SBV Regularity and Hamilton-Jacobi Equations

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I dedicate this thesis
to my beloved mother Alba.

Abstract

Many interesting problems in the physical, biological, engineering and social sciences are modelled by simple but powerful principles. Partial differential equations and in particular here Conservation laws and Hamilton-Jacobi equations are used to describe a very large set of problems. Thus, the mathematical research on these subjects has a big relevance due to the vast set of possible applications. In the first part of this work, we study the regularity properties of the solutions of these two special equations. The last part of the thesis is dedicated to an optimal control problem which is approached with the same techniques of the first part: we try to block the spreading of a contaminating agent minimizing the costs of the operation. We characterize the minimum time function at which one point will be contaminated as a solution of a particular Hamilton-Jacobi equations. The existence of an optimal strategy to block the contamination is then deduced as a Corollary.

Zusammenfassung

Viele interessante Probleme der Physik, Biologie, Ingenieurswesen und in der sozialen Wissenschaft basieren auf einfachen, aber aussagekräftigen Konzepten. Partielle Differentialgleichungen, konkret Erhaltungsgesetze und Hamilton-Jacobi Gleichungen, beschreiben eine grosse Anzahl von Problemen. Aufgrund der riesigen Anzahl an Anwendungen ist die mathematische Forschung auf diesen Gebieten sehr wichtig. Im ersten Teil dieser Arbeit untersuchen wir die Regularität von Lösungen der Gleichungen dieser beiden Typen. Der letzte Teil dieser Doktorarbeit befasst sich mit einem Problem aus der Optimalen Regelung. Wir gehen es mit Techniken und Methoden des ersten Teils an. Wir versuchen die Verbreitung eines kontaminierden Stoffes zu stoppen bei minimalen Kosten. Wir charakterisieren die minimale Zeit bei der ein Punkt kontaminiert wird durch die Lösung einer speziellen Hamilton-Jacobi Gleichung. Die Existenz einer optimalen Strategie zur Eindämmung der Kontamination wird schlussendlich in einem Korollar formuliert.

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Roger Robyr • Zurich, 2010.

Contents

0 Introduction

0.1 Introduction

Systems of Conservation Laws: Many interesting problems in the physical, biological, engineering and social sciences are modelled by simple conservation principles. To begin with, consider a domain $\Omega \subset \mathbb{R}^n$ where a quantity of interest U, defined for all points $x \in \Omega$, evolves in time. For instance, this quantity could be the temperature of a rod, the pressure of a fluid or gas, the density of a human population or the traffic on a road. In all these examples the evolution (in time) of U can be described by the following observation:

- *The time rate change of U in any sub-domain $\omega \subset \Omega$ is equal to the total amount of U produced or destroyed inside ω and the flux of U across the boundary $\partial\omega$.*

This means that the change of U is due to two factors: the *source* or *sink*, which represents the total amount of created and destroyed quantity in ω and the *flux*, which regulates the flow of the quantity on the boundary $\partial\omega$. The mathematical formulation of this principle is an integral equation for U in ω:

$$\int_\omega \frac{\partial U}{\partial t}\,dx = \frac{d}{dt}\int_\omega U\,dx = -\underbrace{\int_{\partial\omega} F\cdot\nu\,d\sigma(x)}_{flux} + \underbrace{\int_\omega S\,dx}_{\text{source}}$$

where ν is the unit outward normal and F and S are the flux and the source respectively. Next, by the divergence Theorem we have that:

$$\int_\omega U_t\,dx + \int_\omega divF\,dx = \int_\omega S\,dx.$$

and since the last equality holds for all subset $\omega \subset \Omega$ we obtain the following differential equation:

$$U_t + divF = S, \qquad \forall(x,t) \in \Omega\times\mathbb{R}^+. \tag{1}$$

Equation (1) is called *balance law* by the fact that the change of U is a balance of the flux and the source. If the source term is not present, i.e. if $S = 0$, we call equation (1) *conservation law*. The simplest example of such type of PDEs is given by the *transport equation*: we consider a conservation law with a simple flux function $F = a(x,t)U$. A relevant example that can be described by the transport equation is the following: let U denote the concentration of a pollutant in a river and assume that we know the velocity field $a(x,t)$ at all points of the river. Thus, the pollutant will be transported in the direction of the velocity and the flux is then given by $F = a(x,t)U$. If we assume that there is no creation and destruction of this chemical agent, for instance if no pollution pipes or no depuration of the water are present, we can set $S = 0$ obtaining:

$$U_t + div(a(x,t)U) = 0.$$

A second interesting scalar example is used to describe the *traffic flow*: if we let ρ be the density of cars and assume that the velocity v of a car depends only on the density, we get:

$$\rho_t + (\rho v)_x = 0.$$

More complex phenomena are described by systems of equations. Perhaps the oldest example is the *Euler equations of gas dynamics*. In the usual macroscopic description of an ideal gas, the three key variables that play a central role are the density ρ, the velocity field U and the internal energy ε. Note that all these three quantities can be measured experimentally. Thus, the Euler equations of gas dynamics are:

$$\rho_t + div(\rho U) = 0, \tag{2}$$
$$(\rho U)_t + div(\rho U \otimes U + pI) = 0, \tag{3}$$
$$E_t + div((E+p)U) = 0. \tag{4}$$

Equation (2) comes from the *conservation of mass*: it is well-known that the total mass of the gas is conserved. The second equation (3) is the *conservation of the momentum*: by Newton's second law of motion, the rate of change of momentum equals force. In absence of forces, the gas pressure exerts the only force on the gas. (Here $\otimes$ denotes the tensor product between any two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ of $\mathbb{R}^3$ and I is the 3×3 identity matrix.) The last equation (4) is the *conservation of energy*, which is the sum of the kinetic and internal energy:

$$E = \frac{1}{2}\rho|U|^2 + \rho\varepsilon.$$

Note that the pressure p is a function $p(\rho, \varepsilon)$.

Hamilton-Jacobi equations: Looking at the classical mechanics (see for instance [30]) or at optimal control problems another powerful principle comes into the play: the minimal least action (or Hamiltonian principle). By this principle we can characterize each mechanical system with the Lagrangian function $L(x, \dot{x}, t)$, which contains information on the conserved quantities, and if we assume that a body is free to move in the space, this object will follow a trajectory that minimizes the *action*:

$$A = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt.$$

By classical variational computations, we can associate to this problem the well-known *Euler-Lagrange equations*, which describe the motion of our body:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x_i}} - \frac{\partial L}{\partial x_i} = 0, \qquad i = 1, \ldots, s;$$

where s is the number of degree of freedom. Next, in prescribed mechanical problems we can take advantage substituting the velocity $\dot{x}$ with the impulse p, which can be subsequentially replaced by $\frac{\partial A}{\partial x}$. Thus, following the same steps in the derivation of the Euler-Lagrange equations we can obtain the *Hamilton-Jacobi equations* for the motion:

$$\frac{\partial A}{\partial t} + H\Big(x, \frac{\partial A}{\partial x}, t\Big) = 0$$

where the *Hamiltonian function* H is connected to the Lagrange function L by the Legendre transformation, i.e $H(p, q, t) = \sum_i^s p_i q_i - L$.

In one space dimension there exists an important connection between conservation laws and Hamilton-Jacobi equations. To illustrate this fact we first suppose that $k \in L^\infty(\mathbb{R})$ and define

$$h(x) := \int_0^x k(y)\, dy \qquad (x \in \mathbb{R}).$$

Next, let w be a regular solution of this initial-value problem for the Hamilton-Jacobi equation:

$$\begin{cases} D_t w(x,t) + H(D_x w(x,t)) = 0, & \text{in } \mathbb{R} \times \mathbb{R}^+; \\ w(x,0) = h(x), & \text{in } \mathbb{R} \times \{t=0\}. \end{cases} \tag{5}$$

If we now differentiate the PDE and its initial condition with respect to x, we obtain that the function w_x is a (weak) solution of initial-value problem for the conservation law:

$$\begin{cases} D_t w_x(x,t) + H(D_x w(x,t))_x = 0, & \text{in } \mathbb{R} \times \mathbb{R}^+; \\ w_x(x,0) = k(x), & \text{in } \mathbb{R} \times \{t=0\}. \end{cases} \tag{6}$$

Singularities of solutions: Consider a single balance laws in one space dimension:

$$D_t u(x,t) + D_x[f(u(x,t),x,t)] + g(u(x,t),x,t) = 0 \qquad \text{in an open set } \Omega \subset \mathbb{R}^2 \tag{7}$$

where f denotes the flux function and g the source term. One of the peculiarities of such type of PDEs is the possibility of formation of singularities even starting from smooth initial function $u(x,0) = u_0(x)$. For instance, we can put $f(u,x,t) = u^2/2$ and $g(u,x,t) = 0$ and couple equation (7) with an initial smooth data:

$$\begin{cases} D_t u(x,t) + D_x \frac{u^2(x,t)}{2} = 0, & \mathbb{R} \times \mathbb{R}^+; \\ u(x,0) = \frac{1}{1+x^2}, & \mathbb{R} \times \{0\}. \end{cases} \tag{8}$$

Then, analyzing the above initial-value problem (the well-known Burgers' equation) we will observe that a discontinuous solution is generated after finite time (see the books [9, 21, 25] for a complete survey on this topic). We can explain the apparition of shocks (or jumps) observing that characteristics with different slope (or speed) may collide after a small time. Thus, we can understand why genuinely nonlinear conservation laws are mostly known as classical cases where the solutions typically lose their (initial) regularity. This means that in general we cannot expect to find global classical solutions for equation (7) and as consequence we have to introduce a suitable definition of weak solution. In order to develop an admissible theory for conservation laws, we need to select an appropriate space of functions, which admits also discontinuous solutions: the space BV of functions of bounded variation will be the suitable one. To give an idea of what a one-dimensional BV function is, we can think to an integrable function with a bound on the total amount of the oscillations and with only at most countably many points of discontinuity. A solution $u(x,t)$ of a scalar conservation law for each fixed $t > 0$ is BV in the space variable and the discontinuity points form the shock curves of $u(x,t)$. Unfortunately there is no unique weak solution to equation (7), but the *entropy condition* permits us to single out a unique one. In the 1950's, the qualitative theory was developed by the Russian school, headed by Oleinik. In particular the decay of positive waves at the rate $O(1/t)$ for the solutions $u(x,t)$ of strictly convex scalar conservation laws is discussed by Oleinik in [34]:

$$u(x+z,t) - u(x,t) \leq \frac{\check{C}}{t} z, \qquad \text{for a } C > 0 \tag{9}$$

holds for all $t > 0$, $x, z \in \mathbb{R}$ where $z > 0$. This decay leads to an uniqueness criterion for convex scalar conservation laws and in the literature is called Oleinik entropy condition. In 1967, assuming that the initial data is continuous and the flux function is convex, Volpert [36] has proved that there is a unique solution that satisfies the following *entropy condition*:

$$u(x-,t) \geq u(x+,t).$$

A new important step in this direction is due to Kruzhkov: in 1970 he proved that there is a unique solution fulfilling the entropy condition, even if the initial does not. At the points of the initial function where the entropy condition is violated, "smooth" rarefaction waves appear after an infinitesimal time.

As consequence of the above relation between (5) and (6) we expect that solutions of conservation laws and of Hamilton-Jacobi equations have similar properties. In particular, if we let

$$L = H^* := \sup_{q\in\mathbb{R}}\{p \cdot q - H(q)\}$$

be the *Legendre transformation*, one standard result in the literature says that the solution of (5) is given by the so-called *Hopf-Lax formula*:

$$w(x,t) := \min_{y\in\mathbb{R}} \Big\{ tL\Big(\frac{x-y}{t}\Big) + h(y) \Big\}. \tag{10}$$

Clearly, a suitable modification of (10) will provide a new formula for the solutions of scalar conservation laws with convex flux function: the *Lax-Oleinik formula*. This means that there is an equivalence between viscosity solutions of Hamilton-Jacobi equations and entropy solutions of scalar conservation laws: $u = w_x$ is an admissible weak solution of (6) if and only if w is a viscosity solution of (5). As viewed, another central point to know when we deal with conservation laws is that solutions may develop discontinuities after finite time, even if the initial data is smooth. By the previous remarks we immediately have that singularities may appear in the derivative of solutions of Hamilton-Jacobi equations.

Now more generally if we consider viscosity solutions u to Hamilton-Jacobi equations

$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n, \tag{11}$$

we will observe that ∇u will lose his (initial) regularity. As it is well known, solutions of the Cauchy problem for (11) develop singularities of the gradient in finite time, even if the initial data $u(0,\cdot)$ is extremely regular. The theory of viscosity solutions, introduced by Crandall and Lions 30 years ago, provides several powerful existence and uniqueness results which allow to go beyond the formation of singularities. Moreover, viscosity solutions are the limit of several smooth approximations of (11). For a review of the concept of viscosity solution and the related theory for equations of type (11) we refer to [10, 16, 31].

Regularity theory: Tough the perspective for the regularity of entropy and gradient of viscosity solutions of (7) and (11) respectively seems to be bad after the creation of the shock waves, a deepest investigation on the subject gives us a surprising result: the nonlinearity exerts a self-regularization effect. Clearly the nonlinearity of the flux function is the cause of the formation of discontinuities after a finite time and this jumps, after the creation, will propagate as shock waves for the eternity. But the nonlinearity has a double face and it may have also a good effect on the solution of conservation laws. For instance, if a very steep slope appears in the solution it will be transformed after a infinitesimal time in a jump (without the nonlinearity this would be transported with no changes). If we look also at the dissipation of the total variation of the solutions we begin to have other elements to hope in solutions, which will not increase without control the total amount of the discontinuities. In applications, regarding at numerical schemes (for example front tracking, finite volume schemes) it is to remark how this self-regularization effects permit to bound the errors created at each step.

The "good" influence of the nonlinearity can also be verified studying in a deeper way the fine properties of entropy solutions of conservation laws. Thus, taking advantage on this self-regularization effect and understanding better the geometry of the problem, in 2004, Ambrosio and De Lellis [3] have shown that entropy solutions $u(x,t)$ of scalar conservation laws

$$D_t u(x,t) + D_x[f(u(x,t))] = 0 \qquad \text{in an open set } \Omega \subset \mathbb{R}^2 \tag{12}$$

with locally uniformly convex flux function $f \in C^2(\mathbb{R})$ and $f'' > 0$, are functions of locally special bounded variation, i.e. the distributional derivative Du has no Cantor part. The canonical example of a function $u \in BV([0,1])$ but not in $SBV([0,1])$ is the Cantor function, also known as Devil's staircase. The Cantor function c on $[0,1]$ is a monotone increasing (or decreasing) continuous function such that $2c(x/3) = c(x)$ and $c(x)+c(1-x) = 1$. In particular it is not absolutely continuous and his derivative Du is a singular nonatomic measure, which contains a fractal structure (the construction of c is indeed based on the classical Cantor ternary set). Observe that any BV function may contain parts of this problematic monotone Cantor function and thus, roughly speaking we can say that a SBV function will be a BV function without "fractal behavior in the derivative of u". Moreover, due to the relation between conservation laws in one space variable and planar Hamilton-Jacobi equations it was possible for the authors to prove an easy corollary: if the hamiltonian $H(.)$ is uniformly convex, then the gradients of viscosity solutions $v(x)$ of

$$H(\nabla v) = 0 \qquad \text{in an open set } \Omega \subset \mathbb{R}^2 \tag{13}$$

have the same property, i.e. they belong to the space $SBV_{loc}(\Omega)$. In [3], we can also find two interesting research directions to extend these results:

(a) Let $u \in BV$ be an admissible entropy solution of a genuinely nonlinear system of conservation laws in one space dimension. Is $u \in SBV_{loc}$?;

(b) Let $v(x)$ be a viscosity solution of a uniformly convex Hamilton-Jacobi PDEs in higher dimension. Is $\nabla v(x) \in SBV_{loc}$?

Regularity results for a bigger class of PDEs: A big part of my doctoral research work has been inspired by the two questions above, and this will bring us to state more general SBV regularity Theorems. More precisely, two parts of this thesis are dedicated to regularity Theorems. The first Theorem, proposed in Chapter 5, increases substantially the class of scalar conservation laws in one space dimension with an admissible SBV solution:

Theorem 0.1. *Let $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a flux function, such that*

$$\{u_i \in \mathbb{R} : f_{uu}(u_i, x, t) = 0\}$$

is at most countable for any fixed (x,t). Let $g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a source term and let $u \in BV(\Omega)$ be an entropy solution of the balance law:

$$D_t u(x,t) + D_x[f(u(x,t),x,t)] + g(u(x,t),x,t) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2.$$

Then there exists a set $S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R} \backslash S$ the following holds:

$$u(.,\tau) \in SBV_{loc}(\Omega_\tau) \qquad \text{with } \Omega_\tau := \{x \in \mathbb{R} : (x,\tau) \in \Omega\}.$$

Moreover, $u(x,t) \in SBV_{loc}(\Omega)$.

Note that we can now get the main Theorem of [3] as a special case of Theorem 0.1.

The second main Theorem of this work, which can be found in Chapter 6, gives a complete answer to question (b):

Theorem 0.2. *Let u be a viscosity solution of*

$$\partial_t u + H(D_x u) = 0 \qquad in\ \Omega \subset [0,T] \times \mathbb{R}^n. \tag{14}$$

Moreover, assume

$$H \in C^2(\mathbb{R}^n) \qquad and \qquad c_H^{-1} Id_n \le D^2 H \le c_H Id_n \quad for\ some\ c_H > 0. \tag{15}$$

and set $\Omega_t := \{x \in \mathbb{R}^n : (t,x) \in \Omega\}$. Then, the set of times

$$S := \{t : D_x u(t,.) \notin SBV_{loc}(\Omega_t)\}$$

is at most countable. In particular $D_x u, \partial_t u \in SBV_{loc}(\Omega)$.

Hamilton-Jacobi equations with obstacles: In Chapter 7 we will deal with a problem in the theory of optimal control introduced by Bressan in [8]. Tough the topics, at first sight, are different, our result is again a combination of the theories of Hamilton-Jacobi equations and SBV functions. To give an idea of what we are looking for, let us imaging a fire that spread in a forest and suppose that we have the possibility to bound the burning area creating barriers clearing the vegetation with a bulldozer. It is then possible to block with walls the burned area, minimizing the costs of the operation? In [8] proposed the following mathematical model for these kind of problems:

A bounded, open set $R_0 \subset \mathbb{R}^2$ is the initial burned portion of the forest and a continuous multifunction $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with compact, convex values will describe the spreading of the fire. Our model prescribe that every continuous trajectory $x(.)$ followed by the fire and which start in R_0 may be characterized by the differential inclusion $\dot{x}(t) \in F(x(t))$. Roughly speaking, this means that at each different point of the forest the velocity and the direction of the spreading of the fire will be influenced by the local properties of the wind, humidity, vegetation. Our purpose is to block the burned area confining it with a wall γ reducing the cost of this operation. We denote by $\gamma(t)$ the portion of the barrier created after the time t and we assume that:

(H1) $\gamma(t_1) \subseteq \gamma(t_2)$ for every $0 \le t_1 \le t_2$;

(H2) $\int_{\gamma(t)} \psi \, d\mathcal{H}^1 \le t$ for every $t \ge 0$, where $\psi \ge \psi_0 > 0$ for a constant $\psi_0 \in \mathbb{R}^+$.

Mathematically the reached set, or in other words the destructed zone, is then given by:

$$R^\gamma(t) := \Big\{ x(t) \,\Big|\, \begin{array}{l} x \in W^{1,1} \cap C([0,t],\mathbb{R}^2)\,,\ x(\tau) \notin \gamma(\tau) \quad \forall \tau\,, \\ x(0) \in R_0 \quad \text{and} \quad \dot{x}(\tau) \in F\big(x(\tau)\big) \text{ for a.e. } \tau \end{array} \Big\}. \tag{16}$$

In [12] the authors studied a general variational problem showing the existence of strategies γ which minimize suitable cost functionals.

We can now introduce the central object of our study: the *minimum time* at which a point is reached by the fire. Given a blocking strategy γ, for any $x \in \mathbb{R}^2$ we set:

$$T^\gamma(x) := \inf\{t > 0 : x \in R^\gamma(t)\}. \tag{17}$$

Clearly, since the initial burned area is R_0 we observe that $T^\gamma|_{R_0} = 0$ and by definition of the reached set we furthermore have that $R^\gamma(t) = \{T^\gamma < +\infty\}$. If $\gamma(t) = \emptyset$ for every t, then T^γ is the minimum time function of a classical control problem. Let us introduce the hamiltonian function related to it.

Definition 0.3. $H(x,p) := \sup_{q \in F(x)} \{p \cdot q\} - 1.$

Next, we will always assume that

(H3) There is a constant $\lambda > 0$ s.t. $B_\lambda(0) \subset F(x)$ for all x.

It is well known that, under (H3) and the assumption $\gamma = \emptyset$, T^γ is a Lipschitz map and satisfies the Hamilton-Jacobi equation

$$H(x, \nabla T^\gamma(x)) \;=\; 0 \qquad \text{for a.e. } x \in \mathbb{R}^2 \setminus R_0. \tag{18}$$

Indeed, T^γ is characterized as the *viscosity solution* of (18) in $\mathbb{R}^2 \setminus \overline{R}_0$ with boundary value equal to 0 (see for instance [26] or [6]).
Assume for the moment that $\gamma_\infty := \cup_t \gamma(t)$ is a sufficiently regular curve. Then T^γ must be a viscosity solution of (18) in $\{T^\gamma < \infty\} \setminus (\overline{R}_0 \cup \gamma_\infty)$. Moreover, T^γ has jump discontinuities on γ_∞. We can regard it as a " viscosity solution of (18) with obstacles γ_∞". In Chapter 7 of this note, we propose a suitable mathematical definition of this concept and use it to characterize T^γ. The strength of our result is its generality, which will give us a few interesting corollaries. For instance, the main Theorem of [12] is recovered as a relatively simple Corollary. Thus, a central step in our study will be the introduction of a suitable definition of *viscosity solution* which characterizes the minimal time function, that now could be viewed as a solution of a Hamilton-Jacobi equation which admits obstacles in the domain. To begin with, we define the class of functions $\mathcal{S}^\gamma$:

Definition 0.4. *Given a measurable function* $u : \mathbb{R}^2 \to [0,\infty]$ *and a* $t \in [0,\infty[$ *we set* $u_t := u \wedge t = \min\{u,t\}$.
For a given strategy γ*, a measurable* $u : \mathbb{R}^2 \to [0,\infty]$ *belongs to the class* $\mathcal{S}^\gamma$ *if the following conditions hold for every* $t \in [0,\infty[$*:*

(a) $u_t \in SBV_{loc}(\mathbb{R}^2)$, $\mathcal{H}^1(J_{u_t} \setminus \gamma(t)) = 0$ *and* $u_t \equiv 0$ *on* R_0*;*

(b) If ∇u_t *denotes the absolutely continuous part of* Du_t*, then*

$$H(x, \nabla u_t(x)) \le 0 \qquad \textit{for a.e. } x\,. \tag{19}$$

By a subtle remark of [**?**] it is possible to optimize a given strategy γ by adding in a canonical way an $\mathcal{H}^1$-negligible amount of walls. We can then generate the so-called *complete strategy* γ^c, which has better properties than γ. A mathematical definition of γ^c may be found in Section 7.1.1.

We can now propose the main Theorem of Chapter 7:

Theorem 0.5. *Let γ be a strategy. Assume (H1), (H2), (H3) and*

(H4) the initial set R_0 is open and ∂R_0 has zero 2–dimensional Lebesgue measure.

Then $T^\gamma \in \mathcal{S}^\gamma$ and T^{γ^c} is the unique maximal element of $\mathcal{S}^\gamma$, that is

$$\text{for every } v \in \mathcal{S}^\gamma \text{ we have } v \le T^{\gamma^c} \text{ a.e.}.$$

A classical result in the literature is the connection between optimal control theory and Hamilton-Jacobi-(Bellman) equations. Here again, as one would expect from an optimal control problem, we are able to characterize the given problem with a suitable Hamilton-Jacobi equation. In the classical control theory, numerical methods for solving these equations yield accurate approximate solutions and therefore it would be interesting to apply our approach to simulate the confinement of a burning forest. Our new description proposed for the problem introduced in [8] and subsequently studied in [12] has new theoretical as well as numerical advantages.

Structure of the Thesis: We now outline the content of the thesis. In the first Chapter, following the book [2], we will give a brief introduction to the main definitions and technical tools on BV and SBV functions. Under suitable assumptions, viscosity solutions of Hamilton-Jacobi equations are semiconcave functions (see the book [16]) and thus they are related to maximal monotone functions, as described in Chapter 2. This theory will be very important to get the results in Chapter 6 and thus, in Chapter 2 we will collect several important technical Propositions on this subject, in particular sfollowing the geometric approach of [1]. Chapters 3 and 4 are dedicated to the theory of genuinely nonlinear balance laws and Hamilton-Jacobi equations. These are the two principal equations studied in this work and for a complete survey on these topics we address the reader to [9, 20, 21, 25] and [10, 16, 25] respectively. Note that in these introductory Sections we will prove only those Lemmas that are directly connected to the results showed in Chapters 5, 6 and 7. Moreover, we have tried to arrange all the material following a natural development and this implies that several new results may be found already in the introductory sections. For instance, the one-sided inequality of section 3.2 is a new Theorem concerning balance laws.

The last three Chapters are the core of this thesis. In particular, the Theorems 0.1, 0.2 and 0.5 are proved, respectively, in the Chapters 5, 6 and 7. These three Chapters will also give a more detailed introduction to the specific results and the related literature.

In the next two paragraphs we give a brief schematical overview of the known results and open questions which are closely related to this work.

0.2 Overview on the SBV regularity

Hyperbolic Conservation Laws:

- 2004 – L.Ambrosio & C.De Lellis. [3]
  The entropy solutions of the scalar equations
  $$D_t u(x,t) + D_x[f(u(x,t))] = 0$$
  with $f \in C^2$ and $f'' > 0$ are locally SBV.

- 2007 – R.Robyr. [35]
  Let $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ changes convexity at most a countable many times and $g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$. Then, the entropy solutions of the scalar balance laws with non-convex flux function:
  $$D_t u(x,t) + D_x[f(u(x,t),x,t)] + g(u(x,t),x,t) = 0,$$
  are locally SBV. [This is Theorem 5.3 of this work.]

- 2008 – C. Dafermos. [22]
  Self-similar BV solutions of genuinely nonlinear strictly hyperbolic systems of conservation laws are SBV.

- **Open question**: Do BV admissible solutions of genuinely nonlinear systems of conservation laws in 1 space dimension belong to the space SBV_{loc}?
  Although, the paper of Dafermos [22] is a first step, there is a lot of work to do in this direction. Unfortunately, it seems that the techniques introduced in [3] and then developed in [35] are not sufficient to obtain a global proof. One can try to adapt these techniques for some special class of systems. But for a more general proof one needs probably a better control on the classical estimates for wave interactions.

Hamilton Jacobi Equations:

- 1997 - P.Cannarsa, A. Mennucci & C.Sinestrari. [15]
  Under strong regularity assumption on the initial functions the viscosity solution u of a first-order Hamilton-Jacobi equation has a gradient $Du \in SBV$. More precisely, for a fixed integer R let $u_0 \in W^{1,\infty}(\mathbb{R}^n) \cap C^{R+1}(\mathbb{R}^n)$ and consider:
  $$\begin{cases} D_t u(x,t) + H(D_x u(x,t),x,t) = 0, & t \in \mathbb{R}^+, x \in \mathbb{R}^n; \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$
  where $H(.,x,t)$ is strictly convex. Then, Du belongs to the class SBV_{loc}, i.e $D^2 u$ is a measure with no Cantor part.

- 2004 - L.Ambrosio & C.De Lellis. [3]
  Let $H \in C^2(\mathbb{R}^2)$ be locally uniformly convex and let $u \in W^{1,\infty}(\Omega)$ be a viscosity solution of $H(Du) = 0$. Then $Du \in SBV_{loc}(\Omega)$.

- 2007 - R.Robyr. [35]
  Let $H(p,x,t) \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be locally uniformly convex in p, i.e. $D_{pp}H > 0$. If $u \in W^{1,\infty}(\Omega)$ is a viscosity solution of
  $$D_t u(x,t) + H(D_x u(x,t),x,t) = 0, \tag{20}$$
  then $Du \in SBV_{loc}(\Omega)$. [This is Corollary 5.4 of Chapter 5.]

- 2010 - S.Bianchini, C.De Lellis & R.Robyr [7]
  Under assumption

  $$H \in C^2(\mathbb{R}^n) \qquad \text{and} \qquad c_H^{-1} Id_n \leq D^2 H \leq c_H Id_n \quad \text{for some } c_H > 0,$$

  the gradient of any viscosity solution u of

  $$H(D_x u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n,$$

  belongs to $SBV_{loc}(\Omega)$. [See also Corollary 6.2 of this thesis.]

- **Open questions**: For some applications in optimal transport theory it would be interesting to develop a regularity theory when the Hamiltonian is just convex. However, the regularization effect is subtler: we cannot in general expect the same regularity as in [7]. Another possible interesting extension is to Hamiltonians depending also on $x \in \mathbb{R}^n$. An interesting geometric problem falls into this class: the distance function from a set in a Riemannian manifold satisfies a PDE of the form $H(x, Du(x)) = 0$.

0.3 Overview on the fire confinement problem

The fire confinement problem was introduced by Bressan in [8] and then studied in several papers [11], [12], [13] and [14]. In these works theoretical and numerical aspects are developed. In Chapter 7 we propose a new proof of the main Theorem of [12] which ensures the existence of an optimal strategy minimizing a given cost functional (see Corollary 7.6).

As above we list here some **open problems** proposed by Bressan.

Isotropic blocking problem On the whole plane, assume:

1. fire propagates with unit speed in all directions
2. wall construction speed is $\sigma \leq 2$.

Prove that NO blocking strategy (i.e. a strategy that confines the fire in a bounded set)exists.

Existence of optimal strategies Select weaker assumptions for Corollary 7.6. In particular, determine whether an optimal strategy exists, in the general case where the velocity sets satisfy $0 \in F(x)$ but without assuming $B_\lambda(0) \subset F(x)$.

Two fires Find the optimal strategy for the isotropic problem (i.e. the fire propagates uniformly in all directions) when R_0 is the union of two discs.

1 The spaces $BV, SBV, GBV, GSBV$

1.1 Basic definitions and properties

It is well-known that in general we cannot find classical smooth solutions for equations (7) and (12): shocks appear in finite time even for smooth initial data $u(x,0) = u_0(x)$. Analogously to *entropy solutions* of conservation laws, the gradient of *viscosity solutions* of Hamilton-Jacobi equations (11) presents the same regularity problems. Thus, in order to study all the possible solutions with jump discontinuities, we take the space of functions of bounded variation BV as working space. We then collect some definitions and theorems about BV and SBV functions. For a complete survey on these topics we address the reader to Chapters 3, 4 of the monograph [2]. We begin with:

Definition 1.1. *Let denote by Ω a generic open set in $\mathbb{R}^n$. Let $u \in L^1(\Omega)$; we say that u is a* function of bounded variation *in Ω if the distributional derivative of u, denoted by Du, is representable by a finite Radon measure in Ω, i.e. if*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega \phi \, dD_i u \qquad \forall \phi \in C_c^\infty(\Omega),\ i = 1, \dots, n$$

for some $\mathbb{R}^n$-valued measure $Du = (D_1 u, \dots, D_n u)$ in Ω. A function $u \in L^1_{loc}(\Omega)$ has locally bounded variation in Ω if for each open set $V \subset\subset \Omega$, u is a function of bounded variation in V. We write $u \in BV(\Omega)$ and $u \in BV_{loc}(\Omega)$ respectively.

Now, we introduce the so-called *variation* $V(u,\Omega)$ of a function $u \in L^1_{loc}(\Omega)$:

Definition 1.2 (Variation)**.** *Let $u \in L^1_{loc}(\Omega)$. The* variation $V(u,\Omega)$ *of u in Ω is defined by*

$$V(u,\Omega) := \sup \left\{ \int_\Omega u \, div \varphi \, dx \ : \ \varphi \in [C_c^1(\Omega)]^n, \|\varphi\|_\infty \leq 1 \right\}.$$

At this point we can characterize the BV space as the class of $L^1(\Omega)$ functions with finite variation (see Proposition 3.6 of [2]):

Proposition 1.3. *Let $u \in L^1(\Omega)$. Then, u belongs to $BV(\Omega)$ if and only if*

$$V(u,\Omega) < \infty.$$

In addition, $V(u,\Omega)$ coincides with $|Du|(\Omega)$ for any $u \in BV(\Omega)$.

We notice that $BV(\Omega)$, endowed with the norm

$$\|u\|_{BV} := \int_\Omega |u| \, dx + |Du|(\Omega)$$

is a Banach space and $|Du|(\Omega)$ will be sometimes called the variation of u in Ω.

Remark 1.4. *Sometimes, given a Radon measure μ on a Borel set $E \subset \mathbb{R}^n$, we will denote its total variation on E by $\|\mu\|_{TV(E)}$.*

If $u \in BV(\Omega)$, then it is possible to split the measure Du into three mutually singular parts:

$$Du = D^a u + D^j u + D^c u.$$

$D^a u$ denotes the absolutely continuous part (with respect to the Lebesgue measure). $D^j u$ denotes the jump part of Du. $D^c u$ is called the *Cantor part* of the gradient and it is the "diffused part" of the singular measure

$$D^s u := D^j u + D^c u = f\nu\mathcal{H}^{n-1} \llcorner J_u + D^c u,$$

where:

- When Ω is a 1-dimensional domain, $D^j u$ consists of a countable sum of weighted Dirac masses, and hence it is also called the atomic part of Du. In higher dimensional domains, $D^j u$ is concentrated on a rectifiable set of codimension 1, which corresponds to the measure-theoretic jump set J_u of u;
- $\mathcal{H}^{n-1} \llcorner J_u$ denotes the measure $\mu(E) := \mathcal{H}^{n-1}(J_u \cap E)$;
- ν is a Borel vector field orthogonal to J_u and with $|\nu| = 1$;
- f is a Borel scalar function;
- the Cantor part $D^c u$ has the property that

$$D^c u(E) = 0 \qquad \text{for any Borel set } E \text{ with } \mathcal{H}^{n-1}(E) < \infty. \tag{21}$$

We are now ready to define the most important space of functions for this thesis:

Definition 1.5. *Let $u \in BV(\Omega)$, then u is a* special function of bounded variation*, and we write $u \in SBV(\Omega)$, if $D^c u = 0$, i.e. if the measure Du has no Cantor part. The more general space $SBV_{loc}(\Omega)$ is defined in the obvious way.*

Remark 1.6. *The classical example of a 1-dimensional continuous function which belongs to BV but not to SBV is the Cantor staircase (cp. with Examples 1.67 and 3.34 of [2]).*

Since in Chapter 7 and more precisely in Subsection 7.6 we will use a compactness Theorem for $GSBV$ functions (see section 4.5 of [2]) we recall here the definitions of the spaces:

Definition 1.7 (GBV and $GSBV$ functions)**.** *Let Ω be an open set of $\mathbb{R}^n$; we say that a function $u : \Omega \to \mathbb{R}$ is a* generalized function of bounded variation*, and write $u \in GBV(\Omega)$, if for every $\phi \in C^1(\mathbb{R})$ with the support of $\nabla\phi$ compact, the composition $\phi \circ u$ belongs to $BV_{loc}(\Omega)$.*
We say that $u \in GSBV(\Omega)$ if for every ϕ as above the composition $\phi \circ u$ belongs to $SBV_{loc}(\Omega)$.

1.2 BV and SBV of one variable

In our proofs we will often deal with one-dimensional functions of bounded variation, therefore we list here more details and selected Theorems about this special case.

The class of BV functions is big, but for every $u \in BV$ we have always the possibility to select a *good representative* $\overline{u}$ with nice properties (for the details on this subject see Chapter 3.2 of [2]):

Definition 1.8. *(Pointwise, essential variation and good representative):*

- *Let* $a, b \in \overline{\mathbb{R}}$ *with* $a < b$ *and* $I = (a, b)$. *For any function* $u : I \to \mathbb{R}$ *the* pointwise variation $pV(u, I)$ *of* u *in* I *is defined by:*

$$pV(u, I) := \sup \left\{ \sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| : \, k \geq 2, \quad a < t_1 < \cdots < t_k < b \right\}.$$

If $\Omega \subset \mathbb{R}$ *is open, the pointwise variation* $pV(u, \Omega)$ *is defined by* $\sum_I pV(u, I)$, *where the sum runs along all the connected components of* Ω.

- *The* essential variation $eV(u, \Omega)$ *is defined by*

$$eV(u, \Omega) := \inf \left\{ pV(v, I) : v = u \quad \mathcal{L}^1 - a.e. \text{ in } \Omega \right\}.$$

- *If* $u \in BV(\Omega)$, *there exists a* good representative $\overline{u}$ *in the class of* u *such that:*

$$pV(\overline{u}, \Omega) = eV(u, \Omega) = V(u, \Omega).$$

As above, using the Radon-Nikodym Theorem we split the Radon measure Du into the absolute continuous part $D^a u$ (with respect to $\mathcal{L}^1$) and the singular part $D^s u$:

$$Du = D^a u + D^s u = Du \llcorner (\Omega \backslash S) + Du \llcorner S \qquad \text{where } S := \left\{ x \in \Omega : \lim_{\rho \downarrow 0} \frac{|Du|(B_\rho(x))}{\rho} = \infty \right\}.$$

Let A denote the set of atoms of Du, i.e. $x \in A$ if and only if $Du(\{x\}) \neq 0$. In the case of one-dimensional BV functions, the jump set $A = J_u$ consists of countably many points. We now split the singular part $D^s u$ into the purely atomic part $D^j u$ and the diffusive part (i.e. without atoms) $D^c u$:

$$Du = D^a u + D^s u = D^a u + D^j u + D^c u = Du \llcorner (\Omega \backslash S) + Du \llcorner A + Du \llcorner (S \backslash A). \tag{22}$$

The above decomposition is unique and the three measures $D^a u, D^j u, D^c u$ are mutually singular. We have

$$\begin{aligned} |Du|(\Omega) &= |D^a u|(\Omega) + |D^j u|(\Omega) + |D^c u|(\Omega) \\ &= \int_\Omega |\overline{u}'| dt + \sum_{t \in A} |\overline{u}(t+) - \overline{u}(t-)| + |D^c u|(\Omega). \end{aligned} \tag{23}$$

where $\overline{u}$ is any good representative of u (see Corollary 3.33 of [2]).

Remark 1.9. *In Chapter 7 the measure* Du *will be denoted by* $\frac{du}{ds}$ *and we will use* u' *for the* L^1 *function* ∇u. *The decomposition above reads then as*

$$\frac{du}{ds} = u' \mathcal{L}^1 + \sum_{s_i \in J_u} f(s_i) \delta_{s_i} + D^c u \, . \tag{24}$$

Each $f(s_i)$ *is, thus, a real number and* $D^c u$ *is the singular nonatomic part of the measure* $\frac{du}{ds}$ *(see Section 3.2 of [2]).*

We are now ready to recall (see for instance Theorem 3.28 and Proposition 3.92 of [2]):

Proposition 1.10. *Let $u \in BV(\Omega)$ and let $\Omega \subset \mathbb{R}$. Let A be the set of atoms of Du. Then:*

(i) Any good representative $\overline{u}$ is continuous in $\Omega \backslash A$ and has a jump discontinuity at any point of A. Moreover, $\overline{u}$ has classical left and right limits (denoted by u^L and u^R) at any $x \in A$.

(ii) $D^c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^0$ and on sets of the form $\overline{u}^{-1}(E)$ with $E \subset \mathbb{R}$ and $\mathcal{L}^1(E) = 0$.

Remark 1.11. *For similar definitions of BV and SBV functions in higher dimensions, we address the reader to Chapters 3 and 4 of [2].*

1.2.1 Fine properties of 1-d BV functions

Motivated by Proposition 1.10, when I is an interval and $u \in BV(I)$, we can change the values of u on a set of zero Lebesgue measure so to gain a function $\tilde{u}$ with the following properties (see Section 3.2 of [2]):

- $\tilde{u}$ is continuous at every point $t \in I \setminus J_u$;
- $u^+(t) = \lim_{\tau \downarrow t} \tilde{u}(\tau)$ and $u^-(t) = \lim_{\tau \uparrow t} \tilde{u}(\tau)$ exist (and are finite) at every $t \in J_u$.

Moreover, the coefficients $f(s_i)$ of (24) satisfy $f(s_i) = u^+(s_i) - u^-(s_i)$. It is customary to set $\tilde{u}(s_i) := (u^+(s_i) + u^-(s_i))/2$. $\tilde{u}$ is then called the *precise representative of* u. The following Proposition is a simple corollary of the properties of the precise representative.

Proposition 1.12. *If I is an interval, $u \in BV(I)$ and $J_u = \emptyset$, then the precise representative $\tilde{u}$ is continuous. If in addition $u \in SBV(I)$, then $\tilde{u} \in W^{1,1} \cap C$ and its distributional derivative is the L^1 function u'.*

1.3 BV and SBV functions in higher dimension: tools for Chapter 7

A technical point needed to our study is the next proposition. This time, however, the statement is a well-known fact for BV functions and we refer to the monograph [2]. In what follows, the derivative of BV functions u, which are Radon measures, will be decomposed into its absolutely continuous part and its singular part, using the notation $Du = \nabla u \mathcal{L}^n + D^s u$.

Theorem 1.13 (Approximate Differentiability)**.** *Let u be a $BV(\Omega)$ function and $Du = \nabla u \mathcal{L}^n + D^s u$. Then, at a.e. $x \in \Omega$ there exists a measurable set B (possibly depending on x) such that:*

$$
\begin{aligned}
(i) \qquad & \lim_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x) \setminus B)}{r^n} \;=\; 0; \\
(ii) \qquad & \lim_{z \to x, z \in B} \frac{u(z) - u(x) - \langle \nabla u(x), (z - x) \rangle}{|z - x|} \;=\; 0\,.
\end{aligned}
$$

Or, in the language of [27], v is approximately differentiable at a.e. x with approximate differential given by $\nabla u(x)$.

1.3.1 SBV functions and slicing

We recall the following slicing theorem (cp. with Section 3.11 of [2]).

Theorem 1.14 (Slicing). *A function $u \in L^1([0,1]^2)$ belongs to BV iff*

1. *The functions $u(y,\cdot)$ and $u(\cdot,y)$ belong to $BV([0,1])$ for a.e. y;*
2. *The following integral is finite*

$$\int \Big(\|\tfrac{d}{ds}u(y,\cdot)\|_{TV([0,1])} + \|\tfrac{d}{ds}u(\cdot,y)\|_{TV([0,1])}\Big)dy\,.$$

The function u belongs to SBV if and only if the two conditions above hold and, in addition

(3) $u(y,\cdot)$ and $u(\cdot,y)$ belong to SBV for a.e. y.

Moreover, if $u \in SBV$ and we write $Du = \nabla u \mathcal{L}^2 + f\nu\mathcal{H}^1 \llcorner J_u$, the following identity is valid for a.e. $y \in [0,1]$:

$$\frac{d}{ds}u(y,\cdot) = \langle \nabla u, (0,1)\rangle \mathcal{L}^1 + \sum_{s_i\in J(y)} \alpha_i \delta_{s_i}\,,$$

where $J(y) := \{s : (y,s) \in J_u\}$ and $\alpha_i = f(y,s_i)\langle \nu(y,s_i),(0,1)\rangle$.

Remark 1.15. *The obvious modification of Theorem 1.14 holds in coordinates which are locally C^1-diffeomorphic to the cartesian ones. For instance the theorem holds in polar coordinates (except at the origin).*

1.3.2 More on fine properties

The properties listed in Section 1.2.1 for 1-d BV functions can be suitably generalized to the higher-dimensional case. In order to do that we must introduce the concept of approximate continuity.

Definition 1.16. *A measurable map $u : \mathbb{R}^n \supset E \to [-\infty,+\infty]$ is said approximately continuous at $x \in E$ if there is a measurable set A such that*

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n((E\setminus A)\cap B_r(x))}{r^n} \;=\; 0;$$

$$\lim_{y\to x, y\in A} u(y) = u(x)\,.$$

We recall, then, the following classical result in real analysis and its improved version for BV functions (we refer to Section 3.7 of [2]).

Proposition 1.17. *Measurable maps are approximately continuous a.e.. If u is a BV map of n variables, then we can redefine it on a set of measure zero so to get a precise representative $\tilde{u}$ which is approximately continuous at every point x which satisfies*

$$\lim_{r\downarrow 0} \frac{|Du|(B_r(x))}{r^{n-1}} \;=\; 0\,. \tag{25}$$

If N denotes the set of points where (25) fails, then $\mathcal{H}^{n-1}(N \setminus J_u) = 0$. Moreover, for every $x \in J_u$, there exist two distinct *values $u^+(x)$ and $u^-(x)$ and a measurable set G such that:*

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x) \setminus G)}{r^n} \;=\; 0;$$

$$\lim_{y \to x,\, y \in G,\, \langle (y-x), \nu(x) \rangle < 0} \tilde{u}(y) \;=\; u^-(x);$$

$$\lim_{y \to x,\, y \in G,\, \langle (y-x), \nu(x) \rangle > 0} \tilde{u}(y) \;=\; u^+(x).$$

Finally, it is useful for our analysis that, roughly speaking, points of approximate continuity of traces of BV functions and points of approximate continuity of the functions themselves, coincide "most of the time". The precise statement is given below. We restrict ourselves to the case of 2-dimensional BV functions, which is the one really needed for our purposes. However, the statement can be suitably generalized to any dimensions.

Proposition 1.18. *Let $u \in BV([0,1]^2)$ and consider the function $\tilde{u}$ of Proposition 1.17. Then, the following property holds for a.e. y:*

- *If $(y, x) \notin J_u \cap (\{y\} \times [0,1])$, then*

$$\lim_{z \to x, (y,z) \notin J_u} \tilde{u}(y, z) \;=\; \tilde{u}(y, x)\,. \tag{26}$$

Proof. First of all, consider the two sets of y's, N_1 and N_2 such that (1) of Theorem 1.14 apply. For each $y \in N_2$, let G^2_y be the set of points y of approximate continuity of $u(\cdot, y)$ and set

$$G^2 \;:=\; \cup_t G^2_t \times \{t\}\,.$$

Finally, let N be the set of Proposition 1.17 and recall that $\mathcal{H}^1(N \setminus J_u) = 0$.
We are now ready to give the set of y's for which the conclusion of the Proposition holds. More precisely, y has to satisfy the following conditions:

(c1) $y \in N_1$ and $(\{y\} \times [0,1]) \cap (N \setminus J_u) = \emptyset$;

(c2) $(y, x) \in G^2$ for a.e. $x \in [0,1]$.

Fix a y satisfying the two conditions above and an x with $(y, x) \notin J_u$. We claim that

(Cl) $v(\cdot) := \widetilde{u}(y, \cdot)$ is approximately continuous at any such x.

Assume for the moment that (Cl) holds. By the classical properties of 1d BV functions (see Section 1.2.1), after redefining v on a set of measure zero, we get a new $\tilde{v}$ which is continuous at every $x \notin J_u$. On the other hand, we must have $v(x) = \tilde{v}(x)$ at every point where v is approximately continuous. So, after having proved (Cl), we conclude that $\tilde{v}$ and $\tilde{u}(y, \cdot)$ coincide at every point x with $(y, x) \notin J_u$. This proves the proposition.
It remains to show (Cl). We argue by contradiction and assume it is false. Then at some x with $(y, x) \notin J_u$, we have a constant $\eta > 0$ with the following property. If we define

$$A_r := \{z \in]x - r, x + r[: |\tilde{u}(y, z) - \tilde{u}(y, x)| \geq \eta\}\ ,$$

then

$$\limsup_{r\downarrow 0} \frac{\mathcal{L}^1(A_r)}{r} \geq \eta\,.$$

Now, set $A'_r := \{z \in A_r : (y,z) \in G^2\}$. By (c2) $\mathcal{L}^1(A_r \setminus A'_r) = 0$. We further restrict A'_r by setting $A''_r := \{z \in A'_r : (\tau, z) \in G^2 \text{ for a.e. } \tau\}$. Then, by Fubini, $\mathcal{L}^1(A'_r \setminus A''_r) = 0$. Hence

$$\limsup_{r\downarrow 0} \frac{\mathcal{L}^1(A''_r)}{r} \geq \eta\,. \tag{27}$$

On the other hand, for $z \in A''_r$, (recalling that $(y,z) \in G^2$) we can write

$$|\tilde{u}(\tau,z) - \tilde{u}(y,z)| \;\leq\; \left|\frac{d}{dt}u(\cdot,z)\right|(]y-r,y+r[) =: g(r,z)$$

for every $\tau \in]y-r,y+r[\in G^2$ (and hence for a.e. $\tau \in]y-r,y+r[$). Since, by (c1), $(y,x) \notin N$, we know that

$$\lim_{r\downarrow 0} \frac{1}{r}\int_{x-r}^{x+r} g(r,z)\,dz \;\leq\; \lim_{r\downarrow 0} \frac{|Du|(B_{2r}(y,x))}{r} \;=\; 0\,.$$

So, for the set

$$C_r := A''_r \cap \{z : g(r,z) < \eta/2\}$$

we have

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^1(A''_r \setminus C_r)}{r} \;=\; 0\,, \qquad \text{which implies} \qquad \limsup_{r\downarrow 0} \frac{\mathcal{L}^1(C_r)}{r} \geq \eta\,. \tag{28}$$

Consider finally the set $D_r := \{(\tau,z) : z \in C_r, |\tau - y| < r\} \cap G^2$. It turns out that:

- $\limsup_{r\downarrow 0} r^{-2}|D_r| \geq \eta/2$;
- $D_r \subset B_{2r}((y,x))$;
- If $(\tau,z) \in D_r$, then

$$|\tilde{u}(\tau,z) - \tilde{u}(y,x)| \geq |\tilde{u}(y,z) - \tilde{u}(y,x)| - |\tilde{u}(\tau,z) - \tilde{u}(y,z)| \geq \eta - \frac{\eta}{2} = \frac{\eta}{2}\,.$$

The existence of the sets D_r obviously contradict the approximate continuity of $\tilde{u}$ at (y,x), which must hold because $(y,x) \notin N$. □

2 Semiconcave functions and the theory of monotone functions

2.1 Semiconcave functions

Since viscosity solutions of Hamilton-Jacobi equations (14) under the assumption (15) are *semiconcave*, we list here definitions and properties about this subject (more details can be found in the book [16]).

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a continuous function $u : \Omega \to \mathbb{R}$ is* semiconcave *if, for any convex $K \subset\subset \Omega$, there exists $C_K > 0$ such that*

$$u(x+h) + u(x-h) - 2u(x) \leq C_K |h|^2, \tag{29}$$

for all $x, h \in \mathbb{R}^n$ with $x, x-h, x+h \in K$. The smallest nonnegative constant C_K such that (29) holds on K will be called semiconcavity constant of u on K*.*

Next, we introduce the concept of superdifferential.

Definition 2.2. *Let $u : \Omega \to \mathbb{R}$ be a measurable function. The set $\partial u(x)$, called the* superdifferential *of u at point $x \in \Omega$, is defined as*

$$\partial u(x) := \Big\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y-x)}{|y-x|} \leq 0 \Big\}. \tag{30}$$

Next, starting form the above definitions we describe some properties of semiconcave functions (see Proposition 1.1.3 of [16]):

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \Omega$ a compact convex set. Let $u : \Omega \to \mathbb{R}$ be a semiconcave function with semiconcavity constant $C_K \geq 0$. Then, the function*

$$\widehat{u} : x \mapsto u(x) - \frac{C_K}{2}|x|^2 \qquad \textit{is concave in } K.$$

In particular, for any given $x, y \in K$, $p \in \partial\widehat{u}(x)$ and $q \in \partial\widehat{u}(y)$ we have that

$$\langle q - p, y - x \rangle \leq 0.$$

From now on and in particular in the analysis of Chapter 6, when u is a semi–concave function, we will denote the set-valued map $x \to \partial\widehat{u}(x) + C_K x$ as ∂u. An important observation is that, being $\widehat{u}$ concave, the map $x \to \partial\widehat{u}(x)$ is a maximal monotone function.
In what follows, when u is a (semi)-concave function, we will denote by $D^2 u$ the distributional hessian of u. Since Du is, in this case, a BV map, the discussion above of Subsection 1.1 applies. In this case we will use the notation $D^2_a u$, $D^2_j u$ and $D^2_c u$ (instead of the usual notation $[D^2]^a u$, $[D^2]^j u$ and $[D^2]^c u$). An important property of $D^2_c u$ is the following regularity property.

Proposition 2.4. *Let u be a (semi)-concave function. If D denotes the set of points where ∂u is not single–valued, then $|D^2_c u|(D) = 0$.*

Proof. By Theorem 2.8 (see below), the set D is $\mathcal{H}^{n-1}$-rectifiable. This means in particular, that it is $\mathcal{H}^{n-1} - \sigma$ finite. By the property (21) we conclude $D^2_c u(E) = 0$ for every Borel subset E of D. Therefore $|D^2_c u|(D) = 0$. □

2.2 Monotone functions in $\mathbb{R}^n$

Following the work of Alberti and Ambrosio [1] we introduce here some results about the theory of monotone functions in $\mathbb{R}^n$. Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued map (or multifunction), i.e. a map which maps every point $x \in \mathbb{R}^n$ into some set $B(x) \subset \mathbb{R}^n$. For all $x \in \mathbb{R}^n$ we define:

- the *domain* of B, $Dm(B) := \{x : B(x) \neq \emptyset\}$,
- the *image* of B, $Im(B) := \{y : \exists x, y \in B(x)\}$,
- the *graph* of B, $\Gamma B := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in B(x)\}$,
- then *inverse* of B, $[B^{-1}](x) := \{y : x \in B(y)\}$.

Definition 2.5. *Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a multifunction, then*

1. *B is a* monotone *function if*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0 \qquad \forall x_i \in \mathbb{R}^n, y_i \in B(x_i), i = 1, 2. \tag{31}$$

2. *A monotone function B is called* maximal *when it is maximal with respect to the inclusion in the class of monotone functions, i.e. if the following implication holds:*

$$A(x) \supset B(x) \text{ for all } x, A \text{ monotone } \Rightarrow A = B.$$

Observe that in this work we assume $\leq$ in (31) instead of the most common $\geq$. However, one can pass from one convention to the other by simply considering $-B$ instead of B. The observation of the previous subsection is then summarized in the following Theorem.

Theorem 2.6. *The supergradient ∂u of a concave function is a maximal monotone function.*

An important tool of the theory of maximal monotone functions, which will play a key role in Chapter 6, is the Hille-Yosida approximation (see Chapters 6 and 7 of [1]):

Definition 2.7. *For every $\varepsilon > 0$ we set $\Psi_\varepsilon(x, y) := (x - \varepsilon y, y)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and for every maximal monotone function B we define B_ε as the multifunction whose graph is $\Psi_\varepsilon(\Gamma B)$, that is, $\Gamma B_\varepsilon = \{(x - \varepsilon y, y) : (x, y) \in \Gamma B\}$. Hence*

$$B_\varepsilon := (\varepsilon Id - B^{-1})^{-1}.$$

In the next Theorems we collect some properties of maximal monotone functions B and their approximations B_ε defined above.

Theorem 2.8. *Let B be a maximal monotone function. Then, the set*

$$S(B) := \{x : B(x) \text{ is not single valued}\}$$

is a $\mathcal{H}^{n-1}$ rectifiable set. Let $\widehat{B} : Dm(B) \to \mathbb{R}^n$ be such that $\widehat{B}(x) \in \widehat{B}(x)$ for every x. Then $\widehat{B}$ is a measurable function and $B(x) = \{\widehat{B}(x)\}$ for a.e. x. If $Dm(B)$ is open, then $D\widehat{B}$ is a measure, i.e. $\widehat{B}$ is a function of locally bounded variation.
If K_i is a sequence of compact sets contained in the interior of $Dm(B)$ with $K_i \downarrow K$, then $B(K_i) \to B(K)$ in the Hausdorff sense. Therefore, the map $\widehat{B}$ is continuous at every $x \notin S(B)$.
Finally, if $Dm(B)$ is open and $B = \partial u$ for some concave function $u : Dm(B) \to \mathbb{R}$, then $\widehat{B}(x) = Du(x)$ for a.e. x (recall that u is locally Lipschitz, and hence the distributional derivative of u coincides a.e. with the classical differential).

Proof. First of all, note that, by Theorem 2.2 of [1], $S(B)$ is the union of rectifiable sets of Hausdorff dimension $n-k$, $k \geq 1$. This guarantees the existence of the classical measurable function $\widehat{B}$. The BV regularity when $Dm(B)$ is open is shown in Proposition 5.1 of [1].

Next, let K be a compact set contained in the interior of $Dm(B)$. By Corollary 1.3(3) of [1], $B(K)$ is bounded. Thus, since $\Gamma B \cap K \times \mathbb{R}^n$s is closed by maximal monotonicity, it turns out that it is also compact. The continuity claimed in the second paragraph of the Theorem is then a simple consequence of this observation.

The final paragraph of the Theorem is proved in Theorem 7.11 of [1]. □

In Section 6 of this work, since we will always consider monotone functions that are the supergradients of some concave functions, we will use ∂u for the supergradient and Du for the distributional gradient. A corollary of Theorem 2.8 is that

Corollary 2.9. *If $u : \Omega \to \mathbb{R}$ is semiconcave, then $\partial u(x) = \{Du(x)\}$ for a.e. x, and at any point where ∂u is single–valued, Du is continuous. Moreover $D^2 u$ is a symmetric matrix of Radon measures.*

Next we state the following important convergence theorem. For the notion of current and the corresponding convergence properties we refer to the work of Alberti and Ambrosio. However, we remark that very little of the theory of currents is needed in to prove the main theorem: what we actually need is a simple corollary of the convergence in (ii), which is stated and proved in Subsection 6.3.2. In (iii) we follow the usual convention of denoting by $|\mu|$ the total variation of a (real-, resp. matrix-, vector- valued) measure μ. The theorem stated below is in fact contained in Theorem 6.2 of [1].

Theorem 2.10. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and let B be a maximal monotone function such that $\Omega \subset Dm(B)$. Let B_ε be the approximations given in Definition 2.7. Then, the following properties hold.*

(i) B_ε is a $1/\varepsilon$-Lipschitz maximal monotone function on $\mathbb{R}^n$ for every $\varepsilon > 0$. Moreover, if $B = Du$, then $B_\varepsilon = Du_\varepsilon$ for the concave function

$$u_\varepsilon(x) \ := \ \inf_{y \in \mathbb{R}^n} \left\{ u(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\} \tag{32}$$

(ii) ΓB and ΓB_ε have a natural structure as integer rectifiable currents, and $\Gamma B_\varepsilon \llcorner \Omega \times \mathbb{R}^n$ converges to $\Gamma B \llcorner \Omega \times \mathbb{R}^n$ in the sense of currents as $\varepsilon \downarrow 0$.

(iii) $DB_\varepsilon \rightharpoonup^* D\widehat{B}$ and $|DB_\varepsilon| \rightharpoonup^* |D\widehat{B}|$ in the sense of measures on Ω.

3 Genuinely Nonlinear Scalar Balance Laws

3.1 Entropy solutions and generalized characteristics

In this Section we focalize our attention on the following first-order hyperbolic PDE:

$$D_t u(x,t) + D_x[f(u(x,t),x,t)] + g(u(x,t),x,t) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2. \tag{33}$$

where the flux f and the source term g are supposed to be enough smooth, i.e. $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ such that it is strictly convex $f_{uu}(.,x,t) > 0$ for fixed (x,t) and $g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$. This equation is usually called *balance law* (or *conservation law* if the source term g is absent). As remarked in the introduction we cannot expect to have global classical solution even if we are looking at the conservation law:

$$D_t u(x,t) + D_x[f(u(x,t))] = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2. \tag{34}$$

with smooth initial data and convex flux function. Thus, by the classical results on hyperbolic conservation laws it is known that with a suitable definition of weak solution and after the introduction of an admissibility condition, the so-called entropy condition, the existence and the uniqueness of an *entropy solution* of equations (34) and (33) are assured. The reader that is interested to have a better overview on these topics can consult the following literature [9, 20, 21, 25]. Firstly we define the space of admissible solutions:

Definition 3.1. *The* entropy solution $u(x,t)$ *of the equation (33) is a locally integrable function which satisfies the following properties:*

1. *For almost all $t \in [0,\infty)$ the one-sided limits $u(x+,t)$ and $u(x-,t)$ exist for all $x \in \mathbb{R}$.*
2. *$u(x,t)$ solves the balance equation (33) in the sense of distributions.*
3. *For almost all $t \in [0,\infty)$ and for all $x \in \mathbb{R}$ we have that*

$$u(x-,t) \geq u(x+,t). \tag{35}$$

Remark 3.2 (On admissibility condition)**.** *Here, some remarks are needed:*

- *Note that for equations (34) we can take the well-known Oleinik estimate as entropy criterion, i.e. a distributional solution $u(x,t)$ of (34) is an entropy solution provided that:*

$$u(x+z,t) - u(x,t) \leq \frac{\breve{C}}{t} z, \qquad \text{for a } C > 0 \tag{36}$$

  holds for all $t > 0$, $x, z \in \mathbb{R}$ where $z > 0$. However we cannot cannot expect to have the same inequality for equation (33), below in Theorem 3.8 we state and prove a weaker but useful one-sided estimate.

- *We have just introduced the entropy condition for balance laws with strictly convex flux. Since we will deal even with more general flux functions f (i.e. convexity of f may change countably many times) we remember from the works of Volpert and Kruzkov, that a locally integrable function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is an entropy solution if*

$$\int\int \{|u-k|\phi_t + sgn(u-k)(f(u)-f(k))\phi_x\} dxdt \geq 0 \tag{37}$$

  for every constant k and every C^1 function $\phi \geq 0$ with compact support contained in $\mathbb{R} \times \mathbb{R}^+$. For convex fluxes we have the equivalence between (36) *and* (37)*.*

Throughout Chapter 5 we shall denote the entropy solution by $u(x,t)$ and we shall write $u(x+,t)$ and $u(x-,t)$ for the one-sided limits of $u(.,t)$ (also denoted by u^R and u^L). Clearly, on shocks it holds that $u^R \neq u^L$. It is important to note that we will be dealing with entropy solutions $u(x,t)$, which belong to the space $BV_{loc}(\Omega)$ and such that for every $t > 0$ the function $u(.,t)$ belongs to $BV_{loc}(\mathbb{R})$. Moreover, we will restrict our analysis on good representative of solutions and then, under our initial hypothesis we follow the works of Dafermos ([19],[20],[21]) giving an introduction to the theory of generalized characteristics and recalling here some results, which we shall use in the sequel.

Definition 3.3. *A Lipschitz continuous curve $\chi(t)$, defined on an interval $I \subset \mathbb{R}^+$, is called a* characteristic *if it solves*

$$\dot{\chi}(t) = f_u(u(\chi(t),t),\chi(t),t)$$

in the sense of Filippov [28], namely

$$\dot{\chi}(t) \in [f_u(u(\chi(t)+,t),\chi(t),t), f_u(u(\chi(t)-,t),\chi(t),t)] \tag{38}$$

for almost all $t \in I$.

By the theory of ordinary differential equations with discontinuous right-hand side like (38) (see [28]), we know that through every fixed point $(y,\tau) \in \mathbb{R} \times \mathbb{R}^+$ passes at least one characteristic. We denote a characteristic either by $\chi(t)$ or by $\chi(t; y, \tau)$ when the point (y,τ) must be specified. Every trajectory is confined between a maximal and a minimal characteristic (not necessarily distinct). Moreover, the speed of a generalized characteristic is not free and more precisely by Theorem 3.1 in [20] a characteristic either propagate at the classical speed or at shock speed:

Theorem 3.4. *Let $\chi : I \to \mathbb{R}$ be a characteristic. Then for almost every $t \in I$*

$$\dot{\chi}(t) = \begin{cases} f_u(u(\chi(t)\pm,t),\chi(t),t), & \text{if } u(\chi(t)-,t) = u(\chi(t)+,t); \\ \frac{f(u(\chi(t)+,t),\chi(t),t) - f(u(\chi(t)-,t),\chi(t),t)}{u(\chi(t)+,t)-u(\chi(t)-,t)}, & \text{if } u(\chi(t)-,t) > u(\chi(t)+,t). \end{cases}$$

A *backward (forward) characteristic* trough any point $(y,\tau) \in \mathbb{R} \times \mathbb{R}^+$, is a characteristic χ defined on $[0,\tau]$ (respectively $[\tau,\infty)$) with $\chi(\tau) = y$. We call *genuine* a characteristic $\chi(t)$ such that $u(\chi(t),t) = u(\chi(t)+,t)$ for almost every t.

At this point, we give a list of properties of generalized characteristics of entropy solutions $u(x,t)$ for the balance laws (33). In Chapter 5, we shall make use of these Theorems and in particular of the No-crossing property of Theorem 3.7, to prove the SBV_{loc} regularity of $u(x,t)$.

Theorem 3.5. *Let $\chi(.)$ be a generalized characteristic for (33), associated with the admissible solution u, which is genuine on $I = [a,b]$. Then there is a C^1 function v defined on I such that:*

1. $u(\chi(a)-,a) \leq v(a) \leq u(\chi(a)+,a)$,
2. $u(\chi(t)-,t) = v(t) = u(\chi(t)+,t)$, *for* $a < t < b$,
3. $u(\chi(b)-,b) \geq v(b) \geq u(\chi(b)+,b)$.

Furthermore, $(\chi(.), v(.))$ satisfy the classical characteristic equations

$$\begin{cases} \dot{\chi}(t) = f_u(v(t), \chi(t), t) \\ \dot{v}(t) = -f_x(v(t), \chi(t), t) - g(v(t), \chi(t), t) \end{cases} \tag{39}$$

on (a, b). In particular, χ is a C^1 function on I.

Remark 3.6. *In [21], the generalized characteristic of Theorem 3.5 is assumed to be "shock-free". In here we state the Theorem for genuine characteristic because under an appropriate normalization, the notions of "shock-free" and "genuine" are equivalent. To conclude: we call* shock-free *a characteristic $\chi(t)$ such that $u(\chi(t)-, t) = u(\chi(t)+, t)$ for almost every t.*

Theorem 3.7. *Given a fixed point $(y, \tau) \in \mathbb{R} \times \mathbb{R}^+$ we have that:*

1. *Through (y, τ) pass a minimal and a maximal backward characteristic denoted, respectively, by $\chi_-(t)$ and $\chi_+(t)$. The characteristics $\chi_-, \chi_+ : [0, \tau] \to \mathbb{R}$ are genuine and are the solutions of the ODEs (39) with the following initial conditions: $\chi_-(\tau) = y$, $v_-(\tau) = u(y-, \tau)$ and $\chi_+(\tau) = y$, $v_+(\tau) = u(y+, \tau)$.*
2. *(No-crossing of characteristics). Two genuine characteristics may intersect only at their end points.*
3. *For $\tau > 0$ through (y, τ) passes a unique forward characteristic. Furthermore, if $u(y+, \tau) < u(y-, \tau)$, then $u(\chi(t)+, t) < u(\chi(t)-, t)$ for all $t \in [\tau, \infty)$.*

3.2 One-sided inequality for the balance law (33)

As mentioned above another problem, due to the presence of the source term and of the (x, t) dependence, is that for equations (33) the Oleinik estimate (36) stop to be true. Moreover, the Oleinik estimate cannot be taken as entropy criterion. What we can do, is to find a suitable generalization of this estimate, i.e. we will prove using the generalized characteristics that:

Theorem 3.8. *Let $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a flux function such that $f_{uu}(.) > 0$. Let $g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a source term and let $u \in L^\infty(\Omega)$ be an entropy solution of the balance law (33). In any fixed compact set $K \subset \Omega$ there exists a positive constant $C > 0$ such that:*

$$u([x+z]+, t) - u(x-, t) \leq Cz, \qquad (z > 0). \tag{40}$$

for every $(x, t), (x + z, t) \in K$.

However, for balance laws it is impossible to recover a constant of the form $C = \breve{C}/t$, where C depends only on the time and on the second derivative of f, estimate (40) is sufficient to obtain all the regularity-results stated in Chaper 5.

3.3 Proof of Theorem 3.8

As far as we know the estimate (40) of Theorem 3.8 has never been proved. In this section we then use the generalized characteristics to obtain that:

Proposition 3.9. *Let f, g be as in the statement of Theorem 3.8, in particular in any compact set $K \subset \Omega$ there exist constants $C_1, C_2 > 0$ with*

$$\|Df\|_{L^\infty(K)}, \|D^2 f\|_{L^\infty(K)}, \|Dg\|_{L^\infty(K)} \leq C_1, \tag{41}$$

and

$$f_{uu}(.) \geq C_2 > 0. \tag{42}$$

If $u(x,t)$ is an entropy solution of the balance equation (33), then for all $\varepsilon > 0$ there exists a constant $C_3 := C_3(C_1, C_2, \varepsilon) > 0$ with

$$u([x+z]+,t) - u(x-,t) \leq C_3 z, \qquad (z > 0)$$

for every fixed $t \in [\varepsilon, 1]$ and for all $x \in \mathbb{R}$ with $(x,t), (x+z,t) \in K$.

3.3.1 Preparatory for the proof of Proposition 3.9

Let $t \in [\varepsilon, 1]$ be fixed and take $x \in \mathbb{R}$ and $z > 0$ with $(x,t), (x+z,t) \in K$. Let us denote by $\chi_-(s)$ the minimal backward characteristic passing through (x,t) and by $\chi_+(s)$ the maximal backward characteristic passing through $(x+z,t)$, (instead of $\chi_-(s;x,t)$ and $\chi_+(s;x+z,t)$). Rewriting the ODEs (39) related to the genuine characteristics for $\chi_-(s), \chi_+(s) : [\varepsilon, t] \to \mathbb{R}$ we obtain

$$\begin{cases} \dot{\chi}_\pm(s) = f_u(v_\pm(s), \chi_\pm(s), s) \\ \dot{v}_\pm(s) = -f_x(v_\pm(s), \chi_\pm(s), s) - g(v_\pm(s), \chi_\pm(s), s) \end{cases} \tag{43}$$

with

$$\begin{cases} \chi_-(t) = x \\ \chi_+(t) = x + z \\ v_-(t) = u(x-,t) =: u^- \\ v_+(t) = u([x+z]+,t) =: u^+ \end{cases} \tag{44}$$

where by the admissible condition (35), $u^- > u^+$. We recall that by the no crossing property of Theorem 3.7 of the genuine characteristics, the distance between the two curves is positive i.e. $\chi_+(s) > \chi_-(s)$ for every $s \in [\varepsilon, t]$.

3.3.2 One Technical Lemma

For the proof of our theorem we will use the following technical lemma:

Lemma 3.10. *Assume that there exists a constant $C_3 > 0$ with*

$$u([x+z]+,t) - u(x-,t) \geq C_3 z, \qquad (z > 0). \tag{45}$$

Then there exists $\delta := \delta(C_1, C_2, \varepsilon)$ with $0 < \delta < \varepsilon$ and such that

$$v_+(s) - v_-(s) \geq \frac{C_3}{16} z, \qquad \forall s \in [t-\delta, t]. \tag{46}$$

Proof. We subdivide the proof into two steps:
Step 1: We claim that:

Claim 3.11. *Let $C_3 > 0$ be sufficiently big and assume that there exists a $\delta > 0$ such that inequality (46) holds. Then, we have*

$$\dot{\chi}_+(s) - \dot{\chi}_-(s) > 0, \qquad \forall s \in [t-\delta, t].$$

If our claim were not true, then it would exists $\tau \in [t-\delta, t]$ such that

$$\begin{cases} \dot\chi_+(s) - \dot\chi_-(s) > 0, & \forall s \in]\tau, t]; \\ \dot\chi_+(\tau) - \dot\chi_-(\tau) = 0. \end{cases} \tag{47}$$

By the equation of the characteristics (39) (or (43)) it follows that:

$$\dot\chi_+(\tau) - \dot\chi_-(\tau) = f_u(v_+(\tau), \chi_+(\tau), \tau) - f_u(v_-(\tau), \chi_-(\tau), \tau) =: U(\tau) + W(\tau). \tag{48}$$

The two terms in (48) are defined as

$$U(\tau) := f_u(v_+(\tau), \chi_+(\tau), \tau) - f_u(v_-(\tau), \chi_+(\tau), \tau) \geq C_2(v_+(\tau) - v_-(\tau)), \tag{49}$$

and

$$W(\tau) := f_u(v_-(\tau), \chi_+(\tau), \tau) - f_u(v_-(\tau), \chi_-(\tau), \tau) \geq -C_1|\chi_+(\tau) - \chi_-(\tau)|. \tag{50}$$

Thus by (48),(49),(50) and (46),(47) we obtain that

$$\begin{aligned} \dot\chi_+(\tau) - \dot\chi_-(\tau) &\geq C_2(v_+(\tau) - v_-(\tau)) - C_1|\chi_+(\tau) - \chi_-(\tau)| \\ &\geq C_2 \frac{C_3}{16} z - C_1 z = \Big(\frac{C_2 C_3}{16} - C_1\Big) z. \end{aligned} \tag{51}$$

Clearly in the last inequality we have used the bounds (41) and (42). So if we choose $C_3 > \frac{16C_1}{C_2}$, then

$$\dot\chi_+(\tau) - \dot\chi_-(\tau) > 0$$

and this is in contradiction with the definition of τ.

Step 2: Let us define the time

$$t_0 := \sup_{s \in [t-\delta, t]} \Big\{ s : v_+(s) - v_-(s) < \frac{C_3}{16} z \Big\}.$$

If $t_0 < t - \eta$ for any $\eta > 0$ then we can conclude the proof, because we can trivially select $\delta = \eta$ in (46). Otherwise from Step 1 we know that

$$0 < \chi_+(s) - \chi_-(s) \leq z, \qquad \forall s \in [t_0, t]. \tag{52}$$

Moreover, by the definition of t_0 we know even that

$$v_+(t_0) - v_-(t_0) = \frac{C_3}{16} z. \tag{53}$$

Using the equations of the characteristics (39) (or see also (43)), the bounds (41), (42) and computing as in Step 1, we can state that for every $s \in [t_0, t]$ it holds that:

$$\begin{aligned} |\dot v_+(s) - \dot v_-(s)| &\leq 2C_1|v_+(s) - v_-(s)| + 2C_1|\chi_+(s) - \chi_-(s)| \\ &\overset{(52)}{\leq} 2C_1\Big(|v_+(s) - v_-(s)| + z\Big) \end{aligned} \tag{54}$$

Putting $E(s) := |v_+(s) - v_-(s)| + z$ from the last inequality we get to:

$$\dot E(s) \leq 2C_1 E(s).$$

Thus, by Gronwall's Lemma we have that for every $s \in [t_0, t]$:

$$E(s) \le e^{2C_1(s-t_0)}E(t_0). \tag{55}$$

Choosing $s = t$ we obtain:

$$(C_3 z + z) \overset{(45)}{\le} E(t) \overset{(55)}{\le} e^{2C_1(t-t_0)}E(t_0) \overset{(53)}{=} e^{2C_1(t-t_0)}\Big(\frac{C_3}{16}z + z\Big).$$

Thus,

$$t - t_0 \ge \frac{1}{2C_1}\log\Big(\frac{16C_3 + 16}{C_3 + 16}\Big)$$

and in particular if C_3 is big enough we conclude that $t - t_0 > 0$. This means that there exists $\delta > 0$ such that $t - \delta \in [t_0, t]$ and the inequality (46) is satisfied. □

3.3.3 Proof of Proposition 3.9

Case 1: Trivially, if it holds

$$u([x+z]+, t) - u(x-, t) \le C_3 z, \qquad (z > 0)$$

our proposition is true.
Case 2: Otherwise, we can choose an $\alpha > C_3$ with

$$\frac{u([x+z]+, t) - u(x-, t)}{z} = \alpha, \qquad (z > 0).$$

By Lemma 3.10 there exists $0 < \delta < \varepsilon$ such that for every $s \in [t-\delta, t]$ we have the following two estimates:

$$0 < \chi_+(s) - \chi_-(s) \le z, \tag{56}$$

$$v_+(s) - v_-(s) \ge \frac{\alpha z}{16}. \tag{57}$$

Now, using the same techniques and bounds as above we compute:

$$\begin{aligned} \chi_+(t) - \chi_-(t) &= \int_{t-\delta}^{t} f_u(v_+(s), \chi_+(s), s) - f_u(v_-(s), \chi_-(s), s)ds + \chi_+(t-\delta) - \chi_-(t-\delta) \\ &\ge \int_{t-\delta}^{t} f_u(v_+(s), \chi_+(s), s) - f_u(v_-(s), \chi_-(s), s)ds \\ &\ge C_2\frac{\alpha\delta}{16}z - \int_{t-\delta}^{t} C_1|\chi_+(s) - \chi_-(s)|ds \ge C_2\frac{\alpha\delta}{16}z - \delta z C_1. \end{aligned} \tag{58}$$

Since $z = \chi_+(t) - \chi_-(t)$ we rewrite (58) as

$$z(1 + C_1\delta) \ge C_2\frac{\alpha\delta}{16}z \qquad \Rightarrow \qquad \alpha \le \frac{16(1 + C_1\delta)}{\delta C_2},$$

and then

$$u([x+z]+, t) - u(x-, t) \le \frac{16(1 + C_1\delta)}{\delta C_2}z, \qquad (z > 0)$$

This conclude the proof.

4 Hamilton-Jacobi equations

4.1 Viscosity solutions and Hopf-Lax Formula

This section is dedicated to the theory of Hamilton-Jacobi equations. For a complete survey on this topic we redirect the reader to the vast literature: for an introduction we suggest the following references [10, 16, 25]. In this work we will often consider Hamilton-Jacobi equations:

$$\partial_t u + H(D_x u) = 0, \qquad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n \ , \tag{59}$$

$$H(D_x u) = 0, \qquad \text{in } \Omega \subset \mathbb{R}^n \ , \tag{60}$$

under the assumption that

A1: The Hamiltonian $H \in C^2(\mathbb{R}^n)$ satisfies:

$$p \mapsto H(p) \text{ is convex and } \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty.$$

Note that this assumption is obviously implied by

$$H \in C^2(\mathbb{R}^n) \qquad \text{and} \qquad c_H^{-1} Id_n \leq D^2 H \leq c_H Id_n \quad \text{for some } c_H > 0. \tag{61}$$

Moreover, as it is well-known, under the assumption (61), any viscosity solution u of (59) is locally semiconcave in x. More precisely, for every $K \subset\subset \Omega$ there is a constant C (depending on K, Ω and c_H) such that the function $x \mapsto u(t,x) - C|x|^2$ is concave on K.

We will usually consider $\Omega = [0,T] \times \mathbb{R}^n$ in (59) and couple it with the initial condition

$$u(0,x) = u_0(x) \tag{62}$$

under the assumption that

A2: The initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous and bounded.

Definition 4.1 (Viscosity solution). *A bounded, uniformly continuous function u is called a* viscosity solution *of* (59) *(resp.* (60)*) provided that*

1. *u is a* viscosity subsolution *of* (59) *(resp.* (60)*): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a maximum at (t_0, x_0) (resp. x_0),*

$$v_t(t_0,x_0) + H(D_x v(t_0,x_0)) \leq 0 \qquad (\text{resp. } H(Dv(x_0)) \leq 0);$$

2. *u is a* viscosity supersolution *of* (59) *(resp.* (60)*): for each $v \in C^\infty(\Omega)$ such that $u - v$ has a minimum at (t_0, x_0) (resp. x_0),*

$$v_t(t_0,x_0) + H(D_x v(t_0,x_0)) \geq 0 \qquad (\text{resp. } H(Dv(x_0)) \geq 0).$$

In addition, we say that u solves the Cauchy problem (59)-(62) *on $\Omega = [0,T] \times \mathbb{R}^n$ if* (62) *holds in the classical sense.*

Theorem 4.2 (The Hopf-Lax formula as viscosity solution)**.** *The unique viscosity solution of the initial-value problem* (59)-(62) *is given by the Hopf-Lax formula*

$$u(t,x) = \min_{y\in\mathbb{R}^n}\Big\{u_0(y) + tL\Big(\frac{x-y}{t}\Big)\Big\} \qquad (t>0, x\in\mathbb{R}^n), \tag{63}$$

where L is the Legendre transform of H:

$$L(q) := \sup_{p\in\mathbb{R}^n}\{p\cdot q - H(p)\} \qquad (q\in\mathbb{R}^n).$$

Remark 4.3. *Due to the strict connection between one-dimensional conservation laws and planar Hamilton-Jacobi equations, a suitable modification of the Hopf-Lax formula (63) could be derived to find a representation formula for entropy solutions of (34). In the literature it's also known as Lax-Oleinik formula (see for instance Section 3.4.2 of [25]).*

In the next Proposition we collect some properties of the viscosity solution defined by the Hopf-Lax formula:

Proposition 4.4. *Let $u(t,x)$ be the viscosity solution of* (59)-(62) *and defined by* (63)*, then*

(i) ***A functional identity:*** *For each $x\in\mathbb{R}^n$ and $0\le s<t\le T$, we have*

$$u(t,x) = \min_{y\in\mathbb{R}^n}\Big\{u(s,y) + (t-s)L\Big(\frac{x-y}{t-s}\Big)\Big\}. \tag{64}$$

(ii) ***Semiconcavity of the solution:*** *For any fixed $\tau>0$ there exists a constant $C(\tau)$ such that the function defined by*

$$u_t:\mathbb{R}^n\to\mathbb{R}^n \text{ with } u_t(x) := u(t,x),$$

is semiconcave with constant less than C for any $t\ge\tau$.

(iii) ***Characteristics:*** *The minimum point y in* (63) *is unique if and only if $\partial u_t(x)$ is single valued. Moreover, in this case we have $y = x - tDH(D_x u(t,x))$.*

(iv) ***The linear programming principle:*** *Let $t>s>0$, $x\in\mathbb{R}^n$ and assume that y is a minimum for* (63)*. Let $z=\frac{s}{t}x+(1-\frac{s}{t})y$. Then y is the* unique *minimum for $u_0(w)+sL((z-w)/s)$.*

Remark 4.5. *For a detailed proof of Theorem 4.2 and Proposition 4.4 we address the reader to Chapter 6 of [16] and Chapters 3, 10 of [25] .*

Next, we state a useful locality property of the solutions of (59).

Proposition 4.6. *Let u be a viscosity solution of* (59) *in Ω. Then u is locally Lipschitz. Moreover, for any $(t_0,x_0)\in\Omega$, there exists a neighborhood U of (t_0,x_0), a positive number δ and a Lipschitz function v_0 on $\mathbb{R}^n$ such that*

(Loc) u coincides on U with the viscosity solution of

$$\begin{cases} \partial_t v + H(D_x v) = 0 & \text{in } [t_0-\delta,\infty[\times\mathbb{R}^n \\ v(t_0-\delta,x) = v_0(x)\,. \end{cases} \tag{65}$$

This property of viscosity solutions of Hamilton-Jacobi equations is obviously related to the finite speed of propagation (which holds when the solution is Lipschitz) and it is well-known. One could prove it, for instance, suitably modifying the proof of Theorem 7 at page 132 of [25]. On the other hand we have not been able to find a complete reference for Proposition 4.6. Therefore, for the reader's convenience, we provide a reduction to some other properties clearly stated in the literature.

Proof. The local Lipschitz regularity of u follows from its local semiconcavity, for which we refer to [16]. As for the locality property (Loc), we let $\delta > 0$ and R be such that $C := [t_0 - \delta, t_0 + \delta] \times \overline{B}_R(x_0) \subset \Omega$. It is then known that the following dynamic programming principle holds for every $(t, x) \in C$ (see for instance Remark 3.1 of [17] or [26]):

$$\begin{aligned} u(t,x) \;=\; \inf\Bigg\{ \int_\tau^t L(\dot\xi(s))\,ds + u(\tau,\xi(\tau)) \,\Bigg|\, & \tau \le t, \xi \in W^{1,\infty}([\tau,t]), \\ & \xi(t) = x \text{ and } (\tau,\xi(\tau)) \in \partial C \Bigg\}. \end{aligned} \tag{66}$$

The Lipschitz regularity of u and the convexity of L ensure that a minimizer exists. Moreover any minimizer is a straight line. Next, assume that $x \in B_\delta(x_0)$. If δ is much smaller than R, the Lipschitz regularity of u ensures that any minimizer ξ has the endpoint $(\tau, \xi(\tau))$ lying in $\{t_0 - \delta\} \times B_R(x_0)$. Thus, for every $(t, x) \in [t_0 - \delta, t_0 + \delta] \times B_\delta(x_0)$ we get the formula

$$u(t,x) \;=\; \min_{y \in \overline{B}_R(x_0)} \left(u(t_0 - \delta, y) + (t - t_0 + \delta) L\left(\frac{x-y}{t - t_0 + \delta} \right) \right). \tag{67}$$

Next, extend the map $\overline{B}_R(0) \ni x \mapsto u(t_0 - \delta, x)$ to a bounded Lipschitz map $v_0 : \mathbb{R}^n \to \mathbb{R}$, keeping the same Lipschitz constant. Then the solution of (65) is given by the Hopf-Lax formula

$$v(t,x) \;=\; \min_{y \in \mathbb{R}^n} \left(v_0(y) + (t - t_0 + \delta) L\left(\frac{x-y}{t - t_0 + \delta} \right) \right). \tag{68}$$

If $(t, x) \in [t_0 - \delta, t_0 + \delta] \times B_\delta(0)$, then any minimum point y in (68) belongs to $\overline{B}_R(0)$, provided δ is sufficiently small (compared to R and the Lipschitz constant of v, which in turn is bounded independently of δ). Finally, since $v_0(y) = u(t_0 - \delta, y)$ for every $y \in \overline{B}_R(0)$, (67) and (68) imply that u and v coincide on $[t_0 - \delta, t_0 + \delta] \times B_\delta(0)$ provided δ is sufficiently small. □

5 SBV regularity of entropy solutions for a class of genuinely nonlinear scalar balance laws with non-convex flux function.

5.1 Introduction

In [3] the authors have shown that entropy solutions $u(x,t)$ of scalar conservation laws

$$D_t u(x,t) + D_x[f(u(x,t))] = 0 \qquad \text{in an open set } \Omega \subset \mathbb{R}^2 \tag{69}$$

with locally uniformly convex flux function $f \in C^2(\mathbb{R})$ and $f'' > 0$, are functions of locally special bounded variation, i.e. the distributional derivative Du has no Cantor part. In the proof proposed by Ambrosio and De Lellis in [3] the good geometric structure of the characteristics field correlated to the entropy solution play an important role and allows to define a geometric functional which jumps every time when a Cantor part of the distributional derivative $Du(.,t)$ appears in the solution. In particular we recall here two significant properties of the characteristics: they are straight lines and two different backward characteristics can cross only at $t = 0$ (the so-called no crossing property). In [3] the one-sided estimate of Oleinik (36) is used as entropy criterion and it is used to get the proof.

In this note we extend this regularity result to a bigger class of hyperbolic conservation laws. At first we again consider scalar conservation laws (69) but allowing the change of convexity of the flux function f at a countable set of points. One of the difficulties in dealing with these equations is that rarefaction waves may appear even for $t > 0$ and consequently the no crossing property used in [3] does not hold. For instance, it is possible to construct a Riemann problem where the flux function has two inflections points and a shock splits into two contact discontinuities (see [33]). As we will see the strategy of the proof is not as complicated as one can expect: using an appropriate covering of Ω and working locally we reduce the problem to the convex or concave case. Thus, our first extension theorem states:

Theorem 5.1. *Let $f \in C^2(\mathbb{R})$ be a flux function, such that $\{u_i \in \mathbb{R} : f''(u_i) = 0\}$ is at most countable. Let $u \in BV(\Omega)$ be an entropy solution of the scalar conservation law (69). Then there exists a set $S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R}\backslash S$ the following holds:*

$$u(.,\tau) \in SBV_{loc}(\Omega_\tau) \qquad \text{with } \Omega_\tau := \{x \in \mathbb{R} : (x,\tau) \in \Omega\}.$$

In the second part of this section we focus our attention on genuinely nonlinear scalar balance laws

$$D_t u(x,t) + D_x[f(u(x,t),x,t)] + g(u(x,t),x,t) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2 \tag{70}$$

where the source term g belongs to $C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$, the flux functions f belongs to $C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ and $f_{uu}(.,x,t) > 0$ for any fixed $(x,t) \in \Omega$. Again the geometric structure of the characteristics is not as easy as in the case treated in [3]: now the characteristics are Lipschitz curves and in general are not straight lines. The different shape of the characteristics are due to the presence of the source term and to the dependence of f on the points $(x,t) \in \Omega$. Fortunately, we can make use of the theory of generalized characteristics introduced by Dafermos (see [19],[20],[21]) to analyze the behavior of the characteristics for entropy solutions of (70). Important for our analysis is the no-crossing property between genuine characteristics. Thanks to this property we can expect to reproduce the geometric proof proposed in [3]. All the definitions and propositions about the theory of generalized characteristics, which are helpful in our work, are listed in Chaper 3. The second theorem on the SBV regularity proposed is:

Theorem 5.2. *Let $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a flux function such that $f_{uu}(.) > 0$. Let $g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a source term and let $u \in L^\infty(\Omega)$ be an entropy solution of the balance law (70). Then there exists a set $S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R} \backslash S$ the following holds:*

$$u(., \tau) \in SBV_{loc}(\Omega_\tau) \qquad \text{with } \Omega_\tau := \{x \in \mathbb{R} : (x, \tau) \in \Omega\}.$$

Combining the two Theorems on the SBV regularity we get a generalized Theorem, which says that also for balance laws with a flux function which changes convexity at most countable many times, the entropy solution is a locally SBV function. Thus, as a consequence of Theorem 5.2 and 5.1 and of the slicing theory of BV functions, we state:

Theorem 5.3. *Let $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a flux function, such that*

$$\{u_i \in \mathbb{R} : f_{uu}(u_i, x, t) = 0\}$$

is at most countable for any fixed (x, t). Let $g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be a source term and let $u \in BV(\Omega)$ be an entropy solution of the balance law (70):

$$D_t u(x,t) + D_x[f(u(x,t), x, t)] + g(u(x,t), x, t) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2.$$

Then there exists a set $S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R} \backslash S$ the following holds:

$$u(., \tau) \in SBV_{loc}(\Omega_\tau) \qquad \text{with } \Omega_\tau := \{x \in \mathbb{R} : (x, \tau) \in \Omega\}.$$

Moreover, $u(x,t) \in SBV_{loc}(\Omega)$.

Scalar conservation laws in one space dimension and Hamilton-Jacobi equations in one dimension are strictly connected: entropy solutions correspond to viscosity solutions (see [25]). Thus, at the end using Theorem 5.2 we obtain also a regularity statement for viscosity solutions u of a class of Hamilton-Jacobi equations: we prove that the gradient Du of such solutions belongs to $SBV_{loc}(\Omega)$.

Corollary 5.4 (Hamilton-Jacobi)**.** *Let $H(u, x, t) \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ be locally uniformly convex in u, i.e. $D_{uu}H > 0$. If $w \in W^{1,\infty}(\Omega)$ is a viscosity solution of*

$$w_t(x,t) + H(w_x(x,t), x, t) = 0, \tag{71}$$

then $Dw \in SBV_{loc}(\Omega)$.

Concluding, it would be interesting to find the same regularity for entropy BV solutions of genuinely nonlinear system of conservation laws in one space dimension. We note that there are analogies between the structure of the generalized characteristics of systems and the one of the balance laws (70) of Theorem 5.3 proposed in here: in both cases the characteristics can intersect at $t \neq 0$ and in general they are not straight lines but Lipschitz curves, which are a.e. differentiable. Although the geometry of the characteristics field of these two problems seems to be similar, the case of systems looks much more difficult. As far as we know the unique regularity result for system of conservation laws states that self-similar solutions are SBV_{loc}: see Dafermos [22].

Looking at the Hamilton-Jacobi equations, in the next Chapter we will show the local SBV regularity for gradients of viscosity solutions of uniformly convex Hamilton-Jacobi PDEs in higher space dimensions. About this regularity, in 1997 in [15] the authors have shown that *under strong regularity assumptions* on the initial functions u_0, the viscosity solution u has a gradient Du, which belongs to the class SBV, i.e. D^2u is a measure with no Cantor part (in fact the regularity theory of [15] and [16] gives stronger conclusions).

5.2 Theorem 5.1

In this part we analyze the regularity of the entropy solutions of the conservation laws (69). We recall that in [3] the flux function $f \in C^2$ was selected to be strictly convex and it was proved that entropy solutions are locally SBV. Here, in our first extension Theorem 5.1 we consider any flux function $f \in C^2$, which can change its convexity. Indeed, f is selected such that $\#\{u_i \in \mathbb{R} : f''(u_i) = 0\}$ is at most countable.

5.2.1 Strictly convex or concave flux function

The first step in trying to extend Theorem 1.2 of [2] is to state the same result also for a conservation law with a strictly concave flux function, i.e. $f'' < 0$.

Lemma 5.5 (Strictly convex or concave flux). *Let $f \in C^2(\mathbb{R})$ be a flux function with $|f''(u)| > 0$. Let $u \in L^\infty(\Omega)$ be an entropy solution of the scalar conservation law (69). Then there exists a set $S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R}\backslash S$ the following holds:*

$$u(.,\tau) \in SBV_{loc}(\Omega_\tau) \qquad \text{with } \Omega_\tau := \{x \in \mathbb{R} : (x,\tau) \in \Omega\}.$$

Since the arguments are quite standard, we propose only a sketch of the proof of the above Lemma:

Proof. If $f'' > 0$, i.e. the flux function is strictly convex, the statement is exactly Theorem 1.2 of [2]. If $f'' < 0$, i.e. the flux function is strictly concave and we may prove the lemma directly using the convex case. The idea is simple: we reflect an entropy solution of the strictly convex case about the t-axis, to obtain an entropy solution of the related strictly concave problem. Let f be strictly convex and assume that $u(x,t)$ is an entropy solution of (69). We define the coordinates transformation $\phi : \Omega \to \widetilde{\Omega}$, $\phi : (x,t) \mapsto (y,t) = (-x,t)$ and the candidate solution $\widetilde{u}(y,t) := u \circ \phi^{-1}(y,t) = u(-x,t)$ of the strictly concave problem. Then, we have:

$$\begin{aligned} D_t\widetilde{u}(y,t) + D_y[\widetilde{f}(\widetilde{u}(y,t))] &= D_t\widetilde{u}(y,t) - D_y[f(\widetilde{u}(y,t))] = D_t\widetilde{u}(-x,t) - (-D_x)[f(\widetilde{u}(-x,t))] = \\ &= D_t u(x,t) + D_x[f(u(x,t))] = 0 \end{aligned}$$

With $\widetilde{f} = -f$ we denote a strictly concave flux. Moreover, for a point of Ω on a shock $x(t)$ with shock speed $\sigma = dx(t)/dt$, we must have that the Rankine-Hugoniot $\sigma[u^R - u^L] = [f(u^R) - f(u^L)]$ and that the Lax-Entropy condition for a strictly convex flux $u^L > u^R$ are satisfied. We set $\widetilde{\sigma} = d(-x(t))/dt = -\sigma$, $\widetilde{u}^L = u^R$, $\widetilde{u}^R = u^L$ and $\widetilde{f} = -f$, then after the reflection we obtain:

$$\sigma[u^R - u^L] = [f(u^R) - f(u^L)] \Leftrightarrow \widetilde{\sigma}[\widetilde{u}^R - \widetilde{u}^L] = [\widetilde{f}(\widetilde{u}^R) - \widetilde{f}(\widetilde{u}^L)].$$

and $\widetilde{u}^R > \widetilde{u}^L$. This means that if $u(x,t)$ is an entropy solution of (69) for a strictly convex flux f, then $\widetilde{u}(y,t) = u(-x,t)$ is an entropy solution of the "correlate" conservation law with strictly concave flux $\widetilde{f} = -f$. In particular by the convex case and the definition of $\widetilde{u}$, there exists $\widetilde{S} = S \subset \mathbb{R}$ at most countable such that $\forall \tau \in \mathbb{R}\backslash\widetilde{S}$ the following holds:

$$\widetilde{u}(.,\tau) \in SBV_{loc}(\widetilde{\Omega}_\tau) \qquad \text{with } \widetilde{\Omega}_\tau := \{x \in \mathbb{R} : (x,\tau) \in \widetilde{\Omega}\}.$$

□

5.2.2 Proof of Theorem 5.1

Step 1: Preliminary remarks.

Let us fix $(\xi,\tau)\in\Omega$ and r such that $B_r(\xi,\tau)\subset\Omega$. Thanks to the finite speed of propagation, there exists a positive ρ such that the values of u in the ball $B_\rho(\xi,\tau)$ depend only on the values of u on the segment $\{t=\tau-2\rho\}\cap B_r(\xi,\tau)$. Thus, if we denote by w the entropy solution of the Cauchy problem

$$\begin{cases} D_t w(x,t)+D_x[f(w(x,t))]=0 & \text{for } t>\tau-2\rho;\\ w(x,\tau-2\rho)=u(x,\tau-2\rho)\mathbf{1}_{B_r(\xi,\tau)}(x,\tau-2\rho) & \text{for every } x\in\mathbb{R}, \end{cases} \tag{72}$$

we get $w(x,t)=u(x,t)$ on $B_\rho(\xi,\tau)$. Moreover, note that $w(.,t)\in BV$ for every $t>\tau-2\rho$. Thus, it suffices to prove the main theorem under the assumptions that $\Omega=\{t>0\}$ and that $u(.,0)$ is a bounded compactly supported BV function. By assumption, an entropy solution u of the conservation laws (69) belongs to the space $BV(\Omega)$. Moreover, u has a better structure than any 2-dimensional BV-function. Introducing some notation, we recall that if u is an entropy solution of (69), then $u_\tau(.):=u(.,\tau)\in BV(\Omega_\tau)$ for all τ, where $\Omega_\tau:=\{x\in\mathbb{R}:(x,\tau)\in\Omega\}$. By Proposition 1.10 for any 1-dim BV function the set of atoms A coincides with the set of the discontinuous points; hence for all τ we introduce the sets of the jumps $J_\tau:=\{x\in\Omega_\tau : u^L_\tau(x)\neq u^R_\tau(x)\}\subset\mathbb{R}$ and $J:=\{(x,t)\in\Omega : x\in J_t\}$. By assumption the set $I_f:=\{u_i\in\mathbb{R}: f''(u_i)=0\}$, which contains all the inflection points of f, is at most countable. We conclude defining the set $F_\tau:=\{x\in\Omega_\tau : u_\tau(x)=u_i, u_i\in I_f\}$.

Step 2: Bad points have measure 0.

Using the above notation, we introduce the sets $C_\tau:=J_\tau\cup F_\tau$ for all τ and $C:=\{(x,t):x\in C_t\}$, i.e. the sets of the "bad" points for which either $u_\tau(.)$ has a jump or f'' vanishes. For this two sets we state:

Claim 5.6. *For any τ we have that $|D^c_x u_\tau|(C_\tau)=0$.*

Proof. For every τ one has

$$|D^c_x u_\tau|(C_\tau)=|D^c_x u_\tau|(J_\tau\cup F_\tau)\leq |D^c_x u_\tau|(J_\tau)+|D^c_x u_\tau|(F_\tau). \tag{73}$$

Observe that for all τ the Cantor part is zero on the jump sets J_τ, since by (22):

$$D^c_x u_\tau = D_x u_\tau \llcorner (S\backslash J_\tau) \qquad \Rightarrow \qquad D^c_x u_\tau(J_\tau)=0 \qquad \Rightarrow \qquad |D^c_x u_\tau|(J_\tau)=0.$$

Using the second statement of Proposition 1.10, we show that even the second term of inequality (73) vanishes. Since I_f is a countable set, we may rewrite this set as countable union of the sets $E_i:=\{u_i\}$, i.e. $I_f=\bigcup_i E_i$. It is clear that $\mathcal{L}^1(E_i)=0$ for each i and that $F_\tau=\bigcup_i u_\tau^{-1}(E_i)=\bigcup_i u_\tau^{-1}(u_i)$. By Proposition 1.10, the Cantor part is zero on sets of the form $F_\tau=u_\tau^{-1}(I_f)$ with $\mathcal{L}^1(I_f)=0$. Hence, we obtain:

$$|D^c_x u_\tau|(F_\tau)=|D^c_x u_\tau|(u_\tau^{-1}(I_f))=0.$$

This concludes the proof of the claim, i.e. $|D^c_x u_\tau|(C_\tau)=0$. □

Step 3: Locally, more precisely in a triangle, we reduce the problem to the cases with strictly convex or concave flux functions. By Lemma 5.5 the SBV-regularity follows.

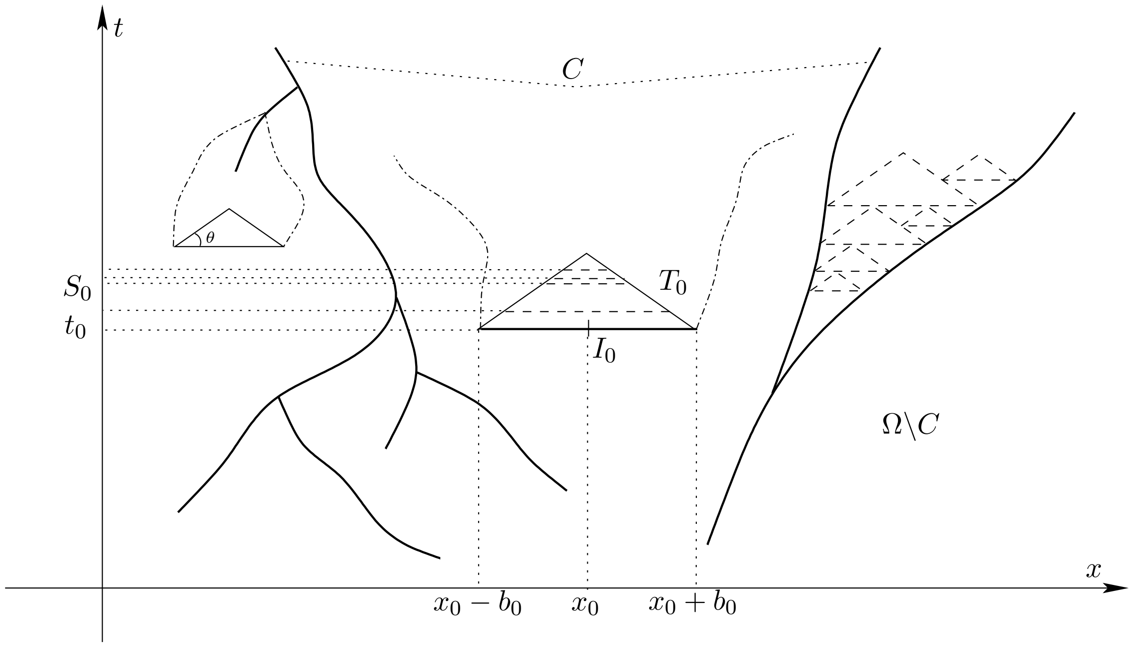

Figure 1: $\Omega\backslash C$ is covered by triangles..

Given any point $(x_0, t_0) \in \Omega\backslash C$ it is always possible to find a positive number $b_0 = b(x_0, t_0) > 0$ and a positive $c_0 = c(x_0, t_0)$, such that the following assertion holds:

$$|f''(u_{t_0}(x))| \geq c_0 > 0 \text{ for every } x \in I_0 :=]-b_0 + x_0, x_0 + b_0[.$$

Since u_{t_0} is a continuous BV-function in $\Omega\backslash C_{t_0}$, we can also assume that there exists a positive $l_0(b_0) > 0$ (which depends on b_0), such that $|u_{t_0}(x) - u_i| \geq l_0 > 0$ for all $x \in I_0$ and for any $u_i \in I_f$. This means that we may select $b_0 > 0$, such that the small variation of u_{t_0} in I_0 allows to consider the Cauchy problem:

$$\begin{cases} D_t w(x,t) + D_x[f_0(w(x,t))] = 0 & \text{for } t > t_0; \\ w(x,t_0) = u_{t_0}(x)\mathbf{1}_{I_0}(x,t_0) & \text{for every } x \in \mathbb{R}, \end{cases} \tag{74}$$

where the flux function f_0 is either strictly concave or strictly convex.

The finite speed of propagation of the characteristics permits to construct an isosceles triangle T_0 with base I_0, s.t. $w(x,t) = u(x,t)$ on the triangle T_0. Let $\theta(f, \|u\|_{L^\infty}) > 0$ be the angle between the base I_0 and the diagonal segment. The angle θ depends only on f and on $\|u\|_{L^\infty}$, since the slope of the maximal or the minimal characteristic of the problem defines this angle. We define the open triangle

$$T_0 := T_{b_0}(x_0, t_0) = \{(x,t) : |x - x_0| < b_0 \text{ and } 0 < t - t_0 < \tan\theta \cdot \min\{x - x_0 + b_0, x_0 + b_0 - x\}\}$$

By the maximum principle, the values of u in T_0 are controlled by the values of u_{t_0} on I_0. Moreover we may apply the statement of Lemma 5.5 on T_0, since the Cauchy problem (74) has either a strictly convex or a concave flux. In particular in our triangle T_0 there exists an at most countable set S_0 consisting of $\tau \in]t_0, t_0 + b_0 \cdot \tan\theta[$ such that the solution is not $SBV(\Omega_\tau \cap T_0)$.

Step 4: Using Step 3, we construct a triangle for all the points of $\Omega \backslash C$. Let B be the set of all points of $\Omega \backslash C$, which are contained in at least one of this triangle and divide $\Omega \backslash C$ into the two subsets B and $C' := \Omega \backslash (C \cup B)$, i.e. $\Omega \backslash C = B \cup C'$.

Claim 5.7. *The set* $\{\tau \in \mathbb{R}^+ : \{t = \tau\} \cap C' \neq \emptyset\}$ *is at most countable.*

Proof. Assume that $\{t : \{t = \tau\} \cap C' \neq \emptyset\}$ is not countable. Let $\{P_\alpha\} = \{(x_\alpha, \tau_\alpha)\} \subset C'$ be a subset of $C' \subset \Omega \backslash C$, such that $\tau_\alpha \neq \tau_\beta$ whenever $\alpha \neq \beta$. Moreover, let $\{P_\alpha^k\} = \{(x_\alpha, \tau_\alpha) : b(x_\alpha, \tau_\alpha) \geq 2^{-k}\} \subset \{P_\alpha\}$ be the subsets of the points, for which the base of the triangle is larger than 2^{-k+1}, where $k \in \mathbb{N}$. By assumption there exists a fixed $K \in \mathbb{N}$ such that $\# P_\alpha^K$ is uncountable and thus the set P_α^K contains an accumulation point $p = (x_p, \tau_p)$. This implies that there exists a sequence $\{p_j\}_j := \{(x_j, \tau_j)\}_{j \in \mathbb{N}} \subset P_\alpha^K$ of points in C', which converges to the accumulation point p. Moreover, any point p_j of this sequence cannot be contained in the triangle $T_p := T_{b(x_p, \tau_p)}(x_p, \tau_p)$, since by definition any point of C' cannot lie into a triangle. We then have that the sequence $\{p_j\}_j$ approaches p from below, i.e. $\tau_j < \tau_p$ for every $j > J$ with J big enough, and the triangles $T_j := T_{b(x_j, \tau_j)}(x_j, \tau_j)$ have a base larger than 2^{-K+1} for each j. Thus, for a $J \in \mathbb{N}$ big enough the accumulation point p belongs to the triangles T_j for all $j > J$. This is a contradiction to $p \in C'$. ☐

Step 5: Cover with triangles.

By definition, every $(x, t) \in B$ lies into at least one triangle T_0 for a $(x_0, \tau_0) \in B$. The set B is then covered by a family of triangles $\{T_\alpha\}_\alpha$. In particular we can find a countable subfamily of triangles $\{T_i\}_i \subset \{T_\alpha\}_\alpha$ which covers B, i.e. $B \subset \bigcup_i T_i$. We now divide Ω using the sets defined above:

$$\Omega = (\Omega \backslash C) \cup C = B \cup C' \cup C \subset \bigcup_i T_i \cup C' \cup C.$$

For every $\tau \in \mathbb{R}$ we have that:
- by Claim 5.7 the set $S_{C'} = \{t : \{t = \tau\} \cap C' \neq \emptyset\}$ is at most countable;
- by Lemma 5.5 for every T_i the set $S_i := \{t : u_t \notin SBV(\{t = \tau\} \cap T_i)\}$ is at most countable.

Thus, for every time $\tau \notin S := S_{C'} \cup \left(\bigcup_i S_i\right)$ we have the following inequality:

$$\begin{aligned} |D_x^c u_\tau|(\Omega \cap \{t = \tau\}) &\leq |D_x^c u_\tau|\Big(\bigcup_i T_i \cap \{t = \tau\}\Big) + |D_x^c u_\tau|(C \cap \{t = \tau\}) \leq \\ &\leq \sum_i |D_x^c u_\tau|\Big(T_i \cap \{t = \tau\}\Big) + |D_x^c u_\tau|(C \cap \{t = \tau\}) = 0. \end{aligned}$$

All terms in the sum vanish by Lemma 5.5 and the second term is equal to zero by Claim 5.6. Letting $\Omega_\tau = \Omega \cap \{t = \tau\}$ we have shown that $\forall \tau \in \mathbb{R} \backslash S$ the following holds:

$$u_\tau(.) = u(., \tau) \in SBV_{loc}(\Omega_\tau).$$

5.3 Preparatory tools for the proof of Theorem 5.2

In the geometric proof of the main Theorem in [3] the characteristics of the entropy solutions of the scalar conservation laws (69) played a fundamental role. Using the good and simple structure of these characteristics it was possible to define a monotone geometric functional, which jumps when a Cantor part appears in the solution. One of the key Lemmas was the No-crossing-Proposition (see Proposition 2.5 of [3]), which implies that two different characteristics $\chi_1 : [0,\tau_1] \to \mathbb{R}$, $\chi_2 : [0,\tau_2] \to \mathbb{R}$ cannot cross for all $t \in (0, max\{\tau_1,\tau_2\})$. The generalized characteristics of a balance laws (70) with source term g are in general no more straight lines, but by Theorem 3.7 we know that the No-crossing property still holds for two distinct genuine characteristics. By the way, for every point $(y,\tau) \in \mathbb{R}\times\mathbb{R}^+$ the minimal and maximal backward characteristics are genuine and then the No-crossing property is assured. This suggests us that the construction proposed in [3] still works even for equations (70) and that we can try to restate the main steps of the original proofs. In particular, a geometric approach to our problem make sense and in this section we shall then introduce a geometric functional defined using the generalized characteristics. We begin by giving some preliminary definitions and propositions, which are inspired to those presented in [3]:

Definition 5.8. *(*Characteristic cones and bases*). Let $\tau > 0$. For $\theta \in [0,\tau]$ the* backward characteristic cone $C^\theta_{y,\tau}$ *emanating from $y \in S_\tau := \{x \in \mathbb{R} : u(x-,\tau) \neq u(x+,\tau)\}$ is defined as the open "triangle" having*

$$(y,\tau), (\chi_-(\theta; y,\tau), \theta), (\chi_+(\theta; y,\tau), \theta)$$

as vertices. The base *of a characteristic cone at time $\theta \in [0,\tau]$ is defined as the open interval:*

$$I^\theta_{y,\tau} :=]\chi_-(\theta; y,\tau), \chi_+(\theta; y,\tau)[.$$

We note that the characteristic cones are confined by minimal and maximal backward characteristics, which are genuine. Then, due to the No-crossing property of genuine characteristics two different cones $C^\theta_{y_1,\tau_1}$ and $C^\theta_{y_2,\tau_2}$ (or two different bases $I^\theta_{y_1,\tau_1}$ and $I^\theta_{y_2,\tau_2}$) are either one contained in the other or disjoint. We define also

$$C^\theta_\tau := \bigcup_{y\in S_\tau} C^\theta_{y,\tau} \qquad \text{and} \qquad I^\theta_\tau := \bigcup_{y\in S_\tau} I^\theta_{y,\tau}.$$

Remark 5.9 (On the parameter θ)**.** *A priori we cannot know if a maximal backward characteristic and minimal one starting from the same point (y,τ) would intersect at $t = 0$. Thus, we introduce the parameter $\theta > 0$ such that all the bases of the characteristic cones $C^\theta_{y,\tau}$ have a strict positive length. In [3] the characteristics are straight lines, then it was possible to select $\theta = 0$. Since here we have to work with characteristics, which in general are curves, we cannot require to have $\theta = 0$ without a deepest investigation.*

It is known that entropy solutions $u(x,t)$ of (70) are BV functions. Moreover, for every t the function $u(.,t)$ is also BV on $\mathbb{R}$ and then by (22) we can split $D_x u(.,t)$ in three mutually singular parts:

$$D_x u(.,t) = D^a_x u(.,t) + D^j_x u(.,t) + D^c_x u(.,t)$$

For convenience we denote by $\mu_t := D^c_x u(.,t)$ the Cantor part and by $\nu_t := D^j_x u(.,t)$ the jump part. Inequality (35) implies that the singular measures μ_t and ν_t are both nonpositive. We recall also the semi-monotonicity of $u(.,t)$ that gives

$$u(y+,t) - u(x-,t) = Du(.,t)([x,y]) \qquad \text{whenever } x < y. \tag{75}$$

Finally we state three technical lemmas, which are to compare with estimates (3.4), (3.10) and Lemma 3.2 of [3].

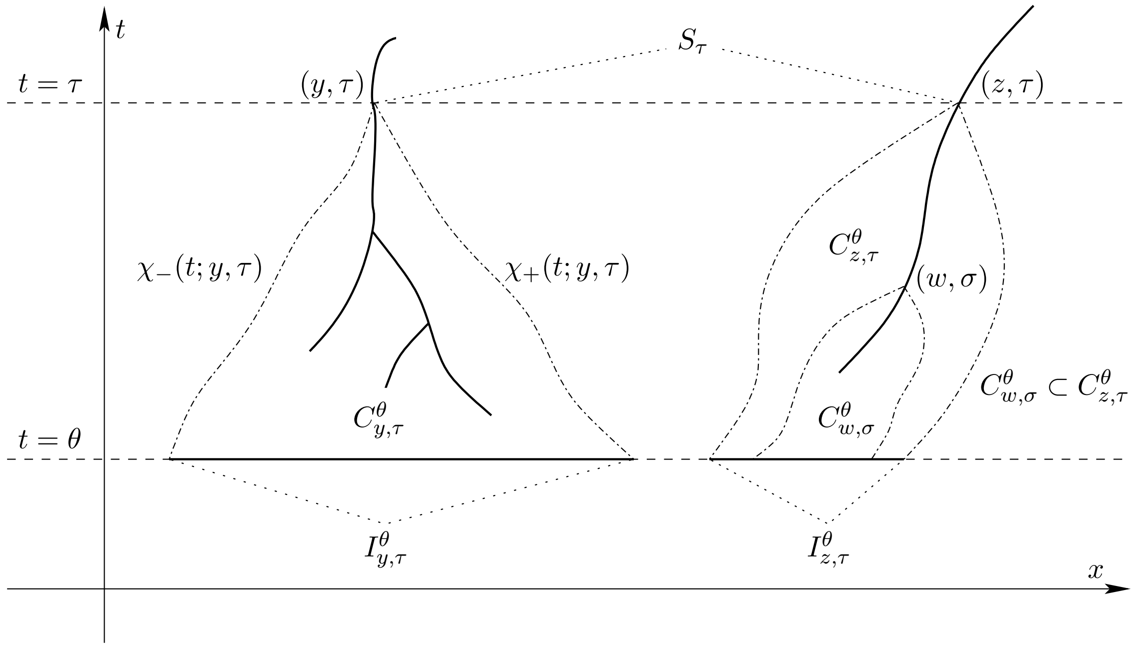

Figure 2: Characteristic cones and bases.

Lemma 5.10. *Let $\tau > 0$. If $y \in S_\tau$, then for all $\theta \in [0, \tau]$ there exists a positive constant c_j such that*

$$\mathcal{L}^1(I^\theta_{y,\tau}) = \chi_+(\theta; y, \tau) - \chi_-(\theta; y, \tau) \leq -c_j \nu_\tau(\{y\}). \tag{76}$$

Lemma 5.11. *Let $\tau_0 > 0$. Then for μ_{τ_0}-a.e. x there exists $\eta := \eta(x, \tau_0, \tau) > 0$ such that*

$$]x - \eta, x + \eta[\subset I^{\tau_0}_\tau \qquad \textit{for } \tau > \tau_0.$$

Definition 5.12. *We denote by E the set of x's for which Lemma 5.11 applies and such that*

$$\lim_{\eta \downarrow 0} \frac{|Du(., \tau_0) - \mu_{\tau_0}|([x - \eta, x + \eta])}{-\mu_{\tau_0}([x - \eta, x + \eta])} = 0. \tag{77}$$

The Besicovitch differentiation theorem gives that $\mu_{\tau_0}(\mathbb{R} \backslash E) = 0$ and with (75) we have for every $x \in E$ that

$$\lim_{\eta \downarrow 0} \frac{u((x - \eta)-, \tau_0) - u((x + \eta)+, \tau_0)}{-\mu_{\tau_0}([x - \eta, x + \eta])} = 1. \tag{78}$$

Lemma 5.13. *Let $\tau_0 > 0$. For every $x \in E$, for every (sufficiently small) $\eta > 0$ such that $x \pm \eta \notin S_{\tau_0}$ and for all $\theta \in (0, \tau_0]$, there exists a positive constant $c_c(\theta)$ such that*

$$\mathcal{L}^1(J^\theta_{x,\eta}) = \chi_+(\theta; x + \eta, \tau_0) - \chi_-(\theta; x - \eta, \tau_0) \geq -c_c(\theta) \mu_{\tau_0}([x - \eta, x + \eta])$$

where $J^\theta_{x,\eta} :=]\chi_-(\theta; x - \eta, \tau_0), \chi_+(\theta; x + \eta, \tau_0)[$.

Note that the proofs of these lemmas are listed in Chapter 5.6 and that to get these results we will use the estimate of Theorem 3.8.

5.4 Proof of Theorem 5.2

Step 1: Preliminary remarks.
Let us fix $(\xi, \tau) \in \Omega$ and a radius r such that the ball $B_r(\xi, \tau) \subset \Omega$. Thanks to the finite speed of propagation, there exists a positive ρ such that the values of $u(x,t)$ in the ball $B_\rho(\xi,\tau)$ depend only on the values of u on the segment $\{t = \tau - 2\rho\} \cap B_r(\xi, \tau)$. If we write w for the entropy solution of the problem

$$\begin{cases} D_t w(x,t) + D_x[f(w(x,t),x,t)] + g(w(x,t),x,t) = 0, & \text{for } t > \tau - 2\rho; \\ w(x, \tau - 2\rho) = u(x, \tau - 2\rho)\mathbf{1}_{B_r(\xi,\tau)}(x, \tau - 2\rho), & \text{for every } x \in \mathbb{R}. \end{cases}$$

we get that $w = u$ on $B_\rho(\xi, \tau)$. We also note that $w(.,t) \in BV(\mathbb{R})$ for every $t > \tau - 2\rho$. Thus it suffices to prove the theorem under the additional assumptions that $\Omega = \{t > 0\}$ and that $u(.,0)$ is a bounded compactly supported BV function. Since the function $u(.,0)$ has compact support, we know that there exist constants R and c_R such that the support of $u(.,t)$ is contained in $\{|x| \leq R\}$ and

$$|Du(.,t)|(\mathbb{R}) \leq c_R.$$

Step 2: Definition of the geometric functional $F^\theta(t)$.
For a fixed $\theta > 0$ we define the functional $F^\theta : [\theta, \infty[\to \mathbb{R}^+$ as follows

$$F^\theta(t) := \mathcal{L}^1(I_t^\theta) = \sum_{y \in S_t} \mathcal{L}^1(I_{y,t}^\theta)$$

where the second equality holds by the No-crossing property of the characteristics. Geometrically the functional $F^\theta(t)$ measures the total length of the bases at time θ of all characteristic cones contained in C_t^θ. With Lemma 5.10 we observe that this functional is bounded from above:

$$F^\theta(t) \leq -c_j \nu_t(\mathbb{R}) \leq c_j |Du(.,t)|(\mathbb{R}) \leq c_j c_R \qquad \text{for every } t \in [\theta, T].$$

Moreover, this geometric functional is nondecreasing. This is a consequence of the fact that for every $\theta \leq t_1 \leq t_2$ we have that $I_{t_1}^\theta \subset I_{t_2}^\theta$. Assuming that there exists $x \in S_{t_1}$ such that $x \in (I_{t_1}^{t_2})^c$, then the shock $s(t)$ passing through (x, t_1) would be not entirely contained in the characteristic cones $C_{t_2}^{t_1}$. But by definition of characteristic cones and the No-crossing property, every shock in $\mathbb{R} \times [t_1, t_2]$ is entirely contained in $C_{t_2}^{t_1}$. Therefore, $S_{t_1} \subset I_{t_2}^{t_1}$ and again due to the No-crossing property we conclude that $I_{t_1}^\theta \subset I_{t_2}^\theta$. We state

Lemma 5.14. *For fixed $T \geq \theta > 0$ the functional $F^\theta : [\theta, T] \to \mathbb{R}^+$ is nondecreasing and bounded from above.*

One of the key remarks of the proof given in [3] was the connection between the jumps of the geometric functional and the Cantor parts created in the solution. We can try to describe this geometric process as follows: when a Cantor part is created in the solution at time τ_0, this part is transformed after an infinitesimal time into several small jumps. The sum of the measures of these jumps give a contribution big enough to make the geometrical functional jump at τ_0. Due to the similar construction of our functional $F^\theta(.)$ and the original functional defined in [3], we can utilize the same idea here. This motivates a first reduction of our problem to the following Lemma:

Lemma 5.15. *For any integer k we have*

$$\tau_0 \geq \frac{T}{k} > \theta \text{ and } \mu_{\tau_0}(\mathbb{R}) \leq -\frac{1}{k} \qquad \Rightarrow \qquad F^\theta(\tau_0+) \geq F^\theta(\tau_0) + c_F(\theta)$$

where $c_F(\theta)$ *is a strictly positive constant which depends on* $\|u\|_\infty, T, k, f, g$ *and on the choice of* $\theta > 0$.

Clearly, Lemma 5.14 and Lemma 5.15 imply that all sets

$$\left\{\tau \in \left[\frac{T}{k}, T\right[: \mu_\tau(\mathbb{R}) \le -\frac{1}{k}\right\}$$

are finite.

Step 3: Proof of Lemma 5.15.

In this step we shall make use of the three technical Lemmas 5.10, 5.11 and 5.13. We fix $\tau > \tau_0 \ge T/k \ge \theta$. Let $x \in E$, where E is the set defined by Lemma 5.10. Consequently to Lemma 5.13, for $\eta > 0$ small enough we have that

$$\mathcal{L}^1(J^\theta_{x,\eta}) \ge -c_c(\theta)\mu_{\tau_0}([x-\eta, x+\eta]) \tag{79}$$

where $J^\theta_{x,\eta} :=]\chi_-(\theta; x-\eta, \tau_0), \chi_+(\theta; x+\eta, \tau_0)[$. By the No-crossing property of characteristics we know that $J^\theta_{x,\eta}$ can only intersect the bases of the cones I^θ_{y,τ_0} emanating from a point $y \in [x-\eta, x+\eta]$, so that recalling Lemma 5.10 it follows that

$$\mathcal{L}^1(J^\theta_{x,\eta} \cap I^\theta_{\tau_0}) = \sum_{y \in S_{\tau_0} \cap [x-\eta, x+\eta]} \mathcal{L}^1(I^\theta_{y,\tau_0}) \le -c_j \nu_{\tau_0}([x-\eta, x+\eta]). \tag{80}$$

Combining (79) and (80) we find that for any $x \in E$ we have that

$$\begin{aligned}\mathcal{L}^1(J^\theta_{x,\eta} \backslash I^\theta_{\tau_0}) &\ge -c_c(\theta)\mu_{\tau_0}([x-\eta, x+\eta]) + c_j \nu_{\tau_0}([x-\eta, x+\eta]) \ge \\ &\ge -c_c(\theta)\mu_{\tau_0}([x-\eta, x+\eta]) - c_j|\nu_{\tau_0}|([x-\eta, x+\eta]).\end{aligned} \tag{81}$$

Finally, invoking the Besicovitch differentiation theorem and in particular (77), we obtain

$$\mathcal{L}^1(J^\theta_{x,\eta} \backslash I^\theta_{\tau_0}) \ge -\frac{c_c(\theta)}{2}\mu_{\tau_0}([x-\eta, x+\eta]), \tag{82}$$

provided that η is small enough. Using the Besicovitch covering lemma, we can cover μ_{τ_0}-a.e. E with pairwise disjoint intervals $K^\theta_{j,\tau_0} := [x_j - \eta_j, x_j + \eta_j]$ such that (82) and the conclusion of Lemma 5.11 both hold for $x = x_j$ and $\eta = \eta_j$. Thanks to the No-crossing property all the intervals $J^\theta_{x_j,\eta_j}$ are pairwise disjoint and recalling Lemma 5.11 we note that all these sets are contained in I^θ_τ. In Lemma 5.15 we assumed that $-\mu_{\tau_0}(\mathbb{R}) \ge 1/k$. Then for all $\theta > 0$ the estimates above imply:

$$\begin{aligned}F^\theta(\tau) - F^\theta(\tau_0) &\ge \sum_j \mathcal{L}^1(J^\theta_{x_j,\eta_j} \backslash I^\theta_{\tau_0}) \ge -\sum_j \frac{c_c(\theta)}{2}\mu_{\tau_0}([x_j-\eta_j, x_j+\eta_j]) \ge \\ &\ge -\frac{c_c(\theta)}{2}\mu_{\tau_0}(E) = -\frac{c_c(\theta)}{2}\mu_{\tau_0}(\mathbb{R}) \ge \frac{c_c(\theta)}{2k} =: c_F(\theta).\end{aligned} \tag{83}$$

Step 4: The end of the proof.

In Step 3 we have shown that for any fixed $\theta > 0$ the interval $(\theta, T]$ contains a set H_θ, which is at most countable and such that for every $\tau \in (\theta, T] \backslash H_\theta$ the following holds:

$$u(.,\tau) \in SBV_{loc}(\Omega_{\tau,\theta}) \qquad \text{with } \Omega_{\tau,\theta} := \{x \in \mathbb{R} : (x,\tau) \in \Omega, \tau \in (\theta, T]\}.$$

Now, we consider the sequence $(\theta_n)_{n\in\mathbb{N}} := T/2^n$ and we define the set S of the main Theorem 5.2 as the countable union of the countable sets $H_{\theta_n} \subset (\theta_n, T]$, which is again countable:

$$S := H_0 = \bigcup_{n\in\mathbb{N}} H_{\frac{T}{2^n}}.$$

This concludes the proof.

5.5 Theorem 5.3 and Corollary 5.4

The proof of Theorem 5.3 combines the ideas of Theorem 5.1 and Theorem 5.2. We repeat it for the reader's convenience.

Proof. **Step 1:** [Strictly concave flux.] As in the proof of Theorem 5.1 let us first show that there exists a locally SBV entropy solution for the problem with a strictly concave flux. By Theorem 5.2 we know that if $f(u(x,t),x,t)$ is strictly convex in u and C^2, the equation

$$D_t u(x,t) + D_x[f(u(x,t),x,t)] + g(u(x,t),x,t) = 0$$

has a locally SBV entropy solution $u(x,t)$. Since $f_1(u(x,t),x,t) := f(u(x,t),-x,t)$ is again C^2 and strictly convex in u and $g_1(u(x,t),x,t) := g(u(x,t),-x,t)$ is C^1, the solution $u_1(x,t)$ of

$$D_t u_1(x,t) + D_x[f_1(u_1(x,t),x,t)] + g_1(u_1(x,t),x,t) = 0 \tag{84}$$

is again a SBV_{loc} function. Next we consider the coordinates transformation $\phi : \Omega \to \widetilde{\Omega}, \phi : (x,t) \mapsto (y,t) = (-x,t)$ and we put $\widetilde{f} := -f, \widetilde{g} := g$. Then we have for $\widetilde{u}(y,t) := u_1 \circ \phi^{-1}(y,t) = u_1(-x,t)$:

$$\begin{aligned}
D_t\widetilde{u}(y,t) &+ D_y[\widetilde{f}(\widetilde{u}(y,t),y,t)] + \widetilde{g}(\widetilde{u}(y,t),y,t) = \\
&= D_t\widetilde{u}(y,t) - D_y[f(\widetilde{u}(y,t),y,t)] + g(\widetilde{u}(y,t),y,t) = \\
&= D_t\widetilde{u}(-x,t) - (-D_x)[f(\widetilde{u}(-x,t),-x,t)] + g(\widetilde{u}(-x,t),-x,t) = \\
&= D_t u_1(x,t) + D_x[f(u_1(x,t),-x,t)] + g(u_1(x,t),-x,t) = \\
&= D_t u_1(x,t) + D_x[f_1(u_1(x,t),x,t)] + g_1(u_1(x,t),x,t) = 0.
\end{aligned} \tag{85}$$

Thus $\widetilde{u}$ is a solution of a balance law with strictly concave flux function and since u_1 is locally SBV this implies that also $\widetilde{u} \in SBV_{loc}$. Moreover, u_1 is an entropy solution of (84) and since we have found the solution $\widetilde{u}$ reflecting u_1 about the t-axis, as in the proof of Lemma 5.5, we can show that even for $\widetilde{u}$ the entropy and the Rankine-Hugoniot conditions hold.

Step 2: [$u(x,t) \in SBV_{loc}(\Omega_\tau)$.] At this point we can repeat the construction given in the proof of Theorem 5.1. Due to the small variation of an entropy solution we can restrict the solution $u(x,t)$ on triangles T_j's, where the flux function f is either strictly convex or strictly concave. Recalling that C is the set of points (x_i,t_i) where the solution has jumps or has a values such that $f_{uu}(u(x_i,t_i),x_i,t_i) = 0$, we cover $\Omega \backslash C$ with a countable family of triangles T_j. For the bad points not contained in one triangle or points of C, we can restate Claim 5.7 and 5.6. We conclude that the entropy solution $u(x,t)$ is locally SBV on Ω_τ for every $\tau \in \mathbb{R}\backslash S$.

Step 3: [$u \in SBV_{loc}(\Omega)$.] The slicing theory says that the 2-dimensional Cantor part of the derivative $D_x u(x,t)$ can be recovered from the corresponding 1-dimensional part. By Theorem 3.108 of [2] we have:

$$D_x^c u(x,t) = \int \mathcal{L}^1 \llcorner \Omega_t \otimes D^c u(.,t) dt$$

where Ω_t is the projection of Ω on $\{x = 0\} \times \mathbb{R}^+$. Since by Step 2 the Cantor part is $D^c u(.,t) = 0$ for every $t \notin S$ and S is at most countable, then also the two dimensional Cantor part $D_x^c u(x,t)$ vanishes. With the Vol'perts chain rule (see Theorem 3.96 of [2]) and equation (70) we get that $D_t^c u(x,t) = 0$. Finally we have obtained that $u \in SBV_{loc}(\Omega)$. □

Next, we use Theorem 5.2 to prove corollary 5.4.

Proof. We differentiate the equation (71) by x (in the sense of distributions):

$$D_x w_t(x,t) + D_x H(w_x(x,t),x,t) = 0 \qquad \Leftrightarrow \qquad D_t w_x(x,t) + D_x H(w_x(x,t),x,t) = 0.$$

Letting $u(x,t) = w_x(x,t)$ and $H(u(x,t),x,t) = f(u(x,t),x,t)$, this is exactly the balance equation of Theorem 5.2, i.e. $u(x,t) = w_x(x,t)$ is $SBV_{loc}(\Omega_\tau)$ for every $\tau \in \mathbb{R}\backslash S$, where S is at most countable. As in Step 3 of the proof of Theorem 5.3, it follows that $u(x,t) \in SBV_{loc}(\Omega)$. □

5.6 Proofs of the three technical lemmas

5.6.1 Proof of Lemma 5.10

Let $\tau > 0$ and $y \in S_\tau$. To simplify the notation, we denote the minimal and the maximal backward characteristics starting from (y,τ) by $\chi_-(t)$ and $\chi_+(t)$ instead of $\chi_-(t;y,\tau)$ and $\chi_+(t;y,\tau)$. Rewriting the ODEs (39) related to the genuine characteristics for $\chi_-(t), \chi_+(t) : [0,\tau] \to \mathbb{R}$ we obtain

$$\begin{cases} \dot{\chi}_\pm(t) = f_u(v_\pm(t), \chi_\pm(t), t) \\ \dot{v}_\pm(t) = -f_x(v_\pm(t), \chi_\pm(t), t) - g(v_\pm(t), \chi_\pm(t), t) \end{cases} \tag{86}$$

with

$$\begin{cases} \chi_-(\tau) = y = \chi_+(\tau) \\ v_-(\tau) = u(y-,\tau) =: u^- \\ v_+(\tau) = u(y+,\tau) =: u^+ \end{cases} \tag{87}$$

where by the admissible condition (35), $u^- > u^+$ and $-\nu_\tau(\{y\}) = u^- - u^+$. Now we change the variable in equations (86) putting $s = \tau - t$:

$$\begin{cases} \dot{\psi}_\pm(s) = -f_u(\omega_\pm(s), \psi_\pm(s), \tau - s) \\ \dot{\omega}_\pm(s) = f_x(\omega_\pm(s), \psi_\pm(s), \tau - s) + g(\omega_\pm(s), \psi_\pm(s), \tau - s) \end{cases} \tag{88}$$

where $\psi_\pm(s) := \chi_\pm(\tau - s)$ and $\omega_\pm(s) := v_\pm(\tau - s)$. Then, using (88) we have

$$\begin{aligned} \frac{d}{ds}|\psi_-(s) - \psi_+(s)| &\le |\dot{\psi}_-(s) - \dot{\psi}_+(s)| = |f_u(\omega_-(s), \psi_-(s), \tau - s) - f_u(\omega_+(s), \psi_+(s), \tau - s)| \\ &\le \underbrace{\|D^2 f\|_{L^\infty(K)}}_{:=c_1}(|\omega_-(s) - \omega_+(s)| + |\psi_-(s) - \psi_+(s)|). \end{aligned} \tag{89}$$

Repeating the computations also for $\omega_\pm(s)$, we find that:

$$\begin{aligned} \frac{d}{ds}|\omega_-(s) - \omega_+(s)| \le & |f_x(\omega_-(s), \psi_-(s), \tau - s) - f_x(\omega_+(s), \psi_+(s), \tau - s)| + \\ & + |g(\omega_-(s), \psi_-(s), \tau - s) - g(\omega_+(s), \psi_+(s), \tau - s)| \le \\ & \le \underbrace{\left[\|D^2 f\|_{L^\infty(K)} + \|Dg\|_{L^\infty(K)}\right]}_{:=c_2}(|\omega_-(s) - \omega_+(s)| + |\psi_-(s) - \psi_+(s)|) \end{aligned} \tag{90}$$

If we choose a compact set

$$K := [-\|u\|_\infty, \|u\|_\infty] \times \left[\min_{t\in[0,\tau]} \chi_-(t), \max_{t\in[0,\tau]} \chi_+(t)\right] \times [0,\tau] \tag{91}$$

then the above constants c_1 and c_2 are positive and finite. Inequalities (89) and (90) together give

$$\frac{d}{ds}\Big(|\psi_-(s)-\psi_+(s)|+|\omega_-(s)-\omega_+(s)|\Big) \le c_3\Big(|\psi_-(s)-\psi_+(s)|+|\omega_-(s)-\omega_+(s)|\Big) \tag{92}$$

where c_3 is again a positive constant depending only on $\|D^2 f\|_{L^\infty(K)}$ and $\|Dg\|_{L^\infty(K)}$. By Gronwall's lemma and (92) we get

$$|\psi_-(s)-\psi_+(s)|+|\omega_-(s)-\omega_+(s)| \le e^{c_3 s}(|\psi_-(0)-\psi_+(0)|+|\omega_-(0)-\omega_+(0)|). \tag{93}$$

Now, we put $s=\vartheta\in[0,\tau]$ into (93) to find the following inequality:

$$\begin{aligned}|\chi_-(\tau-\vartheta)-\chi_+(\tau-\vartheta)| &= |\psi_-(\vartheta)-\psi_+(\vartheta)| \le |\psi_-(\vartheta)-\psi_+(\vartheta)|+|\omega_-(\vartheta)-\omega_+(\vartheta)| \le \\ &\le \underbrace{e^{c_3\tau}}_{:=c_\tau}(|\psi_-(0)-\psi_+(0)|+|\omega_-(0)-\omega_+(0)|) = \\ &= c_\tau(|\chi_-(\tau)-\chi_+(\tau)|+|v_-(\tau)-v_+(\tau)|).\end{aligned} \tag{94}$$

If we set $\theta=\tau-\vartheta$, then by (87) we conclude:

$$\begin{aligned}\chi_+(\theta)-\chi_-(\theta) = |\chi_-(\theta)-\chi_+(\theta)| &\le c_\tau(\underbrace{|\chi_-(\tau)-\chi_+(\tau)|}_{=0}+\underbrace{|v_-(\tau)-v_+(\tau)|}_{=|u^- -u^+|}) \\ &\le c_\tau(u^- -u^+) = -c_\tau\nu_\tau(\{y\}).\end{aligned} \tag{95}$$

5.6.2 Proof of Lemma 5.11

We prove that the conclusion of the lemma holds for any x which satisfy the following conditions:

$$x\notin S_{\tau_0} \qquad \text{and} \qquad \lim_{\eta\downarrow 0}\frac{u(x+\eta,\tau_0)-u(x-\eta,\tau_0)}{2\eta} = -\infty. \tag{96}$$

By the Besicovitch differentiation theorem on intervals, the measure μ_{τ_0} is concentrated on E. In our proof we fix $\tau>\tau_0$ and x such that (96) holds and our goal is to show that for η small enough $\{\tau_0\}\times]x-\eta,x+\eta[\subset I_\tau^{\tau_0}$. To prove that the point x is contained in $I_\tau^{\tau_0}$, we consider all the possible cases:

$$\begin{aligned} I &: x\notin \overline{I_\tau^{\tau_0}} \\ II &: x\in\partial(\overline{I_\tau^{\tau_0}}) \\ III &: \exists\eta>0 \text{ s.t. } (]x-\eta,x+\eta[\cap I_\tau^{\tau_0})^c=\{x\} \\ IV &: x\in I_\tau^{\tau_0}\end{aligned}$$

and in particular we will show that in the first three cases we obtain a contradiction. Then, by IV the point x is in $I_\tau^{\tau_0}$ and since $I_\tau^{\tau_0}$ is open there exists $\eta>0$ such that $\{\tau_0\}\times]x-\eta,x+\eta[\subset I_\tau^{\tau_0}$. The property (96) may be rewritten prescribing that for every positive $\bar\alpha>0$, there exists $\bar\eta>0$ such that for all $0<\eta<\bar\eta$ holds

$$u(x-\eta,\tau_0)-u(x+\eta,\tau_0)>\bar\alpha 2\eta>0. \tag{97}$$

The positive number $\bar\alpha$ will be chosen later, more precisely we will define this constant at the end of case I. Observe also that, if $\{\tau_0\}\times]x-\eta,x+\eta[\subset I_\tau^{\tau_0}$, then $\{\tau_0\}\times]x-\eta,x+\eta[\subset I_t^{\tau_0}$ for every $t>\tau$. Therefore, without loss of generality we can assume

$$\tau-\tau_0 \,<\, 1\,. \tag{98}$$

Case I: Since in this case the distance $dist(x, \overline{I_\tau^{\tau_0}})$ is strictly positive there exists $\eta > 0$ small enough such that $0 < 2\eta < \bar{\eta}$ and $]x-\eta, x+\eta[\cap \overline{I_\tau^{\tau_0}} = \emptyset$. Consequently to the definition of a characteristic cone, we observe that all possible shocks created before τ are contained in $C_\tau^{\tau_0}$ and thus, two different shocks starting at $x_1, x_2 \notin \overline{I_\tau^{\tau_0}}$, where $x_1 \neq x_2$, cannot cross for all $t \in [\tau_0, \tau]$. By the way, also the two characteristics passing trough $(x-\eta, \tau_0)$ and $(x+\eta, \tau_0)$, which are genuine, cannot cross in $[\tau_0, \tau]$. If we denote by $\chi_1(t)$ and $\chi_2(t)$, respectively, the two characteristics $\chi(t; x-\eta, \tau_0)$ and $\chi(t; x+\eta, \tau_0)$, this implies $\chi_1(t) < \chi_2(t)$ for every $t \in [\tau_0, \tau]$. This conclusion is contradicted by the next claim.

Claim 5.16. *If x is a point such that $\exists \eta > 0$ with $]x-\eta, x+\eta[\cap \overline{I_\tau^{\tau_0}} = \emptyset$ and (97) holds then*

$$\chi_1(\tau) - \chi_2(\tau) > 0. \tag{99}$$

Proof. Integrating the ODEs of the characteristics (39) for every $t \in [\tau_0, \tau]$ we obtain:

$$\int_{\tau_0}^t \dot{\chi}(s) ds = \int_{\tau_0}^t f_u(v(s), \chi(s), s) ds$$
$$\chi(t) = \chi(\tau_0) + \int_{\tau_0}^t f_u(v(s), \chi(s), s) ds \tag{100}$$

and

$$\int_{\tau_0}^t \dot{v}(s) ds = -\int_{\tau_0}^t f_x(v(s), \chi(s), s) + g(v(s), \chi(s), s) ds$$
$$v(t) = v(\tau_0) - \int_{\tau_0}^t f_x(v(s), \chi(s), s) + g(v(s), \chi(s), s) ds. \tag{101}$$

Thus, we conclude that the trajectories χ_1 and χ_2 never leave a compact set which can be determined indepedently of η. We set therefore

$$K := [-\|u\|_\infty, \|u\|_\infty] \times \left[\min_{t \in [\tau_0, \tau]} \{\chi_1(t), \chi_2(t)\}, \max_{t \in [\tau_0, \tau]} \{\chi_1(t), \chi_2(t)\} \right] \times [\tau_0, \tau]. \tag{102}$$

Substituting χ_1 and χ_2 to χ in (100) and subtracting the corresponding equations, by the mean value theorem there exists ξ such that:

$$\chi_1(t) - \chi_2(t) = \chi_1(\tau_0) - \chi_2(\tau_0) + \int_{\tau_0}^t f_u(v_1(s), \chi_1(s), s) - f_u(v_2(s), \chi_2(s), s) ds =$$
$$= -2\eta + \int_{\tau_0}^t f_{uu}(\xi)(v_1(s) - v_2(s)) + f_{ux}(\xi)(\chi_1(s) - \chi_2(s)) ds \tag{103}$$

holds.
Note next that, by (97), we have $v_1(\tau_0) = u(x-\eta, \tau_0) > u(x+\eta, \tau_0) = v_2(\tau)$. We therefore introduce the time:

$$\overline{\tau} := \max\{t \in [\tau_0, \tau] : \; v_1(\sigma) \geq v_2(\sigma), \, \forall \sigma \in [\tau_0, t]\},$$

and we remark that, by this definition, we can only have

$$\text{either } v_1(\overline{\tau}) = v_2(\overline{\tau}) \text{ or } \overline{\tau} = \tau. \tag{104}$$

By equation (103) and the no crossing between characteristics, for every $t \in [\tau_0, \overline{\tau}]$ we write the following inequality:

$$0 \leq \chi_2(t) - \chi_1(t) = \chi_2(\tau_0) - \chi_1(\tau_0) + \int_{\tau_0}^{t} \underbrace{f_{uu}(\xi)}_{\geq \gamma > 0} \underbrace{(v_2(s) - v_1(s))}_{\leq 0 \text{ by Def. of } \overline{\tau}} + f_{ux}(\xi)(\chi_2(s) - \chi_1(s)) ds$$

$$\leq \chi_2(\tau_0) - \chi_1(\tau_0) + \|f_{ux}\|_\infty \int_{\tau_0}^{t} |\chi_2(s) - \chi_1(s)| ds. \tag{105}$$

Applying Gronwall's Lemma we get:

$$|\chi_2(t) - \chi_1(t)| \leq e^{\|f_{ux}\|_\infty (t-\tau_0)} |\chi_2(\tau_0) - \chi_1(\tau_0)| \leq C|\chi_2(\tau_0) - \chi_1(\tau_0)| \leq C\eta. \tag{106}$$

where C is a positive constant big enough to satisfy the above estimate. Next, for $t \in [\tau_0, \overline{\tau}]$ we recall the second ODE of (39) and thus for the function $v_1(t) - v_2(t)$ we have:

$$\dot{v}_1(t) - \dot{v}_2(t) = f_x(v_2(t), \chi_2(t), t) - f_x(v_1(t), \chi_1(t), t) + g(v_2(t), \chi_2(t), t) - g(v_1(t), \chi_1(t), t). \tag{107}$$

Using the Lipschitz regularity of f_x and g, the fact that $v_1(t) \geq v_2(t)$ for $t \in [\tau_0, \overline{\tau}]$ and the estimate (106), we conclude

$$\dot{v}_1(t) - \dot{v}_2(t) \geq -\tilde{C}(v_1(t) - v_2(t)) - \tilde{C}\eta \qquad \text{for } t \in [\tau_0, \overline{\tau}]. \tag{108}$$

We note that the constant $\tilde{C}$ is positive and it depends only on the derivatives of f, g, which are bounded on the compact set K, and τ, τ_0. Now, if we define the strict positive function $\alpha(t) := v_1(t) - v_2(t) + \eta$ this last estimate can be rewritten as:

$$\dot{\alpha}(t) \geq -\tilde{C}\alpha(t), \tag{109}$$

and again by Gronwall's Lemma we obtain:

$$\alpha(t) \geq \alpha(\tau_0) e^{-\tilde{C}(t-\tau_0)} \qquad , \forall t \in [\tau_0, \overline{\tau}]. \tag{110}$$

Since $-1 < e^{-\tilde{C}(t-\tau_0)} - 1 \leq 0$, for a suitable positive constant $\overline{C}$ it follows that:

$$v_1(t) - v_2(t) \geq [v_1(\tau_0) - v_2(\tau_0)] e^{-\tilde{C}(t-\tau_0)} + (e^{-\tilde{C}(t-\tau_0)} - 1)\eta$$

$$\geq [v_1(\tau_0) - v_2(\tau_0)] e^{-\tilde{C}(t-\tau_0)} - \overline{C}\eta. \tag{111}$$

Combining property (97):

$$v_1(\tau_0) - v_2(\tau_0) = u(x - \eta, \tau_0) - u(x + \eta, \tau_0) > \alpha 2\eta > \bar{\alpha} 2\eta > 0, \tag{112}$$

with inequality (111) we obtain

$$v_1(t) - v_2(t) \geq [\bar{\alpha} 2 e^{-\tilde{C}(t-\tau_0)} - \overline{C}]\eta. \tag{113}$$

Next, since the constants $\tilde{C}$ and $\overline{C}$ in (111) are independent from the positive number $\overline{\alpha}$ and since $\overline{\alpha}$ is a big constant that can be chosen, looking at estimate (113) we may assume that:

$$v_1(t) - v_2(t) \geq \frac{1}{2}\eta > 0 \qquad , \forall t \in [\tau_0, \overline{\tau}]. \tag{114}$$

By our remark (104), this last estimate implies that $\overline{\tau} \geq \tau$ and thus that inequalities (106) and (111) are valid for all $t \in [\tau_0, \tau]$. At this point we can set $t = \tau$ in equation (103) and compute:

$$\begin{aligned}
\chi_1(\tau) - \chi_2(\tau) &= -2\eta + \int_{\tau_0}^{\tau} \underbrace{f_{uu}(\xi)}_{\geq \gamma > 0} \underbrace{(v_1(s) - v_2(s))}_{\text{use (113)}} ds + \int_{\tau_0}^{\tau} \underbrace{f_{ux}(\xi)}_{\leq \|f_{ux}\|_{L^\infty(K)}} \underbrace{(\chi_1(s) - \chi_2(s))}_{\text{use (106)}} ds \\
&\geq -2\eta + \int_{\tau_0}^{\tau} \gamma \, [\bar{\alpha} 2 e^{-\tilde{C}(s-\tau_0)} - \overline{C}] \eta ds - \int_{\tau_0}^{\tau} C \eta ds \\
&\geq -2\eta + \gamma \bar{\alpha} 2 \eta (\tau - \tau_0) e^{-\tilde{C}(\tau - \tau_0)} - 2(\tau - \tau_0)\eta \Big[C + \gamma \overline{C} \Big] \\
&= 2\eta(\tau - \tau_0) \Big\{ \gamma \bar{\alpha} e^{-\tilde{C}(\tau - \tau_0)} - \frac{1}{\tau - \tau_0} - \Big[C + \gamma \overline{C} \Big] \Big\}. \qquad (115)
\end{aligned}$$

Recall that C, $\tilde{C}$ and $\overline{C}$ are indepedent of η and α and that $\tau - \tau_0 < 1$. Thus, for $\bar{\alpha}$ big enough the right hand side of (115) is positive. □

Case II: If x belongs to $\partial(\overline{I^{\tau}_{\tau_0}})$,then one of the two characteristics $\chi(t; x-\eta, \tau_0)$ or $\chi(t; x+\eta, \tau_0)$ is not contained in $C^{\tau}_{\tau_0}$. Moreover, the characteristic $\chi(t; x, \tau_0)$ is a boundary curve of the characteristic cone $C^{\tau}_{\tau_0}$ and so it is either a minimal or a maximal backward characteristic. Repeating similar computations as in case I it is possible to show that if (96) holds, either $\chi(t; x - \eta, \tau_0)$ or $\chi(t; x + \eta, \tau_0)$ will cross with $\chi(t; x, \tau_0)$ for a $t \in]\tau_0, \tau[$. Recalling again the No-crossing property of genuine characteristic, we get a contradiction.
Case III: By Theorem 3.7 for every $\tau_0 > 0$ trough (x, τ_0) passes a unique forward characteristic. Consequently case III is to discard.
Case IV: In view of the contradictions obtained in the previous cases, the last possible case must be true. In particular $x \in I^{\tau}_{\tau_0}$ and since $I^{\tau}_{\tau_0}$ is open there exists a $\eta > 0$ small enough such that

$$]x - \eta, x + \eta[\subset I^{\tau}_{\tau_0}$$

for $\tau > \tau_0$.

Remark 5.17. *We remark that in [3] Lemma 5.11 was proved using the Hopf-Lax formula. Here we have proposed a more geometrical construction, which make use of the properties of generalized characteristics. This change of strategy is also motivated by the fact that for system of conservation laws the Hopf-Lax does not exists, whereas there is a suitable concept of generalized characteristics (see [21]).*

5.6.3 Proof of Lemma 5.13

Let $\tau_0 > 0$, $x \in E$ and $\eta > 0$ such that $x \pm \eta \notin S_{\tau_0}$. To simplify the notation we write $\chi_-(t)$ and $\chi_+(t)$ instead of $\chi_-(t; x - \eta, \tau_0)$ and $\chi_+(t; x + \eta, \tau_0)$. Our aim is to show that for all $\theta \in]0, \tau_0]$ there exists a positive constant $c_c(\theta)$ such that

$$\chi_+(\theta) - \chi_-(\theta) \geq -c_c(\theta) \mu_{\tau_0}([x - \eta, x + \eta]) \qquad (116)$$

holds. To get our result we will use the same techniques and derive analogous estimates as in the previous proofs. First of all, note that, since $x \in E$ for a small $\eta > 0$ we can assume that:

$$u_- := u(x - \eta, \tau_0) > u(x + \eta, \tau_0) := u_+. \qquad (117)$$

Moreover, by Besicovitch differentiation theorem (78) we have

$$-c\mu_{\tau_0}([x-\eta, x+\eta]) \leq u(x-\eta, \tau_0) - u(x+\eta, \tau_0) = u_- - u_+. \tag{118}$$

It is clear that looking at (116) and inequality (118) we can reduce our Claim to:

Claim 5.18. *There exists a positive constant $C(\theta)$ such that:*

$$\chi_+(\theta) - \chi_-(\theta) \geq C(\theta)(u_- - u_+). \tag{119}$$

We fix $\sigma \in [\theta, \tau_0]$ and we recall the equations of the characteristics (86):

$$\dot{\chi}_\pm(\sigma) = f_u(v_\pm(\sigma), \chi_\pm(\sigma), \sigma) \tag{120}$$

where $v_\pm(\sigma) := u(\chi_\pm(\sigma), \sigma)$. Next, we take the difference between the ODEs of the characteristics and we write:

$$\dot{\chi}_+(\sigma) - \dot{\chi}_-(\sigma) = f_u(v_+(\sigma), \chi_+(\sigma), \sigma) - f_u(v_-(\sigma), \chi_-(\sigma), \sigma) = U(\sigma) + W(\sigma), \tag{121}$$

where we set

$$U(\sigma) := f_u(v_+(\sigma), \chi_+(\sigma), \sigma) - f_u(v_-(\sigma), \chi_+(\sigma), \sigma) \tag{122}$$

and

$$W(\sigma) := f_u(v_-(\sigma), \chi_+(\sigma), \sigma) - f_u(v_-(\sigma), \chi_-(\sigma), \sigma). \tag{123}$$

To estimate $U(\sigma)$ we distinguish two cases:

If $v_+(\sigma) \geq v_-(\sigma)$: Using the one-sided inequality (40) we obtain:

$$U(\sigma) \leq \|f_{uu}\|_{L^\infty}(v_+(\sigma) - v_1(\sigma)) \leq \widetilde{C}(\chi_+(\sigma) - \chi_-(\sigma)).$$

If $v_+(\sigma) \leq v_-(\sigma)$: Recalling that $f_{uu}(.) \geq \gamma > 0$ it holds:

$$U(\sigma) \leq \gamma(v_+(\sigma) - v_1(\sigma)) < 0.$$

Moreover, for the second term $W(\sigma)$ we have:

$$W(\sigma) \leq \|f_{ux}\|_{L^\infty}(\chi_+(\sigma) - \chi_-(\sigma)). \tag{124}$$

Combining equation (121) with these inequalities for $U(\sigma)$ and $W(\sigma)$, we conclude that there exists a positive constant C such that:

$$\dot{\chi}_+(\sigma) - \dot{\chi}_-(\sigma) \leq C(\chi_+(\sigma) - \chi_-(\sigma)) \qquad, \forall \sigma \in [\theta, \tau_0]. \tag{125}$$

By Gronwall's Lemma we get:

$$\chi_+(s) - \chi_-(s) \leq e^{C(s-\sigma)}(\chi_+(\sigma) - \chi_-(\sigma)) \qquad, \forall s \geq \sigma, \tag{126}$$

and in particular putting $\sigma = \theta$ we obtain:

$$\chi_+(s) - \chi_-(s) \leq e^{C(s-\theta)}(\chi_+(\theta) - \chi_-(\theta)) \qquad, \forall s \in [\theta, \tau_0]. \tag{127}$$

At this point, to prove Claim 5.18 it suffices to distinguish between two possible cases:
Case A: We assume that there exists $s \in [\theta, \tau_0]$ such that

$$\chi_+(s) - \chi_-(s) \geq \frac{\gamma}{2\Lambda}(u_- - u_+) > 0, \tag{128}$$

where $\Lambda = \|f_{ux}\|_{L^\infty} > 0$. By estimates (118) and (127) it immediately follows that:

$$\chi_+(\theta) - \chi_-(\theta) \geq e^{-C(s-\theta)}(\chi_+(s) - \chi_-(s)) \geq \frac{e^{-C(s-\theta)}\gamma}{2\Lambda}(u_- - u_+). \tag{129}$$

In this case we can select $C(\theta) = \frac{e^{-C(s-\theta)}\gamma}{2\Lambda}$ to be the constant of Claim 5.18.
Case B: Next, we assume that the reverse inequality is true, i.e. that for every $s \in [\theta, \tau_0]$ we have that

$$\chi_+(s) - \chi_-(s) \leq \frac{\gamma}{2\Lambda}(u_- - u_+), \tag{130}$$

where again we put $\Lambda = \|f_{ux}\|_{L^\infty} > 0$. If we introduce the special time:

$$\overline{\tau} := \min\{\tau : v_-(\sigma) \geq v_+(\sigma)\, \forall \sigma \in [\tau, \tau_0], \tau \geq \theta\}, \tag{131}$$

then for every $\sigma \in [\overline{\tau}, \tau_0]$ and by the equations for the characteristics we have (cp. with Step 2 of the proof of Lemma 3.10)

$$\begin{aligned}\dot{v}_-(\sigma) - \dot{v}_+(\sigma) &\leq \overline{C}[\underbrace{v_-(\sigma) - v_+(\sigma)}_{>0 \text{ by definition of } \overline{\tau}} + \underbrace{\chi_-(\sigma) - \chi_+(\sigma)}_{>0 \text{ by the no-crossing property}}] \\ &\leq \overline{C}(v_-(\sigma) - v_+(\sigma)) + \frac{\overline{C}\gamma}{2\Lambda}(u_- - u_+) \\ &\leq C[v_-(\sigma) - v_+(\sigma) + u_- - u_+]. \end{aligned} \tag{132}$$

Setting $\alpha(\sigma) := v_-(\sigma) - v_+(\sigma) + u_- - u_+$ this last inequality is equivalent to:

$$\dot{\alpha}(\sigma) \leq C\alpha(\sigma).$$

As usual we can apply Gronwall's Lemma to obtain that:

$$\alpha(\tau_0) \leq e^{C(\tau_0 - \sigma)}\alpha(\sigma). \tag{133}$$

If we insert the explicit definition of α in (133) we have that:

$$v_-(\sigma) - v_+(\sigma) \geq (2e^{-C(\tau_0-\sigma)} - 1)[u_- - u_+]. \tag{134}$$

This means that there exists a small time $\phi > 0$ such that:

$$v_-(\sigma) - v_+(\sigma) \geq \frac{3}{4}[u_- - u_+] > 0 \qquad \forall \sigma \in [\tau_0 - \phi, \tau_0]. \tag{135}$$

We note that the selection of ϕ depends only on the bounded interval of time $[\theta, \tau_0]$ and on the sup-norm of the derivatives f and g (which are bounded a-priori because the characteristics take their values on a compact set). In particular ϕ is independent from $u_- - u_+$. Thus, using again the above estimates on $U(\sigma)$ (for the case $v_-(\sigma) \geq v_+(\sigma)$) and $W(\sigma)$, for $\sigma \in [\tau_0 - \phi, \tau_0]$ we get:

$$\begin{aligned}\dot{\chi}_+(\sigma) - \dot{\chi}_-(\sigma) &\leq \gamma(\underbrace{v_+(\sigma) - v_-(\sigma)}_{<0}) + \Lambda(\underbrace{\chi_+(\sigma) - \chi_-(\sigma)}_{\text{use assumption (130)}}) \\ &\overset{(135)}{\leq} -\frac{3}{4}\gamma(u_- - u_+) + \frac{1}{2}\gamma(u_- - u_+) \leq -\frac{\gamma}{4}(u_- - u_+). \end{aligned} \tag{136}$$

Recalling that $\chi_+(\tau_0) - \chi_-(\tau_0) > 0$, we integrate now the last estimate and we have that:

$$-\frac{\phi\gamma}{4}(u_- - u_+) \geq \int_{\tau_0-\phi}^{\tau_0} \dot{\chi}_+(\sigma) - \dot{\chi}_-(\sigma)\, d\sigma \geq -[\chi_+(\tau_0 - \phi) - \chi_-(\tau_0 - \phi)]. \tag{137}$$

Finally we combine equations (127) and (137) to obtain:

$$\begin{aligned} \chi_+(\theta) - \chi_-(\theta) &\geq e^{-C(\tau_0-\phi-\theta)}(\chi_+(\tau_0 - \phi) - \chi_-(\tau_0 - \phi)) \\ &\geq e^{-C(\tau_0-\phi-\theta)}\frac{\phi\gamma}{4}(u_- - u_+). \end{aligned} \tag{138}$$

Thus, even for the second case we have derived the constant $C(\theta)$ of Claim 5.18. This concludes the proof of Lemma 5.13.

6 SBV regularity for Hamilton-Jacobi equations in $\mathbb{R}^n$

6.1 Introduction

In this Chapter we are concerned about the regularity of the gradient of viscosity solutions u to Hamilton-Jacobi equations

$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n. \tag{139}$$

under the following key assumption:

$$H \in C^2(\mathbb{R}^n) \qquad \text{and} \qquad c_H^{-1} Id_n \leq D^2 H \leq c_H Id_n \quad \text{for some } c_H > 0. \tag{140}$$

Note, that under the assumption (140), any viscosity solution u of (139) is locally semiconcave in x. This easily implies that u is locally Lipschitz and that ∇u has locally bounded variation, i.e. that the distributional Hessian $D_x^2 u$ is a symmetric matrix of Radon measures. It is then not difficult to see that the same conclusion holds for $\partial_t D_x u$ and $\partial_{tt} u$. Note that this result is independent of the boundary values of u and can be regarded as an interior regularization effect of the equation.
The rough intuitive picture that one has in mind is therefore that of functions which are Lipschitz and whose gradient is piecewise smooth, undergoing jump discontinuities along a set of codimension 1 (in space and time). A refined regularity theory, which confirms this picture and goes beyond, analyzing the behavior of the functions where singularities are formed, is available under further assumptions on the boundary values of u (we refer to the book [16] for an account on this research topic). However, if the boundary values are just Lipschitz, these results do not apply and the corresponding viscosity solutions might be indeed quite rough, if we understand their regularity only in a pointwise sense.
In this part of the thesis we prove that the BV regularization effect is in fact more subtle and there is a measure-theoretic analog of "piecewise C^1 with jumps of the gradients". As a consequence of our analysis, we know for instance that the singular parts of the Radon measures $\partial_{x_i x_j} u$, $\partial_{x_i t} u$ and $\partial_{tt} u$ are concentrated on a rectifiable set of codimension 1. This set is indeed the measure theoretic jump set $J_{D_x u}$ of $D_x u$ (see Chapter 1 for the precise definition). This excludes, for instance, that the second derivative of u can have a complicated fractal behaviour. Using the language introduced in [23] we say that $D_x u$ and $\partial_t u$ are (locally) special functions of bounded variation, i.e. they belong to the space SBV_{loc} (we refer to the monograph [2] for more details). A typical example of a 1-dimensional function which belongs to BV but not to SBV is the classical Cantor staircase (cp. with Example 1.67 of [2]).

We state now the main Theorem of this Chapter:

Theorem 6.1. *Let u be a viscosity solution of* (139)*, assume* (140) *and set* $\Omega_t := \{x \in \mathbb{R}^n : (t,x) \in \Omega\}$*. Then, the set of times*

$$S := \{t : D_x u(t,.) \notin SBV_{loc}(\Omega_t)\} \tag{141}$$

is at most countable. In particular $D_x u, \partial_t u \in SBV_{loc}(\Omega)$.

Corollary 6.2. *Under assumption* (140)*, the gradient of any viscosity solution u of*

$$H(D_x u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n,$$

belongs to $SBV_{loc}(\Omega)$.

Theorem 6.1 was proved first by Luigi Ambrosio and Camillo De Lellis in the special case $n = 1$ (see [3] and also [35] for the extension to Hamiltonians H depending on (t, x) and u). Some of the ideas of our proof originate indeed in the work [3]. However, in order to handle the higher dimensional case, some new ideas are needed. In particular, a key role is played by the geometrical theory of monotone functions developed by Alberti and Ambrosio in [1].

6.2 Proof of the main Theorem

6.2.1 Preliminary remarks

Let u be a viscosity solution of (139). By Proposition 4.6 and the time invariance of the equation, we can, without loss of generality, assume that u is a solution on $[0, T] \times \mathbb{R}^n$ of the Cauchy-Problem (139) coupled with an initial condition

$$u(0, x) \;=\; u_0(x) \tag{142}$$

under the assumptions

A1: The Hamiltonian $H \in C^2(\mathbb{R}^n)$ satisfies:

$$p \mapsto H(p) \text{ is convex and } \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty.$$

A2: The initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous and bounded.

Clearly, it suffices to show that, for every $j > 0$, the set of times $S \cap]1/j, +\infty[$ is countable. Therefore, by Proposition 4.2 and the time–invariance of the Hamilton–Jacobi equations, we can restrict ourselves to the following case:

$$\exists C \text{ s.t. } u_\tau \text{ is semiconcave ! with constant less than } C \text{ and } |Du_\tau| \le C \;\forall \tau \in [0, T]. \tag{143}$$

Arguing in the same way, we can further assume that

$$T \text{ is smaller than some constant } \varepsilon(C) > 0, \tag{144}$$

where the choice of the constant $\varepsilon(C)$ will be specified later.
Next we consider a ball $B_R(0) \subset \mathbb{R}^n$ and a bounded convex set $\Omega \subset [0, T] \times \mathbb{R}^n$ with the properties that:

- $B_R(0) \times \{s\} \subset \Omega$ for every $s \in [0, T]$;
- For any $(t, x) \in \Omega$ and for any y reaching the minimum in the formulation (63), $(0, y) \in \Omega$ (and therefore the entire segment joining (t, x) to $(0, y)$ is contained in Ω).

Indeed, recalling that $\|Du\|_\infty < \infty$, it suffices to choose $\Omega := \{(x, t) \in \mathbb{R}^n \times [0, T] : |x| \le R + C'(T - t)\}$ where the constant C' is sufficiently large, depending only on $\|Du\|_\infty$ and H. Our goal is now to show the countability of the set S in (141).

6.2.2 A function depending on time

For any $s < t \in [0,T]$, we define the set–valued map

$$X_{t,s}(x) \;:=\; x - (t-s)DH(\partial u_t(x))\,. \tag{145}$$

Moreover, we will denote by $\chi_{t,s}$ the restriction of $X_{t,s}$ to the points where $X_{t,s}$ is single–valued. According to Theorem 2.8 and Proposition 4.4(iii), the domain of $\chi_{t,s}$ consists of those points where $Du_t(\cdot)$ is continuous, which are those where the minimum point y in (64) is unique. Moreover, in this case we have $\chi_{t,s}(x) = \{y\}$.
Clearly, $\chi_{t,s}$ is defined a.e. on Ω_t. With a slight abuse of notation we set

$$F(t) \;:=\; |\chi_{t,0}(\Omega_t)|\,,$$

meaning that, if we denote by U_t the set of points $x \in \Omega_t$ such that (63) has a unique minimum point, we have $F(t) = |X_{t,0}(U_t)|$.
The proof is then split in the following three lemmas:

Lemma 6.3. *The functional F is nonincreasing,*

$$F(\sigma) \geq F(\tau) \qquad \textit{for any } \sigma,\tau \in [0,T] \textit{ with } \sigma < \tau.$$

Lemma 6.4. *If ε in (144) is small enough, then the following holds. For any $t \in]0,T[$ and $\delta \in]0,T-t]$ there exists a Borel set $E \subset \Omega_t$ such that*

(i) $|E| = 0$, and $|D^2_c u_t|(\Omega_t \setminus E) = 0$;

(ii) $X_{t,0}$ is single valued on E (i.e. $X_{t,0}(x) = \{\chi_{t,0}(x)\}$ for every $x \in E$);

(iii) and

$$\chi_{t,0}(E) \cap \chi_{t+\delta,0}(\Omega_{t+\delta}) = \emptyset. \tag{146}$$

Lemma 6.5. *If ε in (144) is small enough, then the following holds. For any $t \in]0,\varepsilon]$ and any Borel set $E \subset \Omega_t$, we have*

$$|X_{t,0}(E)| \geq c_0|E| - c_1 t\int_E d(\Delta u_t)\,, \tag{147}$$

where c_0 and c_1 are positive constants and Δu_t is the Laplacian of u_t.

6.2.3 Proof of Theorem 6.1

The three key lemmas stated above will be proved in the next two sections. We now show how to complete the proof of the Theorem. First of all, note that F is a bounded function. Since F is, by Lemma 6.3, a monotone function, its points of discontinuity are, at most, countable. We claim that, if $t \in]0,T[$ is such that $u_t \notin SBV_{loc}(\Omega_t)$, then F has a discontinuity at t.
Indeed, in this case we have

$$|D^2_c u_t|(\Omega_t) > 0. \tag{148}$$

Consider any $\delta > 0$ and let $B = E$ be the set of Lemma 6.4. Clearly, by Lemma 6.4(i) and (ii), (146) and (147),

$$F(t+\delta) \;\leq\; F(t) + c_1 t\int_E d\,\Delta_s u_t \;\leq\; F(t) + c_1 t\int_{\Omega_t} d\,\Delta_c u_t\,, \tag{149}$$

where the last inequality follows from $\Delta_s u_t = \Delta_c u_t + \Delta_j u_t$ and $\Delta_j u_t \leq 0$ (because of the semiconcavity of u).
Next, consider the Radon–Nykodim decomposition $D^2_c u_t = M|D^2_c u_t|$, where M is a matrix–valued Borel function with $|M| = 1$. Since we are dealing with second derivatives, M is symmetric, and since u_t is semiconcave, $M \leq 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $-M$. Then $1 = |M|^2 = \lambda_1^2 + \ldots + \lambda_n^2$ and $-TrM = \lambda_1 + \ldots + \lambda_n$. Since $\lambda_i \geq 0$, we easily get $-TrM \geq 1$. Therefore,

$$-\Delta_c u_t \;=\; -TrM|D^2_c u_t| \;\geq\; |D^2_c u_t|\,. \tag{150}$$

Hence

$$F(t+\delta) \overset{(149)+(150)}{\leq} F(t) - c_1 t |D^2_c u_t|(\Omega_t)\,.$$

Letting $\delta \downarrow 0$ we conclude

$$\limsup_{\delta\downarrow 0} F(t+\delta) \;<\; F(t)\,.$$

Therefore t is a point of discontinuity of F, which is the desired claim.

6.2.4 Easy corollaries

The conclusion that $D_x u \in SBV(\Omega)$ follows from the slicing theory of BV functions (see Theorem 3.108 of [2]). In order to prove the same property for $\partial_t u$ we apply the Volpert chain rule to $\partial_t u = -H(D_x u)$. According to Theorem 3.96 of [2], we conclude that $[\partial_{x_j t}]_c u = -\sum_i \partial_i H(D_x u)[\partial_{x_j x_i}]_c u = 0$ (because $[D^2_x]_c u = 0$) and $[\partial_{tt}]_c u = -\sum_i \partial_i H(D_x u)[\partial_{x_i t}]_c u = 0$ (because we just concluded $[D^2_{xt}]_c u = 0$).
As for Corollary 6.2, let u be a viscosity solution of (60) and set $\widetilde{u}(t,x) := u(x)$. Then $\tilde{u}$ is a viscosity solution of

$$\partial_t \widetilde{u} + H(D_x \widetilde{u}) = 0$$

in $\mathbb{R} \times \Omega$. By our main Theorem 6.1 the set of times for which $D_x \widetilde{u}(t,.) \notin SBV_{loc}(\Omega)$ is at most countable. Since $D_x \widetilde{u}(t,\cdot) = Du$, for every t, we conclude that $Du \in SBV_{loc}(\Omega)$.

Remark 6.6. *The special case of this Corollary for $\Omega \subset \mathbb{R}^2$ was already proved in [3] (see Corollary 1.4 therein). We note that the proof proposed in [3] was more complicated than the one above. This is due to the power of Theorem 6.1. In [3] the authors proved the 1–dimensional case of Theorem 6.1. The proof above reduces the 2–dimensional case of Corollary 6.2 to the 2+1 case of Theorem 6.1. In [3] the 2-dimensional case of Corollary 6.2 was reduced to the $1+1$ case of Theorem 6.1: this reduction requires a subtler argument.*

6.3 Estimates

In this section we prove two important estimates. The first is the one in Lemma 6.5. The second is an estimate which will be useful in proving Lemma 6.4 and will be stated here.

Lemma 6.7. *If $\varepsilon(C)$ in (144) is sufficiently small, then the following holds. For any $t \in]0,T]$, any $\delta \in [0,t]$ and any Borel set $E \subset \Omega_t$ we have*

$$\Big|X_{t,\delta}(E)\Big| \geq \frac{(t-\delta)^n}{t^n}\Big|X_{t,0}(E)\Big|\,. \tag{151}$$

6.3.1 Injectivity

In the proof of both lemmas, the following remark plays a fundamental role.

Proposition 6.8. *For any $C > 0$ there exists $\varepsilon(C) > 0$ with the following property. If v is a semiconcave function with constant less than C, then the map $x \mapsto x - tDH(\partial v)$ is injective for every $t \in [0, \varepsilon(C)]$.*

Here the injectivity of a set–valued map B is understood in the following natural way

$$x \neq y \qquad \Longrightarrow \qquad B(x) \cap B(y) = \emptyset\,.$$

Proof. We assume by contradiction that there exist $x_1, x_2 \in \Omega_t$ with $x_1 \neq x_2$ and such that:

$$[x_1 - tDH(\partial v(x_1))] \cap [x_2 - tDH(\partial v(x_2))] \neq \emptyset.$$

This means that there is a point y such that

$$\begin{cases} \frac{x_1-y}{t} \in DH(\partial v(x_1)), \\ \frac{x_2-y}{t} \in DH(\partial v(x_2)); \end{cases} \Rightarrow \begin{cases} DH^{-1}(\frac{x_1-y}{t}) \in \partial v(x_1), \\ DH^{-1}(\frac{x_2-y}{t}) \in \partial v(x_2). \end{cases}$$

By the semiconcavity of v we get:

$$M(x_1,x_2) := \Big\langle DH^{-1}\Big(\frac{x_1-y}{t}\Big) - DH^{-1}\Big(\frac{x_2-y}{t}\Big), x_1 - x_2\Big\rangle \leq C|x_1-x_2|^2. \tag{152}$$

On the other hand, $D(DH^{-1})(x) = (D^2H)^{-1}(DH^{-1}(x))$ (note that in this formula, DH^{-1} denotes the inverse of the map $x \mapsto DH(x)$, whereas $D^2H^{-1}(y)$ denotes the matrix A which is the inverse of the matrix $B := D^2H(y)$). Therefore $D(DH^{-1})(x)$ is a symmetric matrix, with $D(DH^{-1})(x) \geq c_H^{-1} Id_n$. It follows that

$$\begin{aligned} M(x_1,x_2) &= t\Big\langle DH^{-1}\Big(\frac{x_1-y}{t}\Big) - DH^{-1}\Big(\frac{x_2-y}{t}\Big), \frac{x_1-y}{t} - \frac{x_2-y}{t}\Big\rangle \geq \\ &\geq \frac{t}{2c_H}\Big|\frac{x_1-y}{t} - \frac{x_2-y}{t}\Big|^2 \geq \frac{1}{2tc_H}|x_1-x_2|^2 \geq \frac{1}{2\varepsilon c_H}|x_1-x_2|^2. \end{aligned} \tag{153}$$

But if $\varepsilon > 0$ is small enough, or more precisely if it is chosen to satisfy $2\varepsilon c_H < \frac{1}{C}$ the two inequalities (152) and (153) are in contradiction. ☐

6.3.2 Approximation

We next consider u as in the formulations of the two lemmas, and $t \in [0, T]$. Then the function $\tilde{v}(x) := u(x) - C|x|^2/2$ is concave. Consider the approximations B_η (with $\eta > 0$) of $\partial\tilde{v}$ given in Definition 2.7. By Theorem 2.10(i), $B_\eta = D\tilde{v}_\eta$ for some concave function $\tilde{v}_\eta$ with Lipschitz gradient. Consider therefore the function $v_\eta(x) = \tilde{v}_\eta(x) + C|x|^2/2$. The semiconcavity constant of v_η is not larger than C.
Therefore we can apply Proposition 6.8 and choose $\varepsilon(C)$ sufficiently small in such a way that the maps

$$x \;\mapsto\; A(x) = x - tDH(\partial u_t) \qquad x \;\mapsto\; A_\eta(x) = x - tDH(Dv_\eta) \tag{154}$$

are both injective. Consider next the following measures:

$$\mu_\eta(E) \;:=\; |(Id - tDH(Dv_\eta))(E)| \qquad \mu(E) \;:=\; |(Id - tDH(\partial u_t))(E)|\,. \tag{155}$$

These measures are well-defined because of the injectivity property proved in Proposition 6.8. Now, according to Theorem 2.10, the graphs ΓDv_η and $\Gamma\partial u_t$ are both rectifiable currents and the first are converging, as $\eta \downarrow 0$, to the latter. We denote them, respectively, by T_η and T. Similarly, we can associate the rectifiable currents S and S_η to the graphs ΓA and ΓA_η of the maps in (154). Note that these graphs can be obtained by composing $\Gamma\partial u_t$ and ΓDv_η with the following global diffeomorphism of $\mathbb{R}^n$:

$$(x, y) \ \mapsto\ \Phi(x, y) = x - tDH(y)\,.$$

In the language of currents we then have $S_\eta = \Phi_\sharp T_\eta$ and $S = \Phi_\sharp T$. Therefore, $S_\eta \to S$ in the sense of currents.
We want to show that

$$\mu_\eta \ \rightharpoonup^* \ \mu\,. \tag{156}$$

First of all, note that S and S_η are rectifiable currents of multiplicity 1 supported on the rectifiable sets $\Gamma A = \Phi(\Gamma\partial u_t)$ and $\Gamma A_\eta = \Phi(\Gamma B_\eta) = \Phi(\Gamma Dv_\eta)$. Since B_η is a Lipschitz map, the approximate tangent plane π to S_η in (a.e.) point $(x, A_\eta(x))$ is spanned by the vectors $e_i + DA_\eta(x)\cdot e_i$ and hence oriented by the n-vector

$$\vec{v} := \frac{(e_1 + DA_\eta(x)\cdot e_1)\wedge\ldots\wedge(e_n + DA_\eta(x)\cdot e_n)}{|(e_1 + DA_\eta(x)\cdot e_1)\wedge\ldots\wedge(e_n + DA_\eta(x)\cdot e_n)|}\,.$$

Now, by the calculation of Proposition 6.8, it follows that $\det DA_\eta \geq 0$. Hence

$$\langle dy_1 \wedge\ldots\wedge dy_n, \vec{v}\rangle \ \geq\ 0\,. \tag{157}$$

By the convergence $S_\eta \to S$, (157) holds for the tangent planes to S as well.
Next, consider a $\varphi \in C^\infty_c(\Omega_t)$. Since both ΓA and ΓA_η are bounded sets, consider a ball $B_R(0)$ such that $\mathrm{supp}\,(\Gamma A), \mathrm{supp}\,(\Gamma A_\eta) \subset \mathbb{R}^n \times B_R(0)$ and let $\chi \in C^\infty_c(\mathbb{R}^n)$ be a cut-off function with $\chi|_{B_R(0)} = 1$. Then, by standard calculations on currents, the injectivity property of Proposition 6.8 and (157) imply that

$$\int \varphi d\mu \ = \ \langle S, \varphi(x)\chi(y) dy_1\wedge\ldots\wedge dy_n\rangle, \tag{158}$$

$$\int \varphi d\mu_\eta \ = \ \langle S_\eta, \varphi(x)\chi(y) dy_1\wedge\ldots\wedge dy_n\rangle\,. \tag{159}$$

Therefore, since $S_\eta \to S$, we conclude that

$$\lim_{\eta\downarrow 0}\int \varphi d\mu_\eta \ = \ \int\varphi d\mu\,.$$

This shows (156).

6.3.3 Proof of Lemma 6.7

First of all we choose ε so small that the conclusions of Proposition 6.8 and those of Subsection 6.3.2 hold.
We consider therefore, the approximations v_η of Subsection 6.3.2, we define the measures μ and μ_η as in (155) and the measures $\tilde\mu$ and $\tilde\mu_\eta$ as

$$\tilde\mu(E) \ :=\ |(Id - (t-\delta)DH(\partial u_t))(E)| \qquad \tilde\mu_\eta(E) \ :=\ |(Id - (t-\delta)DH(Dv_\eta))(E)|\,. \tag{160}$$

By the same arguments as in Subsection 6.3.2, we necessarily have $\tilde{\mu}_\eta \rightharpoonup^* \tilde{\mu}$.
The conclusion of the Lemma can now be formulated as

$$\tilde{\mu} \;\geq\; \frac{(t-\delta)^n}{t^n}\mu\,. \tag{161}$$

By the convergence of the measures μ_η and $\tilde{\mu}_\eta$ to μ and $\tilde{\mu}$, it suffices to show

$$\tilde{\mu}_\eta \;\geq\; \frac{(t-\delta)^n}{t^n}\mu_\eta\,. \tag{162}$$

On the other hand, since the maps $x \mapsto x - tDH(Dv_\eta)$ and $x \mapsto x-(t-\delta)DH(Dv_\eta)$ are both injective and Lipschitz, we can use the area formula to write:

$$\tilde{\mu}_\eta(E) \;=\; \int_E \det\Big(Id_n - (t-\delta)D^2H(Dv_\eta(x))D^2v_\eta(x)\Big)\,dx, \tag{163}$$

$$\mu_\eta(E) \;=\; \int_E \det\Big(Id_n - tD^2H(Dv_\eta(x))D^2v_\eta(x)\Big)\,dx \tag{164}$$

Therefore, if we set

$$\begin{aligned} M_1(x) &:= Id_n - (t-\delta)D^2H(Dv_\eta(x))D^2v_\eta(x) \\ M_2(x) &:= Id_n - tD^2H(Dv_\eta(x))D^2v_\eta(x)\ , \end{aligned}$$

the inequality (161) is equivalent to

$$\det M_1(x) \;\geq\; \frac{(t-\delta)^n}{t^n}\det M_2(x) \qquad \text{for a.e. } x. \tag{165}$$

Note next that

$$\begin{aligned} \det M_1(x) &= \det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1} - (t-\delta)D^2v_\eta(x)\Big) \\ \det M_2(x) &= \det(D^2H(Dv_\eta(x)))\det\Big([D^2H(Dv_\eta(x))]^{-1} - tD^2v_\eta(x)\Big) \end{aligned}$$

Set $A(x) := [D^2H(Dv_\eta(x))]^{-1}$ and $B(x) = D^2v_\eta(x)$. Then it suffices to prove that:

$$\det(A(x)-(t-\delta)B(x)) \;\geq\; \frac{(t-\delta)^n}{t^n}\det(A(x)-tB(x))\,.$$

Note that

$$A-(t-\delta)B \;=\; \frac{\delta}{t}A + \frac{t-\delta}{t}(A-tB)\,.$$

By choosing ε sufficiently small (but only depending on c_H and C), we can assume that $A-tB$ is a positive semidefinite matrix. Since A is a positive definite matrix, we conclude

$$A-(t-\delta)B \;\geq\; \frac{t-\delta}{t}(A-tB)\,. \tag{166}$$

A standard argument in linear algebra shows that

$$\det(A-(t-\delta)B) \;\geq\; \frac{(t-\delta)^n}{t^n}\det(A-tB) \tag{167}$$

which concludes the proof. We include, for the reader convenience, a proof of (166) $\Longrightarrow$(167). It suffices to show that, if E and D are positive semidefinite matrices with $E \geq D$, then $\det E \geq \det D$. Without loss of generality, we can assume that E is in diagonal form, i.e. $E = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, and that $E > D$. Then each λ_i is positive. Define $G := \mathrm{diag}\,(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Then

$$\mathrm{Id}_n \;\geq\; G^{-1} D G^{-1} = \tilde{D}\,.$$

Our claim would follow if we can prove $1 \geq \det \tilde{D}$, that is, if we can prove the original claim for E and D in the special case where E is the identity matrix. But in this case we can diagonalize E and D at the same time. Therefore $D = \mathrm{diag}\,(\mu_1, \dots, \mu_n)$. But, since $E \geq D \geq 0$, we have $0 \leq \mu_i \leq 1$ for each μ_i. Therefore

$$\det E \;=\; 1 \;\geq\; \Pi_i \mu_i \;=\; \det D\,.$$

6.3.4 Proof of Lemma 6.5

As in the proof above we will show the Lemma by approximation with the functions v_η. Once again we introduce the measures μ_η and μ of (155). Then, the conclusion of the Lemma can be formulated as

$$\mu \;\geq\; c_0 \mathcal{L}^n - t c_1 \Delta u_t\,. \tag{168}$$

Since $\Delta v_\eta \rightharpoonup^* \Delta u_t$ by Theorem 2.10(iii), it suffices to show

$$\mu_\eta \;\geq\; c_0 \mathcal{L}^n - t c_1 \Delta v_\eta\,. \tag{169}$$

Once again we can use the area formula to compute

$$\mu_\eta(E) \;=\; \int_E \det(D^2 H(Dv_\eta(x))) \det \Big([D^2 H(Dv_\eta(x))]^{-1} - t D^2 v_\eta(x) \Big) dx \tag{170}$$

Since $D^2 H \geq c_H^{-1} Id_n$ and $[D^2 H]^{-1} \geq c_H^{-1} Id_n$, we can estimate

$$\det(D^2 H(Dv_\eta(x))) \det \Big([D^2 H(Dv_\eta(x))]^{-1} - t D^2 v_\eta(x) \Big) \;\geq\; c_H^{-n} \det \left(\frac{1}{c_H} Id_n - t D^2 v_\eta(x) \right) \tag{171}$$

arguing as in Subsection 6.3.3. If we choose ε so small that $0 < \varepsilon < \frac{1}{2 c_H C}$, then $M(x) := \frac{1}{2c_H} Id_n - t D^2 v_\eta(x)$ is positive semidefinite. Therefore

$$\det(D^2 H(Dv_\eta(x))) \det \Big([D^2 H(Dv_\eta(x))]^{-1} - t D^2 v_\eta(x) \Big) \;\geq\; c_H^{-n} \det \left(\frac{1}{2c_H} Id_n + M(x) \right)\,. \tag{172}$$

Diagonalizing $M(x) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, we can estimate

$$\begin{aligned} \det \left(\frac{1}{2c_H} Id_n + M(x) \right) \;&=\; \left(\frac{1}{2c_H} \right)^n \prod_{i=1}^n (1 + 2 c_H \lambda_i) \;\geq\; \left(\frac{1}{2c_H} \right)^n (1 + 2 c_H \mathrm{Tr}\, M(x)) \\ &=\; c_2 - c_3 t \Delta v_\eta(x)\,. \end{aligned} \tag{173}$$

Finally, by (170), (171), (172) and (173), we get

$$\mu_\eta(E) \;\geq\; \int_E (c_0 - c_1 t \Delta v_\eta(x))\, dx\,.$$

This concludes the proof.

6.4 Proofs of Lemma 6.3 and Lemma 6.4

6.4.1 Proof of Lemma 6.3

The claim follows from the following consideration:

$$\chi_{t,0}(\Omega_t) \subset \chi_{s,0}(\Omega_s) \qquad \text{for every } 0 \le s \le t \le T. \tag{174}$$

Indeed, consider $y \in \chi_{t,0}(\Omega_t)$. Then there exists $x \in \Omega_t$ such that y is the unique minimum of (63). Consider $z := \frac{s}{t}x + \frac{t-s}{t}y$. Then $z \in \Omega_s$. Moreover, by Proposition 4.4(iv), y is the unique minimizer of $u_0(w) + sL((z-w)/s)$. Therefore $y = \chi_{s,0}(z) \in \chi_{s,0}(\Omega_s)$.

6.4.2 Proof of Lemma 6.4

First of all, by Proposition 2.4, we can select a Borel set E of measure 0 such that

- $\partial u_t(x)$ is single-valued for every $x \in E$;
- $|E| = 0$;
- $|D_c^2 u_t|(\Omega_t \setminus E) = 0$.

If we assume that our statement were false, then there would exist a compact set $K \subset E$ such that

$$|D_c^2 u_t|(K) \ > \ 0\,. \tag{175}$$

and $X_{t,0}(K) = \chi_{t,0}(K) \subset \chi_{t+\delta,0}(\Omega_{t+\delta})$. Therefore it turns out that $X_{t,0}(K) = \chi_{t+\delta,0}(\tilde{K}) = X_{t+\delta,0}(\tilde{K})$ for some Borel set $\tilde{K}$.
Now, consider $x \in \tilde{K}$ and let $y := \chi_{t+\delta,0}(x) \in X_{t+\delta,0}(\tilde{K})$ and $z := \chi_{t+\delta,t}(x)$. By Proposition 4.4(iv), y is the unique minimizer of $u_0(y) + tL((z-y)/t)$, i.e. $\chi_{t,0}(z) = y$.
Since $y \in \chi_{t,0}(K)$, there exists z' such that $\chi_{t,0}(z')$. On the other hand, by Proposition 6.8, provided ε has been chosen sufficiently small, $\chi_{t,0}$ is an injective map. Hence we necessarily have $z' = z$. This shows that

$$X_{t+\delta,t}(\tilde{K}) \subset K\,. \tag{176}$$

By Lemma 6.7,

$$|K| \ \ge \ |X_{t+\delta,t}(\tilde{K})| \ \ge \ \frac{\delta^n}{(t+\delta)^n}|X_{t+\delta,0}(\tilde{K})| \ = \ \frac{\delta^n}{(t+\delta)^n}|X_{t,0}(K)|\,. \tag{177}$$

Hence, by Lemma 6.5

$$|K| \ \ge \ c_0|K| - c_1 t\frac{\delta^n}{(t+\delta)^n}\int_K d\,\Delta u_t\,. \tag{178}$$

On the other hand, recall that $K \subset E$ and $|E| = 0$. Thus, $\int_K d\,\Delta_s u_t = \int_K d\,\Delta u_t \ge 0$. On the other hand $\Delta_s u_t \le 0$ (by the semiconcavity of u). Thus we conclude that $\Delta_s u_t$, and hence also $\Delta_c u_t$, vanishes indentically on K. However, arguing as in Subsection 6.2.3, we can show $-\Delta_c u_t \ge |D_c^2 u_t|$, and hence, recalling (175), $-\Delta_c u_t(K) > 0$. This is a contradiction and hence concludes the proof.

7 Hamilton Jacobi equations with obstacles

7.1 Introduction

In the last part of this thesis we dealwith a problem in the theory of optimal control introduced for the first time by Alberto Bressan in [8] and which has been subsequently studied in several papers (see [11], [12], [13] and [14]). The problem models the spread of fire in a forest or that of a contaminating agent. Tough that this topic is different by the previous ones, we will get all the results using the theory of SBV functions and Hamilton-Jacobi equations.

Bressan in [8] proposed the following mathematical model. Consider a continuous multi-function $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with compact, convex values (that is, $F(x)$ is a compact convex set for every x and $F(x_n) \to F(x)$ in the sense of Hausdorff when $x_n \to x$). A bounded, open set $R_0 \subset \mathbb{R}^2$ is the initial contaminated set and F describes the speed at which the contamination might spread. A controller can construct one-dimensional rectifiable sets γ (or "walls") which block the spreading of the contamination, without exceeding a certain length. More precisely, consider a continuous function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}_+$ and a constant ψ_0 with $\psi \geq \psi_0 > 0$. We denote by $\gamma(t) \subset \mathbb{R}^2$ the portion of the wall constructed within time $t \geq 0$ and we make the following assumptions ($\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure):

(H1) $\gamma(t_1) \subseteq \gamma(t_2)$ for every $0 \leq t_1 \leq t_2$;

(H2) $\int_{\gamma(t)} \psi \, d\mathcal{H}^1 \leq t$ for every $t \geq 0$.

A strategy γ satisfying (H1)–(H2) will be called an *admissible strategy*. In what follows, we will always assume that

(H3) There is a constant $\lambda > 0$ s.t. $B_\lambda(0) \subset F(x)$ for all x.

At each time t, the contaminated set consists of the points reached by absolutely continuous trajectories $x(\cdot)$ which start in R_0, solve the differential inclusion $\dot{x} \in F(x)$ and do not cross the walls γ. That is,

$$R^\gamma(t) := \Big\{ x(t) \Big| \quad x \in W^{1,1} \cap C([0,t], \mathbb{R}^2), \; x(\tau) \notin \gamma(\tau) \quad \forall \tau, \quad x(0) \in R_0 \quad \text{and} \quad \dot{x}(\tau) \in F\big(x(\tau)\big) \text{ for a.e. } \tau \Big\}. \tag{179}$$

Moreover, given an admissible strategy γ, for any $x \in \mathbb{R}^2$ we set

$$T^\gamma(x) := \inf\{t > 0 : x \in R^\gamma(t)\}. \tag{180}$$

The purpose of this Section is to study the minimum time function T^γ at which a point x gets contaminated. Obviously T^γ vanishes identically on R_0 and the total contaminated set is given by $\{T^\gamma < +\infty\}$. We will be able to characterize this function via a suitable modification of the usual Hamilton-Jacobi partial differential equation and the related hamiltonian function will be:

Definition 7.1. $H(x,p) := \sup_{q \in F(x)} \{p \cdot q\} - 1.$

In the paper [12] Bressan and De Lellis introduced a variational problem on the set of admissible strategies and proved the existence of a minimizer (this problem is connected to that of confining the fire in a bounded set, see for instance [13]). An interesting byproduct of our analysis is a shorter proof of this existence result. The prize to pay is the use of some more advanced techniques in geometric measure theory.
In order to state our main theorem, we need some notation.

7.1.1 Main Theorem

We start by introducing the "complete strategies", which were first defined in [12]. The definition is motivated by the following example. Assume that γ is an admissible strategy and consider a family of sets $\eta(t)$ satisfying (H1) and $\mathcal{H}^1(\eta(t)) = 0$ for every t. Then $\gamma(t) \cup \eta(t)$ satisfies (H1)–(H2). In other words, given an admissible strategy γ, we can increase its effectiveness by adding an $\mathcal{H}^1$-negligible amount of walls.

Definition 7.2. *An admissible strategy γ is* complete *if*

(i) $\gamma(t) = \bigcap_{s>t} \gamma(s)$*;*

(ii) $\gamma(t)$ *contains all its points of positive upper density, i.e. all x s.t.*

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^1(B_r(x) \cap \gamma(t))}{r} \;>\; 0\,. \tag{181}$$

The following proposition follows from standard geometric measure theory.

Proposition 7.3 (Lemma 4.2 of [12])**.** *Let γ be an admissible strategy. Then there exists a complete admissible strategy γ^c such that*

(iii) $\gamma(t) \subset \gamma^c(t)$*;*

(iv) $\mathcal{H}^1(\gamma^c(t) \setminus \gamma(t)) = 0$ *except for a countable number of times t.*

An interesting byproduct of the results of this note is a proof of the intuitive fact that γ^c has the maximum effectiveness among all strategies which differ from γ by a negligible amount of walls (that is, γ^c has the largest minimum time function in this set of strategies, cp. with Theorem 7.5 below).

We next introduce some notation in order to describe our "viscosity solution" to the Hamilton-Jacobi equation with obstacles.

Definition 7.4. *Given a measurable function $u : \mathbb{R}^2 \to [0, \infty]$ and a $t \in [0, \infty[$ we set $u_t := u \wedge t = \min\{u, t\}$.*
For a given strategy γ, a measurable $u : \mathbb{R}^2 \to [0, \infty]$ belongs to the class $\mathcal{S}^\gamma$ if the following conditions hold for every $t \in [0, \infty[$:

(a) $u_t \in SBV_{loc}(\mathbb{R}^2)$, $\mathcal{H}^1(J_{u_t} \setminus \gamma(t)) = 0$ *and* $u_t \equiv 0$ *on* R_0*;*

(b) If ∇u_t denotes the absolutely continuous part of Du_t, then

$$H(x, \nabla u_t(x)) \leq 0 \qquad \textit{for a.e. } x\,. \tag{182}$$

$SBV_{loc}(\mathbb{R}^2)$ is a linear subspace of $BV_{loc}(\mathbb{R}^2)$ (where the latter is the space of functions having bounded variation on every bounded open subset of $\mathbb{R}^2$). For its precise definition we refer to Chapter 1. We are now ready to state the main result of this Section.

Theorem 7.5. *Let γ be an admissible strategy. Assume (H1), (H2), (H3) and*

(H4) the initial set R_0 is open and ∂R_0 has zero 2–dimensional Lebesgue measure.

Then $T^\gamma \in \mathcal{S}^\gamma$ and T^{γ^c} is the unique maximal element of $\mathcal{S}^\gamma$, that is

$$\textit{for every } v \in \mathcal{S}^\gamma \textit{ we have } v \leq T^{\gamma^c} \textit{ a.e.}.$$

7.1.2 A variational problem

Besides its intrinsic interest, Theorem 7.5, together with the SBV compactness theorem of Ambrosio and De Giorgi, yields a direct proof of the existence of minima for the variational problem first studied in [12]. More precisely, consider two continuous, non-negative functions $\alpha, \beta : \mathbb{R}^2 \mapsto \mathbb{R}_+$ and define

$$R^\gamma_\infty := \bigcup_{t>0} R^\gamma(t)\,, \qquad \gamma_\infty := \bigcup_{t>0} \gamma(t) \qquad \textit{and} \tag{183}$$

$$J(\gamma) := \int_{R^\gamma_\infty} \alpha \, d\mathcal{L}^2 + \int_{\gamma_\infty} \beta \, d\mathcal{H}^1\,, \tag{184}$$

Note that the functional J is well defined: the set R^γ_∞ is indeed measurable by Theorem 7.5 because $R^\gamma_\infty = \{T^\gamma < \infty\}$ (however, the measurability of R^γ_∞ can also be proved directly; cp. with Lemma 3.1 of [12]). As a consequence of Theorem 7.5 we have the following.

Corollary 7.6 (Cp. with Theorem 1.1 of [12]). *In addition to (H1)–(H4) assume that:*

(H5) $\alpha \geq 0$, $\beta \geq 0$, *α is locally integrable and β is lower semicontinuous.*

Then, there exists a strategy that minimizes J among all the admissible ones.

7.2 Preliminaries on BV functions

This section will be devoted to prove the following technical proposition, which is a key point of our proof. We refer to Chapter 1 for the definition of approximate continuity.

Proposition 7.7. *Let $u \in \mathcal{S}^\gamma$ and assume γ is a complete strategy. Then there is a measurable function $\tilde{u}$ having the following properties:*

(i) $u = \tilde{u}$ *a.e. (i.e. $\tilde{u}$ is a representative of u);*

(ii) $\tilde{u}_t$ *is approximately continuous at every* $x \notin \gamma(t)$*;*

(iii) If $\Phi : [0,1]\times[0,1] \to \mathbb{R}^2$ is a C^1 diffeomorphism (of $[0,1]^2$ with its image) and α_τ denotes the curve $\{\Phi(\tau, s) : s \in [0,1]\}$, then the following holds for a.e. τ and for every t:

$$\left.\begin{array}{ll} \textit{If } \alpha_\tau \cap \gamma(t) = \emptyset, \textit{ then } w(\cdot) := \tilde{u}_t(\Phi(\tau,\cdot)) \textit{ is Lipschitz and} & \\ \dot{w}(s) = \nabla u_t(\Phi(\tau,s)) \cdot \partial_s \Phi(\tau,s) & \textit{for a.e. } s \\ H(\Phi(\tau,s), \nabla u_t(\Phi(\tau,s))) \leq 0 & \textit{for a.e. } s\,. \end{array}\right\} \tag{185}$$

In the proposition above it is crucial that the Lipschitz regularity holds for w in its pointwise definition: we do not need to redefine it on a set of measure zero!

Proof of Proposition 7.7. Consider for any t the SBV map u_t. Consider now the precise representative $\widetilde{u_t}$ of u_t, given by Proposition 1.17. $\widetilde{u_t}$ and u_t differ on a set of measure zero L_t. Moreover, $\widetilde{u_t}$ is approximately continuous at all points x for which

$$\lim_{r\downarrow 0} \frac{|Du_t|(B_r(x))}{r} = 0\,. \tag{186}$$

On the other hand, by the definition of $\mathcal{S}^\gamma$, we have $Du_t = \nabla u_t \mathcal{L}^2 + f\nu\mathcal{H}^1 \llcorner \gamma(t)$. Now, since $0 \leq u_t \leq t$ a.e., it is a standard fact that $|f| \leq t$. Moreover, since $H(x, \nabla u_t(x)) \leq 0$ for a.e. x, assumption (H3) implies that $|\nabla u_t(x)| \leq \lambda^{-1}$. Thus $|Du_t| \leq \lambda^{-1}\mathcal{L}^2 + t\mathcal{H}^1 \llcorner \gamma(t)$ and, if (186) fails, we necessarily have

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^1(\gamma(t) \cap B_r(x))}{r} > 0\,.$$

The completeness of γ, implies that:

$$\widetilde{u_t} \text{ is approximately continuous at every } x \notin \gamma(t). \tag{187}$$

Obviously, if $t < \tau$, then $\widetilde{u_t}(x) \leq \widetilde{u_\tau}(x)$ for a.e. x. Moreover, if x is a point of approximate continuity of $\widetilde{u_t}$ and $\widetilde{u_t}(x) < t$, then

(a) x is a point of approximate continuity for $\widetilde{u_\tau}$ for every τ;

(b) $\widetilde{u_\tau}(x) = \widetilde{u_t}(x)$ for every $\tau > t$ and $\widetilde{u_\tau}(x) \leq \widetilde{u_t}(x)$ for every $\tau \leq t$.

Set then $\widetilde{u}(x) := \sup_t \widetilde{u_t}(x)$.

Step 1 First we prove assertion (i), that is $\tilde{u} = u$ a.e.. Indeed, consider first the set $A_N := \{\tilde{u} < N\}$, where $N \in \mathbb{N}$. Then $\tilde{u} = \widetilde{u_N}$ on the set $A'_N \subset A_N$ of points of approximate continuity for $\widetilde{u_N}$ and $\tilde{u}$. Indeed, at such a point x we have $\widetilde{u_N}(x) \leq \tilde{u}(x) < N$. Thus we can apply (a) and (b), from which we conclude $\tilde{u}(x) = \sup_\tau \widetilde{u_\tau}(x) = \widetilde{u_N}(x)$. Observe next that $|A_N \setminus A'_N| = 0$ and that $\widetilde{u_N} = u_N$ on a set $A''_N \subset A'_N$ with $|A'_N \setminus A''_N| = 0$. On the other hand, on every $x \in A''_N$ we have $u_N(x) < N$ and thus $u(x) = u_N(x) = \widetilde{u_N}(x) = \tilde{u}(x)$. So, $u = \tilde{u}$ a.e. on A_N.
Since $\cup_N A_N = \{\tilde{u} < \infty\}$, it remains to show that $u = \infty$ a.e. on $A := \{\tilde{u} = \infty\}$. Consider now the subset $A' \subset A$ of points x where all $\widetilde{u_N}$ are approximately continuous. Clearly $|A \setminus A'| = 0$. On the other hand, on each $x \in A'$ we necessarily have $\widetilde{u_N}(x) = N$. Otherwise, by (a) and (b) we would have $\tilde{u}(x) = \sup_\tau \widetilde{u_\tau}(x) = \widetilde{u_N}(x) < N$, contradicting $\tilde{u}(x) = \infty$. Consider next the set $A'' \subset A'$ of points x where $\widetilde{u_N}(x) = u_N(x)$ for every N. Again $|A' \setminus A''| = 0$.
Hence, for every $x \in A''$ we have $u_N(x) = \widetilde{u_N}(x) = N$. Letting $N \uparrow \infty$ we conclude $u(x) = \infty$ for every $x \in A''$.

Step 2 We claim next that, if $\widetilde{u_t}$ is approximately continuous at x, so is $\tilde{u}_t$ (observe that $\widetilde{u_t}$ is the precise representative of u_t, whereas $\tilde{u}_t = \tilde{u} \wedge t$). Assume indeed that $\widetilde{u_t}$ is approximately continuous at x. Let then E be a measurable set satisfying the requirements of Definition 1.16. Obviously, if we reduce further E taking all the points $y \in E$ of approximate continuity for $\widetilde{u_t}$, the new set still satisfies the requirements of Definition 1.16. With a slight abuse of notation, we keep the name E for this second set. Next, if $y \in E$, either $\widetilde{u_t}(y) < t$, and hence $\tilde{u}(y) = \widetilde{u_t}(y)$ (because $\widetilde{u_t}$ is approximately continuous at y and hence (b) applies), or $\widetilde{u_t}(y) = t$ and hence $\tilde{u}(y) \geq t$. In both cases, $\tilde{u}_t(y) = \widetilde{u_t}(y)$. For the same reasons $\tilde{u}_t(x) = \widetilde{u_t}(x)$. We therefore conclude that

$$\lim_{y\in E, y\to x} \tilde{u}_t(y) \;=\; \lim_{y\in E, y\to x} \widetilde{u_t}(y) \;=\; \widetilde{u_t}(x) = \tilde{u}_t(x)\,.$$

This shows that all the points of approximate continuity of $\widetilde{u_t}$ are points of approximate continuity of $\tilde{u}_t$. Thus assertion (ii) follows from (187). Finally, assertion (iii) follows easily from Proposition 1.18, Theorem 1.14 and assertion (ii). □

7.3 Zig-zag construction and faster trajectories

7.3.1 Zig-zag constructions

In this section we outline a crucial construction for our proof of Theorem 7.5. The basic idea is borrowed from [12], but we require several technical improvements. We assume that

(Z1) γ is an admissible strategy, not necessarily complete;

(Z2) $t \in]0, \infty[$ and x_0 is a point such that

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^1(B_r(x_0) \cap \gamma(t))}{r} \;=\; 0\,. \tag{188}$$

Lemma 7.8 (Zig-zag)**.** *Assume (Z1)–(Z2) and let ε be any given positive number. Then there is a set G of radii such that*

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^1([0,r] \setminus G)}{r} \;=\; 0 \tag{189}$$

and the following property holds.
If $B_\varepsilon(v) \subset F(x_0)$, $\mu|v| \in G$ and $\tau < t - \mu$, then there exists a Lipschitz trajectory $z : [\tau, \tau+\mu] \to \mathbb{R}^2$ satisfying the following assumptions

(z1) $z(\tau) = x_0$, $z(\tau + \mu) = x_0 + \mu v$;

(z2) $\dot{z}(s) \in F(z(s))$ *for a.e.* s;

(z3) $z(s) \notin \gamma(t)$ *for* every s.

Assume in addition that γ is a complete strategy, $u \in \mathcal{S}^\gamma$ and $\tilde{u}$ is the function given by Proposition 7.7. Then, we can require the following additional property:

(z4) $w(s) := \tilde{u}_t(z(s))$ *is Lipschitz, u_t is approximately differentiable at $z(s)$ for a.e. s and the following identities hold:*

$$\left\{ \begin{array}{rcl} \dot{w}(s) &=& \nabla u_t(z(s)) \cdot \dot{z}(s) \\[2ex] H(z(s), \nabla u_t(z(s))) &\leq& 0 \end{array} \right. . \tag{190}$$

For v and μ as above and $\tau < t$ there exists a trajectory $z : [\tau - \mu, \tau] \to \mathbb{R}^2$ enjoying (z2)–(z4) but with $z(\tau - \mu) = x_0 - \mu v$ and $z(\tau) = x_0$.

Proof. The proof of the first assertion of the Theorem follows essentially from the same arguments proving the second assertion. We assume therefore that the strategy γ is complete and prove the existence of a set G satisfying (189) (and of the corresponding trajectories satisfying (z1)–(z4)).
Without loss of generality we assume $v = (1, 0)$ and $x_0 = 0$. Observe also that (by the continuity of the multifunction F) there is a $\delta > 0$ such that:

$$B_{\varepsilon/2}((\cos\theta, \sin\theta)) \subset F(x) \qquad \text{if } |x| < \delta \text{ and } |\theta| \leq \delta. \tag{191}$$

By the properties of $\tilde{u}$, we know that $\tilde{u}_t$ is approximately continuous at 0. Let therefore A be a measurable set such that

(AC1) $r^{-2}|B_r \setminus A| \to 0$ for $r \downarrow 0$;

(AC2) $\tilde{u}_t(x) \to \tilde{u}_t(0)$ if $x \in A$ and $x \to 0$.

Next, fix a small positive number $\alpha < \delta$ to be chosen later. For every r consider the arc of circle $\eta_r := \{r(\cos\theta, \sin\theta) : |\theta| \leq \alpha\}$. We denote by H the set of radii r such that $\gamma(t) \cap \eta_r = \emptyset$. By (Z2) it easily follows that

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^1([0,r] \setminus H)}{r} \;=\; 0\,. \tag{192}$$

On the other hand, by Proposition 7.7 we can conclude that, for a.e. $r \in H$:

(G1) $w = \tilde{u}_t|_{\eta_r}$ is Lipschitz;

(G2) the derivative of w at $p \in \eta_r$ is the tangential component of $\nabla u_t(p)$ for $\mathcal{H}^1$–a.e. $p \in \eta_r$;

(G3) $H(p, \nabla u_t(p)) \leq 0$ for $\mathcal{H}^1$–a.e. $p \in \eta_r$.

We define G as the set of elements $r \in H$ which satisfy (G1)–(G3) and which are smaller than a positive constant c_0 (to be chosen later). Then (189) holds. Next, for every $N \in \mathbb{N}$ and any angle $\theta \in]-\alpha, \alpha[$ consider the segment

$$\sigma_{\theta,N} \;:=\; \{\rho(\cos\theta, \sin\theta) : 2^{-(N+2)} \leq \rho \leq 2^{-N}\}\,.$$

We say that (θ, N) is good if

(G4) The conditions corresponding to (G1)–(G3) are satisfied for $\tilde{u}|_{\sigma_{\theta,N}}$;

(G5) There is a $\rho = \rho(N,\theta)$ between $\frac{3}{8}2^{-N}$ and 2^{-N-1} such that

$$\rho(N,\theta)(\cos\theta, \sin\theta) \in A\,.$$

Obviously, again by (Z2) and by (AC1), there is a constant c_0 such that, for every N with $2^{-N} \leq c_0$ there always exists an angle θ_N for which (θ_N, N) is good.
It is also easy to conclude that, by possibly choosing c_0 smaller, there is always a radius $r_N \in]2^{-(N+2)}, \frac{3}{8}2^{-N}[$ belonging to H. Assume therefore that $\mu \in G$. Let N_0 be the largest natural number such that $2^{-N_0} \geq \mu$. We construct a piecewise smooth curve joining $\mu(1,0)$ and $(0,0)$ as follows.

- We first let p_0 be the intersection of $\sigma_{\theta_{N_0},N_0}$ with the arc η_μ and we let ψ_0 be the arc contained in η_μ joining $\mu(1,0)$ and p_0.
- We then let $q_0 := \sigma_{\theta_{N_0},N_0} \cap \eta_{r_{N_0}}$ and denote by σ_0 the segment with endpoints p_0 and q_0;
- We let $p_1 := \sigma_{\theta_{N_0+1},N_0+1} \cap \eta_{r_{N_0}}$ and let ψ_1 be the arc contained in $\eta_{r_{N_0}}$ joining q_0 and p_1.

We proceed inductively. The trajectory consists of infinitely many radial segments σ_i and of infinitely many arcs ψ_i. We call their union Ψ. The sum the lengths of σ_i is exactly μ. The sum of the lengths of ψ_i is bounded from above by $C\alpha\mu$, where C is a geometric constant independent of α and μ. We can go at all speeds up to $1+\varepsilon/2$ along the segments σ_i (by (191)) and at all speeds up to λ along the arcs ψ_i (by (H3)).

Therefore, it is surely possible to go along the trajectory Ψ with a map $z : [\tau, \tau + \mu] \to \Psi$ satisfying (z1) and (z2) if the following inequality holds:

$$\mu \left(1 + \frac{\varepsilon}{2}\right)^{-1} + C\alpha\frac{\mu}{\lambda} \le \mu\,.$$

However, this is certainly the case if α is chosen sufficiently small. Next, since $\Psi \cap \gamma(t) = \emptyset$, z obviously satisfies (z3).
Now, the function $w = \tilde{u}_t \circ z$ is obviously locally Lipschitz on $]\tau, \tau + \mu]$ because of (G1)–(G4). Moreover, (190) is satisfied, and therefore the Lipschitz constant of w on any interval $[\tau + \nu, \tau + \mu]$ is bounded by a constant C independent of $\nu > 0$ (recall indeed that, by (H3), if $H(x, p) \le 0$, then $|p| \le \lambda^{-1}$). This means that w extends to a continuous function $\tilde{w}$ on $[\tau, \tau + \mu]$ and, in order to conclude the proof, it suffices to check that $\tilde{w}(\tau) = w(\tau)$. Note that by our construction, the points $\rho(i, \theta_i)(\cos\theta_i, \sin\theta_i)$ belong to the trajectory Ψ and they are hence equal to $z(\tau_i)$ for some sequence $\tau_i \downarrow \tau$. But then $z(\tau_i) \in A$, and by (AC2), we have that $w(\tau_i) = \tilde{u}_t(z(\tau_i))$ converges to $\tilde{u}_t(0) = w(\tau)$. This completes the proof. □

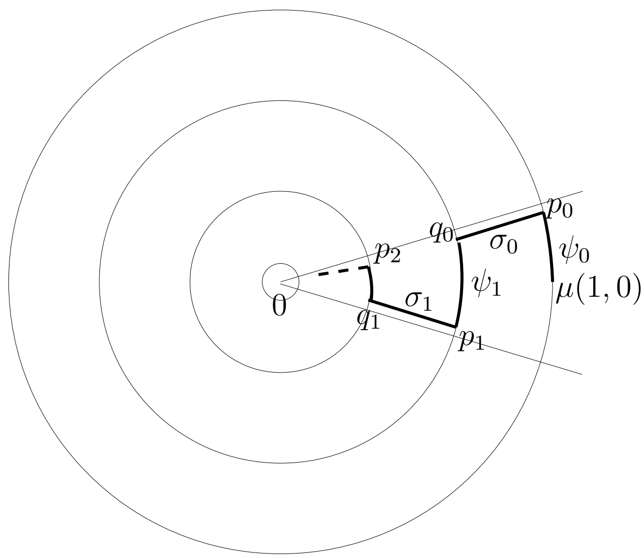

Figure 3: The zig-zag curve constructed in the proof of Lemma 7.8.

7.3.2 Faster trajectories

The last technical tool needed comes again from an idea of [12] (cp. to Lemma 7.1 therein). The obvious proof is left to the reader.

Lemma 7.9 (Faster trajectory)**.** *Let $x : [0, T] \to \mathbb{R}^2$ be an admissible trajectory, i.e.:*

- $\dot{x}(t) \in F(x(t))$ *for a.e. t;*
- $x(t) \notin \gamma(t)$ *for every t;*
- $x(0) \in R_0$.

Let $0 < \varepsilon < \delta$ and consider the trajectory $x^\sharp : [0, T - \varepsilon] \to \mathbb{R}^2$ given by

$$x^\sharp(t) \;=\; x\left(\frac{T}{T+\delta+\varepsilon}(t+\delta+2\varepsilon)\right) .$$

For δ and ε appropriately small, we have

- $B_{2\varepsilon}(\dot{x}^\sharp(t)) \subset F(x^\sharp(t))$ *for a.e. t;*
- $x^\sharp(t) \notin \gamma(t+\varepsilon)$ *for every t;*
- $x^\sharp(0) \in R_0$.

7.4 Proof of Theorem 7.5: Part I

In this section we prove that T^γ belongs to $\mathcal{S}^\gamma$ under the only assumption that γ is an admissible strategy. Thus we have to show that T^γ satisfies the requirements (a) and (b) of Definition 7.4.

7.4.1 Condition (a)

Obviously $T^\gamma \equiv 0$ on R_0.

Step 1 We fix $t > 0$ and start by showing that T^γ_t belongs to SBV_{loc}. For an arbitrary $x \in \mathbb{R}$, we set $l_x := \{(x, y) : y \in \mathbb{R}\}$ and $l_{x,\gamma} := l_x \cap \gamma(t)$. We claim that

(Cl) T^γ_t is locally Lipschitz on the interior of $l_x \setminus l_{x,\gamma}$, with Lipschitz constant smaller than λ^{-1} (where λ is the constant in (H3)).

We will prove this claim later. Obviously the same proof gives the following symmetric statement, where $l'_y := \{(x, y) : x \in \mathbb{R}\}$ and $l'_{y,\gamma} = l'_y \cap \gamma(t)$:

(Cl') T^γ_t is locally Lipschitz on the interior of $l'_y \setminus l'_{y,\gamma}$ with constant smaller than λ^{-1}.

First of all, (Cl) and (Cl') imply the measurability of T^γ_t. Indeed, recall that γ is rectifiable and hence Borel measurable. Therefore, for every fixed integer $j > 0$ it is possible to find a closed set $\Gamma^j \subset \gamma(t)$ such that $\mathcal{H}^1(\gamma(t) \setminus \Gamma^j) < \frac{1}{j}$. Let $V_j, H_j \subset \mathbb{R}$ be the projections of the set $\gamma(t) \setminus \Gamma^j$ respectively on the horizontal and the vertical axis. (Cl) and (Cl') imply that T^γ_t is locally Lipschitz on

$$C_j := [((\mathbb{R} \setminus H_j) \times \mathbb{R}) \cap (\mathbb{R} \times (\mathbb{R} \setminus V_j))] \setminus \Gamma^j .$$

Indeed, fix $(x_1, y_1) \in C_j$. Since Γ^j is closed, there is a ball B centered at (x_1, y_1) such that $B \cap \Gamma^j = \emptyset$. Consider any other point $(x_2, y_2) \in B$ and let σ and η be the segments joining, respectively, (x_1, y_1) with (x_1, y_2) and (x_1, y_2) with (x_2, y_2). Since $x_1 \notin H_j$ and $y_2 \notin V_j$, the intersections $\eta \cap \gamma(t)$ and $\sigma \cap \gamma(t)$ must be contained in Γ^j. On the other hand the segments σ and η are also contained in B and thus we conclude that $\eta \cap \gamma(t) = \sigma \cap \gamma(t) = \emptyset$. Therefore (Cl) and (Cl') imply that

$$|T^\gamma_t(x_1, y_1) - T^\gamma_t(x_2, y_2)| \le \frac{|x_1 - x_2| + |y_1 - y_2|}{\lambda}$$

Observe next that $\mathcal{L}^1(H_j) + \mathcal{L}^1(V_j) < 2/j$. Thus, $\mathbb{R}^2 \setminus \bigcup C_j$ has zero Lebesgue measure and, having concluded that T^γ_t is locally Lipschitz on each set C_j, we infer that T^γ_t is measurable.

Note that, if $l_{x,\gamma}$ is finite, (Cl) clearly implies that the restriction $T_t^\gamma|_{l_x}$ is an SBV function with finitely many jumps. On the other hand we have the coarea formula

$$\int \sharp(l_{x,\gamma})\, dx \ \leq\ \mathcal{H}^1(\gamma(t)) < \infty\,, \tag{193}$$

which implies that $(l_{x,\gamma})$ is finite for a.e. x. Since $0 \leq T_t^\gamma \leq t$, each jump has size at most t and we therefore bound

$$\int_{-R}^{R} \|\tfrac{d}{dy} T_t^\gamma(x,\cdot)\|_{TV(]-R,R[)}\, dx \ \leq\ \int_{-R}^{R} (\lambda^{-1} + t\,\sharp(l_{x,\gamma}))\, dx \ \overset{(193)}{<}\ +\infty\,. \tag{194}$$

The same argument applies if we fix the y coordinate and let x vary. We can therefore apply Theorem 1.14 to conclude that $T_t^\gamma \in SBV(]-R,R[^2)$ for every positive R. This shows that $T_t^\gamma \in SBV_{loc}$.

We now come to the proof of (Cl). We fix $Y = (x,y) \in l_x \setminus l_{x,\gamma}$ and distinguish two cases:

Case 1: $\tau := T_t^\gamma(x,y) < t$. In this case $T_t^\gamma(x,y) = T^\gamma(Y)$. We fix $\varepsilon < \frac{t-\tau}{2}$ and

$$\delta < \min\left\{\varepsilon, \lambda^{-1}\mathrm{dist}\left((x,y), l_{x,\gamma}\right)\right\}\,.$$

Let $Z = (x,z)$. When $|z-y| < \delta$ we consider the path $\varphi : [0, \lambda^{-1}|z-y|] \to \mathbb{R}^2$ given by

$$\varphi(s) \ =\ \left(x, y + \frac{z-y}{|z-y|}\lambda s\right) = Y + \frac{Z-Y}{|Z-Y|}\lambda s.$$

It is easy to see that $\dot\varphi \in F(\varphi)$ (because of (H3)) and that $\varphi(s) \notin \gamma(t)$. On the other hand, if T is a given time in $]\tau, \tau+\varepsilon[$, there is an admissible path $\psi : [0,T] \to \mathbb{R}^2$ which starts from a point $\psi(0) \in R_0$ and reaches $Y = (x,y)$. If we join the paths ψ and φ in the obvious way, then we obtain an admissible path which reaches $Z = (x,z)$ at a time $T + \lambda^{-1}|z-y|$. Since T can be chosen arbitrarily close to $\tau = T^\gamma(x,y)$, we conclude

$$T^\gamma(x,z) \ \leq\ T^\gamma(x,y) + \frac{1}{\lambda}|z-y|\,. \tag{195}$$

On the other hand, a symmetric argument shows

$$T^\gamma(x,z) \ \geq\ T^\gamma(x,y) - \frac{1}{\lambda}|z-y|\,, \tag{196}$$

which therefore completes the proof of the claim.

Case 2: $T^\gamma(x,y) \geq t$. In this case $T_t^\gamma(x,y) = t$ and, since $T_t^\gamma \leq t$, it suffices to show

$$T^\gamma(x,z) \ \geq\ t - \lambda^{-1}|z-y| \tag{197}$$

for any z sufficiently close to y. On the other hand, if (197) were false for a sufficiently close z, we could argue as in (195) reversing the roles of z and y and finding

$$T^\gamma(x,y) \ \leq\ T^\gamma(x,z) + \lambda^{-1}|z-y| \ <\ t\,,$$

which contradicts our assumption $T^\gamma(x,y) \geq t$.

Step 2 To complete the proof that (a) in Definition 7.4 is satisfied, we must show that the jump set J of T_t^γ is contained in $\gamma(t)$. Let A be the set of x's such that $\sharp l_{x,\gamma} < \infty$ and B the set of y's for which $\sharp l'_{y,\gamma} < \infty$. In the previous subsection we have shown that $\mathcal{L}^1(\mathbb{R}\setminus A) = 0$ and that for any $x \in A$ the jump set J_x of $T^\gamma_t|_{l_x}$ is contained in $\gamma(t)$. By Theorem 1.14, there is a further set $A' \subset A$ with $\mathcal{L}^1(A\setminus A') = 0$ such that $J_x = J \cap l_x$ for every $x \in A'$. We thus conclude that $J\cap(A'\times\mathbb{R}) \subset \gamma(t)$ and $\mathcal{L}^1(\mathbb{R}\setminus A') = 0$. Arguing similarly for the y coordinates, we conclude the existence of a set B' with $\mathcal{L}^1(\mathbb{R}\setminus B') = 0$ such that

$$J \;\subset\; \gamma(t) \cup \Big(\big((\mathbb{R}\setminus A')\times\mathbb{R}\big) \cap \big(\mathbb{R}\times(\mathbb{R}\setminus B')\big)\Big)\,. \tag{198}$$

On the other hand $\Big(\big((\mathbb{R}\setminus A')\times\mathbb{R}\big) \cap \big(\mathbb{R}\times(\mathbb{R}\setminus B')\big)\Big) = (\mathbb{R}\setminus A')\times(\mathbb{R}\setminus B')$. But, since J is a 1-d rectifiable set, $\mathcal{H}^1(J_{T^\gamma_t} \cap ((\mathbb{R}\setminus A')\times(\mathbb{R}\setminus B')) = 0$.

7.4.2 Condition (b)

We start by observing that (182) holds a.e. on $\{T^\gamma_t = t\}$. Indeed, if this set has measure zero, then there is nothing to prove. Otherwise, using Theorem 1.13 and the Lebesgue Theorem it is easy to show that $\nabla T^\gamma_t = 0$ a.e. on $\{T^\gamma_t = t\}$. Since (H3) implies that $H(X,0) < 0$ for every X, this proves our claim. The same observation shows that (182) holds at every $X \in R_0$.
We fix next a point X such that

- $T^\gamma(X) = T^\gamma_t(X) < t$;
- T^γ_t is approximately differentiable with differential $\nabla T^\gamma_t(X)$;
- $X \notin \overline{R}_0$ and

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(\gamma(t)\cap B_r(X))}{r} \;=\; 0\,. \tag{199}$$

Clearly, a.e. $X \in \mathbb{R}^2 \setminus (R_0 \cup \{T^\gamma_t = t\})$ satisfies these requirements. Our aim is to show

$$\nabla T^\gamma_t(X)\cdot w \le 1, \qquad \text{for every } w \in \overset{\circ}{F}(X)\,. \tag{200}$$

From this easily follows that:

$$H(X, \nabla T^\gamma_t(X)) \;=\; \sup_{w\in \overset{\circ}{F}(X)} \nabla T^\gamma_t(X)\cdot w - 1 \le 0.$$

We now show (200) and fix, therefore, $w \in \overset{\circ}{F}(X)$. Choose $\varepsilon \in]0,1/2[$ so that $B_{2\varepsilon}(w) \subset F(X)$ and $T^\gamma(X)+2\varepsilon < t$. Apply Lemma 7.8 with $x_0 = X$, t, ε and $u = T^\gamma$. Let $\tau \in]T^\gamma(X), T^\gamma(X)+\varepsilon[$ and v a vector in $B_\varepsilon(w)$. G is the set given by Lemma 7.8. If μ is such that $\mu|v| \in G$ and $\mu < \varepsilon$, let $z : [\tau, \tau+\mu]$ be the trajectory given by the first assertion of Lemma 7.8. Since $\tau \in]T^\gamma(X), T^\gamma(X)+\varepsilon[$, there exists a trajectory $x : [0,\tau] \to \mathbb{R}^2$ such that

- $x(0) \in R_0$, $x(\tau) = X$;
- $\dot{x}(s) \in F(x(s))$ for a.e. s;
- $x(s) \notin \gamma(s)$ for every s.

Obviously, if we extend x to $[0,\tau+\mu]$ by setting $x(s)=z(s)$ for $s\in[\tau,\tau+\mu]$, x continues to enjoy the same properties. This implies that $T^\gamma(X+\mu v)<\tau+\mu$. Let now τ converge to $T^\gamma(X)$ to conclude

$$T^\gamma_t(X+\mu v)\leq T^\gamma(X+\mu v)\leq T^\gamma(X)+\mu=T^\gamma_t(X)+\mu\,.$$

Since T^γ_t is approximately differentiable at X, we find a set B satisfying (i) and (ii) of Theorem 1.13. Clearly, for every $\eta>0$, there are $\mu<\eta$ and $v\in B_\varepsilon(w)$ such that $X+\mu v\in B$ and $\mu|v|\in G$.
We thus conclude that, for every $\varepsilon>0$ and $\kappa>0$, we find $\mu<\varepsilon$ and $v\in B_\varepsilon(w)$ such that

$$\nabla T^\gamma_t(X)\cdot v\;\leq\;\frac{T^\gamma_t(X+\mu v)-T^\gamma_t(X)}{\mu}+\kappa\;\leq\;1+\kappa\,.$$

We thus can estimate

$$\begin{aligned}\nabla T^\gamma_t(X)\cdot w\;&\leq\;\nabla T^\gamma_t(X)\cdot v+|\nabla T^\gamma_t(X)||w-v|\\&\leq\;|\nabla T^\gamma_t(X)|\varepsilon+1+\kappa\,.\end{aligned}\tag{201}$$

Letting κ and ε go to 0 we conclude

$$\nabla T^\gamma_t(X)\cdot w\leq 1\,.$$

7.5 Proof of Theorem 7.5: Part II

In this section we prove the second part of Theorem 7.5. We first claim that $\mathcal{S}^\gamma=\mathcal{S}^{\gamma^c}$. The inclusion $\mathcal{S}^\gamma\subset\mathcal{S}^{\gamma^c}$ is obvious. In order to show the opposite inclusion, recall that there is a countable set C of t's such that $\mathcal{H}^1(\gamma^c(t)\setminus\gamma(t))=0$ for every $t\notin C$. Thus, let $u\in\mathcal{S}^{\gamma^c}$. The only thing we need to show is that $\mathcal{H}^1(J_{u_t}\setminus\gamma(t))=0$ for $t\in C$, since for $t\notin C$ this identity is trivial. Fix therefore a $t\in C$ and a point x in J_{u_t}. Let $u^-_t(x)$ and $u^+_t(x)$ be the left and right approximate values of u_t at x, according to Proposition 1.17. To fix ideas, assume $u^+_t(x)>u^-_t(x)$ (recall that the two values are necessarily different!). Then, for $\tau>u^-_t(x)$, we obviously conclude that x is not a point of approximate continuity for τ. Choose a sequence $\{\tau_i\}\subset\mathbb{R}\setminus C$ with $\tau_i\uparrow t$. According to Proposition 1.17, our argument shows

$$\mathcal{H}^1\left(J_{u_t}\setminus\bigcup_i J_{u_{\tau_i}}\right)\;=\;0\,.$$

On the other hand $\mathcal{H}^1(J_{u_{\tau_i}}\setminus\gamma^c(\tau_i))=0$, $\mathcal{H}^1(\gamma^c(\tau_i)\setminus\gamma(\tau_i))=0$ and $\gamma(\tau_i)\subset\gamma(t)$. Therefore we conclude $\mathcal{H}^1(J_{u_t}\setminus\gamma(t))=0$.

Having proved that $\mathcal{S}^\gamma=\mathcal{S}^{\gamma^c}$, we can assume that γ itself is a complete strategy and aim at proving that T^γ is the maximal element of $\mathcal{S}^\gamma$. Thus we consider an arbitrary $u\in\mathcal{S}^\gamma$ and, to simplify the notation, we assume that $u=\tilde u$, where $\tilde u$ is the function of Proposition 7.7. Our goal is to show that $u\leq T^\gamma$ a.e.. This condition is obvious on R_0 and on the set $\{T^\gamma=+\infty\}$. Thus, we can assume that

- $X\notin\overline{R}_0$, $X\notin\gamma_\infty$, u is approximately continuous at X and $T^\gamma(X)<\infty$.

We fix therefore such a point X and we will show that, for every positive ε, $u(X)\leq T^\gamma(X)+\varepsilon$.

Using Lemma 7.9 we can assume that, for some positive $T<T^\gamma(X)+\varepsilon$ and some $\delta>0$, there exists a trajectory $x:[0,T]\to\mathbb{R}^2$ such that

- $x(0) \in R_0$;
- $B_{2\delta}(\dot{x}(t)) \subset F(x(t))$ for a.e. t;
- $x(t) \notin \gamma(t+\delta)$ for every t;
- $x(T) = X$.

We next define a set $\mathcal{P} \subset [0,T]$: s belongs to $\mathcal{P}$ if and only if there is a trajectory $y : [0,s] \to \mathbb{R}^2$ with the following properties:

(P1) $y(0) = x(0)$ and $y(s) = x(s)$;

(P2) $\dot{y}(\sigma) \in F(y(\sigma))$ for a.e. σ;

(P3) $w := u_{T+\delta} \circ y$ is Lipschitz and for a.e. σ we have

$$\text{either } \dot{w}(\sigma) = 0 \text{ or } \left\{ \begin{array}{l} u_{T+\delta} \text{ is approximately differentiable at } y(\sigma) \\ \dot{w}(\sigma) \;=\; \nabla u_{T+\delta}(y(\sigma)) \cdot \dot{y}(\sigma) \\ H(y(\sigma), \nabla u_{T+\delta}(y(\sigma))) \;\leq\; 0 \end{array} \right\}. \tag{202}$$

We will show below that:

- $\mathcal{P}$ has a maximal element;
- the maximal element of $\mathcal{P}$ is necessarily T.

We assume, for the moment, these two facts and conclude our proof. Since $T \in \mathcal{P}$, there is a trajectory $y : [0,T] \to \mathbb{R}^2$ satisfying (P1)–(P3). Note that, in a neighborhood of 0, the trajectory y takes values in R_0, where $u_{T+\delta}$ vanishes identically. Hence $w(0) = 0$. Moreover, for a.e. σ, either $\dot{w}(\sigma) = 0$ or

$$\begin{aligned} \dot{w}(\sigma) \;&=\; \nabla u_{T+\delta}(y(\sigma)) \cdot \dot{y}(\sigma) \;\leq\; \sup_{v \in F(y(\sigma))} \nabla u_{T+\delta}(y(\sigma)) \cdot v \\ &=\; 1 + H(y(\sigma), \nabla u_{T+\delta}(y(\sigma))) \;\leq\; 1\,. \end{aligned} \tag{203}$$

Therefore we conclude

$$u_{T+\delta}(X) \;=\; w(T) \;=\; \int_0^T \dot{w}(\tau)\,d\tau \;\leq\; T\,. \tag{204}$$

But this implies $u(X) = u_{T+\delta}(X) < T^{\gamma}(X) + \varepsilon$, which is the desired conclusion.

Step 1. $\mathcal{P}$ has a maximal element.
Let $S := \sup \mathcal{P}$. If $x(S) = x(0)$, then the assertion is trivial. Therefore, without loss of generality, we assume $X := x(S) \neq x(0)$. We let $\{s_i\}$ be a sequence in $\mathcal{P}$ converging to S and we denote by y_i the corresponding trajectories satisfying the conditions (P1)–(P3). The idea is that, for i sufficiently large, we will be able to prolong the trajectory to reach X. This will be done by adding a zig-zag curve to a portion of y_i.
Next, we set

$$a_i \;:=\; \frac{x(S) - x(s_i)}{S - s_i}$$

and, passing to a subsequence, we assume that a_i converges to some point. We set a equal to this limit if it is different from 0 (we call this the *principal case*). If not, we distinguish two possibilities. If $x(s_i) = x(S)$ for some i, then we trivially have $S \in \mathcal{P}$. Indeed, it suffices to put $y(\tau) = y_i(\tau)$ for $\tau \le s_i$ and $y(\tau) = x(s_i) = x(S)$ for $\tau \in [s_i, S]$ to get a trajectory y satisfying (P1), (P2) and (P3). Otherwise, we can assume (passing to a subsequence) that

$$\frac{x(S) - x(s_i)}{|x(S) - x(s_i)|}$$

converges to some limit $\tilde{a}$ with $|\tilde{a}| = 1$. In this case we set $a := \lambda\tilde{a}/2$ and we call it *secondary case*. It will be clear from the proof below that this situation is just a variant of the principal case. We therefore assume that $a \neq 0$ is the limit of the a_i and leave to the reader the obvious modifications for the secondary case.

Note that, by our assumptions on F, it follows easily that $B_{2\delta}(a) \subset F(x(S))$. Next choose $v = (1+\kappa)a$, where κ is a positive constant, chosen so that $B_\delta(v) \subset F(x(S))$. To fix ideas, assume $a = (1,0)$ and $x(S) = 0$. Fix moreover $\alpha > 0$ (to be chosen later), set $\tau_i = S - s_i$ and consider, for every i and for every $\beta \in]\alpha/2, \alpha[$ the set $Q_{i,\beta}$ delimited by

- the segments
$$d^+ \ = \ [\tau_i(1-\beta)(\cos\beta, \sin\beta), \tau_i(1+\beta)(\cos\beta, \sin\beta)]$$
and
$$d^- \ = \ [\tau_i(1-\beta)(\cos\beta, -\sin\beta), \tau_i(1+\beta)(\cos\beta, -\sin\beta)]\,;$$

- the arcs ar^- and ar^+ with radii, respectively, $\tau_i(1-\beta)$ and $\tau_i(1+\beta)$ and delimited, respectively, by the pair of points
$$\tau_i(1-\beta)(\cos\beta, -\sin\beta) \qquad \tau_i(1-\beta)(\cos\beta, \sin\beta)$$
and by the pair of points
$$\tau_i(1+\beta)(\cos\beta, -\sin\beta) \qquad \tau_i(1+\beta)(\cos\beta, \sin\beta)\,.$$

See Figure 4.

Observe that 0, u and $\tau = S$ satisfy the assumptions of the Lemma 7.8 if we choose $t = S + \delta$. Let, therefore G be the set of radii given by the Lemma. We want, for i sufficiently large, choose a β such that the following conditions hold:

(a) $\tau_i(1-\beta)|a| = \tau_i(1-\beta)(1+\kappa)^{-1}|v|$ belongs to G, so that there exists a trajectory as in Lemma 7.8;

(b) The restriction on $\partial Q_{i,\beta}$ of the function u_t is a Lipschitz function ζ;

(c) u_t is approximately differentiable at $\mathcal{H}^1$–a.e. point $x \in \partial Q_{i,\beta}$, and satisfies $H(x, \nabla u_t(x)) \le 0$;

(d) The derivative of ζ corresponds, $\mathcal{H}^1$–a.e. on $x \in \partial Q_{i,\beta}$, to the tangential component of ∇u_t.

According to Proposition 7.7, the last three conditions are satisfied for a.e. β such that $\partial Q_{i,\beta} \cap \gamma(t) = \emptyset$. Since

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(B_r(0) \cap \gamma(t))}{r} \ = \ 0$$

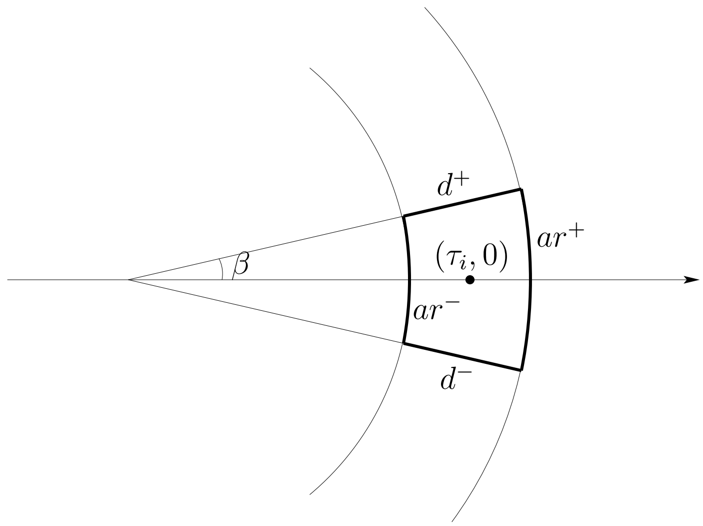

Figure 4: The set $Q_{i,\beta}$

and

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^1(G\cap[0,r])}{r} \;=\; 0$$

the existence of such a β is guaranteed if i is sufficiently large.
Now, we choose such a $\beta = \beta(i)$ for every i and set $Q^i := Q_{i,\beta(i)}$. Note that $y_i(s_i) \in Q^i$ if i is large enough. Moreover, since $y_i(0) = x(0)$ and $x(0) \neq 0$, we have $y_i(0) \notin Q^i$, for any i large enough. Thus, for large i's, there is a $\tilde{s}_i < s_i$ such that $y_i(\tilde{s}_i) \in \partial Q^i$. Now we let $z : [S-\tau_i(1-\beta)(1+\kappa)^{-1}, S] \to \mathbb{R}^2$ be the trajectory given by the last assertion of Lemma 7.8, which is joining the points $z(S-\tau_i(1-\beta)(1+\kappa)^{-1}) = x(S)-\tau_i(1-\beta)(1,0)$ and $0 = x(S)$. Note that the first point belongs to ∂Q^i.
Next, observe that the perimeter of Q^i can be bounded by $10\tau_i\beta$. If α is chosen sufficiently small, the number

$$\omega := S-\tau_i(1-\beta)(1+\kappa)^{-1}-\tilde{s}_i$$

is larger than $5\beta\tau_i/\lambda$. Indeed, we have the inequalities

$$5\beta\tau_i\lambda^{-1} \leq 5\alpha\tau_i\lambda^{-1}$$

$$\omega \;\geq\; S-\tau_i(1-\alpha)(1+\kappa)^{-1}-s_i \;=\; \tau_i\left[1-(1-\alpha)(1+\kappa)^{-1}\right]\,.$$

Hence the inequality $\omega \geq 5\beta\tau_i\lambda^{-1}$ holds whenever

$$\frac{\kappa+\alpha}{1+\kappa} \geq \frac{5\alpha}{\lambda}\,.$$

Thus, the choice of α depends only on κ and λ, which were fixed a priori.
Having chosen α accordingly small, we can find a trajectory

$$\varphi : [\tilde{s}_i, S-\tau_i(1-\beta)(1+\kappa)^{-1}] \quad\to\quad \partial Q^i$$

which joins $\varphi(\tilde{s}_i) = y_i(\tilde{s}_i)$ and

$$\varphi(S-\tau_i(1-\beta)(1+\kappa)^{-1}) = z(S-\tau_i(1-\beta)(1+\kappa)^{-1})$$

and satisfies $\dot\varphi(\sigma) \in F(\varphi(\sigma))$ for every σ.
We join z and φ into a single trajectory z on $[\tilde s_i, S]$, for which we have the following conclusions:

- $w = u_t \circ z$ is Lipschitz;
- for a.e. σ, either $\dot z(\sigma) = 0$ or u_t is approximately differentiable at $z(\sigma)$ and the approximate differential satisfies $H(z(\sigma), \nabla u_t(z(\sigma))) \le 0$;
- for a.e. σ, either $\dot w(\sigma) = 0$ or $\frac{d}{d\sigma} u_t \circ z(\sigma) = \nabla u_t(z(\sigma)) \cdot \dot z(\sigma)$ for a.e..

Next join the trajectory $y_i|_{[0,\tilde s_i]}$ to the trajectory z in order to build a new trajectory y. We claim that y satisfies the requirements (P1)–(P3), thus showing that $S \in \mathcal{P}$. Indeed, y satisfies all the requirements with $u_t = u_{S+\delta}$ in place of $u_{T+\delta}$. Thus, the computations (203) and (204) are still valid if we replace T with S and we infer $u_{S+\delta}(y(\sigma)) \le \sigma \le S < S + \delta$ for every σ. Therefore, the properties (P1)–(P3) with the desired value $T \ge S$ can be easily inferred from the following facts, which are easy consequences of the definitions of approximate differentiability and approximate continuity. Assume $a \in \mathbb{R}$ and $u_a(x) < a$. Then

- If u_a is approximately continuous at x, so is any u_b with $b > a$;
- If u_a is approximately differentiable at x, so is any u_b with $b > a$ and the corresponding approximate differentials coincide.

This completes the proof that $S \in \mathcal{P}$.

Step 2. The maximal element of $\mathcal{P}$ is T. Let S be the maximal element. Then, it is obvious that $x(s) \neq x(S)$ for every $s > S$. In particular, if $S < T$, we must have $x(T) \neq x(S)$. Assume by contradiction that $S < T$ and, for $s > S$, consider the vectors

$$v(s) \;:=\; \frac{x(s) - x(S)}{s - S}\,.$$

Recall that $B_{2\delta}(\dot x(\sigma)) \in F(x(\sigma))$. By our assumptions on the multifunction F, it follows easily that $B_\delta(x(s)) \subset F(x(S))$ provided s is sufficiently close to S. Therefore, we can apply Lemma 7.8. Given the set of radii G, it follows that, for any $\varepsilon > 0$, there is $0 < s < S + \varepsilon$ with $|s - S||v(s)| \in G$. We can therefore construct a zig-zag curve $z : [S, s] \to \mathbb{R}^2$ satisfying the assumptions of the Lemma with $t = S + \delta$, with $z(S) = x(S)$ and $z(s) = z(S) + (s - S)v(s) = x(s)$. Now, since $S \in \mathcal{P}$, there is a trajectory $y : [0, S] \to \mathbb{R}^2$ satisfying (P1), (P2) and (P3) with $y(S) = x(S)$. On the other hand, joining z and y into one single trajectory $\tilde y$, we can argue as in the previous step to conclude that $\tilde y : [0, s] \to \mathbb{R}^2$ satisfies (P1), (P2) and (P3). Since $\tilde y(s) = x(s)$, this implies that $s \in \mathcal{P}$, thus contradicting the maximality of S.

7.6 Proof of Corollary 7.6

Let $\{\gamma^k\}$ be a minimizing sequence of admissible strategies for the functional J. Consider the completions η^k of γ^k. Then, $R_\infty^{\gamma^k} \supset R_\infty^{\eta^k}$ (because, by Theorem 7.5 $T^{\gamma^k} \le T^{\eta^k}$). Moreover, $\mathcal{H}^1(\eta^k_\infty \setminus \gamma^k_\infty) = 0$. Thus, we conclude $J(\gamma^k) \ge J(\eta^k)$. Therefore, without loss of generality we can assume that the minimizing sequence of strategies $\{\gamma^k\}$ consists of complete strategies.
Consider the corresponding maximum time functions $T^k := T^{\gamma^k}$. Note that the functions T^k belong to the space of functions $GSBV$ (see Section 4.5 of [2]; this space is just a variant of the space of SBV functions introduced by Ambrosio and De Giorgi). Note also that $|DT^k_t| \le \lambda^{-1}\mathcal{L}^2 + t\mathcal{H}^1 \llcorner \gamma(t)$. This uniform bound allows to apply the compactness theorem for GSBV

functions (see Theorem 4.36 of [2]), which is just a variant of the SBV compactness Theorem of Ambrosio and De Giorgi. Hence, after passing to a subsequence, we can assume that T^k converges pointwise a.e. to a function u satisfying the following properties:

(a) u_t is an SBV function for every t;

(b) J_{u_t} is a rectifiable set and

$$\int_{J_{u_t}} \psi \, d\mathcal{H}^1 \;\leq\; \liminf_k \int_{J_{T_t^k}} \psi \, d\mathcal{H}^1 \;\leq\; t$$

(see Theorem 5.22 of [2]);

(c) ∇T_t^k converges weakly, in every L^p with $p < \infty$, to ∇u_t (see Corollary 5.31 of [2]).

For each t, denote by $\gamma(t)$ the set of points where the precise representative of u_t is not approximately continuous. It is not difficult to see that $\gamma(t) \subset \gamma(s)$ for every $s > t$. Moreover, by Proposition 1.17, $\mathcal{H}^1(\gamma(t) \setminus J_{u_t}) = 0$. It follows, therefore, from (b) that $\gamma(t)$ satisfies (H2) and, hence, it is an admissible strategy.
Note next that, H is a continuous function and that $H(x, \cdot)$ is convex for every x. Then, the property $H(x, \nabla T_t^k(x)) \leq 0$ for a.e. x implies, by (c), $H(x, \nabla u_t(x)) \leq 0$ for a.e. x. Thus, $u \in \mathcal{S}^\gamma$. So, if we consider the completion γ^c of γ, we conclude $T^{\gamma^c} \geq u$.
Since T^k converges pointwise a.e. to u, we conclude that

$$\mathbf{1}_{\{u<\infty\}}(x) \leq \liminf_{k\uparrow\infty} \mathbf{1}_{\{T^k<\infty\}}(x) \qquad \text{for a.e. } x.$$

Thus, recall that $\alpha \geq 0$ and use Fatou's Lemma to conclude

$$\begin{aligned} \int_{R_\infty^{\gamma^c}} \alpha \, d\mathcal{L}^2 \;&=\; \int_{\{T^{\gamma^c}<\infty\}} \alpha \, d\mathcal{L}^2 \;\leq\; \int_{\{u<\infty\}} \alpha \, d\mathcal{L}^2 \\ &\leq\; \liminf_{k\uparrow\infty} \int_{\{T^k<\infty\}} \alpha \, d\mathcal{L}^2 \;=\; \liminf_{k\uparrow\infty} \int_{R_\infty^{\gamma^k}} \alpha \, d\mathcal{L}^2 \,. \end{aligned} \tag{205}$$

On the other hand, by the Semicontinuity Theorem for SBV functions (see again Theorem 5.22 of [2]),

$$\int_{J_{u_t}} \beta \, d\mathcal{H}^1 \;\leq\; \liminf_{k\uparrow\infty} \int_{J_{T_t^k}} \beta \, d\mathcal{H}^1 \;\leq\; \liminf_{k\uparrow\infty} \int_{\gamma_\infty^k} \beta \, d\mathcal{H}^1 \,.$$

Since

$$\int_{\gamma_\infty^c} \beta \, d\mathcal{H}^1 \;=\; \sup_{t<\infty} \int_{J_{u_t}} \beta \, d\mathcal{H}^1 \,,$$

we conclude that

$$\int_{\gamma_\infty^c} \beta \, d\mathcal{H}^1 \;\leq\; \liminf_{k\uparrow\infty} \int_{\gamma_\infty^k} \beta \, d\mathcal{H}^1 \,. \tag{206}$$

From (205) and (206) it follows trivially that $J(\gamma^c) \leq \liminf_k J(\gamma^k)$. Hence, γ^c is the desired minimizer.

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