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Calculus of Variations

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A C^0 estimate for nearly umbilical surfaces

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Abstract Let $\Sigma \subset \mathbf{R}^3$ be a smooth compact connected surface without boundary. Denote by A its second fundamental form and by $\mathring{A}$ the tensor $A - (\operatorname{tr} A/2)\mathrm{Id}$. In [4] we proved that, if $\|\mathring{A}\|_{L^2(\Sigma)}$ is small, then Σ is $W^{2,2}$-close to a round sphere. In this note we show that, in addition, the metric of Σ is C^0–close to the standard metric of $\mathbf{S}^2$.

1 Introduction

Let $\Sigma \subset \mathbf{R}^3$ be a smooth surface. A point p of Σ is called umbilical if the principal curvatures of Σ at p are equal and the surface Σ is called umbilical if every point $x \in \Sigma$ is umbilical. A classical theorem in differential geometry states that if Σ is a compact connected umbilical surface without boundary, then Σ is a a round sphere. In [2] we proved the following quantitative version. Here:

- Id denotes the identity (1, 1)-tensor and the (0, 2)-tensor naturally associated to it;
- $\mathring{A}$ denotes the traceless part of A, i.e. the tensor $A - \frac{\operatorname{tr} A}{2}\mathrm{Id}$;
- $\mathrm{id} : \mathbf{S}^2 \subset \mathbf{R}^3 \to \mathbf{R}^3$ is the standard isometric embedding of the round sphere.

Theorem 1 *Let $\Sigma \subset \mathbf{R}^3$ denote a smooth compact connected surface without boundary and for convenience normalize the area of Σ by $\mathrm{ar}(\Sigma) = 4\pi$. Then*

$$\|A - \mathrm{Id}\|_{L^2(\Sigma)} \leq C\|\mathring{A}\|_{L^2(\Sigma)}, \tag{1}$$

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where C is a universal constant. If in addition $\|\mathring{A}\|^2_{L^2(\Sigma)} \leq 4\pi$, then there exists a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ and a vector $c_\Sigma \in \mathbf{R}^3$ such that

$$\|\psi - (c_\Sigma + \mathrm{id})\|_{W^{2,2}(\mathbf{S}^2)} \;\leq\; C\|\mathring{A}\|_{L^2(\Sigma)}. \tag{2}$$

Since ψ conformal, if we denote by g the metric of Σ and by σ the standard metric on $\mathbf{S}^2$, then $\psi_\# g = h^2\sigma$ for some positive function h. Hence Theorem 1 gives

$$\|h-1\|_{W^{1,2}(\mathbf{R}^2)} \;\leq\; C\|\mathring{A}\|_{L^2(\mathbf{S}^2)}. \tag{3}$$

Therefore, by Sobolev embeddings, for every $p < \infty$ there exists a constant C_p such that

$$\|h-1\|_{L^p(\mathbf{S}^2)} \;\leq\; C_p\|\mathring{A}\|_{L^2(\mathbf{S}^2)},$$

From (3) we cannot get a similar estimate for $\|h-1\|_{L^\infty}$. Nonetheless in this paper we show that such an estimate holds.

Theorem 2 *There exists a universal constant C with the following property. Let Σ be any given compact connected surface of $\mathbf{R}^3$ without boundary, such that* $\mathrm{ar}(\Sigma) = 4\pi$ *and* $\|\mathring{A}\|_{L^2(\Sigma)} \leq 8\pi$*. Then the conformal parameterization ψ of Theorem 1 enjoys the bound*

$$\|h-1\|_{C^0(\mathbf{S}^2)} \;\leq\; C\|\mathring{A}\|_{L^2(\Sigma)}. \tag{4}$$

We prove this estimate by suitably modifying techniques and ideas from [4]. There the authors showed bounds for $\|h\|_\infty$ when $A \in L^2$, by proving suitable bounds for $\det A$ in the Hardy space $\mathcal{H}^1$. These Hardy bounds were achieved through the $\mathbf{R}^n$ theory of [1] after locally lifting the Gauss map $N : \Sigma \to \mathbf{S}^2$ to a suitable map $M : \Sigma \to \mathbf{S}^5$. The same strategy can be implemented using $\mathbf{S}^3$–liftings. The core of Theorem 2 consists in showing that when $\|\mathring{A}\|_{L^2}$ is small, these liftings can be chosen $W^{1,2}$–close to suitable liftings of the identity map.

Estimate (4) is crucial to conclude that some geometric constants of Σ are close to the corresponding ones of $\mathbf{S}^2$. For instance it implies that the spectrum of the Laplace–Beltrami operator of Σ is close to that of $\mathbf{S}^2$. More precisely, given a compact surface Γ without boundary, we denote by $\lambda_i(\Gamma)$ the i-th eigenvalue of the Laplace–Beltrami operator, with the following conventions: $\lambda_0(\Gamma) = 0$ and if a is an eigenvalue with multiplicity n, then it appears n times in the sequence $\{\lambda_i(\Gamma)\}$ (e.g. $\lambda_1(\mathbf{S}^2) = \lambda_2(\mathbf{S}^2) = \lambda_3(\mathbf{S}^2) = 2$).

Corollary 1 *For each i there exists a constant C_i with the following property. Let Σ be any given compact connected surface of $\mathbf{R}^3$ without boundary, such that* $\mathrm{ar}(\Sigma) = 4\pi$ *and* $\|\mathring{A}\|_{L^2(\Sigma)} \leq 4\pi$*. Then*

$$|\lambda_i(\Sigma) - \lambda_i(\mathbf{S}^2)| \;\leq\; C_i\|\mathring{A}\|_{L^2(\Sigma)}. \tag{5}$$

2 Hardy bounds

We denote by

- N the Gauss map on Σ;
- M the map $M := N \circ \psi$;

– K_Σ the Gauss curvature $\det dN$;
– K the function $K := K_\Sigma \circ \psi$;
– ω the standard volume form on $\mathbf{S}^2$.

In order to simplify the notation, for every 2-form α on $\mathbf{S}^2$ and every function space H, we denote by $\|\alpha\|_H$ the number $\|f\|_H$, where $f\omega = \alpha$.

Then Theorem 2 follows from the following Hardy bound.

Proposition 1 *There exist positive constants C and ε such that the following holds. If $M : \mathbf{S}^2 \to \mathbf{S}^2$ is a map such that $\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \leq \varepsilon$, then*

$$\|M^*\omega - \omega\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)}. \tag{6}$$

Proof (Proof of Theorem 2) Since h is a positive function there exists a unique function u such that $h = e^u$. Set

$$\delta := \|\mathring{A}\|_{L^2(\Sigma)}. \tag{7}$$

From Proposition 3.2 of [2] we have that, under the assumptions of Theorem 2, there exists a universal constant C_1 such that

$$\|u\|_{C^0} + \|u\|_{W^{2,1}} \leq C_1. \tag{8}$$

Thus it suffices to prove the existence of positive constants η and C_2 such that

$$\|u\|_{C^0} \leq C_2\delta \quad \text{whenever } \delta < \eta. \tag{9}$$

Thanks to Theorem 1 and to the bounds (8), there exists a universal constant C_3 such that

$$\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \leq C_3\delta\,. \tag{10}$$

Let ε be the constant of Proposition 1 and $\delta < \eta = \varepsilon/C_3$. Then we have

$$\|M^*\omega - \omega\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C_4\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \leq C_5\delta\,. \tag{11}$$

Note that $Ke^{2u}\omega = M^*\omega$ and hence (11) gives

$$\|Ke^{2u} - 1\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C_5\delta\,. \tag{12}$$

Recall that u satisfies

$$-\Delta_{\mathbf{S}^2} u = Ke^{2u} - 1\,. \tag{13}$$

Since the only harmonic functions on $\mathbf{S}^2$ are the constants, the bound (12) and the results of [3] imply that

$$\|u - c\|_{C^0} \leq C_6\delta \quad \text{for some constant } c. \tag{14}$$

The conformality of ψ gives $4\pi = \mathrm{ar}(\Sigma) = \int_{\mathbf{S}^2} e^{2u}$ and (8) implies

$$\left|e^{2(u-c)} - 1\right| \leq C_7|u - c|\,,$$

for some constant C_7. Therefore we have

$$4\pi|e^{2c} - 1| = e^{2c}\left|\int_{\mathbf{S}^2}\left(e^{2(u-c)} - 1\right)\right| \leq C_7C_6 4\pi\delta.$$

Hence there is a constant C_8 such that $|c| \leq C_8\delta$. From this and (14) we get (9). □

The Hardy bound of Proposition 1 is proved by "locally" lifting the maps M and id to maps into $\mathbf{S}^3$ via the Hopf fibration $\pi : \mathbf{S}^2 \to \mathbf{S}^3$. The reason why we cannot argue globally is that there is no such smooth lifting for the identity. Let $p \in \mathbf{S}^2$ and denote by $D_{\pi/2+1}(p)$ the geodesic disk of $\mathbf{S}^2$ with center p and radius $\pi/2+1$. Then in the next two sections we will prove the following proposition.

Proposition 2 (Hardy bound) *Let $\Psi \in C^\infty(\mathbf{S}^2, \mathbf{S}^2)$ be a fixed map with $\Psi(\mathbf{S}^2) \subset D_{\pi/2+1}(p)$. There exist positive constants C and ε, depending only on $\|\Psi\|_{C^2}$, such that:*

(HB) If $M \in C^\infty(\mathbf{S}^2, \mathbf{S}^2)$ satisfies $\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} \leq \varepsilon$, then

$$\|M^*\omega - \Psi^*\omega\|_{\mathcal{H}^1(\mathbf{S}^2)} \;\leq\; C\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}\,. \tag{15}$$

Note that, since $\Psi(\mathbf{S}^2) \subset D_{\pi/2+1}(p)$, there exists a smooth lifting of Ψ through the Hopf fibration (see Proposition 3). This lifting exists under the weaker assumption $\int_{\mathbf{S}^2} \Psi^*\omega = 0$. However, the stronger assumption $\Psi(\mathbf{S}^2) \subset D_{\pi/2+1}(p)$ will be crucial later in order to prove some estimates on the lifting (compare with the Second Step of the proof of Lemma 1).

From Proposition 2 one concludes Proposition 1 with a "cut and paste" procedure.

Proof (Proof of Proposition 1) First of all we introduce some notation. We let p be any point of $\mathbf{S}^2 \subset \mathbf{R}^3$. Then we let

$$D \;:=\; D_{\pi/2+1/2}(p) \quad \tilde{D} \;:=\; D_{\pi/2+1}(p).$$

We claim that if M is a smooth map and $\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)}$ is sufficiently small, then there exist two maps $M', \Psi' : \mathbf{S}^2 \to \mathbf{S}^2$ such that:

- $M' = M$ and $\Psi' = \mathrm{id}$ on D;
- $\Psi'(\mathbf{S}^2) \subset \tilde{D}$;
- The following estimates hold for some universal constant C:

$$\|M' - \Psi'\|_{W^{1,2}(\mathbf{S}^2)} \;\leq\; C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \quad \|\Psi'\|_{C^2} \;\leq\; C. \tag{16}$$

This fact, combined with Proposition 2, yields the the existence of two positive constants C and ε such that

$$\|M^*\omega - \omega\|_{\mathcal{H}^1(D_{\pi/2+1/2}(p))} \;\leq\; C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \tag{17}$$

for all $p \in \mathbf{S}^2$ and all M with $\|M - \mathrm{id}\|_{W^{1,2}} < \varepsilon$. Note that if p and q are two antipodal points, then

$$D_{\pi/2+1/2}(p) \cup D_{\pi/2+1/2}(q) \;=\; \mathbf{S}^2.$$

Therefore from (17) we would get

$$\|M^*\omega - \omega\|_{\mathcal{H}^1(\mathbf{S}^2)} \;\leq\; C'\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)}, \tag{18}$$

which is the desired conclusion. It remains to prove the existence of the maps M' and Ψ'.

First Step By Fubini's Theorem, there exists a universal constant C with the following property: There exists $\rho \in [\pi/2 + 1/2, \pi/2 + 3/4]$ such that

$$\begin{aligned}\|M - \mathrm{id}\|_{W^{1,2}(\partial D_\rho(p))} &\leq \|M - \mathrm{id}\|_{L^2(\partial D_\rho(p))} + \|D(M - \mathrm{id})\|_{L^2(\partial D_\rho(p))} \\ &\leq C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} . \end{aligned} \tag{19}$$

Now let us fix radial coordinates θ, r on $\tilde{D}$. We define $\tilde{M}, \tilde{\Psi} := \tilde{D} \to \mathbf{S}^2$ as

$$\tilde{M}(\theta, r) = \begin{cases} M(\theta, r) & \text{if } r < \rho \\ M(\theta, \pi/2 + 3/4) & \text{if } r \geq \rho \end{cases}$$

$$\tilde{\Psi}(\theta, r) = \begin{cases} \mathrm{id}(\theta, r) & \text{if } r < \rho \\ \mathrm{id}(\theta, \pi/2 + 3/4) & \text{if } r \geq \rho \ . \end{cases}$$

Clearly $\|\tilde{M} - \tilde{\Psi}\|_{W^{1,2}(\tilde{D})} \leq C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)}$ for some universal constant C.

Second Step We claim the existence of positive constants ε and η with the following property. If $\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \leq \varepsilon$, then there exists a point $q \in \mathbf{S}^2 \setminus \tilde{D}$ such that

$$\mathrm{dist}\,(q, \tilde{M}(\partial\tilde{D})) + \mathrm{dist}\,(q, \partial\tilde{D}) \ \geq \ \eta.$$

This claim will be proved later. Assuming it, we set $\zeta := \min\{1/8, \eta/2\}$. Using such a point q we can construct a C^2 map

$$R \ : \ \left[\pi/2 + 3/4, \pi/2 + 1\right] \times \left\{\mathbf{S}^2 \setminus D_\zeta(q)\right\} \ \to \ \mathbf{S}^2$$

such that:

- $R(t, \cdot)$ maps $\tilde{D}$ into $\tilde{D}$ for every t;
- $R(\pi/2 + 1, \cdot)$ maps $\mathbf{S}^2 \setminus D_\zeta(q)$ onto p;
- $\|R\|_{C^2}$ is bounded by a universal constant depending only on ζ.

Given such an R we define the maps $M', \Psi' : \tilde{D} \to \mathbf{S}^2$ as

$$M'(\theta, r) = \begin{cases} \tilde{M}(\theta, r) & \text{if } r < \pi + 3/4 \\ R(r, \tilde{M}(\theta, \pi/2 + 3/4)) & \text{if } r \geq \pi + 3/4 \end{cases}$$

$$\Psi'(\theta, r) = \begin{cases} \tilde{\Psi}(\theta, r) & \text{if } r < \pi + 3/4 \\ R(r, \tilde{\Psi}(\theta, \pi/2 + 3/4)) & \text{if } r \geq \pi + 3/4. \end{cases}$$

Finally, we extend both Ψ' and M' to $\mathbf{S}^2$ by setting $\Psi' = M' = p$ on $\mathbf{S}^2 \setminus \tilde{D}$. Then M' and Ψ' would satisfy all the requirements of the Lemma. Therefore, in order to conclude the proof it suffices to show the existence of the point q.

Third Step For any regular value $\tilde{q} \in \mathbf{S}^2 \setminus \tilde{M}(\partial\tilde{D})$ we define the degree $\deg(\tilde{q}, \tilde{M}, \tilde{D})$ in the usual way. It is a classical fact that deg is constant in the connected component of $\mathbf{S}^2 \setminus \tilde{M}(\partial\tilde{D})$. Hence we extend it to $\mathbf{S}^2 \setminus \tilde{M}(\partial\tilde{D})$ by continuity and we set

$$U_0 := \left\{\tilde{q} \in \mathbf{S}^2 \ : \ \deg(\tilde{q}, \tilde{M}, \tilde{D}) \ = \ 0\right\}.$$

It turns out that U_0 is an open set with boundary contained in the curve

$$\gamma = \tilde{M}(\partial \tilde{D}) = M(\partial \tilde{D}).$$

By (19) the length of γ is less than $C + C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)}$, for some universal constant C.

Consider the open set $U := U_0 \setminus \tilde{D}$. Clearly, $\{\mathbf{S}^2 \setminus [\tilde{M}(\tilde{D}) \cup \tilde{D}]\} \subset U$. Moreover, by construction we have $\tilde{M}(\tilde{D}) \subset M(\tilde{D})$. From the area formula it follows that

$$\mathrm{ar}\big(M(\tilde{D}) \setminus \tilde{D}\big) \;\le\; C\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)}.$$

Therefore, there exist positive universal constants C_1, C_2, C_3 such that, if $\|M - \mathrm{id}\|_{W^{1,2}(\mathbf{S}^2)} \le C_1$ then U is an open set with the following properties:

– ∂U is contained in the union of two connected curves $\gamma = \tilde{M}(\partial \tilde{D})$ and $\tilde{\gamma} = \partial \tilde{D}$;
– $\mathrm{ar}(U) \ge C_2$ and $\mathrm{len}\,(\gamma) + \mathrm{len}\,(\tilde{\gamma}) \le C_3$.

An elementary argument shows the existence of a positive constant η such that every U satisfying the conditions above contains a disk of radius η (see for instance Lemma C.1 of [2]). The center of this disk is the desired point q. □

3 Liftings through Hopf fibration

Denote by $\pi : \mathbf{S}^3 \subset \mathbb{C}^2 \to \mathbf{S}^2$ the Hopf fibration. Note that if we choose ε small enough in Proposition 2, then we have

$$\left| \int_{\mathbf{S}^2} (M^*\omega - \Psi^*\omega) \right| < 1\,. \tag{20}$$

From classical topological arguments we know that $\int_{\mathbf{S}^2} M^*\omega$ is an integer and that $\int_{\mathbf{S}^2} \Psi^*\omega = 0$ (this last equality follows from the assumption $\Psi(\mathbf{S}^2) \subset \tilde{D}$). Therefore $\int_{\mathbf{S}^2} M^*\omega = 0$.

The condition $\int_{\mathbf{S}^2} \Psi^*\omega = \int_{\mathbf{S}^2} M^*\omega = 0$ implies that the maps Ψ and M are homotopically trivial. Therefore there exist smooth maps $\Phi, F : \mathbf{S}^2 \to \mathbf{S}^3$ with $\pi \circ \Phi = \Psi$ and $\pi \circ F = M$. One main idea of [4] is that one can prove an Hardy bound $\|M^*\omega\|_{\mathcal{H}^1}$ by showing that the lifting Ψ can be chosen with bounded $W^{1,2}$ norm. (In passing we remark that in the paper [4] the authors used liftings to $\mathbf{S}^5$; however this is only a technical difference, mainly due to the fact that in [4] this technique is applied to the case of 2–dimensional surfaces in $\mathbf{R}^n$.) Therefore one naturally expects that, if the liftings Ψ and M can be chosen $W^{1,2}$–close, then one gets the bound (15).

Proposition 3 *Let Ψ and M be as in Proposition 2. Then there exist two maps $\Phi, F : \mathbf{S}^2 \to \mathbf{S}^3$ such that*

– $\Psi = \pi \circ \Phi$, $M = \pi \circ F$;
– $\|\Phi\|_{C^1} \le C$, $\|F - \Phi\|_{W^{1,2}(\mathbf{S}^2)} \le C\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}$.

The constant C depends only on $\|\Psi\|_{C^2}$ and not on M.

Building on this proposition, the proof of Proposition 2 is a short argument. However we set first a bit of notation. We fix coordinates on $\mathbb{C}^2$ so that

$$\mathbf{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1|^2 + |z_2|^2 = 1\}$$
$$\mathbf{S}^2 = \{(z, t) \in \mathbb{C} \times \mathbf{R} |\ |z|^2 + t^2 = 1\}.$$

Then the Hopf fibration is given by $\pi(z_1, z_2) = (2\bar{z}_1 z_2, |z_1|^2 - |z_2|^2)$. Note that if $p = (z_1, z_2) \in \mathbf{S}^3$, then the fiber

$$F_p := \{(w_1, w_2) |\ \pi(w_1, w_2) = \pi(z_1, z_2)\} \tag{21}$$

is given by $\{(e^{i\theta} z_1, e^{i\theta} z_2), \theta \in \mathbf{R}\}$.

Proof (Proof of Proposition 2) Let Φ and F be the liftings of Proposition 3. Using the coordinates above we write $\Phi = (\Phi_1, \Phi_2)$ and $F = (F_1, F_2)$. The following identities can be easily checked:

$$\begin{aligned} 2\Psi^*\omega &= 2\Phi^*\pi^*\omega = i(d\Phi_1 \wedge d\bar{\Phi}_1 + d\Phi_2 \wedge d\bar{\Phi}_2) \\ 2M^*\omega &= 2F^*\pi^*\omega = i(dF_1 \wedge d\bar{F}_1 + dF_2 \wedge d\bar{F}_2). \end{aligned} \tag{22}$$

Note that

$$\begin{aligned} 2(\Psi^*\omega - M^*\omega) = i\{ & d\Phi_1 \wedge d(\bar{\Phi}_1 - \bar{F}_1) + d(\Phi_1 - F_1) \wedge d\bar{F}_1 \\ & + d\Phi_2 \wedge d(\bar{\Phi}_2 - \bar{F}_2) + d(\Phi_2 - F_2) \wedge d\bar{F}_2\} \end{aligned}$$

Hence, using the results of [1] we get

$$\|\Psi^*\omega - M^*\omega\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C\big(\|dF\|_{L^2(\mathbf{S}^2)} + \|d\Phi\|_{L^2(\mathbf{S}^2)}\big)\|dF - d\Phi\|_{L^2(\mathbf{S}^2)}.$$

Therefore the bounds satisfied by F and Φ yield the desired estimate. □

The rest of the paper is devoted to prove the existence of the liftings claimed in Proposition 3. First we introduce a suitable norm on differentials of maps with target in $\mathbf{S}^3$, see (24). This norm is invariant under the action of $\mathbf{S}^3$ on itself as Lie group.

We recall that $\mathbb{C}^2$ can be identified to the field of quaternions $\mathbb{H}$. We denote by $\times$ the multiplication between quaternions and we recall that the usual norm $|\cdot|$ has the property that $|a \times b| = |a||b|$. Hence, $\times$ naturally induces a Lie group structure on $\mathbf{S}^3$ and the maps $l^w : \mathbf{S}^3 \to \mathbf{S}^3$ given by $l^w(a) = w \times a$ are isometries of $\mathbf{S}^3$. The same holds for the maps $r^w : \mathbf{S}^3 \to \mathbf{S}^3$ given by $r^w(a) = a \times w$.

Definition 1 Given $a, b \in \mathbf{S}^3$ and $\xi \in T_a\mathbf{S}^3$ we denote by $b\xi$ the vector of $T_{b\times a}\mathbf{S}^3$ given by $dl^b|_a(\xi)$. In a similar way we define ξb as $dr^b|_a(\xi) \in T_{a\times b}\mathbf{S}^3$.

The diffeomorphisms l^x allow to define an "intrinsic" notion of distance between vectors belonging to $T_a\mathbf{S}^3$ and $T_b\mathbf{S}^3$. This allows a natural way to compare the differential of two distinct maps with target in $\mathbf{S}^3$.

Definition 2 Given $\xi \in T_b\mathbf{S}^3$, $\zeta \in T_a\mathbf{S}^3$ we denote by $|\xi - \zeta|_{\mathcal{L}}$ the nonnegative real number

$$|a^{-1}\xi - b^{-1}\eta| \;=\; |(b \times a^{-1})\xi - \eta| \;=\; |\xi - (a \times b^{-1})\zeta|\,,$$

where, for vectors $\lambda, \mu \in T_p\mathbf{S}^3$, $|\lambda - \mu|$ denotes the usual Hilbert norm (that is, the norm induced by the Riemann structure of $\mathbf{S}^3$ as submanifold of $\mathbf{R}^4$).

Given a riemannian manifold Ω and smooth maps $F, \Phi : \Omega \to \mathbf{S}^3$, we define

$$|dF|_p - d\Phi|_p|_{\mathcal{L}} \;:=\; \sup_{|\xi|=1} |dF|_p(\xi) - d\Phi|_p(\xi)|_{\mathcal{L}} \tag{23}$$

$$|||dF - d\Phi|||_{L^2(\Omega)} \;:=\; \left(\int_\Omega |dF - d\Phi|^2_{\mathcal{L}}\right)^{1/2} . \tag{24}$$

The proof of Proposition 3 is based on two lemmas. The first one, Lemma 1, shows the existence of liftings for which one can estimate the norm $|||dF - d\Phi|||_{L^2(D_r)}$ as in (26). The second, Lemma 2, is a Poincare' type inequality. With the help of this inequality, one can absorb the second term of (26), provided r is smaller than a universal constant. This gives an estimate of the form

$$|||dF - d\Phi|||_{L^2(D_r)} \;\leq\; C\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} . \tag{25}$$

The number of disks D_r needed to cover $\mathbf{S}^2$ is smaller than a universal constant. Therefore we can bound $|||dF - d\Phi|||_{L^2(\mathbf{S}^2)}$. We then use again Lemma 2 to show the existence of a new lifting $\tilde{F}$ such that

$$|||d\tilde{F} - d\Phi|||_{L^2(\mathbf{S}^2)} + \|\tilde{F} - \Phi\|_{L^2} \;\leq\; C\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} .$$

Finally it is not difficult to show that

$$\|\tilde{F} - \Phi\|_{W^{1,2}(\mathbf{S}^2)} \;\leq\; |||d\tilde{F} - d\Phi|||_{L^2(\mathbf{S}^2)} + \|F - \Phi\|_{L^2} .$$

Lemma 1 *Let M and Ψ be as in Proposition 2 and choose ε sufficiently small so that M is homotopically trivial. Then there exists a universal constant C and two maps $F, \Phi : \mathbf{S}^2 \to \mathbf{S}^3$ such that:*

- $\Psi = \pi \circ \Phi$, $M = \pi \circ F$ *and* $\|\Phi\|_{C^1} \leq C$;
- *For every disk $D_r \subset \mathbf{S}^2$ we have the estimate*

$$|||dF - d\Phi|||_{L^2(D_r)} \;\leq\; C\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} + C\min_\theta \|F - e^{i\theta}\Phi\|_{L^2(D_r)} \tag{26}$$

Lemma 2 *Let D_r be a geodesic disk of $\mathbf{S}^3$ and $\Phi, F : D_r \to \mathbf{S}^3$ two smooth maps. Then*

$$\min_{w \in \mathbf{S}^3} \|F - w \times \Phi\|_{L^2(D_r)} \;\leq\; Cr|||dF - d\Phi|||_{L^2(D_r)} , \tag{27}$$

for some universal constant C.

The proof of Lemma 1 is given in the next Section. Hereby we prove Lemma 2 and we show how to conclude Proposition 2.

Proof (Proof of Lemma 2) Let $G : D_r \to \mathbf{S}^3 \subset \mathbb{H}$ be given by $G(p) = F(p) \times \Phi(p)^{-1}$. Using the notation of Definition 1 we write

$$dG_p(\xi) = (dF|_p(\xi))\Phi(p)^{-1} - [F(p)\Phi(p)^{-1}](d\Phi|_p(\xi))\Phi(p)^{-1}.$$

Since the multiplication from the right is an isometry, we get $|\zeta b - \xi b| = |\zeta - \xi|$ for every $\xi \in T_a\mathbf{S}^3$, $\zeta \in T_a\mathbf{S}^3$. Hence

$$|dG|_p(\xi)| = |dF|_p(\xi) - [F(p)\Phi(p)^{-1}](d\Phi|_p(\xi))|. \tag{28}$$

We remark that the right hand side of (28) is precisely the definition of $|dF|_p(\xi) - d\Phi|_p(\xi)|_{\mathcal{L}}$. Thus,

$$\|dG\|_{L^2(D_r)} = |||dF - d\Phi|||_{L^2(D_r)}.$$

Hence, by the usual Poincaré inequality on Euclidean spaces, there exists $w \in \mathbb{H} = \mathbb{C}^2$ such that

$$\|G - w\|_{L^2(D_r)} \leq Cr\|dG\|_{L^2(D_r)} = Cr|||dF - d\Phi|||_{L^2(D_r)}.$$

Note that

$$\begin{aligned}\pi r^2|1 - |w||| = \int_{D_r} ||G| - |w|| &\leq \|G - w\|_{L^1(D_r)} \\ \leq C_1 r\|G - w\|_{L^2(D_r)} &\leq C_2 r^2|||dF - d\Phi|||_{L^2(D_r)}.\end{aligned} \tag{29}$$

Set $\tilde{w} := w/|w|$. Then, by (29), we have $|\tilde{w} - w| = |1 - |w|| \leq C_3|||dF - d\Phi|||_{L^2(D_r)}$. Hence

$$\|G - \tilde{w}\|_{L^2(D_r)} \leq C_4 r|\tilde{w} - w| + C_5\|w - G\|_{L^2(D_r)} \leq C_6 r|||dF - d\Phi|||_{L^2(D_r)}. \tag{30}$$

Since $\tilde{w} \in \mathbf{S}^3$, this gives the desired inequality.

Proof (Proof of Proposition 2) We start from the liftings F and Φ provided by Lemma 1 and we break the proof into two steps.

First Step In this step we show that

$$|||dF - d\Phi|||_{L^2(D_r)} \leq C_2\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} \quad \text{if } r \leq C_1, \tag{31}$$

for some universal constant C_1. Since $\mathbf{S}^2$ is compact (31) implies

$$|||dF - d\Phi|||_{L^2(\mathbf{S}^2)} \leq C\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \tag{32}$$

Recall the Poincaré inequality proved in Lemma 2:

$$\min_{w\in\mathbf{S}^3} \|F - w \times \Phi\|_{L^2(D_r)} \leq Cr|||dF - d\Phi|||_{L^2(D_r)} \tag{33}$$

Let w be a point where the minimum in the left hand side of (33) is attained and let θ_0 be a point where $f(\theta) = |w - e^{i\theta}|$ attains its minimum. Recall that the quaternionic multiplication by an element of $\mathbf{S}^3$ is an isometry of $\mathbf{S}^3$. Thus, for every $a \in \mathbf{S}^3$, the function $f_a(\theta) = |w \times a - e^{i\theta}a|$ attains its minimum in θ_0.

It is not difficult to check that

$$\min_\theta |w \times a - e^{i\theta} a| \;\le\; C_1 |\pi(w \times a) - \pi(a)|\,,$$

for some universal constant C_1. Moreover, recall that π is Lipschitz and call C_2 its Lipschitz constant. Thus

$$\begin{aligned}\|w \times \Phi - e^{i\theta_0}\Phi\|_{L^2(D_r)} &\le C_1\|\pi(w \times \Phi) - \pi(\Phi)\|_{L^2(D_r)}\\ &\le C_1\|\pi(w \times \Phi) - \pi(F)\|_{L^2(D_r)} + C_1\|\pi(F) - \pi(\Phi)\|_{L^2(D_r)}\\ &\le C_1C_2\|w \times \Phi - F\|_{L^2(D_r)} + C_1\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \qquad (34)\end{aligned}$$

Combining (34) and (33) we get

$$\min_\theta \|F - e^{i\theta}\Phi\|_{L^2(D_r)} \;\le\; C_3 r|||dF - d\Phi|||_{L^2(D_r)} + C_4\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \qquad (35)$$

Plugging (35) into (26) we get

$$|||dF - d\Phi|||_{L^2(D_r)} \;\le\; C_5\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} + C_6 r|||dF - d\Phi|||_{L^2(D_r)}. \qquad (36)$$

Thus it is sufficient to choose $r \le (2C_6)^{-1}$ to get

$$|||dF - d\Phi|||_{L^2(D_r)} \;\le\; 2C_7\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \qquad (37)$$

This gives (31) and hence completes the proof of (32).

Second Step Conclusion

Let $\xi \in T_a\mathbf{S}^3$, $\zeta \in T_b\mathbf{S}^3$. The following elementary inequality holds:

$$|\xi - \zeta| \;\le\; C|\xi|\big|a - b\big| + C|\xi - \zeta|_{\mathcal{L}}. \qquad (38)$$

Indeed, since the map

$$\mathbf{S}^3 \times T\mathbf{S}^3 \ni (w, a, \xi) \;\to\; w\xi \in T_{w\times a}\mathbf{S}^3 \subset \mathbb{C}^2$$

is Lipschitz on compact sets, we have

$$|\xi - (b \times a^{-1})\xi|_{\mathcal{L}} \;\le\; C|1 - b \times a^{-1}| \;=\; C|a - b| \quad \text{for } |\xi| \le 1.$$

Thus, if we define $\tilde{\xi} = \xi/|\xi|$ we get

$$\begin{aligned}|\xi - \zeta| &\le |(b \times a^{-1})\xi - \zeta| + |(b \times a^{-1})\xi - \xi|\\ &= |\xi - \zeta|_{\mathcal{L}} + |\xi||(b \times a^{-1})\tilde{\xi} - \tilde{\xi}|\\ &\le |\xi - \zeta|_{\mathcal{L}} + C|\xi||b - a|.\end{aligned}$$

Let θ_0 be a point where the expression

$$g(\theta) = \|e^{i\theta}F - \Phi\|_{L^2(\mathbf{S}^2)}$$

attains its minimum. Set $\tilde{F} = e^{i\theta_0}F$. Replacing D_r with $\mathbf{S}^2$ in (35) we get

$$\begin{aligned}\|\tilde{F} - \Phi\|_{L^2(\mathbf{S}^2)} &\le C_1|||dF - d\Phi|||_{L^2(\mathbf{S}^2)} + C_1\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}\\ &\le C_2\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} \qquad (39)\end{aligned}$$

From (38) we get

$$|d\tilde{F} - d\Phi|^2 \ \le\ 2|d\tilde{F} - d\Phi|^2_{\mathcal{L}} + 2C^2|d\Phi|^2|\tilde{F} - \Phi|^2 .$$

Integrating this inequality we get

$$\begin{aligned}\|d\tilde{F} - d\Phi\|^2_{L^2(\mathbf{S}^2)} \ &\le\ 2C^2 \int_{\mathbf{S}^2} |d\Phi|^2|\tilde{F} - \Phi|^2 + |||dF - d\Phi|||^2_{L^2(\mathbf{S}^2)} \\ &\overset{(32)}{\le}\ C_3\|\Phi\|^2_{C^1}\|\tilde{F} - \Phi\|^2_{L^2} + C_4\|M - \Psi\|^2_{W^{1,2}(\mathbf{S}^2)} \\ &\overset{(39)}{\le}\ C_5\|M - \Psi\|^2_{W^{1,2}(\mathbf{S}^2)} . \end{aligned} \tag{40}$$

This concludes the proof.

4 Proof of Lemma 1

Recall the definition of F_p given in (21) and note that the vector tangent to F_p in $p = (z_1, z_2)$ is (iz_1, iz_2). Thus, we decompose $T_p\mathbf{S}^3$ into two orthogonal subspaces:

$$TF_p \ =\ \{t(iz_1, iz_2)|\ t \in \mathbf{R}\} \quad TN_p \ =\ \{w \in T_p\mathbf{S}^3|\ w \cdot (iz_1, iz_2) = 0\}, \tag{41}$$

where the hermitian product $(a_1, a_2) \cdot (b_1, b_2)$ is given by $\mathrm{Re}\,(a_1\bar{b}_1 + a_2\bar{b}_2)$.

Definition 3 If $\Phi : \Omega \to \mathbf{S}^3$ is a smooth map, we we write $d\Phi = d_1\Phi + d_2\Phi$, where

- $d_1\Phi|_q(\xi)$ is the projection of $d\Phi|_q(\xi)$ on $TF_{\Phi(q)}$,
- $d_2\Phi|_q(\xi)$ is the projection of $d\Phi|_q(\xi)$ on $TN_{\Phi(q)}$.

Proof (Proof of Lemma 1)
First Step In this step we derive a preliminary estimate on $\big\||dF| - |d\Phi|\big\|_{L^2(\mathbf{S}^2)}$, provided F and Φ are chosen in a suitable way.

First of all fix any pair of liftings (F, Φ). It can be easily checked that $|d_2F| = |dM|$ and $|d_2\Phi| = |d\Psi|$. Moreover, if we define the 1–form $\alpha := -i\bar{z}_1dz_2 - i\bar{z}_2dz_2$, then we get

$$d_1F \ =\ (iF_1, iF_2)F^*\alpha \quad d_1\Phi \ =\ (i\Phi_1, i\Phi_2)\Phi^*\alpha. \tag{42}$$

Thus

$$\int_{\mathbf{S}^2} ||d_1F| - |d_1\Phi||^2 \ =\ \int_{\mathbf{S}^2} |F^*\alpha - \Phi^*\alpha|^2.$$

We will show that the liftings F and Φ can be chosen so that

$$\int_{\mathbf{S}^2} |F^*\alpha - \Phi^*\alpha|^2 \ =\ \|M^*\omega - \Psi^*\omega\|^2_{W^{-1,2}}.$$

Indeed, fix a lifting $\tilde{F} : \mathbf{S}^2 \to \mathbf{S}^3$ of M and set $\beta = \tilde{F}^*\alpha$. We can use the standard Hodge decomposition to write

$$\beta \ =\ d\theta + *d\psi$$

where θ and ψ are smooth functions on $\mathbf{S}^2$. If we set $F = e^{-i\theta}\tilde{F}$ we get $F^*\alpha = *d\psi$. We can make a similar choice for Φ and note that since $\Psi \in C^2$, standard linear theory for elliptic PDEs gives that our Φ is in C^1. Thus we get

$$F^*\alpha - \Phi^*\alpha \;=\; *df \quad \text{for some function } f.$$

This implies that

$$\int_{\mathbf{S}^2} |F^*\alpha - \Phi^*\alpha|^2 \;=\; \|d*df\|^2_{W^{-1,2}} \;=\; \|F^*d\alpha - \Phi^*d\alpha\|^2_{W^{-1,2}}. \qquad (43)$$

By (22) we have $2F^*d\alpha = M^*\pi^*(idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2) = 2M^*\omega$ and $2\Phi^*d\alpha = 2\Psi^*\omega$. Thus, we conclude that

$$\begin{aligned}\||dF| - |d\Phi|\|_{L^2} &\leq \||d_1F| - |d_1\Phi|\|_{L^2} + \||d_2F| - |d_2\Phi|\|_{L^2}\\ &= \|M^*\omega - \Psi^*\omega\|_{W^{-1,2}} + \||dM| - |d\Psi|\|_{L^2}. \qquad (44)\end{aligned}$$

Second Step In this step we show how to estimate $\|M^*\omega - \Psi^*\omega\|_{W^{-1,2}}$.

Recall that $\Psi(\mathbf{S}^2) \subset \tilde{D}$, which is the geodesic disk $D_{\pi/2+1}(p)$. Denote by n the antipodal of p. From the area formula there exists a constant C_1 such that

$$\text{ar}(M(\mathbf{S}^2) \cap D_{1/2}(n)) \leq C_1\|M - \Psi\|^2_{L^2(\mathbf{S}^2)}.$$

Therefore if $\|M - \Psi\|^2_{L^2(\mathbf{S}^2)}$ is sufficiently small, $\text{ar}(D_{1/2}(n) \setminus M(\mathbf{S}^2)) \geq C_2$ for some positive constant C_2. We claim the existence of a 1-form η such that:

- $\omega = d\eta$ on $\tilde{D} \cup M(\mathbf{S}^2)$;
- $\|\eta\|_{L^\infty} \leq \frac{C}{C_2}$;
- $|\eta(x) - \eta(y)| \leq C|x - y|$ for every $x, y \in \tilde{D}$;

where C is a universal constant.

We construct η in the following way. First, for every $x \in \mathbf{S}^2$ we take the form $\eta_x \in C^\infty(\mathbf{S}^2 \setminus \{x\}) \cap L^1(\mathbf{S}^2)$ defined in 3.5.1 of [4]. This "canonical" form has a singularity in x but satisfies $d\eta_x = \omega$ on $\mathbf{S}^2 \setminus \{x\}$.

Then we take a closed set $E \subset D_{1/2}(n) \setminus M(\mathbf{S}^2)$ such that

$$\text{ar}(E) \;=\; \frac{1}{2}\text{ar}\,(D_{1/2}(n) \setminus M(\mathbf{S}^2))$$

and we define

$$\eta \;:=\; \frac{1}{\text{ar}(E)} \int_{x\in E} \eta_x$$

Clearly $d\eta = \omega$ on $D \cup M(\mathbf{S}^2) \subset \mathbf{S}^2 \setminus E$. Moreover, η is smooth on the closure of $\tilde{D}$. The estimate $\|\eta\|_{L^\infty} \leq C(\text{ar}(E))^{-1}$ can be proved as in 3.5.5 of [4]. Finally we compute

$$\begin{aligned}\|M^*\omega - \Psi^*\omega\|_{W^{-1,2}(\mathbf{S}^2)} &= \|d(M^*\eta - \Psi^*\eta)\|_{W^{-1,2}}\\ &= \sup_{\|\varphi\|_{W^{1,2}}=1} \int_{\mathbf{S}^2} \varphi\, d(M^*\eta - \Psi^*\eta)\\ &= \sup_{\|\varphi\|_{W^{1,2}}=1} \int_{\mathbf{S}^2} d\varphi \wedge (M^*\eta - \Psi^*\eta).\end{aligned}$$

Now, write $\eta = f_1 dx_1 + f_2 dx_2$ in some local coordinates and note that

$$\begin{aligned}\Psi^*(f_i dx_i) - M^*(f_i dx_i) &= f_i(\Psi)\, d\Psi_i - f_i(M)\, dM_i \\ &= [f_i(\Psi) - f_i(M)] d\Psi_i + f_i(M) d[\Psi_i - M_i]. \end{aligned} \quad (45)$$

Set $\mathbf{S}^b := \{p | M(p) \notin \tilde{D}\}$ and $\mathbf{S}^g := \mathbf{S}^2 \setminus \mathbf{S}^b$. Then we have

$$|\Psi^*\eta - M^*\eta| \le \begin{cases} C|d\Psi|\big|\Psi - M\big| + C|d\Psi - dM| & \text{on } \mathbf{S}^g \\ 2C|d\Psi| + C|d\Psi - dM| & \text{on } \mathbf{S}^b. \end{cases}$$

Thus we can estimate

$$\begin{aligned}&\left|\int_{\mathbf{S}^2} d\varphi \wedge (\Psi^*\eta - M^*\eta)\right| \\ &\le C\|\Psi\|_{C^1} \int_{\mathbf{S}^g} |d\varphi||\Psi - M| + 2C\|\Psi\|_{C^1} \int_{\mathbf{S}^b} |d\varphi| + C \int_{\mathbf{S}^2} |d\varphi||d(\Psi - M)| \\ &\le C\|\Psi\|_{C^1} \|d\varphi\|_{L^2} \|\Psi - M\|_{L^2} + 2C\|\Psi\|_{C^1} \|d\varphi\|_{L^2} (\mathrm{ar}(\mathbf{S}^b))^{1/2} \\ &\quad + C\|d\varphi\|_{L^2} \|d\Psi - dM\|_{L^2}. \end{aligned}$$

Recalling that $\|d\varphi\|_{L^2} \le \|\varphi\|_{W^{1,2}} = 1$ and that $(\mathrm{ar}(\mathbf{S}^b))^{1/2} \le C\|\Psi - M\|_{L^2}$, we derive

$$\|M^*\omega - \Psi^*\omega\|_{W^{-1,2}} \le C_1 \|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \quad (46)$$

This, together with (44), gives

$$\||dF| - |d\Phi|\|_{L^2} \le C_2 \|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \quad (47)$$

Moreover, for a later use, we remark that (46) and (43) give

$$\|F^*\alpha - \Phi^*\alpha\|_{L^2(\mathbf{S}^2)} \le C_3 \|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)}. \quad (48)$$

Third Step We now come to the proof of (26).

We begin with the following pointwise inequality:

$$|dF - d\Phi|^2_{\mathcal{L}} \le C_2(|dF| + |d\Phi|)^2 |F - \Phi|^2 + 4|dM - d\Phi|^2 + 4|F^*\alpha - \Phi^*\alpha|^2, \quad (49)$$

where α is the differential form $-i\bar{z}_1 dz_2 - i\bar{z}_2 dz_2$, which satisfies (42).

In order to prove (49), for every $\xi \in T_a\mathbf{S}^3$, $\zeta \in T_b\mathbf{S}^3$ we define a distance $d(\xi, \zeta)$ in the following way. We write $\xi = \tilde{\xi} + tia$ and $\zeta = \tilde{\zeta} + \tau ib$, where $\tilde{\xi} \in TN_a$, $\tilde{\zeta} \in TN_b$ and $t, \tau \in \mathbf{R}$ (see (41)). Then we set

$$d(\xi, \eta) := \sqrt{|d\pi_a(\xi) - d\pi_b(\zeta)|^2 + |\tau - t|^2}.$$

Now, construct the function $f : T_a\mathbf{S}^3 \times T_b\mathbf{S}^3 \to \mathbf{R}$ given by

$$f(a, b, \xi, \eta) = ||\xi - \eta|_{\mathcal{L}} - d(\xi, \eta)|.$$

Note that both d and $|\cdot|_{\mathcal{L}}$ are locally Lipschitz in a, b, ξ, and ζ. Moreover

$$d(\xi, \eta) = |\xi - \eta| = |\xi - \eta|_{\mathcal{L}} \quad \text{for } \xi, \eta \in T_a\mathbf{S}^3,$$

which translates into $f(a, a, \xi, \eta) = 0$. This condition and the locally Lipschitz property of f gives the existence of a constant C such that:

$$f(a, b, \xi, \eta) \ \le \ C|a - b| \quad \text{for } |\xi| + |\eta| \le 2. \tag{50}$$

Given *any* ξ, η we define $M := \max\{|\xi|, |\zeta|\}$ and $\hat{\xi} := \xi/M$, $\hat{\zeta} := \zeta/M$. Then we can compute

$$\begin{aligned} |\xi - \eta|_{\mathcal{L}} = M|\hat{\xi} - \hat{\zeta}|_{\mathcal{L}} \ &\le \ M d(\hat{\xi}, \hat{\zeta}) + CM|a - b| \\ &\le d(\xi, \eta) + C(|\xi| + |\eta|)|a - b| \,. \end{aligned} \tag{51}$$

From this we easily get (49). Integrating (49) and recalling (48) we get the inequality

$$\begin{aligned} & |||dF - d\Phi|||^2_{L^2(D_r)} \\ & \quad \le C_1 \int_{D_r} (|dF| + |d\Phi|)^2 |F - \Phi|^2 + C_2 \|M - \Psi\|^2_{W^{1,2}(\mathbf{S}^2)} \,. \end{aligned} \tag{52}$$

Moreover, we have

$$\begin{aligned} \int_{D_r} (|dF| + |d\Phi|)^2 |F - \Phi|^2 &\le \int_{D_r} \left(8|d\Phi|^2 + 2\big||dF| - |d\Phi|\big|^2\right)|F - \Phi|^2 \\ &\le 4\||dF| - |d\Phi|\|^2_{L^2(D_r)} + 8\|\Phi\|^2_{C^1}\|F - \Phi\|^2_{L^2(D_r)} \\ &\overset{(47)}{\le} C_3\|M - \Psi\|^2_{W^{1,2}(\mathbf{S}^2)} + C_4\|F - \Phi\|^2_{L^2(D_r)} \end{aligned} \tag{53}$$

Plugging (53) into (52) we derive

$$|||dF - d\Phi|||_{L^2(D_r)} \ \le \ C_5\|M - \Psi\|_{W^{1,2}(\mathbf{S}^2)} + C_6\|F - \Phi\|_{L^2(D_r)} \,. \tag{54}$$

Given $\theta \in \mathbf{R}$, define $\tilde{\Phi} = e^{i\theta}\Phi$. Then, clearly $|d\Phi - d\tilde{\Phi}|_{\mathcal{L}} = 0$. Note that $\tilde{\Phi}$ is a lifting of Ψ and that all the estimates derived for Φ holds for $\tilde{\Phi}$ as well. Hence from (54) we get (26).

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