > M$. The strategy of the proof is to decompose f into several pieces in order to apply the inductive hypothesis. To that end, we first *collapse* the mean $\bar g$, by applying Lemma 3.8 of [15] to $T = \bar g$. For any $\epsilon \in (0, 1)$, we obtain $S = \sum_{j=1}^J k_j[\![S_j]\!] \in \mathcal{A}_Q$ which satisfies

$$\beta M \le \beta d(\bar g) \le s(S) < +\infty, \tag{3.45}$$

$$\mathcal{G}(S, \bar g) \le \epsilon s(S). \tag{3.46}$$

Here $\beta = \beta(\epsilon, Q)$ is the constant in Lemma 3.8. The fact that $s(S) < +\infty$ means $J \ge 2$. Recall (3.44) (this estimate is independent of the assumption on $d(\bar g)$), we get

$$\min S \le \min \bar g + \mathcal{G}(S, \bar g) \le C + \epsilon s(S). \tag{3.47}$$

Assume without loss of generality that $|S_1| = \min S$. We let

$$\widetilde S := k_1[\![0]\!] + \sum_{j=2}^J k_j[\![S_j]\!].$$

By (3.47),

$$\mathcal{G}(S, \widetilde S) \le \sqrt{k_1|S_1|^2} < \sqrt{Q}\,\mathrm{dist}(\mathrm{spt}\,(S), 0) \le C\sqrt{Q} + \epsilon\sqrt{Q}s(S). \tag{3.48}$$

We fix ϵ with $\epsilon\sqrt{Q} = \frac{1}{64}$; we may also choose $M = M(Q,\beta(\epsilon, Q))$ sufficiently large

$$C \le \epsilon\beta M \le \epsilon s(S). \tag{3.49}$$

Thus it follows from (3.45) that

$$\mathcal{G}(S, \widetilde S) < 2\epsilon\sqrt{Q}s(S) = \frac{1}{32}s(S).$$

Combined with (3.46), we have

$$\mathcal{G}(\bar g, \widetilde S) \le \sqrt{2\mathcal{G}(S, \bar g)^2 + 2\mathcal{G}(S, \widetilde S)^2} < \frac{1}{16}s(S). \tag{3.50}$$

On the other hand, we also have

$$s(\widetilde{S}) \geq (1-2\epsilon)s(S). \tag{3.51}$$

In fact, either $s(\widetilde{S}) = |S_i - S_j|$ for $i, j \neq 1$, in which case $s(\widetilde{S}) \geq s(S)$ by definition; or $s(\widetilde{S}) = |S_i|$ for some $i \neq 1$, then by (3.47) and (3.49)

$$s(\widetilde{S}) = |S_i| \geq |S_i - S_1| - |S_1| \geq s(S) - \min S \geq (1-2\epsilon)s(S).$$

Let

$$\vartheta : \mathcal{A}_Q \to \overline{B_{s(\widetilde{S})/8}(\widetilde{S})}$$

be the retraction given by Lemma 3.9. We define $h \in W^{1,2}(B_{1-\eta})$ by

$$h(x) := \vartheta\left(f\left(\frac{x}{1-\eta}\right)\right),$$

where η is a small parameter to be determined later. By Lemma 3.9 (iii), $h(x)$ contains a zero element for every $x \in B^-_{1-\eta}$. By removing one zero element in the lower half space we may consider h as a function in $W^{1,2}(B_{1-\eta}, \mathcal{A}^{\pm}_Q)$. Therefore by Theorem 4.2 in [8] there exists a Dir-minimizer $\hat{h} \in W^{1,2}(B_{1-\eta}, \mathcal{A}^{\pm}_Q)$ with interface (γ, φ), such that $\hat{h} = h$ on $\partial B_{1-\eta} \setminus \gamma$. Almost everywhere on $\partial B_{1-\eta}$, $\hat{h}$ takes value in $\overline{B_{s(\widetilde{S})/8}(\widetilde{S})}$. Therefore by Proposition 3.16 $\hat{h}$ can be decomposed into the sum of h_1 and h_2, where h_1 is a K-valued function and Dir-minimizer, h_2 is an L-valued function and Dir-minimizer with interface (γ, φ), and $K, L \leq Q-1$. By Proposition 3.10 of [15] and the inductive hypothesis, we have

$$\mathrm{Dir}(h_1, B_{1-\eta}) \leq \left(\frac{1}{m-2} - \gamma_i\right)(1-\eta)\ \mathrm{Dir}(h_1, \partial B_{1-\eta}),$$

$$\mathrm{Dir}(h_2, B_{1-\eta}) \leq \left(\frac{1}{m-2} - \gamma_b\right)(1-\eta)\ \mathrm{Dir}(h_2, \partial B_{1-\eta}).$$

Hence

$$\begin{aligned}
\mathrm{Dir}(\hat{h}, B_{1-\eta}) &\leq \left(\frac{1}{m-2} - \gamma\right)(1-\eta)\ \mathrm{Dir}(h, \partial B_{1-\eta}) \\
&= \left(\frac{1}{m-2} - \gamma\right)(1-\eta)^{m-2}\ \mathrm{Dir}(g, \partial B_1) \\
&< \left(\frac{1}{m-2} - \gamma\right).
\end{aligned}$$

Here $\gamma_0 = \min\{\gamma_i, \gamma_b\} > 0$ is a constant depending on m and Q. We consider the following competitor

$$\hat{f} = \begin{cases} \hat{h}, & \text{in } B_{1-\eta} \\ \text{interpolation between } \hat{h} \text{ and } g \text{ as in Lemma 3.31}, & \text{in } B_1 \setminus B_{1-\eta}. \end{cases}$$

By the estimate (3.33),

$$\mathrm{Dir}(\hat{f}, B_1) \leq \mathrm{Dir}(\hat{h}, B_{1-\eta}) + C\eta\left(\mathrm{Dir}(\hat{h}, \partial B_{1-\eta}) + \mathrm{Dir}(g, \partial B_1)\right) + \frac{C}{\eta}\int_{\partial B_1} \mathcal{G}(g, \vartheta(g))^2$$

$$< \frac{1}{m-2} - \gamma_0 + 2C\eta + \frac{C}{\eta}\int_{\partial B_1} \mathcal{G}(g, \vartheta(g))^2. \tag{3.52}$$

Now we estimate the last term in the right hand side of (3.52). By the definition of the retraction ϑ, g and $\vartheta(g)$ only differ on the set

$$E := \left\{ x \in \partial B_1 : g(x) \notin \overline{B_{s(\widetilde{S})/8}(\widetilde{S})} \right\}.$$

For every $x \in E$, by (3.50) and the properties of ϑ,

$$\mathcal{G}(\vartheta \circ g(x), \bar{g}) = \mathcal{G}(\vartheta \circ g(x), \vartheta(\bar{g})) < \mathcal{G}(g(x), \bar{g}),$$

and

$$\mathcal{G}(g(x), \bar{g}) \geq \mathcal{G}(g(x), \widetilde{S}) - \mathcal{G}(\bar{g}, \widetilde{S}) > \frac{1}{16} s(\widetilde{S}).$$

Hence

$$\int_{\partial B_1} \mathcal{G}(g, \vartheta(g))^2 \leq 2\int_E \mathcal{G}(g(x), \bar{g})^2 + \mathcal{G}(\vartheta \circ g(x), \bar{g})^2 \leq 4\int_E \mathcal{G}(g(x), \bar{g})^2 \leq C|E|^{\frac{2}{m-1}}. \tag{3.53}$$

Recall (3.51) and (3.49),

$$s(\widetilde{S}) \geq (1-2\epsilon)s(S) \geq (1-2\epsilon)\beta M.$$

We may estimate the measure of E by Chebyshev inequality

$$|E| \leq \int_{B_1} \left(\frac{\mathcal{G}(g(x), \bar{g})}{s(\widetilde{S})/16} \right)^2 \leq \frac{C}{M^2}.$$

Combined with (3.52) and (3.53), we conclude that

$$\mathrm{Dir}(\hat{f}, B_1) \leq \frac{1}{m-2} - \gamma_0 + C'\eta + \frac{C'}{\eta M^2},$$

where the constants γ_0, C' only depend on Q and m. We first choose η so that $C'\eta = \frac{\gamma_0}{3}$, then we choose M so that $\frac{C'}{\eta M^2} = \frac{\gamma_0}{3}$. Therefore by the minimality of f

$$\mathrm{Dir}(f, B_1) \leq \mathrm{Dir}(\hat{f}, B_1) \leq \frac{1}{m-2} - \frac{\gamma_0}{3}.$$

Step 3. Conclusion. With the value of M fixed, Step 1 shows that if $d(\bar{g}) \leq M$, there exists $\gamma = \gamma(Q) > 0$ such that

$$\mathrm{Dir}(f, B_1) \leq \frac{1}{m-2} - \gamma.$$

Assuming the inductive hypothesis, Step 2 shows that if $d(\bar{g}) > M$,

$$\mathrm{Dir}(f, B_1) \leq \frac{1}{m-2} - \frac{\gamma_0}{3}.$$

This concludes the proof. □

3.6. **Proof of Theorem 3.1.** We want to prove the following decay of Dirichlet energy

$$\mathrm{Dir}(f, B_r) \leq C r^{m-2+2\beta}(\mathrm{Dir}(f, B_1) + \|D\varphi\|^2_{C^0}) \tag{3.54}$$

for every $y \in B_{\frac{1}{2}}$ and almost every $0 < r \leq \frac{1}{2}$.

First of all observe that the estimate follows from Proposition 3.38 for $y \in \gamma$. Indeed in that case, if we let $h(r) = \int_{B_r(y)} |Df|^2$, then h is absolutely continuous and

$$h'(r) = \int_{\partial B_r(y)} |Df|^2 \geq \int_{\partial B_r(y)} |D_\tau f|^2 =: \mathrm{Dir}(f, \partial B_r(y)) \quad \text{for almost every } r.$$

Combined with (3.39) we have

$$h(r) \leq C(m) r h'(r) + C r^m \|D\varphi\|^2_{C^2} \leq \frac{r h'(r)}{m-2+2\beta} + C r^m \|D\varphi\|^2_{C^0},$$

(where β is assumed to be smaller than 1). We next define $k(r) := h(r) + A r^m$ and compute

$$\begin{aligned} k(r) = h(r) + A r^m &\leq \frac{r}{m-2+2\beta} h'(r) + C\|D\varphi\|^2_{C^0} r^m + A r^m \\ &\leq \frac{r}{m-2+2\beta} k'(r) + C\|D\varphi\|^2_{C^0} r^m - A\left(\frac{m}{m-2+2\beta} - 1\right) r^m . \end{aligned}$$

Since

$$\frac{m}{m-2+2\beta} - 1 > 0,$$

for $A = C'\|D\varphi\|^2_{C_0}$ with C' sufficiently large we conclude

$$k(r) \leq \frac{r}{m-2+2\beta} k'(r)$$

and integrating the latter inequality in the interval $[r, 1/2]$ we get the desired estimate

$$\begin{aligned} \mathrm{Dir}(f, B_r(y)) \leq k(r) \leq r^{m-2+2\beta} k\left(\frac{1}{2}\right) &\leq r^{m-2+2\beta}\left(\mathrm{Dir}(f, B_{1/2}(y)) + C'\|D\varphi\|^2_{C^0}\right) \\ &\leq C r^{m-2+2\beta}\left(\mathrm{Dir}(f, B_1) + \|D\varphi\|^2_{C^0}\right) . \end{aligned}$$

Consider now a point $y \in B_{1/2} \setminus \gamma$. If $r \geq \frac{1}{4}$ the estimate (3.54) is then obvious. Hence we assume $r < \frac{1}{4}$. Let next $\rho := \mathrm{dist}(y, \gamma)$. If $r \geq \rho$, consider $x \in \gamma$ such that $|x - y| = \mathrm{dist}(y, \gamma)$ and observe that $B_{2r}(x) \supset B_r(y)$. The estimate follows then from the one for $y \in \gamma$. Otherwise, we have two possibilities. If $\rho \geq \frac{1}{4} > r$, we then can use the decay estimate for Q-valued Dir-minimizers to infer

$$\mathrm{Dir}(f, B_r(y)) \leq C r^{m-2+2\beta} \mathrm{Dir}(f, B_{1/4}(y)) \leq C r^{m-2+2\beta} \mathrm{Dir}(f, B_1) .$$

If $r < \rho < \frac{1}{4}$ we can then proceed in two steps to prove

$$\mathrm{Dir}(f, B_r(y)) \leq \left(\frac{r}{\rho}\right)^{m-2+2\beta} \mathrm{Dir}(f, B_\rho(y)) \leq C r^{m-2+2\beta}\left(\mathrm{Dir}(f, B_1) + \|D\varphi\|^2_{C^0}\right) .$$

Having finally proved the decay (3.54), the Hölder continuity follows from the Campanato-Morrey estimate.

4. First variations and monotonicity of the frequency function

In this section we address a main tool to prove Theorem 1.6, the monotonicity of the frequency function. The original frequency function was introduced by Almgren in [2] for Dir-minimizing Q-valued map, cf. also [15]. The one for $(Q-\frac{1}{2})$-valued maps with interface $(\gamma, 0)$ in $\mathbb{R}^m$ was introduced in [8] and requires a subtle argument. Since our Theorem 1.6 is 2-dimensional, we can take advantage of the reduction to Theorem 2.1 and restrict our attention the model situation in which the interface is $(\mathbb{R}, 0)$. Under such assumption the statement and proof of the relevant formulae is just a straightforward adaptation of the arguments in [15], which we give below for the reader's convenience (the issue in [8] is that in dimension $m \geq 3$ it is not possible to "rectify" a general γ with a conformal change of coordinates).

Definition 4.1 (The frequency function)**.** Assume $f = (f^+, f^-)$ is a $(Q-\frac{1}{2})$-valued map on $\Omega \subset \mathbb{R}^m$ with interface (γ, φ) and consider a ball $B_r(x) \subset \Omega$ with $x \in \gamma$. We define

$$D_{x,f}(r) = \mathrm{Dir}\,(f, B_r(x)), \quad H_{x,f}(r) = \int_{\partial B_r(x)} |f|^2 := \int_{\partial B_r^+(x)} |f^+|^2 + \int_{\partial B_r^-(x)} |f^-|^2. \tag{4.2}$$

When $H_{x,f}(r) > 0$, we define the frequency function

$$I_{x,f}(r) = \frac{r D_{x,f}(r)}{H_{x,f}(r)}. \tag{4.3}$$

When x and f are clear from the context, we often use the shorthand notation $D(r), H(r)$ and $I(r)$.

Proposition 4.4 (First variations)**.** *Assume $f = (f^+, f^-) \in W^{1,2}(\Omega, \mathcal{A}_Q^\pm)$ is Dir-minimizing on $\Omega \subset \mathbb{R}^2$ with interface $(\mathbb{R}, 0)$ and let $B_r \subset \Omega$. Then*

$$\int_{\partial B_r(x)} |Df|^2 = 2\left(\int_{\partial B_r^+(x)} \sum_{j=1}^{Q} |\partial_\nu f_j^+|^2 + \int_{\partial B_r^-(x)} \sum_{j=1}^{Q-1} |\partial_\nu f_j^-|^2\right) \tag{4.5}$$

and

$$\iint_{B_r(x)} |Df|^2 = \int_{\partial B_r^+(x)} \sum_{j=1}^{Q} \langle \partial_\nu f_j^+, f_j^+ \rangle + \int_{\partial B_r^-(x)} \sum_{j=1}^{Q-1} \langle \partial_\nu f_j^-, f_j^- \rangle\,. \tag{4.6}$$

Here ∂_ν denotes the outer unit normal on the boundary of the given ball, and $f^+ = \sum_{j=1}^{Q} [\![f_j^+]\!]$ and $f^- = \sum_{j=1}^{Q-1} [\![f_j^-]\!]$ are measurable selections of f^+ and f^-, given by Proposition 0.4 of [15].

Remark 4.7. Identity (4.5) implies that the integral of the square of the tangential derivative on the circle ∂B_r equals the integral of the square of the normal derivative.

Proof. The proof follows the same computations of [15, Proof of Proposition 3.2]. It just suffices to observe the following two facts:

- (4.5) is derived by comparing the Dirichlet energy of f with competitors of the form $f \circ \Phi_\varepsilon$, where $\{\Phi_\varepsilon\}$ are some specific one-parameter families of diffeomorphisms. It easy to check that the ones used in [15, Proof of Proposition 3.2] map $\mathbb{R}$, Ω^+ and Ω^- onto themselves and hence give an addmissible family of competitors for our variational problem as well.
- Similarly, (4.6) is derived by comparing the Dirichlet energy of f with competitors of the form
$$f^{\varepsilon,\pm}(x) := \sum_j [\![f_j^\pm(x) + \varepsilon\psi(x, f_j^\pm(x))]\!]$$
where $\psi(x,u) = \phi(|x|)u$ satisfies $\psi(x,0) = 0$. Therefore the functions $f^{\varepsilon,\pm}$ have also interface $(\mathbb{R}, 0)$ and they are in the class of admissible competitors. $\square$

The above first variation formulae impliy:

Theorem 4.8 (Monotonicity of the frequency, analogue of Theorem 3.15 [15])**.** *Let f be a $(Q-\frac{1}{2})$-valued Dir-minimizing map with interface $(\mathbb{R}, 0)$ in an open set $\Omega \subset \mathbb{R}^2$ containing the origin and assume that $f^+(0) = Q[\![0]\!]$. Either there exists $\delta > 0$ such that*
$$f^+|_{B_\delta^+(0)} \equiv Q[\![0]\!], \quad f^-|_{B_\delta^-(0)} \equiv (Q-1)[\![0]\!];$$
or $I_{0,f}(r)$ is an absolutely continuous nondecreasing positive function on $(0, \mathrm{dist}(0,\partial\Omega))$.

Proof. If $H(r) = 0$ for some $r > 0$, then $f^+ = Q[\![0]\!]$ a.e. on $\partial B_r^+(0)$ and $f^- = (Q-1)[\![0]\!]$ a.e. on $\partial B_r^-(0)$. For such boundary data the only minimizer is the pair which is constant on the respective $B_r^\pm(0)$. From now on we assume therefore that $H(r) > 0$ for every $r \in (0,1)$.

D is absolutely continuous and

$$D'(r) = \int_{\partial B_r} |Df|^2 \text{ for almost every } r. \tag{4.9}$$

Since $f^+, f^- \in W^{1,2}$ are approximate differentiable almost everywhere, we can apply the chain-rule formulas (see Propositions 1.12 and 2.8 [15]) and justify the following computations:

$$\begin{aligned}
H'(r) &= \frac{d}{dr}\int_{\partial B_1^+} r|f^+(ry)|^2 dy + \frac{d}{dr}\int_{\partial B_1^-} r|f^-(ry)|^2 dy \\
&= \int_{\partial B_1} |f(ry)|^2 dy + \int_{\partial B_1^+} r\frac{\partial}{\partial r}|f^+(ry)|^2 dy + \int_{\partial B_1^-} r\frac{\partial}{\partial r}|f^-(ry)|^2 dy \\
&= \frac{1}{r}\int_{\partial B_r} |f|^2 + 2\int_{\partial B_r^+}\sum_{j=1}^{Q}\langle\partial_\nu f_j^+, f_j^+\rangle + 2\int_{\partial B_r^-}\sum_{j=1}^{Q-1}\langle\partial_\nu f_j^-, f_j^-\rangle = \frac{1}{r}H(r) + 2D(r),
\end{aligned} \tag{4.10}$$

by the outer variation formula (4.6). In fact, since both $H(r)$ and $D(r)$ are continuous, we have $H \in C^1$ and the above inequality holds pointwise. Therefore

$$I'(r) = \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - rD(r)\frac{H'(r)}{H(r)^2}$$

$$
\begin{aligned}
&= \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - \frac{D(r)}{H(r)} - 2r\frac{D(r)^2}{H(r)^2}\\
&= \frac{rD'(r)}{H(r)} - 2r\frac{D(r)^2}{H(r)^2}.
\end{aligned}
$$

Again by the inner and outer variations formula (4.5), (4.6), we conclude, it follows that

$$
\begin{aligned}
I'(r) = \frac{2r}{H(r)^2}\Bigg[&\left(\int_{\partial B_r^+}\sum_{j=1}^{Q}|\partial_\nu f_j^+|^2 + \int_{B_r^-}\sum_{j=1}^{Q-1}|\partial_\nu f_j^-|^2\right)\cdot\left(\int_{\partial B_r^+}\sum_{j=1}^{Q}|f_j^+|^2 + \int_{B_r^-}\sum_{j=1}^{Q-1}|f_j^-|^2\right)\\
&-\left(\int_{\partial B_r^+}\sum_{j=1}^{Q}\langle\partial_\nu f_j^+, f_j^+\rangle + \int_{B_r^-}\sum_{j=1}^{Q-1}\langle\partial_\nu f_j^-, f_j^-\rangle\right)^2\Bigg]\,. \qquad (4.11)
\end{aligned}
$$

We can now choose a measurable selection of the various multifuctions involved and extend such selections $f_j^\pm, \partial_\nu f_j^\pm$ to 0 respectively on $B_r^\mp$. The Cauchy-Schwartz inequality will then imply:

$$
\begin{aligned}
I'(r) = \frac{2r}{H(r)^2}\Bigg[&\int_{\partial B_r}\left(\sum_{j=1}^{Q}|\partial_\nu f_j^+|^2 + \sum_{j=1}^{Q-1}|\partial_\nu f_j^-|^2\right)\cdot\int_{\partial B_r}\left(\sum_{j=1}^{Q}|f_j^+|^2 + \sum_{j=1}^{Q-1}|f_j^-|^2\right)\\
&-\left(\int_{\partial B_r}\sum_{j=1}^{Q}\langle\partial_\nu f_j^+, f_j^+\rangle + \sum_{j=1}^{Q-1}\langle\partial_\nu f_j^-, f_j^-\rangle\right)^2\Bigg] \geq 0\,. \qquad (4.12)
\end{aligned}
$$

□

Corollary 4.13. *Let f be as in Theorem 4.8. $I_{0,f}(r) \equiv \alpha$ if and only if (f^+, f^-) is α-homogeneous, i.e.*

$$
f^\pm(x) = \sum_i \left[\!\left[|x|^\alpha f_i^\pm\left(\frac{x}{|x|}\right)\right]\!\right]\,.
$$

In the rest of the note, when dealing with a Q-valued funcion $f = \sum_i [\![f_i]\!]$, we will use the notation λf for the multifunction $\sum_i [\![\lambda f_i]\!]$. Similarly, for a $(Q-\frac{1}{2})$-valued function $f = (f^+, f^-)$, λf will denote $(\lambda f^+, \lambda f^-)$. In particular, the homogeneity of f in the corollary above will be expressed by the formula

$$
f(x) = |x|^\alpha f\left(\frac{x}{|x|}\right)\,.
$$

Proof. Suppose $\alpha = 0$. $I_{0,f}(r) \equiv 0$ if and only if each $f_j^\pm$ is constant, so clearly (f^+, f^-) is 0-homogeneous. If (f^+, f^-) is 0-homogeneous, then each $f_j^\pm$ is constant on the ray starting from the origin. Thus by the continuity of f near the origin, each $f_j^\pm$ is constant on its domain and $I_{0,f}(r) \equiv 0$.

Suppose $\alpha > 0$. Then by the proof of the above theorem, $I(r)$ is a constant if and only if equality occurs in (4.12), i.e. if and only if there exists constants $\lambda_r \in \mathbb{R}$ such that

$$
\partial_\nu f_j^\pm(x) = \lambda_r f_j^\pm(x) \text{ for almost every } r \text{ and almost every } x \text{ with } |x| = r.
$$

Moreover,

$$\alpha = I(r) = \frac{rD(r)}{H(r)} = \frac{r\int_{\partial B_r}\sum\langle\partial_\nu f_j^\pm, f_j^\pm\rangle}{\int_{\partial B_r}\sum|f_j^\pm|^2} = r\lambda_r.$$

Therefore $I(r) \equiv \alpha$ if and only if

$$\partial_\nu f_j^\pm(x) = \frac{\alpha}{|x|} f_j^\pm(x) \text{ for almost every } x. \tag{4.14}$$

If f is α-homogeneous, clearly (4.14) holds. On the other hand, suppose (4.14) holds, we want to show that f is α-homogeneous. To that end we let $x \in \partial B_1$ and $\sigma_x = \{rx : 0 < r \leq 1\}$ be the ray from the origin through x. Note that $f|_{\sigma_x} \in W^{1,2}$ for almost every $x \in \partial B_1$. For those x (4.14) implies

$$\frac{d}{dr}\frac{f_j^\pm(rx)}{r^\alpha} \equiv 0.$$

Thus $f_j^\pm(rx) = r^\alpha f_j^\pm(x)$ for all $0 < r \leq 1$ and almost every $x \in \partial B_1$. □

Corollary 4.15 (analogue of Corollary 3.18 [15]). *Let f be as in Theorem 4.8. Suppose $H(r) \neq 0$ for every $r \in (0, \operatorname{dist}(\partial\Omega, 0))$. Then*

(i) for almost every $r < 1$,

$$\frac{d}{dr}\log\left(\frac{H(r)}{r^{m-1}}\right) = \frac{2I(r)}{r}, \tag{4.16}$$

and thus for almost every $s \leq t < 1$,

$$\left(\frac{s}{t}\right)^{2I(t)}\frac{H(t)}{t^{m-1}} \leq \frac{H(s)}{s^{m-1}} \leq \left(\frac{s}{t}\right)^{2I(s)}\frac{H(t)}{t^{m-1}}; \tag{4.17}$$

(ii) for almost every $s \leq t < 1$, if $I(t) > 0$, then

$$\frac{I(s)}{I(t)}\left(\frac{s}{t}\right)^{2I(t)}\frac{D(t)}{t^{m-2}} \leq \frac{D(s)}{s^{m-2}} \leq \left(\frac{s}{t}\right)^{2I(s)}\frac{D(t)}{t^{m-2}}. \tag{4.18}$$

Proof. (4.16) follows from (4.10). (4.17) follows from (4.16) and the monotonicity of the frequency function. Finally, (4.18) follows from (4.17) and the definition of the frequency function. □

5. Compactness and tangent functions in planar domains

The monotonicity of the frequency function provides a way of studying the asymptotic behaviour of a minimizer at small scales around a given point with highest multiplicity.

Definition 5.1. Let f be a Dir-minimizing $(Q - \frac{1}{2})$-valued map on a planar domain Ω with interface $(\mathbb{R}, 0)$. Let y be a point at the inteface $\mathbb{R}$ and assume that $\operatorname{Dir}(f, B_\rho(y)) > 0$ for every ρ. We define the following rescalings of f at y:

$$f_{y,\rho}(x) = \frac{f(\rho x + y)}{\sqrt{\operatorname{Dir}(f, B_\rho(y))}}. \tag{5.2}$$

The key point is that, up to subsequences, the latter rescalings converge locally strongly to nontrivial Dir-minimizers.

Theorem 5.3 (Compactness, analogue of Theorem 3.19 in [15])**.** *Let $f \in W^{1,2}(\Omega, \mathcal{A}_Q^{\pm})$ be a Dir-minimizing map on a planar domain Ω with interface $(\mathbb{R}, 0)$. Assume $f^+(0) = Q[\![0]\!]$ and* $\mathrm{Dir}(f, B_\rho) > 0$ *for every $\rho \in (0, \mathrm{dist}(0, \partial\Omega))$. Then for any sequence $\{f_{\rho_k}\}$ with $\rho_k \searrow 0$, a subsequence, not relabelled, converges locally uniformly to a function $g : \mathbb{R}^2 \to \mathcal{A}_Q^{\pm}$ satisfying the following properties:*

(a) $\mathrm{Dir}(g, B_1) = 1$ *and $g|_U$ is Dir-minimizing with interface $(\mathbb{R}, 0)$ for any bounded set $U \subset \mathbb{R}^2$;*
(b) $g(x) = |x|^\alpha g\left(\frac{x}{|x|}\right)$*, where $\alpha = I_{0,f}(0) > 0$ is the frequency of f at* 0.

From now, any limit of a sequence of rescalings $\{f_{\rho_k}\}_k$ with $\rho_k \downarrow 0$ will be called a *tangent function*. A feature of the 2-dimensional setting is that the compactness theorem above can be considerably strengthened: analogously to the "interior case", cf. [15, Theorem 5.3], we can prove that the tangent function at a given point is unique and that the rescaling converge at a suitable rate to it. The key is to first show a suitable rate of convergence for the frequency function.

Proposition 5.4 (Rate of convergence, analogue of Proposition 5.2 in [15])**.** *Let f be as in Theorem 5.3 and set $\alpha = I_{0,f}(0)$. Then there exist constants $r_0, \beta, C, H_0, D_0 > 0$ such that for every $0 < r \leq r_0$,*

$$0 \leq I(r) - \alpha \leq C r^\beta, \tag{5.5}$$

$$0 \leq \frac{H(r)}{r^{2\alpha+1}} - H_0 \leq C r^\beta, \quad 0 \leq \frac{D(r)}{r^{2\alpha}} - D_0 \leq C r^\beta. \tag{5.6}$$

Theorem 5.7 (Unique tangent map, analogue of Theorem 5.3 [15])**.** *Let $f \in W^{1,2}(\mathbb{D}, \mathcal{A}_Q^{\pm})$ be as in Theorem 5.3 and denote by β the exponent of the decay estimate (5.5). Then the tangent function f_0 to f at* 0 *is unique and, moreover,*

$$\|\mathcal{G}(f_{0,\rho}, f_0)\|^2_{L^2(\mathbb{S}^1)} \leq C\rho^\beta\,. \tag{5.8}$$

5.1. **Compactness and tangent functions: Proof of Theorem 5.3.** Let B_R denote a ball of sufficiently large radius $R \gg 1$. By the definition (5.2),

$$\mathrm{Dir}(f_\rho, B_R) = \frac{D(\rho R)}{D(\rho)} \leq R^{m-2+2I(\rho R)} \frac{I(\rho)}{I(\rho R)} \leq R^{m-2+2(\alpha+1)},$$

where we use the estimate (4.18) for the first inequality, and the properties of the frequency function for the second. Since f_ρ's are all Dir-minimizing with interface $(\mathbb{R}, 0)$, by Theorem 3.1 they are locally equi-Hölder continuous. The assumption $f^+(0) = Q[\![0]\!]$ implies $f^+_\rho(0) = Q[\![0]\!]$ and $f^-_\rho(0) = (Q-1)[\![0]\!]$ for all ρ. Thus f_ρ's are locally uniformly bounded, and by Azelà-Ascoli Theorem a subsequence (not relabelled) converges uniformly on compact sets to a continuous function $g = (g^+, g^-)$. (To use Azelà-Ascoli Theorem, we may add to f^-_ρ a zero sheet to get functions valued in the metric space $\mathcal{A}_Q$.) In particular $g^+|_\gamma = g^-|_\gamma + [\![0]\!]$; moreover $f^+_{\rho_k}$ converges weakly in $W^{1,2}_{loc}(\mathbb{R}^2_+, \mathcal{A}_Q)$ to g^+, and $f^-_{\rho_k}$ converges weakly in $W^{1,2}_{loc}(\mathbb{R}^2_-, \mathcal{A}_{Q-1})$ to g^- (see Definition 2.9 in [15]). By (1.1) it follows then that $\mathrm{Dir}(g, B_r) \le \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, B_r)$ for all $r > 0$.

Proof of (a). Let $R > 0$ be fixed. We will show that for any $0 < r \le R$,

$$\mathrm{Dir}(g, B_r) = \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, B_r) \text{ and } g|_{B_r} \text{ is Dir-minimizing with interface } (\mathbb{R}, 0). \tag{5.9}$$

For any $R > 0$, we will show (5.9) holds for all $r \le R$.

By Fatou's Lemma,

$$\int_0^R \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, \partial B_r)\, dr \le \liminf_{k\to\infty} \int_0^R \mathrm{Dir}(f_{\rho_k}, \partial B_r)\, dr \le \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, B_R) \le C < +\infty.$$

Hence for almost every $r \in (0, R)$, the integrand of the first term is finite, and moreover by weak convergence

$$\mathrm{Dir}(g, \partial B_r) \le \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, \partial B_r) \le M < +\infty. \tag{5.10}$$

For the sake of contradiction, assume that either one of the statement in (5.9) fails for such r, then there exists a function $h \in W^{1,2}(B_r, \mathcal{A}^\pm_Q)$ with interface $(\mathbb{R}, 0)$ such that

$$h|_{\partial B_r} = g|_{\partial B_r} \text{ and } \mathrm{Dir}(h, B_r) < \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, B_r). \tag{5.11}$$

Let $\delta = 1/N < 1/2$ to be fixed later, and consider the functions $\widetilde{f_k}$ on B_r defined by

$$\widetilde{f_k} = \begin{cases} \left(\frac{1}{1-\delta}\right)^{\frac{m-2}{2}} h\left(\frac{x}{1-\delta}\right), & \text{for } x \in B_{(1-\delta)r}, \\ h_k(x), & \text{for } x \in B_r \setminus B_{(1-\delta)r}, \end{cases}$$

where the h_k's are the interpolation functions provided by Lemma 3.31 between $f_{\rho_k} \in W^{1,2}(\partial B_r, \mathcal{A}^\pm_Q)$ and $h\left(\frac{x}{1-\delta}\right) \in W^{1,2}(\partial B_{(1-\delta)r}, \mathcal{A}^\pm_Q)$. Notice that h_k's satisfy the boundary condition $h^+_k|_\gamma = h^-_k|_\gamma + [\![0]\!]$. By the minimality of f_{ρ_k}, (3.33) and changes of variables, we have

$$\begin{aligned} \mathrm{Dir}(f_{\rho_k}, B_r) &\le \mathrm{Dir}(\widetilde{f_k}, B_r) \\ &\le \mathrm{Dir}\left(\left(\frac{1}{1-\delta}\right)^{\frac{m-2}{2}} h\left(\frac{x}{1-\delta}\right), B_{(1-\delta)r}\right) + \mathrm{Dir}(h_k, B_r \setminus B_{(1-\delta)r}) \\ &\le \mathrm{Dir}(h, B_r) + C\delta r\, \mathrm{Dir}(f_{\rho_k}, \partial B_r) + C\delta r\, \mathrm{Dir}(h, \partial B_r) + \frac{C}{\delta} \sup_{x\in\partial B_r} \mathcal{G}(h(x), f_{\rho_k}(x)) \\ &\le \mathrm{Dir}(h, B_r) + C\delta R\, \mathrm{Dir}(f_{\rho_k}, \partial B_r) + C\delta R\, \mathrm{Dir}(g, \partial B_r) + \frac{C}{\delta} \sup_{x\in\partial B_r} \mathcal{G}(g(x), f_{\rho_k}(x)). \end{aligned}$$

Passing $k \to \infty$, by the uniform convergence of f_{ρ_k} to g, (5.10) and the assumption (5.11), we get

$$\limsup_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, B_r) \le \mathrm{Dir}(h, B_r) + 2C\delta RM < \liminf_{k\to\infty} \mathrm{Dir}(f_{\rho_k}, B_r) + 2C\delta RM. \tag{5.12}$$

We get a contradiction by choosing δ arbitrarily small. Therefore (5.9) holds for almost every $r \in (0, R)$. By the upper semi-continuity of $\mathrm{Dir}(g, B_r)$ in r, it follows that (5.9) holds for all $r \le R$.

Proof of (b). We observe that for every $r > 0$,

$$I_g(r) = \frac{r\,\mathrm{Dir}(g, B_r)}{\int_{\partial B_r} |g|^2} = \liminf_{k\to\infty} \frac{r\,\mathrm{Dir}(f_{\rho_k}, B_r)}{\int_{\partial B_r} |f_{\rho_k}|^2} = \liminf_{k\to\infty} \frac{r\rho_k\,\mathrm{Dir}(f, B_{r\rho_k})}{\int_{\partial B_{r\rho_k}} |f|^2} = I_f(0).$$

Since g is Dir-minimizing, by Corollary 4.13 it is a homogeneous function with homogenity $\alpha = I_f(0)$. If $\alpha = 0$, a continuous 0-homogeneous function with $g(0) = Q[\![0]\!]$ is necessarily $g \equiv Q[\![0]\!]$. This is in contradiction with $\mathrm{Dir}(g, B_1) = \lim_k \mathrm{Dir}(f_{\rho_k}, B_1) = 1$, and thus $\alpha > 0$.

5.2. **Rate of convergence: Proof of Proposition 5.4.** *Step 1.* We claim the following estimate holds for some $\beta > 0$:

$$I'(r) \geq \frac{2}{r}\,(\alpha + \beta - I(r))\,(I(r) - \alpha). \tag{5.13}$$

Recall (4.10), we have

$$I'(r) = \frac{rD'(r)}{H(r)} - \frac{2I^2(r)}{r}. \tag{5.14}$$

Thus (5.13) is reduced to prove

$$(2\alpha + \beta)D(r) \leq \frac{rD'(r)}{2} + \frac{\alpha(\alpha+\beta)H(r)}{r}. \tag{5.15}$$

Let r be fixed, and let $g(\theta) := f(re^{i\theta})$. Consider the decomposition of $g(\theta)$ as in Proposition $g = g_0 + \sum_{j=1}^J g_j$, where $g_0 \in W^{1,2}(\mathbb{S}^1, \mathcal{A}^{\pm}_{Q_0})$ and $g_j \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_{Q_j})$ are irreducible maps. Recall that for each irreducible g_j, we can find $\zeta_j \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^n)$ such that

$$g_j(\theta) = \sum_{i=1}^{Q_j} \left[\!\!\left[\zeta_j\left(\frac{\theta + 2\pi i}{Q_j}\right) \right]\!\!\right]. \tag{5.16}$$

We write the Fourier expansions of ζ_j's as

$$\zeta_j(\theta) = \frac{a_{j,0}}{2} + \sum_{l=1}^{\infty} \left(a_{j,l}\cos(l\theta) + b_{j,l}\sin(l\theta)\right), \quad \theta \in [0, 2\pi]. \tag{5.17}$$

Suppose g_0 unrolls to a function $\zeta_0 : \mathbb{S}^1 \to \mathbb{R}^n$, as in Lemma 2.11. The boundary condition $g_0^+|_\gamma = g_0^-|_\gamma + [\![0]\!]$ implies $\zeta_0(0) = \zeta_0(2\pi) = 0$. Hence we write the Fourier expansion of ζ_0 as

$$\zeta_0(\theta) = \sum_{l=1}^{\infty} c_l \sin\left(\frac{l\theta}{2}\right), \quad \theta \in [0, 2\pi].$$

Recall (4.5) and Lemma 2.11, we get

$$D'(r) = 2\,\mathrm{Dir}(f, \partial B_r) = \frac{2}{r}\mathrm{Dir}(g, \mathbb{S}^1) = \frac{2}{r}\sum_{j=0}^{J} \mathrm{Dir}(g_j, \mathbb{S}^1)$$

$$= \frac{2}{r}\left(\frac{2}{2Q_0-1}\operatorname{Dir}(\zeta_0,\mathbb{S}^1)+\sum_{j=1}^{J}\frac{1}{Q_j}\operatorname{Dir}(\zeta_j,\mathbb{S}^1)\right)$$

$$= \frac{2\pi}{r}\left(\frac{1}{2(2Q_0-1)}\sum_l c_l^2 l^2+\sum_{j=1}^{J}\sum_l \frac{\left(a_{j,l}^2+b_{j,l}^2\right)l^2}{Q_j}\right), \tag{5.18}$$

and

$$H(r)=\int_{\partial B_r}|f|^2 = r\int_{\mathbb{S}^1}|g|^2 = r\left(\frac{2Q_0-1}{2}\int_{\mathbb{S}^1}|\zeta_0|^2+\sum_{j=1}^{J}Q_j\int_{\mathbb{S}^1}|\zeta_j|^2\right)$$

$$= \pi r\left[\frac{2Q_0-1}{2}\sum_l c_l^2+\sum_{j=1}^{J}Q_j\left(\frac{a_{j,0}^2}{2}+\sum_l\left(a_{j,l}^2+b_{j,l}^2\right)\right)\right]. \tag{5.19}$$

On the other hand, consider a $W^{1,2}$-extension of ζ_0 on B_1

$$\overline{\zeta}_0(\rho,\theta)=\sum_{l=1}^{\infty}\rho^{\frac{l}{2}}c_l\sin\left(\frac{l\theta}{2}\right), \tag{5.20}$$

(note that $\overline{\zeta_0}$ is not harmonic at the origin), and the harmonic extension of each ζ_j:

$$\overline{\zeta}_j(\rho,\theta)=\frac{a_{j,0}}{2}+\sum_{l=1}^{\infty}\rho^l\left(a_{j,l}\cos(l\theta)+b_{j,l}\sin(l\theta)\right),\quad j=1,\cdots,J,$$

where $\rho\le 1$ and $\theta\in[0,2\pi]$. We then construct a competitor h of f in B_r as follows: $h(\rho e^{i\theta})=\widehat{h}(\rho e^{i\theta}/r)$, where $\widehat{h}$, a function defined on B_1, is given by

$$\widehat{h}(\rho e^{i\theta}):=\left(h_0^+(\rho e^{i\theta}),h_0^-(\rho e^{i\theta})\right)+\sum_{j=1}^{J}\sum_{i=0}^{Q_j}\left[\!\left[\overline{\zeta}_j\left(\rho^{\frac{1}{Q_j}},\frac{\theta+2\pi i}{Q_j}\right)\right]\!\right],$$

and

$$h_0^+(\rho e^{i\theta})=\overline{\zeta}_0\left(\rho^{\frac{2}{2Q_0-1}},\frac{2\theta}{2Q_0-1}+\frac{4\pi}{2Q_0-1}(i-1)\right),\quad \theta\in[0,\pi], i=1,\cdots,Q_0,$$

$$h_0^-(\rho e^{i\theta})=\overline{\zeta}_0\left(\rho^{\frac{2}{2Q_0-1}},\frac{2\theta}{2Q_0-1}+\frac{4\pi}{2Q_0-1}(i-1)\right),\quad \theta\in[\pi,2\pi], i=1,\cdots,Q_0-1.$$

Notice that by definition (5.20),

$$h_0^+|_\gamma = h_0^-|_\gamma+[\![0]\!],$$

hence $\widehat{h}$ has interface (γ,φ). A simple computation, combined with Lemma 2.11 and Lemma 3.12 of [15], shows

$$\operatorname{Dir}(h,B_r)=\operatorname{Dir}(\widehat{h},B_1)=\sum_{j=0}^{J}\iint_{B_1}|D\overline{\zeta}_j|^2=\frac{\pi}{2}\sum_l c_l^2 l+\pi\sum_{j=1}^{J}\sum_l(a_{j,l}^2+b_{j,l}^2)l.$$

Thus by the minimality of f, we know

$$D(r) \le \mathrm{Dir}(h, B_r) = \frac{\pi}{2}\sum_l c_l^2 l + \pi \sum_{j=1}^J \sum_l (a_{j,l}^2 + b_{j,l}^2) l. \tag{5.21}$$

We combine (5.18), (5.19) and (5.21) and plug into the left and right hand sides of (5.15). After simplifications, we show that it is enough to find $\beta > 0$ satisfying

$$[l - \alpha(2Q_0 - 1)]\left[l - (\alpha+\beta)(2Q_0 - 1)\right] \ge 0, \tag{5.22}$$

$$\left(l - \alpha Q_j\right)\left[l - (\alpha+\beta)Q_j\right] \ge 0, \quad j = 1, \cdots, J, \tag{5.23}$$

for every $l \in \mathbb{N}$. This is equivalent to say the intervals $(\alpha(2Q_0 - 1), (\alpha+\beta)(2Q_0 - 1))$ and $(\alpha Q_j, (\alpha+\beta)Q_j)$ do not contain integer points. This is verified, if we choose

$$\beta = \min_{1\le k\le Q}\left\{\frac{\lfloor \alpha k\rfloor + 1 - \alpha k}{k}, \frac{\lfloor \alpha(2k-1)\rfloor + 1 - \alpha(2k-1)}{2k-1}\right\} > 0.$$

Step 2. Since $I(r)$ is monotone decreasing, there exists $r_0 > 0$ such that $I(r) \le \alpha + \frac{\beta}{2}$ for all $r \le r_0$. Hence (5.13) implies that

$$I'(r) \ge \frac{\beta}{r}\left(I(r) - \alpha\right) \text{ for almost every } r \le r_0. \tag{5.24}$$

Integrating the differential inequality, we get the desired estimate

$$I(r) - \alpha \le \left(\frac{r}{r_0}\right)^{\beta}(I(r_0) - \alpha) \le C r^{\beta}. \tag{5.25}$$

Recall that the derivative of H satisfies (4.10). In particular when $m = 2$ we have

$$\left(\frac{H(r)}{r}\right)' = \frac{2D(r)}{r}.$$

This implies

$$\left(\log\frac{H(r)}{r^{2\alpha+1}}\right)' = \left(\log\frac{H(r)}{r} - \log r^{2\alpha}\right)' = \left(\log\frac{H(r)}{r}\right)' - \frac{2\alpha}{r} = \frac{2}{r}\left(I(r) - \alpha\right) \ge 0. \tag{5.26}$$

Hence $\frac{H(r)}{r^{2\alpha+1}}$ is monotone increasing. In particular, its limit exists as $r \to 0+$:

$$\frac{H(r)}{r^{2\alpha+1}} \ge \lim_{r\to 0}\frac{H(r)}{r^{2\alpha+1}} =: H_0 \ge 0. \tag{5.27}$$

On the other hand combining (5.26) and (5.25) we get

$$\left(\log\frac{H(r)}{r^{2\alpha+1}}\right)' \le 2Cr^{\beta-1}, \quad \text{thus} \left(\log\frac{H(r)e^{-C_\beta r^\beta}}{r^{2\alpha+1}}\right)' \le 0.$$

Hence $\frac{H(r)e^{-C_\beta r^\beta}}{r^{2\alpha+1}}$ is monotone decreasing, and

$$\frac{H(r)e^{-C_\beta r^\beta}}{r^{2\alpha+1}} \le \lim_{r\to 0} \frac{H(r)e^{-C_\beta r^\beta}}{r^{2\alpha+1}} = \lim_{r\to 0} \frac{H(r)}{r^{2\alpha+1}} = H_0.$$

In particular $H_0 > 0$. Moreover

$$\frac{H(r)}{r^{2\alpha+1}}\left(1 - C_\beta r^\beta\right) \le \frac{H(r)e^{-C_\beta r^\beta}}{r^{2\alpha+1}} \le H_0,$$

and we conclude that

$$\frac{H(r)}{r^{2\alpha+1}} - H_0 \le C_\beta \frac{H(r)}{r^{2\alpha+1}} r^\beta \le C_\beta \frac{H(r_0)}{r_0^{2\alpha+1}} r^\beta \le C r^\beta.$$

The last inequality of (5.6) follows from:

$$\frac{D(r)}{r^{2\alpha}} - D_0 = (I(r) - I_0)\frac{H(r)}{r^{2\alpha+1}} + I_0 \left(\frac{H(r)}{r^{2\alpha+1}} - H_0\right),$$

where $I_0 = \alpha$ and $D_0 = I_0 H_0$.

5.3. **Uniqueness of the tangent map: Proof of Theorem 5.7.** Without loss of generality, we assume $D_0 = 1$. By the estimate (5.6) and the definition of blow-up, it follows that

$$f_\rho(r,\theta) = \rho^{-\alpha} f(r\rho,\theta)\left(1 + O\left(\rho^{\beta/2}\right)\right). \tag{5.28}$$

It suffices to show the existence of a uniform limit for the dominant function $h_\rho(r,\theta) = \rho^{-\alpha} f(r\rho,\theta)$. Note that the function h_ρ is homogeneous:

$$h_\rho(r,\theta) = \rho^{-\alpha} f(r\rho,\theta) = r^\alpha h_{r\rho}(1,\theta),$$

it is enough to prove the existence of a uniform limit for $h_\rho|_{\mathbb{S}^1}$. Each function

$$h_\rho|_{\mathbb{S}^1} = h_\rho(1,\theta) = \rho^{-\alpha} f(\rho,\theta)$$

is Dir-minimizing, so by Theorem 3.1 it is Hölder continuous with a uniform constant. We first show that $h_\rho|_{\mathbb{S}^1}$ has an L^2 limit.

Let $\{T_i\}$ and $\{T_i'\}$ be countable dense subsets of $\mathcal{A}_Q(\mathbb{R}^n)$ and $\mathcal{A}_{Q-1}(\mathbb{R}^n)$ respectively. We consider $r/2 \le s \le r$ and estimate

$$\begin{aligned}
\int_0^{2\pi} \mathcal{G}(h_r,h_s)^2 d\theta &= \int_0^{2\pi} \mathcal{G}\left(\frac{f(r,\theta)}{r^\alpha}, \frac{f(s,\theta)}{s^\alpha}\right)^2 d\theta \\
&= \int_0^{\pi} \sup_i \left|\mathcal{G}\left(\frac{f^+(r,\theta)}{r^\alpha}, T_i\right) - \mathcal{G}\left(\frac{f^+(s,\theta)}{s^\alpha}, T_i\right)\right|^2 d\theta + \\
&\qquad \int_\pi^{2\pi} \sup_i \left|\mathcal{G}\left(\frac{f^-(r,\theta)}{r^\alpha}, T_i'\right) - \mathcal{G}\left(\frac{f^-(s,\theta)}{s^\alpha}, T_i'\right)\right|^2 d\theta \\
&\le (r-s)\int_0^{\pi} \sup_i \int_s^r \left|\frac{\partial}{\partial t}\mathcal{G}\left(\frac{f^+(t,\theta)}{t^\alpha}, T_i\right)\right|^2 dt\, d\theta +
\end{aligned}$$

$$
\begin{aligned}
&(r-s)\int_{\pi}^{2\pi}\sup_i\int_s^r\left|\frac{\partial}{\partial t}\mathcal{G}\left(\frac{f^-(t,\theta)}{t^\alpha},T_i'\right)\right|^2 dt\,d\theta\\
\leq\ &(r-s)\int_0^{\pi}\int_s^r\left|\frac{\partial}{\partial t}\left(\frac{f^+(t,\theta)}{t^\alpha}\right)\right|^2 dt\,d\theta+\\
&(r-s)\int_0^{\pi}\int_s^r\left|\frac{\partial}{\partial t}\left(\frac{f^-(t,\theta)}{t^\alpha}\right)\right|^2 dt\,d\theta\\
=\ &(r-s)\int_0^{\pi}\int_s^r\sum_j\left\{\alpha^2\frac{|f_j^+|^2}{t^{2\alpha+2}}+\frac{|\partial_\nu f_j^+|^2}{t^{2\alpha}}-2\alpha\frac{\langle\partial_\nu f_j^+,f_j^+\rangle}{t^{2\alpha+1}}\right\} dt\,d\theta+\\
&(r-s)\int_0^{\pi}\int_s^r\sum_j\left\{\alpha^2\frac{|f_j^-|^2}{t^{2\alpha+2}}+\frac{|\partial_\nu f_j^-|^2}{t^{2\alpha}}-2\alpha\frac{\langle\partial_\nu f_j^-,f_j^-\rangle}{t^{2\alpha+1}}\right\} dt\,d\theta\\
=\ &(r-s)\int_s^r\left\{\alpha^2\frac{H(t)}{t^{2\alpha+3}}+\frac{D'(t)}{2t^{2\alpha+1}}-2\alpha\frac{D(t)}{t^{2\alpha+2}}\right\}dt\\
=\ &(r-s)\int_s^t\left\{\frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}\right)'+\alpha\frac{H(t)}{2t^{2\alpha+3}}\left(\alpha-I_{0,f}(t)\right)\right\}dt\\
\leq\ &(r-s)\int_s^t\frac{1}{2t}\left(\frac{D(t)}{t^{2\alpha}}-D_0\right)'dt\\
=\ &(r-s)\left[\frac{1}{2r}\left(\frac{D(r)}{r^{2\alpha}}-D_0\right)-\frac{1}{2s}\left(\frac{D(s)}{s^{2\alpha}}-D_0\right)\right]+\\
&(r-s)\int_s^t\frac{1}{2t^2}\left(\frac{D(t)}{t^{2\alpha}}-D_0\right)dt\\
\leq\ &Cr^\beta.
\end{aligned}
$$

Let $s\leq r$ be arbitrary, and let l be a positive integer such that $r/2^{l+1}<s\leq r/2^l$. Iterating the above estimate we get

$$
\|\mathcal{G}(h_r,h_s)\|_{L^2(\mathbb{S}^1)}\leq\sum_{k=0}^{l}\|\mathcal{G}\left(h_{r/2^k},h_{r/2^{k+1}}\right)\|_{L^2(\mathbb{S}^1)}+\|\mathcal{G}(h_{r/2^l},h_s)\|_{L^2(\mathbb{S}^1)}\leq\sum_{k=0}^{l}C\left(\frac{r}{2^k}\right)^{\frac{\beta}{2}}\leq C'r^{\frac{\beta}{2}}.
$$

This shows that $h_\rho|_{\mathbb{S}^1}$ is a Cauchy sequence in L^2, and thus converges to a limit function $h\in L^2(\mathbb{S}^1)$. Moreover, since h_ρ is equi-Hölder continuous on $\mathbb{S}^1$, it follows that h_ρ converges uniformly to h on $\mathbb{S}^1$. In other words, f_ρ converges locally uniformly to an α-homogeneous function g, with $g(z)=|z|^\alpha h\left(\frac{z}{|z|}\right)$.

6. Homogeneous Dir-minimizers

In this section we study homogeneous Dir-minimizers in planar domains. We do not really give a complete characterization, but rather a set of necessary conditions that they have to satisfy.

Proposition 6.1 (Characterization of tangent maps)**.** *Let $\alpha > 0$ and let $f \in W^{1,2}_{loc}(\mathbb{R}^2, \mathcal{A}^{\pm}_Q)$ be a nontrivial α-homogeneous Dir-minimizer with interface $(\mathbb{R}, 0)$. If $Q > 1$ and f satisfies (2.3), then the following alternative holds:*

(a) *either*

$$f^+ = \left[\!\left[r^l \vec{c} \sin(l\theta) \right]\!\right] + \sum_{j=1}^J k_j \left[\!\left[r^l \left(\vec{a_j} \cos(l\theta) + \vec{b_j} \sin(l\theta) \right) \right]\!\right] =: f_0^+ + \sum_{j=1}^J k_j f_j,$$

$$f^- = \sum_{j=1}^J k_j \left[\!\left[r^l \left(\vec{a_j} \cos(l\theta) + \vec{b_j} \sin(l\theta) \right) \right]\!\right] =: \sum_{j=1}^J k_j f_j, \quad J \geq 1;$$

for some $l \in \mathbb{N}$;

(b) *or*

$$f^+ = \left[\!\left[r^{\frac{2}{3}} \vec{c} \sin\left(\tfrac{2}{3}\theta\right) \right]\!\right] + \left[\!\left[r^{\frac{2}{3}} \vec{c} \sin\left(\tfrac{2}{3}(\theta + 2\pi)\right) \right]\!\right] + \sum_{j=1}^J k_j \sum_{\substack{z^3 = x \\ z = (r,\theta)}} \left[\!\left[r^2 \left(\vec{a_j} \cos(2\theta) + \vec{b_j} \sin(2\theta) \right) \right]\!\right]$$

$$=: f_0^+ + \sum_{j=1}^J k_j f_j,$$

$$f^- = \left[\!\left[r^{\frac{2}{3}} \vec{c} \sin\left(\tfrac{2}{3}\theta\right) \right]\!\right] + \sum_{j=1}^J k_j \sum_{\substack{z^3 = x \\ z = (r,\theta)}} \left[\!\left[r^2 \left(\vec{a_j} \cos(2\theta) + \vec{b_j} \sin(2\theta) \right) \right]\!\right]$$

$$=: f_0^- + \sum_{j=1}^J k_j f_j, \quad J \geq 0.$$

In both cases $k_j \geq 1$ and $\vec{c}, \vec{a_j}, \vec{b_j}$ are constants in $\mathbb{R}^n$ for all j. Moreover, in both cases the supports of $f_i(x)$ and $f_j(x)$ are disjoint for any $i \neq j \in \{0, 1, \cdots, J\}$ and any x not at the origin.

Remark 6.2. It would be interesting to know whether the second case does indeed occur, namely whether the map (f_0^+, f_0^-) of case (b) is a $\frac{3}{2}$-valued minimizer.

Proof. We decompose $g \in W^{1,2}(\mathbb{S}^1, \mathcal{A}^{\pm}_Q)$ into irreducible pieces as described in Proposition 2.7:

$$g = g_0 + \widetilde{g} = g_0 + \sum_{j=1}^J g_j,$$

where $g_0 = (g_0^+, g_0^-) \in W^{1,2}(\mathbb{S}^1, \mathcal{A}^{\pm}_{Q_0})$ is a irreducible map with interface (γ, φ). In other words f decompose as follows:

$$f = f_0 + \widetilde{f} := r^\alpha g_0 + r^\alpha \widetilde{g}.$$

(If the map g is irreducible itself, we just have $f = f_0$.) By Proposition 2.7 there exists a function $\zeta : \mathbb{S}^1 \to \mathbb{R}^n$ satisfying $\zeta(0) = 0$ such that g_0 unwinds to ζ. Let

$$\zeta(\rho, \theta) = \rho^{\alpha \frac{2Q_0 - 1}{2}} \zeta(\theta)$$

be an extension of ζ to the disk $\mathbb{D}$. By (2.12) and (2.13) f unwinds to $\zeta(\rho, \theta)$, and thus $\mathrm{Dir}(f, \mathbb{D}) = \int_{\mathbb{D}} |D\zeta|^2$ by Lemma 2.11. We consider the function $\overline{\zeta}(z) := \zeta(z^2) : \mathbb{D}^+ \to \mathbb{R}^n$. By definition $\overline{\zeta}|_\gamma \equiv 0$. Since conformal maps do not change Dirichlet energy, we have

$$\iint_{\mathbb{D}^+} |D\overline{\zeta}|^2 = \iint_{\mathbb{D}} |D\zeta|^2 = \mathrm{Dir}(f, \mathbb{D}). \tag{6.3}$$

For any function $\eta : \mathbb{D}^+ \to \mathbb{R}^n$ satisfying $\eta|_{\partial\mathbb{D}^+} = \overline{\zeta}|_{\partial\mathbb{D}^+}$, we can wind the function $\eta(\sqrt{z}) : \mathbb{D} \to \mathbb{R}^n$ by the formula (2.12), (2.13) and find a corresponding function $h : \mathbb{D} \to \mathcal{A}_Q^{\pm}$ such that $h|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}$ and $h^+|_\gamma = h^-|_\gamma + [\![0]\!]$. By the minimality of f with interface (γ, φ), we have

$$\mathrm{Dir}(f, \mathbb{D}) \le \mathrm{Dir}(h, \mathbb{D}) = \iint_{\mathbb{D}^+} |D\eta|^2.$$

This combined with (6.3) shows that $\overline{\zeta}$ is a Dir-minimizer in $\mathbb{D}^+$ with fixed boundary value on γ. Thus $\overline{\zeta}$ is a harmonic function in $\mathbb{D}^+$. On the other hand $\overline{\zeta}$ is $\alpha(2Q_0 - 1)$-homogeneous. By spherical harmonics we know $\alpha(2Q_0 - 1) = l \in \mathbb{N}$ and $\overline{\zeta}(r, \theta) = \vec{c} r^l \sin(l\theta)$ with some constant $\vec{c} \in \mathbb{R}^n$. Therefore $\zeta(\theta) = \vec{c} \sin\left(\frac{l\theta}{2}\right)$ on $\mathbb{S}^1$.

Claim: Suppose a $\left(Q_0 - \frac{1}{2}\right)$-valued map g_0 unwinds to $\zeta(\theta) = \vec{c}\sin\left(\frac{l\theta}{2}\right)$ on $\mathbb{S}^1$, then g_0 is irreducible if and only if either $Q_0 = 1$ or $l = Q_0 = 2$. In the first case $g_0^+ = \vec{c}\sin(l\theta)$ for any integer $l \in \mathbb{N}$; in the second case

$$g_0^+ = \left[\!\!\left[\vec{c}\sin\left(\frac{2}{3}\theta\right)\right]\!\!\right] + \left[\!\!\left[\vec{c}\sin\left(\frac{2}{3}(\theta + 2\pi)\right)\right]\!\!\right], \quad \theta \in [0, \pi], \tag{6.4}$$

$$g_0^- = \left[\!\!\left[\vec{c}\sin\left(\frac{2}{3}\theta\right)\right]\!\!\right], \quad \theta \in [\pi, 2\pi]. \tag{6.5}$$

Proof of the claim. When $Q_0 = 1$, the condition (i) in Proposition 2.7 holds trivially, thus g_0 is irreducible. Now assume $Q_0 > 1$. The condition (i) fails if we can find $\theta \in [0, 2\pi]$ and $k \in \mathbb{N}$ such that

$$\zeta(\theta) = \zeta\left(\theta + \frac{4\pi}{2Q_0 - 1}k\right), \quad 0 \le \theta, \theta + \frac{4\pi}{2Q_0 - 1}k < 2\pi. \tag{6.6}$$

We denote $\beta = l\theta/2$, then (6.6) becomes

$$\sin(\beta) = \sin\left(\beta + \frac{2\pi lk}{2Q_0 - 1}\right), \quad 0 \le \beta, \beta + \frac{2\pi lk}{2Q_0 - 1} < l\pi. \tag{6.7}$$

To rephrase it slightly different, (6.7) is equivalent to find $\beta_1, \beta_2 \in [0, l\pi)$ such that $\sin(\beta_1) = \sin(\beta_2)$ and they are $\frac{2k}{2Q_0 - 1} l\pi$ distance apart for some $k \in \mathbb{N} \setminus \{0\}$. For all odd integers $l \in \mathbb{N}$, we can always find $\beta_1, \beta_2 \in [0, l\pi)$ with arbitrary distance in the range $[0, l\pi)$ satisfying $\sin(\beta_1) = \sin(\beta_2)$; for all even integers $l \in \mathbb{N}$, we can always find $\beta_1, \beta_2 \in [0, l\pi)$ having the same sinus and with arbitrary distance in the range $[0,(l-1)\pi]$. In the latter case, the only way (i) could be satisfied is that there is an even integer $l \geq 2$ so that

$$\frac{2}{2Q_0-1} l\pi > (l-1)\pi \text{ for some } Q_0 > 1.$$

Namely we are looking for an even $l \geq 2$ and a natural $Q_0 > 1$ such that $\frac{2}{2Q_0-1} > \frac{l-1}{l}$. Clearly, $\frac{l-1}{l} \geq \frac{1}{2}$. On the other hand $\frac{2}{2Q_0-1} \leq \frac{2}{5} < \frac{1}{2}$ when $Q_0 \geq 3$. The only possibility is thus $l = Q_0 = 2$.

Now we analyze the two cases separately. *Case I: $Q_0 = 1$ and $l \in \mathbb{N}$ is arbitrary.* In this case $\alpha = l$. We consider the $(Q-1)$-valued map $\widetilde{f}$. (Note that $Q-1 > 0$ by our assumption.) Its center of mass $\phi := \boldsymbol{\eta} \circ \widetilde{f}$ is an l-homogeneous harmonic function on $\mathbb{D}$, and thus $\phi(r,\theta) = r^l \left(\vec{a_1}\cos(l\theta) + \vec{b_1}\sin(l\theta)\right)$. The function $\sum_{j=1}^{Q-1} [\![\widetilde{f}(x) - \phi(x)]\!]$ has center of mass zero. Either it is trivial, i.e. $\widetilde{f} = (Q-1)[\![\phi]\!]$; or it satisfies the assumption of Proposition 5.1 in [15].

In the first case $\widetilde{f} = (Q-1)[\![\phi]\!]$, for contradiction we assume

$$\operatorname{spt} \widetilde{f}(r,\theta_0) \cap \operatorname{spt} f_0^+(r,\theta_0) \neq \emptyset \text{ for some } \theta_0 \in [0,\pi].$$

If $\theta_0 \in (0,\pi)$, then either the entire ray $\{(r,\theta_0) : 0 < r < 1\}$ is contained in the interior singular set of the Dir-minimizer (without boundary) $\sum_{j=0}^{Q-1} f_j^+$ on $\mathbb{D}^+$, or ϕ and f_0^+ agree on all of $\mathbb{D}^+$. The former is impossible due to the isolation of interior singular set (c.f. Theorem 0.12 of [15]). If the latter holds, then

$$f^+(r,\theta) = Q[\![f_0^+(r,\theta)]\!] = Q[\![r^l \vec{c}\sin(l\theta)]\!],$$
$$f^-(r,\theta) = (Q-1)[\![r^l \vec{c}\sin(l\theta)]\!].$$

The symmetric assumption (2.3) then implies that

$$Q\, r^l \vec{c}\sin(l\theta) = (Q-1)\, r^l \vec{c}\sin\left(l(2\pi-\theta)\right) \text{ for all } \theta \in [0,\pi].$$

Hence $\vec{c} = 0$ and f is trivial, contradiction. If $\theta_0 = 0$ (the case when $\theta_0 = \pi$ is similar), then $\phi(\cdot,0) = 0$. Thus $\vec{a_0} = 0$ and $\phi = r^l \vec{b_0}\sin(l\theta)$. In particular $\phi(\cdot,\pi) = 0$ as well, and $f = f_0^+ + \widetilde{f}$ collapses at the interface. Theorem 4.5 of [8] implies 0 is a boundary regular point. We again deduce from the symmetry (2.3) that f is trivial, contradiction.

For the second case we apply Proposition 5.1 of [15] and get $n^* = l$, $Q^* = 1$, and

$$\widetilde{f} - (Q-1)[\![\phi]\!] = k_1[\![0]\!] + \sum_{j=2}^{J} k_j \left[\!\!\left[r^l \left(\vec{a_j}\cos(l\theta) + \vec{b_j}\sin(l\theta)\right)\right]\!\!\right]. \tag{6.8}$$

In other words

$$\widetilde{f} = \sum_{j=1}^{J} k_j \left[\!\!\left[r^l \left(\vec{a_j}\cos(l\theta) + \vec{b_j}\sin(l\theta)\right)\right]\!\!\right] =: \sum_{j=1}^{J} k_j f_j. \tag{6.9}$$

Moreover $J \geq 2$, and the supports of $f_i(x)$ and $f_j(x)$ are disjoint for any $i \neq j$ and $z \neq 0$. Similar as argued in the previous case, the support of any f_j can not intersect $\operatorname{spt} f_0^+$ unless they completely agree, in which scenario f is trivial.

Case II: $l = Q_0 = 2$ and g_0 is of the form (6.4)*,* (6.5). Either $f = f_0 = r^{\frac{2}{3}} g_0$ is irreducible (that is, $Q = 2$), or f can be decomposed as the sum of f_0 and $\widetilde{f}$. In the first case we have shown that $\operatorname{card} \operatorname{spt} f_0^+ = 2$ and we are done. Hence we may assume $f = f_0 + \widetilde{f}$. Since $\alpha = \frac{2}{3}$, the center of mass $\phi := \boldsymbol{\eta} \circ \widetilde{f}$ ought to be a $\frac{2}{3}$-homogeneous harmonic function in $\mathbb{D}$. This is only possible if $\phi \equiv 0$. Thus either $\widetilde{f}$ is trivial, or we may apply Proposition 5.1 of [15]. When $\widetilde{f}$ is trivial, again we get $f = f_0 + \widetilde{f}$ has a ray or interior singular points in $\mathbb{D}^-$, or $\vec{c} = 0$ and f is trivial. This is impossible. Alternatively we deduce $n^* = 2, Q^* = 3$ and

$$\widetilde{f}(x) = k_1 [\![0]\!] + \sum_{j=2}^{J} k_j \sum_{\substack{z^3 = x \\ z=(r,\theta)}} \left[\!\!\left[r^2 \left(\vec{a_j} \cos(2\theta) + \vec{b_j} \sin(2\theta) \right) \right]\!\!\right] =: \sum_{j=1}^{J} k_j f_j. \tag{6.10}$$

Moreover, the supports of $f_i(x)$ and $f_j(x)$ are disjoint for any $i \neq j$ and any $x \neq 0$. Again by homogeneity and the isolation of interior singular point, the support of any f_j does not intersect that of f_0 at $\theta_0 \in (0, \pi) \cup (\pi, 2\pi)$. We also note that since $f_0^- \left(\cdot, \frac{3\pi}{2}\right) = 0$, by the same reason $k_1 = 0$. Suppose $\operatorname{spt} f_j$ and $\operatorname{spt} f_0$ intersect at $\theta_0 = 0$, then the point of intersection $\operatorname{spt} f_j(\cdot, 0) \cap \operatorname{spt} f_0(\cdot, 0)$ is either (i) $r^{\frac{2}{3}} \vec{c} \sin(0) = 0$ or (ii) $r^{\frac{2}{3}} \vec{c} \sin\left(\frac{4\pi}{3}\right) = -r^{\frac{2}{3}} \vec{c} \frac{\sqrt{3}}{2}$. Consider a selection of f_j

$$f_j = \sum_{k=1}^{3} [\![f_{j,k}]\!],$$

and without loss of generality we assume the first sheet satisfies $f_{j,1}(\cdot, 0)$ is the point of intersection, that is, $f_{j,1}(\cdot, 0) \in \operatorname{spt} f_j(\cdot, 0) \cap \operatorname{spt} f_0(\cdot, 0)$. By the formula of f_j in (6.10), we remark that $f_{j,k}$ and the sheets of f_0 have the same profile, with possibly different phases and amplitude. For both scenarios (i) and (ii) we have that from this point on, $f_{j,1}$ either has the same profile, but possibly different magnitude (that is, $f_{j,1} = \vec{c}' r^{\frac{2}{3}} \sin\left(\frac{3\theta}{2}\right)$ or $\vec{c}' r^{\frac{2}{3}} \sin\left(\frac{2\theta}{3} + \frac{4\pi}{3}\right)$), as f_0, or is symmetric with f_0. In the first scenario, it then follows that $\operatorname{spt} f_{j,1}$ and $\operatorname{spt} f_0^-$ would also intersect at $\theta_0 = \frac{3\pi}{2}$ with value 0. We again argue by the isolation of interior singular set. In the second scenario and when $f_{j,1}$ has the same profile, by tracing $f_{j,1}$ and f_0^- backwards we find that $\operatorname{spt} f_{j,3}$ and $\operatorname{spt} f_0^-$ also intersect at $\theta_0 = \frac{3\pi}{2}$ with value 0. In the second scenario and when the profile of $f_{j,1}$ is symmetric with respect to f_0, by symmetry we may trace $f_{j,1}$ and f_0 backwards and find that $\operatorname{spt} f_{j,3}$ and $\operatorname{spt} f_0^+$ intersect at $\theta_0 = \frac{\pi}{2}$ with value $r^{\frac{2}{3}} \vec{c} \frac{\sqrt{3}}{2}$. We conclude that $\operatorname{spt} f_j$ and $\operatorname{spt} f_0$ do not intersect, other than at the origin. □

7. Proof of Theorem 1.6: Discreteness of the singular set

The proof of the main theorem is by induction on the number of values Q. The basic step $Q = 1$ is clearly trivial, because f^- does not exist in that case and f^+ is a classical harmonic function. . Now we assume $Q > 1$ and, as induction hypotheses, that the theorem holds for every $Q' < Q$.

We argue by contradiction and assume the existence of a Dir-minimizing $\left(Q - \frac{1}{2}\right)$-valued planar function with real analytic interface (γ, φ) whose singular set is not discrete. As shown in Section 2 we can assume, without loss of generality that f is in Theorem 2.1, namely $Q\eta+ =$

$(Q-1)\eta^-$ and the interface is $(\mathbb{R},0)$. Under our assumptions the singular set must have an accumulation point x_0. The latter cannot be in the interior, and thus belongs to the interface. Without loss of generality we can assume that $x_0 = 0$.

Next, we must have $f^+(0) = Q\,[\![0]\!]$. Otherwise we have $f^+(0) = Q_1\,[\![0]\!] + T$ with $T \in \mathcal{A}_{Q_2}(\mathbb{R}^n)$, where $Q_1 + Q_2 = Q$, $1 \le Q_1 \le Q-1$ and $\mathrm{spt}(T)$ does not contain the origin. By the Hölder continuity theorem, in a neighborhood U of the origin there would be a Q_2-valued map $h \in W^{1,2}(U)$ and a $(Q_1 - \frac{1}{2})$-valued map $g = (g^+, g^-) \in W^{1,2}(U)$, with disjoint supports and such that $f^\pm = g^\pm + h$. Then the singular set of f in U would be the union of the singular set of h and of the singular set of f. Moreover, both must be Dir-minimizing. Hence the singular set of h is discrete by the interior regularity theory, whereas the singular set of g is discrete by the inductive assumption. This is however not possible because we know that 0 is an accumulation point of the singular set of f.

Note next that it must be $D(r) > 0$ for every r in a positive interval, otherwise we would have $f^+ \equiv Q\,[\![0]\!]$ and $f^- \equiv (Q-1)\,[\![0]\!]$ in some neighborhood of 0. Thus $I_f(r)$ is well-defined for every $r > 0$ sufficiently small. Let g be the (homogeneous) tangent function to f at 0, given by Theorem 5.7. By the characterization in Proposition 6.1 g has the following decomposition:

$$g^+ = g_0^+ + \sum_{j=1}^{J} k_j g_j, \quad g^- = g_0^- + \sum_{j=1}^{J} k_j g_j$$

where:

- In the alternative (a) of Proposition 6.1 (g_0^+, g_0^-) is $\frac{1}{2}$-valued, namely g_0^+ is a classical harmonic function which vanishes at $\mathbb{R}$ and g_0^- does not exist.
- In the alternative (b) $(g_0^+, g_0^-) \in W^{1,2}(\mathbb{R}^2, \mathcal{A}_2^\pm)$.

In the alternative (b) g_0^+ is 2-valued, namely $g_0^+ = [\![(g_0^+)_1]\!] + [\![(g_0^+)_2]\!]$ and we define

$$d_0 := \min_{x\in\mathbb{S}^1_+} \mathrm{sep}(g_0^+(x)) = \min_{x\in\mathbb{S}^+_1} \left|(g_0^+)_1(x) - (g_0^+)_2(x)\right| .$$

Note that d_0 is positive. In the alternative (a) we set $d_0 = +\infty$.

For each $j \in \{1, \cdots, J\}$ we define

$$d_{0,j} := \min\left\{ \min_{x\in\mathbb{S}^1_+} \mathrm{dist}\left(\mathrm{spt}(g_0^+(x)), \mathrm{spt}(g_j(x))\right), \min_{x\in\mathbb{S}^1_-} \mathrm{dist}\left(\mathrm{spt}(g_0^-(x)), \mathrm{spt}(g_j(x))\right) \right\},$$

and define for each pair $i \neq j \in \{1, \cdots, J\}$

$$d_{i,j} := \min_{x\in\mathbb{S}^1} \mathrm{dist}\left(\mathrm{spt}(g_i(x)), \mathrm{spt}(g_j(x)\right) .$$

By Proposition 6.1 we know $d_0, d_{0,j}, d_{i,j} > 0$ for all i, j. Let

$$\epsilon = \frac{1}{4} \min\left\{ d_0, \min_j d_{0,j}, \min_{i\neq j} d_{i,j} \right\} > 0.$$

We claim that there exists $r_0 > 0$ such that

$$\mathcal{G}(f(x), g(x)) \le \epsilon |x|^\alpha \quad \text{for every } |x| \le r_0, \tag{7.1}$$

where $\alpha = I_{0,f}(0) > 0$. In fact, recall the uniform convergence of the blow-ups f_r to g:

$$\mathcal{G}(f_r(\theta), g(\theta)) \to 0 \text{ uniformly in } \theta \in \mathbb{S}^1 \text{ as } r \to 0.$$

Recall (5.28), the blow-ups satisfy

$$\frac{f(x)}{|x|^\alpha} = f_{|x|}\left(\frac{x}{|x|}\right)\left(1 + O\left(|x|^{\frac{\beta}{2}}\right)\right).$$

Hence

$$\mathcal{G}\left(\frac{f(r,\theta)}{r^\alpha}, g(\theta)\right) \to 0 \text{ uniformly in } \theta \in \mathbb{S}^1 \text{ as } r \to 0.$$

Recall that g is as α-homogeneous map, i.e. $g(x) = |x|^\alpha g(\frac{x}{|x|})$. We have thus showed (7.1).

The choice of ϵ implies the existence of functions h_j with $j \in \{0, 1, \cdots, J\}$, such that:

- $h_0 = (h_0^+, h_0^-) \in W^{1,2}(B_{r_0}, \mathcal{A}^\pm_{Q_0})$ with interface (γ, φ) and $Q_0 = 1$ or 2, depending on whether alternative (a) or (b) in Proposition 6.1 holds, and in particular $\operatorname{card} \operatorname{spt}(h_0^+(x)) = Q_0$ for all $x \in B^+_{r_0} \setminus \{0\}$;
- each h_j is in $W^{1,2}(B_{r_0}, \mathcal{A}_{k_j Q_j})$, and

$$f|_{B_{r_0}} = (h_0^+, h_0^-) + \sum_{j=1}^J h_j; \tag{7.2}$$

- For every $x \in B_{r_0} \setminus \{0\}$ and every $i > j > 0$ we have $\operatorname{spt}(h_j(x)) \cap \operatorname{spt}(h_i(x)) = \emptyset$;
- For every $x \in B^+_{r_0} \setminus \{0\}$ and every $i > 0$ we have $\operatorname{spt}(h_i(x)) \cap \operatorname{spt}(h_0^+(x)) = \emptyset$;
- For every $x \in B^-_{r_0} \setminus \{\}$ and every $i > 0$ we have $\operatorname{spt}(h_i(x)) \cap \operatorname{spt}(h_0^-(x)) = \emptyset$.

In particular:

- h_0 is a Dir-minimizer with interface $(\mathbb{R}, 0)$, and each h_j is a Dir-minimizer;
- The singular set of f in B_{r_0} is given by 0 and the union of the singular sets of h_0^+, h_0^- and the h_j's.

Suppose $J = 0$. Recall Proposition 6.1, this may only occur in the alternative (b), i.e. when $f|_{B_{r_0}} = (h_0^+, h_0^-)$ is a $\frac{3}{2}$-valued map. By the separation of sheets of h_0^+, the singular set of f in B_{r_0} is just the origin and we get a contradiction. Suppose $J \geq 1$, in other words the sum (7.2) contains at least two terms, so h_0^+ takes strictly less than Q values and we can use our inductive hypothesis to conclude that the singular set of h_0 is discrete. On the other hand, the singular set of each h_j with $j > 0$ is discrete by Theorem 0.12 of [15]. We conclude that the singular set of f in B_{r_0} is discrete as well, contradicting the assumption that the origin was an accumulation point for it.

References

[1] W. K. Allard. On the first variation of a varifold: boundary behavior. *Ann. of Math. (2)*, 101:418–446, 1975.

[2] J. F. J. Almgren. *Almgren's big regularity paper*, volume 1 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

[3] S. X. Chang. Two-dimensional area minimizing integral currents are classical minimal surfaces. *J. Amer. Math. Soc.*, 1(4):699–778, 1988.

[4] R. Courant. The existence of minimal surfaces of given topological structure under prescribed boundary conditions. *Acta Math.*, 72:51–98, 1940.

[5] E. De Giorgi. *Frontiere orientate di misura minima*. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61. Editrice Tecnico Scientifica, Pisa, 1961.

[6] E. De Giorgi, F. Colombini, and L. C. Piccinini. *Frontiere orientate di misura minima e questioni collegate*. Scuola Normale Superiore, Pisa, 1972.

[7] C. De Lellis, G. De Philippis, and J. Hirsch. Forthcoming.

[8] C. De Lellis, G. De Philippis, J. Hirsch, and A. Massaccesi. On the boundary behavior of mass-minimizing integral currents. *arXiv e-prints*, page arXiv:1809.09457, Sep 2018.

[9] C. De Lellis, J. Hirsch, A. Marchese, and S. Stuvard. Forthcoming.

[10] C. De Lellis, J. Hirsch, A. Marchese, and S. Stuvard. Forthcoming.

[11] C. De Lellis, E. Spadaro, and L. Spolaor. Regularity theory for 2-dimensional almost minimal currents I: Lipschitz approximation. *ArXiv e-prints. To appear in Trans. Amer. Math. Soc.*, Aug. 2015.

[12] C. De Lellis, E. Spadaro, and L. Spolaor. Regularity theory for 2-dimensional almost minimal currents III: blowup. *ArXiv e-prints. To appear in Jour. of Diff. Geom*, Aug. 2015.

[13] C. De Lellis, E. Spadaro, and L. Spolaor. Regularity Theory for 2-Dimensional Almost Minimal Currents II: Branched Center Manifold. *Ann. PDE*, 3(2):3:18, 2017.

[14] C. De Lellis, E. Spadaro, and L. Spolaor. Uniqueness of tangent cones for two-dimensional almost-minimizing currents. *Comm. Pure Appl. Math.*, 70(7):1402–1421, 2017.

[15] C. De Lellis and E. N. Spadaro. Q-valued functions revisited. *Mem. Amer. Math. Soc.*, 211(991):vi+79, 2011.

[16] J. Douglas. Minimal surfaces of higher topological structure. *Ann. of Math. (2)*, 40(1):205–298, 1939.

[17] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[18] W. H. Fleming. An Example in the Problem of Least Area. *P. Am. Math. Soc.*, 7:1063–1074, 1956.

[19] R. Hardt and L. Simon. Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. of Math. (2)*, 110(3):439–486, 1979.

[20] L. Spolaor. Almgren's type regularity for Semicalibrated Currents. *ArXiv e-prints*, Nov. 2015.

[21] B. White. Classical area minimizing surfaces with real-analytic boundaries. *Acta Math.*, 179(2):295–305, 1997.

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