Vectorial Problems in the Calculus of Variations and Fluid Dynamics

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ABSTRACT

In this thesis we collect problems that arise in different contexts: PDEs, Calculus of Variations and Fluid Dynamics. We have sorted them in three groups, corresponding to three different parts:

(a) **Part i:** *Differential Inclusions related to Geometric Problems.* This chapter contains some partial results towards the proof of Allard's celebrated regularity theorem [1] for varifolds that are stationary for more general functionals than the area functional. We focus in particular on stationary varifolds that are given by graphs. The point of view we adopt here is the one of differential inclusions, that has been successfully exploited in the last twenty years to produce counterexamples using the so-called *convex integration* methods. The results we present seem to exclude the possibility to construct counterexample to regularity through convex integration, at least with the methods available at present;

(b) **Part ii:** *Divergence-Free Matrix Fields in* $\mathrm{Sym}^+(n)$. Here we study some properties of divergence-free matrix fields from the n dimensional torus with values in the non-negative symmetric matrices, thus continuing the study started in [76];

(c) **Part iii:** *Sharp Energy Regularity for Euler Equations.* Using convex integration methods introduced in the last years by De Lellis and Székelyhidi, we prove that if $\theta < \frac{1}{3}$, $\frac{2\theta}{1-\theta}$ is the optimal Hölder regularity for the energy of C^θ solutions to the incompressible Euler equations on $\mathbb{T}^3$;

The first similarity these problems share is that they are essentially vectorial, in the sense that either their one dimensional counterpart would lose meaning, as in (b)-(c), or the one-dimensional version of the problem has already been solved, as in (a). Other than this immediate similarity, we give a deeper explanation of the connection among the aforementioned problems through the study of the associated Tartar's wave-cone.

ZUSAMMENFASSUNG

In dieser Arbeit führen wir Probleme aus verschiedenen Bereichen zusammen: PDGs, Variationsrechnung und Fluiddynamik. Wir unterteilen sie in drei Gruppen, welche jeweils ein Kapitel der Arbeit ausmachen:

(a) **Part i:** *Differential Inklusionen mit Bezug zu geometrischen Problemen.* Dieses Kapitel enthält einige Teilresultate für die Anpassung des Beweis von Allards berühmtem Regularitätssatz [1] für Varifolds, welche nicht nur für das Flächenfunktional stationär sind, sondern auch für allgemeinere Funktionale. Wir legen den Schwerpunkt insbesondere auf stationäre Varifolds, die durch Graphen beschrieben werden. Wir betrachten das Problem aus der Sicht der Differential Inklusionen. Diese wurden in den letzten zwanzig Jahren erfolgreich benutzt, um Gegenbeispiele zu konstruieren, indem man die sogenannte emphkonvexe Integrationsmethode verwendet hat. Unsere Resultate schliessen die Möglichkeit aus, solche Gegenbeispiele mit den heute verfügbaren Methoden durch konvexe Integration analog zu Allards Beweis zu konstruieren.

(b) **Part ii:** *Divergenzfreie Matrixfelder in* $\mathrm{Sym}^+(n)$*.* Hier untersuchen wir einige Eigenschaften von divergenzfreien Matrixfelder vom n-dimensionalen Torus mit Werten im Raum der postiv semi-definiten symmetrischen Matrizen. Wir führen also die Untersuchung von [76] fort.

(c) **Part iii:** *Optimale Energie Regularität für Euler Gleichungen.* Indem wir die konvexen Integrationsmethoden verwenden, welche De Lellis und Székelyhidi in den letzten Jahren eingeführt haben, beweisen wir, dass, falls θ, dann ist θ die optimale Hölder Regularität der Energie von C^θ-Lösungen der inkompressiblen Euler Gleichungen auf $\mathbb{T}^3$.

Die erste Gemeinsamkeit dieser Probleme ist, dass sie alle im Grunde vektoriell sind. Entweder würde ihr eindimensionales Gegenstück jegliche Bedeutung verlieren (wie in (b)-(c)), oder die eindimensionale Version des Problems wurde bereits gelöst (wie in (a)). Zusätzlich zu dieser unmittelbaren Gemeinsamkeit erklären wir später die Verbindung zwischen den vorher genannten Problemen durch eine Untersuchung der zugehörigen Wellenkegeln von Tartar.

CONTENTS

1

INTRODUCTION

This thesis is divided into three parts:

(a) *Differential Inclusions related to Geometric Problems*;

(b) *Divergence-Free Matrix Fields in* $\mathrm{Sym}^+(n)$;

(c) *Sharp Energy Regularity for Euler Equations*.

We introduce separately the three problems in the next sections, and we conclude the introduction with a section where we give some heuristic explanations on why these results are connected. We have tried to keep technicalities at the minimum, but anyway even the non-technical description of the results of the thesis contained in the next sections require some notation and terminology. In order to help the reader who is not familiar with all of them, these have been collected in Chapter A of the Appendix.

1.1 DIFFERENTIAL INCLUSIONS RELATED TO GEOMETRIC PROBLEMS

Let $\Omega \subset \mathbb{R}^m$ be open and $f \in C^1(\mathbb{R}^{n\times m}, \mathbb{R})$ be a (strictly) polyconvex function, i.e. such that there is a (strictly) convex $g \in C^1$ for which $f(X) = g(\Phi(X))$, where $\Phi(X)$ denotes the vector of subdeterminants of X of all orders. We then consider the following *energy* $\mathbb{E} : \mathrm{Lip}(\Omega, \mathbb{R}^n) \to \mathbb{R}$:

$$\mathbb{E}(u) \doteq \int_\Omega f(Du)dx\,. \tag{1.1}$$

For a map $\bar{u} \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$, the one-parameter family of functions $\bar{u} + \varepsilon v$ will be called *outer variations* and $\bar{u}$ will be called *critical for* $\mathbb{E}$ if

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbb{E}(\bar{u} + \varepsilon v) = 0, \qquad\qquad \forall v \in C^\infty_c(\Omega, \mathbb{R}^n)\,.$$

Given a vector field $\Phi \in C^1_c(\Omega, \mathbb{R}^m)$ we let X_ε be its flow[1]. The one-parameter family of functions $u_\varepsilon = \bar{u} \circ X_\varepsilon$ will be called an *inner variation*. A critical point $\bar{u} \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ is *stationary* for $\mathbb{E}$ if

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbb{E}(u_\varepsilon) = 0, \qquad\qquad \forall \Phi \in C^1_c(\Omega, \mathbb{R}^m)\,.$$

Simple computations show that the conditions above are equivalent to, respectively,

$$\int_\Omega \langle Df(D\bar{u}), Dv\rangle\, dx = 0, \qquad\qquad \forall v \in C^1_c(\Omega, \mathbb{R}^n). \tag{1.2}$$

and

$$\int_\Omega \langle Df(D\bar{u}), D\bar{u}D\Phi\rangle dx - \int_\Omega f(D\bar{u})\,\mathrm{div}\,\Phi dx = 0, \qquad\qquad \forall \Phi \in C^1_c(\Omega, \mathbb{R}^m)\,. \tag{1.3}$$

The graphs of Lipschitz functions can be naturally given the structure of integer rectifiable currents (without boundary in $\Omega \times \mathbb{R}^m$) and of integral varifold, cf. [33, 78, 38]. For the definition of rectifiable varifold, see Section A.4. In particular, the graph of any stationary point $\bar{u} \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ for a polyconvex energy $\mathbb{E}$ can be thought as a stationary point for a corresponding elliptic energy, in the space of integer rectifiable currents and in that of integral varifolds, respectively, see [39,

1 Namely $X_\varepsilon(x) = \gamma_x(\varepsilon)$, where γ_x is the solution of the ODE $\gamma'(t) = \Phi(\gamma(t))$ subject to the initial condition $\gamma(0) = x$.

Chapter 1, Section 2]. Note that a particular example of polyconvex energy is given by the area integrand

$$\mathcal{A}(X) = \sqrt{\det(\mathrm{id}_{\mathbb{R}^{m\times m}} + X^T X)}\,. \tag{1.4}$$

The latter is *strongly polyconvex* when restricted to any ball $B_R \subset \mathbb{R}^{n\times m}$, namely there is a positive constant $\varepsilon(R)$ such that $X \mapsto \mathcal{A}(X) - \varepsilon(R)|X|^2$ is still polyconvex on B_R.

When $n = 1$ strong polyconvexity reduces to locally uniform convexity and any Lipschitz critical point is therefore $C^{1,\alpha}$ by the De Giorgi-Nash theorem, see [15] and [68] respectively. The same regularity statement holds in the much simpler dual case $m = 1$, where criticality implies that the vector valued map $\bar{u}$ satisfies an appropriate system of ODEs. L. Székelyhidi in [82] proved the existence of smooth strongly polyconvex integrands $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ for which the corresponding energy has Lipschitz critical points which are nowhere C^1. The paper [82] is indeed an extension of a previous groundbreaking result of S. Müller and V. Šverák [62], where the authors constructed a Lipschitz critical point to a smooth strongly quasiconvex energy (cf. [62] for the relevant definition) which is nowhere C^1. A precursor of such examples can be found in the pioneering PhD thesis of V. Scheffer, [74]. On the other hand, minimizers of strongly quasiconvex functions have been proved to be regular almost everywhere, see [28, 52, 74]. Note that the *geometric* counterpart of the latter statement is Almgren's celebrated regularity theorem for integral currents minimizing strongly elliptic integrands [4]. Let us remark that stationary points need not to be local minimizers for the energy. This is proved, for instance, in the case $f(X) = \mathcal{A}(X)$ for $n = m = 2$, by H. Lawson and R. Osserman in [54, Theorem 5.3]. Standard computations show, on the other hand, that every minimizer for an energy is a stationary point. Moreover, combining the uniqueness result in [84] and [62, Theorem 4.1], it is easy to see that there exist critical points that are not stationary.

Other than the result in [84], not much is known about the properties of stationary points, in particular it is not known whether they must be C^1 on a set of full measure. Observe that Allard's ε-regularity theorem applies when f is the area integrand and allows to answer positively to the latter question for f as in (1.4). The validity of an Allard-type ε-regularity theorem for general elliptic energies is however widely open, even though in the last years there have been important contributions. Indeed, in [21], G. De Philippis, A. De Rosa and F. Ghiraldin characterize in terms of an appropriate condition on the integrand (called *atomic condition*, cf. [21, Definition 1.1]) those energies for which rectifiability of stationary points hold. For this reason, integrands satisfying the atomic condition have good *ellipticity* properties, and seem to be the most likely to allow for an ε-regularity Theorem.

A first interesting question is whether one could extend the examples of Müller and Šveràk and Székelyhidi to provide counterexamples. Both in [62] and [82], the starting point of the construction of irregular solutions is rewriting the condition (1.2) as a differential inclusion, and then finding a so-called T_N-configuration ($N = 4$ in the first case, $N = 5$ in the latter) in the set defining the differential inclusion. In [17], it is shown that such a strategy fails in the case of stationary points. More precisely:

(a) We show that $\bar{u}$ solves (1.2), (1.3) if and only if there exists an L^∞ matrix field A that solves a certain system of linear, constant coefficients, PDEs and takes almost everywhere values in a fixed set of matrices, which we denote by K_f and call *inclusion set*, cf. Lemma 2.3. The latter system of PDEs will be called a *div-curl* differential inclusion, in order to distinguish them from classical differential inclusions, which are PDE of type $Du \in K$ a.e., and from *divergence differential inclusions* as for instance considered in [18].

(b) We give the appropriate generalization of T_N configurations for div-curl differential inclusions, which we will call T_N' configurations, cf. Definition 2.7. As in the *classical* case, the latter are subsets of cardinality N of the set K_f which satisfy a particular set of conditions.

(c) We then prove the following nonexistence result, contained in [17]:

Theorem 1.1. *If $f \in C^1(\mathbb{R}^{n\times m})$ is strictly polyconvex, then K_f does not contain any set $\{A_1,\dots,A_N\}$ which induces a T'_N configuration.*

Instead of giving the proof of Theorem 1.1, we will deduce it as a corollary of a stronger result, that we explain now. The Constancy Theorem (see, for instance, [78, Theorem 8.4.1]), asserts that if an m-dimensional integer rectifiable varifold $V = (\Gamma,\theta)$ is stationary with respect to the area functional and is contained in a C^2, m-dimensional submanifold M of $\mathbb{R}^{n+m}$, then V must be the varifold given by the integration over M and θ must be constant. This has to be interpreted as a regularizing effect of the area functional. It has been proved in [27] that the smoothness assumption can be dropped, and the manifold can be taken to be only Lipschitz. Moreover, in [22], the authors show that for codimension 1 varifolds the same assertion is true for more general functionals than the area. The same question can be asked for higher codimension varifolds, and some results on this problem will appear in [41]. In [41], we use again differential inclusions to study stationary varifolds given by graphs with (real) multiplicity, and we therefore study a suitable inclusion set $C_f \subset \mathbb{R}^{(2n+m)\times m}$. In Chapter 2 we present the following result, that will be part of [41]:

Theorem 1.2. *If $f \in C^1(\mathbb{R}^{n\times m})$ is a strictly polyconvex function, then C_f does not contain any set $\{A_1,\dots,A_N\} \subset \mathbb{R}^{(2n+m)\times m}$ which induces a T'_N configuration, provided that $f(X_1) \ge 0,\dots,f(X_N) \ge 0$, if*

$$A_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}, \quad X_i, Y_i \in \mathbb{R}^{n\times m}, Z_i \in \mathbb{R}^{m\times m}, \forall i \in \{1,\dots,N\}.$$

We will also show how Theorem 1.1 follows from this last result. Preliminarily to this result, we will show that positivity of f also allows us to discard another class of matrices in C_f that could be used to build counterexamples. For $m = 2$, these are equivalent to the well-known *rank-one connections*, while for $m > 2$ one needs to introduce the Λ-cone of a certain differential operator to define them, as described in the final section of this introduction. In the case of graphs with multiplicity one the polyconvexity of f is sufficient to avoid them. This is what leads the aforementioned authors to consider in [62, 82] more complicated sets of matrices such as T_N configurations. In the case of variable multiplicity, the result is to the best of our knowledge not known and a proof will be given in Proposition 2.22. These results are the content of Chapter 2. In Chapter 3, we will provide the aforementioned link between stationary points for polyconvex energies and stationary varifolds for anisotropic integrands.

Theorem 1.1 serves as an indication of the fact that partial regularity is possible, but it is far from clear if this information can be used to give a proof of regularity of stationary objects. In Chapter 4 we have collected the results of [87], where we have anyway proved a regularity theorem for two-dimensional stationary points for functionals sufficiently close to the area, namely the following[2]:

Theorem 1.3. *For every $R > 0$, there exists $\alpha = \alpha(R) > 0$ such that, if f is a $C^k(\mathbb{R}^{n\times 2})$ function, $k \ge 2$, with the property that*

$$\|f - \mathcal{A}\|_{C^2(B_{2R}(0))} \le \alpha, \tag{1.5}$$

and $u : \Omega \to \mathbb{R}^n$ is a Lipschitz stationary point of f with

$$\|Du\|_\infty \le R,$$

then $u \in C^{k-1,\rho}(\Omega)$, for some positive $\rho > 0$.

2 Notice that we have chosen to state this result in a slightly different language with respect to Theorem 4.20, but the result is completely equivalent. We chose to simplify the statement to avoid the technical discussion on how to pass from *div-curl* to purely *curl* differential inclusion. This is postponed to Section 4.1.

This result is obtained thanks to the analysis of the regularity of differential inclusions carried out by V. Šverák in [88]. In particular, we prove that the differential inclusion associated to the two dimensional area functional

$$DU(x) \in K_{\mathcal{A}},$$

is *elliptic in the sense of Šverák*, meaning that there exist constants $c_{ij} \in \mathbb{R}, \varepsilon > 0$ such that

$$\sum_{i<j} c_{ij} \det_{ij}(X - Y) \geq \varepsilon \|X - Y\|^2, \ \forall X, Y \in K_{\mathcal{A}},$$

where $\det_{ij} : \mathbb{R}^{(2n+2)\times 2} \to \mathbb{R}$ is the function

$$\det_{ij}(X) = \det \begin{pmatrix} x_{i1} & x_{i2} \\ x_{j1} & x_{j2} \end{pmatrix}.$$

This new ellipticity result on $K_{\mathcal{A}}$ is sufficient to ensure that stationary points to the area functional are smooth and allows us to prove Theorem 1.3.

From a merely analytical point of view, it is still unclear how stationarity with respect to the inner variations can help in the proof of a partial regularity theorem. An interesting viewpoint is to interpret the system of PDEs arising from the inner variations as an additional *conservation law*, as it happens for instance in the context of Euler's equations with the energy, see Section 1.3. As in the case of C^1 solutions to the Euler's equations, one can see that if one assumes $u \in C^2(\Omega, \mathbb{R}^n)$ and satisfying (1.2), then (1.3) follows automatically and does not give additional information. Hence they can be useful only if one assumes to start with a solution with low regularity, for instance $u \in W^{1,p}$. In very specific cases, the additional information inner variations carry is a monotonicity formula for the solutions. This happens for the area functional and for harmonic maps, see [16, 29]. For general problems no monotonicity formula is available, and at present is quite unclear which form it should have (on the topic, see [2]).

Classical proofs of regularity, as the one of Evans of [28], are based on showing that an estimate of the form

$$\fint_{B_{\tau r}(x)} \|Du(y) - (Du)_{x,\tau r}\|^2 \, dy \lesssim \tau^2 \fint_{B_r(x)} \|Du(y) - (Du)_{x,r}\|^2 \, dy, \tag{1.6}$$

holds at every x of Ω, for some $\tau \in (0,1)$. In (1.6), $(Du)_{x,r}$ denotes the average of Du on $B_r(x)$. From (1.6) partial regularity follows, and the key point in every regularity proof is to use the properties of the problem to show that (1.6) holds. For instance, in [28], (1.6) is deduced from a Caccioppoli inequality, i.e., an inequality of the form

$$\int_{B_r(x)} |Du - Da|^2 \lesssim \int_{B_{2r}(x)} |u - a|^2 \tag{1.7}$$

for all affine functions $a(x) = b + Ax$, see [28, Lemma 3.1]. In [28], minimality is used in an essential way to deduce the last inequality, therefore the same proof cannot be used in our case. Nonetheless, even without (1.7), one can try to prove (1.6). This is usually achieved by contradiction, i.e. supposing that there exists a sequence of maps $u_j : B_1(0) \to \mathbb{R}^n$ equibounded in $W^{1,2}$, solving (1.2) and (1.3) for the functional induced by

$$f_j(X) = \frac{f(A_j + \lambda_j X) - f(A_j) - \lambda_j \langle Df(A_j), X\rangle}{\lambda_j^2},$$

where A_j is a convergent sequence of matrices and $\lambda_j \to 0$ such that (1.6) fails for u_j. These solutions u_j converge weakly in $W^{1,2}(B_1(0), \mathbb{R}^n)$ to a solution $\bar{u}$ of the *linearized* problem, i.e. that solves (1.2) and (1.3) for the functional $D^2 f(A)[X, X]$, A being the limit of $(A_j)_j$. The crucial point is to turn this weak convergence into strong convergence. We learned from J. Hirsch that the fact that $(u_j)_j$ is a solution merely of the outer variations equations is sufficient to guarantee that

$(Du_j)_j$ converges pointwise a.e., or, equivalently, eliminates oscillations in the sequence of the gradients. In order to ensure strong convergence, one also needs to show that the sequence of measures $\mu_j := \|Du_j\|^2\,dx$ does not *concentrate* in $B_1(0)$, i.e. one would like to prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|E| \le \delta$, then

$$\sup_{j\in\mathbb{N}} \mu_j(E) \le \varepsilon.$$

It is tempting to conjecture that the inner variations should be sufficient to avoid this phenomenon, but there is no proof for this at the moment.

Inner variations appear also in similar, yet different, problems arising from elasticity. Consider the functional $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$

$$f(X) := \frac{\|X\|^2}{2} + g(\det(X)), \tag{1.8}$$

where $g(t)$ is a convex function on $(-\infty, +\infty)$ with $g(t) \equiv +\infty$ for $t \le 0$ and $\lim_{t\to 0} g(t) = +\infty$. The question one asks in this case is, as usual, the one of regularity: for any $\varphi \in C^\infty(\partial B_1, \mathbb{R}^2)$, are minimizers in the class $\mathcal{D}_\varphi := \{u \in W^{1,2}(B_1, \mathbb{R}^2) : \det(Du) > 0, u|_{\partial B_1} = \varphi|_{\partial B_1}\}$ for the corresponding energy (1.1) smooth? In this problem outer variations cannot be used, since variations of the form $\bar u + \varepsilon v$ do not respect the the determinant constraint in $\mathcal{D}_\varphi$. Nonetheless, it can be shown that they satisfy the inner variation equations (1.3). Thus, one might ask whether (1.3) and the positivity of the determinant are sufficient to prove regularity for minimizers. This is a long-standing open problem in the field, see for instance [8] and references therein. In this direction, in [47], the authors prove for a certain class of energies that critical points for inner variations with non-negative Jacobian are Lipschitz regular but not necessarily C^1. In [79], the authors construct for a large class of functionals solutions to the inner variations singular solution lying in $W^{1,p}$ for $1 \le p < n$. At the end of Chapter 4, we will show how a result of [51] allows to construct a nowhere C^1 solution to (1.3) in the case in which f is the Dirichlet energy or the area functional, at least if the hypothesis on the non-negativity of the Jacobian is dropped.

1.2 DIVERGENCE-FREE MATRIX FIELDS IN $\mathrm{Sym}^+(n)$

In this part of the thesis and its appendix we collect some results of [24, 23]. D. Serre proved in [76, Theorem 2.1] the following:

Theorem. *Let the divergence-free, non-negative definite matrix field $x \mapsto A(x)$ be Γ-periodic, with $A \in L^1(\mathbb{R}^n/\Gamma)$. Then*

$$\det(A) \in L^{\frac{1}{n-1}}(\mathbb{R}^n/\Gamma)$$

and there holds

$$\fint_{\mathbb{R}^n/\Gamma} \det(A(x))^{\frac{1}{n-1}}\,dx \le \det\left(\fint_{\mathbb{R}^n/\Gamma} A(x)\,dx\right)^{\frac{1}{n-1}}. \tag{1.9}$$

Here Γ is a lattice of $\mathbb{R}^n$ (one can imagine $\Gamma = \mathbb{Z}^n$, i.e. $\mathbb{R}^n/\Gamma = \mathbb{T}^n$). This theorem shows an improvement in the integrability of the function $x \mapsto \det(A)^{\frac{1}{n-1}}(x)$, with respect to the straightforward one $\det(A) \in L^{\frac{1}{n}}(\mathbb{R}^n/\Gamma)$. In [76, Open Question 2.1], it was asked:

Open Question 2.1: Let $x \mapsto A(x)$ be Γ-periodic, taking values in $\mathrm{Sym}^+(n)$. Let A and $\mathrm{div}(A)$ belong to $L^p(\mathbb{R}^n/\Gamma)$ with $1 < p < n$. Defining $\frac{1}{p'} = \frac{1}{p} - \frac{1}{n}$, is it true that

$$\det(A)^{\frac{1}{n}} \in L^{p'}(\mathbb{R}^n/\Gamma)\,?$$

The answer to the question is negative, and is the content of the first chapter of Part ii. The proof involves the construction of a family of *approximate counterexamples* in Lemma 5.3, and then an application of Baire's Theorem A.4 in Theorem 5.1 to find actual counterexamples that are also topologically typical. See Section A.6 for the terminology we adopt concerning Baire Theorem. Even though in our case the situation is quite simple since our family of starting approximate counterexample is explicit, notice that these two steps are common to all the so-called convex integration schemes. One can compare, for instance, [51, Proposition 4.17]. To state the result in more precise terms, let Ω be an open subset of $\mathbb{R}^n$ and define

$$\begin{aligned} Y_{p,K} \doteq \{&A \in L^p(\Omega, \mathrm{Sym}^+(n)) : \mathrm{div}(A) \in L^p(\Omega, \mathbb{R}^n), \\ &A \equiv \bar{A} \text{ outside } K, \text{ for some fixed } \bar{A} \in \mathrm{Sym}^+(n)\}, \end{aligned}$$

for any compact $K \subset \Omega$ with $\mathrm{clos}(\mathrm{int}(K)) = K \neq \emptyset$. We consider the following distance on $Y_{p,K}$, that turns it into a complete metric space:

$$d(A, B) \doteq \|A - B\|_{L^p} + \|\,\mathrm{div}(A - B)\|_{L^p}.$$

The main theorem is:

Theorem 1.4. *Let* $p^* \doteq \max\left\{0, \frac{p(n-1)-n}{p(n-1)}\right\}$. *The set*

$$D_{p,K} \doteq \{A \in Y_{p,K} : \det(A)^{\frac{1}{n-1}} \in L^{\frac{1}{1-p^*}}(\Omega) \setminus L^{\frac{1}{1-p^*}+\varepsilon}(\Omega), \forall \varepsilon > 0\}$$

is residual in $Y_{p,K}$.

The idea behind this improvement of integrability, that can be read in [76, Proposition 1.2] and the discussion before and after the proposition, is that if one couples a PDE constraint, such as $\mathrm{div}(A) = 0$, with some other constraint, as $A \in \mathrm{Sym}^+(n)$, it is possible to show some *elliptic regularizations* of the solutions, i.e. some improvement in properties of the solutions. In the particular case when the operator is the divergence, one can show that, on matrices with maximal rank, $A \mapsto \det(A)$ is *elliptic*, see for instance [76, Proposition 1.2]. We will discuss more on this point in the last section of the introduction. What sparked our interest in this problem is the fact that for some functional arising in the Calculus of Variations and Fluid Dynamics, the inner variation tensor is symmetric and non-negative definite. This happens for instance for the area functional. Some conclusions in this directions are made by Serre in [76, Section 1].

Serre's result has another consequence. Inequality (1.9) is a generalized Jensen inequality for non-concave functions, and can be viewed as a div-quasiconcavity property. Let us explain the link between quasiconcavity and upper semi-continuity of the related functional by considering the dual of these objects, namely quasiconvexity and lower-semicontinuity, that have received much more attention in the literature. We will use as a domain the n-dimensional torus $\mathbb{T}^n$ simply because it is the domain we will use throughout the paper, but more generally one could consider any $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$. The general question one poses is the following: given a continuous integrand $f : \mathbb{R}^N \to \mathbb{R}$ with growth

$$|f(z)| \leq C(1 + \|z\|^p), \tag{1.10}$$

under which conditions is the functional

$$\mathbb{E}(z) \doteq \int_{\mathbb{T}^n} f(z(x))dx,$$

defined, for instance, for $z \in L^q(\mathbb{T}^n, \mathbb{R}^N)$, $q \leq p$, sequentially weakly lower semi-continuous? In the case we are interested in, the functional is given by

$$\mathbb{D}(A) \doteq \int_{\mathbb{T}^n} \det(A(x))^{\frac{1}{n-1}} dx,$$

and one is interested in its upper-semicontinuity with respect to its weak topology in

$$X_p \doteq \left\{A \in L^p(\mathbb{T}^n, \mathrm{Sym}^+(n)) : \mathrm{div}\, A \in \mathcal{M}(\mathbb{T}^n, \mathbb{R}^n)\right\}.$$

The first example of these lower-semicontinuity problems was studied by C.B. Morrey in the case in which $N = m \times n$, $z(x) = Du(x)$, where $u : \mathbb{T}^n \to \mathbb{R}^m$ is a $W^{1,q}$ function. In [58], he introduced the notion of *quasiconvexity*, that is:

$$f(A) \le \int_{\mathbb{T}^n} f(A + D\phi(x))dx, \qquad \forall \phi \in C^\infty(\mathbb{T}^n, \mathbb{R}^m), \forall A \in \mathbb{R}^{m\times n}. \tag{1.11}$$

It can be proved that (1.10) and (1.11) imply the weak lower semi-continuity of the functional $\mathbb{E}(\cdot)$, when $q < p$, see for instance [48] for more information and results. It is easy to see using Jensen's inequality that every polyconvex function, introduced above, is also quasiconvex. In other problems, one is interested in maps $z : \mathbb{T}^n \to \mathbb{R}^N$ satisfying more general constraints than $z(x) = Du(x)$. The general framework, considered for instance in [36, 35], consists in taking a differential operator of order k with smooth coefficients, usually denoted by $\mathscr{A}$, of the form

$$\mathscr{A} \doteq \sum_{|\alpha|\le k} A_\alpha \partial_\alpha, \quad A_\alpha \in C^\infty(\mathbb{T}^n, \mathbb{R}^{\ell\times N}).$$

In [36] it is proved that f is weakly lower-semicontinuous on $L^q(\mathbb{T}^n, \mathbb{R}^N) \cap \ker(\mathscr{A})$, $q < p$, provided that $\mathscr{A}$ satisfies Murat's constant rank condition (see [36] or [63] for the definition), f satisfies (1.10) and is $\mathscr{A}$-quasiconvex, in the sense that

$$f(A) \le \int_{\mathbb{T}^n} f(A + z(x))dx, \qquad \forall A \in \mathbb{R}^N, \forall z \in C^\infty(\mathbb{T}^n, \mathbb{R}^m) \text{ with } \mathscr{A}z = 0.$$

From the proof of [36], that exploits linearity of certain operators on the Fourier coefficients of the maps, it is unclear that one can add the non-linear constraint $z(x) \in \operatorname{Sym}^+(n)$, thus it is not straightforward[3] to see that it can be used to prove upper-semicontinuity of $\mathbb{D}$ on X_p. In Chapter 6 we show, using a different proof than the one in [36] the following:

Theorem 1.5. *Let $p > \frac{n}{n-1}$ and $\{A_k\}_k \subset X_p$ be such that $A_k \rightharpoonup A$ in X_p. Then*

$$\limsup_k \mathbb{D}(A_k) \le \mathbb{D}(A).$$

Moreover we show its failure for $p \le \frac{n}{n-1}$. We also briefly discuss some applications to the multi-dimensional Burgers equation.

1.3 CONVEX INTEGRATION AND ENERGY REGULARITY FOR EULER EQUATIONS

This part and its appendix contain results appeared in [25]. In the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, we consider the incompressible Euler equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + Dp = 0, \\ \operatorname{div} v = 0, \end{cases} \qquad \text{in } \mathbb{T}^3 \times [0, T], \tag{1.12}$$

where $v : \mathbb{T}^3 \times [0, T] \to \mathbb{R}^3$ represents the velocity of an incompressible fluid, $p : \mathbb{T}^3 \times [0, T] \to \mathbb{R}$ is the hydrodynamic pressure, with the constraint $\int_{\mathbb{T}^3} p\, dx = 0$ which guarantees its uniqueness. A weak solution of the system (1.12) is a vector field $v \in L^2(\mathbb{T}^3 \times [0, T]; \mathbb{R}^3)$ such that

$$\int_0^T \int_{\mathbb{T}^3} (v \cdot \partial_t \varphi + v \otimes v : D\varphi)\; dx\, dt = 0,$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (0, T); \mathbb{R}^3)$ such that $\operatorname{div} \varphi = 0$. The pressure does not appear in the weak formulation because it can be recovered as the unique 0-average solution of

$$-\Delta p = \operatorname{div}\operatorname{div}(v \otimes v).$$

3 Nonetheless, in a very recent work, [80], J. Skipper and E. Wiedemann have managed to modify the proof of [36] to obtain this result.

Multiplying by v the first equation in (1.12) and integrating by parts on $\mathbb{T}^3$, one gets that, at least for smooth solutions,

$$\frac{d}{dt}e_v(t) := \frac{d}{dt}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = 0, \qquad \forall t \in [0,T].$$

For weak solutions $v \in L^\infty((0,T);C^\theta(\mathbb{T}^3))$ it is known, and was previously conjectured by Lars Onsager, that the threshold for the energy conservation is $\theta = 1/3$. The first proof of the conservation in the range $\theta > 1/3$ was given in [13], while in [44] P. Isett proved the existence of dissipative solutions for any $\theta < 1/3$ using the convex integration techniques introduced by C. De Lellis and L. Székelyhidi in [18, 19, 20, 10].

As proved in [43], given any solution $v \in L^\infty((0,T);C^\theta(\mathbb{T}^3))$, it can be shown that the associated kinetic energy e_v satisfies

$$|e_v(t) - e_v(s)| \leq C\,|t-s|^{\frac{2\theta}{1-\theta}}, \qquad \forall t,s \in [0,T], \tag{1.13}$$

which in particular implies the conservation if $\theta > 1/3$, but also shows a peculiar Hölder regularity of the energy (see also [12] for an alternative proof). It is a natural question to ask whether this regularity is optimal. In [46, Conjecture 1], Isett and S. Oh formulated the following:

Conjecture. *For any $\theta < \frac{1}{3}$, there exists a solution to (1.12) in the class $v \in C^\theta(\mathbb{R}\times\mathbb{T}^n)$ whose energy profile $e(t)$ fails to have any regularity above the exponent $\frac{2\theta}{1-\theta}$, in the sense that $e_v(t) \notin W^{\frac{2\theta}{1-\theta}+\rho,p}(I)$, for every $\rho > 0$, $p \geq 1$ and every open time interval $I \subset \mathbb{R}$. Furthermore, the set of all such solutions v with the above property is residual (in the sense of category) within the space of all weak solutions to (1.12) in the class $e_v \in C^\theta(\mathbb{R}\times\mathbb{T}^n)$ when the latter space is endowed with the topology from the C^θ norm.*

The result we present here answers to the first part of the conjecture. In particular, we start by proving, in the spirit of [11]:

Theorem 1.6. *Fix $\gamma > 0$ and $\theta \in (0,1/3)$ such that $\frac{2\theta}{1-\theta} + \gamma < 1$. For every strictly positive $e \in C^{\frac{2\theta}{1-\theta}+\gamma}([0,T])$, there exists a vector field $v \in C^\theta(\mathbb{T}^3 \times [0,T])$ that solves (1.12) in the distributional sense and such that*

$$e(t) = \int_{\mathbb{T}^3} |v|^2(x,t)\,dx, \qquad \forall t \in [0,T].$$

This result shows how it is possible to construct *approximate* counterexamples to the fact that the energy is more regular than $C^{\frac{2\theta}{1-\theta}}$. This type of result already appeared in [45] for any $\theta \in (0,1/5)$. Once again, Baire category argument, Theorem A.4, allows us to construct exact counterexamples. We define

$$X_\theta = \overline{\left\{ v \in \bigcup_{\theta'>\theta} C^{\theta'}(\mathbb{T}^3\times[0,T]) \;:\; v \text{ weakly solves (1.12)} \right\}}^{\|\cdot\|_{C^\theta_{x,t}}}, \tag{1.14}$$

endowed with the distance

$$\mathrm{d}(u,v) := \|u - v\|_{C^\theta_{x,t}}.$$

It is clear that (X_θ, d) is a complete metric space. We also define

$$Y_\theta = \left\{ v \in X_\theta \;:\; e_v \in C^{\frac{2\theta}{1-\theta}}([0,T]) \setminus \bigcup_{\gamma>0} W^{\frac{2\theta}{1-\theta}+\gamma,1}(I), \textit{for any interval } I \subset [0,T] \right\}.$$

Finally we prove our main Theorem:

Theorem 1.7. *For any $\theta \in (0,1/3)$, the set Y_θ is residual in X_θ.*

The previous Theorem yields some immediate corollaries. First, it implies that the typical solution in X_θ is not of bounded variation, thus not monotonic, in any open subset of $[0, T]$. Therefore, Theorem 1.7 shows a very irregular behaviour of the energy of solutions, in sharp contrast with the conservation of the energy in the case $\theta > 1/3$. We refer the reader to [46, 45] for further discussions. A second immediate corollary of Theorem 1.7 is that, for every $\theta \in (0, 1/3)$, there exists a weak solution v of (1.12) such that $e_v \in C^{\theta^*}([0, T])$ but $e_v \notin C^{\theta^*+\gamma}([0, T])$, for any $\gamma > 0$. Let us note in passing that this also yields a weak $C^\theta(\mathbb{T}^3 \times [0, T])$ solution of (1.12) that is not in $C^{\theta+\gamma}(\mathbb{T}^3 \times [0, T])$, for any γ. Indeed, from (1.13) it is clear that Y_θ can not contain solutions v that are more Hölder regular than $C^\theta(\mathbb{T}^3 \times [0, T])$. While the residuality property implies that the kinetic energy of many $C^\theta(\mathbb{T}^3 \times [0, T])$ solutions enjoys the sharp regularity (1.13), it must be noted that X_θ might not contain all the $C^\theta(\mathbb{T}^3 \times [0, T])$ solutions of Euler, since in general not all the $C^\theta(\mathbb{T}^3 \times [0, T])$ functions can be obtained as limit of more regular ones. The argument used to prove Theorem 1.7 does not work with the choice

$$X_\theta = \{u \in C^\theta([0, T]) : u \text{ solve } (1.12)\},$$

or[4]

$$X_\theta = \{u \in c^\theta([0, T]) : u \text{ solve } (1.12)\},$$

hence it does not answer to the second part of the above conjecture. In Section 7.5, we comment more on the technical reasons why this is the case.

1.4 CONNECTIONS AMONG THE PROBLEMS

In this section we give a heuristic explanation on why many of the results of the previous sections can be thought as consequence of a quite general principle. These ideas have their root in the classical theory of *compensated compactness* of F. Murat and L. Tartar, see [64, 65, 85, 86]. Let us first give the following:

Definition 1.8. Let $\mathscr{A}$ be a given constant coefficient differential operator of order 1 acting on maps from a domain $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^{n\times m}$. Namely

$$\mathscr{A} \doteq \sum_{\alpha=1}^{m} A_\alpha \partial_\alpha,$$

where $A_\alpha \in \mathbb{R}^{r\times(n\times m)}$, $r \in \mathbb{N}$. The *wave cone* of $\mathscr{A}$ is defined as

$$\Lambda_{\mathscr{A}} \doteq \bigcup_{\|\xi\|=1} \ker \mathbb{A}(\xi), \text{ where } \mathbb{A}(\xi) \doteq \sum_{\alpha=1}^{m} \xi_\alpha A_\alpha.$$

This is not the most general definition, as one could allow for differential operators of order k with continuous coefficients, but for our presentation this is completely sufficient. An equivalent definition of the wave cone can be given in terms of *plane wave solutions*, i.e. maps of the form

$$z(x) = h(x \cdot \xi)a, \tag{1.15}$$

solving

$$\mathscr{A} z = 0, \tag{1.16}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* $\Lambda_{\mathscr{A}}$ is exactly given by the states $a \in \mathbb{R}^{n\times m}$ for which there is a vector $\xi \neq 0$ such that for any choice of the profile h the function (1.15) solves (1.16), that is,

$$\Lambda_{\mathscr{A}} = \left\{ a \in \mathbb{R}^{n\times m} : \exists \xi \in \mathbb{R}^m \setminus \{0\} \quad \text{with} \quad \sum_{\alpha=1}^{m} \xi_\alpha A_\alpha a = 0 \right\}. \tag{1.17}$$

4 For the definition of the function space c^θ (or C^θ), see Section A.3.

As explained in [71], $\Lambda_{\mathscr{A}}$ essentially captures the directions in which a sequence of maps z_n equibounded in L^1 and solving (1.16) can develop oscillations. This is justified by the definition of the wave-cone through plane-wave solutions, (1.17). Indeed, consider a segment $[A, B]$, for two matrices A, B such that $A - B = C \in \Lambda_{\mathscr{A}}$. Then, the family of maps

$$z_\varepsilon(x) = \frac{A+B}{2} + \sin\left(\frac{(x,\xi)}{\varepsilon}\right)\frac{C}{2},$$

gives a highly oscillatory sequence of plane-wave solutions of the differential inclusion

$$z_\varepsilon(x) \in [A, B], \quad \text{for a.e. } x \in Q_1.$$

The fact that $C \in \Lambda_{\mathscr{A}}$ implies moreover that $\mathscr{A} z_\varepsilon = 0$, for every $\varepsilon > 0$. This family fails to converge strongly as $\varepsilon \to 0$, due to the oscillations in direction C. Let us make another important example. Let $\mathscr{A} = \operatorname{curl}$, so that $z \in L^1$ solving $\mathscr{A} z = 0$ on the square $Q_1 \subset \mathbb{R}^m$ equals to say $z = Du$ for some $u \in W^{1,1}(Q_1, \mathbb{R}^n)$. In this case, one has

$$\Lambda_{\text{curl}} = \{M \in \mathbb{R}^{n\times m} : \operatorname{rank}(M) \le 1\}.$$

Given $A, B \in \mathbb{R}^{n\times m}$, if $A - B \in \Lambda_{\text{curl}}$, then one can construct a so-called *simple laminate*, i.e. for every $\lambda \in [0, 1]$, $\varepsilon > 0$, one can find a map $z_{\varepsilon,\lambda} : Q_1 \to \mathbb{R}^n$ such that

$$|\{x \in Q_1 : z_{\varepsilon,\lambda}(x) = A\}| = \lambda, \; |\{x \in Q_1 : z_{\varepsilon,\lambda}(x) = B\}| = 1 - \lambda,$$

and that moreover still solves (1.16). These oscillatory solutions can be constructed in such a way that they jump between A and B arbitrarily fast, i.e. in strips of size ε. It is easy to see that the family $(z_{\varepsilon,\lambda})_{\varepsilon>0}$ converges weakly but not strongly in L^p.

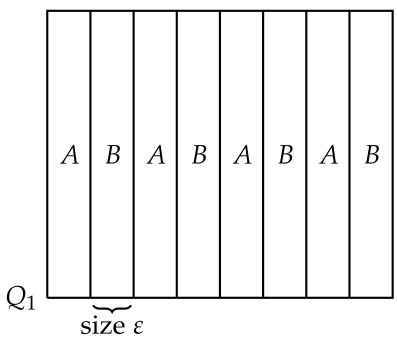

Figure 1: An example of simple laminate in $\mathbb{R}^2$ with $\xi = e_1$ and $\lambda = \frac{1}{2}$.

The examples above explain why, when looking at differential inclusions of the form

$$u \in \operatorname{Lip}(Q_1, \mathbb{R}^n), \; Du(x) \in K \subset \mathbb{R}^{n\times m} \text{ a.e.}, \tag{1.18}$$

strong compactness is doomed by the existence of rank-one connections in K, i.e. matrices $A, B \in K$ such that $A - B \in \Lambda_{\text{curl}}$. This holds more generally when considering inclusions of the form

$$W \in L^\infty(Q_1, \mathbb{R}^n), \; W(x) \in K \subset \mathbb{R}^{n\times m} \text{ a.e.}, \; \mathscr{A} W = 0,$$

and K contains matrices A, B such that $A - B \in \Lambda_{\mathscr{A}}$.

A deep recent result by G. De Philippis and F. Rindler, [71, Theorem 1.1], asserts that if $\mu \in \mathcal{M}(Q_1, \mathbb{R}^N)$ solves

$$\mathscr{A}\mu = 0,$$

then the singular part of μ with respect to the Lebesgue measure, μ^{sing}, has the property that

$$\frac{d\mu^{\text{sing}}}{d\|\mu^{\text{sing}}\|}(x) \in \Lambda_{\mathscr{A}}, \ \|\mu\|^{\text{sing}}\text{-a.e.},$$

that proves formally the idea that $\Lambda_{\mathscr{A}}$ encodes the directions of the oscillations of an equibounded L^1 sequence of $\mathscr{A}$-free maps. This result, together with its extension [6], allows to prove as a corollary many celebrated results, such as Alberti's Rank One Theorem, see [40], and the aforementioned recent extension of Allard's rectifiability result, [21].

Most of the results contained in the literature study the case of differential inclusion of the form (1.18). In [88], it is proved that the absence of rank-one connections and connectedness of K in the case $n = m = 2$ yields *compactness* of the differential inclusion, i.e. a sequence $u_n \in \text{Lip}(Q_1, \mathbb{R}^n)$ with equibounded Lipschitz norm solving $Du_n \in K, \forall n$, admits a subsequence converging strongly in L^p. If one drops the hypothesis of connectedness of K, this is false. Indeed Tartar discovered a *special* set of 4 matrices in $\mathbb{R}^{2\times 2}$ nowadays called *Tartar's square*, $\mathcal{E} = \{A_1, A_2, A_3, A_4\}$, such that there exists a sequence of equibounded Lipschitz maps u_n with

$$\mathrm{d}(Du_n(x), \mathcal{E}) \to 0 \text{ pointwise a.e.},$$

but such that no subsequence of Du_n converges strongly in L^p. This set $\mathcal{E}$ is called a T_4-configurations, and we will describe it in detail in Part i. In $\mathbb{R}^{2\times 2}$, a deep result of Faraco and Székelyhidi, [32], states that if $K \subset \mathbb{R}^{2\times 2}$ does not contain rank-one connections and T_4 configurations, then K has such compactness properties.

The interest in these properties for the set defining a differential inclusion stems from the fact that usually a lack of compactness for the differential inclusion allows for the construction of irregular solutions. This procedure is nowadays called *convex integration*, and it was introduced by Gromov extending the groundbreaking result of Nash on the existence of isometric embeddings, see [67]. As said, the presence of T_4 (resp. T_5) configurations in the differential inclusion considered in [62] (resp. [82]) yields the existence of very irregular solutions. The first part of this thesis, Part i, is devoted to results in this direction concerning the differential inclusion in K_f.

Similarly, in [18], Euler equations are rewritten as a differential inclusion of the form

$$z(x) \in K, \ \operatorname{div}(z) = 0, \ \text{ on } \mathbb{T}^n,$$

for a suitable set $K \subset \mathbb{R}^{n\times n}$. For the divergence operator we have

$$\Lambda_{\text{div}} = \{M \in \mathbb{R}^{n\times n} : \text{rank}(M) \leq n-1\}.$$

In [18], the authors exploit the richness of rank- (n - 1) segments in the differential inclusion defining Euler equations to find weak solutions to Euler equations with the required properties, see [18, Section 2]. Refining these methods, De Lellis and Székelyhidi made the progresses in the solution of Onsager conjecture that finally led to its solution, ultimately inventing the convex integration scheme that is used in Part iii.

On the other hand, if a matrix-field z satisfying $\mathscr{A}z = 0$ is in some sense *far* from $\Lambda_{\mathscr{A}}$, one expects some elliptic estimates. Examples are given by the improvement in the integrability of the determinant for Hessians of convex functions, see Appendix C, or the regularizing properties of mappings of bounded distorsion, see [72, II.1.2]. The result by Serre that was mentioned in Section 1.2 follows this line, indeed to a divergence-free matrix field z with values in $\text{Sym}^+(n)$, one can always add $\varepsilon\,\text{id}$ to have

$$\mathrm{d}(z(x) + \varepsilon\,\text{id}_n, \Lambda_{\text{div}}) \geq \varepsilon, \text{ for a.e. } x \in \mathbb{T}^n.$$

As discussed in Part ii, one gets an elliptic improvement in the integrability of the function $x \mapsto \det^{\frac{1}{n-1}}(\cdot)$, and these are stable when letting $\varepsilon \to 0$. Nonetheless, the result presented in the first chapter of Part ii shows how these L^1 estimates cannot be turned into L^p estimates, and ultimately that we are still far from a complete understanding of the relation between the wave cone and elliptic estimates in the form of functional inequalities.

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Part I

DIFFERENTIAL INCLUSIONS RELATED TO GEOMETRIC PROBLEMS

2

ABSENCE OF T'_N CONFIGURATIONS

This chapter is devoted to the proof of the following:

Theorem 2.1. *If $f \in C^1(\mathbb{R}^{n\times m})$ is a strictly polyconvex function, then C_f does not contain any set $\{A_1,\dots,A_N\} \subset \mathbb{R}^{(2n+m)\times m}$ which induces a T'_N configuration, provided that $f(X_1) \geq 0,\dots,f(X_N) \geq 0$, if*

$$A_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}, \quad X_i, Y_i \in \mathbb{R}^{n\times m}, Z_i \in \mathbb{R}^{m\times m}, \forall i \in \{1,\dots,N\}.$$

This result is part of the forthcoming paper [41]. As a consequence of this, we obtain as a corollary the following main result of [17]

Theorem 2.2. *If $f \in C^1(\mathbb{R}^{n\times m})$ is strictly polyconvex, then K_f does not contain any set $\{A_1,\dots,A_N\}$ which induces a T'_N configuration.*

The chapter is organized as follows: in Section 2.1 we rewrite the Euler Lagrange defining stationary points as a div-curl differential inclusion and we determine its wave cone. We then introduce the inclusion sets C_f and K_f appearing in the statements of the previous theorems and, after recalling the definition of T_N configurations for classical differential inclusions, we define corresponding T'_N configurations for div-curl differential inclusions. In Section 2.2 we give a small extension of a key result of [83] on classical T_N configurations. In Section 2.3 we consider arbitrary sets of N matrices and give an algebraic characterization of those sets which belong to an inclusion set K_f for some strictly polyconvex f. In Section 2.4 we then prove the Theorem 2.1 and deduce Theorem 2.2.

2.1 DIV-CURL DIFFERENTIAL INCLUSIONS, WAVE CONES AND INCLUSION SETS

As written in Section 1.1, the Euler-Lagrange conditions for energies $\mathbb{E}$ of the form

$$\mathbb{E} = \int_\Omega f(Du(x))dx \tag{2.1}$$

are given by:

$$\begin{cases} \displaystyle\int_\Omega \langle Df(Du), Dv\rangle\, dx = 0 & \forall v \in C^1_c(\Omega,\mathbb{R}^n) \\ \displaystyle\int_\Omega \langle Df(Du), DuD\Phi\rangle dx - \int_\Omega f(Du)\operatorname{div}\Phi dx = 0 & \forall \Phi \in C^1_c(\Omega,\mathbb{R}^m), \end{cases} \tag{2.2}$$

Here we rewrite the system (2.2) as a differential inclusion. To do so, it is sufficient to notice that the left hand side of the second equation can be rewritten as

$$\begin{aligned}\int_\Omega \langle Df(Du), DuD\Phi\rangle dx - \int_\Omega f(Du)\operatorname{div}\Phi dx &= \int_\Omega \langle Du^T Df(Du), D\Phi\rangle - \langle f(Du)\,\mathrm{id}, Dg\rangle dx \\ &= \int_\Omega \langle Du^T Df(Du) - f(Du)\,\mathrm{id}, D\Phi\rangle dx\end{aligned}$$

Hence, the inner variation equation is the weak formulation of

$$\operatorname{div}(Du^T Df(Du) - f(Du)\,\mathrm{id}) = 0.$$

Since also the outer variation is the weak formulation of a PDE in divergence form, namely

$$\operatorname{div}(Df(Du)) = 0,$$

we consider the following *div-curl differential inclusion* for a triple of maps $X, Y \in L^\infty(\Omega, \mathbb{R}^{n\times m})$ and $Z \in L^\infty(\Omega, \mathbb{R}^{m\times m})$:

$$\operatorname{curl} X = 0, \qquad \operatorname{div} Y = 0, \qquad \operatorname{div} Z = 0\,, \tag{2.3}$$

$$W \doteq \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in K_f \doteq \left\{ A \in \mathbb{R}^{(2n+m)\times m} : A = \begin{pmatrix} X \\ Df(X) \\ X^T Df(X) - f(X)\,\mathrm{id} \end{pmatrix} \right\}, \tag{2.4}$$

where $f \in C^1(\mathbb{R}^{n\times m})$ is a fixed function.

Moreover, we also consider the following more general system of PDEs, for $u \in \operatorname{Lip}(\Omega, \mathbb{R}^n)$ and a Borel map $\beta \in L^\infty(\Omega, (0, +\infty))$:

$$\begin{cases} \int_\Omega \langle Df(Du), Dv\rangle \beta \, dx = 0 & \forall v \in C^1_c(\Omega, \mathbb{R}^n) \\ \int_\Omega \langle Df(Du), DuD\Phi\rangle \beta \, dx - \int_\Omega f(Du) \operatorname{div} \Phi \beta \, dx = 0 & \forall \Phi \in C^1_c(\Omega, \mathbb{R}^m). \end{cases} \tag{2.5}$$

This system is equivalent to the stationarity in the sense of varifolds of the varifold $V = (\Gamma_u, \beta)$, where Γ_u is the graph of u. This will be discussed in detail in Chapter 3. The div-curl differential inclusion associated to this system is, again for a triple of maps $X, Y \in L^\infty(\Omega, \mathbb{R}^{n\times m})$ and $Z \in L^\infty(\Omega, \mathbb{R}^{m\times m})$:

$$\operatorname{curl} X = 0, \qquad \operatorname{div} Y = 0, \qquad \operatorname{div} Z = 0\,, \tag{2.6}$$

$$W \doteq \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in C_f \tag{2.7}$$

where

$$C_f \doteq \left\{ C \in \mathbb{R}^{(2n+m)\times m} : C = \begin{pmatrix} X \\ \beta Df(X) \\ \beta X^T Df(X) - \beta f(X)\,\mathrm{id} \end{pmatrix}, \text{ for some } \beta > 0 \right\},$$

The following lemma is then an obvious consequence of the above discussion

Lemma 2.3. *Let $f \in C^1(\mathbb{R}^{n\times m})$. A map $u \in \operatorname{Lip}(\Omega, \mathbb{R}^n)$ is a stationary point of the energy (2.1) if and only there are matrix fields $Y \in L^\infty(\Omega, \mathbb{R}^{n\times m})$ and $Z \in L^\infty(\Omega, \mathbb{R}^{m\times m})$ such that $W = (Du, Y, Z)$ solves the div-curl differential inclusion (2.3)-(2.4). Moreover, the couple $(u, \beta) \in \operatorname{Lip}(\Omega, \mathbb{R}^n) \times L^\infty(\Omega, (0, +\infty))$ solves (2.5) if and only there are matrix fields $Y \in L^\infty(\Omega, \mathbb{R}^{n\times m})$ and $Z \in L^\infty(\Omega, \mathbb{R}^{m\times m})$ such that $W = (Du, Y, Z)$ solves the div-curl differential inclusion (2.6)-(2.7).*

Motivated by the arguments of Section 1.4, we introduce here the Λ-cone, recall Definition 1.8, of the mixed *div-curl* operator we are considering in this chapter.

Lemma 2.4. *The wave cone of the system $\operatorname{curl} X = 0$ is given by rank one matrices, whereas the wave cone for the system (2.3) is given by triple of matrices (X, Y, Z) for which there is a unit vector $\xi \in \mathbb{S}^{m-1}$ and a vector $u \in \mathbb{R}^n$ such that $X = u \otimes \xi$, $Y\xi = 0$ and $Z\xi = 0$.*

Motivated by the above lemma we then define

Definition 2.5. The cone $\Lambda_{dc} \subset \mathbb{R}^{(2n+m)\times m}$ consists of the matrices in block form

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

with the property that there is a direction $\xi \in \mathbb{S}^{m-1}$ and a vector $u \in \mathbb{R}^n$ such that $X = u \otimes \xi$, $Y\xi = 0$ and $Z\xi = 0$.

2.1.1 T_N *configurations*

We start definining T_N configurations for *classical* curl-type differential inclusions.

Definition 2.6. An ordered set of $N \geq 2$ matrices $\{X_i\}_{i=1}^N \subset \mathbb{R}^{n\times m}$ of distinct matrices is said to *induce a* T_N *configuration* if there exist matrices $P, C_i \in \mathbb{R}^{n\times m}$ and real numbers $k_i > 1$ such that:

(a) Each C_i belongs to the wave cone of $\operatorname{curl} X = 0$, namely $\operatorname{rank}(C_i) \leq 1$ for each i;

(b) $\sum_i C_i = 0$;

(c) $X_1, \dots, X_N$, P and $C_1, \dots, C_N$ satisfy the following N linear conditions

$$\begin{aligned} X_1 &= P + k_1 C_1, \\ X_2 &= P + C_1 + k_2 C_2, \\ &\dots \\ &\dots \\ X_N &= P + C_1 + \dots + k_N C_N\,. \end{aligned} \tag{2.8}$$

In the rest of the chapter we will use the word T_N configuration for the data

$$P, C_1, \dots, C_N, k_1, \dots k_N.$$

We will moreover say that the configuration is *nondegenerate* if $\operatorname{rank}(C_i) = 1$ for every i.

Note that our definition is more general that the one usually given in the literature (cf. [62, 82, 83]) because we drop the requirement that there are no rank one connections between distinct X_i and X_j. Moreover, rather than calling $\{X_1, \dots, X_N\}$ a T_N configuration, we prefer to say that it *induces* a T_N configuration, namely we regard the whole data $X_1, \dots, X_N, C_1, \dots, C_N, k_1, \dots, k_N$ since it is not at all clear that given an ordered set $\{X_1, \dots, X_N\}$ of distinct matrices there can be more than one choice of the matrices $C_1, \dots, C_N$ and of the coefficients $k_1, \dots, k_N$ satisfying the conditions above (if we drop the condition that the set is ordered, then it is known that there is more than one choice, see [37]). We observe that the definition of T_N configuration could be split into two parts. A *geometric part*', namely the points (b) and (c), can be considered as characterizing a certain *arrangement of* $2N$ *points* in the space of matrices, consisting of:

- A closed piecewise linear loop, loosely speaking a polygon (not necessarily *planar*) with vertices $P_1 = P + C_1, P_2 = P + C_1 + C_2, \dots, P_N = P + C_1 + \dots + C_N = P$;
- N additional "arms" which extend the sides of the polygon, ending in the points $X_1, \dots, X_N$.

See Figure 2 for a graphical illustration of these facts in the case $N = 4$.
The closing condition in Definition 2.6(b) is a necessary and sufficient condition for the polygonal line to "close". Condition (c) determines that each X_i is a point on the line containing the segment $P_{i-1}P_i$. Note that the inequality $k_i > 1$ ensures that X_i is external to the segment, "on the side of P_i". The "nondegeneracy" condition is equivalent to the vertices of the polygon being all distinct. Note moreover that, in view of our definition, we include the possibility $N = 2$. In the latter case the T_2 configuration consists of a single rank one line and of 4 points X_1, X_2, C_1, C_2 lying on it. We have decided to follow this convention, even though this is an unusual choice compared to the literature. The second part of the Definition, namely condition (a), is of algebraic nature and related to the fact that T_N configurations are used to study "classical differential inclusions", namely PDEs of the form $\operatorname{curl} X = 0$. The condition prescribes simply that each vector $X_i - P_i$ belongs to the wave cone of $\operatorname{curl} X = 0$.

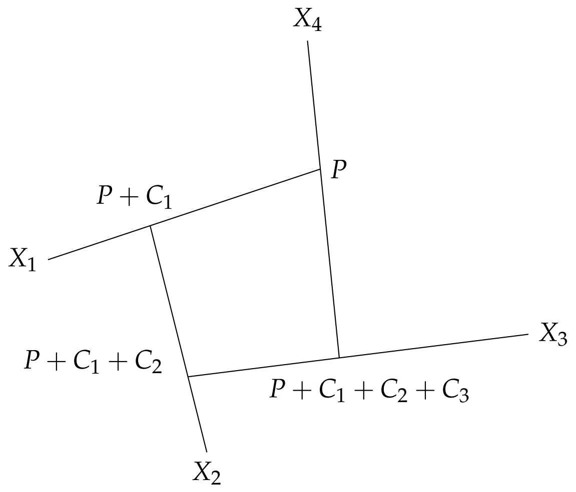

Figure 2: The geometric arrangement of a T_4 configuration.

2.1.2 T'_N *configurations*

In this section we generalize the notion of T_N configuration to div-curl differential inclusions. The geometric arrangement remains the same, while the wave cone condition is replaced by the one dictated by the new PDE (2.3).

Definition 2.7. A family $\{A_1, \dots, A_N\} \subset \mathbb{R}^{(2n+m)\times m}$ of $N \geq 2$ *distinct*

$$A_i \doteq \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

induces a T'_N *configuration* if there are matrices $P, Q, C_i, D_i \in \mathbb{R}^{n\times m}$, $R, E_i \in \mathbb{R}^{m\times m}$ and coefficients $k_i > 1$ such that

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \begin{pmatrix} C_1 \\ D_1 \\ E_1 \end{pmatrix} + \cdots + \begin{pmatrix} C_{i-1} \\ D_{i-1} \\ E_{i-1} \end{pmatrix} + k_i \begin{pmatrix} C_i \\ D_i \\ E_i \end{pmatrix} \tag{2.9}$$

and the following properties hold:

(a) each element (C_i, D_i, E_i) belongs to the wave cone Λ_{dc} of (2.3);

(b) $\sum_\ell C_\ell = 0$, $\sum_\ell D_\ell = 0$ and $\sum_\ell E_\ell = 0$.

We say that the T'_N configuration is *nondegenerate* if $\operatorname{rank}(C_i) = 1$ for every i.

We collect here some simple consequences of the definition above and of the discussion on T_N configurations.

Proposition 2.8. *Assume $A_1, \dots, A_N$ induce a T'_N configuration with P, Q, R, C_i, D_i, E_i and k_i as in Definition 2.7. Then:*

(i) $\{X_1, \dots, X_N\}$ induce a T_N configuration of the form (2.8), if they are distinct; moreover the T'_N configuration is nondegenerate if and only if the T_N configuration induced by $\{X_1, \dots, X_N\}$ is nondegenerate;

(ii) For each i there is an $n_i \in \mathbb{S}^{m-1}$ and a $u_i \in \mathbb{R}^n$ such that $C_i = u_i \otimes n_i$, $D_i n_i = 0$ and $E_i n_i = 0$;

(iii) $\operatorname{tr} C_i^T D_i = \langle C_i, D_i \rangle = 0$ for every i.

Proof. (i) and (ii) are an obvious consequence of Definition 2.7 and of Definition 2.5. After extending n_i to an orthonormal basis $\{n_i, v_2^j, \dots v_m^j\}$ of $\mathbb{R}^m$ we can explicitely compute

$$\langle C_i, D_i \rangle = (D_i n_i, C_i n_i) + \sum_{j=2}^{m} (D_i v_i^j, C_i v_i^j) = 0\,,$$

where $(\cdot, \cdot)$ denotes the Euclidean scalar product. □

2.1.3 *Strategy*

Before starting with the proof of the main result of this chapter, it is convenient to explain the strategy we intend to follow. In order to do so, let us consider the simplest case $n = m = 2$, $N = 5$. Suppose by contradiction that there exists a strictly polyconvex function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$, $f(X) = g(X, \det(X))$ and a T_5' configuration A_1, A_2, A_3, A_4, A_5,

$$A_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}, \quad \forall i \in \{1, \dots, 5\},$$

where X_i, Y_i, Z_i fulfill the relations of (2.9), i.e.

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \begin{pmatrix} C_1 \\ D_1 \\ E_1 \end{pmatrix} + \dots + \begin{pmatrix} C_{i-1} \\ D_{i-1} \\ E_{i-1} \end{pmatrix} + k_i \begin{pmatrix} C_i \\ D_i \\ E_i \end{pmatrix}.$$

For convenience, let us consider $P = 0$. We will prove in Lemma 2.23 that this can be done without loss of generality. It is convenient to think of the relations $A_i \in C_f$, $\forall i$ as two separate pieces of information:

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \in K_f' = \left\{ A \in \mathbb{R}^{4\times 2} : A = \begin{pmatrix} X \\ \beta Df(X) \end{pmatrix}, \beta > 0, X \in \mathbb{R}^{2\times 2} \right\} \tag{2.10}$$

and

$$Z_i = X_i^T Y_i - \beta_i f(X_i)\,\mathrm{id}\,. \tag{2.11}$$

Let us denote with $c_i \doteq f(X_i)$. Similarly to the procedure of [82], we exploit the polyconvexity of f to rewrite (2.10) in terms of inequalities involving X_i, Y_i, c_i, d_i, where $d_i \doteq \partial_{y_5} g(y_1, y_2, y_3, y_4, y_5)|_{(X_i, \det(X_i))}$, of the form

$$c_i - c_j + \frac{1}{\beta_i} \langle Y_i, X_j - X_i \rangle - d_i \det(X_i - X_j) < 0. \tag{2.12}$$

This is the content of Proposition 2.19. The final goal is to prove that these inequalities can not be fulfilled at the same time. The previous expression can be considerably simplified by the structure result on T_N configurations in $\mathbb{R}^{2\times 2}$ of [83, Proposition 1]. This asserts, in the specific case of the ongoing example, the existence of 5 vectors $(t_1^i, \dots, t_5^i), i \in \{1, \dots, 5\}$ with positive components, such that

$$\sum_{j=1}^{5} t_j^i \det(X_j - X_i) = 0. \tag{2.13}$$

If we use this result in (2.12), we can eliminate from the expression the variable d_i, thus obtaining

$$\begin{aligned} \nu_i &\doteq \sum_{j=1}^{5} t_j^i (c_i - c_j + \frac{1}{\beta_i} \langle Y_i, X_j - X_i \rangle - d_i \det(X_i - X_j)) \\ &= \sum_{j=1}^{5} t_j^i (c_i - c_j + \frac{1}{\beta_i} \langle Y_i, X_j - X_i \rangle) < 0, \quad \forall i \in \{1, \dots, 5\}, \end{aligned}$$

compare Corollary 2.20. Section 2.2 is devoted to extending relations (2.13) to general T_N configurations in $\mathbb{R}^{n\times m}$. Despite being very useful, this simplification can not conclude the proof. Indeed, up to now we have exploited (2.10) and the fact that $\{X_1,\dots,X_5\}$ induce a T_5 configuration, but, if $\beta_i = 1, \forall i$, this is the exact same situation of [82]. Since from that paper we know the existence of T_5 configurations in K'_f, clearly we can not reach a contradiction at this point of the strategy. This is where the inner variations come into play. In the proof of Theorem 2.1, we rewrite (2.11) using the definition of T'_5 configuration and, after some manipulations, we find that the numbers

$$\mu_i \doteq \sum_{j=1}^{5} t^i_j(\langle X_j - X_i, Y_i\rangle - \beta_i c_i + \beta_j c_j)$$

must all be 0. For the index I such that $\beta_I = \min_i \beta_i$, and essentially using the positivity of c_j, we find that

$$0 = \mu_I = \sum_{j=1}^{5} t^i_j(\langle X_j - X_i, Y_i\rangle - \beta_i c_i + \beta_j c_j) \le \sum_{j=1}^{5} t^i_j(\langle X_j - X_i, Y_i\rangle - \beta_i c_i + \beta_I c_j) = \nu_I,$$

which is in contradiction with the negativity of ν_I.

2.2 PRELIMINARIES ON CLASSICAL T_N CONFIGURATIONS

This section is devoted to a generalization of a powerful machinery introduced in [83] to study T_N configurations.

2.2.1 *Székelyhidi's characterization of T_N configurations in $\mathbb{R}^{2\times 2}$*

We start with the following elegant characterization.

Proposition 2.9. *([83, Proposition 1]) Given a set $\{X_1,\dots,X_N\} \subset \mathbb{R}^{2\times 2}$ and $\mu \in \mathbb{R}$, we let A^μ be the following $N \times N$ matrix:*

$$A^\mu \doteq \begin{pmatrix} 0 & \det(X_1 - X_2) & \det(X_1 - X_3) & \dots & \det(X_1 - X_N) \\ \mu\det(X_1 - X_2) & 0 & \det(X_2 - X_3) & \dots & \det(X_2 - X_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu\det(X_1 - X_N) & \mu\det(X_2 - X_N) & \mu\det(X_3 - X_N) & \dots & 0 \end{pmatrix}.$$

Then, $\{X_1,\dots,X_N\}$ induces a T_N configuration if and only if there exists a vector $\lambda \in \mathbb{R}^N$ with positive components and $\mu > 1$ such that

$$A^\mu\lambda = 0.$$

Even though not explicitely stated in [83], the following Corollary is part of the proof of Proposition 2.9 and it is worth stating it here again, since we will make extensive use of it in the sequel.

Corollary 2.10. *Let $\{X_1,\dots,X_N\} \subset \mathbb{R}^{2\times 2}$ and let $\mu > 1$ and $\lambda \in \mathbb{R}^N$ be a vector with positive entries such that $A^\mu\lambda = 0$. Define the vectors*

$$t^i \doteq \frac{1}{\xi_i}(\mu\lambda_1,\dots,\mu\lambda_{i-1},\lambda_i,\dots,\lambda_N), \text{ for } i \in \{1,\dots,N\} \tag{2.14}$$

where $\xi_i > 0$ is a normalizing constant so that $\|t^i\|_1 \doteq \sum_j |t^i_j| = 1, \forall i$. Define the matrices C_j with $j \in \{1,\dots,N-1\}$ and P by solving recursively

$$\sum_{j=1}^{N} t^i_j X_j = P + C_1 + \dots + C_{i-1} \tag{2.15}$$

and set $C_N \doteq -C_1 - \ldots - C_{N-1}$*. Finally, define*

$$k_i = \frac{\mu\lambda_1 + \cdots + \mu\lambda_i + \lambda_{i+1} \cdots + \lambda_N}{(\mu - 1)\lambda_i}. \tag{2.16}$$

Then $P, C_1, \ldots, C_N$ *and* $k_1, \ldots k_N$ *give a* T_N *configuration induced by* $\{X_1, \ldots, X_N\}$ *(i.e. (2.8) holds).*
Moreover, the following relation holds for every i:

$$\det\left(\sum_{j=1}^{N} t_j^i X_j\right) = \sum_{j=1}^{N} t_j^i \det(X_j). \tag{2.17}$$

Remark 2.11. Observe that the relations (2.16) can be inverted in order to compute μ and λ (the latter up to scalar multiples) in terms of $k_1, \ldots, k_N$. In fact, let us impose

$$\|\lambda\|_1 = \lambda_1 + \cdots + \lambda_N = 1.$$

Then, regarding μ as a parameter, the equations (2.16) give a linear system in triangular form which can be explicitely solved recursively, giving the formula

$$\lambda_j = \frac{k_1 k_2 \cdots k_{j-1}}{(\mu - 1)(k_1 - 1)(k_2 - 1) \cdots (k_j - 1)}. \tag{2.18}$$

The following identity can easily be proved by induction:

$$\frac{1}{k_1 - 1} + \frac{k_1}{(k_1 - 1)(k_2 - 1)} + \cdots + \frac{k_1 \cdots k_{j-1}}{(k_1 - 1) \cdots (k_j - 1)} = \frac{k_1 \cdots k_j}{(k_1 - 1) \cdots (k_j - 1)} - 1.$$

Hence, summing (2.18) and imposing $\sum_j \lambda_j = 1$ we find the equation

$$1 = \frac{1}{\mu - 1}\left(\frac{k_1 \cdots k_N}{(k_1 - 1) \cdots (k_N - 1)} - 1\right),$$

which determines uniquely μ as

$$\mu = \frac{k_1 \cdots k_N}{(k_1 - 1) \cdots (k_N - 1)}. \tag{2.19}$$

A second corollary of the computations in [83] is that

Corollary 2.12. *Assume* $\{X_1, \ldots, X_N\} \in \mathbb{R}^{2\times 2}$ *induce the* T_N *configuration of form (2.8) and let* μ *and* λ *be as in (2.18) and (2.19). Then* $A^\mu \lambda = 0$*.*

2.2.2 *A characterization of* T_N *configurations in* $\mathbb{R}^{n\times m}$

We start with a straightforward consequence of the results above. Let us first introduce some notation concerning multi-indexes. We will use I for multi-indexes referring to ordered sets of rows of matrices and J for multi-indexes referring to ordered sets of columns. In our specific case, where we deal with matrices in $\mathbb{R}^{n\times m}$ we will thus have

$$\begin{aligned} & I = (i_1, \ldots, i_r), && 1 \le i_1 < \cdots < i_r \le n, \\ \text{and} \quad & J = (j_1, \ldots, j_s), && 1 \le j_1 < \cdots < j_s \le m \end{aligned}$$

and we will use the notation $|I| \doteq r$ and $|J| \doteq s$. In the sequel we will always have $r = s$.

Definition 2.13. We denote by $\mathcal{A}_r$ the set

$$\mathcal{A}_r = \{(I, J) : |I| = |J| = r\}, \qquad 2 \le r \le \min(n, m).$$

For a matrix $M \in \mathbb{R}^{n\times m}$ and for $Z \in \mathcal{A}_r$ of the form $Z = (I, J)$, we denote by M^Z the squared $r \times r$ matrix obtained by A considering just the elements a_{ij} with $i \in I$, $j \in J$ (using the order induced by I and J).

Given a set $\{X_1, \dots, X_N\} \subset \mathbb{R}^{n\times m}$, $\mu \in \mathbb{R}$ and $Z \in \mathcal{A}_r$, we introduce the matrix

$$A_Z^\mu \doteq \begin{pmatrix} 0 & \det(X_2^Z - X_1^Z) & \det(X_3^Z - X_1^Z) & \dots & \det(X_N^Z - X_1^Z) \\ \mu\det(X_1^Z - X_2^Z) & 0 & \det(X_3^Z - X_2^Z) & \dots & \det(X_N^Z - X_2^Z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu\det(X_1^Z - X_N^Z) & \mu\det(X_2^Z - X_N^Z) & \mu\det(X_3^Z - X_N^Z) & \dots & 0 \end{pmatrix}.$$

Proposition 2.14. *A set $\{X_1, \dots, X_N\} \subset \mathbb{R}^{n\times m}$ induces a T_N configuration if and only if there is a real $\mu > 1$ and a vector $\lambda \in \mathbb{R}^N$ with positive components such that*

$$A_Z^\mu \lambda = 0 \qquad \forall Z \in \mathcal{A}_2\,.$$

Moreover, if we define the vectors t^i as in (2.14), the coefficients k_i through (2.16) and the matrices P and C_i through (2.15), then $P, C_1, \dots, C_N$ and $k_1, \dots, k_N$ give a T_N configuration induced by $\{X_1, \dots, X_N\}$.

For this reason and in view of Remark 2.11, we can introduce the following terminology:

Definition 2.15. Given a T_N-configuration $P, C_1, \dots, C_N$ and $k_1, \dots, k_N$ we let μ and λ be given by (2.18) and (2.19) and we call $(\lambda, \mu) \in \mathbb{R}^{N+1}$ the *defining vector* of the T_N configuration.

Proof of Proposition 2.14. **Direction** $\Longleftarrow$. Fix a set $\{X_1, \dots, X_N\}$ of matrices with the property that there is a common $\mu > 1$ and a common λ with positive entries such that $A_Z^\mu \lambda = 0$ for every $Z \in \mathcal{A}_2$. For each Z we consider the corresponding set $\{X_1^Z, \dots, Z_N^Z\}$ and we use formulas (2.14), (2.16) and (2.15) to find $k_1, \dots, k_N$, $P(Z)$ and $C_i(Z)$ such that

$$X_i^Z = P(Z) + C_1(Z) + \dots C_{i-1}(Z) + k_i C_i(Z)\,.$$

Since the coefficients k_i are independent of Z, the formulas give that the matrices $C_i(Z)$ (and $P(Z)$) are compactible, in the sense that, if $j\ell$ is an entry common to Z and Z', then $(C_i(Z))_{j\ell} = (C_i(Z'))_{j\ell}$. In particular there are matrics C_i's and P such that $C_i(Z) = C_i^Z$ and $P(Z) = P^Z$ and thus (2.8) holds. Moreover, we also know from Proposition 2.9 that $\operatorname{rank}(C_i^Z) \le 1$ for every Z and thus $\operatorname{rank}(C_i) \le 1$. We also know that $C_1^Z + \dots + C_N^Z = 0$ for every Z and thus $C_1 + \dots + C_N = 0$.

Direction $\Longrightarrow$. Assume $X_1, \dots, X_N$ induce a T_N configuration as in (2.8). Then $X_1^Z, \dots, X_N^Z$ induce a T_N configuration with corresponding $P^Z, C_1^Z, \dots, C_N^Z$ and $k_1, \dots, k_N$, where the latter coefficients are independent of Z. Thus, by Corollary 2.12, $A_Z^\mu \lambda = 0$. □

2.2.3 *Computing minors*

We end this section with a further generalization, this time of (2.17): we want to extend the validity of it to any minor.

Proposition 2.16. *Let $\{X_1, \dots, X_N\} \subset \mathbb{R}^{n\times m}$ induce a T_N configuration as in (2.8) with defining vector (λ, μ). Define the vectors $t^1, \dots, t^N$ as in (2.14) and for every $Z \in \mathcal{A}_r$ of order $r \le \min\{n, m\}$ define the minor $\mathcal{S} : \mathbb{R}^{n\times m} \ni X \mapsto \mathcal{S}(X) \doteq \det(X^Z) \in \mathbb{R}$. Then*

$$\sum_{j=1}^N t_j^i \mathcal{S}(X_j) = \mathcal{S}\left(\sum_{j=1}^N t_j^i X_j\right) = \mathcal{S}(P + C_1 + \dots + C_{i-1})\,. \tag{2.20}$$

and $A_Z^\mu \lambda = 0$.

We will need the following elementary linear algebra fact, which in the literature is sometimes called Matrix Determinant Lemma:

Lemma 2.17. *Let A, B be matrices in $\mathbb{R}^{m\times m}$, and let $\operatorname{rank}(B) \le 1$. Then,*

$$\det(A+B) = \det(A) + \langle \operatorname{cof}(A)^T, B\rangle$$

Moreover, we need another elementary computation, which is essentially contained in [83] and for which we report the proof at the end of the section for the reader's convenience.

Lemma 2.18. *Assume the real numbers $\mu > 1$, $\lambda_1, \dots, \lambda_N > 0$ and $k_1, \dots, k_N > 1$ are linked by the formulas (2.16). Assume $v, v_1, \dots, v_N, w_1, \dots, w_N$ are elements of a vector space satisfying the relations*

$$w_i = v + v_1 + \ldots + v_{i-1} + k_i v_i \tag{2.21}$$

$$0 = v_1 + \ldots + v_N\,. \tag{2.22}$$

If we define the vectors t^i as in (2.14), then

$$\sum_j t^i_j w_j = v + v_1 + \ldots + v_{i-1}\,. \tag{2.23}$$

Proof of Proposition 2.16. Fix the Z of the statement of the proposition. $X_1^Z, \dots, X_N^Z$ induces T_N with the same coefficients $k_1, \dots k_N$. This reduces therefore the statement to the case in which $m = n$, $Z = ((1, \dots n), (1, \dots, n))$ and the minor $\mathcal{S}$ is the usual determinant.

We first prove (2.20). In order to do this we specialize (2.23) to $w_\ell = \det(X_\ell)$, $v = \det(P)$, $v_\ell = \langle \operatorname{cof}^T(P + C_1 + \dots + C_{\ell-1}), C_\ell\rangle$. To simplify the notation set

$$P^{(1)} = P, \text{ and } P^{(\ell)} = P + C_1 + \dots + C_{\ell-1} \qquad \forall \ell \in \{1, \dots, N+1\}.$$

We want to show that

$$v + v_1 + \dots + v_{i-1} = \det(P^{(i)}) \text{ and } v_1 + \dots + v_N = 0,$$

and this would conclude the proof of (2.20) because of Lemma 2.18. A repeated application of Lemma 2.17 yields:

$$v + v_1 + \dots + v_{i-1} = \underbrace{\underbrace{\det(P) + \langle \operatorname{cof}^T(P), C_1\rangle}_{\det(P^{(2)})} + \langle \operatorname{cof}^T(P^{(2)}), C_2\rangle}_{\det(P^{(3)})} +$$

$$+ \dots + \langle \operatorname{cof}^T(P^{(i)}), C_{i-1}\rangle = \det(P^{(i)}) = \det(P + C_1 + \dots + C_{i-1}).$$

As a consequence of Lemma 2.17, we also have $v_\ell = \det(P^{(\ell+1)}) - \det(P^{(\ell)})$. Therefore:

$$v_1 + \dots + v_N = \sum_{\ell=1}^N \left(\det(P^{(\ell+1)}) - \det(P^{(\ell)})\right) = \det(P^{(N+1)}) - \det(P^{(1)}).$$

Since $\sum_\ell C_\ell = 0$ and $\det(P^{(N+1)}) = \det(P + \sum_\ell C_\ell)$, we have

$$\det(P^{(N+1)}) - \det(P^{(1)}) = \det(P + \sum_\ell C_\ell) - \det(P) = \det(P) - \det(P) = 0,$$

and the conclusion is thus reached.

To prove the second part of the statement notice that $A^\mu_Z \lambda = 0$ is equivalent to the following N equations:

$$\sum_{j=1}^N t^i_j \det(X_j - X_i) = 0 \qquad \forall i \in \{1, \dots, N\}.$$

Fix $i \in \{1, \dots, N\}$ and define matrices $Y_j \doteq X_j - X_i, \forall j$. $\{Y_1, \dots, Y_N\}$ is still a T_N configuration of the form

$$Y_i = P' + \sum_{\ell=1}^{i-1} C_\ell + k_i C_i,$$

and $P' = -X_i$ (recall that $P = 0$). Apply now (2.20) to find that

$$\sum_j t^i_j \det(X_j - X_i) = \sum_j t^i_j \det(Y_j)$$
$$= \det\left(P' + \sum_{\ell=1}^{i-1} C_\ell\right) = \det\left(-X_i + \sum_{\ell=1}^{i-1} C_\ell\right) = \det(-k_i C_i) = 0$$

and conclude the proof. □

2.2.4 *Proof of Lemma 2.18*

It is sufficient to compute separately $\sum_{j=1}^N t^1_j w_j = \sum_{j=1}^N \lambda_j w_j$ and $\sum_{j=1}^{i-1} \lambda_j w_j$. In fact,

$$\sum_j^N t^i_j w_j = \frac{1}{\xi_i}\left[\sum_{j=1}^N \lambda_j w_j + (\mu - 1)\sum_{j=1}^{i-1} \lambda_j w_j\right]. \tag{2.24}$$

We can write

$$\sum_j \lambda_j w_j = v + a_1 v_1 + \dots + a_N v_N,$$

being, $\forall \ell \in \{1, \dots, N\}, a_\ell = k_\ell \lambda_\ell + \dots + \lambda_N$. Recalling that the defining vector and the numbers k_i are related through (2.16), we compute

$$\begin{aligned} a_\ell = k_\ell \lambda_\ell + \dots + \lambda_N &= \frac{\mu\lambda_1 + \dots + \mu\lambda_\ell + \lambda_{\ell+1} + \dots \lambda_N}{\mu - 1} + \lambda_{\ell+1} + \dots + \lambda_N \\ = \frac{\mu(\lambda_1 + \dots + \lambda_N)}{\mu - 1} &= \frac{\mu}{\mu - 1} =: a. \end{aligned} \tag{2.25}$$

Hence

$$\sum_{j=1}^N \lambda_j w_j = v + \frac{\mu}{\mu - 1}(v_1 + \dots + v_N).$$

On the other hand,

$$\sum_{j=1}^{i-1} \lambda_j w_j = b_1 v + b_2 v_1 + \dots + b_i v_{i-1},$$

and

$$\begin{aligned} &b_1 = \lambda_1 + \dots + \lambda_{i-1} =: c, \\ &b_\ell = k_\ell \lambda_\ell + \dots + \lambda_{i-1} = \frac{\mu(\lambda_1 + \dots + \lambda_\ell) + \sum_{j=\ell+1}^N \lambda_j + (\mu - 1)\sum_{j=\ell+1}^{i-1} \lambda_j}{\mu - 1} = \\ &= \frac{\mu(\sum_{j=1}^{i-1} \lambda_j) + \sum_{j=i}^N \lambda_j}{\mu - 1} =: b, \forall \ell \in \{2, \dots, i\}. \end{aligned}$$

Also,

$$\xi_i = \|(\mu\lambda_1, \dots, \mu\lambda_{i-1}, \lambda_i, \dots, \lambda_N)\|_1 = (\mu - 1)(\lambda_1 + \dots + \lambda_{i-1}) + 1 = (\mu - 1)b = 1 + (\mu - 1)c.$$

We can now compute (2.24):

$$\begin{aligned} &\frac{1}{\xi_i}\left[\sum_{j=1}^N \lambda_j w_j + (\mu - 1)\sum_{j=1}^{i-1} \lambda_j w_j\right] = \\ &\frac{1}{\xi_i}\left[v + a_1 v_1 + \dots + a_N v_N + (\mu - 1)(b_1 v + b_2 v_1 + \dots + b_i v_{i-1})\right] = \end{aligned}$$

$$\frac{1}{\xi_i}\left[(\mu-1)b(v+v_1+\cdots+v_{i-1})+a(v_1+\cdots+v_N)\right]=$$
$$v+v_1+\cdots+v_{i-1}+\frac{a}{(\mu-1)b}(v_1+\cdots+v_N)$$

We use the just obtained identity

$$\sum_{j=1}^{N} t^i_j w_j = v+v_1+\cdots+v_{i-1}+\frac{a}{(\mu-1)b}(v_1+\cdots+v_N) \tag{2.26}$$

Using that $v_1+\ldots+v_N=0$ we conclude the desired identity.

2.3 inclusions sets relative to polyconvex functions

In this section we consider the following question. Given a set of distinct matrices $A_i \in \mathbb{R}^{(2n)\times m}$

$$A_i \doteq \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \tag{2.27}$$

do they belong to a set of the form

$$K'_f \doteq \left\{ \begin{pmatrix} X \\ Df(X) \end{pmatrix} : X \in \mathbb{R}^{n\times m} \right\} \tag{2.28}$$

for some strictly polyconvex function $f:\mathbb{R}^{n\times m}\to\mathbb{R}$? In order to answer, we first need to introduce the following notation. Let $f:\mathbb{R}^{n\times m}\to\mathbb{R}$ be a strictly polyconvex function of the form $f(X)=g(\Phi(X))$, where $g\in C^1(\mathbb{R}^k)$ is strictly convex and Φ is the vector of all the subdeterminants of X, i.e.

$$\Phi(X)=(X,v_1(X),\ldots,v_{\min(n,m)}(X)),$$

and

$$v_s(X)=(\det(X_{Z_1}),\ldots,\det(X_{Z_{\#\mathcal{A}_s}}))$$

for some fixed (but arbitrary) ordering of all the elements $Z\in\mathcal{A}_s$. Variables of $\mathbb{R}^k$, and hence partial derivatives in $\mathbb{R}^k$, are labeled using the ordering induced by Φ. The first nm partial derivatives, corresponding in $\Phi(X)$ to X, are collected in a $n\times m$ matrix denoted with $D_X g$. The j-th partial derivative, $mn+1\le j\le k$, is instead denoted by $\partial_Z g$, where Z is the element of $\mathcal{A}_s$ corresponding to the j-th position of Φ. Let us make an example in low dimension: if $n=3, m=2$, then $k=9$, and we choose the ordering of Φ to be

$$\Phi(X)=(X,\det(X_{(12,12)}),\det(X_{(13,12)}),\det(X_{(23,12)})).$$

In this case, $y\in\mathbb{R}^k$ has coordinates

$$y=(y_{11},y_{12},y_{21},y_{22},y_{31},y_{32},y_{(12,12)},y_{(13,12)},y_{(23,12)}).$$

The partial derivatives with respect to the first 6 variables are collected in the 3×2 matrix:

$$D_X g=\begin{pmatrix} \partial_{11}g & \partial_{12}g \\ \partial_{21}g & \partial_{22}g \\ \partial_{31}g & \partial_{32}g \end{pmatrix}$$

The partial derivatives with respect to the remaining variables are denoted as $\partial_{(12,12)}g$, $\partial_{(13,12)}g$ and $\partial_{(23,12)}g$, i.e. following the ordering induced by Φ. We are ready to state the following

Proposition 2.19. *Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be a strictly polyconvex function of the form $f(X) = g(\Phi(X))$, where $g \in C^1$ is strictly convex and Φ is the vector of all the subdeterminants of X, i.e.*

$$\Phi(X) = (X, v_1(X), \dots, v_{\min(n,m)}(X)),$$

and

$$v_s(X) = (\det(X_{Z_1}), \dots, \det(X_{Z_{\#\mathcal{A}_s}}))$$

for some fixed (but arbitrary) ordering of all the elements $Z \in \mathcal{A}_s$. If $A_i \in K'_f$ and $A_i \neq A_j$ for $i \neq j$, then X_i, $Y_i = Df(X_i)$ and $c_i = f(X_i)$ fulfill the following inequalities for every $i \neq j$:

$$c_i - c_j + \langle Y_i, X_j - X_i\rangle - \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\langle \operatorname{cof}(X^Z_i)^T, X^Z_j - X^Z_i\rangle - \det(X^Z_j) + \det(X^Z_i)\right) < 0, \tag{2.29}$$

where $d^i_Z = \partial_Z g(\Phi(X_i))$.

We now introduce the set

$$C'_f \doteq \left\{ C' \in \mathbb{R}^{2n\times m} : C' = \begin{pmatrix} X \\ \beta Df(X) \end{pmatrix}, \text{ for some } \beta > 0 \right\}.$$

Notice that C'_f is the projection of C_f on the first $2n \times m$ coordinates. We immediately obtain from the previous proposition and the definition of C'_f that

$$A_i \in C'_f, \quad \forall i \in \{1, \dots, N\}$$

if and only if there exist numbers $\beta_i > 0, \forall i$, such that

$$c_i - c_j + \frac{1}{\beta_i}\langle Y_i, X_j - X_i\rangle - \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\langle \operatorname{cof}(X^Z_i)^T, X^Z_j - X^Z_i\rangle - \det(X^Z_j) + \det(X^Z_i)\right) < 0. \tag{2.30}$$

The expressions in (2.30) can be considerably simplied when the matrices $X_1, \dots, X_N$ induce a T_N configuration:

Corollary 2.20. *Let f be a strictly polyconvex function and let $A_1, \dots, A_N$ be distinct elements of K'_f with the additional property that $\{X_1, \dots, X_N\}$ induces a T_N configuration of the form (2.8) with defining vector (μ, λ). Then,*

$$c_i - \sum_j t^i_j c_j - \frac{k_i}{\beta_i}\langle Y_i, C_i\rangle < 0, \tag{2.31}$$

where the t^i's are given by (2.14).

To prove the previous corollary, we will use the following lemma, that is an easy consequence of the results of Section 2.2

Lemma 2.21. *Assume $X_1, \dots, X_N$ induces a T_N configuration of the form (2.8) and associated vectors t^i, $i \in \{1, \dots, N\}$. Then, $\forall i \in \{1, \dots, N\}$, $\forall r \in \{2, \dots, \min(m,n)\}$, $\forall Z \in \mathcal{A}_r$,*

$$\sum_j t^i_j \left(\langle \operatorname{cof}(X^Z_i)^T, X^Z_j - X^Z_i\rangle - \det(X^Z_j) + \det(X^Z_i)\right) = 0. \tag{2.32}$$

2.3.1 *Proof of Proposition 2.19*

The strict convexity of g yields, for $i \neq j$,

$$\langle Dg(\Phi(X_i)), \Phi(X_j) - \Phi(X_i)\rangle < g(\Phi(X_j)) - g(\Phi(X_i)). \tag{2.33}$$

A simple computation shows that for the function $\det(\cdot) : \mathbb{R}^{r\times r} \to \mathbb{R}$:

$$D(\det(X))|_{X=Y} = \operatorname{cof}(Y)^T.$$

In the following equation, we will write, for an $n \times m$ matrix M and for $Z \in \mathcal{A}_r$, $\overline{\operatorname{cof}(M^Z)^T}$ to denote the $n \times m$ matrix with 0 in every entry, except for the rows and columns corresponding to the multiindex $Z = (I, J)$, which will be filled with the entries of the matrix $\operatorname{cof}(M^Z)^T \in \mathbb{R}^{r\times r}$, namely, if $i \notin I$ or $j \notin J$, then $(\overline{\operatorname{cof}(M^Z)^T})_{ij} = 0$ and, if we eliminate all such coefficients, the remaining $r \times r$ matrix equals $\operatorname{cof}(M^Z)^T$. Moreover, we will identify the differential of a map from $\mathbb{R}^{n\times m}$ to $\mathbb{R}$ with the obvious associated matrix. We thus have the formula, recalling the notation introduced in at the beginning of Section 2.3,

$$Df(X) = D(g(\Phi(X))) = D_X g(\Phi(X)) + \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} \partial_Z g(\Phi(X))\overline{\operatorname{cof}(X^Z)^T}$$

When evaluated at $X = X_i$,

$$Y_i = D_X g(\Phi(X_i)) + \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} \partial_Z g(\Phi(X_i))\overline{\operatorname{cof}(X_i^Z)}$$

In order to simplify the notation set now $d^i_Z \doteq \partial_Z g(\Phi(X_i))$. The previous expression yields:

$$\begin{aligned}
&\langle Dg(\Phi(X_i)),\Phi(X_j) - \Phi(X_i)\rangle \\
&\quad= \langle D_X g(\Phi(X_i)), X_j - X_i\rangle + \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\det(X_j^Z) - \det(X_i^Z)\right) \\
&\quad= \left\langle Y_i - \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \overline{\operatorname{cof}(X_i^Z)^T}, X_j - X_i \right\rangle + \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\det(X_j^Z) - \det(X_i^Z)\right).
\end{aligned}$$

Since

$$g(\Phi(X_j)) - g(\Phi(X_i)) = f(X_j) - f(X_i) = c_j - c_i,$$

(2.33) becomes:

$$\langle Y_i, X_j - X_i\rangle - \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\langle\overline{\operatorname{cof}(X_i^Z)^T}, X_j - X_i\rangle - \det(X_j^Z) + \det(X_i^Z)\right) < c_j - c_i.$$

Finally, summing $c_i - c_j$ on both sides:

$$c_i - c_j + \langle Y_i, X_j - X_i\rangle - \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\langle\overline{\operatorname{cof}(X_i^Z)^T}, X_j - X_i\rangle - \det(X_j^Z) - \det(X_i^Z)\right) < 0 \tag{2.34}$$

Using the fact that $\langle\overline{\operatorname{cof}(X_i^Z)^T}, X_j - X_i\rangle = \langle \operatorname{cof}(X_i^Z)^T, X_j^Z - X_i^Z\rangle$, we see that the previous inequality implies the conclusion

$\forall i \neq j,$

$$c_i - c_j + \langle Y_i, X_j - X_i\rangle - \sum_{r=2}^{\min(m,n)} \sum_{Z\in\mathcal{A}_r} d^i_Z \left(\langle \operatorname{cof}(X_i^Z)^T, X_j^Z - X_i^Z\rangle - \det(X_j^Z) + \det(X_i^Z)\right) < 0.$$

2.3.2 *Proof of Lemma 2.21*

The result is a direct consequence of Lemma 2.17 and Proposition 2.16. First of all, by Proposition 2.16 we have

$$\sum_j t^i_j \det(X^Z_j) = \det\left(\sum_j t^i_j X^Z_j\right) = \det\left(P^Z_1 + \cdots + C^Z_{i-1}\right) \tag{2.35}$$

Moreover, by (2.15), we get

$$\sum_j t^i_j \langle \mathrm{cof}(X^Z_i)^T, X^Z_j - X^Z_i\rangle = \langle \mathrm{cof}(X^Z_i)^T, P^Z + C^Z_1 + \cdots + C^Z_{i-1} - X^Z_i\rangle = -k_i\langle \mathrm{cof}(X^Z_i)^T, C^Z_i\rangle\,. \tag{2.36}$$

Finally, apply Lemma 2.17 to $A = X^Z_i$ and $B = -k_i C^Z_i$ to get

$$\det\left(P^Z + \cdots + C^Z_{i-1}\right) = \det(X^Z_i) - k_i\langle \mathrm{cof}(X^Z_i)^T, C^Z_i\rangle\,. \tag{2.37}$$

These three equalities together give (2.32).

2.3.3 *Proof of Corollary 2.20*

Multiply (2.30) by t^i_j and sum over j. Using Lemma 2.21 and taking into account $\sum_j t^i_j = 1$ we get

$$c_i - \sum_j t^i_j c_j + \frac{1}{\beta_i}\left\langle Y_i, \sum_j t^i_j X_j - X_i\right\rangle < 0\,.$$

Since

$$\sum_j t^i_j X_j = P + C_1 + \ldots + C_{i-1}$$

and

$$X_i = P + C_1 + \ldots + C_{i-1} + k_i C_i\,,$$

we conclude that (2.31) holds.

2.4 PROOF OF THE MAIN RESULTS

As explained in Section 1.4, before checking whether the inclusion set contains T_N or T'_N configurations, we need to exclude more basic building block for bad solutions, such as rank-one connections or, as in this case, Λ_{dc}-connections in C_f. It is rather easy to see, compare for instance [82], that if f is strictly polyconvex, then for $A, B \in K_f$ it is not possible to have

$$A - B \in \Lambda_{dc}.$$

Indeed the same result holds even considering K'_f. To prove this, it is sufficient to observe that if $X, Y \in \mathbb{R}^{n\times m}$ are rank-one connected, i.e. for some $u \in \mathbb{S}^{m-1}$

$$(X - Y)v = 0,\ \forall v \perp u, \tag{2.38}$$

and

$$(Df(X) - Df(Y))u = 0, \tag{2.39}$$

then

$$\langle Df(X) - Df(Y), X - Y\rangle = \sum_{i=1}^m ((Df(X) - Df(Y))u_i, (X - Y)u_i)$$

$$\overset{(2.38)}{=} ((Df(X) - Df(Y))u, (X - Y)u) \overset{(2.39)}{=} 0,$$

where $\{u_1, \dots, u_m\}$ is an orthonormal basis of $\mathbb{R}^m$ with $u_1 = u$. On the other hand, since f is strictly polyconvex, it is easy to see that

$$\langle Df(X) - Df(Y), X - Y \rangle > 0$$

if $\operatorname{rank}(X - Y) = 1$. The first result of this section shows that this result holds also for C_f, provided f is positive.

Proposition 2.22. *Let f be strictly polyconvex. If*

$$A = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \; B = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} \in C_f,$$

and $f(X) \geq 0, f(X') \geq 0$, then

$$A - B \notin \Lambda_{dc}.$$

Proof. Suppose by contradiction that there exist

$$A = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in C_f, \; B = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} X + C \\ Y + D \\ Z + E \end{pmatrix} \in C_f,$$

with $c \doteq f(X) \geq 0, c' \doteq f(X') \geq 0$, and there is a vector $\xi \in \mathbb{R}^m$ with $\|\xi\| = 1$ such that for every $v \perp \xi$,

$$Cv = 0, \; D\xi = 0, \; E\xi = 0.$$

Using the expressions found in (2.30) with

$$A_1 = \begin{pmatrix} X \\ Y \end{pmatrix}, \; A_2 = \begin{pmatrix} X + C \\ Y + D \end{pmatrix},$$

it is easy to see using the Matrix Determinant Lemma 2.17 that we have the two simple inequalities

$$c - c' - \frac{1}{\beta}\langle X - X', Y \rangle < 0, \tag{2.40}$$

$$c' - c - \frac{1}{\beta'}\langle X' - X, Y' \rangle < 0. \tag{2.41}$$

Moreover by assumption $(Z' - Z)\xi = 0$, i.e.

$$(Z' - Z)\xi = 0 = (X')^T Y'\xi - X^T Y\xi - (c'\beta' - c\beta)\xi.$$

Thus, using $(Y' - Y)\xi = 0$,

$$0 = (X' - X)^T Y'\xi - (c'\beta' - c\beta)\xi = \langle C, Y \rangle \xi - (c'\beta' - c\beta)\xi,$$

that yields, since $\|\xi\| = 1$,

$$\langle C, Y \rangle = c'\beta' - c\beta. \tag{2.42}$$

In the previous lines we have used the fact that

$$(X' - X)^T Y'\xi = C^T (Y + D)\xi = C^T Y\xi,$$

and, since C is of rank one with $Cv = 0, \forall v \perp \xi$,

$$C^T Y\xi = \langle C, Y \rangle \xi.$$

Exploiting (2.42), we rewrite (2.40) as

$$c - c' - \frac{1}{\beta}\langle X - X', Y\rangle = c - c' + \frac{1}{\beta}\langle C, Y\rangle = c - c' + \frac{1}{\beta}(c'\beta' - c\beta) < 0, \tag{2.43}$$

and (2.41) as

$$c' - c - \frac{1}{\beta'}\langle C, Y\rangle = c' - c - \frac{1}{\beta'}(c'\beta' - c\beta) < 0 \tag{2.44}$$

From (2.43), we infer

$$\beta c - \beta c' + (c'\beta' - c\beta) < 0 \Leftrightarrow c'(\beta' - \beta) < 0$$

and from (2.44)

$$\beta' c' - \beta' c - (c'\beta' - c\beta) < 0 \Leftrightarrow c(\beta - \beta') < 0.$$

Since $c > 0$ and $c' > 0$, we get a contradiction. □

Now that we have excluded Λ_{dc} - connections, we can ask ourselves the same question concerning T_N' configurations. In particular we want to prove the main Theorem of the chapter, Theorem 2.1, that we recall here for the reader's convenience

Theorem. *If $f \in C^1(\mathbb{R}^{n\times m})$ is a strictly polyconvex function, then C_f does not contain any set $\{A_1, \dots, A_N\} \subset \mathbb{R}^{(2n+m)\times m}$ which induces a T_N' configuration, provided that $f(X_1) \geq 0, \dots, f(X_N) \geq 0$, if*

$$A_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}, \quad X_i, Y_i \in \mathbb{R}^{n\times m}, Z_i \in \mathbb{R}^{m\times m}, \forall i \in \{1, \dots, N\}.$$

Let us fix the notation. We will always consider T_N' configurations of the following form:

$$A_i \doteq \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}, \quad X_i, Y_i \in \mathbb{R}^{n\times m}, Z_i \in \mathbb{R}^{m\times m}, \tag{2.45}$$

with:

$$X_i = P + \sum_{j=1}^{i-1} C_j + k_i C_i,\ Y_i = Q + \sum_{j=1}^{i-1} D_j + k_i D_i,\ Z_i = R + \sum_{j=1}^{i-1} E_j + k_i E_i,$$

and we denote with $n_i \in \mathbb{S}^{m-1}$ the vectors such that

$$D_i n_i = 0, E_i n_i = 0, C_i v = 0, \quad \forall v \perp n_i,\ \forall 1 \leq i \leq N.$$

2.4.1 *Idea of the Proof*

Before proving the Theorem, let us give an idea of the key steps of the proof. First of all, in Lemma 2.23, we will see that without loss of generality we can choose $P = 0$. As already seen in Subsection 2.1.3, we want to prove that the system of inequalities

$$\nu_i \doteq \beta_i c_i - \sum_j \beta_i t_j^i c_j - k_i \langle Y_i, C_i\rangle < 0, \forall i\,, \tag{2.46}$$

cannot be fulfilled at the same time. This gives a contradiction with Corollary 2.20. In particular, we show that for the index σ such that $\beta_\sigma = \min_j \beta_j$,

$$\nu_\sigma \geq 0.$$

To do so, we prove that the quantities

$$\mu_i \doteq \beta_i c_i - \sum_j \beta_j t_j^i c_j - k_i \langle Y_i, C_i\rangle$$

equal to 0 for every i. Then, choosing σ as above and exploiting the positivity of $c_j, \forall j$, we estimate

$$0 = \mu_\sigma = \beta_\sigma c_\sigma - \sum_j \beta_j t_j^\sigma c_j - k_\sigma \langle Y_\sigma, C_\sigma \rangle \leq \beta_\sigma c_\sigma - \sum_j \beta_\sigma t_j^\sigma c_j - k_\sigma \langle Y_\sigma, C_\sigma \rangle = \nu_\sigma.$$

This will then yield the required contradiction. In order to show $\mu_i = 0, \forall 1 \leq i \leq N$, we consider N matrices M_i defined as

$$M_i \doteq \mu \sum_{j \leq i-1} \alpha_j C_j^T D_j + \sum_{j \geq i} \alpha_j C_j^T D_j,$$

where $\mu > 1$ is part of the defining vector of the T_N configuration $\{X_1, \dots, X_N\}$, compare Definition 2.15, and α_j are real numbers. We prove that for numbers $\xi_j > 0$, a subset $\mathcal{I}_i \subset \{\xi_1\mu_1, \dots, \xi_N\mu_N\}$ is made of generalized eigenvalues of M_i, see (2.57). This is achieved thanks to Lemma 2.24. Since M_i is trace-free, as can be seen by the structure of C_j and D_j, we will find N relations of the form

$$\sum_{\xi_j\mu_j \in \mathcal{I}_i} \xi_j \mu_j = 0.$$

This can be read as the equations for the kernel for a specific matrix $N \times N$ matrix, W. Proving that W has trivial kernel will yield $\xi_j\mu_j = 0, \forall j$, and thus $\mu_j = 0$ since $\xi_j > 0$. The proof of the invertibility of W is the content of the last Lemma 2.25.

2.4.2 *Proof of Theorem 2.1*

Lemma 2.23. *If f is a strictly polyconvex function such that $A_i \in C_f, \forall 1 \leq i \leq N$ and $f(X_i) \geq 0, \forall 1 \leq i \leq N$, then there exists another strictly polyconvex function F such that the T_N' configuration B_i defined as*

$$B_i = \begin{pmatrix} X_i - P \\ Y_i \\ Z_i - P^T Y_i \end{pmatrix}$$

satisfies $B_i \in C_F$, for every $1 \leq i \leq N$ and moreover $F(X_i - P) \geq 0, \forall i$.

Proof. Simply define $F(X)$ by $F(X) \doteq f(X + P)$. Clearly the newly defined family $\{B_1, \dots B_N\}$ still induces a T_N' configuration, and it is straightforward that $B_i \in C_F$. Moreover, this does not affect positivity, in the sense that $F(X_i - P) = f(X_i - P + P) = f(X_i) \geq 0$. □

Lemma 2.24. *Suppose $A_i \in C_f$, $\forall i$, and $P = 0$. Then, for every $i \in \{1, \dots, N\}$:*

$$\sum_{j=1}^N k_j(k_j - 1) t_j^i C_j^T D_j n_i = \left(k_i \langle C_i, Y_i \rangle - \beta_i c_i + \sum_{j=1}^N \beta_j t_j^i c_j \right) n_i, \quad \forall i = 1, \dots, N,$$

where t^i are the vectors defined in (2.14).

Proof. We need to compute the following sums:

$$\sum_j t_j^i Z_j = \sum_j t_j^i X_j^T Y_j - \sum_j t_j^i c_j \beta_j \,\mathrm{id}\,. \tag{2.47}$$

Let us start computing the sum for $i = 1$, $\sum_j \lambda_j X_j^T Y_j$. First, notice that

$$\sum_j \lambda_j X_j^T Y_j = \sum_j \lambda_j X_j^T (Y_j - Q) + \sum_j \lambda_j X_j^T Q = \sum_j \lambda_j X_j^T (Y_j - Q),$$

since

$$\sum_j \lambda_j X_j^T Q = P^T Q = 0.$$

We rewrite it in the following way:

$$\begin{aligned}\sum_j \lambda_j X_j^T Y_j &= \sum_j \lambda_j X_j^T (Y_j - Q) \\ &= \sum_{j=1}^N \lambda_j \left(\sum_{1\le a,b\le j-1} C_a^T D_b + k_j \sum_{1\le a\le j-1} C_a^T D_j + k_j \sum_{1\le b\le j-1} C_j^T D_b + k_j^2 C_j^T D_j \right) \\ &= \sum_{i,j} g_{ij} C_i^T D_j,\end{aligned} \tag{2.48}$$

where we collected in the coefficients g_{ij} the following quantities:

$$g_{ij} = \begin{cases} \lambda_i k_i + \sum_{r=i+1}^N \lambda_r, & \text{if } i \neq j \\ \lambda_i k_i^2 + \sum_{r=i+1}^N \lambda_r, & \text{if } i = j. \end{cases}$$

We have:

$$g_{ij} = g_{ji} = \lambda_i k_i + \sum_{r=i+1}^N \lambda_r = \frac{\mu}{\mu-1},$$

On the other hand,

$$g_{ii} = k_i^2 \lambda_i + \sum_{r=i+1}^N \lambda_r = k_i(k_i-1)\lambda_i + \frac{\mu}{\mu-1}.$$

Using the equalities $\sum_\ell C_\ell = 0 = \sum_\ell D_\ell$, then also $\sum_{i,j} C_i^T D_j = 0$, and so $\sum_{i\neq j} C_i^T D_j = -\sum_i C_i^T D_i$. Hence, (2.48) becomes

$$\sum_{i,j} g_{ij} C_i^T D_j = \frac{\mu}{\mu-1} \sum_{i\neq j} C_i^T D_j + \sum_i \left(k_i(k_i-1)\lambda_i + \frac{\mu}{\mu-1} \right) C_i^T D_i = \sum_i k_i(k_i-1)\lambda_i C_i^T D_i.$$

We just proved that

$$\sum_j \lambda_j X_j^T Y_j = \sum_j k_j(k_j-1)\lambda_j C_j^T D_j. \tag{2.49}$$

We also have:

$$\sum_j \lambda_j Z_j = \sum_j \lambda_j X_j^T Y_j - \sum_j \lambda_j c_j \beta_j \,\mathrm{id} \Rightarrow \sum_i k_i(k_i-1)\lambda_i C_i^T D_i = R + \sum_j \lambda_j c_j \beta_j \,\mathrm{id}\,.$$

Recall the definition of t^i, namely

$$t^i = \frac{1}{\xi_i}(\mu\lambda_1, \dots, \mu\lambda_{i-1}, \lambda_i, \dots, \lambda_N)\,.$$

By the previous computation ($i = 1$), it is convenient to rewrite (2.47) as

$$R + \sum_j^{i-1} E_j = \frac{1}{\xi_i} \left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j + (\mu-1) \sum_{j=1}^{i-1} \lambda_j X_j^T Y_j \right) - \sum_j t_j^i c_j \beta_j \,\mathrm{id}\,. \tag{2.50}$$

Once again, let us express the sum up to $i-1$ in the following way:

$$\sum_{j=1}^{i-1} \lambda_j X_j^T Y_j = \sum_{j=1}^{i-1} \lambda_j X_j^T Q + \sum_{k,j}^{i-1} s_{kj} C_k^T D_j.$$

A combinatorial argument analogous to the one in the previous case gives

$$s_{\ell\ell} = k_\ell^2 \lambda_\ell + \cdots + \lambda_{i-1}$$

$$
\begin{aligned}
&= (k_\ell^2 - k_\ell)\lambda_\ell + k_\ell\lambda_\ell + \cdots + \lambda_{i-1}, \\
s_{\alpha\beta} &= k_\alpha\lambda_\alpha + \cdots + \lambda_{i-1}, \qquad \alpha > \beta \\
s_{\beta\alpha} &= k_\beta\lambda_\beta + \cdots + \lambda_{i-1}, \qquad \alpha < \beta.
\end{aligned}
$$

Now

$$k_r\lambda_r + \cdots + \lambda_{i-1} = \frac{\mu(\sum_{j=1}^{i-1}\lambda_j) + \sum_{j=i}^{N}\lambda_j}{\mu - 1}$$

and so

$$k_r\lambda_r + \cdots + \lambda_{i-1} = \frac{(\mu-1)(\sum_{j=1}^{i-1}\lambda_j) + 1}{\mu - 1} = \frac{\xi_i}{\mu-1} =: b_{i-1}$$

Hence

$$\sum_{j=1}^{i-1}\lambda_j X_j^T Y_j = \sum_{j=1}^{i-1}\lambda_j X_j^T Q + \sum_{k,j}^{i-1} s_{kj} C_k^T D_j = \sum_{j=1}^{i-1}\lambda_j X_j^T Q + b_{i-1}\sum_{k,j}^{i-1} C_k^T D_j + \sum_{j=1}^{i-1} k_j(k_j-1)\lambda_j C_j^T D_j.$$

We rewrite (2.50) as

$$
\begin{aligned}
R + \sum_{j=1}^{i-1} E_j = \frac{1}{\xi_i}&\left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j + \xi_i\sum_{k,j}^{i-1} C_k^T D_j + (\mu-1)\sum_{j=1}^{i-1}(k_j(k_j-1)\lambda_j C_j^T D_j + \lambda_j X_j^T Q)\right) \\
&- \sum_j \beta_j t_j^i c_j \,\mathrm{id}
\end{aligned}
\tag{2.51}
$$

E_i is readily computed using (2.51) and the definition of Z_i:

$$
\begin{aligned}
k_i E_i + \frac{1}{\xi_i}&\left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j + \xi_i\sum_{k,j}^{i-1} C_k^T D_j + (\mu-1)\sum_{j=1}^{i-1}(k_j(k_j-1)\lambda_j C_j^T D_j + \lambda_j X_j^T Q)\right) \\
&- \sum_j \beta_j t_j^i c_j \,\mathrm{id} = X_i^T Y_i - \beta_i c_i \,\mathrm{id}
\end{aligned}
$$

Multiply by n_i the previous expression

$$
\begin{aligned}
\frac{1}{\xi_i}&\left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j n_i + \xi_i\sum_{k,j}^{i-1} C_k^T D_j n_i + (\mu-1)\sum_{j=1}^{i-1}(k_j(k_j-1)\lambda_j C_j^T D_j n_i + \lambda_j X_j^T Q n_i)\right) \\
&- \sum_j \beta_j t_j^i c_j n_i = X_i^T Y_i n_i - \beta_i c_i n_i.
\end{aligned}
\tag{2.52}
$$

Now notice that

$$
\begin{aligned}
X_i^T Y_i n_i &= X_i^T Q n_i + \sum_{j,k}^{i-1} C_k^T D_j n_i + k_i\sum_{j=1}^{i} C_i^T D_j n_i + k_i\sum_{j=1}^{i} C_j^T D_i n_i + k_i^2 C_i^T D_i n_i \\
&= X_i^T Q n_i + \sum_{j,k}^{i-1} C_k^T D_j n_i + k_i\sum_{j=1}^{i} C_i^T D_j n_i.
\end{aligned}
$$

Thus (2.52) becomes

$$
\begin{aligned}
\frac{1}{\xi_i}&\left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j n_i + (\mu-1)\sum_{j=1}^{i-1}(k_j(k_j-1)\lambda_j C_j^T D_j n_i + \lambda_j X_j^T Q n_i)\right) \\
&- \sum_j \beta_j t_j^i c_j n_i = X_i^T Q n_i + k_i\sum_{j=1}^{i-1} C_i^T D_j n_i - \beta_i c_i n_i.
\end{aligned}
\tag{2.53}
$$

Now we need to compute

$$\sum_{j=1}^{i-1} \lambda_j X_j = \sum_{j=1}^{i-1} y_j C_j,$$

and

$$y_j = k_j \lambda_j + \cdots + \lambda_{i-1} = \frac{\xi_i}{\mu - 1}, \ \forall j \in \{1, \ldots, i-1\}.$$

With this, (2.53) becomes

$$\begin{aligned} &\frac{1}{\xi_i}\left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j n_i + (\mu-1)\sum_{j=1}^{i-1} k_j(k_j-1)\lambda_j C_j^T D_j n_i\right) + \sum_{j=1}^{i-1} C_j^T Q n_i - \sum_j \beta_j t_j^i c_j n_i = \\ &X_i^T Q n_i + k_i \sum_{j=1}^{i-1} C_i^T D_j n_i - \beta_i c_i n_i. \end{aligned} \tag{2.54}$$

Exploiting the definition of t_j^i, we see that we can rewrite

$$\frac{1}{\xi_i}\left(\sum_j k_j(k_j-1)\lambda_j C_j^T D_j n_i + (\mu-1)\sum_{j=1}^{i-1} k_j(k_j-1)\lambda_j C_j^T D_j n_i\right) = \sum_j k_j(k_j-1) t_j^i C_j^T D_j n_i,$$

and

$$\begin{aligned} &X_i^T Q n_i + k_i \sum_{j=1}^{i-1} C_i^T D_j n_i - \sum_{j=1}^{i-1} C_j^T Q n_i \\ &\quad = \sum_{j=1}^{i-1} C_j^T Q n_i + k_i C_i^T Q n_i + k_i \sum_{j=1}^{i-1} C_i^T D_j n_i - \sum_{j=1}^{i-1} C_j^T Q n_i = k_i C_i^T Y_i n_i. \end{aligned}$$

Thus (2.54) can be rewritten as

$$\sum_j k_j(k_j-1) t_j^i C_j^T D_j n_i - \sum_j \beta_j t_j^i c_j n_i = k_i \sum_{j=1}^{i-1} C_i^T Y_i n_i - \beta_i c_i n_i.$$

We finally obtain the statement of the Lemma:

$$\sum_{j=1}^{N} k_j(k_j-1) t_j^i C_j^T D_j n_i = \left(k_i\langle C_i, Y_i\rangle - \beta_i c_i + \sum_{j=1}^{N} \beta_j t_j^i c_j\right) n_i, \forall i = 1, \ldots, N.$$

□

We are finally in position to prove the main Theorem.

Proof of Theorem 2.1. Assume by contradiction the existence of a T_N' configuration induced by matrices $\{A_1, \ldots, A_N\}$ of the form (2.45) which belong to the inclusion set C_f of some stictly polyconvex function $f \in C^1(\mathbb{R}^{n\times m})$ and $f(X_i) \geq 0$ for every i. We can assume, without loss of generality by Lemma 2.23, that

$$P = 0.$$

Using the previous Lemma, we find

$$\sum_{j=1}^{N} k_j(k_j-1) t_j^i C_j^T D_j n_i = \left(k_i\langle C_i, Y_i\rangle - \beta_i c_i + \sum_{j=1}^{N} \beta_j t_j^i c_j\right) n_i, \forall i. \tag{2.55}$$

Define $\alpha_j \doteq k_j(k_j-1)\lambda_j > 0$, and

$$M_i \doteq \mu \sum_{j\le i-1} \alpha_j C_j^T D_j + \sum_{j\ge i} \alpha_j C_j^T D_j,$$

for $i \in \{1,\dots,N\}$. Define moreover for convenience

$$M_i \doteq \mu M_{i-N}, \forall i \in \{N+1,\dots,2N\}.$$

Then, (2.55) can be rewritten as

$$M_i n_i = \xi_i \mu_i n_i, \forall i \in \{1,\dots,N\}. \tag{2.56}$$

We define $n_s \doteq n_{s-N}$, for $s \in \{N+1,\dots,N\}$. As explained in Subsection 2.4.1, the idea is to show that a subset of the vectors n_j are *generalized eigenvectors* and ξn_i *are generalized eigenvalues* of M_i. In particular, we want to show the following equalities:

$$\begin{cases} M_i n_{i+a} = \xi_{i+a}\mu_{i+1} n_{i+a} + v_{i,a}, & \text{if } i \le i+a \le N \\ M_i n_{i+a} = \mu\xi_{i+a}\mu_{i+1} n_{i+a} + v_{i,a}, & \text{if } N+1 \le i+a \le N+i-1, \end{cases} \tag{2.57}$$

where $v_{i,a} \in \operatorname{span}\{n_i,\dots,n_{i+a-1}\}$. From now on, we fix $i \in \{1,\dots,N\}$. To prove (2.57), first we rewrite

$$M_i n_{i+a} = (M_i - M_{i+a})n_{i+a} + M_{i+a} n_{i+a}, \tag{2.58}$$

and then we use (2.56) to obtain

$$(M_i - M_{i+a})n_{i+a} + M_{i+a}n_{i+a} = \begin{cases} \xi_{i+a}\mu_{i+a}n_{i+a} + (M_i - M_{i+a})n_{i+a}, & \text{if } i+a \le N, \\ \mu\xi_{i+a}\mu_{i+a}n_{i+a} + (M_i - M_{i+a})n_{i+a}, & \text{if } i+a > N. \end{cases}.$$

To conclude the proof of (2.57), we only need to show that

$$(M_i - M_{i+a})n_{i+a} \in \operatorname{span}\{n_i,\dots,n_{i+a-1}\}. \tag{2.59}$$

To do so, we compute $M_i - M_{i+a}$. Let us start from the case $i+a \le N$:

$$\begin{aligned} M_i - M_{i+a} &= \mu \sum_{j<i} \alpha_j C_j^T D_j + \sum_{j\ge i} \alpha_j C_j^T D_j - \mu \sum_{j<i+a} \alpha_j C_j^T D_j - \sum_{j\ge i+a} \alpha_j C_j^T D_j \\ &= \sum_{i+a>j\ge i} \alpha_j C_j^T D_j - \mu \sum_{i<j<i+a} \alpha_j C_j^T D_j. \end{aligned}$$

On the other hand, if $N+1 \le i+a \le i+N-1$, then

$$\begin{aligned} M_i - M_{i+a} &= M_i - \mu M_{i+a-N} \\ &= \mu \sum_{j<i} \alpha_j C_j^T D_j + \sum_{j\ge i} \alpha_j C_j^T D_j - \mu^2 \sum_{j<i+a-N} \alpha_j C_j^T D_j - \mu \sum_{j\ge i+a-N} \alpha_j C_j^T D_j \\ &= \mu \sum_{j<i+a-N} \alpha_j C_j^T D_j - \mu \sum_{j\ge i} \alpha_j C_j^T D_j + \sum_{j\ge i} \alpha_j C_j^T D_j - \mu^2 \sum_{j<i+a-N} \alpha_j C_j^T D_j. \end{aligned}$$

Now the crucial observation is that, due to the structure of C_j, the image of $C_j^T D_j$ is contained in the line $\operatorname{span}(n_j)$, for every $j \in \{1,\dots,N\}$. Therefore, the previous computations prove (2.59) and hence (2.57). Now we introduce

$$V_i \doteq \{n_i, n_{i+1}, n_{i+2}, \dots, n_N, n_{N+1}, \dots, n_{N+i-1}\}.$$

We can extract a basis for $\operatorname{span}(V_i)$ in the following way. First, choose indexes

$$\overline{S}_i \doteq \{k : k = i \text{ or } i < k \le N+i-1, n_k \notin \operatorname{span}(n_i,\dots,n_{k-1})\}. \tag{2.60}$$

Then, consider the basis $\mathcal{B}_i \doteq \{n_k : k \in \overline{S}_i\}$ for $\text{span}(V_i)$. Since

$$\text{span}(\mathcal{B}_i) = \text{span}(\{n_1, \dots, n_N\}), \forall i,$$

then $\#S_i = C \le \min\{m, N\}$. Indexes in $\overline{S}_i$ lie in the set $\{1, \dots, 2N\}$. For technical reasons, we also need to consider the *modulo* N counterpart of $\overline{S}_i$, that is

$$S_i \doteq \{k \in \{1, \dots, N\} : k \in \overline{S}_i \text{ or } k + N \in \overline{S}_i\}. \tag{2.61}$$

In S_i, consider furthermore $S'_i \doteq S_i \cap \{i, \dots, N\}$, $S''_i \doteq S_i \cap \{1, \dots, i-1\}$. If necessary, complete $\mathcal{B}_i$ to a basis of $\mathbb{R}^m$ made with elements γ_j orthogonal to the ones of $\mathcal{B}_i$. Note that, since $\text{Im}(C_i^T D_i) \subset \{n_i\}$, then $\text{Im}(M_i) \subset \{n_1, \dots, n_N\}$. Then, the associated matrix to M_i with respect to $\mathcal{B}_i$ is

$$M_i = \left(\begin{array}{ccccc|c} a_{1i} & * & * & \dots & * & \\ 0 & a_{2i} & * & \dots & * & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{T} \\ 0 & 0 & 0 & \dots & a_{Ci} & \\ \hline & & \mathbf{0}_{m-C,C} & & & \mathbf{0}_{m-C,m-C} \end{array}\right). \tag{2.62}$$

We denoted with $\mathbf{0}_{c,d}$ the zero matrix with c rows and d columns, with $\mathbf{T}$ the $C \times (m - C)$ matrix of the coefficients of $M_i\gamma_j$ with respect to $\{n_s : s \in \overline{S}_i\}$, and with $*$ numbers we are not interested in computing explicitely. Finally, we have chosen an enumeration $s_1 < s_2 < \dots < s_\ell < \dots < s_C$ of the elements of $\overline{S}_i$, and we have defined

$$a_{\ell i} = \begin{cases} \xi_{s_\ell}\mu_{s_\ell}, & \text{if } s_\ell \in S'_i, \\ \mu\xi_{s_\ell}\mu_{s_\ell}, & \text{if } s_\ell - N \in S''_i. \end{cases}$$

The triangular form of the matrix representing M_i is exactly due to (2.57). Now, $\text{tr}\,(M_i) = 0, \forall i$, since $C_i^T D_i$ is trace-free for every i. This implies that the matrix in (2.62) must be trace-free, hence:

$$0 = \text{tr}\,(M_i) = \sum_{\ell=1}^{C} a_{\ell i} = \sum_{a \in S'_i} \xi_a\mu_a + \mu \sum_{b \in S''_i} \xi_{b+N}\mu_{b+N}. \tag{2.63}$$

We have thus reduced the problem to the following Linear Algebra simple statement: we wish to show that, if W is the $N \times N$ matrix defined as

$$W_{ij} = \begin{cases} 1, & \text{if } j \in S'_i, \\ \mu, & \text{if } j \in S''_i, \\ 0, & \text{if } j \notin S_i, \end{cases}$$

then, $Wx = 0 \Rightarrow x = 0$. By (2.63), the vector $x \in \mathbb{R}^N$ defined as $x_j \doteq \xi_j\mu_j, \forall 1 \le j \le N$, is such that $Wx = 0$, thus if the statement is true we get $\xi_j\mu_j = 0, \forall 1 \le j \le N$, and since $\xi_j > 1$, also $\mu_j = 0, \forall 1 \le j \le N$. As discussed at the beginning of the proof, this is sufficient to conclude. Therefore, we only need to show that $Wx = 0 \Rightarrow x = 0$. This proof will be given in Lemma 2.25. □

Before giving the proof of the final Lemma, let us make some examples of possible matrices W arising from the previous construction. For the sake of illustration, let us take N to be as small as possible, i.e. $N = 4$.

E.g. 2.1. Consider the case in which $C = 2$. This corresponds, for instance, to the case $m = 2$. Then, by Proposition 2.22, the only possible form of W is

$$W = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mu & 0 & 0 & 1 \end{pmatrix}, \quad Wx = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \\ \mu x_1 + x_4 \end{pmatrix} = 0.$$

Let W_i be the i-th row of W. We notice that for $i = 1, 2, 3$, W_{i+1} differs from W_i by exactly two elements, while W_4 does not differ with W_1 by only two elements. It does, though, with μW_1. Hence we rewrite equivalently the system $Wx = 0$ as $W_i - W_{i+1}$, $W_4 - \mu W_1$:

$$0 = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_4 \\ x_3 - \mu x_1 \\ x_4 - \mu x_2 \end{pmatrix}, \text{ i.e. } x_i = a_i x_{h(i)}, a_i = \begin{cases} 1, & \text{if } h(i) > i, \\ \mu, & \text{if } h(i) \le i, \end{cases}$$

For a function $h : \{1, \dots, 4\} \to \{1, \dots, 4\}$. Since $\mu > 1$, this immediately implies $x_i = 0, \forall i$.

E.g. 2.2. Consider the case in which $C = 4$, corresponding to n_1, n_2, n_3, n_4 linearly independent. Then,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & 1 & 1 & 1 \\ \mu & \mu & 1 & 1 \\ \mu & \mu & \mu & 1 \end{pmatrix}, \; Wx = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ \mu x_1 + x_2 + x_3 + x_4 \\ \mu x_1 + \mu x_2 + x_3 + x_4 \\ \mu x_1 + \mu x_2 + \mu x_3 + x_4 \end{pmatrix} = 0.$$

As in the previous example, for $i = 1, 2, 3$, W_{i+1} differs from W_i by exactly one element, while W_4 does the same with μW_1. Thus as before we rewrite equivalently the system $Wx = 0$ as $W_i - W_{i+1}$, $W_4 - \mu W_1$:

$$0 = \begin{pmatrix} (\mu - 1)x_1 \\ (\mu - 1)x_2 \\ (\mu - 1)x_3 \\ (\mu - 1)x_4 \end{pmatrix}, \text{ i.e. } x_i = a_i x_{h(i)}, a_i = \begin{cases} 1, & \text{if } h(i) > i, \\ \mu, & \text{if } h(i) \le i, \end{cases}$$

In this case, $h(i) = i, \forall i \in \{1, \dots, 4\}$. Clearly also in this case $\mu > 1$, implies $x_i = 0, \forall i$.

Finally, let us show a less symmetric example:

E.g. 2.3. Consider the case in which $C = 3$. Then, a possible matrix is:

$$W = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \mu & 0 & 1 & 1 \\ \mu & \mu & 0 & 1 \end{pmatrix}, \; Wx = \begin{pmatrix} x_1 + x_2 + x_4 \\ x_2 + x_3 + x_4 \\ \mu x_1 + x_3 + x_4 \\ \mu x_1 + \mu x_2 + x_4 \end{pmatrix} = 0.$$

First, let us comment on the fact that this is a possible matrix appearing in the proof of the previous Theorem. Indeed, let's consider the first two lines:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The fact that $W_{13} = 0$ means that $n_3 \in \text{span}(n_1, n_2)$, since $3 \notin S_1$. On the other hand, Proposition 2.22 ensures that n_3 is not a multiple of n_2, hence $n_3 \in S_2$, and $W_{23} = 1 \neq 0$. For this reason, the matrix

$$W = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ \mu & 0 & 1 & 1 \\ \mu & \mu & 0 & 1 \end{pmatrix}$$

would for instance have been non-admissible. Now, in order to prove $Wx = 0 \Rightarrow x = 0$, we work as in the previous examples, by noticing that for $i = 1, 2, 3$, W_{i+1} differs from W_i by at most two elements, while W_4 must be compared with μW_1. Thus we write $W_i - W_{i+1}$, $W_4 - \mu W_1$:

$$0 = \begin{pmatrix} x_1 - x_3 \\ x_2 - \mu x_1 \\ x_3 - \mu x_2 \\ (\mu - 1)x_4 \end{pmatrix}, \text{ i.e. } x_i = a_i x_{h(i)}, a_i = \begin{cases} 1, & \text{if } h(i) > i, \\ \mu, & \text{if } h(i) \le i. \end{cases}$$

It is an elementary computation to show that $x_i = 0, \forall i$.

Even though the examples we have given are too simple to appreciate the usefulness of the function h such that $x_i = a_i x_{h(i)}$, this will be crucial in the proof of the Lemma.

Lemma 2.25. *Let W be the matrix defined in the proof of Theorem 2.1. Then,* $\mathrm{Ker}(W) = \{0\}$.

Proof. Throughout the proof, we always consider a given vector $x \in \mathbb{R}^N$ such that $Wx = 0$. The strategy of the proof, partially suggested by the previous examples, consists in the following steps. First, we show that the rows of W, W_i and W_{i+1} (if $i = N$, we compare W_N with μW_1) differ for at most two elements, and one of them is always x_i. This immediately yields the existence of a function $h : \{1, \dots, N\} \to \{1, \dots, N\}$ such that $x_i = a_i x_{h(i)}$. We will then use this and the crucial fact that $\mu > 1$ to conclude that $x_i = 0, \forall i$. Let us make the following claims, and see from them how to conclude the proof of the present Lemma. We will use freely the notation introduced at the end of the proof of Theorem 2.1.

$\boxed{\text{Claim 1:}}$ Let $i \in \{1, \dots, N\}$. Then $\overline{S}_i$ differs from $\overline{S}_{i+1}$ (if $i = N$, $\overline{S}_{i+1} = \overline{S}_1$) of at most two elements, in the sense that

$$\overline{S}_i \Delta \overline{S}_{i+1} \doteq \overline{S}_i \setminus \overline{S}_{i+1} \cup \overline{S}_{i+1} \setminus \overline{S}_i$$

contains at most 2 elements. Moreover, if $\overline{S}_i \Delta \overline{S}_{i+1} \neq \emptyset$, then $i \in \overline{S}_i \Delta \overline{S}_{i+1}$, and if $\overline{S}_i \Delta \overline{S}_{i+1} = \{i, I(i)\}$ with $I(i) \neq i$, then $I(i) \in \overline{S}_{i+1} \setminus \overline{S}_i$.

$\boxed{\text{Claim 2:}}$ Let $i \in \{1, \dots, N-1\}$. The couple of rows W_i,W_{i+1} and μW_1, W_N differ at most by two elements, in the sense that if $W_i = (W_{i1}, \dots, W_{iN})$ and $W_{i+1} = (W_{(i+1)1}, \dots, W_{(i+1)N})$, then there are at most two indexes j_1, j_2 such that $W_{ij_1} - W_{(i+1)j_1} \neq 0$ and $W_{ij_2} - W_{(i+1)j_2} \neq 0$ (and analogously for μW_1 and W_N).

Finally, with this claim at hand, we are going to prove

$\boxed{\text{Claim 3:}}$ There exists a function $h : \{1, \dots, N\} \to \{1, \dots, N\}$ and numbers $a_i, i \in \{1, \dots, N\}$, such that

$$x_i = a_i x_{h(i)} \tag{2.64}$$

with the property

$$a_i = \begin{cases} 1, & \text{if } h(i) > i, \\ \mu, & \text{if } h(i) \leq i. \end{cases}$$

Let us show how the proof of the Lemma follows from Claim 3, and postpone the proofs of the claims. Fix $i \in \{1, \dots, N\}$ and use (2.64) recursively to find

$$x_i = a_i a_{h(i)} \dots a_{h^{(n-1)}(i)} x_{h^{(n)}(i)},$$

where $h^{(n)}$ denotes the function obtained by applying h to itself n times. We will also use the notation $h^{(0)}$ to denote the identity function: $h^{(0)}(i) = i, \forall i \in \{1, \dots, N\}$. By the properties of a_j, we have, $\forall r \in \{0, \dots, n-1\}$,

$$a_{h^{(r)}(i)} = \begin{cases} 1, & \text{if } h^{(r)}(i) > h^{(r-1)}(i), \\ \mu, & \text{if } h^{(r)}(i) \leq h^{(r-1)}(i). \end{cases}$$

Fix $k \in \mathbb{N}$, and let $r \in \{k+1, \dots, k+N+1\}$. Then, $h^{(r)}(i) > h^{(r-1)}(i)$ can occur at most N times in this range, since otherwise we would find

$$1 \leq h^{(k)}(i) < h^{(k+1)}(i) < h^{(k+2)}(i) < \dots < h^{(k+N+1)}(i) \leq N,$$

and this is impossible since we would have $N+1$ distinct elements in the set $\{1, \dots, N\}$. Now clearly this observation implies that for every fixed $l \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that

$$x_i = \mu^t x_{h^{(s)}(i)}, \text{ for some } t \geq l.$$

This can only happen if $x_i = 0$. Since i is arbitrary, the conclusion follows.

Let us now turn to the proof of the claims.

$\boxed{\text{Proof of claim 1:}}$ To prove the claim, we need to use the definition of S_i. Let us recall the definition of $\overline{S}_i$, given in (2.60). To build $\overline{S}_i$ what we do is consider the ordered set $\{n_i, n_{i+1}, \dots, n_{i+1-N}\}$ and select from it a basis of span$\{n_1, \dots, n_N\}$ starting from n_i and then at the step $i+1 \le k \le N$ deciding whether to insert the vector n_k in our collection based on the fact that it is linear dependent or not from the previous ones. Recall also that S_i is the *modulo* N version of $\overline{S}_i$, see (2.61), and that we define $n_j \doteq n_{j-N}$, for $j \in \{N+1, \dots, 2N\}$. Hence now fix $i \in \{1, \dots, N\}$ and consider S_i. If $S_i = \{1, \dots, N\}$, then $\#S_i = N$, thus $S_j = \{1, \dots, N\}, \forall 1 \le j \le N$ and the claim holds. Otherwise, let $i+1 < I = I(i) \le i+N-1$ be the first element in $(\overline{S}_i)^c$. There are two cases:

1. $n_I \in \text{span}(n_i, \dots, n_{I-1}) \setminus \text{span}(n_{i+1}, \dots, n_{I-1})$;
2. $n_I \in \text{span}(n_{i+1}, \dots, n_{I-1})$.

At the same time, consider what happens in $\overline{S}_{i+1}$: the span in the $(i+1)$-th case starts with one vector less than the one of the i-th case, simply because the collection of indexes in $\overline{S}_{i+1}$ starts from n_{i+1}. Hence, since I is the first missing index in $\overline{S}_i$, I is also the first possible missing index for $\overline{S}_{i+1}$. Therefore, consider the first case

$$n_I \in \text{span}(n_i, \dots, n_{I-1}) \setminus \text{span}(n_{i+1}, \dots, n_{I-1}).$$

This implies that $I \in \overline{S}_{i+1}$. Moreover, we are now adding n_I to the set of vectors $n_{i+1}, \dots, n_{I-1}$, and $n_I \in \text{span}(n_i, \dots, n_{I-1}) \setminus \text{span}(n_{i+1}, \dots, n_{I-1})$, hence n_I adds to the previous vectors the component relative to n_i, in the sense that

$$\text{span}(n_{i+1}, \dots, n_I) = \text{span}(n_i, \dots, n_{I-1}).$$

This moreover implies that $j \in \overline{S}_i \Leftrightarrow j \in \overline{S}_{i+1}$, $\forall I \le j < N+i-1$. Since $n_i \in \text{span}(n_{i+1}, \dots, n_I)$, $i \notin \overline{S}_{i+1}$. Thus $\overline{S}_i$ and $\overline{S}_{i+1}$ differ by at most two elements, and we have $i \in \overline{S}_i \setminus \overline{S}_{i+1}$ and $I = I(i) \in \overline{S}_{i+1} \setminus \overline{S}_i$. This concludes the case

$$n_I \in \text{span}(n_i, \dots, n_{I-1}) \setminus \text{span}(n_{i+1}, \dots, n_{I-1}).$$

If instead $n_I \in \text{span}(n_{i+1}, \dots, n_{I-1})$, then we see that $I \notin \overline{S}_{i+1}$, and we can iterate this reasoning from there, in the sense that we look for the next index I' such that $I' \notin \overline{S}_i$ and divide again into the two cases above. Clearly, for the indexes $i+1 \le j < I'_1$, we have $j \in \overline{S}_{i+1}$ and $j \in \overline{S}_i$. Either this iteration enters in case 1 of the previous subdivision for some element $I \notin \overline{S}_i$, or we conclude $\overline{S}_i = \overline{S}_{i+1}$. This concludes the proof of the claim.

$\boxed{\text{Proof of claim 2:}}$

Note that nonzero elements of W_i are found in positions corresponding to elements of S_i. Hence now fix $i \in \{1, \dots, N-1\}$ and consider W_i and W_{i+1}. If $S_i = S_{i+1}$, then $W_{ij} = 0 \Leftrightarrow W_{(i+1)j} = 0$. Moreover, we introduce the *modulo* N counterpart of the number $I(i)$ found in Claim 2, i.e. $I'(i) = I(i)$ if $I(i) \in \{1, \dots, N\}$, and $I'(i) = I(i) - N$ if $I(i) \in \{N+1, \dots, 2N\}$. Thus using the definition of W, we can deduce

$$\begin{cases} W_{(i+1)j} = W_{ij} = 0, & \text{if } j \notin S_i \\ W_{(i+1)j} = W_{ij} = \mu, & \text{if } j \in S_i, j < i \\ W_{(i+1)j} = W_{ij} = 1, & \text{if } j \in S_i, j > i \\ W_{(i+1)i} = \mu, W_{ii} = 1, & \text{otherwise,} \end{cases} \tag{2.65}$$

and the claim holds in this case. Finally, if $S_i \Delta S_{i+1} = \{i, I'(i)\}$, then:

$$\begin{cases} W_{(i+1)j} = W_{ij} = 0, & \text{if } j \notin S_i, j \neq I'(i) \\ W_{(i+1)j} = 1, W_{ij} = 0, & \text{if } j = I'(i) > i+1 \\ W_{(i+1)j} = \mu, W_{ij} = 0, & \text{if } j = I'(i) < i-1 \\ W_{(i+1)j} = W_{ij} = \mu, & \text{if } j \in S_i, j < i \\ W_{(i+1)j} = W_{ij} = 1, & \text{if } j \in S_i, j > i \\ W_{(i+1)i} = 0, W_{ii} = 1, & \text{otherwise.} \end{cases} \tag{2.66}$$

This concludes the proof of the claim if $i \in \{1, \dots, N-1\}$. If $i = N$, then we need to compare W_N with μW_1, and we obtain two cases, in analogy with the previous situation:

$$\text{if } S_N \Delta S_1 = \emptyset, \text{ then : } \begin{cases} \mu W_{1j} = W_{Nj} = 0, & \text{if } j \notin S_N \\ \mu W_{1j} = W_{Nj} = \mu, & \text{if } j \in S_N, j < N \\ \mu W_{1N} = \mu, W_{NN} = 1, & \text{otherwise,} \end{cases} \tag{2.67}$$

and

$$\text{if } S_N \Delta S_1 = \{x_N, x_{I'(N)}\}, \text{ then : } \begin{cases} \mu W_{1j} = W_{Nj} = 0, & \text{if } j \notin S_N, j \neq I'(N) \\ \mu W_{1j} = 0, W_{Nj} = \mu, & \text{if } j \notin S_N, j = I'(N) \\ \mu W_{1j} = W_{Nj} = \mu, & \text{if } j \in S_N, j < N \\ \mu W_{1N} = 0, W_{NN} = 1, & \text{otherwise.} \end{cases} \tag{2.68}$$

$\boxed{\text{Proof of Claim 3:}}$ Fix $i \in \{1, \dots, N\}$. We want to consider the equations given by

$$(W_{i+1} - W_i)x = 0, \text{ if } i \in \{1, \dots, N-1\}, \text{ and } (W_N - \mu W_1)x = 0.$$

If we consider $i \in \{1, \dots, N-1\}$, we see from (2.65) and (4.6) that

$$0 = (W_i - W_{i+1})x = \sum_{j=1}^{N} (W_{ij} - W_{(i+1)j})x_j = \begin{cases} (1-\mu)x_i, & \text{if } S_i \Delta S_{i-1} = \emptyset \\ x_i - x_{I'(i)}, & \text{if } S_i \Delta S_{i-1} = \{i, I'(i)\}, I'(i) > i+1 \\ x_i - \mu x_{I'(i)}, & \text{if } S_i \Delta S_{i-1} = \{i, I'(i)\}, I'(i) < i-1 \end{cases}$$

and from (2.67) and (2.68) we infer

$$0 = (W_N - \mu W_1)x = \begin{cases} (1-\mu)x_N, & \text{if } S_N \Delta S_1 = \emptyset \\ x_N - x_{I'(N)}, & \text{if } S_N \Delta S_1 = \{N, I'(N)\}. \end{cases}$$

From these equations we see that (2.64) holds with the choice $h(i) \doteq I'(i)$, when i is such that $S_i \Delta S_{i+1} \neq \emptyset$, and $h(i) \doteq i$ otherwise.

□

We end this section by showing that Theorem 2.1 implies Theorem 2.2.

Proof of Theorem 2.2. Assume by contradiction that there exists a family of matrices

$$\{A_1, \dots, A_N\} \subset K_f$$

inducing a T'_N configuration of the form (2.45). We show that then there exists another T'_N configuration $\{B_1, \dots, B_N\}$ such that $B_i \in K_F \subset C_F, \forall 1 \leq i \leq N$ for some strictly polyconvex F with

$$F(X'_i) \geq 0, \; \forall 1 \leq i \leq N,$$

if

$$B_i = \begin{pmatrix} X_i' \\ Y_i' \\ Z_i' \end{pmatrix}, \ \forall 1 \le i \le N.$$

This is a contradiction with Theorem 2.1. To accomplish this, it is is sufficient to define $F(X) \doteq f(X) - \min_i f(X_i)$. This function is clearly strictly polyconvex, since f is. Moreover, we define

$$X_i' \doteq X_i,\ Y_i' \doteq Y_i \text{ and } Z_i' \doteq Z_i - \min_i f(X_i)\,\mathrm{id}\,.$$

In this way, B_i is still a T_N' configuration. Moreover, $B_i \in K_F$, $\forall 1 \le i \le N$. To see this, it is sufficient to notice that, since $A_i \in K_f$,

$$Y_i' = Y_i = Df(X_i) = DF(X_i'),\ \forall 1 \le i \le N,$$

and

$$Z_i' = Z_i + \min_i f(X_i)\,\mathrm{id} = X_i^T Y_i - f(X_i)\,\mathrm{id} + \min_i f(X_i)\,\mathrm{id} = (X_i')^T Y_i' - F(X_i)\,\mathrm{id}\,.$$

This finishes the proof. □

Remark 2.26. It is a natural question to ask whether the hypothesis of positivity is actually necessary in the hypothesis of Theorem 2.1. In [41], we will give an answer to this question, in the sense that we will construct a polyconvex function such that C_f contains a family of matrices inducing a T_5' configuration, and from that we will deduce that the constancy Lemma is false for *geometric* functionals that are allowed to change sign. We also remark that it is rather easy already to violate Proposition 2.22 without the hypothesis of positivity of f.

3 STATIONARY GRAPHS AND STATIONARY VARIFOLDS

The aim of this chapter is to provide the link between stationary points for energies defined on graphs with multiplicity, that we recall being equivalent to the following system

$$\begin{cases} \displaystyle\int_\Omega \langle Df(Du), Dv\rangle \beta\, dx = 0 & \forall v \in C^1_c(\Omega,\mathbb{R}^n) \\ \displaystyle\int_\Omega \langle Df(Du), DuD\Phi\rangle \beta\, dx - \int_\Omega f(Du)\operatorname{div}\Phi\beta\, dx = 0 & \forall \Phi \in C^1_c(\Omega,\mathbb{R}^m), \end{cases} \tag{3.1}$$

and stationary varifolds for *geometric* energies. In Section 3.1 we will give the definitions we will need, and then in Section 3.2 we will prove, after having shown several technical lemmas, the main result of this chapter:

Proposition. *Assume that $f \in C^1(\mathbb{R}^{n\times m})$ admits an extension $\Psi \in C^1(\mathbb{G}(m, m+n))$, in the sense that (6.4) holds for every $X \in \mathbb{R}^{n\times m}$. Fix any $m \le p \le +\infty$, $1 \le q < +\infty$ and a Lipschitz, bounded, open set $\Omega \subset \mathbb{R}^m$. If a map $u \in W^{1,p}(\Omega,\mathbb{R}^n)$ and a Borel function $\beta \in L^\infty(\Omega,(0,+\infty))$ satisfy, for every $v \in C^1_c(\Omega,\mathbb{R}^n)$ and $\Phi \in C^1_c(\Omega,\mathbb{R}^m)$,*

$$\begin{cases} \left|\displaystyle\int_\Omega \langle Df(Du), Dv\rangle \beta\, \mathrm{d}x\right| \le C\|v\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q \\ \left|\displaystyle\int_\Omega \langle Df(Du), DuD\Phi(x)\rangle \beta\, \mathrm{d}x - \int_\Omega f(Du)\operatorname{div}(\Phi)\beta dx\right| \le C\|\Phi\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q, \end{cases} \tag{3.2}$$

for some $C \ge 0$, then the rectifiable varifold of $\mathbb{R}^{m+n}$ $\theta[\![\mathcal{G}_u]\!]$, where $\theta(x,y) = \beta(x)$, satisfies

$$|\delta_\Psi(\theta[\![\mathcal{G}_u]\!])(g)| \le C'\|g\theta^{\frac{1}{q}}\|_{q,\mathcal{G}_u} \qquad \forall g \in C^1_c(\Omega\times\mathbb{R}^n,\mathbb{R}^{m+n}), \tag{3.3}$$

for some number $C' = C'(C,m,p,q) \ge 0$. Conversely, if (3.3) holds for some C', then (3.2) holds for some $C = C(C',m,p,q)$. Moreover, $C' = 0$ if and only $C = 0$, namely u is satisfies (3.1) if and only if $\theta[\![\mathcal{G}_u]\!]$ is stationary for the energy Σ.

3.1 NOTATION AND PRELIMINARY DEFINITIONS

Recall that general m-dimensional varifolds in $\mathbb{R}^{m+n}$ (introduced by L.C. Young in [89] and pioneered in geometric measure theory by Almgren [3] and Allard [1]) are nonnegative Radon measures on the Grassmaniann of $\mathbb{G}(m, m+n)$ of (unoriented) m-dimensional planes of $\mathbb{R}^{m+n}$. In our specific case we are interested on a subclass, namely rectifiable varifolds, for which we can take the simpler Definition 3.1 below. A quick reference for the terminology used in this section is [16], whereas comprehensive introductions can be found in the foundational paper [1] and in the book [78].

Definition 3.1. A *rectifiable varifold* V of dimension m is a couple (Γ,θ), where $\Gamma \subset \mathbb{R}^{m+n}$ is a m-rectifiable set in $\mathbb{R}^N$, and $\theta : \Gamma \to (0,+\infty)$ is a Borel map. If θ has values in $\mathbb{N}\setminus\{0\}$, then the varifold is called integer rectifiable.

It is customary to denote (Γ,θ) as $\theta[\![\Gamma]\!]$ and to call θ the multiplicity of the varifold.

Definition 3.2. Let U be an open set of $\mathbb{R}^{m+n}$, and let $\Phi : \mathbb{R}^{m+n} \to U$ be a diffeomorphism. The *pushforward* of a rectifiable varifold $V = \theta[\![\Gamma]\!]$ through Φ is defined as $\Phi_\# V = \theta\circ\Phi^{-1}[\![\Phi(\Gamma)]\!]$.

For a rectifiable varifold $\theta[\![\Gamma]\!]$, it is customary to introduce a notion of approximate tangent plane, which exists for $\mathcal{H}^m$-a.e. point of Γ, we refer to [78, Theorem 3.1.8] for the relevant details.

Provided it exists, the tangent plane at the point $y \in \Gamma$ will be denoted with $T_y\Gamma$ and it is an element of $\mathbb{G}(m, m+n)$. In the following, we will identify the Grassmanian manifold with a suitable subset of orthogonal projections, i.e. for every $L \in \mathbb{G}(m, m+n)$ we consider the linear map $P : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ which is the orthogonal projection onto L. With this identification we have

$$\mathbb{G}(m, m+n) \sim \left\{ P \in \mathbb{R}^{(m+n)\times(m+n)} : P = P^T, P^2 = P, \operatorname{rank}(P) = \operatorname{tr}(P) = m \right\}.$$

We are interested in graphs of maps $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, and we always consider $\mathbb{R}^m = \operatorname{span}\{e_1, \dots, e_m\}$, where $\{e_1, \dots, e_{n+m}\}$ is the canonical basis of $\mathbb{R}^{n+m}$. In other words, we are interested in sets of the form $\Gamma_u = \{z \in \mathbb{R}^{n+m} : z = (x, u(x)), x \in \Omega\}$. For this reason, we need to characterize the space of orthogonal projections on tangent planes to graphs (on the plane $\operatorname{span}\{e_1, \dots, e_m\}$). Since at the differentiability point x_0, we have

$$D((x, u(x)))|_{x=x_0} = \begin{pmatrix} \mathrm{id}_m \\ Du(x_0) \end{pmatrix} \in \mathbb{R}^{(m+n)\times m},$$

it is convenient to introduce the following notation:

$$M(X) \doteq \begin{pmatrix} \mathrm{id}_m \\ X \end{pmatrix}.$$

Therefore, every tangent plane to a graph Γ_u is of the form

$$\tau(X) = \operatorname{span}\{M(X)^T e_1, \dots, M(X)^T e_{n+m}\}. \tag{3.4}$$

With the notation above, the tangent plane of Γ_u at x_0 is $\tau(Du(x_0))$. The orthogonal projection on $\tau(X)$ is given by the formula

$$h(X) \doteq M(X)S(X)M(X)^T$$

where

$$S(X) \doteq (M(X)^T M(X))^{-1},$$

or, more explicitely,

$$h(X) = \left(\begin{array}{c|c} h_1(X) & h_3(X) \\ \hline h_2(X) & h_4(X) \end{array} \right) = \left(\begin{array}{c|c} S(X) & S(X)X^T \\ \hline XS(X) & XS(X)X^T \end{array} \right). \tag{3.5}$$

In particular, using the notation above, we remark that $T_{x_0}\Gamma_u = h(Du(x_0))$. This discussion motivates the following

Definition 3.3. We denote by $\mathbb{G}_0(m, m+n) \doteq h(\mathbb{R}^{n\times m}) \subset \mathbb{G}(m, m+n)$ the set of orthogonal projections of tangent planes to graphs of maps defined on $\operatorname{span}\{e_1, \dots, e_m\} \subset \mathbb{R}^{n+m}$.

We will use for any matrix $M \in \mathbb{R}^{(m+n)\times(m+n)}$ the same splitting as in (3.5):

$$M = \left[\begin{array}{c|c} M_1 & M_3 \\ \hline M_2 & M_4 \end{array} \right] \tag{3.6}$$

with $M_1 \in \mathbb{R}^{m\times m}$, $M_4 \in \mathbb{R}^{n\times n}$. Using this notation, it is not difficult to verify that

$$h^{-1}(P) = P_2 P_1^{-1}. \tag{3.7}$$

The map h is therefore a smooth diffeomorphism between $\mathbb{R}^{n\times m}$ and the open subset $\mathbb{G}_0$.

In this section, we will use freely the following fact. Recall that, by (1.4), for every $X \in \mathbb{R}^{n\times m}$ the area element is given by

$$\mathcal{A}(X) = \sqrt{\det(\mathrm{id}_m + X^T X)}.$$

By the Cauchy-Binet formula, [5, Proposition 2.69],

$$\mathcal{A}(X) = \sqrt{1 + \|X\|^2 + \sum_{r=2}^{\min\{m,n\}} \sum_{Z \in \mathcal{A}_r} \det(X^Z)^2},$$

where we used the notation introduced in Definition 2.13.

Finally, throughout the chapter, we use the following notation:

- if $z \in \mathbb{R}^m \times \mathbb{R}^n$, then we will write $z = (x, y), x \in \mathbb{R}^m, y \in \mathbb{R}^n$;
- $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection on the first factor, i.e. $\pi(z) = \pi((x, y)) = x$.

3.1.1 *Graphs and varifolds*

If $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^m$ and $p > m$, Morrey's embedding theorem shows the existence of a precise representative of u which is Hölder continuous. In what follows we will always assume that the map u is given pointwise by such (Hölder) continuous precise representative. As done above, we use the notation Γ_u for the (set-theoretic) graph $\{(x, u(x)) : x \in \Omega\}$, which is a relatively closed subset of $\Omega \times \mathbb{R}^n$. The classical area formula (see for instance [38, Cor. 2, Ch. 3]) implies that Γ_u is m-rectifiable and its $\mathcal{H}^m$ measure is given by

$$\int_\Omega \mathcal{A}(Du) \,.$$

We can thus consider the corresponding varifold $[\![\Gamma_u]\!]$.

If $u \in W^{1,m}(\Omega, \mathbb{R}^n)$, then u has a precise representative which is however defined only up to a set of m-capacity 0 (but not *everywhere*). Moreover, if for maps $u \in W^{1,m} \cap C(\Omega, \mathbb{R}^n)$, for which the set-theoretic graph Γ_u could be defined classically, it can be proven that Γ_u does not necessarily have locally finite $\mathcal{H}^m$-measure, in spite of the fact that $\mathcal{A}(Du)$ belongs to L^1_{loc}. In particular the area formula fails. For this reason, following the notation and terminology of [38, Sec. 1.5, 2.1], we introduce the *rectifiable part of the graph of* u, which will be denoted by $\mathcal{G}_u$ (the notation in [38] is in fact $\mathcal{G}_{u,\Omega}$: we will omit the domain Ω since in our case it is always clear from the context).

First we denote the set of Lebesgue points of u by $\mathcal{L}_u$ and we introduce the set

$$A_D(u) \doteq \{x \in \Omega : u \text{ is approximately differentiable at } x\}.$$

For the definition of approximate differentiability, see [38, Sec. 1.4, Def. 3]. We also set

$$\mathcal{R}_u \doteq A_D(u) \cap \mathcal{L}_u.$$

Notice that, since $u \in W^{1,m}(\Omega, \mathbb{R}^n)$, then $|\Omega \setminus \mathcal{R}_u| = 0$. From now on, we always assume that u so that $u(x)$ is the Lebesgue value at every point $x \in \mathcal{R}_u$. The *rectifiable part of the graph* of u is then

$$\mathcal{G}_u \doteq \{(x, u(x)) : x \in \mathcal{R}_u\}\,.$$

By [38, Sec. 1.5, Th. 4], $\mathcal{G}_u$ is m-rectifiable and

$$\mathcal{H}^m(\mathcal{G}_u) = \int_\Omega \mathcal{A}(Du(x))dx\,. \tag{3.8}$$

Since $\mathcal{A}(Du) \in L^1$, this allows us to introduce the integer rectifiable varifold [1] $[\![\mathcal{G}_u]\!]$. When $u \in W^{1,p}$ for $p > m$, the Lusin property (namely the fact that $v(x) \doteq (x, u(x))$ maps sets of

1 In fact the $\mathcal{G}_u$ can be oriented to give an integer rectifiable current of multiplicity 1 and without boundary in $\Omega \times \mathbb{R}^n$, see [38, Pr. 1, Sec 2.1]. The varifold that we consider is then the one induced by the current in the usual sense.

Lebesgue measure zero in sets of $\mathcal{H}^m$-measure zero, cf. again [38]) and Morrey's embedding imply $\mathcal{G}_u \subset \Gamma_u$ and $\mathcal{H}^m(\Gamma_u \setminus \mathcal{G}_u) = 0$. In particular $[\![\mathcal{G}_u]\!] = [\![\Gamma_u]\!]$.

By [38, Sec. 1.5, Th. 5], the approximate tangent plane $T_y\mathcal{G}_u$ coincides for $\mathcal{H}^m$-a.e. $z_0 = (x_0, u(x_0)) \in \mathcal{G}_u$ with the orthogonal projection on $\tau(Du(x_0))$, introduced in (3.4). Recall that this orthogonal projection is denoted with $h(Du(x_0))$. The following proposition allows then to pass from functionals defined on varifolds to classical functionals in the vectorial calculus of the variations (and viceversa). The proof of this result will be given in Appendix B.

Proposition 3.4. *Let $u \in W^{1,m}(\Omega, \mathbb{R}^n)$, and define $v(x) \doteq (x, u(x))$. Let $\beta \in L^\infty(\Omega, (0, +\infty))$ and define $\theta(x, y) \doteq \beta(x), \forall (x, y) \in \mathbb{R}^{m+n}$. The following holds*

$$\theta[\![\mathcal{G}_u]\!](\varphi) \doteq \int_{\mathcal{G}_u} \varphi(z, T_z\mathcal{G}_u)\theta(z)d\mathcal{H}^m(z) = \int_\Omega \varphi(v(x), h(Du(x)))\mathcal{A}(Du(x))\beta(x)\,\mathrm{d}x \tag{3.9}$$

for every $\varphi \in C_b(\Omega \times \mathbb{R}^n \times \mathbb{G}_0)$.

Consider therefore a functional

$$\mathbb{E}(u, \beta) \doteq \int_\Omega f(Du(x))\beta(x)dx\,,$$

for some $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ with

$$\frac{f(X)}{\mathcal{A}(X)} \in C_b(\mathbb{R}^{n\times m}).$$

Define moreover $F, G : \mathbb{G}_0 \to \mathbb{R}$ as

$$F(M) \doteq f(h^{-1}(M)),\ G(M) \doteq \mathcal{A}(h^{-1}(M)).$$

Finally consider the map Ψ on the open subset $\mathbb{G}_0$ of the Grassmanian $\mathbb{G}(m, m+n)$ as

$$\Psi(h(X)) \doteq \frac{F(h(X))}{G(h(X))} = \frac{f(X)}{\mathcal{A}(X)}. \tag{3.10}$$

We can apply (3.9) to write:

$$\begin{aligned}\int_\Omega f(Du(x))\beta(x)\,dx &= \int_\Omega F(h(Du(x)))\beta(x)\,dx\\ &= \int_\Omega \frac{F(h(Du(x)))}{G(h(Du(x)))}\mathcal{A}(Du(x))\beta(x)\,dx = \int_{\mathcal{G}_u} \Psi(T_z\mathcal{G}_u)\theta(z)d\mathcal{H}^m(z).\end{aligned}$$

We are thus ready to introduce the following functional

Definition 3.5. Let $V = \theta[\![\Gamma]\!]$ be an m-dimensional rectifiable varifold in $\mathbb{R}^{m+n}$ with the property that the approximate tangent $T_z\Gamma$ belongs to $\mathbb{G}_0$ for $\mathcal{H}^m$-a.e. $z \in \Gamma$. Then

$$\Sigma(V) = \int_\Gamma \Psi(T_z\Gamma)\theta(z)d\mathcal{H}^m(z)\,.$$

The above discussion then proves the following

Proposition 3.6. *If $\Omega \subset \mathbb{R}^m$, $u \in W^{1,m}(\Omega, \mathbb{R}^n)$ and $\theta(x, u(x)) = \beta(x) \in L^\infty(\Omega, (0, +\infty))$, then $\Sigma(\theta[\![\mathcal{G}_u]\!]) = \mathbb{E}(u, \beta)$. Moreover, if $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > m$, then $\Sigma(\theta[\![\Gamma_u]\!]) = \mathbb{E}(u, \beta)$.*

3.1.2 *First variations*

We do not address here the issue of extending the functional Σ to general varifolds (namely of extending Ψ to all of $\mathbb{G}(m, m+n)$). Rather, assuming that such an extension exists, we wish to show that the usual stationarity of varifolds with respect to the functional Σ is equivalent to stationarity with respect to two particular classes of deformations, which reduce to inner and outer variations in the case of graphs. We start recalling the usual stationarity condition.

Definition 3.7. Let $\Psi : \mathbb{G}(m, m+n) \to [0, \infty]$ be a continuous function. Fix a vector field $g \in C^1_c(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$ and define X_ε as the flow generated by g, namely $X_\varepsilon(x) = \gamma_x(\varepsilon)$, if γ_x is the solution of the following system

$$\begin{cases} \gamma'(t) = g(\gamma(t)) \\ \gamma(0) = x. \end{cases}$$

We define the variation of V with respect to the vector field $g \in C^1_c(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$ as

$$[\delta_\Psi V](g) \doteq \lim_{\varepsilon \to 0} \frac{\Sigma((X_\varepsilon)_\# V) - \Sigma(V)}{\varepsilon}.$$

V is said to be *stationary* if $[\delta_\Psi V](g) = 0, \forall g \in C^1_c(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$.

Before continuing, let us give a rough explanation of why the equivalence of stationarity for graphs and stationarity for varifolds should hold. Suppose for the moment $\beta \equiv 1$. Recall that the first equation of (3.1) is given differentiating in ε the variation $u(x) + \varepsilon v(x)$. The latter corresponds, at the infinitesimal level, to the one-parameter family of deformations of the graph induced by the vector field $g(x, y) = (0, v(x))$. Similarly, the second equation of (3.1) is given differentiating the variation $u \circ X_\varepsilon$ where $X_\varepsilon(x) = x + \varepsilon \Phi(x)$, which corresponds to the one-parameter family of deformations of the graph induced by the vector field $(-\Phi(x), 0)$. These remarks can be used in order to show rigorously that, if $[\![\mathcal{G}_u]\!]$ is stationary in the sense of varifolds (for the energy corresponding to $\mathbb{E}$), then u satisfies (3.1). The converse is less obvious: even though any vector field $g(x, y)$ can be decomposed into a horizontal and vertical part $(g_1(x, y), 0) + (0, g_2(x, y))$, there is still the issue that the g_i's depend on the variable y as well. When the graph u is smooth, we can simply argue that variations of the graph along the vector field $g(x, y)$ are equal to variations along $\tilde{g}(x) \doteq g(x, u(x))$. This however creates several technical difficulties if we only assume Sobolev regularity for u. Nonetheless the conclusion is still correct. We conclude this section with a rather general equivalence statement between stationarity of graphs and stationarity of varifolds, for which we need first some suitable terminology and notation. The (somewhat lengthy) proof is postponed to the next section.

Given an orthogonal projection $P \in \mathbb{G}(m, m+n)$, we denote $P^\perp \doteq \mathrm{id}_{m+n} - P$. The notation $P^\perp$ is due to the fact that, if P represents the orthogonal projection onto the m-plane $\tau \subset \mathbb{R}^{n+m}$, $\mathrm{id}_{n+m} - P$ is the element in $\mathbb{G}(n, m+n)$ representing the orthogonal projection onto the n-plane $\tau^\perp \subset \mathbb{R}^{n+m}$. From [21, Lemma A.2], we know that, for $V = \theta[\![\Gamma]\!]$,

$$[\delta_\Psi(V)](g) = \int_\Gamma \langle B_\Psi(T_x\Gamma), Dg(x)\rangle \theta(x) d\mathcal{H}^m(x), \forall g \in C^1_c(\mathbb{R}^{m+n}, \mathbb{R}^{m+n}) \tag{3.11}$$

where $B_\Psi(\cdot) : \mathbb{G}(m, m+n) \to \mathbb{R}^{(m+n)\times(m+n)}$ is defined through the relation

$$\begin{aligned} \langle B_\Psi(P), L\rangle &= \Psi(P)\langle P, L\rangle + \langle d\Psi(P), P^\perp L P + (P^\perp L P)^T\rangle, \\ &\qquad \forall P \in \mathbb{G}(m, m+n), \forall L \in \mathbb{R}^{(m+n)\times(m+n)}, \end{aligned} \tag{3.12}$$

We are now ready to state our desired equivalence between stationarity of the map u for the energy $\mathbb{E}$ and stationarity of the varifold $[\![\mathcal{G}_u]\!]$ for the corresponding functional Σ. In what follows, given a function g on $\mathcal{G}_u$ we will use the shorthand notation $\|g\|_{q,\mathcal{G}_u}$ for the norm $\|g\|_{L^q(\mathcal{H}^m \llcorner \mathcal{G}_u)}$.

Proposition 3.8. *Assume that $f \in C^1(\mathbb{R}^{n\times m})$ admits an extension $\Psi \in C^1(\mathbb{G}(m, m+n))$, in the sense that (6.4) holds for every $X \in \mathbb{R}^{n\times m}$. Fix any $m \le p \le +\infty$, $1 \le q < +\infty$ and a Lipschitz, bounded, open set $\Omega \subset \mathbb{R}^m$. If a map $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and a Borel function $\beta \in L^\infty(\Omega, (0, +\infty))$ satisfy, for every $v \in C^1_c(\Omega, \mathbb{R}^n)$ and $\Phi \in C^1_c(\Omega, \mathbb{R}^m)$,*

$$\begin{cases} \left| \int_\Omega \langle Df(Du), Dv\rangle \beta \,\mathrm{d}x \right| \le C\|v\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q \\ \left| \int_\Omega \langle Df(Du), DuD\Phi(x)\rangle \beta \,\mathrm{d}x - \int_\Omega f(Du)\,\mathrm{div}(\Phi)\beta dx \right| \le C\|\Phi\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q, \end{cases} \tag{3.13}$$

for some $C \geq 0$, then the rectifiable varifold of $\mathbb{R}^{m+n}$ $\theta[\![\mathcal{G}_u]\!]$, where $\theta(x,y) = \beta(x)$, satisfies

$$|\delta_\Psi(\theta[\![\mathcal{G}_u]\!])(g)| \leq C'\|g\theta^{\frac{1}{q}}\|_{q,\mathcal{G}_u} \qquad \forall g \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^{m+n}), \tag{3.14}$$

for some number $C' = C'(C,m,p,q) \geq 0$. Conversely, if (3.14) holds for some C', then (3.13) holds for some $C = C(C',m,p,q)$. Moreover, $C' = 0$ if and only $C = 0$, namely u is satisfies (3.1) if and only if $\theta[\![\mathcal{G}_u]\!]$ is stationary for the energy Σ.

Remark 3.9. As already noticed, when $p > m$ we can replace $[\![\mathcal{G}_u]\!]$ with $[\![\Gamma_u]\!]$. Moreover, under such stronger assumption, the proposition holds also for $q = \infty$, provided we set $\mathcal{A}(Du)^{\frac{1}{q}} \doteq 1$ in that case. Finally, if $p = \infty$, then we can drop the request that f admits a C^1 extension Ψ, and the same proof would work if we extended Ψ as in (6.4) as $\Psi(T) \equiv +\infty$, if $T \notin \mathbb{G}_0(m, m+n)$.

The proof of the previous proposition is a consequence of a few technical lemmas and will be given in the next section.

3.2 PROOF OF PROPOSITION 3.8

Let $f \in C^1(\mathbb{R}^{n\times m})$ be of the form $f(X) = \Psi(h(X))\mathcal{A}(X)$. In the next lemma we study the growth of the matrix-fields associated to the inner and the outer variations, i.e.

$$A(X) \doteq Df(X) \tag{3.15}$$

$$B(X) \doteq f(X)\,\mathrm{id}_m - X^T Df(X). \tag{3.16}$$

Define also the matrix-field $V_f : \mathbb{R}^{n\times m} \to \mathbb{R}^{(m+n)\times(m+n)}$ to be

$$V_f(X) \doteq \frac{1}{\mathcal{A}(X)}\left(\begin{array}{c|c} B(X) & B(X)X^T \\ \hline A(X) & A(X)X^T \end{array}\right). \tag{3.17}$$

In Lemma 3.11, we will prove that

$$B_\Psi(h(X)) = V_f(X),\ \forall X \in \mathbb{R}^{n\times m}.$$

Combining Lemma 3.10 and 3.11 with the area formula we obtain Lemma 3.12, from which we will infer Proposition 3.8.

Lemma 3.10. *Let $\Psi \in C^1(\mathbb{G}(m,m+n))$ and let $f(X) = \Psi(h(X))\mathcal{A}(X)$, where h is the map defined in (3.5). Then,*

$$\|A(X)\| \lesssim 1 + \|X\|^{\min\{m,n\}-1}, \|B(X)\| \lesssim 1 + \|X\|^{\min\{n,m-1\}}. \tag{3.18}$$

In the statement of the Lemma and in the proof, the symbol $\Lambda \lesssim \Xi$ means that there exist a non-negative constant C depending only on n, m and on $\|\Psi\|_{C^1(\mathbb{G}(m,m+n))}$ such that

$$\Lambda \leq C\,\Xi.$$

The lemma above is needed to get reach enough summability in order to justify the integral formulas in (the statement and the proof of) Lemma 3.12. In some sense it is thus less crucial than the next lemma, which contains instead the core computations. For these reasons, the argument of Lemma 3.10, which contains several lengthy computations is given in Appendix B.

Lemma 3.11. *For every $X \in \mathbb{R}^{n\times m}$,*

$$B_\Psi(h(X)) = V_f(X).$$

Lemma 3.12. *Let $f(X) = \Psi(h(X))\mathcal{A}(X)$ be a function of class $C^1(\mathbb{R}^{n\times m})$. Let moreover $\theta(x,u(x)) = \beta(x) \in L^\infty(\Omega,(0,+\infty))$. Then, for every $g = (g^1,\dots,g^{m+n}) \in C^1_c(\Omega\times\mathbb{R}^n)$, the following equality holds:*

$$\begin{aligned}&\delta_\Psi(\theta[\![\mathcal{G}_u]\!])(g)\\ &= \int_\Omega \langle B(Du(x)), D(g_1(x,u(x)))\rangle\beta(x)dx + \int_\Omega \langle A(Du(x)), D(g_2(x,u(x)))\rangle\beta(x)dx,\end{aligned} \tag{3.19}$$

where $g_1(x,y) \doteq (g^1(x,y),\dots,g^m(x,y))$, $g_2(x,y) \doteq (g^{m+1}(x,y),\dots,g^{m+n}(x,y))$ *and* $A(X)$ *and* $B(X)$ *are as in* (3.15) *and* (3.16)*.*

We next prove Lemma 3.11 and Lemma 3.12 and hence end the section showing how to use Lemma 3.12 to conclude the desired Proposition 3.8.

3.2.1 *Proof of Lemma 3.11*

For a map $g : \mathbb{G}(m+n,m) \to \mathbb{R}^\ell$, $\ell \geq 1$, of class C^1, we denote the differential at the point $P \in \mathbb{G}(m+n,m)$ with the symbol $d_P g$. For $H \in T_P\mathbb{G}(m+n,m)$, and for $\gamma : (-1,1) \to \mathbb{G}(m+n,m)$ with $\gamma(0) = P$, $\gamma'(0) = H$, we denote

$$d_P g(P)[H] \doteq \lim_{t \to 0} \frac{g(\gamma(t)) - g(P)}{t}.$$

If $\ell = 1$, we identify $d_P g(P)$ with the $\mathbb{R}^{(m+n)\times(m+n)}$ associated matrix representing the differential, and we denote $d_P g(P)[H]$ with $\langle d_P g(P), H\rangle$. Moreover, we recall the splitting introduced in (3.6), namely for any matrix $M \in \mathbb{R}^{(m+n)\times(m+n)}$ we denote

$$M = \left(\begin{array}{c|c} M_1 & M_3 \\ \hline M_2 & M_4 \end{array}\right)$$

with $M_1 \in \mathbb{R}^{m\times m}$, $M_4 \in \mathbb{R}^{n \times n}$. In this proof, we will use the following facts:

- The tangent plane of $\mathbb{G}(m,m+n)$ at the point P is given by

  $$T_P\mathbb{G}(m,m+n) = \{M \in \mathbb{R}^{(m+n)\times(m+n)} : M = P^\perp L P + (P^\perp L P)^T, \text{for some } L \in \mathbb{R}^{(m+n)\times(m+n)}\},$$

  as proved in [21, Appendix A].

- Let $h : \mathbb{R}^{n\times m} \to \mathbb{G}_0$ be the map defined in (3.5). Recall that its inverse is given by $h^{-1}(P) = P_2 P_1^{-1}$. For every $H \in T_P\mathbb{G}(m,m+n)$, one has:

  $$d_P(h^{-1})(P)[H] = (H_2 - P_2 P_1^{-1} H_1) P_1^{-1} \in \mathbb{R}^{n\times m}. \tag{3.20}$$

- Recall that the area functional is defined as

  $$\mathcal{A}(X) = \sqrt{\det(M(X)^T M(X))} \qquad \text{where } \; M(X) = \begin{pmatrix} \mathrm{id}_m \\ X \end{pmatrix}.$$

  Hence, for every $X, Y \in \mathbb{R}^{n\times m}$, we have

  $$\langle D\mathcal{A}(X), Y\rangle = \frac{1}{2}\mathcal{A}(X)\mathrm{tr}\,[(M(X)^T M(X))^{-1}(Y^T X + X^T Y)]. \tag{3.21}$$

Recall the definition of $B_\Psi(P)$ given in (3.12). Since

$$\Psi(P) = \frac{f(h^{-1}(P))}{\mathcal{A}(h^{-1}(P))},$$

for every $H \in T_P\mathbb{G}(m,m+n)$ we have

$$\begin{aligned}&\langle d_P \Psi(P), H\rangle \\ &\quad = \frac{1}{\mathcal{A}(X)}\langle Df(h^{-1}(P)), d_P(h^{-1})(P)[H]\rangle - \frac{f(h^{-1}(P))}{\mathcal{A}^2(h^{-1}(P))}\langle D\mathcal{A}(h^{-1}(P)), d_P(h^{-1})(P)[H]\rangle.\end{aligned}$$

When evaluated at $P = h(X)$, the previous expression reads

$$\begin{aligned}&\langle d_P\Psi(h(X)), H\rangle \\ &\quad = \frac{1}{\mathcal{A}(X)}\langle Df(X), d_P(h^{-1})(h(X))[H]\rangle - \frac{f(X)}{\mathcal{A}^2(X)}\langle D\mathcal{A}(X), d_P(h^{-1})(h(X))[H]\rangle.\end{aligned} \tag{3.22}$$

By (3.12), we know that, for every $L \in \mathbb{R}^{(m+n)\times(m+n)}$,

$$\langle B_\Psi(h(X)), L\rangle = \Psi(h(X))\langle h(X), L\rangle + \langle d_P\Psi(h(X)), h(X)^\perp L h(X) + (h(X)^\perp L h(X))^T\rangle.$$

Therefore, we want to compute (3.22) when

$$H = h(X)^\perp L h(X) + (h(X)^\perp L h(X))^T = Lh(X) - h(X)Lh(X) + h(X)L^T - h(X)L^T h(X).$$

We wish to find an expression for

$$d_P(h^{-1})(h(X))[h(X)^\perp L h(X) + h(X)L^T h(X)^\perp].$$

Using the decomposition introduced in (3.6) of L in 4 submatrices, we compute

$$Lh(X) = \left(\begin{array}{c|c} L_1 & L_3 \\ \hline L_2 & L_4\end{array}\right)\left(\begin{array}{c|c} S & SX^T \\ \hline XS & XSX^T\end{array}\right) = \left(\begin{array}{c|c} L_1S + L_3XS & L_1SX^T + L_3XSX^T \\ \hline L_2S + L_4XS & L_2SX^T + L_4XSX^T\end{array}\right) \tag{3.23}$$

and

$$\begin{aligned}&h(X)Lh(X) = \\ &\left(\begin{array}{c|c} S(L_1 + L_3X + X^TL_2 + X^TL_4X)S & S(L_1 + L_3X + X^TL_2 + X^TL_4X)SX^T \\ \hline XS(L_1 + L_3X + X^TL_2 + X^TL_4X)S & XS(L_1 + L_3X + X^TL_2 + X^TL_4X)SX^T\end{array}\right)\end{aligned} \tag{3.24}$$

Combining (3.20) with (3.24), we get

$$\begin{aligned} d_P(h^{-1})(h(X))[Lh(X)] &= (L_2S + L_4XS - XSS^{-1}L_1S - XSS^{-1}L_3XS)S^{-1} \\ &= L_2 + L_4X - XL_1 - XL_3X,\end{aligned} \tag{3.25}$$

$$\begin{aligned}&d_P(h^{-1})(h(X))[h(X)Lh(X)] \\ &\qquad = XS(L_1 + L_3X + X^TL_2 + X^TL_4X - S^{-1}SL_1 - S^{-1}SL_3X \\ &\qquad - S^{-1}SX^TL_2 - S^{-1}SX^TL_4X)SS^{-1} \\ &\qquad = XS(L_1 + L_3X + X^TL_2 + X^TL_4X - L_1 - L_3X - X^TL_2 - X^TL_4X) = 0\end{aligned} \tag{3.26}$$

and

$$\begin{aligned} d_P(h^{-1})(h(X))[h(X)L^T] &= d_P(h^{-1})(h(X))[(L\circ h(X))^T] \\ &= (XSL_1^T + XSX^TL_3^T - XSL_1^T - XSX^TL_3^T)S^{-1} = 0.\end{aligned} \tag{3.27}$$

Combining (3.25), (3.26) and (3.27), we get that

$$\begin{aligned} d_P(h^{-1})(h(X))[h(X)^\perp Lh(X) + h(X)L^Th(X)^\perp] &= d_P(h^{-1})(h(X))(Lh(X)) \\ &= L_2 + L_4X - XL_1 - XL_3X.\end{aligned}$$

Now define the matrix:

$$C \doteq L_2 + L_4X - XL_1 - XL_3X.$$

To expand (3.22), we now need to rewrite

$$\langle D\mathcal{A}(X), d_P(h^{-1})(h(X))[H]\rangle.$$

First, we must compute the trace part coming from (3.21):

$$\begin{aligned}\operatorname{tr}\,[S(C^TX+X^TC)] &= \operatorname{tr}\,[S(L_2^TX+X^TL_4^TX-L_1^TX^TX-X^TL_3^TX^TX)]\\ &\quad+\operatorname{tr}\,[S(X^TL_2+X^TL_4X-X^TXL_1-X^TXL_3X)]\\ &=2\operatorname{tr}\,(SX^TL_2)+2\operatorname{tr}\,(SX^TL_4X)-2\operatorname{tr}\,(SX^TXL_1)-2\operatorname{tr}\,(SX^TXL_3X).\end{aligned}$$

Hence, if $H=h(X)^{\perp}Lh(X)+h(X)L^Th(X)^{\perp}$, we have just proved that:

$$\begin{aligned}\langle d_P\Psi(h(X)),H\rangle &= \frac{1}{\mathcal{A}(X)}\langle Df(X),L_2+L_4X-XL_1-XL_3X\rangle\\ &-\frac{f(X)}{\mathcal{A}(X)}(\operatorname{tr}\,(SX^TL_2)+\operatorname{tr}\,(SX^TL_4X)-\operatorname{tr}\,(SX^TXL_1)-\operatorname{tr}\,(SX^TXL_3X)).\end{aligned}\tag{3.28}$$

To conclude, we also need to compute

$$\begin{aligned}\Psi(h(X))\langle h(X),L\rangle &= \frac{f(X)}{\mathcal{A}(X)}(\langle L_1,S\rangle+\langle L_2,XS\rangle+\langle L_3,SX^T\rangle+\langle L_4,XSX^T\rangle)\\ &=\frac{f(X)}{\mathcal{A}(X)}(\operatorname{tr}\,(SL_1)+\operatorname{tr}\,(SX^TL_2)+\operatorname{tr}\,(XSL_3)+\operatorname{tr}\,(XSX^TL_4)).\end{aligned}\tag{3.29}$$

Now we sum (3.28) and (3.29) to get $\langle B_\Psi(h(X)),L\rangle$. Using that $S^{-1}(X)=X^TX+\mathrm{id}_m$ and the invariance of the trace under cyclic permutations, we rewrite

$$\begin{aligned}&\operatorname{tr}\,(SL_1)+\operatorname{tr}\,(SX^TL_2)+\operatorname{tr}\,(XSL_3)+\operatorname{tr}\,(XSX^TL_4)\\ &\quad-\operatorname{tr}\,(SX^TL_2)-\operatorname{tr}\,(SX^TL_4X)+\operatorname{tr}\,(SX^TXL_1)+\operatorname{tr}\,(SX^TXL_3X)=\operatorname{tr}\,(L_1)+\operatorname{tr}\,(L_3X).\end{aligned}$$

Combining our previous computations, we find

$$\begin{aligned}\langle B_\Psi(h(X)),L\rangle &= \frac{f(X)}{\mathcal{A}(X)}(\operatorname{tr}\,(L_1)+\operatorname{tr}\,(L_3X))+\frac{1}{\mathcal{A}(X)}\langle Df(X),L_2+L_4X-XL_1-XL_3X\rangle\\ &=\frac{1}{\mathcal{A}(X)}[-\langle X^TDf(X)+f(X)\,\mathrm{id}_m,L_1\rangle+\langle Df(X),L_2\rangle\\ &\quad+\langle f(X)X^T-X^TDf(X)X^T,L_3\rangle+\langle Df(X)X^T,L_4\rangle].\end{aligned}$$

Since L was arbitrary, we conclude that

$$B_\Psi(h(X))=\frac{1}{\mathcal{A}(X)}\left(\begin{array}{c|c}B(X) & B(X)X^T\\ \hline A(X) & A(X)X^T\end{array}\right).$$

3.2.2 *Proof of Lemma 3.12*

Fix g as in the statement of the Lemma. By (3.11), we know that

$$\delta_\Psi(\theta[\![\mathcal{G}_u]\!])(g)=\int_{\mathcal{G}_u}\langle B_\Psi(T_z\mathcal{G}_u),Dg(z)\rangle\theta(z)d\mathcal{H}^m(z).$$

Now define $F(z,T)\doteq\langle B_\Psi(T),Dg(z)\rangle$ and $\bar F(x,u(x))\doteq\langle B_\Psi(h(Du(x))),Dg(x,u(x))\rangle$. We have $F\in C_c(\mathcal{G}_u\times\mathbb{G}_0)$ and we apply Proposition 3.4 to find the equality

$$\int_{\mathcal{G}_u}\langle B_\Psi(T_z\Gamma_u),Dg(z)\rangle\theta(z)d\mathcal{H}^m(z)=\int_\Omega\mathcal{A}(Du(x))\bar F(x,u(x))\beta(x)dx.\tag{3.30}$$

By Lemma 3.11,

$$\bar F(x,u(x))=\langle V_f(Du(x)),Dg(x,u(x))\rangle$$

a.e. in Ω. Moreover, since

$$\mathcal{A}(Du(x))V_f(Du(x)) = \left(\frac{B(Du(x)) \mid B(Du(x))Du(x)^T}{A(Du(x)) \mid A(Du(x))Du(x)^T}\right),$$

we have

$$\begin{aligned}\mathcal{A}(Du(x))\bar{F}(x,u(x)) &= \langle D_x g^1(x,u(x)), B(Du(x))\rangle + \langle B(Du(x))Du^T(x), D_y g^1(x,u(x))\rangle\\ &+ \langle D_x g^2(x,u(x)), A(Du(x))\rangle + \langle A(Du(x))Du^T(x), D_y g^2(x,u(x))\rangle\\ &= \langle B(Du(x)), D(g^1(x,u(x)))\rangle + \langle A(Du(x)), D(g^2(x,u(x)))\rangle.\end{aligned}$$

The previous equality and (3.30) yield the conclusion.

3.2.3 *Proof of Proposition 3.8*

First, assume (3.13), and fix any $g \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^{m+n})$, $g = (g^1, \dots, g^{m+n})$. Define

$$\begin{aligned}\bar{\Phi}(x) &\doteq (g^1(x,u(x)), \dots, g^m(x,u(x)))\\ \bar{v}(x) &\doteq (g^{m+1}(x,u(x)), \dots, g^{m+n}(x,u(x))\,.\end{aligned}$$

We have $\bar{\Phi} \in L^\infty \cap W^{1,m}_0(\Omega,\mathbb{R}^m)$ and $\bar{v} \in L^\infty \cap W^{1,m}_0(\Omega,\mathbb{R}^n)$. Notice that we require (3.13) to hold only for C^1 maps with compact support, but Lemma 3.10 implies through an approximation argument that

$$\begin{cases}\left|\displaystyle\int_\Omega \langle A(Du), Dv\rangle \beta(x)\,dx\right| \le C\|v\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q, \forall v \in L^\infty \cap W^{1,m}_0(\Omega,\mathbb{R}^n)\\ \left|\displaystyle\int_\Omega \langle B(Du), D\Phi\rangle \beta(x) dx\right| \le C\|\Phi\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q, \forall \Phi \in L^\infty \cap W^{1,m}_0(\Omega,\mathbb{R}^m).\end{cases} \tag{3.31}$$

Indeed, to prove, for instance, that the first inequality holds for any $v \in L^\infty \cap W^{1,m}_0$, pick a sequence $v_k \in C^\infty_c(\Omega,\mathbb{R}^n)$ such that $\|v_k\|_{L^\infty}$ is equibounded and $v_k \to v$ in $W^{1,m}$, L^q and pointwise a.e.. The fact that

$$\int_\Omega \langle A(Du), Dv_k\rangle \beta(x)\,dx \to \int_\Omega \langle A(Du), Dv\rangle \beta(x)\,dx$$

is an easy consequence of the $W^{1,m}$ convergence of v_k to v and the fact that $A(Du)\beta(x) \in W^{\frac{m}{m-1}}(\Omega,\mathbb{R}^{n\times m})$ by Lemma 3.10. Moreover, the quantity

$$\|v_k\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q \to \|v\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q$$

by the dominated convergence theorem. Indeed, we required the pointwise convergence of v_k to v and moreover we can bound for every k and almost every $x \in \Omega$:

$$\|v_k\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|^q(x) \le \sup_k \|v_k\|^q_{L^\infty}\mathcal{A}(Du(x))\beta(x) \in L^1(\Omega).$$

Hence (3.31) with v_k instead of v implies the same inequality for v by taking the limit as $k \to \infty$. The proof of the second inequality of (3.31) is analogous. We combine (3.31) with (3.19) to write

$$\begin{aligned}|\delta_\Psi(\theta[\![\mathcal{G}_u]\!])(g)| &\le \left|\int_\Omega \langle A(Du), D\bar{v}\rangle \beta\,dx\right| + \left|\int_\Omega \langle B(Du), D\bar{\Phi}\rangle \beta dx\right|\\ &\le C(\|\bar{v}\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q + \|\bar{\Phi}\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q).\end{aligned}$$

Notice that, since $\bar{v}(\cdot,u(\cdot))\beta(\cdot) \in L^\infty(\Omega,\mathbb{R}^n)$ and $\bar{\Phi}(\cdot,u(\cdot))\beta(\cdot) \in L^\infty(\Omega,\mathbb{R}^n)$, we have

$$\|\bar{v}(\cdot,u(\cdot))\|^q\mathcal{A}(Du(\cdot))\beta(\cdot) + \|\bar{\Phi}(\cdot,u(\cdot))\|^q\mathcal{A}(Du(\cdot))\beta(\cdot) \in L^1(\Omega).$$

Now we use the trivial estimate $\|\bar{v}(x,y)\| \le \|g(x,y)\|$ for all $x \in \Omega, y \in \mathbb{R}^n$, and area formula (3.9) to conclude

$$\|\bar{v}\mathcal{A}^{\frac{1}{q}}(Du)\beta\|_q^q = \int_\Omega \|\bar{v}(x,u(x))\|^q \mathcal{A}(Du(x))\beta(x)dx \le \int_\Omega \|g(x,u(x))\|^q \mathcal{A}(Du(x))\beta(x)dx$$
$$= \int_{\mathcal{G}_u} \|g\|^q(z)\theta(z)d\mathcal{H}^m(z) = \|g\theta^{\frac{1}{q}}\|^q_{L^q(\mathcal{G}_u)}.$$

With analogous estimates, we also find

$$\|\bar{\Phi}\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q^q \le \|g\theta^{\frac{1}{q}}\|^q_{L^q(\mathcal{G}_u)}.$$

Therefore, (3.14) holds with constant $C' = 2C$. Now assume (3.14). Choose the following sequence $g_k \in C^1_c(\Omega \times \mathbb{R}^n)$:

$$g_k(x,y) \doteq G(x)\chi_k(y),$$

where $G \in C^1_c(\Omega, \mathbb{R}^{n+m})$, and $\chi_k \in C^\infty_c(\mathbb{R}^n)$ with $0 \le \chi_k(y) \le 1, \forall y \in \mathbb{R}^n$, $\chi_k \equiv 1$ on $B_k(0)$, $\chi_k \equiv 0$ on $B^c_{2k}(0)$ and $\|D\chi_k(y)\| \le \frac{1}{k}$, for all $y \in \mathbb{R}^n$. Using again area formula (3.9), we write

$$\|g_k\theta^{\frac{1}{q}}\|^q_{L^q(\mathcal{G}_u)} = \int_\Omega \|g_k(x,u(x))\|^q \mathcal{A}(Du(x))\beta^{\frac{1}{q}}(x)dx.$$

Monotone convergence theorem implies

$$\lim_k \|g_k\theta^{\frac{1}{q}}\|^q_{L^q(\mathcal{G}_u)} = \|G\mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q^q.$$

Now we want to use (3.19). Using the same notation as in the statement of Lemma 3.12, i.e. splitting G into $G_1 = (G^1, \dots, G^m)$ and $G_2 = (G^{n+1}, \dots, G^{n+m})$, we have

$$\int_\Omega \langle B(Du(x)), D((g_k)_1(x,u(x)))\rangle \beta(x)\,dx = \int_\Omega \langle B(Du(x)), D(\chi_k(u(x))G_1(x))\rangle \beta(x)\,dx$$
$$= \int_\Omega \chi_k(u(x))\langle B(Du(x)), DG_1(x)\rangle \beta(x)\,dx + \int_\Omega \langle B(Du(x)), G_1(x) \otimes (D\chi_k(u(x))Du(x))\rangle \beta(x)\,dx$$

By Lemma 3.10 and the regularity of G_1, we have that

$$\|DG_1\|\|B(Du)\|\beta \in L^1(\Omega) \text{ and } \|G_1\|\|B(Du)\|\|Du\|\beta \in L^1(\Omega). \tag{3.32}$$

Since

$$\chi_k(u(x))\langle B(Du(x)), DG_1(x)\rangle\beta(x) \to \langle B(Du(x)), DG_1(x)\rangle\beta(x)$$

pointwise a.e. as $k \to \infty$, (3.32) tells us that we can apply dominated convergence theorem to infer

$$\lim_{k\to\infty} \int_\Omega \chi_k(u(x))\langle B(Du(x)), DG_1(x)\rangle\beta(x)\,dx = \int_\Omega \langle B(Du(x)), DG_1(x)\rangle\beta(x)dx.$$

Moreover using the pointwise bound $\|D\chi_k(u(x))\| \le \frac{1}{k}$,

$$\left|\int_\Omega \langle B(Du(x)), G_1(x) \otimes (D\chi_k(x)Du(x))\rangle\beta(x)\,dx\right| \le \frac{1}{k}\int_\Omega \|B(Du(x))\|\|G_1(x)\|Du(x)\|\beta(x)dx.$$

Again through (3.32), we infer that the last term converges to 0. This implies that

$$\int_\Omega \langle B(Du(x)), D((g_k)_1(x,u(x)))\rangle\,dx \to \int_\Omega \langle B(Du(x)), DG_1(x)\rangle\,dx \text{ as } k \to \infty.$$

In a completely analogous way,

$$\int_\Omega \langle A(Du(x)), D((g_k)_2(x,u(x)))\rangle\beta(x)\,dx \to \int_\Omega \langle A(Du(x)), DG_2(x)\rangle\beta(x)\,dx \text{ as } k \to \infty.$$

Now (3.19) and the previous computations yield

$$\begin{aligned}
&\int_\Omega \langle A(Du(x)), DG_2(x)\rangle \beta(x)\,dx + \int_\Omega \langle B(Du(x)), DG_1(x)\rangle \beta(x)\,dx \\
&= \lim_{k\to\infty} \left[\int_\Omega \langle A(Du(x)), D(g_k)_2(x)\rangle \beta(x)\,dx + \int_\Omega \langle B(Du(x)), D(g_k)_1(x)\rangle dx \beta(x) \right] \\
&= \lim_{k\to\infty} \delta_\Psi(\theta[\![\mathcal{G}_u]\!])(g_k) \leq C' \lim_k \|g_k \theta^{\frac{1}{q}}\|_{L^q(\mathcal{G}_u)} = C' \|G \mathcal{A}^{\frac{1}{q}}(Du)\beta^{\frac{1}{q}}\|_q,
\end{aligned}$$

and it is immediate to see that this implies (3.13) with constant $\bar{C}' = C'$.

4 TOWARDS REGULARITY IN TWO DIMENSIONS

In this chapter we study the differential inclusion associated to the area function $\mathcal{A}(X)$ in two dimensions. The main result of this chapter is Theorem 4.20:

Theorem. *For every $R > 0$, there exists $\alpha = \alpha(R) > 0$ such that, if f is a $C^k(\mathbb{R}^{2n+2\times 2})$ function, $k \geq 2$, with the property that*

$$\|f - \mathcal{A}\|_{C^2(B_{2R}(0))} \leq \alpha, \tag{4.1}$$

and $\mathcal{U} : \Omega \to \mathbb{R}^{2n+2}$ is a Lipschitz solution of

$$D\mathcal{U}(x) \in C_f, \text{ for a.e. } x \in \Omega \tag{4.2}$$

with

$$\|D\mathcal{U}\|_\infty \leq R,$$

then $\mathcal{U} \in C^{k-1,\rho}(\Omega)$, for some positive $\rho > 0$.

This is obtained as a consequence of several preliminary results, in particular it relies on classical regularity results for solutions of the Monge-Ampére equations, Proposition 4.4, the estimate of algebraic nature of Theorem 4.6 and a slight generalization of the result of [88], Proposition 4.15. As a consequence of the aforementioned results, we will also prove a compactness result, Theorem 4.10. In the final section, Section 4.6, we use an example of [51] to show that there exist irregular points for the inner variation equations for the area functional, namely we show Theorem 4.21.

4.1 TWO-DIMENSIONAL DIFFERENTIAL INCLUSIONS AND THE AREA FUNCTIONAL

In this section we rewrite the partial differential system defining a stationary graph for the area functional as a differential inclusion. This is done in an analogous way to what we have done in Section 2.1. The only technical difference is that, instead considering a mixed div-curl system, we use the rotation given by the symplectic matrix

$$J \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

to translate the problem into a *classical* differential inclusion. This simply amounts to realizing that, if $v = (v_1, v_2) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, then

$$\operatorname{div}(v) = \operatorname{curl}(vJ). \tag{4.3}$$

For this reason, we will study the following modification of (2.4) for any polyconvex $f \in C^2(\mathbb{R}^{n\times 2})$:

$$D\mathcal{U}(x) \in C_f = \begin{pmatrix} X \\ A_f(X) \\ B_f(X) \end{pmatrix}, \text{ for a.e. } x \in \Omega \tag{4.4}$$

where $\mathcal{U} : \Omega \to \mathbb{R}^{2n+2}$ is a function in a Sobolev space (its regularity will be discussed at the end of this section) and[1]

$$A_f(X) = Df(X)J, \quad B_f(X) = X^T Df(X)J - f(X)J.$$

1 Notice that A_f and B_f already appeared in the previous chapter to denote the same matrices, up to the multiplication by J. Here we want to remark the dependence on f in order to distinguish them from the fields A and B associated to the area, so the notation differs slightly for the introduction of the subscript f.

We will always use the following notation for a map $\mathcal{U}$ with the property (4.4):

$$\mathcal{U} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \; u, v : \Omega \to \mathbb{R}^n, w : \Omega \to \mathbb{R}^2,$$

so that $\mathcal{U}$ satisfies (4.4) if and only if $Dv(x) = A_f(Du(x))$ and $Dw(x) = B_f(Du(x))$ for a.e. $x \in \Omega$. Through (4.3), it is immediate to see that u solves (2.2) if and only if there exists $\mathcal{U}$ as above such that $D\mathcal{U}(x) \in C_f$ for a.e. $x \in \Omega$. If $f = \mathcal{A}$, then we drop the f in A_f and B_f and we simply write

$$D\mathcal{U}(x) \in C_{\mathcal{A}} = \begin{pmatrix} X \\ A(X) \\ B(X) \end{pmatrix}, \text{ for a.e. } x \in \Omega. \tag{4.5}$$

We need to compute explicitely $A(X)$ and $B(X)$. Recall that the area on 2-dimensional graphs is given by $\mathcal{A} : \mathbb{R}^{n\times 2} \to \mathbb{R}$:

$$\mathcal{A}(X) = \sqrt{1 + \|X\|^2 + \sum_{1\le a\le b\le n} \det(X^{ab})^2},$$

where X^{ab} is the 2×2 submatrix obtained from X considering just the a-th and the b-th rows. We compute

$$D\mathcal{A}(X) = \frac{X + \sum_{1\le a\le b\le n} \det(X^{ab}) C_{ab}(X)}{\mathcal{A}(X)}, \tag{4.6}$$

where[2] $C_{ab}(X)$ denotes the $n \times 2$ matrix defined as

$$(C_{ab}(X))_{ij} = \begin{cases} 0, & \text{if } i \neq a \text{ or } i \neq b \\ (\operatorname{cof}(X^{ab})^T)_{ij}, & \text{otherwise.} \end{cases}$$

From (4.6), it also follows that

$$X^T D\mathcal{A}(X) - \mathcal{A}(X)\,\mathrm{id} = \frac{X^T X + \sum_{a\le b} \det(X^{ab})^2\,\mathrm{id} - \mathcal{A}^2(X)\,\mathrm{id}}{\mathcal{A}(X)} = \frac{X^T X - (1 + \|X\|^2)\,\mathrm{id}}{\mathcal{A}(X)}.$$

Let us make some preliminary computations that we will need in the chapter. Namely:

Lemma 4.1. *The following hold*

1. $\|A(X)\| \le 2\|X\|$;
2. $\frac{1+\|X\|^2}{2\mathcal{A}(X)} \le \|B(X)\| \le 2(1 + \|X\|)$.

Proof. In this proof, we will make use of the Cauchy-Binet Theorem (see [5, Proposition 2.69]), that asserts the identity $\sum_{1\le a\le b\le n} \det(X^{ab})^2 = \det(X^T X)$. To prove 1, we write

$$\begin{aligned} \|A(X)\|^2 &= \frac{\|X + \sum_{1\le a\le b\le n} \det(X^{ab}) C^T_{ab}(X)\|^2}{(1 + \|X\|^2 + \det(X^T X)} \\ &\le 2\frac{\|X\|^2 + \sum_{1\le a\le b\le n} \det(X^{ab})^2 \|C^T_{ab}(X)\|^2}{1 + \|X\|^2 + \det(X^T X)} \\ &\le 2\frac{\|X\|^2 + \sum_{1\le a\le b\le n} \det(X^{ab})^2 \|X^{ab}\|^2}{1 + \|X\|^2 + \det(X^T X)} \\ &\le 2\|X\|^2 \frac{1 + \sum_{1\le a\le b\le n} \det(X^{ab})^2}{1 + \|X\|^2 + \det(X^T X)} \end{aligned}$$

2 The notation introduced here for C_{ab} differs from the one introduced in the proof of Proposition 2.19, but we prefer to denote the matrix C_{ab} in this way in this chapter since we deal with an explicit number of dimensions of the domain and this allows for a simplified notation.

$$= 2\|X\|^2 \frac{1+\det(X^TX)}{1+\|X\|^2+\det(X^TX)} < 2\|X\|^2.$$

To prove 2, we again write

$$\begin{aligned} \|B(X)\|^2 &= \frac{\|X^TX - (1+\|X\|^2)\,\mathrm{id}_2\,\|^2}{1+\|X\|^2+\det(X^TX)} \\ &= \frac{\|X^TX\|^2 + 2(1+\|X\|^2)^2 - 2\|X\|^2 - 2\|X\|^4}{1+\|X\|^2+\det(X^TX)} \\ &= \frac{\|X^TX\|^2 + 2 + 2\|X\|^2}{1+\|X\|^2+\det(X^TX)}. \end{aligned}$$

It is easy to see that

$$\frac{1}{4}\|X\|^4 \leq \|X^TX\|^2 \leq 4\|X\|^4.$$

Therefore, we get the estimates

$$\frac{4^{-1}\|X\|^4 + 2 + 2\|X\|^2}{1+\|X\|^2+\det(X^TX)} \leq \|B(X)\|^2 \leq \frac{4\|X\|^4 + 2 + 2\|X\|^2}{1+\|X\|^2+\det(X^TX)},$$

and we deduce that

$$\frac{1}{4}\frac{\|X\|^4 + 1 + 2\|X\|^2}{1+\|X\|^2+\det(X^TX)} \leq \|B(X)\|^2 \leq 4\frac{\|X\|^4 + 1 + 2\|X\|^2}{1+\|X\|^2+\det(X^TX)}.$$

Using the fact that $\det(X^TX) \geq 0$, rewriting $\|X\|^4 + 1 + 2\|X\|^2 = (1+\|X\|^2)^2$, and taking the square root of the terms of the inequalities, we get

$$\frac{1+\|X\|^2}{2\mathcal{A}(X)} \leq \|B(X)\| \leq \sqrt{4}\frac{1+\|X\|^2}{\sqrt{1+\|X\|^2}} = 2\sqrt{1+\|X\|^2}.$$

Hence also the second estimate is proven. □

With the previous lemma, we immediately get

Corollary 4.2. *For any $p \geq 1$, if $u \in W^{1,p}(\Omega)$, and $\mathcal{U}$ satisfies (4.5), then $\mathcal{U} \in W^{1,p}(\Omega)$.*

4.2 PROPERTIES OF $B(\cdot)$

In this section, we prove some properties of the matrix field $B(X)$. In Proposition 4.4 we show how these imply the smoothness of the function w in (4.5). We recall that

$$B(X) = \frac{X^TXJ - (1+\|X\|^2)J}{\mathcal{A}(X)}. \tag{4.7}$$

Lemma 4.3. *The following properties hold:*

(i) *tr* $(B(X)) = 0, \forall X$;

(ii) $B(X)_{12} < 0, B(X)_{21} > 0, \forall X$;

(iii) $\det(B(X)) = 1$.

Proof. Let us write $B(X)$ explicitely. Denote with X^1, X^2 the column vectors of $\mathbb{R}^n$ representing the columns of the matrix X. First,

$$X^T XJ = \begin{pmatrix} \|X^1\|^2 & (X^1, X^2) \\ (X^1, X^2) & \|X^2\|^2 \end{pmatrix} J = \begin{pmatrix} -(X^1, X^2) & \|X^1\|^2 \\ -\|X^2\|^2 & (X^1, X^2) \end{pmatrix}.$$

Therefore,

$$\mathcal{A}(X)B(X) = X^T XJ - (1 + \|X\|^2)J = \begin{pmatrix} -(X^1, X^2) & \|X^1\|^2 \\ -\|X^2\|^2 & (X^1, X^2) \end{pmatrix} - \begin{pmatrix} 0 & 1 + \|X\|^2 \\ -1 - \|X\|^2 & 0 \end{pmatrix},$$

and

$$\mathcal{A}(X)B(X) = X^T XJ - (1 + \|X\|^2)J = \begin{pmatrix} -(X^1, X^2) & -1 - \|X^2\|^2 \\ 1 + \|X^1\|^2 & (X^1, X^2) \end{pmatrix}. \tag{4.8}$$

Since $\mathcal{A}(X)$ is always positive, we can divide the previous expressions by $\mathcal{A}(X)$ to infer (i) and (ii). In order to prove the third property, we compute:

$$\mathcal{A}^2(X)\det(B(X)) = 1 + \|X\|^2 + \|X^1\|^2\|X^2\|^2 - (X^1, X^2)^2 = 1 + \|X\|^2 + \det(X^T X) = \mathcal{A}^2(X).$$

Again the positivity of $\mathcal{A}(X)$ implies the conclusion of (6.20). □

We now consider properties of the differential inclusion

$$Dw(x) = B(Du(x)), \text{ for a.e. } x \in \Omega \tag{4.9}$$

for $w \in W^{1,2}(\Omega)$. By (i) of Lemma 4.3 we have $\operatorname{div}(w) = 0$. Therefore, $w = (w_1, w_2)$ can be rewritten as

$$w = (-\partial_2 z, \partial_1 z)$$

for some $z \in W^{2,2}(\Omega)$. Consequently, (4.9) is rewritten as

$$\begin{pmatrix} -\partial_{12} z & -\partial_{22} z \\ \partial_{11} z & \partial_{12} z \end{pmatrix} = B(Du).$$

Using properties (ii) and (6.20) of Lemma 4.3, we find that z enjoys the following properties

$$\begin{cases} \det(D^2 z) = 1 \text{ a.e.}, \\ \Delta z > 0 \text{ a.e.}, \\ z \in W^{2,2}(\Omega). \end{cases} \tag{4.10}$$

In the next Proposition, we will exploit some fundamental results concerning solutions to the Monge-Ampère equation. We refer the reader to [34] for the definitions and the results we will use. In particular, we refer the reader to [34, Definition 2.1] for the definition of Monge-Ampère measure.

Proposition 4.4. *Suppose z solves (4.10). Then, z is smooth.*

Proof. We just need to prove that z is an Alexandrov solution of the Monge-Ampère equation, and then apply the classical regularity results for the Monge-Ampère equation. It is not restrictive to prove the result on balls $B_r(\bar{x}) \subset \Omega$ such that $B_r(\bar{x}) \subset B_R(\bar{x}) \subset \Omega$. Consider a standard mollification kernel ρ_ε, i.e. $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^2)$, $\operatorname{spt}(\rho_\varepsilon) \subset B_\varepsilon(0)$, $\rho_\varepsilon \geq 0$, $\int_{\mathbb{R}^2} \rho_\varepsilon(x)\, dx = 1$ for every $\varepsilon > 0$. Finally, define $z_\varepsilon(x) \doteq (z \star \rho_\varepsilon)(x)$, for $\varepsilon \leq \frac{R-r}{2}$. We exploit the embedding

$$C^0(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \subset W^{2,2}(\Omega) \tag{4.11}$$

to argue that z is continuous in $B_r(\bar{x})$. We also prove that it is convex on $B_r(\bar{x})$. For every $x \in B_r(\bar{x})$ and for every $v \in \mathbb{R}^2$, we compute

$$(D^2 z_\varepsilon(x)v, v) = \int_{\mathbb{R}^2} \rho_\varepsilon(y + x)(D^2 z(y)v, v)\, dy > 0.$$

Therefore, z_ε is a sequence of convex functions converging in the $C^0(B_r(\bar{x}))$ topology to z. Thus, z must be convex too. Denote with μ_z and μ_{z_ε} the Monge-Ampère measures associated to z and z_ε respectively. We need to show that

$$\mu_z = \det(D^2 z)\mathcal{L}^2 \llcorner B_r(\bar{x}).$$

To do so, first we notice that the $W^{2,2}$ convergence of z_ε to z imply that $\det(D^2 z_\varepsilon) \to \det(D^2 z)$ in the L^1- norm. Moreover we use [34, Proposition 2.6] to infer that the Monge-Ampère measures associated to z_ε converge weakly in the sense of measures to the Monge-Ampère measure associated to z. From the regularity of z_ε, we infer $\mu_{z_\varepsilon} = \det(D^2 z_\varepsilon)\mathcal{L}^2 \llcorner B_r(\bar{x})$, hence for every $g \in C_c(B_r(\bar{x}))$ we have:

$$\int_{B_r(\bar{x})} g d\mu_\varepsilon = \int_{B_r(\bar{x})} g(x)\det(D^2 z_\varepsilon)(x)\,dx \to \int_{B_r(\bar{x})} g\det(D^2 z)\,dx$$

and

$$\int_{B_r(\bar{x})} g d\mu_\varepsilon \to \int_{B_r(\bar{x})} g d\mu.$$

We infer $\mu = \det(D^2 z)\mathcal{L}^2 \llcorner B_r(\bar{x}) = \mathcal{L}^2 \llcorner B_r(\bar{x})$. Hence, z is an Alexandrov solution to $\det(D^2 z) = 1$. It follows that z is strictly convex by [34, Theorem 2.19] and smooth by [34, Theorem 3.10]. $\square$

Let us conclude this section with another important property of $B(X)$, that follows a direct computation:

Proposition 4.5. *For all $R > 0$, there exists $\mu = \mu(R) > 0$ such that if $\|X\|, \|Y\| \leq R$, then*

$$\det(B(X) - B(Y)) \leq -\mu\|B(X) - B(Y)\|^2. \tag{4.12}$$

4.3 BOUNDS ON THE SUBDETERMINANTS AND REGULARITY

Theorem 4.6. *For every number $k \geq 0$ there exists positive numbers $C(k), \delta(k) > 0$ such that for every couple $(X, Y) \in \mathbb{R}^{n\times 2} \times \mathbb{R}^{n\times 2}$ the following holds:*

$$-\langle (A(X) - A(Y))J, X - Y\rangle + C\|B(X) - B(Y)\| \min\{\|Y\|, \|X\|\}\|X - Y\| \geq \delta\|X - Y\|^2, \tag{4.13}$$

provided that

$$\max\{\|B(X)\|, \|B(Y)\|\} \leq k.$$

Remark 4.7. Let us use the following notation: $\alpha(X) \doteq -B_{12}(X)$, $\beta(X) \doteq B_{21}(X)$, $\gamma(X) \doteq -B_{11}(X)$. These functions were explicitly written in Lemma 4.3. Notice that, as it was proved in (6.20) of Lemma 4.3:

$$\alpha(X)\beta(X) - \gamma^2(X) = \det(B(X)) = 1, \forall X \in \mathbb{R}^{n\times 2} \tag{4.14}$$

Proof. For a matrix $M \in \mathbb{R}^{n\times 2}$, we use the notation

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ \dots & \dots \\ m_{n1} & m_{n2} \end{pmatrix},$$

and we write M^1, M^2 for the first and second column of M, respectively, i.e.

$$M^1 = \begin{pmatrix} m_{11} \\ m_{21} \\ \dots \\ m_{n1} \end{pmatrix} \quad \text{and} \quad M^2 = \begin{pmatrix} m_{12} \\ m_{22} \\ \dots \\ m_{n2} \end{pmatrix}$$

First of all, we compute

$$\begin{aligned}\mathcal{A}(X)D\mathcal{A}(X)_{j1} &= x_{j1} - \sum_{i=1}^{j-1} x_{i2}(x_{i1}x_{j2} - x_{i2}x_{j1}) + \sum_{i=j}^{n} x_{i2}(x_{j1}x_{i2} - x_{j2}x_{i1}) \\ &= x_{j1} - \sum_{i=1}^{j-1} x_{i2}(x_{i1}x_{j2} - x_{i2}x_{j1}) + \sum_{i=j}^{n} x_{i2}(x_{j1}x_{i2} - x_{j2}x_{i1}) \\ &= x_{j1}(1 + \|X^2\|^2) - (X^1, X^2)x_{j2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}(X)D\mathcal{A}(X)_{j2} &= x_{j2} + \sum_{i=1}^{j-1} x_{i1}(x_{i1}x_{j2} - x_{i2}x_{j1}) - \sum_{i=j}^{n} x_{i1}(x_{j1}x_{i2} - x_{j2}x_{i1}) \\ &= x_{j2}(1 + \|X^1\|^2) - (X^1, X^2)x_{j1}.\end{aligned}$$

Using the notation of Remark 4.7

$$D\mathcal{A}(X)_{j1} = \beta(X)x_{j1} - \gamma(X)x_{j2} \text{ and } D\mathcal{A}(X)_{j2} = \alpha(X)x_{j2} - \gamma(X)x_{j1}.$$

Assume, without loss of generality, that $\|X\| \geq \|Y\|$. We can write

$$\begin{aligned}&(D\mathcal{A}(X)_{j1} - D\mathcal{A}(Y)_{j1})(x_{j1} - y_{j1}) \\ &= (\beta(X)x_{j1} - \beta(Y)y_{j1})(x_{j1} - y_{j1}) - (\gamma(X)x_{j2} - \gamma(Y)y_{j2})(x_{j1} - y_{j1}) \\ &= \beta(X)(x_{j1} - y_{j1})^2 + (\beta(X) - \beta(Y))y_{j1}(x_{j1} - y_{j1}) - (\gamma(X)x_{j2} - \gamma(Y)y_{j2})(x_{j1} - y_{j1}) \\ &= \beta(X)(x_{j1} - y_{j1})^2 + (\beta(X) - \beta(Y))y_{j1}(x_{j1} - y_{j1}) - \gamma(X)(x_{j2} - y_{j2})(x_{j1} - y_{j1}) \\ &+ (\gamma(Y) - \gamma(X))y_{j2}(x_{j1} - y_{j1})\end{aligned}$$

and

$$\begin{aligned}&(D\mathcal{A}(X)_{j2} - D\mathcal{A}(Y)_{j2})(x_{j2} - y_{j2}) \\ &= (\alpha(X)x_{j2} - \alpha(Y)y_{j2})(x_{j2} - y_{j2}) - (\gamma(X)x_{j1} - \gamma(Y)y_{j1})(x_{j2} - y_{j2}) \\ &= \alpha(X)(x_{j2} - y_{j2})^2 + (\alpha(X) - \alpha(Y))y_{j2}(x_{j2} - y_{j2}) - (\gamma(X)x_{j1} - \gamma(Y)y_{j1})(x_{j2} - y_{j2}) \\ &= \alpha(X)(x_{j2} - y_{j2})^2 + (\alpha(X) - \alpha(Y))y_{j2}(x_{j2} - y_{j2}) - \gamma(X)(x_{j1} - y_{j1})(x_{j2} - y_{j2}) \\ &+ (\gamma(Y) - \gamma(X))y_{j1}(x_{j2} - y_{j2}).\end{aligned}$$

Therefore

$$\begin{aligned}&-\langle (A(X) - A(Y))J, X - Y\rangle = \langle D\mathcal{A}(X) - D\mathcal{A}(Y), X - Y\rangle \\ &= \sum_{j=1}^{n} (D\mathcal{A}(X)_{j1} - D\mathcal{A}(Y)_{j1})(x_{j1} - y_{j1}) + \sum_{j=1}^{n} (D\mathcal{A}(X)_{j2} - D\mathcal{A}(Y)_{j2})(x_{j2} - y_{j2}) \\ &= \sum_{j} \beta(X)(x_{j1} - y_{j1})^2 - 2\gamma(X)(x_{j2} - y_{j2})(x_{j1} - y_{j1}) + \alpha(X)(x_{j2} - y_{j2})^2 \qquad (4.15) \\ &+ (\gamma(Y) - \gamma(X))y_{j2}(x_{j1} - y_{j1}) + (\alpha(X) - \alpha(Y))y_{j2}(x_{j2} - y_{j2}) \\ &+ (\gamma(Y) - \gamma(X))y_{j1}(x_{j2} - y_{j2}) + (\beta(X) - \beta(Y))y_{j1}(x_{j1} - y_{j1}).\end{aligned}$$

First, we claim that there exists a constant $\delta = \delta(k)$ independent of X such that, for every X for which $\|B(X)\| \leq k$ and for every $a, b \in \mathbb{R}$

$$-2|\gamma(X)|ab + \beta(X)a^2 + \alpha(X)b^2 \geq \delta(a^2 + b^2). \tag{4.16}$$

Fix X. Since $\alpha(X) + \beta(X) \geq 2$, either $\beta(X) \geq 1$ or $\alpha(X) \geq 1$. Without loss of generality, we can suppose $\beta(X) \geq 1$. Therefore, if $b = 0$, we can choose any $\delta < 1$. If $b \neq 0$, we divide the expression by b^2 and claim (4.16) becomes equivalent to

$$(\beta(X) - \delta)x^2 - 2|\gamma(X)|x + (\alpha(X) - \delta) \geq 0, \forall x \in \mathbb{R}.$$

Taking into account (4.14), i.e. $\gamma^2 = \alpha\beta - 1$, the discriminant of the previous equation becomes

$$\Delta(X)_\delta = 4\gamma^2 - 4(\alpha(X) - \delta)(\beta(X) - \delta) = -4 - 4\delta^2 + 4\delta(\alpha(X) + \beta(X)).$$

Since $\beta(X)$ and $\alpha(X)$ are both uniformly bounded, we can choose some small $\delta < 1$ depending only on k (so, in particular, independent of X) for which $\Delta(X)_\delta < 0$ for every X such that $\|B(X)\| \leq k$. This implies that the polynomial $x \mapsto (\beta(X) - \delta)x^2 - 2\gamma(X)x + (\alpha(X) - \delta)$ has no real root. Since $\beta(X) \geq 1 > \delta$ by assumption, then the polynomial is positive for large values of x, therefore it is positive everywhere, as we wanted. Having shown the claim, we can apply inequality (4.16) with $a = \sqrt{\sum_j (x_{j1} - y_{j1})^2}$ and $b = \sqrt{\sum_j (x_{j2} - y_{j2})^2}$ to deduce that

$$\begin{aligned}
&\sum_j (\beta(X)(x_{j1} - y_{j1})^2 - 2\gamma(X)(x_{j2} - y_{j2})(x_{j1} - y_{j1}) + \alpha(X)(x_{j2} - y_{j2})^2) \\
&\geq \beta(X)\sum_j (x_{j1} - y_{j1})^2 + \alpha(X)\sum_j (x_{j2} - y_{j2})^2 - 2|\gamma(X)|\sqrt{\sum_j (x_{j1} - y_{j1})^2}\sqrt{\sum_j (x_{j2} - y_{j2})^2)} \\
&\geq \delta \sum_j ((x_{j1} - y_{j1})^2 + (x_{j2} - y_{j2})^2) = \delta\|X - Y\|^2.
\end{aligned} \tag{4.17}$$

We also estimate:

$$\begin{aligned}
(\gamma(Y) - \gamma(X))y_{j2}(x_{j1} - y_{j1}) &\geq -|\gamma(Y) - \gamma(X)|\|Y\|\|X - Y\|, \\
(\alpha(X) - \alpha(Y))y_{j2}(x_{j2} - y_{j2}) &\geq -|\alpha(Y) - \alpha(X)|\|Y\|\|X - Y\|, \\
(\gamma(Y) - \gamma(X))y_{j1}(x_{j2} - y_{j2}) &\geq -|\gamma(Y) - \gamma(X)|\|Y\|\|X - Y\|, \\
(\beta(X) - \beta(Y))y_{j1}(x_{j1} - y_{j1}) &\geq -|\beta(Y) - \beta(X)|\|Y\|\|X - Y\|.
\end{aligned} \tag{4.18}$$

By the definition of α, β and γ, $2|\gamma(Y) - \gamma(X)| + |\alpha(Y) - \alpha(X)| + |\beta(Y) - \beta(X)| \leq C_1\|B(X) - B(Y)\|$, where $C_1 > 0$ is an universal constant. Combining (4.17) and (4.18), we finally estimate in (4.15):

$$\begin{aligned}
&\sum_j \left(\beta(X)(x_{j1} - y_{j1})^2 - 2\gamma(X)(x_{j2} - y_{j2})(x_{j1} - y_{j1}) + \alpha(X)(x_{j2} - y_{j2})^2\right) \\
&+ \sum_j (\gamma(Y) - \gamma(X))y_{j2}(x_{j1} - y_{j1}) + \sum_j (\alpha(X) - \alpha(Y))y_{j2}(x_{j2} - y_{j2}) \\
&+ \sum_j (\gamma(Y) - \gamma(X))y_{j1}(x_{j2} - y_{j2}) + \sum_j (\beta(X) - \beta(Y))y_{j1}(x_{j1} - y_{j1}) \\
&\geq \delta\|X - Y\|^2 - nC_1\|B(X) - B(Y)\|\|Y\|\|X - Y\|.
\end{aligned}$$

This estimate completes the proof of (4.13). □

4.3.1 *Regularity of the Differential Inclusion*

The regularity of $W^{1,2}$ solutions of (4.5) is surely a well-known result to the experts of the field. Since we could not find a reference of this fact in the literature and the argument is very short, we give a proof here.

Proposition 4.8. *Every $W^{1,2}$ solution $\mathcal{U}$ of (4.5) is smooth.*

Proof. From the proof of the previous theorem, we know that

$$D\mathcal{A}(X)_{j1} = \beta(X)x_{j1} - \gamma(X)x_{j2} \text{ and } D\mathcal{A}(X)_{j2} = \alpha(X)x_{j2} - \gamma(X)x_{j1}.$$

The equation

$$\operatorname{div}(D\mathcal{A}(Du)) = 0$$

reads, for every $j \in \{1,\dots,n\}$,

$$\partial_1(\beta(Du)\partial_1 u^j - \gamma(Du)\partial_2 u^j) + \partial_2(\alpha(Du)\partial_2 u^j - \gamma(Du)\partial_1 u^j) = 0, \tag{4.19}$$

where $u = (u^1,\dots,u^n)$. The previous equation has to be intended in the weak sense. In (4.4) it is showed that $\alpha(Du)$, $\beta(Du)$, $\gamma(Du)$ are smooth functions. Moreover, the matrix

$$M(Du) = (BJ)^T(Du) = \begin{pmatrix} \beta(Du) & -\gamma(Du) \\ -\gamma(Du) & \alpha(Du) \end{pmatrix}$$

is locally bounded in the sense of quadratic forms above and below by

$$c_1 \operatorname{id} \le M(Du(x)) \le c_2 \operatorname{id} \tag{4.20}$$

for two positive constants $c_1 \le c_2$. The argument to prove (4.20) is exactly the same as the one used to prove (4.16). Therefore, every u^j is the weak solution to a second order elliptic equation with smooth coefficients, (6.12). It is well known that solutions to this class of equations are smooth. □

Remark 4.9. This is not the first time that regularity results for the Monge-Ampère equation have been exploited to obtain regularity for the minimal surface equation. In [69], this connection is used to prove Bernstein's theorem (i.e., that the only solution to the minimal surface equation/system in the whole $\mathbb{R}^2$ are affine functions) for 2-dimensional minimal graphs in $\mathbb{R}^3$. We remark that, in view of the well-known Bernstein property for solutions of Monge-Ampère equation (see [69]), Proposition 4.4 and Proposition 4.8 immediately give Bernstein's property for $W^{1,\infty}$ 2-dimensional minimal graphs in $\mathbb{R}^{n+2}$.

4.4 COMPACTNESS OF THE DIFFERENTIAL INCLUSION IN $W^{1,p}$, $p > 2$

The main result of this section is Theorem 4.10, where we prove the compactness of the differential inclusion (4.5). We will make use of the following identity, that can be easily checked by direct computation

$$\langle X, YJ\rangle = -\sum_{i=1}^m \det\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \tag{4.21}$$

for every $X, Y \in \mathbb{R}^{m\times 2}$, where X_i, Y_i are the i-th rows of the matrices X and Y.

Theorem 4.10 (Compactness of the differential inclusion)**.** *Suppose $\mathcal{U}_n : \Omega \to \mathbb{R}^{2n+2}$ is an equibounded sequence in $W^{1,p}(\Omega;\mathbb{R}^{2n+2})$ for $p > 2$. If*

$$\int_\Omega \mathrm{d}(D\mathcal{U}_n(x), C_{\mathcal{A}})\eta(x) \to 0,\ \forall \eta \in C_c^\infty(\Omega), \tag{4.22}$$

then, up to a (non-relabeled) subsequence, $\mathcal{U}_n$ converges strongly in $W^{1,\bar{p}}$ to a function $\mathcal{U} : \Omega \to \mathbb{R}^{2n+2}$, for every $1 \le \bar{p} < p$. Moreover, $D\mathcal{U}(x) \in C_{\mathcal{A}}$ for a.e. $x \in \Omega$.

Proof. Throughout the proof, we will use the splitting

$$\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{pmatrix},\ \Lambda_1, \Lambda_2 \in \mathbb{R}^{n\times 2}, \Lambda_3 \in \mathbb{R}^{2\times 2} \tag{4.23}$$

for every $\Lambda \in \mathbb{R}^{(2n+2)\times 2}$. We can assume that $\mathcal{U}_n$ converges weakly in $W^{1,p}$ to $\mathcal{U}$, and that $D\mathcal{U}_n$ converges in the sense of Young measures to $\{\nu_x\}_x$. We claim that, for almost every $x \in \Omega$, we have

(i) $\operatorname{spt}(\nu_x) \subseteq C_{\mathcal{A}}$;

(ii) $\int_{\mathbb{R}^{2n+2}} \det(\Lambda^{ab}) d\nu_x(\Lambda) = \det\left(\left(\int_{\mathbb{R}^{2n+2}} \Lambda d\nu_x\right)^{ab}\right), \forall 1 \le a \le b \le 2n+2.$

To prove the previous claim, it just suffices to apply the definition of Young measure generated by $\mathcal{U}_n$. Indeed to show (i) consider the function $f \in C(\mathbb{R}^{(2n+2)\times 2})$ defined as $f(\Lambda) \doteq \mathrm{d}(\Lambda, C_{\mathcal{A}})$. The proof of (ii) is analogous to the one given in [88, Theorem 1]. Moreover, using the equality

$$\det(M_1 + M_2) = \det(M_1) + \det(M_2) + \langle M_1, \operatorname{cof}^T(M_2)\rangle, \tag{4.24}$$

valid for every matrices $M_1, M_2 \in \mathbb{R}^{2\times 2}$, and (ii) of the previous claim, it is easy to see that

$$\int_{\mathbb{R}^{(2n+2)\times 2}\times\mathbb{R}^{(2n+2)\times 2}} \det((\Lambda - \Gamma)^{ab}) d(\nu_x(\Lambda)\otimes \nu_x(\Gamma)) = 0 \text{ for a.e. } x \in \Omega,$$

where $\nu_x \otimes \nu_x$ denotes the standard product measure constructed with ν_x. Clearly this implies that for any collection of numbers $t_{ab} \in \mathbb{R}$,

$$\sum_{1\le a\le b\le 2n+2} t_{ab} \int_{\mathbb{R}^{(2n+2)\times 2}\times\mathbb{R}^{(2n+2)\times 2}} \det((\Lambda - \Gamma)^{ab}) d(\nu_x(\Lambda)\otimes \nu_x(\Gamma)) = 0. \tag{4.25}$$

First, we choose $t_{ab} = 0$ for every $1 \le a \le b \le 2n$ and $t_{ab} = 1$ if $a = 2n+1, b = 2n+2$. Using (i) of the claim and (4.12), we infer that $\nu_x \otimes \nu_x$ is supported in the set of matrices

$$C_{\mathcal{A}} \times C_{\mathcal{A}} \cap \{(\Lambda', \Lambda'') \in \mathbb{R}^{(2n+2)\times 2} \times \mathbb{R}^{(2n+2)\times 2} : \Lambda'_3 = \Lambda''_3\}.$$

Thus, we obtain the existence of a 2×2 matrix B_x such that $B(\Lambda_1) = B_x$ for a.e. $x \in \Omega$ and for ν_x-a.e. $\Lambda \in \mathbb{R}^{2n+2}$. Let us remark that the matrix B_x possibly depends on $x \in \Omega$ but not on $\Lambda \in \mathbb{R}^{(2n+2)\times 2}$. To finish the proof, apply (4.21) to find coefficients t_{ab} such that

$$\sum_{1\le a\le b\le 2n+2} t_{ab} \det((\Lambda - \Gamma)^{ab}) = \langle (A(\Lambda_1) - A(\Gamma_1))J, \Lambda_1 - \Gamma_1\rangle, \forall \Lambda, \Gamma \in \mathbb{R}^{(2n+2)\times 2}.$$

Now we can use (4.13) to infer that for a.e. $x \in \Omega$, there exists a number $\delta(x) > 0$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{(2n+2)\times 2}\times\mathbb{R}^{(2n+2)\times 2}} \langle (A(\Lambda_1) - A(\Gamma_1))J, \Lambda - \Gamma\rangle d(\nu_x(\Lambda)\otimes\nu_x(\Gamma)) \\ &\ge \int_{\mathbb{R}^{(2n+2)\times 2}\times\mathbb{R}^{(2n+2)\times 2}} \delta(x)\|\Lambda_1 - \Gamma_1\|^2 d(\nu_x(\Lambda)\otimes\nu_x(\Gamma)). \end{aligned}$$

This yields $\nu_x = \delta_{D\mathcal{U}(x)}$ for a.e. $x \in \Omega$. Corollary A.3 implies that $D\mathcal{U}_n$ converges in measure to $D\mathcal{U}$ and therefore strongly for every $1 \le \bar{p} < p$.

$\square$

4.5 PERTURBATIVE RESULT

We will prove that solutions with fixed Lipschitz constant of the differential inclusion (4.4) for functionals sufficiently near to the area functional are actually as smooth as the functional under consideration. The strategy is the following. In Lemma 4.11, we prove inequality (4.26), through which we bound the norm of the difference of two matrices with a linear combination of subdeterminants of $C_{\mathcal{A}}$. Next, in Lemma 4.13, we show that, if we fix $R > 0$, there exists a number $\varepsilon(R) > 0$ such that, if $f : \mathbb{R}^{n\times 2} \to \mathbb{R}$ is a C^2 functional with $\|f - \mathcal{A}\|_{C^2(B_{2R})} \le \varepsilon(R)$, then for f the same kind of inequality holds (see (4.27)). In Theorem 4.14 and Proposition 4.15, we show how inequality (4.27) implies Hölder continuity of gradients of functions $\mathcal{U}$ satisfying

$$D\mathcal{U}(x) \in C_f, \text{ for a.e. } x \in \Omega.$$

Finally, in Subsection 4.5.1, we will improve the Hölder continuity of the gradient of the solution to higher regularity.

Lemma 4.11. *For every $R > 0$, there exist constants $\lambda(R), \delta(R) > 0$ such that, $\forall X, Y \in B_{\frac{3R}{2}}(0)$, we have*

$$-\langle (A(X) - A(Y))J, X - Y\rangle - \lambda \det(B(X) - B(Y)) \geq \delta\|X - Y\|^2, \tag{4.26}$$

Proof. We note that for $(X, Y) \in B_{\frac{3R}{2}}(0) \times B_{\frac{3R}{2}}(0)$ the assumptions of Theorem 4.6 are fulfilled. Therefore, we find constants $C = C(R)$ and $c = c(R)$ such that

$$-\langle (A(X) - A(Y))J, X - Y\rangle + C\|B(X) - B(Y)\| \min\{\|Y\|, \|X\|\}\|X - Y\| \geq c\|X - Y\|^2.$$

Using the hypothesis, we estimate $\min\{\|Y\|, \|X\|\} \leq \max\{\|Y\|, \|X\|\} \leq \frac{3R}{2}$. Moreover Young inequality yields

$$-\langle (A(X) - A(Y))J, X - Y\rangle + \frac{3CR\tau}{4}\|X - Y\|^2 + \frac{3CR}{4\tau}\|B(X) - B(Y)\|^2 \geq c\|X - Y\|^2.$$

Clearly, we can choose $\tau = \tau(R)$ such that $c - \frac{3CR\tau}{4} \geq \frac{c}{2}$. Therefore, define $\delta \doteq \frac{c}{2}$. Finally by (4.12) we find a constant $\mu = \mu(R) \geq 0$ such that

$$\|B(X) - B(Y)\|^2 \leq -\frac{1}{\mu}\det(B(X) - B(Y)), \forall X, Y \in B_{\frac{3R}{2}}(0).$$

This finally concludes the proof of the present Lemma, with $\lambda(R) \doteq \frac{3CR}{4\tau\mu}$. $\square$

Remark 4.12. Notice that inequality (4.26) can be interpreted as some sort of *generalized convexity* of the area functional. Indeed, for a function $f \in C^2(\mathbb{R}^{n\times 2})$, the inequality

$$\langle Df(X) - Df(Y), X - Y\rangle = -\langle (A_f(X) - A_f(Y))J, X - Y\rangle \geq \delta\|X - Y\|^2$$

is equivalent to convexity. It can be checked that when $n > 1$, the area functional is not convex, hence the previous inequality cannot hold. The previous Lemma shows that adding the term $-\lambda \det(B(X) - B(Y))$ we can nonetheless bound from above the quantity $\|X - Y\|^2$. The key point here is that the determinant is a null Lagrangian and therefore it still allows to prove a regularity result as Proposition 4.15.

Lemma 4.13. *Fix $R > 0$. Recall that $A_f(X) = Df(X)J$ and $B_f(X) = X^T Df(X)J - f(X)J$. There exists $\varepsilon = \varepsilon(R)$ and $c = c(f, R) > 0$ such that if*

$$\|f - \mathcal{A}\|_{C^2(B_{2R})} \leq \varepsilon,$$

then, for the same constant λ of formula (4.26),

$$-\langle (A_f(X) - A_f(Y))J, X - Y\rangle - \lambda \det(B_f(X) - B_f(Y)) \geq c\|X - Y\|^2, \textit{for every } X, Y \in B_{\frac{3R}{2}}(0). \tag{4.27}$$

Proof. The proof is by contradiction. Assume we can find a sequence of functions f_n, a sequence of numbers c_n and sequences of matrices X_n and Y_n such that

(i) $\|f_n - \mathcal{A}\|_{C^2(B_{2R})} \leq \frac{1}{n}$;

(ii) $c_n \to 0$;

(iii) $X_n \to X$, $Y_n \to Y$, $\frac{X_n - Y_n}{\|X_n - Y_n\|} \to Z$;

(iv) $-\langle (A_{f_n}(X_n) - A_{f_n}(Y_n))J, X_n - Y_n\rangle - \lambda \det(B_{f_n}(X_n) - B_{f_n}(Y_n)) \le c_n \|X_n - Y_n\|^2$.

First, suppose $X \neq Y$. Then, in the limit we find a contradiction with (4.26)

$$\delta \|X - Y\|^2 \le -\langle (A(X) - A(Y))J, X - Y\rangle - \lambda \det(B(X) - B(Y)) \le 0.$$

Now suppose $X = Y$. Define

$$T_n \doteq \frac{A_{f_n}(X_n) - A_{f_n}(Y_n)}{\|X_n - Y_n\|} \text{ and } B_n \doteq \frac{B_{f_n}(X_n) - B_{f_n}(Y_n)}{\|X_n - Y_n\|}.$$

Then, for every n, (iv) yields:

$$-\left\langle T_n J, \frac{X_n - Y_n}{\|X_n - Y_n\|}\right\rangle - \lambda \det(B_n) \le 0. \tag{4.28}$$

We have

$$T_n = \frac{A_{f_n}(X_n) - A_{f_n}(Y_n)}{\|X_n - Y_n\|} = \frac{\int_0^1 DA_{f_n}(tX_n + (1-t)Y_n)[X_n - Y_n]dt}{\|X_n - Y_n\|} \to DA(X)[Z]$$

and, analogously,

$$B_n \to DB(X)[Z].$$

The convergence of T_n and B_n are a direct consequence of (i). Consequently, in the limit (4.28) becomes

$$-\langle DA(X)[Z]J, Z\rangle - \lambda \det(DB(X)[Z]) \le 0. \tag{4.29}$$

Now, by (4.26) and for every n,

$$-\langle (A(X_n) - A(Y_n))J, X_n - Y_n\rangle - \lambda \det(B(X_n) - B(Y_n)) \ge \delta \|X_n - Y_n\|^2,$$

so that, if we divide by $\|X_n - Y_n\|^2$ and pass to the limit, we obtain a contradiction with (4.29). $\square$

Theorem 4.14. *Let $k \ge 2$. For every $R > 0$, there exists $\varepsilon = \varepsilon(R) > 0$ for which, if $f : \mathbb{R}^{n\times 2} \to \mathbb{R}$ is a function of class C^k with*

$$\|f - \mathcal{A}\|_{C^2(B_{2R})} \le \varepsilon,$$

then, for every $\mathcal{U} \in W^{1,\infty}(\Omega; \mathbb{R}^{2n+2})$, $\|D\mathcal{U}\|_{L^\infty} \le R$, such that

$$D\mathcal{U}(x) \in C_f \text{ for a.e. } x \in \Omega,$$

it holds $\mathcal{U} \in W^{2,2+\rho}(\Omega)$, for some positive ρ.

The proof of the previous Theorem is a consequence of the following result, that in turn is a simple generalization of [88, Theorem 3].

Proposition 4.15. *Consider differential inclusions of the following form, for $\mathcal{V} \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{R}^{r+m})$,*

$$D\mathcal{V}(x) \in C = \left\{ Y \in \mathbb{R}^{r+m,2} : Y = \begin{pmatrix} X \\ F(X) \end{pmatrix} \right\}, \text{ for a.e. } x \in \Omega, \tag{4.30}$$

where $F \in C^k(\mathbb{R}^{r\times 2}; \mathbb{R}^{m\times 2})$, $k \ge 1$. Consider moreover the splitting $\mathcal{V} = \begin{pmatrix} u \\ v \end{pmatrix}$, with $u : \Omega \to \mathbb{R}^r$ and $v : \Omega \to \mathbb{R}^m$. Suppose there exist constants $c_{ab} \in \mathbb{R}$ such that

$$\|X - Y\|^2 \le \sum_{1 \le a \le b \le m+r} c_{ab} \det(M^{ab} - N^{ab}), \tag{4.31}$$

for every couple of $M, N \in K$ of the form

$$M = \begin{pmatrix} X \\ F(X) \end{pmatrix}, \quad N = \begin{pmatrix} Y \\ F(Y) \end{pmatrix}.$$

Then, $u \in W^{2,2+\rho}_{\text{loc}}(\Omega; \mathbb{R}^n)$, for some $\rho > 0$.

Proof. From now on, we fix open sets $\Omega' \subset \Omega'' \subset \Omega$, each with compact closure in the other. For any couple a, b with $1 \le a \le b \le m + r$, denote $w_{ab} \doteq \begin{pmatrix} \mathcal{V}_a \\ \mathcal{V}_b \end{pmatrix}$. Take any nonnegative $\eta \in C_c^\infty(\Omega'')$, $q_{ab} \in \mathbb{R}^2$ constant vectors, and $h \in \mathbb{R}^2$ with $\|h\| \le \frac{d(\partial\Omega'', \partial\Omega)}{2}$, and moreover denote, for any function $g : \Omega' \to \mathbb{R}^m$,

$$g^h(x) = \frac{g(x+h) - g(x)}{\|h\|}.$$

Since the determinant is a null Lagrangian and η has compact support

$$\sum_{ab} c_{ab} \int_\Omega \det(D(\eta(x)(w^h_{ab}(x) - q_{ab}))\,dx = 0.$$

Equation (4.24) yields

$$\begin{aligned}0 &= \sum_{ab} c_{ab} \int_\Omega \det(D(\eta(x) w^h_{ab}(x) - q_{ab}))\,dx = \\ &= \sum_{a,b} c_{ab} \int_\Omega \eta^2(x) \det(D w^h_{ab}(x))\,dx + \sum_{a,b} c_{ab} \int_\Omega \eta(x) \langle \operatorname{cof}^T((w^h_{ab}(x) - q_{ab}) \otimes D\eta(x)), D w^h_{ab}(x)\rangle.\end{aligned}$$

Hence, by (4.31) and our previous computation, we can write

$$\begin{aligned}\int_\Omega \eta^2(x)\|Du^h(x)\|^2\,dx &= \frac{1}{\|h\|^2}\int_\Omega \eta^2(x)\|D(u(x+h) - u(x))\|^2\,dx \\ &\le \frac{1}{\|h\|^2}\sum_{ab} c_{ab}\int_\Omega \eta^2(x)\det(D(w_{ab}(x+h) - w_{ab}(x)))\,dx \\ &= -\sum_{a,b} c_{ab}\int_\Omega \eta(x)\langle \operatorname{cof}^T((w^h_{ab}(x) - q_{ab}) \otimes D\eta(x)), Dw^h_{ab}(x)\rangle\,dx \\ &\le \sum_{a,b}|c_{ab}|\int_\Omega \eta(x)\|w^h_{ab}(x) - q_{ab}\|\|D\eta(x)\|\|Dw^h_{ab}(x)\|\,dx\,.\end{aligned}$$

Since F is C^1, it is locally Lipschitz. In particular, if $\|u\|_{W^{1,\infty}} \le R$, this implies that, for some constant $c \ge 0$ depending on R,

$$\|Dw^h_{ab}(x)\| \le c\|Du^h(x)\|, \text{ a.e. }.$$

From now on, we will not keep track of the constants, and we will simply denote them by C. Continuing our computation, we readily obtain through Hölder's inequality that

$$\int_\Omega \eta^2(x)\|Du^h(x)\|^2\,dx \le C\sum_{a,b}\int_\Omega \|w^h_{ab}(x) - q_{ab}\|^2\|D\eta(x)\|^2\,dx\,. \tag{4.32}$$

Choose $q_{ab} = 0$ for every a, b and $\eta \equiv 1$ on Ω'. Using the fact that $\mathcal{V}$ is Lipschitz, we get

$$\int_{\Omega'} \|Du^h(x)\|^2\,dx \le C(R, \Omega'), \text{ for every sufficiently small } h.$$

By standard results about Sobolev spaces (see [9, Proposition 9.3]), this implies that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. To conclude the proof, we show higher integrability of the Hessian of u, namely $D^2u \in L^{2+\rho}$, for some $\rho > 0$. To do so, consider again (4.32). This time, consider any square $Q \subset \Omega'$ such that $2Q \subset \Omega'$, where $2Q$ is the square of side s centered at the center of Q but with twice the side. We take $\eta \in C_c^\infty(\sqrt{2}Q)$ with $\eta \equiv 1$ on Q, and

$$\eta \equiv 1 \text{ on } Q \text{ and } \|D\eta\|(x) \le \frac{C}{s} \text{ on } \sqrt{2}Q,$$

for some $C > 0$ independent on x and s. Then, (4.32) becomes

$$\int_Q \|Du^h(x)\|^2\,dx \le C\sum_{a,b}\int_{\sqrt{2}Q}\|w^h_{ab}(x)-q_{ab}\|^2\|D\eta(x)\|^2\,dx \le \frac{C}{s^2}\sum_{a,b}\int_{\sqrt{2}Q}\|w^h_{ab}(x)-q_{ab}\|^2\,dx\,. \tag{4.33}$$

Now, using [50, Theorem 3.6], we can estimate the last term with a Sobolev-type inequality, using $p = 2$ and $p^* = 1$, once we have chosen suitably q_{ab}:

$$\sum_{a,b}\int_{\sqrt{2}Q}\|w^h_{ab}(x)-q_{ab}\|^2\,dx \le C\sum_{a,b}\left(\int_{2Q}\|Dw^h_{ab}\|\,dx\right)^2.$$

Once again, $\|Dw^h_{ab}\| \le C\|Du^h\|$ pointwise a.e., where C depends only on the Lipschitz constant of F (that in turn depends only on the Lipschitz constant of α). In this way, (4.33) can be rewritten as

$$\int_Q \|Du^h\|^2\,dx \le \frac{C}{s^2}\left(\int_{2Q}\|Du^h\|\,dx\right)^2.$$

Passing to the limit as $h \to 0$, we finally get

$$\left(\fint_Q \|D^2u\|^2\,dx\right)^{\frac{1}{2}} \le C\fint_{2Q}\|D^2u\|\,dx\,.$$

We can apply Gehring's Lemma as stated, for instance, in [49, Theorem 1.5], to deduce the higher integrability of the Hessian of our function. □

4.5.1 *Higher regularity*

By Theorem 4.14, we know that for every R, there exists $\varepsilon(R) > 0$ such that

$$D\mathcal{U} \in C_f \Rightarrow D\mathcal{U} \in W^{2,2+\rho}_{\text{loc}}(\Omega)$$

provided that $\|f - \mathcal{A}\|_{C^2(B_{2R})} \le \varepsilon$. In this subsection, we show that, possibly taking a smaller ε, if $f \in C^k$, for $k \ge 2$, then $\mathcal{U} \in C^{k-1}$. The procedure here is quite stardard (see, for instance, [88, Corollary]) and we describe it for the reader's convenience. To show the improvement of regularity, we exploit the results of [57, 58]. Suppose that

$$f \in C^k(\mathbb{R}^{n\times 2},\mathbb{R}), \quad k \ge 2$$

satisfies the following Legendre-Hadamard condition (briefly, LH), i.e. there exists a constant $\mu > 0$ such that

$$D^2f(X)[Y,Y] \ge \mu\|Y\|^2,\ \forall X,Y \in \mathbb{R}^{n\times 2}, \text{rank}(Y) = 1, \tag{4.34}$$

where

$$D^2f(X)[Y,Y] \doteq \frac{d^2}{dt^2}|_{t=0}f(X+tY).$$

Then, applying [58, Theorem 6.2.5], we infer that the $W^{2,2+\rho}$ solutions of

$$\text{div}(Df(Du)) = 0$$

belong to $C^{k-1,\alpha}_{\text{loc}}$, for some α depending on ρ. In order to apply [58, Theorem 6.2.5], we need to prove that functionals close to the area satisfies the LH condition. In Lemma 4.16 we prove that the area satisfy a local LH condition, and in Lemma 4.17 we extend this to functions close to the area. To apply Morrey's [58, Theorem 6.2.5], we need to prove a global LH condition for these functionals. Nevertheless, since we are just interested in Lipschitz solution of constant $R > 0$, it will be sufficient to prove that there exists an extension of the function f under consideration to the whole $\mathbb{R}^{n\times 2}$ that satisfies the LH condition. This extension is the content of Lemma 4.19.

Lemma 4.16. *For every $R > 0$, there exists a constant $\tau(R) > 0$ such that*

$$D^2\mathcal{A}(X)[Y,Y] \geq \tau\|Y\|^2,\ \forall X, Y \in \mathbb{R}^{n\times 2}, X \in B_{\frac{3R}{2}}(0), \operatorname{rank}(Y) = 1.$$

Proof. Fix $X \in \mathbb{R}^{n\times 2}$, $\|X\| \leq R$, and $Y \in \mathbb{R}^{n\times 2}$ with $\|Y\| = 1$ and $\operatorname{rank}(Y) = 1$. Define the function

$$g(t) \doteq \mathcal{A}(X + tY).$$

The thesis is equivalent to

$$g''(0) \geq \tau(R).$$

Since $\operatorname{rank}(Y) = 1$

$$g(t) = \sqrt{1 + \|X + tY\|^2 + \sum_{a,b}(\det(X^{ab}) + t\langle X^{ab}, \operatorname{cof}^T(Y^{ab})\rangle)^2}.$$

Therefore,

$$g'(t) = \frac{s(t)}{g(t)},$$

where

$$s(t) = \langle X + tY, Y\rangle + \sum_{a,b}(\det(X^{ab}) + t\langle X^{ab}, \operatorname{cof}^T(Y^{ab})\rangle)\langle X^{ab}, \operatorname{cof}^T(Y^{ab})\rangle.$$

This implies

$$g''(t) = \frac{s'(t)}{g(t)} - \frac{s(t)g'(t)}{g^2(t)} = \frac{s'(t)}{g(t)} - \frac{s^2(t)}{g^3(t)} = \frac{s'(t)g^2(t) - s^2(t)}{g^3(t)}.$$

Finally:

$$g''(0) = \frac{s'(0)g^2(0) - s^2(0)}{g^3(0)}.$$

We will now show that $s'(0)g^2(0) - s^2(0) \geq 1$, and this concludes the proof. To simplify the notation, define

$$A \doteq \sum_{a,b}\langle X^{ab}, \operatorname{cof}^T(Y^{ab})\rangle^2,$$
$$B \doteq \sum_{a,b}\det(X^{ab})\langle X^{ab}, \operatorname{cof}^T(Y^{ab})\rangle.$$

Recall that we are assuming $\|Y\| = 1$, and that $\sum_{a,b}(\det(X^{ab}))^2 = \det(X^TX)$. Therefore:

$$s'(0)g^2(0) - s^2(0) = 1 + \|X\|^2 + \det(X^TX) + A + A\|X\|^2 + A\det(X^TX) - (\langle X, Y\rangle + B)^2, \tag{4.35}$$

and

$$\begin{aligned}&\|X\|^2 + \det(X^TX) + A\|X\|^2 + A\det(X^TX) - (\langle X, Y\rangle + B)^2 \\ &= (\|X\|^2 - \langle X, Y\rangle^2) + (A\det(X^TX) - B^2) + (\det(X^TX) + A\|X\|^2 - 2\langle X, Y\rangle B).\end{aligned}$$

We claim that the terms in brackets of the previous expression are all nonnegative. This would conclude the proof, since then, considering (4.35)

$$s'(0)g^2(0) - s^2(0) \geq 1 + A$$

and $A \geq 0$. Let us prove the claim. First, we need to show that

$$\|X\|^2 - \langle X, Y\rangle^2 \geq 0.$$

Cauchy-Schwartz inequality and the fact that $\|Y\| = 1$ imply

$$\|X\|^2 - \langle X, Y\rangle^2 \geq \|X\|^2 - \|X\|^2\|Y\|^2 = \|X\|^2 - \|X\|^2 = 0.$$

The second inequality we need is

$$B^2 \le A \det(X^T X).$$

By the definition of B and applying again Cauchy-Schwartz inequality:

$$B^2 = \left(\sum_{a,b} \det(X^{ab}) \langle X^{ab}, \operatorname{cof}^T(Y^{ab}) \rangle \right)^2 \le \sum_{a,b} \det(X^{ab})^2 \sum_{a,b} \langle X^{ab}, \operatorname{cof}^T(Y^{ab}) \rangle^2 = A \det(X^T X).$$

Finally, we prove that

$$2\langle X, Y \rangle B \le A \|X\|^2 + \det(X^T X).$$

By Cauchy-Schwartz and Young inequality:

$$\begin{aligned} 2\langle X, Y \rangle B &= 2\langle X, Y \rangle \sum_{a,b} (\det(X^{ab}) \langle X^{ab}, \operatorname{cof}^T(Y^{ab}) \rangle) \\ &\le 2|\langle X, Y \rangle| \sqrt{\sum_{a,b} \det(X^{ab})^2} \sqrt{\sum_{a,b} \langle X^{ab}, \operatorname{cof}^T(Y^{ab}) \rangle^2} \\ &= 2|\langle X, Y \rangle| \det(X^T X)^{\frac{1}{2}} A^{\frac{1}{2}} \\ &\le A |\langle X, Y \rangle|^2 + \det(X^T X) \le A\|X\|^2 + \det(X^T X). \end{aligned}$$

□

Lemma 4.17. *For every $R > 0$, there exists $\varepsilon'(R) > 0$ such that, if $f \in C^2(\mathbb{R}^{n\times 2})$ and*

$$\|f - \mathcal{A}\|_{C^2(B_{2R})} \le \varepsilon'(R),$$

then there exists a constant $\tau' = \tau'(R)$ such that

$$D^2 f(X)[Y, Y] \ge \tau' \|Y\|^2, \forall X, Y \in \mathbb{R}^{n\times 2}, \|X\| \le \frac{3R}{2}, \operatorname{rank}(Y) = 1.$$

Proof. Suppose by contradiction that the thesis is false. Then, we can find a sequence of functions f_n, a sequence of positive numbers c_n and sequences of matrices X_n, Y_n such that:

(i) $\|f_n - \mathcal{A}\|_{C^2(B_{2R})} \le \frac{1}{n}$;

(ii) $c_n \to 0$;

(iii) $X_n \to X$;

(iv) $\|Y_n\| = 1$, $\operatorname{rank}(Y_n) = 1$, and $Y_n \to Y \in \mathbb{R}^{n\times 2}, \|Y\| = 1, \operatorname{rank}(Y) = 1$

(v) $D^2 f_n(X_n)[Y_n, Y_n] \le c_n$.

Passing to the limit in (v), we immediately get a contradiction with Lemma 4.16. □

In order to prove the next lemma we need to introduce a new:

Definition 4.18. Let $\mu \ge 0$. The function $h : \mathbb{R}^{n\times 2} \to \mathbb{R}$ is μ-rank-one convex if and only if for every $X, Y \in \mathbb{R}^{n\times 2}$, $\operatorname{rank}(Y) = 1$,

$$\phi(t) \doteq h(X + tY)$$

is a uniformly convex function with constant μ, i.e.

$$\phi(at_1 + bt_2) \le t_1\phi(a) + t_2\phi(b) - t_1 t_2 \mu |a - b|^2, \ \forall a, b, t_1, t_2 \in \mathbb{R}, t_1 + t_2 = 1, t_1, t_2 \ge 0.$$

If $\mu = 0$, the function h is simply called rank-one convex.

It is not difficult to see that if $h \in C^2(\mathbb{R}^{n\times 2})$, then h is μ-rank-one convex if and only if it satisfies the LH condition with constant μ (i.e., (4.34) holds). Therefore we will say that a C^2 function h is μ- rank-one convex in $B_r(0)$ for some $r > 0$ if and only if (4.34) holds for every $X \in B_r(0) \subset \mathbb{R}^{n\times 2}$ and for every $Y \in \mathbb{R}^{n\times 2}$ with $\operatorname{rank}(Y) = 1$.

Lemma 4.19. *Let $f \in C^k(B_{2R})$, $k \geq 2$, be a μ-rank-one convex function on B_{2R}. Then, there exists a function F such that*

- *$F = f$ on $B_{\frac{3R}{2}}$;*
- *$F \in C^k(\mathbb{R}^{n\times 2})$;*
- *F is $\frac{\mu}{2}$ rank-one convex.*

Proof. Choose any $R_1 \in \left(\frac{3R}{2}, 2R\right)$. Moreover, define $f'(X) \doteq f(X) - \frac{3\mu\|X\|^2}{4}$. Notice that, by our hypothesis, f' is still rank-one convex on $B_{2R}(0)$. Apply [62, Lemma 2.3] to find a rank one convex function $F' : \mathbb{R}^{n\times 2} \to \mathbb{R}$ such that F' coincides with f' on B_{R_1}. The function

$$F''(X) \doteq F'(X) + \frac{3\mu\|X\|^2}{4}$$

is $\frac{3\mu}{4}$- rank-one convex on the whole $\mathbb{R}^{n\times 2}$ and on B_{R_1} it coincides with $f(X)$. We take any family of mollifiers ρ_ε on $\mathbb{R}^{n\times 2}$ with $\operatorname{spt}(\rho_\varepsilon) \subset B_\varepsilon(0)$ and $\rho_\varepsilon(X) \geq 0$ for every $X \in \mathbb{R}^{n\times 2}$, and define

$$F_\varepsilon(X) \doteq (F'' \star \rho_\varepsilon)(X), \forall X \in \mathbb{R}^{n\times 2}.$$

The convolution is well defined since rank-one convexity implies that F'' is locally Lipschitz. Through a direct computation, it is easy to see that F_ε is still $\frac{3\mu}{4}$-rank one convex. Consider any $R_2 \in \left(\frac{3R}{2}, R_1\right)$ and take a function $\eta \in C^\infty_c(\mathbb{R}^{n\times 2})$ such that $0 \leq \eta(X) \leq 1, \forall X$, $\eta \equiv 1$ on $B_{R_2+\delta}$ and $\eta \equiv 0$ on $B^c_{R_1-\delta}$, with $0 < \delta \doteq \frac{R_1 - R_2}{10}$. Next, define

$$G_\varepsilon(X) \doteq \eta(X)F''(X) + (1 - \eta(X))F_\varepsilon(X).$$

We claim that there exists $\varepsilon > 0$ such that $G_\varepsilon(X)$ has the desired properties. Indeed, for every $\varepsilon > 0$, G_ε is a $C^k(\mathbb{R}^{n\times 2})$ function that coincides with F'' and therefore f on $B_{\frac{3R}{2}}$. Moreover, by the properties of the support of η and the $\frac{3\mu}{4}$-rank one convexity of F'' and F_ε, it holds

$$D^2G_\varepsilon(X)[Y, Y] \geq \frac{3\mu}{4}\|Y\|^2$$

for every $\varepsilon > 0$, $Y \in \mathbb{R}^{n\times 2}$ with $\operatorname{rank}(Y) = 1$ and $X \in \mathcal{B} \doteq \bar{B}_{R_2+\frac{\delta}{2}} \cup B^c_{R_1-\frac{\delta}{2}}$. Therefore, to conclude the proof, we need to show that for $\varepsilon > 0$ sufficiently small,

$$D^2G_\varepsilon(X)[Y, Y] \geq \frac{\mu}{2}\|Y\|^2, \text{ for } X \in \mathcal{B}^c.$$

Take $\varepsilon < \frac{R_1 - R_2}{100}$. In this case, we see that for every $X \in \mathcal{B}^c$

$$DF_\varepsilon(X) = (DF'' \star \rho_\varepsilon)(X) \text{ and } D^2F_\varepsilon(X) = (D^2F'' \star \rho_\varepsilon)(X), \tag{4.36}$$

since F'' coincides with the C^k ($k \geq 2$) function f on B_{R_1}. We obtain

$$\begin{aligned} D^2G_\varepsilon &= F''D^2\eta + (D\eta \otimes DF'' + DF'' \otimes D\eta) + \eta D^2F'' \\ &\quad - F_\varepsilon D^2\eta - (D\eta \otimes DF_\varepsilon + DF_\varepsilon \otimes D\eta) + (1 - \eta)D^2F_\varepsilon. \end{aligned}$$

Define

$$V_\varepsilon \doteq F''D^2\eta + (D\eta \otimes DF'' + DF'' \otimes D\eta)$$

$$- F_\varepsilon D^2\eta - (D\eta \otimes DF_\varepsilon + DF_\varepsilon \otimes D\eta).$$

For every tensor $W = (W_{abcd}), a, c \in \{1, \dots n\}, b, d \in \{1, 2\}$, denote with

$$W[Y,Y] \doteq \sum_{a,b,c,d} W_{abcd} y_{ab} y_{cd}, \forall Y = (y_{ij}) \in \mathbb{R}^{n\times 2}.$$

Exploiting (4.36) and the regularity of F'', we see that there exists a constant $C > 0$ independent of X such that

$$|V_\varepsilon(X)[Y,Y]| \leq C\varepsilon \|Y\|^2,$$

for every $X \in \mathcal{B}^c$ and every $Y \in \mathbb{R}^{n\times 2}$ (non necessarily with $\operatorname{rank}(Y) = 1$). We can choose any number $0 < \varepsilon \leq \frac{\mu}{4C}$. Let it be ε_0, and call $F(X) \doteq G_{\varepsilon_0}(X)$. F has the three properties listed in the statement of the Lemma. □

We can summarize the result of this section in the following

Theorem 4.20. *For every $R > 0$, there exists $\alpha = \alpha(R) > 0$ such that, if f is a $C^k(\mathbb{R}^{2n+2\times 2})$ function, $k \geq 2$, with the property that*

$$\|f - \mathcal{A}\|_{C^2(B_{2R}(0))} \leq \alpha, \tag{4.37}$$

and $\mathcal{U} : \Omega \to \mathbb{R}^{2n+2}$ is a Lipschitz solution of

$$D\mathcal{U}(x) \in C_f, \text{ for a.e. } x \in \Omega \tag{4.38}$$

with

$$\|D\mathcal{U}\|_\infty \leq R,$$

then $\mathcal{U} \in C^{k-1,\rho}(\Omega)$, for some positive $\rho > 0$.

Proof. Fix $R > 0$. Choose $\alpha(R) \doteq \min\{\varepsilon(R), \varepsilon'(R)\}$, where ε and ε' are defined in Lemma 4.13 and Lemma 4.17 respectively. Take any f satisfying (4.37) and a R-Lipschitz $\mathcal{U}$ satisfying (4.38). By our choice of α, $\mathcal{U}$ belongs to $W^{2,2+\rho}_{\text{loc}}(\Omega)$ by Theorem 4.14. Again, by the choice of α, by Lemma 4.17 we have that f satisfies the LH condition in B_{2R}. Using Lemma 4.19, we can consider $F \in C^k(\mathbb{R}^{n\times 2})$ that extends f outside $B_{\frac{3R}{2}}$ and that satisfies the LH condition on the whole $\mathbb{R}^{n\times 2}$. Since $\|D\mathcal{U}\|_\infty \leq R$,

$$\operatorname{div}(DF(Du)) = \operatorname{div}(Df(Du)) = 0, \text{a.e. in } \Omega.$$

$\mathcal{U}$ has the desired regularity by [58, Theorem 6.2.5], as described at the beginning of this subsection. □

4.6 IRREGULAR CRITICAL POINTS FOR INNER VARIATIONS

The purpose of this section is to show the following

Theorem 4.21. *Let Ω be an open and bounded subset of $\mathbb{R}^2$. There exists a map $\psi \in W^{1,p}(\Omega, \mathbb{R}^2)$ for some $p > 2$ that solves*

$$\operatorname{curl}(B(D\psi)) = 0,$$

and such that for every open $\mathcal{V} \subset \Omega$, ψ is not in $C^1(\mathcal{V})$.

The proof of this Theorem is achieved by combining a simple Linear Algebra lemma, Lemma 4.22, with the counterexample constructed in [51, Example 4.41]. First, let us define

$$H_1 \doteq \left\{X \in \mathbb{R}^{2\times 2} : X = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right\}$$

and

$$H_2 \doteq \left\{X \in \mathbb{R}^{2\times 2} : X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right\}.$$

Lemma 4.22. *For every $X \in H_1 \cup H_2$, we have*

$$A(X) = XJ$$

and

$$B(X) = J$$

Proof. Let us consider the matrix

$$X = \begin{pmatrix} a & \alpha b \\ b & \beta a \end{pmatrix},$$

with $\alpha = \pm 1$, $\alpha\beta = -1$, $a, b \in \mathbb{R}$. Clearly, every matrix in $H_1 \cup H_2$ is of this form. We have

$$\|X\|^2 = 2(a^2 + b^2), \ \det(X)^2 = (a^2 + b^2)^2,$$

hence

$$\mathcal{A}(X) = 1 + a^2 + b^2.$$

Moreover,

$$\operatorname{cof}(X)^T = \begin{pmatrix} \beta a & -b \\ -\alpha b & a \end{pmatrix},$$

thus

$$X + \det(X)\operatorname{cof}(X)^T = \begin{pmatrix} a & \alpha b \\ b & \beta a \end{pmatrix} + (\beta a^2 - \alpha b^2)\begin{pmatrix} \beta a & -b \\ -\alpha b & a \end{pmatrix} = \mathcal{A}(X)\begin{pmatrix} a & \alpha b \\ b & \beta a \end{pmatrix} = \mathcal{A}(X)X.$$

Therefore, $A(X) = XJ$. We now prove that $B(X) = -J$. To do so, we compute

$$X^T X = \begin{pmatrix} a & b \\ \alpha b & \beta a \end{pmatrix}\begin{pmatrix} a & \alpha b \\ b & \beta a \end{pmatrix} = (a^2 + b^2)\,\mathrm{id} = \frac{\|X\|^2}{2}\,\mathrm{id}\,.$$

Hence

$$\mathcal{A}(X)B(X) = -\left(1 + \frac{\|X\|^2}{2}\right)J = -\mathcal{A}(X)J.$$

This concludes the lemma. □

Remark 4.23. Similar computations show that the previous Theorem holds true also in the case $f(X) = \|X\|^2$, i.e. when considering the Dirichlet energy.

In [51, Example 4.41] it is shown that there exists a Sobolev map $\psi \in W^{1,p}(\Omega, \mathbb{R}^2)$, $p > 2$, such that $D\psi$ belongs, at almost every point of Ω, to $H_1 \cup H_2$, and moreover

$$|\{x \in \Omega : D\psi(x) = 0\}| > 0$$

but ψ is non-constant. By Lemma 4.22, we immediately deduce that this function ψ solves

$$\operatorname{curl}(B(D\psi(x))) = \operatorname{curl}(-J) = 0,$$

hence it is a solution to the inner variations equations for the area function. We want to construct such a ψ by using the same methods of [51, Example 4.41], but we moreover want to construct it in such a way that for every open subset $\mathcal{V} \subset \Omega$

$$|\{y \in \mathcal{V} : D\psi(y) = 0\}| > 0$$

but ψ is non-constant in $\mathcal{V}$. In this way, we would deduce that ψ cannot be C^1 on any open set. In fact, suppose by contradiction that there exists a connected open set $\mathcal{V}$ such that $\psi \in C^1(\mathcal{V})$. Let $\mathcal{W} \subset \mathcal{V}$ be an open, compactly contained subset of $\mathcal{V}$. Since H_1, H_2 are closed, we obtain that

$$A_i \doteq \{y \in \overline{\mathcal{W}} : D\psi(y) \in H_i\}$$

are closed sets, contained in $\mathcal{W}$, for $i = 1, 2$, and that moreover

$$\overline{\mathcal{W}} = A_1 \cup A_2.$$

There are two cases: A_1 does not contain any ball or there exists $B_r(y) \subset A_1$. If $\mathrm{int}(A_1) = \emptyset$, then A_2 is dense in $\overline{\mathcal{W}}$. Since it is also closed, then $\overline{\mathcal{W}} = A_2$. In particular, on the open set $\mathcal{W}$, one has $D\psi \in H_2$. This implies that ψ is harmonic and smooth. It is well-known, see for instance [66], that for a non-constant harmonic function ψ

$$|\{y \in \mathcal{W} : D\psi(y) = 0\}| = 0,$$

which is a contradiction with $|\{y \in \mathcal{W} : D\psi(y) = 0\}| > 0$. Therefore, we are left with the case $B_r(y) \subset A_1$. But then, exactly the same reasoning applied with $B_r(y)$ instead of $\mathcal{W}$ yields the same contradiction.

This discussion motivates the fact that, in order to conclude that we can find a solution that is not C^1 in any open set of Ω, we need the following

Lemma 4.24. *There exists an open set Ω and a $W^{1,p}$, $p > 2$, map $\psi : \Omega \to \mathbb{R}^2$ with the property for every open set $\mathcal{V} \subset \Omega$,*

- *ψ is non-constant on $\mathcal{V}$;*
- *$|\mathcal{V} \cap \{y \in \Omega : D\psi(y) = 0\}| > 0$.*

To prove Lemma 4.24, it is sufficient to show the following Lemma:

Lemma 4.25. *There exists a Lipschitz map $f : B_1(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:*

- *$Df(x) \in \{A_1, \dots, A_5\}$ for five 2×2 matrices $A_1, \dots, A_5$ (explicitly written in [51, Example 4.41]), for a.e. $x \in B_1(0)$;*
- *If $\mathcal{A}_i \doteq \{x \in B_1(0) : Du(x) = A_i\}$, then for every open subset of $B_1(0)$, B, it holds*

$$|B \cap \mathcal{A}_i| \neq 0, \forall i = 1, \dots, 5.$$

If Lemma 4.24 holds, then the previous discussion constitutes the proof of Theorem 4.21. Let us now explain how Lemma 4.25 implies Lemma 4.24.

Proof of Lemma 4.24. This proof is exactly the same described in [51, Example 4.41], and we report it here for the reader's convenience. Suppose a map f as the one of Lemma 4.25 exists. We can define the mapping ψ as in [51, Example 4.41], i.e. $\psi(x) \doteq f(F^{-1}(x))$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a suitable $W^{1,p}$, $p > 2$ quasiregular homeomorphism. Since we do not need to explicitly introduce quasiregular maps or Beltrami equations, we will not enter in the details of this theory. We refer the interested reader to the references given in [51, Example 4.41]. The open set Ω is $\Omega \doteq F(B_1(0))$. The map F satisfies a suitable Beltrami equation, introduced in such a way that for a.e. $y \in F(\mathcal{A}_1 \cup \mathcal{A}_2)$, we have $D\psi(y) \in H_1$, while for a.e. $y \in F(\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5)$, we have $D\psi(y) \in H_2$. Moreover, by the computations of [51, Example 4.41] (in particular, by the equation following (4.10)), we find that

$$y \in F(\mathcal{A}_1) \Rightarrow D\psi(y) \neq 0. \tag{4.39}$$

Now $\mathcal{V} \subset \Omega$ open. We want to show that ψ is non-constant on $\mathcal{V}$ and

$$|\{x \in \mathcal{V} : D\psi(x) = 0\}| > 0.$$

We claim

$$|\mathcal{V} \cap F(\mathcal{A}_i)| > 0, \quad \forall i \in \{1, \dots 5\}. \tag{4.40}$$

Indeed, if for some i we had

$$|\mathcal{V} \cap F(\mathcal{A}_i)| = 0,$$

then, making repeated use of the fact that F is bijective,

$$0 = |F(F^{-1}(\mathcal{V})) \cap F(\mathcal{A}_i)| = |F(F^{-1}(\mathcal{V}) \cap \mathcal{A}_i)|. \tag{4.41}$$

From [7, Corollary 3.7.6] we have that F has the N^{-1} property, i.e. for every Borel set A,

$$|A| = 0 \Rightarrow |F^{-1}(A)| = 0.$$

With this, we can infer from (4.41) that there exists i such that

$$|F^{-1}(\mathcal{V}) \cap \mathcal{A}_i| = 0.$$

Since F^{-1} is an open mapping, then $F^{-1}(\mathcal{V})$ is an open set, hence the previous equality is in contradiction with the properties of the map f. Using (4.40), we immediately see that on $\mathcal{V}$ ψ cannot be constant since, as noted in (4.39), $D\psi \neq 0$ on $F(\mathcal{A}_1)$. On the other hand, $y \in F(\mathcal{A}_2) \Rightarrow DF(y) = 0$. This implies, again by (4.40), that $DF(y) = 0$ on a set of positive measure inside $\mathcal{V}$, but F is not constant on $\mathcal{V}$. □

In the next and final subsection we will show Lemma 4.25.

4.6.1 *Convex integration: proof of Lemma 4.25*

To prove Lemma 4.25, we use the Baire Cathegory arguments of [51]. First, we need to recall the following:

Definition 4.26. Let $\mathcal{U} \subset \mathbb{R}^{n\times m}$ be bounded and $K \subset \mathbb{R}^{n\times m}$ be closed. We say that gradients in $\mathcal{U}$ are stable only near K if for every $\varepsilon > 0$, one can find $\delta = \delta(\varepsilon) > 0$ such that, if $A \in \mathcal{U}$ and $\mathrm{d}(A, K) > \varepsilon$, then there exists a piecewise affine map $\varphi \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m)$ with bounded support such that

- $D\varphi(x) + A \in \mathcal{U}$ for a.e. $x \in \mathbb{R}^n$;
- $\displaystyle\int \|D\varphi\|\,dx \geq \delta|\operatorname{spt}(\varphi)|$.

The reason why this definition is useful is given by the following result, see [51, Proposition 3.17, Corollary 3.18]. Let

$$\mathcal{P} \doteq \{u \in \mathrm{Lip}(\Omega, \mathbb{R}^n) : u \text{ piecewise affine, } Du(x) \in \mathcal{U} \text{ a.e. in } \Omega\}$$

and define the complete metric space

$$X \doteq \overline{\mathcal{P}}^{\|\cdot\|_{L^\infty}}. \tag{4.42}$$

Proposition 4.27. *Let the gradients of $\mathcal{U}$ be stable only near a closed set K. Then the typical map $u \in X$ has the property*

$$Du \in K \text{ a.e.}.$$

We now show Lemma 4.25, but first we need to explain how to obtain the matrices $\{A_1, \dots, A_5\}$ in the statement of the Lemma. These matrices are obtained from another set of five symmetric matrices $K \doteq \{P_{F_0}, P_{B_0}, P_{R_0}, P_{L_0}, P_{H_0}\}$ simply by considering $M(K - P_{F_0}) = \{A_1, \dots, A_5\}$, where M is a suitable 2×2 matrix. The importance of the set K, found by Kirchheim and D. Preiss in [51, Construction 4.38], is due to the fact that it is the first example in the literature of a set of five *non-rigid* matrices, i.e. such that there exists a non-affine map $u \in \mathrm{Lip}(B_1(0), \mathbb{R}^2)$ that fulfills

$$Du(x) \in K$$

for a.e. $x \in B_1(0)$. The strategy they use is to find an open subset $\mathcal{U}$ of $\mathrm{Sym}(2)$ such that gradients of $\mathcal{U}$ are stable only near K, see [51, Construction 4.38]. We can now start the:

Proof of Lemma 4.25. Following the previous notation we consider $K = \{P_{F_0}, P_{B_0}, P_{R_0}, P_{L_0}, P_{H_0}\}$ and $\mathcal{U}$ be the open subset of $\mathrm{Sym}(2)$ found by Kirchheim and D. Preiss in [51, Construction 4.38]. We consider X defined as in (4.42). Now enumerate the points with rational coordinates in $B_1(0)$, $\{q_i\}_{i\in\mathbb{N}}$, and define the sets

$$X_{q_i,r,j} \doteq \{u \in X : u \text{ is affine in } B_r(q_i)\}.$$

for rational $0 < r < \mathrm{d}(q_i, \partial B_1(0))$ and $1 \le j \le 5$. We aim to show $Y \doteq \bigcup_{i,r,j} X_{q_i,r,j}$ is meager. If this is the case, then $Z \doteq Y^c \cap \{u \in X : Du(x) \in K, \text{ for a.e. } x \in \Omega\}$ is residual in X. Baire Theorem A.4 tells us that the latter set is non-empty, and obviously for any $u \in Z$, one has

$$Du(x) \in K = \{P_{F_0}, P_{B_0}, P_{R_0}, P_{L_0}, P_{H_0}\}, \text{ for a.e. } x \in \Omega.$$

Considering $f(x) \doteq M(u(x) - P_{F_0}x)$, where M was introduced before the proof of the present Lemma, we get

$$Df(x) \in \{A_1, A_2, A_3, A_4, A_5\}, \text{ a.e..}$$

Moreover, for every $1 \le j \le 5$, $q \in \mathbb{Q}^2 \cap \Omega$, rational radius $0 < r < \mathrm{d}(x, \partial\Omega)$,

$$|\mathcal{A}_j \cap B_r(q)| > 0. \tag{4.43}$$

Indeed, if $|\mathcal{A}_j \cap B_r(q)| = 0$, by the rigidity for the four gradients problem, see [51, Theorem 4.33], we get that f is necessarily affine on $B_r(q)$, against the definition of Z. Since (4.43) is clearly equivalent to

$$|\mathcal{A}_j \cap \mathcal{V}| > 0$$

for every open subset $\mathcal{V} \subset \Omega$ and $1 \le j \le 5$, we would then conclude the proof. In order to show that Y is meager, we prove that $X_{q_i,r,j}$ are closed sets with empty interior. The closedness inside the complete metric space X is straighforward, since a sequence of affine functions converging in L^∞ need to converge to an affine function. Now suppose by contradiction that for some i, r, j, $X_{q_i,r,j}$ has non-empty interior. In particular, we suppose we have that for some $\alpha > 0$ and $u \in X$,

$$\{v \in X : \|u - v\|_\infty < \alpha\} \subset X_{q_i,r,j}.$$

Since $u \in X$, we can pick a function $\bar{u} \in \mathcal{P}$ such that $\|\bar{u} - u\| \le \frac{\alpha}{4}$ and $D\bar{u} \in \mathcal{U}$. We also know, by assumption, that $\bar{u}$ is affine on $B_r(q_i)$, say $\bar{u} = Ax + b$ on $B_r(q_i)$ with $A \in \mathcal{U}$. Since $A \in \mathcal{U}$, that is an open subset of $\mathrm{Sym}(2)$, as follows by the construction of [51], then we can easily find two matrices B and C in $\mathcal{U}$ such that $\mathrm{rank}(B - C) = 1$ and $\frac{B+C}{2} = A$. For instance, one can take

$$B \doteq A + \lambda E_{11}, \quad C \doteq A - \lambda E_{11},$$

where $\lambda > 0$ is a sufficiently small parameter and

$$E_{11} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

By [51, Proposition 3.4], recalled below, for every $\varepsilon > 0$ we can find a Lipschitz and piecewise affine map $w : B_{\frac{r}{2}}(q_i) \to \mathbb{R}^2$ with

- $Dw(x) \in \mathcal{U}$ a.e.;
- $w(x) = Ax$ on $\partial B_{r/2}(q_i)$;
- $\|w - A\|_\infty \le \varepsilon$.

Of course, if we traslate w with $\overline{w} \doteq w + b$, we have

- $D\overline{w}(x) \in \mathcal{U}$ a.e.;
- $\overline{w}(x) = \bar{u}(x)$ on $\partial B_{r/2}(q_i)$;

- $\|\bar{w} - \bar{u}\|_{L^\infty(B_{r/2}(q_i))} \leq \varepsilon$.

Moreover, the same proposition yields the following property

$$|\{x \in B_{r/2}(q_i) : D\bar{w}(x) = B\}| \geq \frac{(1-\varepsilon)}{2}|B_{r/2}(q_i)|$$

and

$$|\{x \in B_{r/2}(q_i) : D\bar{w}(x) = C\}| \geq \frac{(1-\varepsilon)}{2}|B_{r/2}(q_i)|.$$

In particular, this implies that $\bar{w}$ cannot be affine on $B_{r/2}(q_i)$. We finally get a contradiction, because the map

$$z(x) \doteq \begin{cases} \bar{u}(x), & \text{if } x \in \Omega \setminus B_{r/2}(q_i) \\ \bar{w}(x), & \text{if } x \in B_{r/2}(q_i) \end{cases}$$

is piecewise affine, Lipschitz, $\|z - \bar{u}\|_{L^\infty(\Omega} \leq \varepsilon$ and $Dz(x) \in \mathcal{U}$, for a.e. $x \in \Omega$. If $\varepsilon < \frac{\alpha}{4}$, then we would obtain that z is affine on $B_{r/2}(q_i)$, against the construction of $\bar{w}$. This concludes the proof. □

We recall here [51, Proposition 3.4],

Proposition 4.28. *Let $A, B, C \in \mathrm{Sym}(n)$, with $\mathrm{rank}(B - C) = 1$, and $A = tB + (1-t)C$, for some $t \in [0,1]$. Let also $\Omega \subset \mathbb{R}^n$ be a fixed open domain. Then, for every $\varepsilon > 0$, one can find a Lipschitz piecewise affine map $f : \Omega \to \mathbb{R}^n$ such that*

- *$f(x) = Ax$ on $\partial\Omega$ and $\|f - A\|_\infty \leq \varepsilon$;*
- *$Df(x) \in \mathrm{Sym}(n) \cap B_\varepsilon([B, C])$;*
- *$|\{x \in \Omega : Df(x) = B\}| \geq (1-\varepsilon)t|\Omega|$ and $|\{x \in \Omega : Df(x) = C\}| \geq (1-\varepsilon)(1-t)|\Omega|$.*

Remark 4.29. Notice that Proposition (4.28) asserts the construction of the simple laminates (with the simmetry constraint on the gradient matrix) that we have already mentioned in Section 1.4.

Remark 4.30. To the best of our knowledge, there are various open problems related to the one of this proof. For instance, one might ask whether a solution $u \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$ to

$$\mathrm{curl}(A(Du)) = 0$$

needs to be regular or not. Moreover, in our example we have essentially used that $\{D\psi = 0\}$ is a set of positive Lebesgue measure. It is unclear to us if one can find a counterexample satisfying $\|D\psi\| \geq \delta > 0$ at a.e. point, or $\det(D\psi) > 0$ a.e..

Part II

DIVERGENCE-FREE MATRIX FIELDS IN $\mathrm{Sym}^+(n)$

5 INTEGRABILITY OF $\det^{\frac{1}{n-1}}(\cdot)$

In this chapter we give negative answer to [76, Open Question 2.1], that we recall here:

Open Question 2.1: Let $x \mapsto A(x)$ be Γ-periodic, taking values in $\mathrm{Sym}^+(n)$. Let A and $\mathrm{div}(A)$ belong to $L^p(\mathbb{R}^n/\Gamma)$ with $1 < p < n$. Defining $\frac{1}{p'} = \frac{1}{p} - \frac{1}{n}$, is it true that

$$\det(A)^{\frac{1}{n}} \in L^{p'}(\mathbb{R}^n/\Gamma)\,?$$

The answer is the content of the main theorem of this chapter, Theorem 5.1. Let us first introduce some notation that we will need in the next two chapters.

NOTATIONS

For symmetric matrices $A, B \in \mathrm{Sym}^+(n)$, we use the standard partial order relation

$$A \geq B \Leftrightarrow (Av, v) \geq (Bv, v), \quad \forall v \in \mathbb{R}^n.$$

Recall the basic monotonicity property of the determinant

$$A \geq B \Rightarrow \det(A) \geq \det(B).$$

For a matrix A, we denote with $P_A(\lambda)$ its characteristic polynomial, i.e.

$$P_A(\lambda) \doteq \det(\lambda\,\mathrm{id} - A).$$

Let us define, for a matrix $A \in \mathrm{Sym}^+(n)$ with eigenvalues $\lambda_1, \dots, \lambda_n$,

$$M_i(A) \doteq \sum_{1 \leq j_1 \leq \cdots \leq j_i \leq n} \lambda_{j_1} \dots \lambda_{j_i}, \quad \forall i \in \{1, \dots, n\},\ M_0(A) \doteq 1.$$

It is a basic Linear Algebra fact that, if $0 \leq i \leq n$ the i-th coefficient of $P_A(\lambda)$ is given by $(-1)^{i+n} M_{n-i}(A)$. Notice in particular that $M_n(A) = \det(A)$.

5.1 MAIN RESULT

Let Ω be an open subset of $\mathbb{R}^n$. Let

$$\begin{aligned} Y_{p,K} \doteq \{&A \in L^p(\Omega, \mathrm{Sym}^+(n)) : \mathrm{div}(A) \in L^p(\Omega, \mathbb{R}^n), \\ &A \equiv \bar{A} \text{ outside } K,\ \text{for some fixed } \bar{A} \in \mathrm{Sym}^+(n)\}, \end{aligned}$$

for any compact $K \subset \Omega$ with $\mathrm{clos}(\mathrm{int}(K)) = K \neq \emptyset$. We consider the following distance on $Y_{p,K}$, that turns it into a complete metric space:

$$d(A, B) \doteq \|A - B\|_{L^p} + \|\mathrm{div}(A - B)\|_{L^p}.$$

We prove the following

Theorem 5.1. *Let $p^* \doteq \max\left\{0, \frac{p(n-1)-n}{p(n-1)}\right\}$. The set*

$$D_{p,K} \doteq \{A \in Y_{p,K} : \det(A)^{\frac{1}{n-1}} \in L^{\frac{1}{1-p^*}}(\Omega) \setminus L^{\frac{1}{1-p^*}+\varepsilon}(\Omega), \forall \varepsilon > 0\}$$

is residual in $Y_{p,K}$.

Remark 5.2. The same result (without modifying the proof) would have held if we had required $\operatorname{div}(A) = 0$ in the definition of $Y_{p,K}$, or if we had chosen instead of $Y_{p,K}$,

$$X_p = \{A \in L^p(\Omega, \mathrm{Sym}^+(n)) : \operatorname{div}(A) \in L^p,$$
$$A\nu \equiv \bar{A}\nu \text{ on } \partial\Omega, \text{ for some fixed } \bar{A} \in \mathrm{Sym}^+(n)\},$$

or, as in Serre's original question

$$S_p = \{A \in L^p(\mathbb{R}^n/\Gamma, \mathrm{Sym}^+(\mathbb{R}^n)) : \operatorname{div}(A) \in L^p(\mathbb{R}^n/\Gamma, \mathrm{Sym}^+(\mathbb{R}^n))\}.$$

Let us explain how this result gives negative answer to [76, Open Question 2.1]. If $p \le \frac{n}{n-1}$, we obtain the existence of one (in fact, many) divergence free, non-negative definite tensor fields A such that

$$\det(A)^{\frac{1}{n-1}} \in L^1 \setminus L^{1+\varepsilon}, \quad \forall \varepsilon > 0,$$

thus proving the optimality of Serre's results. The existence of this tensor field is guaranteed by Baire Theorem, Theorem A.4. Moreover, also in the supercritical case, i.e. $p > \frac{n}{n-1}$, Theorem 5.1 tells us that for many divergence free, non-negative definite A, $\det^{\frac{1}{n-1}}(A) \in L^{\frac{p(n-1)}{n}} \setminus L^{\frac{p(n-1)}{n}+\varepsilon}$, thus proving that there can be no general gain in the integrability of the determinant with respect to the general estimate $\det(A) \in L^{\frac{p}{n}}$.

In order to prove Theorem 5.1, we make use of the classical fact that $\operatorname{div}(\operatorname{cof}^T(Du)) = 0$, for $u \in C^\infty(\Omega, \mathbb{R}^n)$. This is proved in [30, Ch. 8, Th. 2]. By approximation, it is easy to see that this holds also for maps $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$. We exploit this result when building *approximate* counterexamples to Open Question 2.1, in Lemma 5.3. In that Lemma, we consider a suitable family of convex functions of class $W^{2,p(n-1)}$, denoted by $\varphi_{\beta,\delta,\varepsilon,x_0}$. The matrix-field obtained by taking $x \mapsto \operatorname{cof}(H\varphi_{\beta,\delta,\varepsilon,x_0})(x)$ will then be non-negative definite, by convexity, and divergence-free, by the aforementioned result. With this family we are also able to show the optimal integrability of $x \mapsto \det(H\phi)(x)$, where ϕ is a convex function in the Sobolev class. Since this Chapter is devoted to the study of divergence-free tensor fields, we have moved this discussion in Appendix C.

Lemma 5.3. *Fix $p \ge 1$. For every $\beta > 0, \delta > 0, \varepsilon > 0, x_0 \in \Omega$ there exists a convex function $\varphi_{\beta,\delta,\varepsilon,x_0} \in W^{2,p(n-1)}_{\mathrm{loc}}(\Omega)$ and a matrix $S_{\beta,\delta,\varepsilon,x_0} \in \mathrm{Sym}^+(n)$ such that*

(i) $\varphi_{\beta,\delta,\varepsilon,x_0} \equiv x^T S_{\beta,\delta,\varepsilon,x_0} x$ *outside* $B_\beta(x_0)$;

(ii) $\|\operatorname{cof}(H\varphi_{\beta,\delta,\varepsilon,x_0})\|_{L^p(\Omega)} \le \delta$;

(iii) $\det^{\frac{1}{n-1}}(\operatorname{cof}(H\varphi_{\beta,\delta,\varepsilon,x_0})) \notin L^{\frac{1}{1-p^*}+\varepsilon}(B_r(x_0)), \forall r > 0.$

Proof. We divide the proof in four steps:

$\boxed{\text{Step 1: Definition and properties of the starting function.}}$

For $\alpha \ge 0$, define the function

$$f_\alpha(x) \doteq \begin{cases} \|x\|^{1+\alpha} + b, & \text{if } \|x\| \le 1, \\ a\|x\|^2, & \text{if } \|x\| > 1, \end{cases}$$

where $a, b \in \mathbb{R}$ are chosen in such a way that the function f_α is in $C^1(\mathbb{R}^n \setminus \{0\})$, i.e. we need to solve

$$1 + b = a \text{ and } 1 + \alpha = 2a.$$

Therefore

$$f_\alpha(x) \doteq \begin{cases} \|x\|^{1+\alpha} + \frac{\alpha-1}{2}, & \text{if } \|x\| \le 1, \\ \frac{1+\alpha}{2}\|x\|^2, & \text{if } \|x\| > 1. \end{cases} \tag{5.1}$$

It is easy to see that f_α defined in this way is convex. We compute its pointwise Hessian (except for the points $x \in \mathbb{R}^n$ such that $\|x\| = 0$ or $\|x\| = 1$):

$$Hf_\alpha(x) \doteq \begin{cases} (1+\alpha)\left(\|x\|^{\alpha-1}\,\mathrm{id}_n + (\alpha-1)\|x\|^{\alpha-3} x \otimes x\right), & \text{if } 0 < \|x\| < 1, \\ (1+\alpha)\,\mathrm{id}_n, & \text{if } \|x\| > 1. \end{cases} \tag{5.2}$$

Step 2: L^p estimates on Hf_α.

We can estimate, for some constant $C_{\alpha,n} > 0$,

$$\|Hf_\alpha\|(x) \le \begin{cases} C_{\alpha,n}\|x\|^{\alpha-1}, & \text{if } 0 < \|x\| < 1, \\ (1+\alpha)\sqrt{n}, & \text{if } \|x\| > 1. \end{cases} \tag{5.3}$$

The Matrix Determinant Lemma, Lemma 2.17, tells us that:

$$\det(A+B) = \det(A) + \langle B, \mathrm{cof}^T(A)\rangle, \quad \forall A, B \in \mathbb{R}^{n\times n},\ \mathrm{rank}(B) = 1.$$

We can use it to compute explicitely the pointwise determinant of the Hessian of f_α:

$$\det(Hf_\alpha)(x) = \begin{cases} \alpha(1+\alpha)^n\|x\|^{n(\alpha-1)}, & \text{if } 0 < \|x\| < 1, \\ (1+\alpha)^n, & \text{if } \|x\| > 1. \end{cases} \tag{5.4}$$

From (5.3), we find that $Hf_\alpha \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \ge 0$ if $p < n$ and for $\alpha > \frac{p-n}{p}$ if $p \ge n$. For these values of α, we also get $f_\alpha \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$, as proved in Lemma 5.4, and that Hf_α is not only the pointwise Hessian of f_α but also its distributional Hessian.

Step 3: Integrability of the determinant and the cofactors of Hf_α.

Define

$$A_\alpha(x) \doteq \mathrm{cof}(Hf_\alpha)(x). \tag{5.5}$$

In view of the equality $\det^{\frac{1}{n-1}}(A_\alpha) = \det(Hf_\alpha)$ and (5.4),

$$\det{}^{\frac{1}{n-1}}(A_\alpha) \in L^{\frac{1}{1-\alpha}-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n), \quad \forall \varepsilon > 0, \tag{5.6}$$

but

$$\det{}^{\frac{1}{n-1}}(A_\alpha) \notin L^{\frac{1}{1-\alpha}}(B_r(0)) \text{ for any } r > 0. \tag{5.7}$$

Moreover, by (5.3),

$$\|A_\alpha\|(x) = \|\mathrm{cof}(Hf_\alpha)\|(x) \le c_n\|Hf_\alpha\|^{n-1}(x) \overset{(5.3)}{\le} C'_{\alpha,n}\max\{\|x\|^{(n-1)(\alpha-1)}, 1\},$$

for some constant $C'_{\alpha,n} > 0$. Hence, if $(n-1)(1-\alpha)p < n$, i.e. if $\alpha > p^*$, then $A_\alpha \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. The same computation shows, in particular, that for $\alpha > p^*$ one has $f_\alpha \in W^{2,p(n-1)}_{\mathrm{loc}}(\mathbb{R}^n)$.

Step 4: Construction of $\varphi_{\beta,\delta,\varepsilon,x_0}$.

Fix $p, \beta, \delta, \varepsilon, x_0$ as in the statement of the Lemma. Choose $\alpha = \alpha(\varepsilon) > 0$ such that

$$\frac{1}{1-\alpha} = \frac{1}{1-p^*} + \varepsilon,$$

that in particular implies $\alpha > p^*$. Finally define, for a constant $c_{\beta,\delta,\varepsilon} > 0$ to be fixed later,

$$\varphi_{\beta,\delta,\varepsilon,x_0}(x) \doteq c_{\beta,\delta,\varepsilon}\left[f_\alpha\left(\frac{2}{\beta}(x-x_0)\right) - 2\left(\frac{1+\alpha}{\beta^2}\right)\left(\|x_0\|^2 - 2(x, x_0)\right)\right]. \tag{5.8}$$

By the definition of f_α, we get (i). Moreover, (iii) is a consequence of our choice of α and (5.7). Finally, since $\alpha > p^*$, A_α belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, as proved in the previous step. Therefore, we can choose $c_{\beta,\delta,\varepsilon}$ small enough so that (ii) is fulfilled. □

Lemma 5.4. *The function f_α defined in (5.1) is in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \geq 0$ if $p < n$ and for $\alpha > \frac{p-n}{p}$ if $p \geq n$. Moreover, its pointwise Hessian, computed in (5.2), coincides with its distributional Hessian.*

Proof. To see this write, for any $\eta \in C^\infty_c(\mathbb{R}^n)$ and $i, j \in \{1, \dots, n\}$,

$$\int_{\mathbb{R}^n} f_\alpha \partial^2_{ij}\eta = \lim_{R\to 0}\left[\int_{\mathbb{R}^n\setminus(B_R(0)\cup S_R)} f_\alpha \partial^2_{ij}\eta\right],$$

where $S_R = B^c_{1-R}(0) \cap B_{1+R}(0)$. Integrating by parts we get

$$\int_{\mathbb{R}^n} f_\alpha \partial^2_{ij}\eta = \lim_{R\to 0}\left[\int_{\partial S_R\cup\partial B_R(0)} f_\alpha\, \partial_i\eta\, \nu^j - \int_{\mathbb{R}^n\setminus(B_R(0)\cup S_R)} \partial_j f_\alpha \partial_i\eta\right],$$

and since $f_\alpha \in C^0(\mathbb{R}^n)$ the first term vanishes. Thus we are left with the second one, which again integrating by parts can be written as

$$\lim_{R\to 0} - \int_{\mathbb{R}^n\setminus(B_R(0)\cup S_R)} \partial_j f_\alpha \partial_i \eta = \lim_{R\to 0}\left[-\int_{\partial S_R\cup\partial B_R(0)} \partial_j f_\alpha\, \eta\, \nu^i + \int_{\mathbb{R}^n\setminus(B_R(0)\cup S_R)} \eta\, \partial^2_{ij} f_\alpha\right].$$

Note that for every $\alpha \geq 0$, $\partial_j f_\alpha \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $\partial_j f_\alpha$ is continuous in $\mathbb{R}^n \setminus \{0\}$. Thus we have

$$\lim_{R\to 0}\int_{\partial S_R\cup\partial B_R(0)} \partial_j f_\alpha\, \eta\, \nu^i = \lim_{R\to 0}\left[\int_{\partial S_R} \partial_j f_\alpha\, \eta\, \nu^i + \int_{\partial B_R(0)} \partial_j f_\alpha\, \eta\, \nu^i\right] = 0\,.$$

Finally by Step 2 of Lemma 5.3, we know that $\partial^2_{ij} f_\alpha$ is in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \geq 0$ if $p < n$ and for $\alpha > \frac{p-n}{p}$ if $p \geq n$. Thus, for the ranges of α and p we are considering, we have $Hf_\alpha \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, and by dominated convergence we conclude

$$\int_{\mathbb{R}^n} f_\alpha \partial^2_{ij}\eta = \lim_{R\to 0}\int_{\mathbb{R}^n\setminus(B_R(0)\cup S_R)} \eta\, \partial^2_{ij} f_\alpha = \int_{\mathbb{R}^n} \eta\, \partial^2_{ij} f_\alpha\,.$$

□

We can finally prove our main result.

Proof of Theorem 5.1. First observe that

$$D^c_{p,K} = \{A \in Y_{p,K} : \det(A)^{\frac{1}{n-1}} \in L^{\frac{1}{1-p^*}+\varepsilon} \text{ for some } \varepsilon > 0\},$$

which is true for $p < \frac{n}{n-1}$ because of Serre's result [76, Theorem 2.4], while for $p \geq \frac{n}{n-1}$ it is just a consequence of the definition of p^* and the fact that $\det(A)^{\frac{1}{n-1}} \in L^{\frac{p(n-1)}{n}}, \forall A \in Y_{p,K}$.

We want to write $D^c_{p,K}$ as a countable union of closed sets with empty interior. To do so, consider

$$C_{k,j} = \{A \in Y_{p,K} : \|\det(A)^{\frac{1}{n-1}}\|_{\frac{1}{1-p^*}+\frac{1}{k}} \leq j\}.$$

For every k, j, $C_{k,j}$ is closed in $(Y_{p,K}, d)$, as can be easily seen through Fatou's Lemma. Moreover,

$$\bigcup_{k,j} C_{k,j} = D^c_{p,K}.$$

Finally, suppose that for some k, j, $C_{k,j}$ has non-empty interior. This means that we can find $\bar{A} \in C_{k,j}$ and a ball (in the d-topology on $Y_{p,K}$) of radius ρ, $\mathcal{N}_\rho(\bar{A})$, such that $\mathcal{N}_\rho(\bar{A}) \subset C_{k,j}$. In particular this implies that

$$\det(B)^{\frac{1}{n-1}} \in L^{\frac{1}{1-p^*}+\frac{1}{k}}(\Omega), \forall B \in \mathcal{N}_\rho(\bar{A}). \tag{5.9}$$

Fix $x_0 \in \mathrm{int}(K) \subset \Omega$ and let $r > 0$ be such that $B_r(x_0) \subset \mathrm{int}(K)$. Consider $\varphi_{\beta,\delta,\varepsilon,x_0}$ of Lemma 5.3, with $\varepsilon = \frac{1}{k}$, $\beta = \frac{r}{2}$ and $\delta = \frac{\rho}{2}$. Define also

$$\overline{M}_{\beta,\delta,\varepsilon,x_0}(x) \doteq \mathrm{cof}(H\varphi_{\beta,\delta,\varepsilon,x_0}),$$

and finally take

$$B \doteq \bar{A} + \overline{M}_{\beta,\delta,\varepsilon,x_0}.$$

Observe that $\overline{M}_{\beta,\delta,\varepsilon,x_0}$ is a divergence-free non-negative definite tensor field, that is constant outside K. The fact that $\overline{M}_{\beta,\delta,\varepsilon,x_0}$ is divergence-free is because it is the cofactor matrix of the Hessian of a map $\varphi \in W^{2,p(n-1)}_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore, our choice of β, δ and (5.9) imply $\det(B)^{\frac{1}{n-1}} \in L^{\frac{1}{1-p^*}+\frac{1}{k}}$. Since the determinant is monotone on the cone of non-negative symmetric matrices, we have

$$\det(B) = \det(\bar{A} + \overline{M}_{\beta,\delta,\varepsilon,x_0}) \geq \det(\overline{M}_{\beta,\delta,\varepsilon,x_0}) \geq 0,$$

that would imply $\det(\overline{M}_{\beta,\delta,\varepsilon,x_0})^{\frac{1}{n-1}} \in L^{\frac{1}{1-p^*}+\frac{1}{k}}(B_\beta(x_0))$ but this contradicts (iii) of Lemma 5.3 by our choice of ε. □

Remark 5.5. The situation for diagonal matrices is less rigid. If $A = \mathrm{diag}(f_1, \dots, f_n)$, $f_i \in L^p(\mathbb{R}^n)$, compactly supported, and $\mathrm{div}(A) \in L^p(\mathbb{R}^n)$, then $|\det|^{\frac{1}{n-1}}(A) \in L^p(\mathbb{R}^n)$, and

$$\|(\det A)^{\frac{1}{n-1}}\|_{L^p} \leq C\|\,\mathrm{div}(A)\|_{L^p}^{\frac{n}{n-1}},$$

for some constant $C > 0$ which depends on the size of the support of A. Note that one does not even need the non-negativity of A to be satisfied. The proof of the inequality is as follows. We have that $\partial_i f_i \in L^p(\mathbb{R}^n)$. Therefore

$$|f_i|(x_1, \dots, x_n) = \left|\int_{-\infty}^{x_i} \partial_i f_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)dt\right|$$

and

$$|f_i|^p(x_1, \dots, x_n) \leq C\int_{-\infty}^{\infty} |\partial_i f_i|^p(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)dt,$$

where $C = C(p, \mathrm{diam}(\mathrm{spt}(A)))$. Define

$$g_i(\hat{x}_i) \doteq \int_{-\infty}^{\infty} |\partial_i f_i|^p(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)dt.$$

We have $g_i \in L^1(\mathbb{R}^{n-1})$, hence $g_i^{\frac{1}{n-1}} \in L^{n-1}$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^n} |\det(A)|^{\frac{p}{n-1}}(x)\,dx &\leq C\int_{\mathbb{R}^n}\prod_i g_i^{\frac{1}{n-1}}(\hat{x}_i)\,dx \\ &\leq C\prod_i \|g_i\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \leq C\|\,\mathrm{div}(A)\|_{L^p}^{\frac{np}{(n-1)}}.\end{aligned}$$

The second inequality can be found in [9, Lemma 9.4]. Since this inequality is sharp, it is easy to find counterexamples to the statement $\det(A) \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for exponents $q > \frac{p}{n-1}$.

6 UPPER-SEMICONTINUITY OF $\mathbb{D}(\cdot)$

In this chapter we show weak upper-semicontinuity of the functional

$$\mathbb{D}(A) \doteq \int_{\mathbb{T}^n} \det(A(x))^{\frac{1}{n-1}}\, dx$$

in the space

$$X_p \doteq \left\{A \in L^p(\mathbb{T}^n, \mathrm{Sym}^+(n)) : \operatorname{div} A \in \mathcal{M}(\mathbb{T}^n, \mathbb{R}^n)\right\},$$

with respect to its weak topology, when $p > \frac{n}{n-1}$. This is the content of Theorem 6.1. In Proposition 6.6, we show its failure for $p \leq \frac{n}{n-1}$, and at the end we discuss some applications to the multi-dimensional Burgers equation.

6.1 THE CASE $p > \frac{n}{n-1}$

In this section we prove weak upper semi-continuity of the functional $\mathbb{D}(\cdot)$. Fix $p \in [1, \infty]$. Consider the space

$$X_p \doteq \left\{A \in L^p(\mathbb{T}^n, \mathrm{Sym}^+(n)) : \operatorname{div} A \in \mathcal{M}(\mathbb{T}^n, \mathbb{R}^n)\right\}.$$

We say that $A_k \rightharpoonup A$ in X_p if $A_k \rightharpoonup A$ in L^p ($A_k \overset{*}{\rightharpoonup} A$ if $p = \infty$) and $\operatorname{div} A_k \overset{*}{\rightharpoonup} \operatorname{div} A$ in $\mathcal{M}(\mathbb{T}^n, \mathbb{R}^n)$. We prove the following

Theorem 6.1. *Let $p > \frac{n}{n-1}$ and $\{A_k\}_k \subset X_p$ be such that $A_k \rightharpoonup A$ in X_p. Then*

$$\limsup_k \mathbb{D}(A_k) \leq \mathbb{D}(A).$$

To prove Theorem 6.1 we follow the argument of [36], indeed we will prove that the Young measure $\nu = (\nu_x)_{x \in \mathbb{T}^n}$ generated by the sequence $\{A_k\}_k$, satisfies

$$\langle \nu_x, \det(\cdot)^{\frac{1}{n-1}} \rangle \leq \det(A(x))^{\frac{1}{n-1}}, \tag{6.1}$$

for almost every $x \in \mathbb{T}^n$. Indeed, by the Fundamental Theorem of Young Measures, Theorem A.1, and (6.1), we would conclude

$$\limsup_k \mathbb{D}(A_k) = \lim_k \mathbb{D}(A_k) = \int_{\mathbb{T}^n} \langle \nu_x, \det(\cdot)^{\frac{1}{n-1}} \rangle\, dx \overset{(6.1)}{\leq} \mathbb{D}(A),$$

i.e. the weak upper semi-continuity of $\mathbb{D}(\cdot)$ on X_p, where in the first equality we used the fact that up to a subsequence we can further suppose that $\limsup_k \mathbb{D}(A_k) = \lim_k \mathbb{D}(A_k)$. The argument to obtain (6.1) is different to the one given in [36] and heavily relies on the ideas of [76, Proof of Theorem 2.2]. First we make the following remarks of technical nature.

Remark 6.2. Iit is sufficient to prove the theorem in the case in which $A_k, A \geq \varepsilon\, \mathrm{id}_n$ for some $\varepsilon > 0$. Indeed, in the general case one can consider $A_k^\varepsilon = A_k + \varepsilon\, \mathrm{id}_n$, for which one proved weak upper semi-continuity of $\mathbb{D}$, meaning that

$$\limsup_k \mathbb{D}(A_k^\varepsilon) \leq \mathbb{D}(A^\varepsilon).$$

By monotonicity of the determinant on the cone of positive definite matrices, we also have

$$\limsup_k \mathbb{D}(A_k) \leq \limsup_k \mathbb{D}(A_k^\varepsilon) \leq \mathbb{D}(A^\varepsilon),$$

thus the theorem in the general case follows by letting $\varepsilon \to 0$.

Remark 6.3. We can also suppose that the sequence $\{A_k\}_k$ is smooth. Indeed for any $A_k \in X_p$ there exists a smooth matrix field $\tilde{A}_k \in X_p$ such that

(i) $\|A_k - \tilde{A}_k\|_{L^p(\mathbb{T}^n)} \le \frac{1}{k}$;

(ii) $\int_{\mathbb{T}^n} \|\operatorname{div}(\tilde{A}_k)\|(x)\,dx \le \|\operatorname{div}(A_k)\|_{\mathcal{M}(\mathbb{T}^n,\mathbb{R}^n)}$ for every k;

(iii) $\tilde{A}_k \rightharpoonup A$ in X_p.

To construct it, consider a standard family of mollifiers $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$, where

$$\rho \in C_c^\infty(B_1(0)), \quad \int_{\mathbb{R}^n} \rho(x)\,dx = 1, \quad \rho(x) \ge 0, \quad \forall x \in \mathbb{R}^n$$

and consequently $A_{k,\varepsilon}(x) \doteq A_k * \rho_\varepsilon(x)$. Clearly $A_{k,\varepsilon} \in X_p$ and is smooth $\forall k, \varepsilon$. As $\varepsilon \to 0$, we have that $A_{k,\varepsilon} \to A_k$ for fixed k in $L^p(\mathbb{T}^n, \mathrm{Sym}^+(n))$. Hence, for every k we can choose ε_k such that (i) is fulfilled. Define $\tilde{A}_k \doteq A_{k,\varepsilon_k}$. We need to show (ii) and (iii). Since mollification does not increase the total mass, we have

$$\|\tilde{A}_k\|_{L^p} \le \|A_k\|_{L^p}, \quad \|\operatorname{div}(\tilde{A}_k)\|_{\mathcal{M}(\mathbb{T}^n,\mathbb{R}^n)} \le \|\operatorname{div}(A_k)\|_{\mathcal{M}(\mathbb{T}^n,\mathbb{R}^n)}, \forall k \in \mathbb{N}.$$

The second inequality is exactly (ii). Moreover, by the weak convergence in X_p, both $\|A_k\|_{L^p}$ and $\|\operatorname{div} A_k\|_{\mathcal{M}(\mathbb{T}^n,\mathbb{R}^n)}$ are equibounded sequences, hence $\tilde{A}_k$ is precompact in X_p, in the sense that for every subsequence, there exists a further subsequence converging in X_p to some tensor field $B \in X_p$. By (i), any limit point of this sequence with respect to the topology of X_p must be the same as the one of A_k, namely A, hence (iii) follows. Thus, if Theorem 6.1 is true for a smooth sequence, we have

$$\begin{aligned}\limsup_k \mathbb{D}(A_k) &= \limsup_k \left(\mathbb{D}(A_k) - \mathbb{D}(\tilde{A}_k) + \mathbb{D}(\tilde{A}_k)\right) \\ &\le \limsup_k \left(\mathbb{D}(A_k) - \mathbb{D}(\tilde{A}_k)\right) + \limsup_k \mathbb{D}(\tilde{A}_k) \le \mathbb{D}(A).\end{aligned} \tag{6.2}$$

Let us justify the last inequality. We can estimate, using the Hölder property of $t \mapsto t^{\frac{1}{n-1}}$,

$$|\mathbb{D}(A_k) - \mathbb{D}(\tilde{A}_k)| \le \int_{\mathbb{T}^n} |\det(A_k(x))^{\frac{1}{n-1}} - \det(\tilde{A}_k(x))^{\frac{1}{n-1}}|\,dx \le \int_{\mathbb{T}^n} |\det(A_k(x)) - \det(\tilde{A}_k(x))|^{\frac{1}{n-1}}\,dx\,.$$

Moreover, a simple estimate valid for every couple of matrices $X, Y \in \mathbb{R}^{n\times n}$ gives, for some dimensional constant $c > 0$,

$$|\det(X) - \det(Y)| \le c(\|X\|^{n-1} + \|Y\|^{n-1})\|X - Y\|.$$

Therefore, using this inequality and the subadditivity of $t \mapsto t^{\frac{1}{n-1}}$

$$\begin{aligned}&\int_{\mathbb{T}^n} |\det(A_k(x)) - \det(\tilde{A}_k(x))|^{\frac{1}{n-1}}\,dx \\ &\le c^{\frac{1}{n-1}} \int_{\mathbb{T}^n} \left(\|A_k(x)\|^{n-1} + \|\tilde{A}_k(x)\|^{n-1}\right)^{\frac{1}{n-1}} \|A_k(x) - \tilde{A}_k(x))\|^{\frac{1}{n-1}}\,dx \\ &\le c^{\frac{1}{n-1}} \int_{\mathbb{T}^n} (\|A_k(x)\| + \|\tilde{A}_k(x)\|)\,\|A_k(x) - \tilde{A}_k(x))\|^{\frac{1}{n-1}}\,dx \\ &\le c^{\frac{1}{n-1}} \left(\int_{\mathbb{T}^n} (\|A_k(x)\| + \|\tilde{A}_k(x)\|)^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \left(\int_{\mathbb{T}^n} \|A_k(x) - \tilde{A}_k(x))\|^{\frac{n}{n-1}}\,dx\right)^{\frac{1}{n}},\end{aligned}$$

the last inequality being Hölder inequality with exponents $\frac{n}{n-1}$ and n. The previous inequality and (i) justify the last estimate of (6.2).

Proof of Theorem 6.1. First notice that up to (non-relabeled) subsequences we can suppose

$$\limsup_k \mathbb{D}(A_k) = \lim_k \mathbb{D}(A_k)$$

and that $\{A_k\}_k$ generates the Young measure $\nu = (\nu_x)_{x\in\mathbb{T}^n}$. From Remark 6.2 and Remark 6.3, we can further suppose that both $A_k, A \geq \varepsilon\,\mathrm{id}_n$ for some $\varepsilon > 0$ and A_k are smooth.

$\boxed{\text{Step 1: definition of the main objects}}$

Let $\mu_k \in \mathcal{M}_+(\mathbb{T}^n)$ be the finite Radon measures defined by $\mu_k(E) \doteq \int_E \|\operatorname{div}(A_k)\|(x)\,dx$ and call μ its weak-$*$ limit (that we can always suppose to exist up to further subsequences). Notice that, for every $i \in \{1,\dots,n\}$, the map

$$x \mapsto M_i^{\frac{1}{n-1}}(A_k(x))$$

is equibounded in $L^{\frac{p(n-1)}{i}}(\mathbb{T}^n)$. Since $p > \frac{n}{n-1}$ and $i \leq n$, these sequences fulfill the hypotheses of Theorem A.1, hence

$$M_i^{\frac{1}{n-1}}(A_k(x)) \rightharpoonup \langle \nu_x, M_i^{\frac{1}{n-1}}(\cdot)\rangle \quad \text{in } L^1(\mathbb{T}^n).$$

Consider $T' \subset \mathbb{T}^n$ to be the set of points $a \in \mathbb{T}^n$ such that

- $\|A(a)\| < \infty$;
- $\langle \nu_a, M_i^{\frac{1}{n-1}}(\cdot)\rangle < +\infty, \forall i \in \{0,\dots,n\}$;
- a is a Lebesgue point for $x \mapsto A(x)$;
- a is a Lebesgue point for $x \mapsto \langle \nu_x, M_i^{\frac{1}{n-1}}(\cdot)\rangle$, for $i \in \{1,\dots,n\}$.

Since these are $L^1(\mathbb{T}^n)$ functions, we get $|\mathbb{T}^n \setminus T'| = 0$. Let $\mu = g\,dx + \mu^s$ be the Lebesgue decomposition of the weak-* limit of μ_k, and define $T'' \subset \mathbb{T}^n$ to be the set of points that are both Lebesgue points for g and density 0 points for μ^s. By [31, Theorem 1.31], $|\mathbb{T}^n \setminus T''| = 0$. Finally, define $T \doteq T' \cap T'' \cap (0,1)^n$. As explained before the proof of the theorem, we want to prove (6.1), namely

$$\langle \nu_a, \det(\cdot)^{\frac{1}{n-1}}\rangle \leq \det(A(a))^{\frac{1}{n-1}}, \quad \forall a \in T.$$

Therefore, from now on we fix $a \in T$. Consider a cut-off function $\varphi \in C_c^\infty((0,1)^n)$, $0 \leq \varphi \leq 1$. For $k \in \mathbb{N}$ and $R > 0$, we define $B_{k,R}$ over $(0,1)^n$ by

$$B_{k,R}(x) \doteq \varphi(x)A_k(a + Rx) + (1 - \varphi(x))A(a).$$

Remark that $B_{k,R} \equiv A(a)$ over the boundary of $[0,1]^n$, therefore $B_{k,R}$ can be extended smoothly by periodicity to $\mathbb{R}^n$. This defines $B_{k,R}$ over $\mathbb{T}^n$. Notice moreover that $B_{k,R}$ takes values in $\mathrm{Sym}^+(n)$.

$\boxed{\text{Step 2: Monge-Ampère and the main inequality}}$

The argument of this step is the same as the one of [76, Theorem 2.2]. Let $\phi_{k,R} : \mathbb{T}^n \to \mathbb{R}$ be the solution of

$$\det(H\phi_{k,R} + S_{k,R}) = \det(B_{k,R})^{\frac{1}{n-1}} \doteq f_{k,R}, \tag{6.3}$$

where $H\phi_{k,R}(x) + S_{k,R}(x) \in \mathrm{Sym}^+(n), \forall x \in \mathbb{T}^n$, with the constraint

$$\det(S_{k,R}) = \int_{\mathbb{T}^n} f_{k,R}(x)\,dx\,. \tag{6.4}$$

From [55, Theorem 2.2], it is known that the latter is a necessary and sufficient condition to solve the Monge Ampère type equation (6.3). Note that (6.3) is equivalent to

$$\det(H\psi_{k,R}) = f_{k,R}, \tag{6.5}$$

where $H\psi_{k,R}(x)$ is positive definite $\forall x \in \mathbb{T}^n$ and $\psi_{k,R}(x) \doteq \frac{1}{2}x^T S_{k,R}x + \phi_{k,R}(x)$. We can, and will, assume that

$$\psi_{k,R}(a) = \phi_{k,R}(a) = 0, \quad \forall k, R, \tag{6.6}$$

since the solution of (6.5) is determined up to constants (see again [55, Theorem 2.2]). We have

$$f_{k,R} = (f_{k,R}\det(B_{k,R}))^{\frac{1}{n}} = (\det(H\psi_{k,R}B_{k,R}))^{\frac{1}{n}}.$$

Since, for every $x \in \mathbb{T}^n$, $k \in \mathbb{N}$, $R > 0$, $H\psi_{k,R}(x)B_{k,R}(x)$ is the product of two symmetric and positive definite matrices, their product is diagonalizable with positive eigenvalues (see [75, Proposition 6.1]). Dropping the dependence of k, R, x, if we call these eigenvalues $\lambda_1, \dots, \lambda_n$ we can write

$$f_{k,R} = (\det(H\psi_{k,R}B_{k,R}))^{\frac{1}{n}} = (\lambda_1 \dots \lambda_n)^{\frac{1}{n}} \le \frac{\sum_{i=1}^n \lambda_i}{n},$$

where in the last inequality we use the arithmetic-geometric mean inequality. Hence,

$$f_{k,R} \le \frac{\operatorname{tr}(H\psi_{k,R}B_{k,R})}{n}.$$

Using the definition of $\psi_{k,R}$ and rewriting

$$\operatorname{tr}(H\phi_{k,R}B_{k,R}) = \operatorname{div}(B_{k,R}D\phi_{k,R}) - (\operatorname{div}(B_{k,R}), D\phi_{k,R}),$$

we finally get

$$f_{k,R} \le \frac{1}{n}(\operatorname{tr}(B_{k,R}S_{k,R}) + \operatorname{div}(B_{k,R}D\phi_{k,R}) - (\operatorname{div}(B_{k,R}), D\phi_{k,R})). \tag{6.7}$$

We consider $S_{k,R}$ of the form

$$S_{k,R} = \lambda_{k,R}\operatorname{cof}\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right).$$

By (6.4)

$$\lambda_{k,R} = \frac{\left(\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx\right)^{\frac{1}{n}}}{\left(\det(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx)\right)^{\frac{n-1}{n}}}. \tag{6.8}$$

Observing that $\int_{\mathbb{T}^n}\operatorname{div}(B_{k,R}D\phi_{k,R})\,dx = 0$, we integrate (6.7), getting

$$\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}\,dx \le \frac{1}{n}\int_{\mathbb{T}^n}\operatorname{tr}(B_{k,R}S_{k,R})\,dx - \frac{1}{n}\int_{\mathbb{T}^n}(\operatorname{div}(B_{k,R}), D\phi_{k,R}))\,dx. \tag{6.9}$$

We rewrite

$$\begin{aligned}\int_{\mathbb{T}^n}\operatorname{tr}(B_{k,R}S_{k,R})\,dx &= \operatorname{tr}\left(\left(\int_{\mathbb{T}^n} B_{k,R}\,dx\right)S_{k,R}\right) = \lambda_{k,R}\operatorname{tr}\left(\left(\int_{\mathbb{T}^n} B_{k,R}\,dx\right)\operatorname{cof}\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right)\\ &= n\lambda_{k,R}\det\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\\ &\overset{(6.8)}{=} n\left(\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx\right)^{\frac{1}{n}}\left(\det\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right)^{\frac{1}{n}}.\end{aligned}$$

Finally, define also $\gamma_{k,R} \doteq \left(\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx\right)^{\frac{1}{n}}$. By the monotonicity of the determinant and the fact that $A_k(x) \ge \varepsilon\,\mathrm{id}_n, \forall x \in \mathbb{T}^n, \forall k \in \mathbb{N}$, and $A(a) \ge \varepsilon\,\mathrm{id}_n$, we have $B_{k,R} \ge \varepsilon\,\mathrm{id}_n, \forall k, R$, that implies

$$\gamma_{k,R} \ge \varepsilon^{\frac{1}{n-1}}, \quad \forall k, R. \tag{6.10}$$

We divide by $\gamma_{k,R}$ in (6.9), to obtain:

$$\left(\int_{\mathbb{T}^n} \det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx\right)^{\frac{n-1}{n}} \leq \left(\det\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right)^{\frac{1}{n}} - \frac{1}{n\gamma_{k,R}}\int_{\mathbb{T}^n}(\operatorname{div}(B_{k,R}), D\phi_{k,R}))\,dx \tag{6.11}$$

By monotonicity of the determinant we have

$$\int_{\mathbb{T}^n} \varphi(x)^{\frac{n}{n-1}} \det(A_k(a+Rx))^{\frac{1}{n-1}}\,dx \leq \int_{\mathbb{T}^n} \det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx,$$

so that (6.11) becomes

$$\begin{aligned}&\left(\int_{\mathbb{T}^n} \varphi(x)^{\frac{n}{n-1}} \det(A_k(a+Rx))^{\frac{1}{n-1}}\,dx\right)^{\frac{n-1}{n}} \\ &\leq \left(\det\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right)^{\frac{1}{n}} - \frac{1}{n\gamma_{k,R}}\int_{\mathbb{T}^n}(\operatorname{div}(B_{k,R}), D\phi_{k,R}))\,dx\end{aligned} \tag{6.12}$$

thus by denoting

$$I_{k,R} \doteq \int_{\mathbb{T}^n} \varphi(x)^{\frac{n}{n-1}} \det(A_k(a+Rx))^{\frac{1}{n-1}}\,dx,$$

$$II_{k,R} \doteq \det\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right),$$

$$III_{k,R} \doteq \int_{\mathbb{T}^n}(\operatorname{div}(B_{k,R}), D\phi_{k,R}))\,dx,$$

we can put (6.12) in a more compact form:

$$I_{k,R}^{\frac{n-1}{n}} \leq II_{k,R}^{\frac{1}{n}} - \frac{1}{n\gamma_{k,R}} III_{k,R}. \tag{6.13}$$

We will first let $k \to +\infty$ and then $R \to 0$. To this aim, we study separately the three terms.

Step 3: $I_{k,R}$

Denoting $Q_R = a + [0,R]^n$ we have

$$I_{k,R} = \int_{Q_R} \varphi^{\frac{n}{n-1}}\left(\frac{y-a}{R}\right) \det(A_k(y))^{\frac{1}{n-1}} \frac{dy}{R^n}.$$

Since the sequence A_k generates the Young measure ν, by letting $k \to \infty$, we get

$$\lim_{k\to\infty} I_{k,R} = \int_{Q_R} \varphi^{\frac{n}{n-1}}\left(\frac{y-a}{R}\right) \langle \nu_y, \det(\cdot)^{\frac{1}{n-1}}\rangle \frac{dy}{R^n} = \int_{\mathbb{T}^n} \varphi^{\frac{n}{n-1}}(x)\, \langle \nu_{a+Rx}, \det(\cdot)^{\frac{1}{n-1}}\rangle\,dx\,.$$

Finally, since $a \in (0,1)^n$ was a Lebesgue point for the function $x \mapsto \langle \nu_x, \det(\cdot)^{\frac{1}{n-1}}\rangle$, letting $R \to 0$ we achieve

$$\lim_{R\to 0}\lim_{k\to\infty} I_{k,R} = \langle \nu_a, \det(\cdot)^{\frac{1}{n-1}}\rangle \int_{\mathbb{T}^n} \varphi^{\frac{n}{n-1}}(x)\,dx\,.$$

Step 4: $II_{k,R}$

Since $A_k \rightharpoonup A$ in $L^p(\mathbb{T}^n)$, we have

$$\lim_{k\to\infty} II_{k,R} = \det\left(\int_{\mathbb{T}^n} \varphi(x) A(a+Rx)\,dx + A(a)\int_{\mathbb{T}^n} 1-\varphi(x)\,dx\right), \tag{6.14}$$

and since

$$\int_{\mathbb{T}^n} \varphi(x)A(a+Rx)\,dx = \int_{Q_R} \varphi\left(\frac{y-a}{R}\right) A(y)\frac{dy}{R^n}$$

and $|\varphi(x)| \leq 1\ \forall x \in \mathbb{T}^n$, we also get that

$$\left\| \int_{\mathbb{T}^n} \varphi(x)A(a+Rx)\,dx - A(a)\int_{\mathbb{T}^n} \varphi(x)\,dx \right\| \leq \int_{Q_R} \|A(y)-A(a)\|\frac{dy}{R^n}.$$

The last expression tends to 0 as $R \to 0^+$, since a is a Lebesgue point for $x \mapsto A(x)$. Thus, by letting $R \to 0$ in (6.14), we conclude that

$$\lim_{R\to 0}\lim_{k\to\infty} II_{k,R} = \det(A(a)).$$

Step 5: $III_{k,R}$

To prove (6.1), we are just left to show that $\lim_{R\to 0}\lim_{k\to\infty} III_{k,R} = 0$. To do this, we first compute

$$\operatorname{div}(B_{k,R}) = \varphi(x)R\operatorname{div}(A_k)(a+Rx) + (A_k(a+Rx)-A(a))D\varphi(x).$$

Therefore:

$$\begin{aligned} III_{k,R} = R &\int_{\mathbb{T}^n} \varphi(x)(\operatorname{div}(A_k)(a+Rx), D\phi_{k,R})\,dx \\ &+ \int_{\mathbb{T}^n} ((A_k(a+Rx)-A(a))D\varphi, D\phi_{k,R})\,dx\,. \end{aligned}$$

We can use the divergence theorem to rewrite more conveniently the second term:

$$\begin{aligned} &\int_{\mathbb{T}^n} ((A_k(a+Rx)-A(a))D\varphi, D\phi_{k,R})\,dx = \\ &\sum_{i,j}\int_{\mathbb{T}^n} ((A_k)_{ij}(a+Rx)-A_{ij}(a))\partial_j\varphi\partial_i\phi_{k,R}\,dx = \\ &-\sum_{i,j}\int_{\mathbb{T}^n} \partial_i((A_k)_{ij}(a+Rx)-A_{ij}(a))\partial_j\varphi\phi_{k,R}\,dx \\ &-\sum_{i,j}\int_{\mathbb{T}^n} ((A_k)_{ij}(a+Rx)-A_{ij}(a))\partial_{ij}\varphi\phi_{k,R}\,dx = \\ &-R\sum_{i,j}\int_{\mathbb{T}^n} (\partial_i A_k)_{ij}(a+Rx)\partial_j\varphi\phi_{k,R}\,dx \\ &-\sum_{i,j}\int_{\mathbb{T}^n} ((A_k)_{ij}(a+Rx)-A_{ij}(a))\partial_{ij}\varphi\phi_{k,R}\,dx = \\ &-R\int_{\mathbb{T}^n} ((\operatorname{div} A_k)(a+Rx), D\varphi)\phi_{k,R}\,dx \\ &-\int_{\mathbb{T}^n} (A_k(a+Rx)-A(a)), H\varphi)\phi_{k,R}\,dx. \end{aligned}$$

Summarizing, we have

$$\begin{aligned} III_{k,R} = R &\int_{\mathbb{T}^n} \varphi(x)(\operatorname{div}(A_k)(a+Rx), D\phi_{k,R})\,dx \\ &-R\int_{\mathbb{T}^n} ((\operatorname{div} A_k)(a+Rx), D\varphi)\phi_{k,R}\,dx \\ &-\int_{\mathbb{T}^n} (A_k(a+Rx)-A(a)), H\varphi)\phi_{k,R}\,dx\,. \end{aligned}$$

We will denote with:

$$III^1_{k,R} \doteq R\int_{\mathbb{T}^n} \varphi(x)(\operatorname{div}(A_k)(a+Rx), D\phi_{k,R})\,dx,$$
$$III^2_{k,R} \doteq R\int_{\mathbb{T}^n} ((\operatorname{div} A_k)(a+Rx), D\varphi)\phi_{k,R}\,dx,$$
$$III^3_{k,R} \doteq \int_{\mathbb{T}^n} (A_k(a+Rx) - A(a)), H\varphi)\phi_{k,R}\,dx\,.$$

$\boxed{\text{Step 6: estimates on } \phi_{k,R}}$

As remarked in [76, Section 5.2], $\psi_{k,R}$ is convex, for every k, R, and moreover the estimate

$$\|D\phi_{k,R}\|_{L^\infty(\mathbb{T}^n)} \le C\|S_{k,R}\| \tag{6.15}$$

holds for every $k \in \mathbb{N}$ and $R > 0$. We will now show that

$$\limsup_{R\to 0^+}\limsup_{k\to+\infty} \|S_{k,R}\| < +\infty. \tag{6.16}$$

If we do this, we find, through a diagonal argument, a subsequence k_j such that $\phi_{k_j,\frac{1}{m}}$ converges uniformly to a function $\phi_{\frac{1}{m}}$ as $j \to \infty$. Moreover we find a constant $\lambda > 0$ such that

$$\|\phi_{\frac{1}{m}}\|_{C^0(\mathbb{T}^n)} \le \lambda, \quad \forall m \in \mathbb{N}. \tag{6.17}$$

Let us first show how (6.16) implies this last claim. By (6.6) we have $\phi_{k,\frac{1}{m}}(a) = 0, \forall k, m$, and the estimate (6.15) combined with (6.16) tells us that for every fixed m, $\{\phi_{k,\frac{1}{m}}\}_{k\in\mathbb{N}}$ is a precompact subset of $C^0(\mathbb{T}^n)$, hence there exists a diagonal subsequence $\phi_{k_j,\frac{1}{m}}$ that converges uniformly to $\phi_{\frac{1}{m}}$ for every m as $j \to \infty$. Moreover, estimate (6.15) implies that

$$\|\phi_{k_j,\frac{1}{m}}\|_{C^0(\mathbb{T}^n)} \le C\|S_{k_j,\frac{1}{m}}\|, \quad \forall j, m.$$

Therefore, in the limit as $j \to \infty$, we also infer

$$\|\phi_{\frac{1}{m}}\|_{C^0(\mathbb{T}^n)} \le C\limsup_{k\to\infty} \|S_{k,\frac{1}{m}}\|, \quad \forall m$$

and finally

$$\limsup_{m\to\infty} \|\phi_{\frac{1}{m}}\|_{C^0(\mathbb{T}^n)} \le C\limsup_{m\to\infty}\limsup_{k\to\infty} \|S_{k,\frac{1}{m}}\| \overset{(6.16)}{<} +\infty,$$

which finally implies (6.17). Let us prove (6.16). By its definition, we have

$$S_{k,R} = \lambda_{k,R}\operatorname{cof}\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right).$$

Therefore it suffices to prove separately that

$$\limsup_{R\to 0}\limsup_{k\to\infty} \left\|\operatorname{cof}\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right\| < +\infty \tag{6.18}$$

and

$$\limsup_{R\to 0}\limsup_{k\to\infty} \lambda_{k,R} < +\infty. \tag{6.19}$$

We start with (6.18). The weak convergence of A_k implies, as in (6.14) and the subsequent computations, that

$$\lim_{R\to 0}\lim_{k\to\infty} \int_{\mathbb{T}^n} B_{k,R}(x)\,dx = A(a),$$

since $a \in T'$. Hence

$$\limsup_{R\to 0}\limsup_{k\to\infty}\left\|\operatorname{cof}\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right\| = \lim_{R\to 0}\lim_{k\to\infty}\left\|\operatorname{cof}\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right\| = \|\operatorname{cof}(A(a))\| < +\infty,$$

where the last inequality is again justified by $a \in T'$. Finally, we compute (6.19). By definition

$$\lambda_{k,R} = \frac{\left(\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx\right)^{\frac{1}{n}}}{\left(\det(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx)\right)^{\frac{n-1}{n}}}.$$

Analogously to the estimate of $\gamma_{k,R}$ of (6.10), we have

$$\left(\det\left(\int_{\mathbb{T}^n} B_{k,R}(x)\,dx\right)\right)^{\frac{n-1}{n}} \geq \varepsilon^{n-1}.$$

Therefore, to conclude the proof, we just need to show that

$$\limsup_{R\to 0}\limsup_{k\to\infty}\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx < +\infty.$$

First note that

$$A(a) \leq \|A(a)\|\,\mathrm{id}_n,$$

and consequently estimate

$$\begin{aligned}\det(B_{k,R}) &\leq \det(\varphi(x)A_k(a+Rx) + (1-\varphi(x))\|A(a)\|\,\mathrm{id})\\ &= P_{-\varphi(x)A_k(a+Rx)}((1-\varphi(x))\|A(a)\|),\end{aligned}$$

where $P_{-\varphi(x)A_k(a+Rx)}$ is the characteristic polynomial of $-\varphi(x)A_k(a+Rx)$. By the structure of the characteristic polynomial and the subadditivity of the function $t \mapsto t^{\frac{1}{n-1}}$, we can bound

$$\begin{aligned}\det(B_{k,R})^{\frac{1}{n-1}}(x) &\leq |P|^{\frac{1}{n-1}}_{-\varphi(x)A_k(a+Rx)}((1-\varphi(x))\|A(a)\|)\\ &\leq \sum_{i=0}^{n}\left[(1-\varphi(x))^i\|A(a)\|^i M_{n-i}(\varphi(x)A_k(a+Rx))\right]^{\frac{1}{n-1}}.\end{aligned}$$

Since M_{n-i} is $n-i$ homogeneous, $M_{n-i}(\varphi(x)A_k(a+Rx)) = \varphi^{n-i}(x)M_{n-i}(A_k(a+Rx))$. Hence

$$\det(B_{k,R})^{\frac{1}{n-1}}(x) \leq \sum_{i=0}^{n}\left[(1-\varphi(x))^i\|A(a)\|^i\varphi^{n-i}(x)M_{n-i}(A_k(a+Rx))\right]^{\frac{1}{n-1}}.$$

Now observe that, for every $i \in \{0,1,\dots,n\}$,

$$\int_{\mathbb{T}^n}\left[(1-\varphi)^i\varphi^{n-i}M_{n-i}(A_k(a+Rx))\right]^{\frac{1}{n-1}}dx \to \int_{\mathbb{T}^n}\left[(1-\varphi)^i\varphi^{n-i}\right]^{\frac{1}{n-1}}\langle\nu_{a+Rx}, M_{n-i}^{\frac{1}{n-1}}(\cdot)\rangle\,dx$$

as $k\to\infty$, by the Fundamental Theorem of Young measures. Letting $R\to 0^+$, since a is a Lebesgue point for $x \mapsto \langle\nu_x, M^{\frac{1}{n-1}}_{n-i}(\cdot)\rangle$, we find that

$$\begin{aligned}\limsup_{R\to 0^+}\limsup_{k\to\infty}\int_{\mathbb{T}^n}\det(B_{k,R})^{\frac{1}{n-1}}(x)\,dx &\leq \sum_{i=0}^{n}\langle\nu_a, M^{\frac{1}{n-1}}_{n-i}(\cdot)\rangle\int_{\mathbb{T}^n}\left[(1-\varphi(x))^i\|A(a)\|^i\varphi^{n-i}(x)\right]^{\frac{1}{n-1}}dx\\ &\leq \sum_{i=0}^{n}\langle\nu_a, M^{\frac{1}{n-1}}_{n-i}(\cdot)\rangle\|A(a)\|^i,\end{aligned}$$

the last inequality being true since $0 \leq \varphi(x) \leq 1, \forall x \in \mathbb{T}^n$. Clearly the last term is equibounded by our choice $a \in T'$. We are now going to prove that the three terms of $III_{k_j,\frac{1}{m}}$ converge to 0 as

$j \to \infty$ and $m \to \infty$.

Step 7: $III^1_{k_j,\frac{1}{m}}$ and $III^2_{k_j,\frac{1}{m}}$

By (6.15), we know that $\|D\phi_{k_j,\frac{1}{m}}\|_{L^\infty(\mathbb{T}^n)} \leq C\|S_{k_j,\frac{1}{m}}\|$. Hence

$$\begin{aligned}
|III^1_{k_j,\frac{1}{m}}| &= \left|\frac{1}{m}\int_{\mathbb{T}^n} \varphi(x)(\operatorname{div}(A_{k_j})\left(a+\frac{x}{m}\right), D\phi_{k_j,\frac{1}{m}})\, dx\right| \\
&\leq \frac{\|D\phi_{k_j,\frac{1}{m}}\|}{m} \fint_{Q_{\frac{1}{m}}(a)} \varphi\left(m(x-a)\right) \|\operatorname{div}(A_{k_j})\|(x)dx \\
&\leq \frac{C\|S_{k_j,\frac{1}{m}}\|}{m} \fint_{Q_{\frac{1}{m}}(a)} \|\operatorname{div}(A_{k_j})\|(x)dx.
\end{aligned}$$

Recall that we use the notation $\mu_k(E) = \int_E \|\operatorname{div}(A_k)\|(x)\, dx$, for every Borel set $E \subset \mathbb{T}^n$ and for every $k \in \mathbb{N}$. By weak-* convergence of measures, since $Q_{\frac{1}{m}}(a)$ is a compact set, we have (see [31, Theorem 1.40])

$$\begin{aligned}
\limsup_{j\to\infty} \frac{C}{m}\frac{\mu_{k_j}(Q_{\frac{1}{m}}(a))}{(\frac{1}{m})^n} &\leq \frac{C}{m}\frac{\mu(Q_{\frac{1}{m}}(a))}{(\frac{1}{m})^n} \leq \frac{C'}{m}\frac{\mu(B_{\sqrt{2}/m}(a))}{|B_{\sqrt{2}/m}(a)|} \\
&= \frac{C'}{m}\fint_{B_{\sqrt{2}/m}(a))} g(x)\, dx + \frac{C'}{m}\frac{\mu^s(B_{\sqrt{2}/m}(a))}{|B_{\sqrt{2}/m}(a)|},
\end{aligned}$$

for some positive constant C'. Since we chose $a \in T''$, we get that the previous expression converges to 0 as $m \to \infty$. Finally, by (6.16), we also know that

$$\limsup_{R\to 0^+}\limsup_{j\to\infty} \|S_{k_j,R}\| < +\infty,$$

hence $\limsup_{m\to\infty}\limsup_{j\to\infty} III^1_{k_j,\frac{1}{m}} = 0$. The term $III^2_{k_j,\frac{1}{m}}$ is completely analogous.

Step 8: $III^3_{k_j,\frac{1}{m}}$

We finally prove that $\lim_{m\to\infty}\lim_{j\to\infty} III^3_{k_j,\frac{1}{m}} = 0$. We have

$$\begin{aligned}
III^3_{k_j,\frac{1}{m}} &= \int_{\mathbb{T}^n} \left(A_{k_j}\left(a+\frac{x}{m}\right) - A(a)), H\varphi\right) \phi_{k_j,\frac{1}{m}}\, dx \\
&= \int_{\mathbb{T}^n} \left(A_{k_j}\left(a+\frac{x}{m}\right) - A(a), H\varphi\right) (\phi_{k_j,\frac{1}{m}} - \phi_{\frac{1}{m}})\, dx \\
&+ \int_{\mathbb{T}^n} \left(A_{k_j}\left(a+\frac{x}{m}\right) - A(a), H\varphi\right) \phi_{\frac{1}{m}}\, dx\,.
\end{aligned}$$

The first term can be estimated as

$$\begin{aligned}
&\left|\int_{\mathbb{T}^n} \left(A_{k_j}\left(a+\frac{x}{m}\right) - A(a), H\varphi\right) (\phi_{k_j,\frac{1}{m}} - \phi_{\frac{1}{m}})\, dx\right| \\
&\leq \|\phi_{k_j,\frac{1}{m}} - \phi_{\frac{1}{m}}\|_{C^0(\mathbb{T}^n)}\|H\varphi\|_{C^0(\mathbb{T}^n)} \int_{\mathbb{T}^n} \left\|A_{k_j}\left(a+\frac{x}{m}\right) - A(a)\right\| dx \\
&= \|\phi_{k_j,\frac{1}{m}} - \phi_{\frac{1}{m}}\|_{C^0(\mathbb{T}^n)}\|H\varphi\|_{C^0(\mathbb{T}^n)} m^n \int_{Q_{\frac{1}{m}}(a)} \left\|A_{k_j}(x) - A(a)\right\| dx\,.
\end{aligned}$$

Since $h_j(x) \doteq \|A_{k_j}(x) - A(a)\|$ is equibounded in $L^p(Q_{\frac{1}{m}}(a))$ and by the uniform convergence of $\phi_{k_j,\frac{1}{m}}$ to $\phi_{\frac{1}{m}}$, we infer that the last term converges to 0 as $j \to \infty$. On the other hand, by weak L^p convergence,

$$\int_{\mathbb{T}^n} \left(A_{k_j}\left(a+\frac{x}{m}\right) - A(a), H\varphi\right) \phi_{\frac{1}{m}}\, dx \to \int_{\mathbb{T}^n} \left(A\left(a+\frac{x}{m}\right) - A(a), H\varphi\right) \phi_{\frac{1}{m}}\, dx$$

as $j \to \infty$. Now, since a is a Lebesgue point for A, and we can estimate for some constant $\gamma > 0$

$$\left| \int_{\mathbb{T}^n} \left(A\left(a + \frac{x}{m}\right) - A(a), H\varphi \right) \phi_{\frac{1}{m}}\, dx \right| \leq \gamma \int_{\mathbb{T}^n} \left\| A\left(a + \frac{x}{m}\right) - A(a) \right\| dx\,.$$

By definition of Lebesgue point, the last term converges to 0 as $m \to \infty$. This concludes the proof that $\lim_{m\to\infty} \lim_{j\to\infty} III_{k_j,\frac{1}{m}} = 0$.

$\boxed{\text{Step 9: conclusion}}$

Taking the limits in (6.13), we achieve

$$\langle \nu_a, \det(\cdot)^{\frac{1}{n-1}} \rangle \int_{\mathbb{T}^n} \varphi^{\frac{n}{n-1}}(x)\, dx \leq \det(A(a))^{\frac{1}{n-1}}.$$

By letting the cut-off function φ converge to the characteristic function of the torus, we conclude the validity of (6.1) almost everywhere. $\square$

Remark 6.4. By analyzing the proof, it is moreover clear that one could slightly relax the assumptions of the Theorem. Indeed it would suffice to take a sequence $A_k \rightharpoonup A$ in $L^{\frac{n}{n-1}}(\mathbb{T}^n)$ and $\operatorname{div}(A_k) \overset{*}{\rightharpoonup} \operatorname{div}(A)$ such that the sequence of Radon measures defined by

$$\nu_k(E) = \int_E \det(A_k(x))^{\frac{1}{n-1}}\, dx, \quad \forall \text{ Borel } E \subset \mathbb{T}^n$$

weakly-$*$ converges in the sense of measures to a measure ν that is absolutely continuous with respect to the Lebesgue measure. In this case, calling f the density of ν with respect to the Lebesgue measure on $\mathbb{T}^n$, one would prove that

$$f(x) \leq \det(A(x))^{\frac{1}{n-1}} \quad \text{for a.e. } x \in \mathbb{T}^n,$$

and conclude as in the proof of Theorem (6.1). In particular the sequence $\{A_k\}_k$ does not need to be equibounded in L^p for some $p > \frac{n}{n-1}$.

As a simple consequence of the proof of Theorem 6.1, we obtain the following

Corollary 6.5. *Let $p > \frac{n}{n-1}$ and $\{A_k\}_k \subset X_p$ be such that $A_k \rightharpoonup A$ in X_p. Suppose further that $\det(A_k)^{\frac{1}{n-1}} \rightharpoonup g$ in $L^1(\mathbb{T}^n)$. Then we have*

$$g(x) \leq \det(A(x))^{\frac{1}{n-1}},$$

for almost every $x \in \mathbb{T}^n$.

Proof. Fix $\varphi \in C^\infty(\mathbb{T}^n)$ with $\varphi \geq 0$ and note that the sequence $\tilde{A}_k \doteq \varphi A_k$ is in X_p for every k, and $\tilde{A}_k \rightharpoonup \varphi A$ in X_p. Using the hypothesis $\det^{\frac{1}{n-1}}(A_k) \rightharpoonup g$ and applying Theorem 6.1 to the sequence $\tilde{A}_k$, we get

$$\begin{aligned}
\int_{\mathbb{T}^n} g(x)\varphi^{\frac{n}{n-1}}(x)\, dx &= \lim_k \int_{\mathbb{T}^n} \det(A_k)^{\frac{1}{n-1}} \varphi^{\frac{n}{n-1}}(x)\, dx = \lim_k \mathbb{D}(\varphi A_k) \\
&= \lim_k \mathbb{D}(\tilde{A}_k) \leq \limsup_k \mathbb{D}(\tilde{A}_k) \leq \mathbb{D}(\varphi A) \\
&= \int_{\mathbb{T}^n} \det(A(x))^{\frac{1}{n-1}} \varphi^{\frac{n}{n-1}}(x)\, dx\,.
\end{aligned}$$

Since φ was arbitrary, we conclude the proof. $\square$

6.2 THE CASE $p \le \frac{n}{n-1}$

In this section we prove the optimality of the assumptions of Theorem 6.1 and Remark 6.4, by providing an explicit counterexample. In particular, we prove the following

Proposition 6.6. *For every $\varepsilon > 0$ and for every $x_0 \in \mathbb{R}^n$, there exists a sequence of matrix fields A_k such that*

(i) A_k *is compactly supported in* $B_\varepsilon(x_0), \forall k \in \mathbb{N}$;

(ii) $A_k \rightharpoonup 0$ *in* $L^{\frac{n}{n-1}}(\mathbb{R}^n, \mathrm{Sym}^+(n))$ *and strongly in* $L^p(\mathbb{R}^n, \mathrm{Sym}^+(n))$, $\forall p < \frac{n}{n-1}$;

(iii) $\mathrm{div}(A_k) \in \mathcal{M}(\mathbb{T}^n, \mathbb{R}^n)$, $\forall k$ *and* $\sup_{k\in\mathbb{N}} \|\mathrm{div}(A_k)\|_{\mathcal{M}(\mathbb{T}^n,\mathbb{R}^n)} \le 1$;

(iv) $\mathbb{D}(A_k) = \omega_n$, $\forall k$, *so that in particular* $\mathbb{D}(0) = 0 < \limsup_k \mathbb{D}(A_k) = \omega_n$.

Proof. Fix a point $x_0 \in \mathbb{R}^n$ and consider

$$f_k(x) \doteq 2^{k(n-1)}\chi_{B_{2^{-k}}(x_0)}.$$

Define $A_k(x) \doteq f_k(x)\,\mathrm{id}_n$. First we note that $\mathrm{spt}(A_k) \subset B_{2^{-k}}(x_0)$, $\forall k$, so that once $\varepsilon > 0$ is fixed, we can pick k_0 such that if $k \ge k_0$, then (i) is fulfilled by choosing as a sequence $\{A_{k+k_0}\}_{k\in\mathbb{N}}$. Now note that the Hölder conjugate exponent of $\frac{n}{n-1}$ is n. Hence, to see (ii), we compute for any $\varphi \in L^n(\mathbb{R}^n)$

$$\left|\int_{\mathbb{R}^n} f_k(x)\varphi(x)\,dx\right| = 2^{k(n-1)}\left|\int_{B_{2^{-k}(p)}} \varphi(x)\,dx\right| \le \left(\int_{B_{2^{-k}(p)}} |\varphi|^n(x)\,dx\right)^{\frac{1}{n}} \to 0, \text{ as } k \to \infty \quad (6.20)$$

and, if $1 \le p < \frac{n}{n-1}$,

$$\|f_k\|^p_{L^p(\mathbb{R}^n)} = 2^{k(n-1)}2^{-k\frac{n}{p}} = 2^{-k(\frac{n}{p}-n+1)}. \quad (6.21)$$

The last expression converges to 0 as $k \to \infty$ if $p < \frac{n}{n-1}$, thus proving (ii). We turn to (iii). We observe that

$$\mathrm{div}(A_k) = Df_k,$$

where Df_k is the BV derivative of f_k. To compute it, we use the definition. For every $\Phi \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ and for every $k \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^n} f_k(x)\,\mathrm{div}(\Phi(x))\,dx &= 2^{k(n-1)}\int_{B_{2^{-k}}(x_0)} \mathrm{div}(\Phi(x))\,dx \\ &= 2^{k(n-1)}\int_{\partial B_{2^{-k}}(x_0)} (\Phi(z), \nu_k(z))d\sigma(z),\end{aligned}$$

where $\nu_k(z) = \frac{z-x_0}{\|z-x_0\|}$ is the normal to $\partial B_{2^{-k}}(x_0)$. The previous expression can be bounded with

$$\left|\int_{\mathbb{T}^n} f_k(x)\,\mathrm{div}(\Phi(x))\,dx\right| \le \|\Phi\|_{C^0},$$

hence also (iii) is fulfilled. Finally, we prove (iv):

$$\int_{\mathbb{R}^n} \det(A_k(x))^{\frac{1}{n-1}}\,dx = \int_{\mathbb{R}^n} f_k^{\frac{n}{n-1}}(x)\,dx = \int_{B_{2^{-k}}} \left(2^{k(n-1)}\right)^{\frac{n}{n-1}} dx = \omega_n, \quad \forall k.$$

This concludes the proof. $\square$

6.3 APPLICATION: SCALAR CONSERVATION LAW WHOSE DATUM IS A BOUNDED MEASURE

In a recent work [77], Serre and L. Silvestre considered the Cauchy problem for the multi-dimensional Burgers equation

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial y_1}\frac{u^2}{2}+\cdots+\frac{\partial}{\partial y_d}\frac{u^{d+1}}{d+1}=0,\qquad t>0,y\in\mathbb{R}^d. \tag{6.22}$$

Recall that when the initial datum $u(0,\cdot)$ is bounded and measurable, then the Cauchy problem admits a unique bounded entropy solution, see Kruzkhov [53]. The semi-group $(S_t)_{t\geq 0}$ enjoys several properties, among which a comparison principle:

$$(a\leq b)\Longrightarrow(S_ta\leq S_tb).$$

Above all, we have the contraction in L^1-distance:

$$(b-a\in L^1(\mathbb{R}^d))\Longrightarrow\begin{cases} S_tb-S_ta\in L^1(\mathbb{R}^d)\\ \|S_tb-S_ta\|_{L^1}\leq\|b-a\|_{L^1}.\end{cases}$$

The latter property, plus the fact that $L^1\cap L^\infty$ is dense in L^1, allows us to extend by continuity the semigroup to the whole space $L^1(\mathbb{R}^d)$. At this stage, it is however unclear if $t\mapsto u(t)=S_tu(0)$ is an entropy, or even a distributional solution of (6.22); the problem is whether the fluxes $\frac{u^k}{k}$ are locally integrable or not.

This question was solved in [77], where the authors proved even more. For every $p\in[1,\infty)$, the Kruzkhov's semi-group S_t extends continuously from $L^p\cap L^\infty(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. Instead of using the contraction property in the L^1-norm, their result was based upon dispersive estimates

$$\|u(t)\|_{L^q}\leq c_{p,q,d}t^{-\beta(p,q)}\|u(0)\|_{L^p}^{\alpha(p,q)},\qquad\forall q>p, \tag{6.23}$$

for some exponents $\alpha,\beta>0$. In particular, $u(t)$ is a genuine solution of the Cauchy problem in the distributional sense, since every flux $\frac{u^k}{k}$ is locally integrable. In addition, this solution satisfies the entropy inequalities associated with Kruzkhov's entropies.

Going a step further, we may ask whether the Cauchy problem for (6.22) is still well-posed when the initial data are bounded measures, $u(0)\in\mathcal{M}(\mathbb{R}^d)$. This is a delicate question. For instance, it is known (see [56]) that in one space dimension, $d=1$, the solution for $u(0)=\delta$, the Dirac mass at the origin, is not unique. Instead, it is parametrized by a pair (m_-,m_+) with $m_+-m_-=1$ and $m_\pm\geq 0$; each solution is a so-called N-wave, with mass m_+ travelling to the right, and mass $-m_-$ to the left (see [56, Remark 1.2] and references therein). However uniqueness can be recovered whenever $u(0)$ is a non-negative measure, by making the natural requirement that the solution be non-negative too (see [56, Theorem 1.1]).

6.3.1 *Non-uniqueness in several space dimensions*

When $d\geq 2$, the uniqueness faces another obstacle. Equation (6.22) admits a scale invariance: if u is a solution of (6.22), and $\lambda>0$ is a parameter, then

$$u^\lambda(t,y)\doteq\lambda u(\lambda^{\alpha-1}t,\lambda^{\alpha-2}y_1,\ldots,\lambda^{\alpha-d-1}y_d)$$

is another one, with the same mass provided that

$$\alpha=\frac{(d+1)(d+2)}{2d}\,.$$

In particular the semi-group is equivariant with respect to this scaling. If it admitted an extension, say by weak-star continuity (for the vague topology) to $\mathcal{M}(\mathbb{R}^d)$, then the unique solution associated with the datum $u(0)=\delta$ would be self-similar,

$$u(t,y)=t^{\frac{1}{1-\alpha}}U(t^{\frac{\alpha-2}{1-\alpha}}y_1,\ldots,t^{\frac{\alpha-d-1}{1-\alpha}}y_d).$$

Because of (6.23), the profile $U = u(1)$ would belong to $L^1 \cap L^\infty(\mathbb{R}^d)$. Unfortunately, this description is not compatible with an initial data such as $u(0) = \delta$. For a given test function $\phi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u(t,y)\phi(y)\,dy = \int_{\mathbb{R}^d} U(z)\phi(t^{\frac{\alpha-2}{\alpha-1}} z_1, \dots, t^{\frac{\alpha-d-1}{\alpha-1}} z_d)\,dz.$$

Let us make $t \to 0^+$. When $d = 2$, the quantity above tends to

$$\int_{\mathbb{R}^2} U(z)\phi(0, z_2)\,dz$$

and one finds $u(0) = g(z_2)\delta_{z_1=0}$, where

$$g(z_2) = \int_{-\infty}^{+\infty} U(z_1, z_2)\,dz_1.$$

If instead $d \geq 3$, then the exponent $\frac{\alpha-d-1}{\alpha-1}$ is negative and one has

$$\phi(t^{\frac{\alpha-2}{\alpha-1}} z_1, \dots, t^{\frac{\alpha-d-1}{\alpha-1}} z_d) \to 0$$

almost everywhere as $t \to 0^+$. Therefore the integral tends to 0. This means that the mass of the solution escapes at infinity, instead of concentrating at the origin. We have thus proved

Proposition 6.7. *If $d \geq 2$, the multi-D Burgers equation with initial datum $u(0) = \delta_{y=0}$ does not admit a self-similar solution.*

Corollary 6.8. *If an entropy solution exists for the initial datum δ, then it is not unique, unless $d = 1$.*

Proof. Just apply the scaling transformations $u \mapsto u^\lambda$ to such a solution. □

Corollary 6.9. *If $d \geq 2$, the semi-group $(S_t)_{t>0}$ does not admit a weak-star continuous extension to $\mathcal{M}(\mathbb{R}^d)$.*

In other words, the operators S_t are not continuous over $L^1(\mathbb{R}^d)$ equipped with the vague topology.

6.3.2 *Towards existence*

Although uniqueness fails, there remains the question of whether a solution exists when the datum $u(0)$ is a bounded measure (not necessarily positive). Moreover we will assume that

$$\int_{\mathbb{R}^d} \sum_{j=1}^{d} |y_j|^{\frac{q-1}{j}}\, d\|u(0)\| < \infty, \tag{6.24}$$

for some $q \in (1, \frac{3+d}{2} + \frac{1}{d})$. This is due to some technical issues regarding Proposition 6.10. A natural strategy is to consider a sequence of approximate initial data $u_m(0) \in L^1 \cap L^\infty(\mathbb{R}^d)$, such that $u_m(0) \stackrel{*}{\rightharpoonup} u(0)$ in the vague topology, namely the weak-* topology in the space of measures, and to try to show that solutions $u_m(t)$ also converge in some topology to a solution of the equation with initial datum $u(0)$. We may assume that the approximating sequence $u_m(0)$ has compact support and that the mass is preserved in the limit process, namely

$$\lim_{m\to\infty} \int_{\mathbb{R}^d} u_m(0,y)\,dy = u(0)(\mathbb{R}^d) \tag{6.25}$$

Moreover, by (6.24), we may also assume that

$$\sup_m \int_{\mathbb{R}^d} \sum_{j=1}^{d} |y_j|^{\frac{q-1}{j}} |u_m(0,y)|\,dy < \infty \tag{6.26}$$

Such a sequence is also bounded in L^1, and the dispersion inequalities (6.23) guarantee that the associated solutions u_m form a bounded sequence in the spaces $L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $p \in [1, \infty]$. Moreover, since for every $t > 0$, $u_m(t)$ has compact support, by integrating the Burgers equation all over $\mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d} u_m(t,y)\,dy = \int_{R^d} u_m(0,y)\,dy \qquad \forall t > 0. \tag{6.27}$$

Up to the extraction of a subsequence, we may assume that for every continuous function $f : \mathbb{R} \to \mathbb{R}$, the sequence $f \circ u_m$ converges to some $\bar{f}$ in the weak-* topology of L^∞_{loc}. Its limit $\bar{f}$ is given by a Young measure $(\nu_{t,y})_{t>0, y \in \mathbb{R}^d}$,

$$\bar{f}(t,y) = \langle \nu_{t,y}, f \rangle$$

almost everywhere.

The following proposition will imply that the sequence $(u_m)_{m \in \mathbb{N}}$ is *tight*, where we stress that the exponents $\kappa_j \doteq \frac{q-1}{j}$ are positive. We also define $n \doteq d + 1$.

Proposition 6.10. *Let $a \in L^1 \cap L^\infty(\mathbb{R}^d)$ be a function with compact support. Then for $q \in \left(1, \frac{3+d}{2} + \frac{1}{d}\right)$, the functional*

$$I_q[z] \doteq \int_{\mathbb{R}^d} \sum_{j=1}^d |y_j|^{\frac{q-1}{j}} |z(y)|\,dy$$

satisfies

$$I_q(S_t a) \le e^{c_{d,q} t}(I_q[a] + c_{d,q} t^s), \qquad \forall t > 0, \tag{6.28}$$

where

$$s = 1 - \frac{2d(q-1)}{2+dn} > 0.$$

Hereabove $c_{d,q} < \infty$ is a universal constant.

Proof. Let us denote $u(t) = S_t a$. We start from the entropy inequality

$$\frac{\partial |u|}{\partial t} + \frac{\partial}{\partial y_1}\frac{u|u|}{2} + \cdots + \frac{\partial}{\partial y_d}\frac{u^d|u|}{d+1} \le 0,$$

which we multiply by the weight function $w(y) = \sum_{j=1}^d |y_j|^{\kappa_j}$. Integrating by parts, we obtain

$$\frac{d}{dt} I_q[u(t)] \le \sum_{j=1}^d \frac{\kappa_j}{j+1} \int_{\mathbb{R}^d} |y_j|^{\kappa_j - 1} |u|^{j+1}\,dy.$$

Using Hölder and Young Inequalities, we infer

$$\frac{d}{dt} I_q[u(t)] \le \sum_{j=1}^d \frac{\kappa_j}{j+1} I_q[u(t)]^{1-\frac{1}{\kappa_j}} \|u(t)\|_{L^q}^{\frac{q}{\kappa_j}} \le c_{d,q}(I_q[u(t)] + \|u(t)\|_{L^q}^q).$$

We now apply the dispersion inequality (6.23) with $p = 1$, where we have

$$\alpha(1,q) = \frac{2q+dn}{2+dn} \qquad \beta(1,q) = \frac{2d(q-1)}{q(2+dn)}.$$

We obtain

$$\frac{d}{dt} I_q[u(t)] \le c_{d,q}(I_q[u(t)] + t^{-r}), \qquad r \doteq \frac{2d(q-1)}{2+dn}.$$

We conclude by remarking that $0 < r < 1$, so that t^{-r} is integrable over $(0, T)$ for every finite T, thus the estimate (6.28) is just a consequence of the Grönwall's inequality. $\square$

We use the previous Proposition to show that the mass does not escape at infinity in $\mathbb{R}^d$ in the limit process, and therefore to deduce the following:

Corollary 6.11. *Let $u(0)$ be a finite measure of $\mathbb{R}^d$ satisfying (6.24). Let $u_m(0) \in L^1 \cap L^\infty(\mathbb{R}^d)$ be a sequence of functions with compact support, for which (6.25), (6.26) and (6.27) hold. If $u_m(t,y)$ is the solution of (6.22) and $\bar{u}(t,y)$ is its vague limit, then we have*

$$\int_{\mathbb{R}^d} \bar{u}(t,y)g(y)\,dy = \lim_{m\to+\infty} \int_{\mathbb{R}^d} u_m(t,y)g(y)\,dy, \qquad \forall g \in C^0 \cap L^\infty(\mathbb{R}^d), \forall t > 0. \tag{6.29}$$

Moreover

$$\int_{\mathbb{R}^d} \bar{u}(t,y)\,dy = u(0)(\mathbb{R}^d). \tag{6.30}$$

Proof. Note that, by (6.26), we get that $I_q[u_m]$ is uniformly bounded, which implies that the sequence $u_m(t)$ is tight, i.e. fixed $t > 0$, $\forall \varepsilon > 0$, there exists a number $r = r(\varepsilon) > 0$ such that, if $R > r$, then

$$\|u_m(t)\|(\mathbb{R}^d \setminus B_R(0)) = \int_{\mathbb{R}^d \setminus B_R(0)} |u_m(t,y)|\,dy \le \varepsilon, \forall m \in \mathbb{N}.$$

To see this, simply use (6.28) to see that, choosing $r > 1$,

$$\begin{aligned} cr^{\frac{q-1}{d}} \int_{\mathbb{R}^d \setminus B_r(0)} |u_m(t,y)|\,dy &\le \int_{\mathbb{R}^d \setminus B_r(0)} \sum_{j=1}^{d} |y_j|^{\frac{q-1}{j}} |u_m(t,y)|\,dy \\ &\le I_q(u_m(t,\cdot)) = I_q(S_t u_m(0)) \le e^{c_{d,q}t}(I_q[u_m(0)] + c_{d,q}t^s), \end{aligned}$$

where $c > 0$ is a universal constant. Thanks to (6.26), we see that the right hand side of the previous inequality stays uniformly bounded in m, hence we deduce the tightness of the sequence. Now (6.29) is rather easy to check. For any $R > 0$ and any $\varphi \in C_c^\infty(B_{2R}(0))$ with $0 \le \varphi(y) \le 1, \forall y \in \mathbb{R}^d$ and $\varphi \equiv 1$ on $B_R(0)$. Fix moreover $g \in C^0 \cap L^\infty(\mathbb{R}^d)$ and $t > 0$. We have

$$\left| \int_{\mathbb{R}^d} \bar{u}(t,y)g(y)\varphi(y)\,dy - \int_{\mathbb{R}^d} u_m(t,y)g(y)\varphi(y)\,dy \right| \to 0,$$

by definition of vague limit, and

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \bar{u}(t,y)g(y)(1-\varphi(y))\,dy - \int_{\mathbb{R}^d} u_m(t,y)g(y)(1-\varphi(y))\,dy \right| \\ &\le \int_{\mathbb{R}^d} |\bar{u}(t,y)g(y)(1-\varphi(y))|\,dy + \int_{\mathbb{R}^d} |u_m(t,y)g(y)(1-\varphi(y))|\,dy \\ &\le \|g\|_{C^0} \left(\int_{B_R(0)^c} |\bar{u}(t,y)|\,dy + \int_{B_R(0)^c} |u_m(t,y)|\,dy \right) \end{aligned}$$

can be made arbitrarily small by choosing R suitably and by applying the definition of tightness. Therefore, also

$$\left| \int_{\mathbb{R}^d} \bar{u}(t,y)g(y)\,dy - \int_{\mathbb{R}^d} u_m(t,y)g(y)\,dy \right| \to 0, \text{ as } m \to \infty.$$

By choosing $g \equiv 1$ in (6.29) and by using (6.25) together with (6.27), we finally achieve

$$\int_{\mathbb{R}^d} \bar{u}(t,y)\,dy = \lim_{m\to\infty} \int_{\mathbb{R}^d} u_m(t,y)\,dy = \lim_{m\to\infty} \int_{R^d} u_m(0,y)\,dy = u(0)(\mathbb{R}^d)$$

which concludes the proof. ☐

Applying Corollary 6.5

Of course, the tightness is too weak a property to imply a pointwise convergence of the sequence $(u_m)_{m\in\mathbb{N}}$. Thus passing to the limit in (6.22) tells us only that $\nu = (\nu_{t,y})_{t>0,y\in\mathbb{R}^d}$ is a so-called measure-valued solution, but it does not tell us whether the vague limit $\bar{u}$ is a true entropy solution of (6.22), or not. To go further, we need to establish a compactness property. This is where Corollary 6.5 might be useful, in the spirit of Tartar's strategy by Compensated Compactness [85].

We begin by defining a family of symmetric tensors of size $(d+1)\times(d+1)$. For every continuous function $g:\mathbb{R}\to\mathbb{R}$, we form the map

$$s \mapsto T_g(s) \doteq \left(f_{ij}(s)\right)_{0\le i,j\le d} \in \mathrm{Sym}(n), \qquad f_{ij}(s) = \int_0^s r^{i+j} g(r)\,dr.$$

Our tensors are $A_{g,m} \doteq T_g \circ u_m$. Each row of $A_{g,m}$ is an entropy-entropy flux pair of the Burgers equation, and it is proved in [77] that the sequence $\mathrm{div}_{t,y}\, A_{g,m}$ is bounded in $\mathcal{M}(\mathbb{R}_+\times\mathbb{R}_d)$. Because of the L^∞-bound in (6.23), $A_{g,m}$ is also bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}^d)$. The tensors are positive semi-definite whenever g has the property that the quadratic forms

$$P\in\mathbb{R}_d[X] \longmapsto \int_0^s P(r)^2 g(r)\,dr$$

is positive semi-definite, for every $s\in\mathbb{R}$. Hereabove, $\mathbb{R}_d[X]$ denotes the space of polynomials of degree $\le d$. The set of such functions g is denoted by Z_d^+.

Applying Corollary 6.5 to the sequence $(A_{g,m})_{m\in\mathbb{N}}$ in $\mathbb{R}_+\times\mathbb{R}^d$ (in particular, we have $n=1+d$), we find that the Young measure satisfies

$$\langle \nu_{t,y}, (\det T_g)^{\frac{1}{d}}\rangle \le \left(\det\langle \nu_{t,y}, f_{ij}\rangle_{0\le i,j\le d}\right)^{\frac{1}{d}}, \text{ for a.e. } t>0, y\in\mathbb{R}^d. \tag{6.31}$$

Remembering that $\langle \nu_{t,y}, \mathbf{1}\rangle = 1$ ($\nu_{t,y}$ is a probability measure), the inequality above can be rewritten in terms of the tensor product $\nu\otimes\cdots\otimes\nu$ of n copies of ν, which acts on functions of the variable $(s_0,\dots,s_d)\in\mathbb{R}^n$. To illustrate this claim, we consider the case $d=1$ (hence $n=2$) ; then the inequality above rewrites as

$$\langle \nu\otimes\nu, [f_{00}][f_{11}] - [f_{01}]^2\rangle \le 0, \tag{6.32}$$

where the brackets mean $[f](s_0,s_1) \doteq f(s_1) - f(s_0)$. We point out that

$$[f_{ij}](s_0,s_1) = \int_{s_0}^{s_1} r^{i+j} g(r)\,dr.$$

As observed by several authors, including L. Tartar or G.Q. Chen & Y.-G. Lu, (6.32) can be exploited as follows: if g does not vanish on any interval, then the function $(s_0,s_1)\mapsto [f_{00}][f_{11}] - [f_{01}]^2$ is positive away from the diagonal Δ. Thus the support of $\nu\otimes\nu$ is contained in Δ, and this implies that ν is a Dirac mass. Hence the sequence $(u_m)_{m\in\mathbb{N}}$ converges strongly in every space L^p_{loc} for $1\le p<\infty$. We notice that in this case, it is sufficient to work with a single function g.

When the space dimension d is larger, we still have an inequality of the form

$$\langle \nu^{\otimes n}, F_g\rangle \le 0, \qquad \forall g\in Z_d^+, \tag{6.33}$$

where F_g is a symmetric function, defined in terms of g. The map $g\mapsto F_g$ is homogeneous of degree n. By construction, F_g vanishes along the diagonal Δ. Because of the symmetry, we infer that $DF_g \equiv 0$ along Δ, and that

$$D^2F_g(s,\dots,s) = \alpha(s) I_n + \beta(s) M,$$

where $M_{ij} \doteq 1, \forall 1 \leq i, j \leq n$, and for some α, β that satisfy $\alpha(s) + n\beta(s) = 0$. The latter matrix is positive or negative semi-definite, depending on the sign of $\alpha(s)$. For $d \geq 2$, $\alpha(s)$ turns out to be negative, thus F_g is negative in a neighborhood of Δ. Therefore we cannot argue as in the one-dimensional case.

The situation described above resembles that encountered in the analysis of hyperbolic 2×2 systems of conservation laws, where compensated compactness implies a family of identity, parametrized by the entropy-flux pairs. A single identity does not imply that the Young measure is a Dirac mass, but the whole family does, under a hypothesis of genuine nonlinearity, see [26]. Here a single inequality cannot imply that the Young measure is a Dirac mass, but there is some hope that the whole family (6.33) does.

Part III

SHARP ENERGY REGULARITY FOR EULER EQUATIONS

7 CONVEX INTEGRATION AND ENERGY REGULARITY FOR EULER EQUATIONS

This chapter is devoted to the study of the sharp energy regularity for solutions of the incompressible Euler Equations in the spatial periodic setting $\mathbb{T}^3$:

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + Dp = 0, \\ \operatorname{div} v = 0, \end{cases} \qquad \text{in } \mathbb{T}^3 \times [0,T], \tag{7.1}$$

where $v : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ represents the velocity of an incompressible fluid, $p : \mathbb{T}^3 \times [0,T] \to \mathbb{R}$ is the hydrodynamic pressure, with the constraint $\int_{\mathbb{T}^3} p\,dx = 0$ which guarantees its uniqueness. The kinetic energy of a weak solution $v \in L^2(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3)$ to (7.1) is defined as

$$e_v(t) \doteq \int_{\mathbb{T}^3} \|v\|^2(x)dx.$$

Isett proved in [43] that, given any solution $v \in L^\infty((0,T); C^\theta(\mathbb{T}^3))$, the associated kinetic energy e_v satisfies

$$|e_v(t) - e_v(s)| \leq C\,|t-s|^{\frac{2\theta}{1-\theta}}, \qquad \forall t,s \in [0,T], \tag{7.2}$$

Isett and Oh conjectured in [46, Conjecture 1] that this exponent is optimal:

Conjecture. *For any $\theta < \frac{1}{3}$, there exists a solution to (7.1) in the class $v \in C^\theta(\mathbb{R} \times \mathbb{T}^n)$ whose energy profile $e(t)$ fails to have any regularity above the exponent $\frac{2\theta}{1-\theta}$, in the sense that $e_v(t) \notin W^{\frac{2\theta}{1-\theta}+\rho,p}(I)$, for every $\rho > 0$, $p \geq 1$ and every open time interval $I \subset \mathbb{R}$. Furthermore, the set of all such solutions v with the above property is residual (in the sense of category) within the space of all weak solutions to (7.1) in the class $e_v \in C^\theta(\mathbb{R} \times \mathbb{T}^n)$ when the latter space is endowed with the topology from the C^θ norm.*

The main result of this part is Theorem 7.5, where Baire Theorem A.4 is applied similarly to Theorem 5.1 to construct the desired counterexample. To construct *approximate* counterexamples to the statement in Theorem 7.4, we use a convex integration scheme very close to the one introduced in [11]. We will first give the statement of the main inductive Proposition 7.3, and use it to prove the aforementioned theorems in Section 7.2. After this, we prove the inductive proposition. Finally, in Section 7.5, we explain why the results presented are not sufficient to prove the second part of the above Conjecture, i.e. the residuality in C^θ.

7.1 PRELIMINARIES

We start by introducing the notation and some basic properties of the incompressible Euler equations.

7.1.1 *Notation*

In the following $N \in \mathbb{N}$, $\alpha \in (0,1)$ and κ is a multi-index. We introduce the usual (spatial) Hölder norms as follows. First of all, the supremum norm is denoted by $\|f\|_0 \doteq \sup_{\mathbb{T}^3 \times [0,T]} |f|$. We define the Hölder seminorms as

$$[f]_N = \max_{|\kappa|=N} \|D^\kappa f\|_0\,,$$

$$[f]_{N+\alpha} = \max_{|\kappa|=N} \sup_{x \neq y, t} \frac{|D^\kappa f(x,t) - D^\kappa f(y,t)|}{|x-y|^\alpha}\,,$$

where D^κ are space derivatives only. The Hölder norms are then given by

$$\begin{aligned} \|f\|_N &= \sum_{j=0}^{N} [f]_j \\ \|f\|_{N+\alpha} &= \|f\|_N + [f]_{N+\alpha}. \end{aligned}$$

Moreover, we will write $[f(t)]_\alpha$ and $\|f(t)\|_\alpha$ when the time t is fixed and the norms are computed for the restriction of f to the t-time slice. On the other hand we will explicitly write $\|f\|_{C^\alpha_{x,t}}$ when the Hölder norm is computed in both the space and time variables.

Let $\varphi \in C^\infty_c(B_1(0))$ be a standard non negative kernel such that $\int_{B_1(0)} \varphi(x)\,dx = 1$. For any $\delta > 0$ we define $\varphi_\delta \doteq \delta^{-3}\varphi\left(\frac{x}{\delta}\right)$ and we denote the mollifications of a function f as usual as

$$f_\delta \doteq f * \varphi_\delta.$$

We recall the following standard estimates on the mollification of both Hölder continuous functions and vector fields.

Proposition 7.1. *For any $\theta \in (0,1)$ we have*

$$\|f_\delta - f\|_0 \leq \delta^\theta [f]_\theta. \tag{7.3}$$

Moreover, for any $N \geq 0$, there exists a constant $C > 0$ depending on N, such that

$$\|f_\delta * f_\delta - (f*f)_\delta\|_N \leq C\delta^{2\theta-N}[f]^2_\theta, \tag{7.4}$$

$$\|f_\delta\|_{N+1} \leq C\delta^{\theta-N-1}[f]_\theta. \tag{7.5}$$

In the proof of Theorem 7.5, we will make use of the following technical result that show an improvement of the Hölder time regularity of weak solutions of (7.1). We defer its proof to Appendix D.

Proposition 7.2. *Let $u, v : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ be two weak solutions of (7.1) such that $u, v \in C^0(([0,T]; C^\theta(\mathbb{T}^3))$ for some $\theta \in (0,1)$. Then there exists a constant $C > 0$, depending only on θ, $\|u\|_\theta$ and $\|v\|_\theta$, such that*

$$\|u - v\|_{C^\theta_{x,t}} \leq C\|u - v\|_\theta.$$

Finally, we also recall that equations (7.1) are invariant under the following transformation

$$v(x,t) \mapsto v_\Gamma(x,t) \doteq \Gamma v(x, \Gamma t) \quad \text{and} \quad p(x,t) \mapsto p_\Gamma(x,t) \doteq \Gamma^2 p(x, \Gamma t), \tag{7.6}$$

for any $\Gamma > 0$, meaning that if (v,p) solves (7.1) in $\mathbb{T}^3 \times [0,T]$ then (v_Γ, p_Γ) solves (7.1) in $\mathbb{T}^3 \times [0, T/\Gamma]$.

7.1.2 *Inductive proposition*

As said, the proof of the main results are based on a modification of the convex integration scheme of [11], that we are now going to explain.

Let $q \geq 0$ be a natural number. At a given step q we assume to have a smooth triple $(v_q, p_q, \mathring{R}_q)$ solving the Euler-Reynolds system, namely such that

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0\,, \end{cases} \tag{7.7}$$

to which we add the constraints

$$\operatorname{tr} \mathring{R}_q = 0\,, \tag{7.8}$$

$$\int_{\mathbb{T}^3} p_q(x,t)\, dx = 0\,. \tag{7.9}$$

To measure the size of the approximate solution v_q and the error $\mathring{R}_q$, we use a frequency λ_q and an amplitude δ_q, defined through these relations:

$$\lambda_q = 2\pi\lceil a^{(b^q)}, \rceil \tag{7.10}$$

$$\delta_q = \lambda_q^{-2\beta}, \tag{7.11}$$

where $\lceil x \rceil$ denotes the smallest integer $n \geq x$, $a > 1$ is a large parameter, $b > 1$ is close to 1 and $0 < \beta < 1/3$. The parameters a and b will depend on β and on other quantities. We proceed by induction, assuming the estimates

$$\|\mathring{R}_q\|_0 \leq \delta_{q+1}\lambda_q^{-3\alpha} \tag{7.12}$$

$$\|v_q\|_1 \leq M\delta_q^{1/2}\lambda_q \tag{7.13}$$

$$\|v_q\|_0 \leq 1 - \delta_q^{1/2} \tag{7.14}$$

$$\delta_{q+1}\lambda_q^{-\alpha} \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2\, dx \leq \delta_{q+1} \tag{7.15}$$

where $0 < \alpha < 1$ is a small parameter to be chosen suitably, in dependence of β and other quantities, and M is a universal constant.

For any real number $0 < \beta < 1/3$ we will denote

$$\beta^* = \frac{2\beta}{1-\beta}.$$

Note that β^* is an increasing function of β and it satisfies $0 < \beta^* < 1$. We now state the main inductive proposition

Proposition 7.3. *There exists a universal constant M with the following property. Let $0 < \beta < \eta < 1/3$, $E > 0$, and*

$$1 < b < \sqrt{\frac{\eta^*}{\beta^*}}. \tag{7.16}$$

Then there exists an α_0 depending on β, η and b, such that for any $0 < \alpha < \alpha_0$ there exists an a_0 depending on β, b, α, η, E and M, such that for any $a \geq a_0$ the following holds: given a triple $(v_q, p_q, \mathring{R}_q)$ solving (7.7)-(7.9) and satisfying the estimates (7.12)–(7.15) for some strictly positive $e \in C^{\eta^*}([0,T])$ with

$$\|e\|_{\eta^*} \leq E,$$

there exists a solution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ to (7.7)-(7.9) satisfying (7.12)–(7.15) for the same function e with q replaced by $q+1$. Moreover, we have

$$\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}}\|v_{q+1} - v_q\|_1 \leq M\delta_{q+1}^{1/2}. \tag{7.17}$$

The reader may notice that there are four main differences with respect to [11, Proposition 2.1]. First of all the statement is fomulated in a slightly different way than in [11, Proposition 2.1], in order to highlight the fact that the parameter a_0 is uniform once one has chosen the $C^{\eta^*}([0,T])$ norm of e. Moreover, we drop the smoothness hypothesis on the function e, we allow the parameter a_0 to depend on E and finally we suppose in (7.16) a different relation between the parameters b and β. Notice that our relation (7.16) is more restrictive than the one used in [11], indeed we have

$$1 < b < \sqrt{\frac{\eta^*}{\beta^*}} < \sqrt{\frac{1}{\beta^*}} = \sqrt{\frac{1-\beta}{2\beta}} < \frac{1-\beta}{2\beta}. \tag{7.18}$$

7.1.3 *Main results*

The first Theorem we state shows how it is possible to construct *approximate* counterexamples to the fact that the energy is more regular than $C^{\frac{2\theta}{1-\theta}}$. This is, in Gromov's terminology, an *h-principle*.

Theorem 7.4. *Fix $\gamma > 0$ and $\theta \in (0, 1/3)$ such that $\frac{2\theta}{1-\theta} + \gamma < 1$. For every strictly positive $e \in C^{\frac{2\theta}{1-\theta}+\gamma}([0, T])$, there exists a vector field $v \in C^\theta(\mathbb{T}^3 \times [0, T])$ that solves (7.1) in the distributional sense and such that*

$$e(t) = \int_{\mathbb{T}^3} |v|^2(x, t)\, dx, \qquad \forall t \in [0, T].$$

This type of result already appeared in [45] for any $\theta \in (0, 1/5)$. Once again, Baire Theorem A.4 allows us to construct exact counterexamples. We define

$$X_\theta = \overline{\left\{ v \in \bigcup_{\theta' > \theta} C^{\theta'}(\mathbb{T}^3 \times [0, T]) \,:\, v \text{ weakly solves (7.1)} \right\}}^{\|\cdot\|_{C^\theta_{x,t}}}, \tag{7.19}$$

endowed with the distance

$$\mathrm{d}(u, v) \doteq \|u - v\|_{C^\theta_{x,t}}.$$

It is clear that (X_θ, d) is a complete metric space. We also define

$$Y_\theta = \left\{ v \in X_\theta \,:\, e_v \in C^{\frac{2\theta}{1-\theta}}([0, T]) \setminus \bigcup_{\gamma > 0} W^{\frac{2\theta}{1-\theta}+\gamma, 1}(I), \textit{for any interval } I \subset [0, T] \right\}. \tag{7.20}$$

The main result is:

Theorem 7.5. *For any $\theta \in (0, 1/3)$, the set Y_θ is residual in X_θ.*

An immediate corollary of Theorem 7.5 is that, for every $\theta \in (0, 1/3)$, there exists a weak solution v of (7.1) such that $e_v \in C^{\theta^*}([0, T])$ is the sharp regularity of e_v, or more precisely

$$e_v \notin W^{\theta^* + \gamma, p}(I),$$

for any $\gamma > 0, p \geq 1$, subinterval $I \subset [0, T]$, where we identify $W^{\alpha, \infty}(I) = C^\alpha(I)$. This can be deduced from Theorem 7.5 by exploiting the simple embeddings

$$W^{\alpha+\gamma, p}(I) \subset W^{\alpha + \frac{\gamma}{2}, 1}(I), \ \forall \alpha \in (0, 1),\ p \geq 1,\ \gamma > 0,\ I \subset \mathbb{R}.$$

This result answers to the first part of [46, Conjecture 1].

7.2 PROOF OF THE MAIN THEOREMS

In this section we prove our two main theorems. As in [11], the proof of Theorem 7.4 is a direct consequence of Proposition 7.3 and we are going to prove it for the reader's convenience. Theorem 7.5 will still be an application of the iterative proposition. indeed, through a h-principle comparable to [11, Theorem 1.3], we will be able to write the set Y^c_θ as a countable union of closed set with empty interior.

7.2.1 *Proof of Theorem 7.4*

First of all, fix γ, θ and e as in the statement of the theorem. In order to apply Proposition 7.3 we choose $\eta \in (0, 1/3)$ to be the only solution of $\eta^* = \theta^* + \gamma$ and β such that $\theta < \beta < \eta$. Consequently we also fix the parameters b and α appearing in the statement of Proposition 7.3, the first satisfying (7.16) and the second lower than the threeshold α_0. As done in [11, Proof of

Theorem 1.1], by using the invariance of the Euler equations under the rescaling (7.6) we can further assume that the energy profile satisfies

$$\delta_1 \lambda_0^{-\alpha} \leq \inf_t e(t) \leq \sup_t e(t) \leq \delta_1.$$

Then we can apply inductively Proposition 7.3 starting with the triple $(v_0, p_0, \mathring{R}_0) = (0,0,0)$. indeed v_0 and $\mathring{R}_0$ trivially satisfy estimates (7.12)-(7.14) and by the rescaling on the energy we also get (7.15) for $q = 0$. By (7.17) we have

$$\sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_\theta \lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_0^{1-\theta} \|v_{q+1} - v_q\|_1^\theta \lesssim \sum_{q=0}^{\infty} \delta_{q+1}^{1/2} \lambda_{q+1}^\theta \lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{\theta-\beta} < \infty \tag{7.21}$$

and hence v_q converges in $C^0([0,T]; C^\theta(\mathbb{T}^3))$ to a function v. Moreover, by [12, Theorem 1.1], we have that $v \in C^\theta(\mathbb{T}^3 \times [0,T])$. By taking the divergence of the first equation in (7.7), we get that p_q is the unique 0-average solution of

$$-\Delta p_q = \operatorname{div}\operatorname{div}(v_q \otimes v_q - \mathring{R}_q)$$

and since $v_q \otimes v_q - \mathring{R}_q \to v \otimes v$ uniformly, p_q is also converging to some function p in $L^r(\mathbb{T}^3 \times [0,T])$, for any $r < \infty$. Hence it is clear that the limit couple (v,p) solves (7.1) in the distributional sense. Finally, by (7.15), as $q \to \infty$, we also get

$$e(t) = \int_{\mathbb{T}^3} |v|^2(x,t)\, dx \quad \forall t \in [0,T],$$

which concludes the proof of the theorem.

7.2.2 *Proof of Theorem 7.5*

We want to show that Y_θ^c is meager in X_θ. First, enumerate the intervals with rational endpoints inside $[0,T]$, $(I_r)_{r\in\mathbb{N}}$. By (7.20) we can write

$$Y_\theta^c = \bigcup_{m,n \in \mathbb{N}} C_{m,n,r},$$

where

$$C_{m,n,r} \doteq \left\{ v \in X_\theta \,:\, \|e_v\|_{W^{\theta^* + \frac{1}{m},1}(I_r)} \leq n \right\}.$$

It is easily seen that $C_{m,n,r}$ are closed subsets of X_θ. Suppose, by contradiction, that there exist $\overline{m}, \overline{n}, \overline{r}$ such that $C_{\overline{m},\overline{n},\overline{r}}$ has a nonempty interior. Thus there exists $\varepsilon > 0$ and $u_0 \in C_{\overline{m},\overline{n},\overline{r}}$ such that

$$B_\varepsilon(u_0) \doteq \{v \in X_\theta : \|v - u_0\|_{C^\theta_{x,t}} \leq \varepsilon\} \subset C_{\overline{m},\overline{n},\overline{r}}. \tag{7.22}$$

By the definition of X_θ, we can find a solution of (7.1), $u \in C^{\theta'}(\mathbb{T}^3 \times [0,T])$, $\theta' > \theta$, such that $\|u - u_0\|_{C^\theta_{x,t}} \leq \frac{\varepsilon}{3}$. Moreover, (7.22) implies that

$$B_{\frac{\varepsilon}{2}}(u) \subset C_{\overline{m},\overline{n},\overline{r}}. \tag{7.23}$$

From now on, we assume that

$$\theta^* < (\theta')^* < \theta^* + \frac{1}{2\overline{m}}. \tag{7.24}$$

This can be done simply by choosing a possibly smaller θ' and exploiting the embedding $C^\alpha(\mathbb{T}^3 \times [0,T]) \subset C^\beta(\mathbb{T}^3 \times [0,T])$, for any $\beta \leq \alpha$. Now fix parameters $\theta'', \beta, \eta > 0$ such that $\theta < \theta' < \theta'' < \beta < \eta$ and for which $\eta^* < \theta^* + \frac{1}{2\overline{m}}$. This can be done in view of (7.24). Fix moreover a function (of time only) $f \in C^{\eta^*}([0,T]) \setminus \bigcup_{\gamma>0} W^{\eta^*+\gamma,1}(I_r)$, such that $1/2 \leq f \leq 1$ and set

$$e(t) = \int_{\mathbb{T}^3} |u|^2 \, dx + \frac{\rho}{2} f(t), \tag{7.25}$$

for some small parameter $\rho > 0$. These choices imply that the energy $e = e(t)$ satisfies

$$e \notin W^{\theta^* + \frac{1}{\bar{m}},1}(I_r). \tag{7.26}$$

Now we claim that, if ρ is chosen sufficiently small, depending on $\theta, \theta', \theta'', \beta, \eta$ and $\bar{m}$, then there exists a solution of (7.1) $v \in C^{\theta''}(\mathbb{T}^3 \times [0,T])$ such that

$$\|u - v\|_{C^\theta_{x,t}} \leq \frac{\varepsilon}{3}, \tag{7.27}$$

$$e_v(t) = e(t), \quad \forall t \in [0,T]. \tag{7.28}$$

It is clear that the claim implies a contradiction with (7.23). indeed, since $\theta'' > \theta$, we have $v \in X_\theta$. Therefore, by (7.23) and (7.27), we get $e_v \in W^{\theta^* + \frac{1}{\bar{m}},1}(I_r)$, but this is in contradiction with (7.28) and (7.26). This would conclude the proof of the present theorem, hence we are only left with the proof of the claim.

To prove the claim, we want to apply Proposition 7.3. First, as in the proof of Theorem 7.4, we use the rescaling (7.6) on u with $\Gamma = \min\{(2\|u\|_0)^{-1}, 1\}$ to obtain a new solution $\tilde{u} \in C^{\theta'}(\mathbb{T}^3 \times [0, T/\Gamma])$. If $\|u\|_0 = 0$, we work with the convention that $\Gamma = 1$. For every map $w \in C^{\theta'}(\mathbb{T}^3 \times [0,T])$, we denote with $\tilde{w}$ map obtained through the rescaling (7.6) with Γ defined above. Notice that there exist constants $c_1(\|u\|_0), c_2(\|u\|_0) > 0$ such that

$$c_1 \|\tilde{w}_1 - \tilde{w}_2\|_{C^{\theta'}_{x,t}} \leq \|w_1 - w_2\|_{C^{\theta'}_{x,t}} \leq c_2 \|\tilde{w}_1 - \tilde{w}_2\|_{C^{\theta'}_{x,t}}, \quad \forall w_1, w_2 \in C^{\theta'}(\mathbb{T}^3 \times [0,T]), \tag{7.29}$$

and that

$$e_{\tilde{w}}(t) = \Gamma^2 e_w(\Gamma t), \quad \forall t \in [0, T/\Gamma], \forall w \in C^{\theta'}(\mathbb{T}^3 \times [0,T]). \tag{7.30}$$

Therefore, we also define

$$\tilde{e}(t) \doteq \Gamma^2 e(\Gamma t), \quad \forall t \in [0, T/\Gamma]. \tag{7.31}$$

Moreover, Proposition 7.3 requires a smooth starting triple. For this reason we consider a space-time mollification of $\tilde{u}$, $u_\delta \doteq (\tilde{u} * \varphi_\delta) * \psi_\delta$, where φ_δ and ψ_δ are standard mollifiers in space and time respectively and $\delta > 0$ is a parameter that will be fixed later on. Of course, u_δ is smooth and solves the following Euler-Reynolds system

$$\partial_t u_\delta + \operatorname{div}(u_\delta \otimes u_\delta) + \nabla p_\delta = \operatorname{div} \mathring{R}_\delta,$$

where $\mathring{R}_\delta \doteq u_\delta \mathring{\otimes} u_\delta - (\tilde{u} \mathring{\otimes} \tilde{u})_\delta$ and the trace part of the commutator $u_\delta \otimes u_\delta - (\tilde{u} \otimes \tilde{u})_\delta$ is inside the pressure p_δ.

We now want to take $(u_\delta, p_\delta, R_\delta)$ as a starting point for the iterative scheme given by Proposition 7.3. In order to do so, we need to guarantee estimates (7.12), (7.13), (7.14) and to find $\rho > 0$ for which also (7.15) is satisfied with $q = 0$. Recall the definition of λ_q and δ_q of (7.11) and (7.10). We make the following choice of the parameters

$$\delta \doteq \left(\delta_1 \lambda_0^{-4\alpha}\right)^{\frac{1}{2\theta'}} \quad \text{and } \rho \doteq \frac{\delta_1}{\Gamma^2}.$$

Notice that with this choice, obviously both δ and ρ depend on the parameters appearing in Proposition 7.3. In particular the energy profile depends on a, but this will not be a problem since we will bound $\|e\|_{\eta^*}$ independently of a, see also Remark 7.6 for a more thorough explanation.

Finally, we will use another parameter $\sigma > 0$ to measure the (small) distance between u_δ and the solution given by Proposition 7.3. We start with (7.14). Using (7.3) and the rescaling, we get

$$\|u_\delta\|_0 \leq \|u_\delta - \tilde{u}\|_0 + \|\tilde{u}\|_0 \leq C\delta^{\theta'} + \frac{1}{2} \leq C\lambda_0^{-2\alpha}\delta_1^{1/2} + \frac{1}{2},$$

where $C = C(\|u\|_{C^{\theta'}_{x,t}}) > 0$.It is clear that we can find a sufficiently large a such that

$$C\lambda_0^{-2\alpha}\delta_1^{1/2} + \frac{1}{2} \leq 1 - \delta_1^{1/2}. \tag{7.32}$$

Therefore, (7.14) is fulfilled. Let us now show (7.12) and (7.13). First, by (7.4), we have

$$\|\mathring{R}_\delta\|_0 \lesssim \delta^{2\theta'} = \delta_1\lambda_0^{-4\alpha},$$

so that again if $\alpha > 0$ is fixed, then (7.12) holds for $q = 0$ if a is large enough. Moreover, through (7.3),

$$\|u_\delta\|_1 \lesssim \delta^{\theta'-1} = (\delta_1\lambda_0^{-4\alpha})^{\frac{\theta'-1}{2\theta'}},$$

and using the definition of δ_q and λ_q, one verifies that (7.13) holds if a is large enough and $b > 1$ is chosen in such a way that

$$b < \frac{(\theta')^*}{\beta^*} - \frac{2\alpha}{\beta}. \tag{7.33}$$

But since $\beta < \theta'$, if α is sufficiently small (depending on b, β and θ') there exists $b > 1$ sufficiently close to 1 such that (7.33) holds. We are left with the estimate on the energy (7.15). By using (7.4), we estimate

$$\begin{aligned}\tilde{e}(t) - \int_{\mathbb{T}^3} |u_\delta|^2\,dx &= \int_{\mathbb{T}^3} |\tilde{u}|^2\,dx + \frac{\delta_1}{2}f(\Gamma t) - \int_{\mathbb{T}^3} |u_\delta|^2\,dx = \int_{\mathbb{T}^3} \left(\left(|\tilde{u}|^2\right)_\delta - |u_\delta|^2\right)dx + \frac{\delta_1}{2}f(\Gamma t) \\ &\leq C\delta^{2\theta'} + \frac{\delta_1}{2} \leq C\delta_1\lambda_0^{-4\alpha} + \frac{\delta_1}{2},\end{aligned}$$

where the second equality is true in view of the fact that the mollification preserves the mean of every periodic function. If a is large enough,

$$C\delta_1\lambda_0^{-4\alpha} + \frac{\delta_1}{2} \leq \delta_1,$$

hence the upper bound of (7.15) holds. Similarly we have

$$\int_{\mathbb{T}^3} \left(\left(|\tilde{u}|^2\right)_\delta - |u_\delta|^2\right)dx + \frac{\delta_1}{2}f(\Gamma t) \geq -C\delta^{2\theta'} + \frac{\delta_1}{4} = -C\delta_1\lambda_0^{-4\alpha} + \frac{\delta_1}{2} \geq \delta_1\lambda_0^{-\alpha},$$

where, to guarantee the last inequality, we took again the parameter a large enough. Now we observe that, since $\delta_1 \leq 1$ for any choice of the parameters,

$$\|\tilde{e}\|_{\eta^*} \lesssim \|e_u\|_{\eta^*} + \|f\|_{\eta^*},$$

hence independently of a, there exists a constant $E > 0$ such that

$$\|\tilde{e}\|_{\eta^*} \leq E, \quad \forall a \in (0, +\infty).$$

Therefore we are in place to apply Proposition 7.3 to get a solution $\tilde{v} \in C^{\theta''}(\mathbb{T}^3 \times [0, T/\Gamma])$ of (7.1), for any $\theta < \theta'' < \beta$. Moreover

$$e_{\tilde{v}}(t) = \int_{\mathbb{T}^3} |\tilde{v}|^2\,dx = \tilde{e}(t) \tag{7.34}$$

and, as already done in (7.21), we have the estimate

$$\|\tilde{v} - u_\delta\|_\theta \lesssim \sum_{q\geq 1} \lambda_q^{\theta-\beta} < \sigma, \tag{7.35}$$

provided a is chosen sufficiently large. Of course the choice of a depends on σ, that will be fixed at the end of the proof. By the triangular inequality we also get

$$\|\tilde{v} - \tilde{u}\|_\theta \leq \|\tilde{v} - u_\delta\|_\theta + \|u_\delta - \tilde{u}\|_\theta \lesssim \sigma, \tag{7.36}$$

having once again estimated through (7.3)

$$\|u_\delta - \tilde{u}\|_\theta \lesssim \delta^{\theta'-\theta} = (\delta_1 \lambda_0^{-4\alpha})^{\frac{\theta'-\theta}{2\theta'}} \leq \sigma,$$

the last estimate again being true if a is chosen large enough, depending on σ. Notice that this is possible since $\theta' > \theta$. By Proposition 7.2, we also get

$$\|\tilde{v} - \tilde{u}\|_{C^\theta_{x,t}} \lesssim \sigma. \tag{7.37}$$

In order to finish the proof of the claim, we scale back the map $\tilde{v}$ and the energy $\tilde{e}$ through the rescaling (7.6), with $1/\Gamma$ instead of Γ. We define $v(x,t) \doteq \Gamma^{-1}\tilde{v}(x,\Gamma^{-1}t)$. Now (7.37) and (7.29) yield

$$\|v - u\|_{C^\theta_{x,t}} \lesssim \sigma.$$

We fix $\sigma > 0$ in such a way that

$$\|v - u\|_{C^\theta_{x,t}} \leq \frac{\varepsilon}{3},$$

and this gives us (7.27). Moreover, as $\tilde{v} \in C^{\theta''}(\mathbb{T}^3 \times [0, T/\Gamma])$ was a solution of (7.1), then also $v \in C^{\theta''}(\mathbb{T}^3 \times [0,T])$ is a weak solution of (7.1). The last thing to check for the proof of the claim is (7.28). By (7.34), we have

$$e_{\tilde{v}}(t) = \tilde{e}(t).$$

Using (7.30) and (7.31), we can write

$$\Gamma^2 e_v(\Gamma t) = e_{\tilde{v}}(t) = \tilde{e}(t) = \Gamma^2 e(\Gamma t), \quad \forall t \in [0, T/\Gamma],$$

so that

$$e_v(t) = e(t), \quad \forall t \in [0,T],$$

thus proving (7.28) and hence concluding the proof of the claim.

Remark 7.6. Since the choice in the previous proof of the energy profile depends on a, we wish to clarify in this remark the dependences of the parameters appearing in the proof of the claim. First, we fixed parameters $0 < \beta < \theta' < 1/3$, and we chose $b > 1$ in such a way that at the same time (7.33) and

$$b < \sqrt{\frac{\theta'^*}{\beta^*}}$$

hold. By choosing $\alpha \in (0, \alpha_1)$, where α_1 is small enough, this can be guaranteed. Note that in this way α_1 only depends on β, θ' and b, as stated in Proposition 7.3. Therefore, we can always consider $\alpha_1 \leq \alpha_0$, where α_0 is the number appearing in Proposition 7.3. Next, we have proved that there exists a_1 large enough such that for $a \geq a_1$, we can guarantee estimates (7.12), (7.13), (7.14) and (7.15) for $q = 0$, for any function e of the form (7.25). This a_1 only depends on $\beta, b, \alpha, \theta'$ and u. Moreover, in the last steps it is required to take a large enough so that inequality (7.35) holds. This yields therefore a number $a_2 \geq a_1$ that depends on ε, $E \doteq \|e_u\|_{\eta^*} + \|f\|_{\eta^*}$ and the universal constant C of Proposition 7.2. Therefore a_2 now depends only on $\beta, b, \alpha, \theta'$ and E, since u, ε and C are fixed from the start of the proof of the claim. We can therefore take any $a_2 \geq a_0$, where a_0 is the parameter appearing in Proposition 7.3. Hence we take $\alpha \doteq \frac{\alpha_2}{2}$, $a \doteq 2a_2$. These choices define uniquely e as in (7.25) and allows us to prove the claim.

7.3 PRELIMINARIES TO THE PROOF OF PROPOSITION 7.3

The proof of the main iterative proposition given in [11] is subdivided in three steps

1. mollification: $(v_q, \mathring{R}_q) \mapsto (v_\ell, \mathring{R}_\ell)$;
2. gluing : $(v_\ell, \mathring{R}_\ell) \mapsto (\overline{v}_q, \mathring{\overline{R}}_q)$;
3. perturbation: $(\overline{v}_q, \mathring{\overline{R}}_q) \mapsto (v_{q+1}, \mathring{R}_{q+1})$.

In the proof of [11, Proposition 2.1], the energy function e only appears in the perturbation step and both the mollification and the gluing steps are independent on its choice. Thus, also in our case, given the triple $(v_q, p_q, \mathring{R}_q)$ there will exists a new triple $(\overline{v}_q, \overline{p}_q, \mathring{\overline{R}}_q)$ solving the Euler Reynolds system such that the temporal support of $\mathring{\overline{R}}_q$ is contained in pairwise disjoint intervals I_i of length comparable to

$$\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}.$$

More precisely, for any $n \in \mathbb{Z}$ let

$$t_n = \tau_q n, \qquad I_n = \left[t_n + \frac{1}{3}\tau_q, t_n + \frac{2}{3}\tau_q\right] \cap [0,T], \qquad J_n = \left[t_n - \frac{1}{3}\tau_q, t_n + \frac{1}{3}\tau_q\right] \cap [0,T].$$

We have

$$\operatorname{supp} \mathring{\overline{R}}_q \subset \bigcup_{n\in\mathbb{Z}} I_n \times \mathbb{T}^3.$$

Moreover the following estimates hold

$$\|v_q - \overline{v}_q\|_0 \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\alpha} \tag{7.38}$$
$$\|\overline{v}_q\|_{1+N} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N} \tag{7.39}$$
$$\left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha} \tag{7.40}$$
$$\left\|\partial_t\mathring{\overline{R}}_q + (\overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N-\alpha} \tag{7.41}$$
$$\left|\int_{\mathbb{T}^3} |\overline{v}_q|^2 - |v_\ell|^2\,dx\right| \lesssim \delta_{q+1}\ell^\alpha, \tag{7.42}$$

for any $N \geq 0$, where the small parameter ℓ is defined as

$$\ell = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}}$$

and it comes from the mollification step. We observe that by choosing α sufficiently small and a sufficiently large we can assume

$$\lambda_q^{-3/2} \leq \ell \leq \lambda_q^{-1}. \tag{7.43}$$

We also state another inequality we will need in the following, that is a consequence of (7.4),(7.15), and (7.42) :

$$\frac{\delta_{q+1}}{2\lambda_q^\alpha} \leq e(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2\,dx \leq 2\delta_{q+1}. \tag{7.44}$$

Thus we can pass to the perturbation step. The aim is to find a triple $(v_{q+1}, p_{q+1}, \mathring{R}_q)$ which solves (7.7) with the estimates

$$\|v_{q+1} - \overline{v}_q\|_0 + \lambda_{q+1}^{-1}\|v_{q+1} - \overline{v}_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2} \tag{7.45}$$

$$\|\mathring{R}_{q+1}\|_{\alpha} \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}} \tag{7.46}$$

$$\left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx - \frac{\delta_{q+2}}{2} \right| \leq C \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}} + \frac{\delta_{q+2}}{4}. \tag{7.47}$$

Note that estimates (7.45) and (7.46) are the same stated in [11], while (7.47) is slightly different due to the term $\delta_{q+2}/4$. This does not affect the iteration and Proposition 7.3 is still a direct consequence of estimates (7.45)-(7.47). However, since estimate (7.47) is different than the one used in [11], we give a complete proof of Proposition 7.3.

7.3.1 *Proof of Proposition 7.3*

By using (7.38) and (7.45) we estimate

$$\|v_{q+1} - v_q\|_0 \leq \|v_{q+1} - \overline{v}_q\|_0 + \|\overline{v}_q - v_q\|_0 \leq \frac{M}{2}\delta_{q+1}^{1/2} + C\delta_{q+1}^{1/2}\lambda_q^{-\alpha},$$

where the constant C depends only on α, β and M. Thus if a is chosen sufficiently large we can guarantee

$$\|v_{q+1} - v_q\|_0 \leq M\delta_{q+1}^{1/2}. \tag{7.48}$$

Similarly, by using (7.13), (7.39) and (7.45), we have

$$\|v_{q+1} - v_q\|_1 \leq \|v_{q+1} - \overline{v}_q\|_1 + \|\overline{v}_q\|_1 + \|v_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2}\lambda_{q+1} + (C+M)\,\delta_q^{1/2}\lambda_q.$$

Again, if a is chosen sufficiently large, we can ensure

$$\|v_{q+1} - v_q\|_1 \leq M\delta_{q+1}^{1/2}\lambda_{q+1},$$

which, together with (7.48), gives (7.17). By (7.13), (7.14) and (7.17) we get

$$\|v_{q+1}\|_0 \leq \|v_{q+1} - v_q\|_0 + \|v_q\|_0 \leq \frac{M}{2}\delta_{q+1}^{1/2} + 1 - \delta_q^{1/2} \leq 1 - \delta_{q+1}^{1/2},$$
$$\|v_{q+1}\|_1 \leq \|v_{q+1} - v_q\|_1 + \|v_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2}\lambda_{q+1} + M\delta_q^{1/2}\lambda_q \leq M\delta_{q+1}^{1/2}\lambda_{q+1}$$

where we also chose the parameter a sufficiently large to guarantee the last inequalities of the previous estimates. In particular this shows that v_{q+1} obeys (7.13) and (7.14) in which q is replaced by $q+1$. Estimate (7.12) for $\mathring{R}_{q+1}$ is a direct consequence of (7.46) and the parameters inequality

$$\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} \leq \frac{\delta_{q+2}}{\lambda_{q+1}^{8\alpha}}. \tag{7.49}$$

indeed, by taking the logarithms, the last inequality holds by choosing a sufficiently large if

$$-\beta - \beta b + 1 - b + 2b^2\beta + 8b\alpha < 0,$$

but this is true since $b < \frac{1-\beta}{2\beta}$ (see (7.18)) and α is chosen sufficiently small. We are only left with estimate (7.15) for v_{q+1}. By (7.47) and (7.49) we have

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx \leq \frac{\delta_{q+2}}{2} + C\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}} + \frac{\delta_{q+2}}{4} \leq \frac{3}{4}\delta_{q+2} + C\frac{\delta_{q+2}}{\lambda_{q+1}^{6\alpha}},$$

thus, for a sufficiently large a, we get

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx \leq \delta_{q+2}. \tag{7.50}$$

Finally, again by (7.47) we have

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx \geq \frac{\delta_{q+2}}{2} - C \frac{\delta_q^{1/2} \delta_{q+1}^{1/2} \lambda_q^{1+2\alpha}}{\lambda_{q+1}} - \frac{\delta_{q+2}}{4} \geq \left(\frac{1}{4} - \frac{C}{\lambda_{q+1}^{6\alpha}} \right) \delta_{q+2},$$

and, since for a sufficiently large a we can ensure that

$$\frac{1}{4} - \frac{C}{\lambda_{q+1}^{6\alpha}} \geq \frac{1}{\lambda_{q+1}^{\alpha}},$$

we end up with

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx \geq \delta_{q+2} \lambda_{q+1}^{-\alpha},$$

which together with (7.50) gives (7.15) and concludes the proof of the proposition.

7.4 PERTURBATION

We will now outline the construction of the perturbation w_{q+1}, where

$$v_{q+1} \doteq w_{q+1} + \overline{v}_q \, .$$

The perturbation w_{q+1} is highly oscillatory and will be based on the Mikado flows introduced in [14]. We recall the construction in the following lemma

Lemma 7.7. *For any compact subset $\mathcal{N} \subset\subset \mathcal{S}_+^{3\times 3}$ there exists a smooth vector field*

$$W : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3,$$

such that, for every $R \in \mathcal{N}$

$$\begin{cases} \operatorname{div}_\xi (W(R,\xi) \otimes W(R,\xi)) = 0 \\ \\ \operatorname{div}_\xi W(R,\xi) = 0, \end{cases} \tag{7.51}$$

and

$$\fint_{\mathbb{T}^3} W(R,\xi) \, d\xi = 0, \tag{7.52}$$

$$\fint_{\mathbb{T}^3} W(R,\xi) \otimes W(R,\xi) \, d\xi = R. \tag{7.53}$$

Using the fact that $W(R,\xi)$ is $\mathbb{T}^3$-periodic and has zero mean in ξ, we write

$$W(R,\xi) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} a_k(R) e^{ik\cdot\xi} \tag{7.54}$$

for some smooth functions $R \to a_k(R) \in \mathbb{C}^3$, satisfying $a_k(R) \cdot k = 0$. From the smoothness of W, we further infer

$$\sup_{R \in \mathcal{N}} |D_R^N a_k(R)| \leq \frac{C(\mathcal{N}, N, m)}{|k|^m} \tag{7.55}$$

for some constant C, which depends, as highlighted in the statement, on $\mathcal{N}$, N and m.

Remark 7.8. Later in the proof the estimates (7.55) will be used with a specific choice of the compact set $\mathcal{N}$ and of the integers N and m: this specific choice will then determine the universal constant M appearing in Proposition 7.3.

Using the Fourier representation we see that from (7.53)

$$W(R,\xi)\otimes W(R,\xi) = R + \sum_{k\neq 0} C_k(R)e^{ik\cdot\xi} \tag{7.56}$$

where

$$C_k k = 0 \quad \text{and} \quad \sup_{R\in\mathcal{N}} |D_R^N C_k(R)| \leq \frac{C(\mathcal{N},N,m)}{|k|^m} \tag{7.57}$$

for any $m,N\in\mathbb{N}$. It will also be useful to write the Mikado flows in terms of a potential. We note

$$\operatorname{curl}_\xi\left(\left(\frac{ik\times a_k}{|k|^2}\right)e^{ik\cdot\xi}\right) = -i\left(\frac{ik\times a_k}{|k|^2}\right)\times ke^{ik\cdot\xi} = -\frac{k\times(k\times a_k)}{|k|^2}e^{ik\cdot\xi} = a_k e^{ik\cdot\xi} \tag{7.58}$$

We define the smooth non-negative cut-off functions $\eta_i = \eta_i(x,t)$ with the following properties

(i) $\eta_i \in C^\infty(\mathbb{T}^3\times[0,T])$ with $0\leq\eta_i(x,t)\leq 1$ for all (x,t);

(ii) $\operatorname{supp}\eta_i\cap\operatorname{supp}\eta_j = \emptyset$ for $i\neq j$;

(iii) $\mathbb{T}^3\times I_i \subset \{(x,t):\eta_i(x,t)=1\}$;

(iv) $\operatorname{supp}\eta_i\subset\mathbb{T}^3\times I_i\cup J_i\cup J_{i+1}$;

(v) There exists a positive geometric constant $c_0>0$ such that for any $t\in[0,T]$

$$\sum_i\int_{\mathbb{T}^3}\eta_i^2(x,t)\,dx \geq c_0. \tag{7.59}$$

The next lemma is taken from [11].

Lemma 7.9. *There exists cut-off functions $\{\eta_i\}_i$ with the properties (i)-(v) above and such that for any i and $n,m\geq 0$*

$$\|\partial_t^n\eta_i\|_m \leq C(n,m)\tau_q^{-n}$$

where $C(n,m)$ are geometric constants depending only upon m and n.

Analogously to [11], we will now define the perturbations that are necessary to show (7.45)-(7.47). Since the energy profile is not smooth, we will need to mollify it. To do so we will henceforth consider e to be extended on the whole $\mathbb{R}$ as $e(t)=e(0)$ for all $t<0$ and $e(t)=e(T)$ for all $t>T$, in such a way that the extension is still in $C^{\eta^*}(\mathbb{R})$. With this convention we define

$$e_q(t) \doteq (e*\psi_{\varepsilon_q})(t),$$

where ψ_{ε_q} is a standard mollifier and

$$\varepsilon_q \doteq \left(\frac{\delta_{q+2}}{4E}\right)^{\frac{1}{\eta^*}}. \tag{7.60}$$

Define also

$$\rho_q(t) \doteq \frac{1}{3}\left(e_q(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3}|\overline{v}_q|^2\,dx\right)$$

and

$$\rho_{q,i}(x,t) \doteq \frac{\eta_i^2(x,t)}{\sum_j\int_{\mathbb{T}^3}\eta_j^2(y,t)\,dy}\rho_q(t)$$

Define the backward flows Φ_i for the velocity field $\overline{v}_q$ as the solution of the transport equation

$$\begin{cases}(\partial_t+\overline{v}_q\cdot\nabla)\Phi_i = 0\\[2ex] \Phi_i(x,t_i) = x.\end{cases}$$

Define

$$R_{q,i} \doteq \rho_{q,i} \operatorname{id} - \eta_i^2 \mathring{\overline{R}}_q$$

and

$$\tilde{R}_{q,i} = \frac{\nabla\Phi_i R_{q,i} (\nabla\Phi_i)^T}{\rho_{q,i}}. \tag{7.61}$$

We note that, because of properties (ii)-(iv) of η_i,

- $\operatorname{supp} R_{q,i} \subset \operatorname{supp} \eta_i$;
- on $\operatorname{supp} \mathring{\bar{R}}_q$ we have $\sum_i \eta_i^2 = 1$;
- $\operatorname{supp} \tilde{R}_{q,i} \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1}$;
- $\operatorname{supp} \tilde{R}_{q,i} \cap \operatorname{supp} \tilde{R}_{q,j} = \emptyset$ for all $i \neq j$.

Lemma 7.10. *For $a \gg 1$ sufficiently large we have*

$$\|\nabla\Phi_i - \operatorname{id}\|_0 \leq \frac{1}{2} \qquad \textit{for } t \in \operatorname{supp}(\eta_i). \tag{7.62}$$

Furthermore, for any $N \geq 0$

$$\frac{\delta_{q+1}}{8\lambda_q^\alpha} \leq |\rho_q(t)| \leq \delta_{q+1} \quad \textit{for all } t\,, \tag{7.63}$$

$$\|\rho_{q,i}\|_0 \leq \frac{\delta_{q+1}}{c_0}\,, \tag{7.64}$$

$$\|\rho_{q,i}\|_N \lesssim \delta_{q+1}\,, \tag{7.65}$$

$$\|\partial_t \rho_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\,, \tag{7.66}$$

$$\|\partial_t \rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}\,. \tag{7.67}$$

Moreover, for all (x,t)

$$\tilde{R}_{q,i}(x,t) \in B_{1/2}(\operatorname{id}) \subset \mathcal{S}_+^{3\times 3}\,,$$

where $B_{1/2}(\operatorname{id})$ denotes the metric ball of radius $1/2$ around the identity id *in the space $\mathcal{S}^{3\times 3}$.*

Proof. We write

$$\rho_q(t) = \frac{1}{3}\left(e_q(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2\, dx - \frac{\delta_{q+2}}{2}\right) = \frac{1}{3}\left(e_q(t) - e(t) + e(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2\, dx - \frac{\delta_{q+2}}{2}\right),$$

thus by (7.44) we get

$$\frac{1}{3}\left(\frac{\delta_{q+1}}{2\lambda_q^\alpha} - \frac{\delta_{q+2}}{2} - |e_q(t) - e(t)|\right) \leq |\rho_q(t)| \leq \frac{1}{3}\left(|e_q(t) - e(t)| + 2\delta_{q+1} + \frac{\delta_{q+2}}{2}\right). \tag{7.68}$$

By using (7.3) and the fact that $[e]_{\eta^*} \leq E$, we also get

$$|e_q(t) - e(t)| \leq [e]_{\eta^*}\varepsilon_q^{\eta^*} \leq \delta_{q+2}$$

and, by plugging it into (7.68), we achieve

$$\frac{\delta_{q+1}}{6\lambda_q^\alpha} - \frac{\delta_{q+2}}{2} \leq |\rho_q(t)| \leq \frac{2}{3}\delta_{q+1} + \frac{\delta_{q+2}}{2}.$$

It is easy to show that by choosing a sufficiently large we can guarantee (7.63). Note that by definition of the cut-off function η_i

$$c_0 \leq \sum_i \int_{\mathbb{T}^3} \eta_i^2(x,t)\, dx \leq 2 \tag{7.69}$$

and hence we obtain (7.64). Since $|\nabla^N \eta_j| \lesssim 1$, the bound (7.65) also follows. For the bound (7.62) and the fact that $\tilde{R}_{q,i}(x,t) \in B_{1/2}(\mathrm{id}) \subset \mathcal{S}^{3\times 3}_+$ we refer to [11, Lemma 5.4]. To prove (7.66), we first use (7.39), (7.40) to estimate

$$\left|\frac{d}{dt}\int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx\right| = 2\left|\int_{\mathbb{T}^3} \nabla \overline{v}_q \cdot \mathring{\overline{R}}_q \, dx\right| \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q.$$

Moreover, by (7.5), we have

$$|\partial_t e_q| \le [e]_{\eta^*}\varepsilon_q^{\eta^*-1} \le C\delta_{q+2}^{1-1/\eta^*},$$

where the constant C depends on η and E. Thus (7.66) is implied by the following parameters inequality

$$C\delta_{q+2}^{1-1/\eta^*} \le \delta_{q+1}\delta_q^{1/2}\lambda_q. \tag{7.70}$$

Using the definition of the parameters δ_q and λ_q it can be checked that the last inequality holds if one chose a big enough (depending on b, β, η and E) provided that

$$\left(\frac{1}{\eta^*} - 1\right) b^2 + b - \frac{1}{\beta^*} < 0.$$

Since b satisfies (7.16) we have

$$\left(\frac{1}{\eta^*} - 1\right) b^2 + b - \frac{1}{\beta^*} < \left(\frac{1}{\eta^*} - 1\right)\frac{\eta^*}{\beta^*} + \frac{\eta^*}{\beta^*} - \frac{1}{\beta^*} = 0,$$

thus (7.70) holds. Finally, since $\|\partial_t \eta_j\|_N \lesssim \tau_q^{-1}$ and $\tau_q^{-1} \ge \delta_q^{1/2}\lambda_q$, using (7.69), also the estimate (7.67) follows. □

7.4.1 *The constant M*

The principal term of the perturbation can be written as

$$w_o \doteq \sum_i \left(\rho_{q,i}(x,t)\right)^{1/2} (\nabla\Phi_i)^{-1} W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i) = \sum_i w_{o,i}\,, \tag{7.71}$$

where Lemma 7.7 is applied with $\mathcal{N} = \overline{B}_{1/2}(\mathrm{id})$, namely the closed ball (in the space of symmetric 3×3 matrices) of radius $1/2$ centered at the identity matrix.

From Lemma 7.10 it follows that $W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i)$ is well defined. Using the Fourier series representation of the Mikado flows (7.54) we can write

$$w_{o,i} = \sum_{k\neq 0} (\nabla\Phi_i)^{-1} b_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}\,,$$

where

$$b_{i,k}(x,t) \doteq \left(\rho_{q,i}(x,t)\right)^{1/2} a_k(\tilde{R}_{q,i}(x,t)).$$

By the definition of $w_{o,i}$ and (7.53) we compute

$$\begin{aligned} w_{o,i}\otimes w_{o,i} &= \rho_{q,i}\nabla\Phi_i^{-1}(W\otimes W)(\tilde{R}_{q,i},\lambda_{q+1}\Phi_i)\nabla\Phi_i^{-T} \\ &= \rho_{q,i}\nabla\Phi_i^{-1}\tilde{R}_{q,i}\nabla\Phi_i^{-T} + \sum_{k\neq 0}\rho_{q,i}\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i} \\ &= R_{q,i} + \sum_{k\neq 0}\rho_{q,i}\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}. \end{aligned} \tag{7.72}$$

The following is a crucial point of the construction, which ensures that the constant M of Proposition 7.3 is geometric and in particular independent of all the parameters of the construction.

Lemma 7.11. *There is a geometric constant $\bar{M}$ such that*

$$\|b_{i,k}\|_0 \le \frac{\bar{M}}{|k|^4}\delta_{q+1}^{1/2}\,. \tag{7.73}$$

We are finally ready to define the constant M of Proposition 7.3: from Lemma 7.11 it follows trivially that the constant is indeed geometric and hence independent of all the parameters of the statement of Proposition 7.3.

We can now define the geometric constant M as

$$M = 64\bar{M}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\frac{1}{|k|^4}\,, \tag{7.74}$$

where $\bar{M}$ is the constant of Lemma 7.11.

We also define

$$w_c \doteq \frac{-i}{\lambda_{q+1}}\sum_{i,k\neq 0}\left[\operatorname{curl}\left((\rho_{q,i})^{1/2}\frac{\nabla\Phi_i^T(k\times a_k(\tilde{R}_{q,i}))}{|k|^2}\right)\right]e^{i\lambda_{q+1}k\cdot\Phi_i} =: \sum_{i,k\neq 0} c_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\,.$$

Then by direct computations one can check that

$$w_{q+1} = w_o + w_c = \frac{-1}{\lambda_{q+1}}\operatorname{curl}\left(\sum_{i,k\neq 0}(\nabla\Phi_i)^T\left(\frac{ik\times b_{k,i}}{|k|^2}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\,, \tag{7.75}$$

thus the perturbation w_{q+1} is divergence frE.

7.4.2 *The final Reynolds stress and conclusions*

In order to define the new Reynolds tensor, we recall the operator $\mathcal{R}$ from [11], which can be thought of as an inverse divergence operator for symmetric tracefrE 2-tensors. The operator is defined as

$$\begin{aligned}(\mathcal{R}f)^{ij} &= \mathcal{R}^{ijk}f^k\\ \mathcal{R}^{ijk} &= -\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k - \frac{1}{2}\Delta^{-1}\partial_k\delta_{ij} + \Delta^{-1}\partial_i\delta_{jk} + \Delta^{-1}\partial_j\delta_{ik}.\end{aligned} \tag{7.76}$$

when acting on vectors f with zero mean on $\mathbb{T}^3$, and has the property that $\mathcal{R}f$ is symmetric and $\operatorname{div}(\mathcal{R}f) = f$. Upon letting

$$\overline{R}_q = \sum_i R_{q,i}\,,$$

we define the new Reynolds stress as follows

$$\mathring{R}_{q+1} \doteq \mathcal{R}\left(w_{q+1}\cdot\nabla\overline{v}_q + \partial_t w_{q+1} + \overline{v}_q\cdot\nabla w_{q+1} + \operatorname{div}\left(-\overline{R}_q + w_{q+1}\otimes w_{q+1}\right)\right) \tag{7.77}$$

With this definition one may verify that

$$\begin{cases}\partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) + \nabla p_{q+1} = \operatorname{div}(\mathring{R}_{q+1})\,,\\ \\ \operatorname{div} v_{q+1} = 0\,,\end{cases}$$

where the new pressure is defined by

$$p_{q+1}(x,t) = \bar{p}_q(x,t) - \sum_i \rho_{q,i}(x,t) + \rho_q(t). \tag{7.78}$$

The following proposition is taken from [11].

Proposition 7.12. *For $t \in I_i \cup J_i \cup J_{i+1}$ and any $N \geq 0$*

$$\|(\nabla\Phi_i)^{-1}\|_N + \|\nabla\Phi_i\|_N \lesssim \ell^{-N}, \tag{7.79}$$
$$\|\tilde{R}_{q,i}\|_N \lesssim \ell^{-N}, \tag{7.80}$$
$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}|k|^{-6}\ell^{-N}, \tag{7.81}$$
$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{-1}|k|^{-6}\ell^{-N-1}. \tag{7.82}$$

Moreover assuming a is sufficiently large, the perturbations w_o, w_c and w_q satisfy the following estimates

$$\|w_o\|_0 + \frac{1}{\lambda_{q+1}}\|w_o\|_1 \leq \frac{M}{4}\delta_{q+1}^{1/2} \tag{7.83}$$
$$\|w_c\|_0 + \frac{1}{\lambda_{q+1}}\|w_c\|_1 \lesssim \delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{-1} \tag{7.84}$$
$$\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}}\|w_{q+1}\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2} \tag{7.85}$$

where the constant M depends solely on the constant c_0 in (7.59). In particular, we obtain (7.45).

We are now ready to complete the proof of Proposition 7.3 by proving the remaining estimates (7.47) and (7.46). We start with the energy increment

Proposition 7.13. *The energy of v_{q+1} satisfies the following estimate*

$$\left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \,\mathrm{d}x - \frac{\delta_{q+2}}{2} \right| \leq C\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}} + \frac{\delta_{q+2}}{4}.$$

In particular, (7.47) holds.

Proof. By definition we have $v_{q+1} = \overline{v}_q + w_{q+1} = \overline{v}_q + w_o + w_c$, thus we have

$$\left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx - \frac{\delta_{q+2}}{2} \right| \leq \left| e(t) - \int_{\mathbb{T}^3} |w_o|^2 \, dx - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx \right|$$
$$+ \left| \int_{\mathbb{T}^3} |w_c|^2 \, dx + 2\int_{\mathbb{T}^3} w_o \cdot w_c \, dx + 2\int_{\mathbb{T}^3} w_{q+1} \cdot \overline{v}_q \, dx \right|. \tag{7.86}$$

The estimate on the second term in the right hand side of (7.86) is just a a consequence of (7.39) and Proposition 7.12 and for a complete we refer to [11, Proposition 6.2], in which it is proved that

$$\left| \int_{\mathbb{T}^3} |w_c|^2 \, dx + 2\int_{\mathbb{T}^3} w_o \cdot w_c \, dx + 2\int_{\mathbb{T}^3} w_{q+1} \cdot \overline{v}_q \, dx \right| \lesssim \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}.$$

Now recall that from (7.72) and the definition of $R_{q,i}$ we have

$$\begin{aligned}
\int_{\mathbb{T}^3} |w_o|^2 \, dx &= \sum_i \int_{\mathbb{T}^3} \operatorname{tr} R_{q,i} \, dx + \int_{\mathbb{T}^3} \sum_{i,k\neq 0} \rho_{q,i} \nabla\Phi_i^{-1} \operatorname{tr} C_k(\tilde{R}_{q,i}) \nabla\Phi_i^{-T} e^{i\lambda_{q+1}k\cdot\Phi_i} \, dx \\
&= 3\sum_i \int_{\mathbb{T}^3} \rho_{q,i} \, dx + \int_{\mathbb{T}^3} \sum_{i,k\neq 0} \rho_{q,i} \nabla\Phi_i^{-1} \operatorname{tr} C_k(\tilde{R}_{q,i}) \nabla\Phi_i^{-T} e^{i\lambda_{q+1}k\cdot\Phi_i} \, dx \\
&= 3\rho_q(t) + \int_{\mathbb{T}^3} \sum_{i,k\neq 0} \rho_{q,i} \nabla\Phi_i^{-1} \operatorname{tr} C_k(\tilde{R}_{q,i}) \nabla\Phi_i^{-T} e^{i\lambda_{q+1}k\cdot\Phi_i} \, dx \\
&= e_q(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx + \int_{\mathbb{T}^3} \sum_{i,k\neq 0} \rho_{q,i} \nabla\Phi_i^{-1} \operatorname{tr} C_k(\tilde{R}_{q,i}) \nabla\Phi_i^{-T} e^{i\lambda_{q+1}k\cdot\Phi_i} \, dx .
\end{aligned}$$

As a consequence of (7.57), Lemma 7.10 and Proposition 7.12 we have

$$\left|\int_{\mathbb{T}^3} \sum_{i,k\neq 0} \rho_{q,i} \nabla\Phi_i^{-1} \operatorname{tr} C_k(\tilde{R}_{q,i}) \nabla\Phi_i^{-T} e^{i\lambda_{q+1} k\cdot\Phi_i}\, dx\right| \lesssim \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}.$$

For a detailed proof of the previous estimate we again refer to [11, Proposition 6.2]. Thus we are only left with estimating $|e(t)-e_q(t)|$, but from (7.3), the definition of ε_q in (7.60) and the fact that $[e]_{C^{\eta^*}} \le E$, we get

$$|e(t)-e_q(t)| \le [e]_{\eta^*}\varepsilon_q^{\eta^*} \le \frac{\delta_{q+2}}{4},$$

which concludes the proof of the proposition. □

For the inductive estimate on $\mathring{R}_{q+1}$ we refer to [11, Proposition 6.1]

Proposition 7.14. *The Reynolds stress error $\mathring{R}_{q+1}$ defined in (7.77) satisfies the estimate*

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}\,. \tag{7.87}$$

In particular, (7.46) holds.

7.5 FINAL COMMENTS

In this section, we wish to comment on why we need to introduce the space X_θ (see (7.19)), since clearly the most natural choice for X_θ would have simply been the space of all $C^\theta(\mathbb{T}^3\times[0,T])$ or $c^\theta(\mathbb{T}^3\times[0,T])$ solutions of Euler equation. We believe that such a discussion highlights some interesting features of the convex integration scheme.

The introduction of X_θ is related to the proof of Theorem 7.5 and to intrinsic properties of the iterative scheme of [11]. The proof of Theorem 7.5 uses the following strategy, that is quite standard in arguments involving Baire Theorem. As a first step, we rewrite Y_θ^c as union of closed sets $C_{m,n,r}$. The parameters m,n quantify an improvement in the regularity of elements of $C_{m,n,r}$, while r only indicates its localization in space, hence it is not useful to the purpose of this discussion. Secondly, one needs to prove that $C_{m,n,r}$ has empty interior. Equivalently, every element $u_0\in C_{m,n,r}$ must be approximated in the $C^\theta(\mathbb{T}^3\times[0,T])$ norm with elements $u\in X_\theta\setminus C_{m,n,r}$. This is where the convex integration scheme comes into play. The iterative procedure of [11] tells us, roughly speaking, that given a smooth subsolution $\bar{u}$ and a positive and smooth (or $C^{\theta^*+\gamma}([0,T])$, as proved in the present work) energy profile e, one can find an arbitrarily close solution u such that $e=e_u$, provided some initial estimates are verified. In order to obtain the desired "less regular" approximating sequence, it seems therefore rather natural to try to apply this result to the subsolution obtained by mollifying u_0, and choose an energy profile $e\in C^{\theta^*+1/2m}([0,T])\setminus W^{\theta^*+1/m,1}([0,T])$.

Since one wishes to approximate a $C^\theta(\mathbb{T}^3\times[0,T])$ solution with a sequence of smooth functions in the $C^\theta(\mathbb{T}^3\times[0,T])$ topology, the first natural restriction is to take the complete metric space in which to apply the Baire argument to be a closed subset of $c^\theta(\mathbb{T}^3\times[0,T])$. Once one can guarantee the fact that the mollifications of u_0 are close in the right topology to u_0, the next step is to use the convex integration scheme on a close enough space-time mollification of u_0, let us call it u_δ, $\delta>0$ being the parameter of mollification. Let us moreover denote with R_δ the Reynold stress tensor of u_δ, i.e.

$$R_\delta = u_\delta\otimes u_\delta - (u_0\otimes u_0)_\delta.$$

In order to apply the scheme, one needs to guarantee step 0 of the inductive estimates, i.e. (7.12),(7.13), (7.14), (7.15). We will now show that, by choosing any $\theta<\beta$ in order to have the

$C^\theta(\mathbb{T}^3 \times [0,T])$ closeness of the resulting solution to u_δ (and therefore to u_0), (7.12) and (7.13) become impossible to guarantee using the estimates of Proposition 7.1. Through these estimates, one wishes to find $\delta > 0$ and $\alpha > 0$ for which

$$\|\mathring{R}_\delta\|_0 \lesssim \delta^{2\theta} \leq \delta_1 \lambda_0^{-3\alpha} \text{ and } \|u_\delta\|_1 \lesssim \delta^{\theta-1} \leq M\delta_0^{1/2}\lambda_0.$$

These relations are anyway incompatible for any $\delta, \alpha > 0$ if

$$\delta_q = \lambda_q^{-2\beta} = a^{-2\beta b^q} \tag{7.88}$$

for $a > 0, b > 1$. To see this, notice that a solution δ would need to satisfy also

$$\delta^{2\theta} \lesssim \delta_1 = \lambda_1^{-2\beta} \tag{7.89}$$

Moreover, the estimate on the C^1 norm can be rewritten as

$$\delta_0^{-\frac{1}{2(1-\theta)}} \lambda_0^{-\frac{1}{1-\theta}} \lesssim \delta. \tag{7.90}$$

Combining (7.88), (7.89) and (7.90), one obtains

$$a^{-\frac{1-\beta}{1-\theta}} \lesssim a^{-b\frac{\beta}{\theta}},$$

hence that the function $a \mapsto a^{b\frac{\beta}{\theta}-\frac{1-\beta}{1-\theta}}$ is bounded. Since for every $b > 1$, one has $b\frac{\beta}{\theta} - \frac{1-\beta}{1-\theta} > 0$ because of the inequality $\theta < \beta$, we find that a can not be taken freely in an open unbounded interval $(a_0, +\infty)$, hence Proposition 7.3 can not possibly be true in this setting. Nonetheless, as it is clearly stated in [11], we could have found many $C^\beta(\mathbb{T}^3 \times [0,T])$ solutions of (7.1) $C^\beta(\mathbb{T}^3 \times [0,T])$ close to u_δ, for $\beta < \theta$. This is obviously not sufficient for Theorem 7.5. This feature of the "$\theta - \beta$ gap" was noticed also in the work [42], to which we refer the reader for interesting discussions. On the other hand, if the starting point u_0 can be approximated in the $C^\theta(\mathbb{T}^3 \times [0,T])$ topology by more regular solutions, for instance in $C^{\theta'}(\mathbb{T}^3 \times [0,T])$, $\theta < \theta'$, then by the previous discussion it becomes clear that we can now start the scheme from these more regular points obtaining the desired estimates in $C^\theta(\mathbb{T}^3 \times [0,T])$. This is exactly the reason for introducing the space X_θ.

We conclude this discussion by noting that, even though it could not contain all the $C^\theta(\mathbb{T}^3 \times [0,T])$ solutions of (7.1), X_θ contains many elements. indeed, by [11], for every smooth and positive energy profile e and for every $\theta < \theta' < 1/3$, we find a weak solution $u \in C^{\theta'}(\mathbb{T}^3 \times [0,T])$ of (7.1) with $e = e_u$. Since $\theta' > \theta$, $u \in X_\theta$.

A MAIN NOTATION AND PRELIMINARIES

We collect here some of the known results and notations we use in this thesis. This list is sufficient to read the introduction, but some objects will be defined in the next chapters when needed.

A.1 DOMAINS

We always denote with Ω an open subset of an Euclidean space of finite dimension, for which we use the symbols $\mathbb{R}^m, \mathbb{R}^d$ and $\mathbb{R}^n$. $\mathbb{T}^n$ is the n-dimensional torus identified with $[0,1]^n \subset \mathbb{R}^n$, and we work with the convention that every function $f : \mathbb{T}^n \to \mathbb{R}$ is actually a 1-periodic function on $\mathbb{R}^n$, and we do not denote differently this extension. In $X = \mathbb{R}^n$ or $X = \mathbb{T}^n$, the ball of radius $r > 0$ centered at $x \in X$ is denoted with $B_r(x)$. If the ball is centered at 0, we use the shorter notation B_r. The same conventions are adopted for squares, denoted with Q instead of B. If X is a more general metric space, we will use the notation $\mathcal{N}_r(x)$ to denote the ball of radius $r > 0$ around $x \in X$.

A.2 LINEAR ALGEBRA

For any matrix $A \in \mathbb{R}^{n\times n}$, we denote with $\operatorname{cof}(A)$ the matrix defined as

$$\operatorname{cof}(A)_{ij} = (-1)^{i+j}\det(A^{ji}),$$

where A^{ji} is the $(n-1)\times(n-1)$ matrix obtained from A by eliminating the j-th row and the i-th column. In particular

$$A\operatorname{cof}(A) = \det(A)\,\mathrm{id}_n, \quad \forall A \in \mathbb{R}^{n\times n},$$

where id_n is the identity matrix of size n. If $A \in \mathbb{R}^{n\times m}$, then $A^T \in \mathbb{R}^{m\times n}$ denotes its transpose. $\mathrm{Sym}(n)$ is the space of symmetric matrices of size $n \times n$ and $\mathrm{Sym}^+(n)$ is the space of non-negative definite symmetric matrices;

A.3 DIFFERENTIALS AND FUNCTIONAL SPACES

For functions $f : \Omega \to \mathbb{R}^n$, we denote with Df the distributional gradient, and with Hf or D^2f its distributional Hessian, i.e. the matrix of the second derivatives of f. The Lebesgue and Sobolev spaces are as usual denoted with L^p and $W^{k,p}$, respectively, and their local version by L^p_{loc}, $W^{k,p}_{\mathrm{loc}}$. Only in Part ii we will need to distinguish between the pointwise and the distributional derivatives of f. We will always clarify which one of the two we are using, hence we do not denote them differently. The fractional Sobolev space on an interval $I \subset \mathbb{R}$ is

$$W^{\alpha,p}(I) := \left\{ f \in L^p(I) : \frac{|f(x)-f(y)|}{|x-y|^{\frac{1}{p}+\alpha}} \in L^p(I\times I) \right\}.$$

The set $C^\theta(\Omega)$ is the space of θ-Hölder functions in Ω, while the so-called *little Hölder* space $c^\theta(\Omega)$ is the closure in the $C^\theta(\Omega)$-norm of the space of smooth functions on Ω. The space of continuous functions is denoted by $C^0(\Omega)$, the Lipschitz continuous functions by $\mathrm{Lip}(\Omega)$. $C^0_c(\Omega)$ and $C^0_b(\Omega)$ are the spaces of continuous functions with compact support and the one of bounded continuous functions respectively. We use the classical notation $C^\theta(\Omega, Y)$ to denote the space of Hölder spaces with values in Y, and analogously for the other function spaces.

We will denote by $\mathcal{H}^1_{\mathrm{loc}}(\Omega)$ the local Hardy space. We only need to consider non-negative functions f in this space that are constant outside of a compact set $K \subset \Omega$. A function $f : \Omega \to \mathbb{R}$ with these properties belongs to $\mathcal{H}^1_{\mathrm{loc}}(\Omega)$ if and only if (see [59, Lemma 3], that is a consequence of [81])

$$\|f\|_{\mathcal{H}^1(\Omega)} = \int_\Omega f(x) \log(1 + f(x))\, dx \ < +\infty.$$

A.4 MEASURES AND VARIFOLDS

Let X be a subset of $\mathbb{R}^n$ or $\mathbb{T}^n$. $\mathcal{M}(X, \mathbb{R}^m)$ is the space of bounded signed Radon measures with values in $\mathbb{R}^m$. When $m = 1$, we denote this space by $\mathcal{M}(X)$, and the space of positive Radon measures by $\mathcal{M}_+(X)$. Recall that an element $\mu \in \mathcal{M}(X, \mathbb{R}^m)$ is given by $\mu = \vec{T}\|\mu\|$, where $\|\mu\| \in \mathcal{M}_+(X)$ and $\vec{T}$ is a $\|\mu\|$ measurable vector field on X with values in $\mathbb{R}^m$ and $\|\vec{T}\| = 1$. In particular, we denote

$$\mu(\Phi) = \int_X \langle \vec{T}, \Phi \rangle d\|\mu\|, \forall \Phi \in C_c(X, \mathbb{R}^m).$$

$\mathcal{M}(X, \mathbb{R}^m)$ is a normed space, with norm

$$\|\mu\|_{\mathcal{M}(X,\mathbb{R}^m)} \doteq \sup\{\mu(\Phi) : \Phi \in C^0_c(X, \mathbb{R}^m), \|\Phi\|_\infty \le 1\},$$

The weak-$*$ convergence of $\mu_k \in \mathcal{M}(X, \mathbb{R}^m)$ to $\mu \in \mathcal{M}(X, \mathbb{R}^m)$ is given by

$$\mu_k \overset{*}{\rightharpoonup} \mu \Leftrightarrow \mu_k(\Phi) \to \mu(\Phi),\ \forall \Phi \in C^0_c(X, \mathbb{R}^m).$$

If X is compact, $\mathcal{M}(X, \mathbb{R}^m)$ is sequentially weak-$*$ compact, (see [31, Section 1.9]).

For a set $E \subset \mathbb{R}^n$ or $E \subset X$, we denote with $|E|$ its n-dimensional Lebesgue measure, and with $\mathcal{H}^k$ the k-dimensional Hausdorff measure, so that $\mathcal{H}^n(E) = |E|$. We let

$$\mathrm{spt}(\mu) := \bigcap \{C \subset \mathbb{R}^n : C \text{ is closed and } \|\mu\|(\mathbb{R}^n \setminus C) = 0\}$$

the *support* of μ. For a Borel set $E \subset X$ we use the symbol $\mu \llcorner E$ to denote the measure

$$\mu \llcorner E(A) \doteq \mu(E \cap A), \quad \forall A \text{ Borel subset of } X.$$

The terms *absolutely continuous* and *singular* part of a measure need to be intended with respect to the Lebesgue measure. Analogously, when not specified, *almost everywhere*, abbreviated with a.e., is intended with respect to the Lebesgue measure. For every $\mu \in \mathcal{M}(X, \mathbb{R}^m)$, we consider its Lebesgue decomposition

$$\mu = g\, dx + \mu^s,$$

where $g \in L^1(X, \mathbb{R}^m)$ and $\mu^s \in \mathcal{M}(X, \mathbb{R}^m)$ denotes a singular measure with respect to the Lebesgue measure, i.e. there exists a Borel set $A \subset X$ such that $|A| = 0$ and

$$\mu^s(E) = \mu^s(A \cap E), \quad \text{for every Borel set } E \subset X.$$

We recall that a Lebesgue point for a function $g \in L^1(X, \mathbb{R}^m)$ is a point x such that

$$\fint_{B_r(x)} |g(y) - g(x)|\, dy \to 0 \ \text{ as } r \to 0^+,$$

where

$$\fint_E f(y) dy = \frac{1}{|E|} \int_E f(y) dy,$$

for every $f \in L^1(X)$, E Borel subset of X with $|E| > 0$. It is well known that the set of Lebesgue points of such a function g is of full measure in X (see [31, Theorem 1.33]). More generally, if $\mu \in \mathcal{M}(X, \mathbb{R}^m)$, we call its (upper) density the function

$$D\mu(x) \doteq \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_n r^n},$$

where $\omega_n \doteq |B_1(0)|$. If μ is singular with respect to the Lebesgue measure, then $D\mu(x) = 0$ for a.e. point of X (see [31, Theorem 1.31]).

We will use the concept of Monge-Ampère measure associated to a convex function. For every convex function $\varphi : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open and convex, this is defined as the locally finite measure:

$$\mu_\varphi(E) \doteq \left| \bigcup_{x \in E} \partial\varphi(x) \right|,$$

where $\partial\varphi(x)$ denotes the subdifferential at x of φ. We refer the reader to [34, Section 2] for the basic properties of μ_φ.

Finally, we recall that an *rectifiable varifold V* of dimension m is a couple (Γ, θ), where $\Gamma \subset \mathbb{R}^{m+n}$ is a m-rectifiable set in $\mathbb{R}^{n+m}$, and $\theta : \Gamma \to \mathbb{R}_+ \setminus \{0\}$ is a Borel map. It is customary to denote (Γ, θ) as $\theta[\![\Gamma]\!]$ and to call θ the *multiplicity* of the varifold. If θ has values in $\mathbb{N} \setminus \{0\}$, we call the varifold integer rectifiable.

A.5 YOUNG MEASURES

For a comprehensive introduction to Young Measures, we refer the reader to [61, Section 3]. The results we report here are taken from this reference.

Theorem A.1 (Fundamental Theorem on Young measure)**.** *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with finite measure. Consider a sequence $z_j : E \subset \mathbb{R}^d \to \mathbb{R}^N$ of measurable functions satisfying the condition*

$$\sup_{j \in \mathbb{N}} \int_E \|z_j\|^s < +\infty,$$

for some $s > 0$. Then there exists a subsequence z_{j_k} and a weak- measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^N)$ such that for a.e. $x \in E$, $\nu_x \in \mathcal{M}(\mathbb{R}^N)$ and in addition $\nu_x(\mathbb{R}^N) = 1$. Moreover, for every $A \subset E$, and for every $f \in C(\mathbb{R}^N)$, if*

$$f(z_{j_k}) \text{ is relatively weakly compact in } L^1(A),$$

then,

$$f(z_{j_k}) \rightharpoonup \bar{f} \text{ in } L^1(A), \text{ where } \bar{f}(x) = \langle \nu_x, f \rangle = \int_{\mathbb{R}^N} f(y) d\nu_x(y).$$

In this case, we say that z_{j_k} generates the Young measure ν.

Corollary A.2. *Let $p > 1$ and $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with finite measure. If z_j is weakly convergent in $L^p(E)$ to a function $z \in L^p(E)$ and if it generates the Young measure ν, then, for every $f \in C(\mathbb{R}^N)$ such that*

$$|f(y)| \leq C(1 + \|y\|^q), \text{ for } q < p,$$

the following holds

$$f(z_j) \rightharpoonup \bar{f}, \text{weakly in } L^{\frac{p}{q}}(E).$$

In particular, the choice f such that $f(y) = y$, $\forall y \in \mathbb{R}^N$ yields

$$z(x) = \langle \nu_x, f \rangle.$$

Another result, fundamental to establish compactness, is the following [61, Corollary 3.2]:

Corollary A.3. *Suppose that a sequence z_j of measurable functions from E to $\mathbb{R}^N$ generates the Young measure ν. Then*

$$z_j \to z \text{ in measure if and only if } \nu_x = \delta_{z(x)} \text{ for a.e } x.$$

In particular, if $z_j \in L^p(E)$, for $p > 1$, and the following hold

(i) *z_j is weakly convergent in $L^p(E)$ to a function $z \in L^p(E)$,*

(ii) *z_j generates the Young measure ν,*

(iii) *$\nu_x = \delta_{z(x)}$ for a.e. x.*

Then,

$$z_j \to z \text{ in } L^q(E), \text{ for every } 1 \leq q < p.$$

A.6 BAIRE CATEGORY THEOREM

Let (X, d) be a complete metric space. We say that $Y \subset X$ is meager if it is contained in the union of countably many closed sets with empty interior. A set $Z \subset X$ is residual if it contains the intersection of countably many open and dense sets, or, equivalently, if $X \setminus Z$ is meager. We say that an element in the residual set Z is a *typical* element of X. The version of Baire Category Theorem we will extensively use in the thesis is the following:

Theorem A.4 (Baire Theorem)**.** *A complete metric space X is not meager.*

B APPENDIX TO "DIFFERENTIAL INCLUSIONS RELATED TO GEOMETRIC PROBLEMS"

In this appendix we give the proof of some results stated in the first part of this thesis, namely Proposition 3.4 and Lemma 3.10.

B.1 PROOF OF PROPOSITION 3.4

First, by [38, Sec. 1.5,Th. 1], one has that if $w \in W^{1,m}(\Omega, \mathbb{R}^{m+n})$, then for every measurable set $A \subset \Omega$ and every measurable function $g : \mathbb{R}^{m+n} \to \mathbb{R}$ for which

$$g(w(\cdot))J_w(\cdot) \in L^1(A), \tag{B.1}$$

it holds

$$\int_A g(w(x))J_w(x)\,dx = \int_{\mathbb{R}^{m+n}} g(z)N(w,A,z)d\mathcal{H}^m(z),$$

where

$$J_w(x) = \sqrt{\det(Dw(x)^T Dw(x))} \text{ and } N(w,A,z) \doteq \#\{x : x \in A \cap A_D(w), w(x) = z\}.$$

We want to apply this result with

$$A = \mathcal{L}_u,\ w(x) = v(x),\ g(x,y) \doteq \varphi(v(x), T_{v(x)}\mathcal{G}_u)\theta(x,y), \quad \forall x \in \Omega, y \in \mathbb{R}^n.$$

With θ appearing in the last equation we mean any representative of this Borel, L^∞ function, so that g is a well defined measurable function in $\Omega \times \mathbb{R}^n$. We note that the fact that $\theta(x,u(x)) = \beta(x)$ for a.e. $x \in \Omega$ does not depend on the choice of the representative. Indeed, if θ' and β' are any representatives of θ and β respectively, then

$$\{x \in \Omega : \theta'(x,u(x)) = \beta'(x)\} \supseteq \{x \in \Omega : x \text{ is a Lebesgue point for } \beta\}.$$

This justifies the choice of *any* representative of θ in the definition of g, and our notation θ both for the initial L^∞ function and the representative. We will now proceed with the proof. It is straightforward by the fact that $\mathcal{R}_u = \mathcal{L}_u \cap A_D(u)$ and the definition of $v(x)$ that $N(v,\mathcal{L}_u,z) = 1$ for $\mathcal{H}^m \llcorner \mathcal{G}_u$-a.e. $z \in \mathbb{R}^{m+n}$ and $N(v,\mathcal{L}_u,z) = 0$ if $z \notin \mathcal{G}_u$. Hence:

$$\int_{\mathbb{R}^{m+n}} g(z)N(w,A,z)d\mathcal{H}^m(z) = \int_{\mathcal{G}_u} \varphi(z, T_z\mathcal{G}_u)\theta(z)d\mathcal{H}^m(z).$$

Moreover, since $|\Omega \setminus \mathcal{L}_u| = 0$, $J_w(x) = \mathcal{A}(Du(x))$ and $\theta(x,u(x)) = \beta(x)$, we also find

$$\int_{\mathcal{L}_u} g(w(x))J_w(x)\,dx = \int_\Omega \varphi(v(x), T_{v(x)}\mathcal{G}_u)\mathcal{A}(Du(x))\beta(x)\,dx\,.$$

Since $u \in W^{1,m}(\Omega, \mathbb{R}^n)$, $\varphi \in C_b(\Omega \times \mathbb{R}^n \times \mathbb{G}_0)$ and $\theta \in L^\infty(\Omega \times \mathbb{R}^n)$, (B.1) is fulfilled and we can apply the aforementioned result [38, Sec. 1.5,Th. 1] to obtain the desired equality (3.9).

B.2 PROOF OF LEMMA 3.10

First of all we compute $D\mathcal{A}(X)$. Recall the notation on multi-indices introduced in Definition 2.13 and the definition of the matrix $\overline{\operatorname{cof}(X^Z)^T}$ in the proof of Proposition 2.19. Then, since

$$\mathcal{A}(X) = \sqrt{1 + \|X\|^2 + \sum_{2 \le r \le \min\{m,n\}} \sum_{Z \in \mathcal{A}_r} \det(X_Z)^2},$$

we have

$$D\mathcal{A}(X) = \frac{X + \sum_{2\le r\le \min\{m,n\}} \sum_{Z\in\mathcal{A}_r} \det(X_Z)\overline{\mathrm{cof}(X^Z)^T}}{\mathcal{A}(X)}, \ \forall X \in \mathbb{R}^{n\times m}. \tag{B.2}$$

Next, we observe that by the chain rule

$$D(\Psi(h(X))_{ij} = \sum_{1\le\alpha,\beta\le m+n} (\partial_{\alpha\beta}\Psi)(h(X))\partial_{ij}h_{\alpha\beta}(X),$$

hence

$$D(\Psi(h(X)) = \sum_{1\le\alpha,\beta\le m+n} (\partial_{\alpha\beta}\Psi)(h(X))Dh_{\alpha\beta}(X). \tag{B.3}$$

We can therefore write

$$\begin{aligned} A(X) &= \Psi(h(X))D\mathcal{A}(X) + \mathcal{A}(X)D(\Psi(h(X)) \\ &= \Psi(h(X))D\mathcal{A}(X) + \mathcal{A}(X)\sum_{1\le\alpha,\beta\le m+n} (\partial_{\alpha\beta}\Psi)(h(X))Dh_{\alpha\beta}(X) \end{aligned}$$

and

$$B(X) = \Psi(h(X))(-X^TD\mathcal{A}(X) + \mathcal{A}(X)\,\mathrm{id}_m) + \mathcal{A}(X)\sum_{1\le\alpha,\beta\le m+n} (\partial_{\alpha\beta}\Psi)(h(X))X^TDh_{\alpha\beta}(X).$$

Since $\mathbb{G}(m, m+n)$ is compact, we have that both $\Psi(h(X))$ and $(D\Psi)(h(X))$ are bounded in $L^\infty(\mathbb{R}^{n\times m})$ by a constant $c > 0$ and using (B.2), we can bound

$$\Psi(h(X))\|D\mathcal{A}(X)\| \lesssim \|X\|^{\min\{m,n\}-1}.$$

Moreover, for every $X \in \mathbb{R}^{n\times m}$, $2 \le r \le \min\{m,n\}$ and $Z \in \mathcal{A}_r$, we have

$$X^T\overline{\mathrm{cof}(X^Z)^T} = \det(X_Z)I_Z,$$

where, if δ_{ab} denotes Kronecker's delta and Z has the form $Z = (i_1, \dots, i_r, j_1, \dots, j_r)$, I_Z is the $m \times m$ matrix defined as

$$(I_Z)_{ij} = \begin{cases} 0, & \text{if } i \ne i_a \text{ or } j \ne j_b, \forall a, b, \\ \delta_{ab}, & \text{if } i = i_a, j = j_b. \end{cases}$$

Therefore

$$-X^TD\mathcal{A}(X) + \mathcal{A}(X)\,\mathrm{id}_m = -\frac{X^TX + \sum_{2\le r\le\min\{m,n\}}\sum_{Z\in\mathcal{A}_r}\det^2(X_Z)I_Z - \mathcal{A}^2(X)\,\mathrm{id}_m}{\mathcal{A}(X)}. \tag{B.4}$$

If $n \le m-1$, then the best way to estimate the previous expression is simply

$$\|X^TD\mathcal{A}(X) - \mathcal{A}(X)\,\mathrm{id}_m\| \lesssim 1 + \|X\|^n.$$

On the other hand, if $n \ge m$, then for $Z \in \mathcal{A}_m$ we have $I_Z = \mathrm{id}_m$, hence (B.4) becomes

$$\begin{aligned} -X^TD\mathcal{A}(X) + \mathcal{A}(X)\,\mathrm{id}_m &= -\frac{X^TX + \sum_{2\le r\le\min\{m,n\}}\sum_{Z\in\mathcal{A}_r}\det^2(X_Z)I_Z - \mathcal{A}^2(X)\,\mathrm{id}_m}{\mathcal{A}(X)} \\ &= -\frac{X^TX - (1+\|X\|^2)\,\mathrm{id}_m + \sum_{2\le r\le m-1}\sum_{Z\in\mathcal{A}_r}\det^2(X_Z)(I_Z - \mathrm{id}_m)}{\mathcal{A}(X)}. \end{aligned}$$

In this case

$$\|X^TD\mathcal{A}(X) - \mathcal{A}(X)\,\mathrm{id}_m\| \lesssim 1 + \|X\|^{m-1}.$$

To conclude the proof of the Lemma, we still need to prove that for every $1 \le i, j \le m+n$,

$$\mathcal{A}(X)\|Dh_{ij}(X)\| \lesssim 1 + \|X\|^{\min\{n,m\}-1}, \ \mathcal{A}(X)\|X^TDh_{ij}(X)\| \lesssim 1 + \|X\|^{\min\{m-1,n\}}. \tag{B.5}$$

To perform the computation, we need to divide it into cases corresponding to the four blocks of the matrix $h(X)$ as written in (3.5). To this end, recall the notation

$$S(X) = (\mathrm{id}_m + X^T X)^{-1},$$

and moreover notice that $h(X)$ is symmetric, therefore we just need to prove (B.5) in the case $i \leq j$. Another useful fact is the following. First notice that for every matrices $N \in O(n)$, $M \in O(m)$ ($O(k)$ is the group of orthogonal matrices of order k), one has

$$S(NXM) = M^T S(X) M.$$

From an easy computation we then conclude that, for every $1 \leq i, j \leq m+n$ and for every $X \in \mathbb{R}^{n \times m}, N \in O(n), M \in O(m)$,

$$\|Dh_{ij}(X)\| \lesssim \sum_{1 \leq a,b \leq m+n} \|Dh_{ab}(NXM)\| \tag{B.6}$$

$$\|X^T Dh_{ij}(X)\| \lesssim \sum_{1 \leq a,b \leq m+n} \|(NXM)^T Dh_{ab}(NXM)\|. \tag{B.7}$$

Since also $\mathcal{A}(X) = \mathcal{A}(NXM)$, $\forall X \in \mathbb{R}^{n \times m}, M \in O(m), N \in O(n)$, (B.6)-(B.7) tell us that we can check estimates (B.5) just on matrices $Y \doteq NXM$ with two additional hypotheses. Fix $X \in \mathbb{R}^{n \times m}$, define $Z = XM$ and denote the j-th column of a matrix $A \in \mathbb{R}^{n \times m}$ with A^j. First, by a suitable choice of M, we can make sure that $Y^T Y = Z^T Z = M^T X^T X M$ is diagonal. Once this choice is made, if $n \geq m$, then we choose $N = \mathrm{id}_n$. Otherwise, if $n < m$, then we observe that at most n of the columns of Z are non-zero, let these be $Z^{j_1}, \dots, Z^{j_n}$ and let us define $J \doteq \{j_1, \dots, j_n\}$ with $1 \leq j_1 < j_2 < \cdots < j_n \leq m$. If for some j_k we have $Z^{j_k} = 0$, then we set $N = \mathrm{id}_n$. Otherwise, the $n \times n$ matrix V formed using $Z^{j_1}, \dots, Z^{j_n}$ has columns that are pairwise orthogonal and nonzero, hence there exists $O \in O(n)$ such that

$$V = OD,$$

with D diagonal. In this case, we choose $N = O^T$, so that the resulting Y has the property that

$$Y^j = \begin{cases} y_\ell e_\ell & \text{if } j = j_\ell, j_\ell \in \{j_1, \dots, j_n\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $y_j \in \mathbb{R}$ and e_ℓ are the vectors of the canonical basis of $\mathbb{R}^n$. Notice that this choice of M and N also implies that

$$\mathcal{A}(Y) = \sqrt{\prod_{i=1}^m (1 + \|Y^i\|^2)} \text{ and } S(Y) = \mathrm{diag}((1 + \|Y^1\|^2)^{-1}, \dots, (1 + \|Y^m\|^2)^{-1}).$$

We call (HP) these assumptions on the matrix $Y \in \mathbb{R}^{n \times m}$.

$\boxed{\text{First case, } 1 \leq i \leq j \leq m\text{:}}$ In this case, $h_{ij} = S_{ij}$. We have

$$\sum_{1 \leq k \leq m} S^{-1}_{ik} S_{kj} = \delta_{ij},$$

hence, taking a derivative,

$$\sum_{1 \leq k \leq m} \partial_{ab} S^{-1}_{ik} S_{kj} + \sum_{1 \leq k \leq m} S^{-1}_{ik} \partial_{ab} S_{jk} = 0.$$

We can invert the previous relation to get

$$\partial_{ab} S_{kl} = - \sum_{1 \leq c,d \leq m} S_{kc} S_{ld} \partial_{ab} S^{-1}_{cd}. \tag{B.8}$$

Finally, since $S^{-1}_{ik} = \delta_{ik} + \sum_{1\le l\le m} x_{li}x_{lk}$, we have

$$\partial_{ab}S^{-1}_{ik} = \sum_{1\le c\le m}\delta^{ab}_{ci}x_{ck} + \sum_{1\le c\le m}\delta^{ab}_{ck}x_{ci},$$

where the symbol $\delta^{cd}_{\alpha\beta} = 0$ if $\alpha \neq c$ or $\beta \neq d$, otherwise $\delta^{cd}_{\alpha\beta} = \delta^{\alpha\beta}_{\alpha\beta} = 1$. We can therefore use (B.8) to write

$$\begin{aligned}\partial_{ab}S_{ij} &= -\sum_{1\le k,l\le m} S_{ik}S_{jl}\left(\sum_{1\le c\le m}\delta^{ab}_{ck}x_{cl} + \sum_{1\le c\le m}\delta^{ab}_{cl}x_{ck}\right)\\ &= -\sum_{1\le k,l,c\le m} S_{ik}S_{jl}\delta^{ab}_{ck}x_{cl} - \sum_{1\le k,l,c\le m}\delta^{ab}_{cl}x_{ck}S_{ik}S_{jl} = -\sum_{1\le l\le m}\left(S_{ib}S_{jl}x_{al} + x_{al}S_{il}S_{jb}\right).\end{aligned} \tag{B.9}$$

Moreover,

$$(X^TDS_{ij})_{cd} = \sum_{1\le a\le n} x_{ac}\partial_{ad}S_{ij} = -\sum_{1\le l\le m, 1\le a\le n}\left(S_{id}S_{jl}x_{al}x_{ad} + x_{ad}x_{al}S_{il}S_{jd}\right).$$

Now we use our previous observation (B.6)-(B.7) to consider Y satisfying (HP), so that in particular Y^TY is diagonal. In this case, we have

$$|\partial_{ab}S_{ij}(Y)| \le \sum_{1\le l\le m}\left(|S_{ib}S_{jl}y_{al}| + |y_{al}S_{il}S_{jb}|\right).$$

For every $1 \le i, b, j, l \le m, 1 \le a \le n$,

$$\mathcal{A}(Y)|S_{ib}S_{jl}y_{al}| \le \sqrt{\prod_{c=1}^{m}(1+\|Y^i\|^2)}\frac{|y_{al}|}{(1+\|Y^b\|^2)(1+\|Y^l\|^2)}.$$

Let us explain in detail how to get the desired estimate (B.5) in this case. Notice that either Y^l is 0, and in this case there is nothing to prove, or $Y^l \neq 0$. Thanks to (HP), in Y there are at most $\min\{m, n\}$ non-zero columns. First let $m \le n$, then:

$$\sqrt{\prod_{c=1}^{m}(1+\|Y^c\|^2)}\frac{|y_{al}|}{(1+\|Y^b\|^2)(1+\|Y^l\|^2)} \lesssim \sqrt{\prod_{c=1}^{m}(1+\|Y^c\|^2)}\frac{1}{\sqrt{1+\|Y^l\|^2}} \lesssim 1 + \|Y\|^{m-1}.$$

If $n < m$ and J is the set on indices corresponding to non-zero columns, we are in the hypothesis in which $l \in J$. Therefore we have

$$\sqrt{\prod_{c=1}^{m}(1+\|Y^c\|^2)}\frac{|y_{al}|}{(1+\|Y^b\|^2)(1+\|Y^l\|^2)} \lesssim \sqrt{\prod_{c\in J}(1+\|Y^c\|^2)}\frac{1}{\sqrt{1+\|Y^l\|^2}} \lesssim 1 + \|Y\|^{m-1}.$$

This proves that

$$\|Dh_{ij}(Y)\| \lesssim 1 + \|Y\|^{\min\{m,n\}-1} \text{ for } 1 \le i, j \le m. \tag{B.10}$$

We also have

$$\mathcal{A}(Y)|(Y^TDS_{ij})_{cd}(Y)| \le \mathcal{A}(Y)\sum_{1\le l\le m, 1\le a\le n}\left(|S_{id}S_{jl}y_{al}y_{ad}| + |y_{ad}y_{al}S_{il}S_{jd}|\right).$$

Analogously to the previous case, we estimate for every $1 \le i, d, j, l \le m, 1 \le a \le n$,

$$\mathcal{A}(Y)|S_{id}S_{jl}y_{al}y_{ad}| \le \sqrt{\prod_{c=1}^{m}(1+\|Y^c\|^2)}\frac{|y_{al}||y_{ad}|}{(1+\|Y^d\|^2)(1+\|Y^l\|^2)},$$

and the desired estimate is obtained with a reasoning completely analogous to the one of (B.10). This concludes the proof of this case.

$\boxed{\text{Second case, } 1 \leq i \leq m < m+1 \leq j \leq m+n\text{:}}$ From now on we use $m+j$ rather than j for the corresponding index. We thus have

$$h_{ij+m}(X) = (S(X)X^T)_{ij} = \sum_{k=1}^{m} S_{ik}x_{jk}\,.$$

We compute the derivative using (B.9):

$$\partial_{ab}h_{ij+m}(X) = \sum_{k=1}^{m} \delta_{jk}^{ab} S_{ik} + \sum_{k=1}^{m} x_{jk}\partial_{ab}S_{ik} = \sum_{k=1}^{m} \delta_{jk}^{ab} S_{ik} - \sum_{1\leq l,k\leq m} \left(S_{ib}S_{kl}x_{al}x_{jk} + x_{al}x_{jk}S_{il}S_{kb}\right),$$

and also

$$\begin{aligned}(X^TDh_{ij+m}(X))_{ab} = \sum_{1\leq c\leq n} x_{ca}\partial_{cb}h_{ij} &= \sum_{1\leq k\leq m, 1\leq c\leq n} x_{ca}\delta_{jk}^{cb}S_{ik} \\ &\quad - \sum_{1\leq l,k\leq m, 1\leq c\leq n} \left(x_{ca}S_{ib}S_{kl}x_{cl}x_{jk} + x_{ca}x_{cl}x_{jk}S_{il}S_{kb}\right) \\ &= x_{ja}S_{ib} - \sum_{1\leq l,k\leq m, 1\leq c\leq n} \left(x_{ca}S_{ib}S_{kl}x_{cl}x_{jk} + x_{ca}x_{cl}x_{jk}S_{il}S_{kb}\right)\end{aligned}$$

Since $S^{-1}(X) = \mathrm{id}_m + X^TX$,

$$\delta_{ij} = \sum_{1\leq k\leq m} S_{ik}(\delta_{kj} + \sum_{1\leq c\leq n} x_{ck}x_{cj}) = S_{ij} + \sum_{1\leq k\leq m, 1\leq c\leq n} S_{ik}x_{ck}x_{cj}, \tag{B.11}$$

hence we can rewrite

$$\begin{aligned}(X^TDh_{ij+m}(X))_{ab} &= x_{ja}S_{ib} - \sum_{1\leq l,k\leq m, 1\leq c\leq n} \left(x_{ca}S_{ib}S_{kl}x_{cl}x_{jk} + x_{ca}x_{cl}x_{jk}S_{il}S_{kb}\right) \\ &= x_{ja}S_{ib} - \sum_{k=1}^{m} \left(S_{ib}x_{jk} \sum_{1\leq l\leq m, 1\leq c\leq n} x_{ca}S_{kl}x_{cl} + x_{jk}S_{kb} \sum_{1\leq l\leq m, 1\leq c\leq n} x_{ca}x_{cl}S_{il}\right) \\ &= x_{ja}S_{ib} - \sum_{k=1}^{m} S_{ib}x_{jk}(\delta_{ka} - S_{ka}) - \sum_{k=1}^{m} x_{jk}S_{kb}(\delta_{ai} - S_{ai}) \\ &= \sum_{k=1}^{m} S_{ib}x_{jk}S_{ka} + \sum_{k=1}^{m} x_{jk}S_{kb}\delta_{ai} + \sum_{k=1}^{m} x_{jk}S_{kb}S_{ai}.\end{aligned} \tag{B.12}$$

Now we evaluate the previous expressions at Y satisfying (HP). Using the fact that Y^TY is diagonal, we simplify:

$$\begin{aligned}\partial_{ab}h_{ij+m}(Y) &= \sum_{k=1}^{m} \delta_{jk}^{ab} S_{ik} - \sum_{1\leq l,k\leq m} \left(S_{ib}S_{kl}y_{al}y_{jk} + y_{al}y_{jk}S_{il}S_{kb}\right) \\ &= \delta_{ja}\delta_{ib}S_{ii} - \sum_{1\leq k\leq m} \delta_{ib}S_{ii}S_{kk}y_{ak}y_{jk} - y_{ai}y_{jb}S_{ii}S_{bb}\end{aligned} \tag{B.13}$$

First, let $m \leq n$. Then, using that for every $1 \leq a, j, k \leq n$ we have

$$|S_{kk}y_{ak}y_{jk}| \leq 1,$$

we estimate

$$\mathcal{A}(Y)|\partial_{ab}h_{ij+m}(Y)| \leq \mathcal{A}(Y)|S_{ii}| + \mathcal{A}(Y) \sum_{1\leq k\leq m} |S_{ii}S_{kk}y_{ak}y_{jk}| + \mathcal{A}(Y)y_{ai}y_{jb}S_{ii}S_{bb}$$

$$\lesssim \frac{\mathcal{A}(Y)}{1+\|Y^i\|^2} + \frac{\mathcal{A}(Y)}{\sqrt{1+\|Y^i\|^2}\sqrt{1+\|Y^b\|^2}} \lesssim 1+\|Y\|^{m-1}.$$

If $n < m$, let $J = \{j_1, \dots, j_n\}$ be the set of indices defined in (HP). If there exists ℓ such that $Z^{j_\ell} = 0$, then

$$\mathcal{A}(Y) = \sqrt{\prod_{t\in J}(1+\|Y^t\|^2)} = \sqrt{\prod_{t\in J\setminus\{j_k\}}(1+\|Y^t\|^2)} \lesssim 1+\|Y\|^{n-1}$$

and

$$|\partial_{ab}h_{ij+m}(Y)| \le S_{ii} + \sum_{1\le k\le m} S_{ii}S_{kk}|y_{ak}y_{jk}| + |y_{ai}y_{jb}|S_{ii}S_{bb} \lesssim 1,$$

therefore

$$\mathcal{A}(Y)|\partial_{ab}h_{ij}(Y)| \lesssim 1+\|Y\|^{n-1}.$$

Hence we are just left with the case $n < m$ and $Y^{j_\ell} \neq 0$ for every $1 \le \ell \le n$. If this is the case, (HP) implies that $y_{kj_\ell} = \delta_{k\ell}y_{\ell j_\ell}$, for $1 \le k \le n$, and $y_{kj} = 0$ if $j \notin J$ and $1 \le k \le n$. Therefore, recalling (B.13),

$$\partial_{ab}h_{ij+m}(Y) = \begin{cases} S_{ii} - S_{ii}S_{j_aj_a}y^2_{aj_a} - y_{ai}y_{jb}S_{ii}S_{bb} & \text{if } j=a, i=b,\\ -y_{ai}y_{jb}S_{ii}S_{bb} & \text{otherwise.}\end{cases}$$

In the first case, if $j = a, i = b$, we have

$$S_{j_aj_a} = \frac{1}{1+\|Y^{j_a}\|^2} = \frac{1}{1+y^2_{aj_a}},$$

hence

$$1 - S_{j_aj_a}y^2_{aj_a} = 1 - \frac{y^2_{aj_a}}{1+y^2_{aj_a}} = \frac{1}{1+y^2_{aj_a}} = \frac{1}{1+\|Y^{j_a}\|^2}$$

that implies

$$\partial_{ab}h_{ij+m}(Y) = \frac{1}{1+\|Y^{j_a}\|^2} - \frac{y_{ai}y_{jb}}{(1+\|Y^i\|^2)(1+\|Y^b\|^2)},$$

and it is now easy to see that

$$\mathcal{A}(Y)|\partial_{ab}h_{ij+m}(Y)| \lesssim 1+\|Y\|^{n-1}.$$

Since if $j \neq a$ or $b \neq i$, $\partial_{ab}h_{ij+m}(Y) = -y_{ai}y_{jb}S_{ii}S_{bb}$, the same estimate follows. To finish the second case, we still need to show that

$$\mathcal{A}(Y)|(Y^TDh_{ij+m}(Y))_{ab}| \lesssim 1+\|Y\|^{\min\{m,n\}-1}.$$

To do so, we recall (B.12) to estimate

$$|(Y^TDh_{ij+m}(Y))_{ab}| \le \sum_{k=1}^m S_{ib}|y_{jk}|S_{ka} + \sum_{k=1}^m |y_{jk}|S_{kb}\delta_{ai} + \sum_{k=1}^m |y_{jk}|S_{kb}S_{ai}.$$

With similar computations to the one to prove (B.10), we estimate for $1 \le i,b,a,k \le m, 1 \le j \le n$,

$$\mathcal{A}(Y)S_{ib}|y_{jk}|S_{ka} \le \begin{cases} 0 & \text{if } Y^k = 0 \text{ or } k \neq a,\\ \sqrt{\prod_{l\neq k}(1+\|Y^l\|^2)} & \text{otherwise,}\end{cases}$$

that implies

$$\mathcal{A}(Y)S_{ib}|y_{jk}|S_{ka} \lesssim 1+\|Y\|^{\min\{m,n\}-1}.$$

Finally, since also for every $1 \le j \le n$, $1 \le k, b \le m$

$$\mathcal{A}(Y)|y_{jk}|S_{kb} \le \begin{cases} 0 & \text{if } Y^k = 0 \text{ or } k \neq b, \\ \sqrt{\prod_{l \neq k}(1 + \|Y^l\|^2)} & \text{otherwise,} \end{cases}$$

we find

$$\mathcal{A}(Y)|y_{jk}|S_{kb} \lesssim 1 + \|Y\|^{\min\{m,n\}-1}, \forall 1 \le k, b \le m, 1 \le j \le n.$$

This completes the proof of the second case.

$\boxed{\text{Third case, } m+1 \le i \le j \le m+n\text{:}}$ As above we use $m+i$ and $m+j$ in place of i and j. The indices i and j will then satisfy $1 \le i \le j \le n$ and we have

$$h_{i+m,j+m}(X) = (XS(X)X^T)_{ij} = \sum_{1 \le l,k \le m} x_{il} S_{lk} x_{jk}.$$

We compute the derivative using (B.9):

$$\begin{aligned} \partial_{ab} h_{i+m,j+m}(X) &= \sum_{1 \le l,k \le m} \delta_{il}^{ab} S_{lk} x_{jk} + \sum_{1 \le l,k \le m} \delta_{jk}^{ab} S_{lk} x_{il} + \sum_{1 \le l,k \le m} x_{il} \partial_{ab} S_{lk} x_{jk} \\ &= \sum_{1 \le k \le m} \delta_{ia} S_{bk} x_{jk} + \sum_{1 \le l \le m} \delta_{ja} S_{lb} x_{il} \\ &\quad - \sum_{1 \le l,k,c \le m} S_{lb} S_{kc} x_{ac} x_{il} x_{jk} - \sum_{1 \le l,k,c \le m} x_{ac} S_{lc} S_{kb} x_{il} x_{jk}. \end{aligned}$$

Moreover,

$$\begin{aligned} (X^T D h_{i+m,j+m}(X))_{ab} = \sum_{1 \le d \le n} x_{da} \partial_{db} h_{ij} &= \sum_{1 \le d \le n, 1 \le k \le m} \delta_{id} x_{da} S_{bk} x_{jk} + \sum_{1 \le d \le n, 1 \le l \le m} \delta_{jd} S_{lb} x_{il} x_{da} \\ &\quad - \sum_{1 \le c,l,k \le m, 1 \le d \le n} S_{lb} S_{kc} x_{dc} x_{il} x_{jk} x_{da} \\ &\quad - \sum_{1 \le c,l,k \le m, 1 \le d \le n} x_{da} x_{dc} S_{lc} S_{kb} x_{il} x_{jk}. \end{aligned} \tag{B.14}$$

By (B.11), we have, for every $1 \le i, j \le m$

$$\sum_{1 \le k \le m, 1 \le d \le n} S_{ik} x_{dk} x_{dj} = \delta_{ij} - S_{ij}.$$

Hence we can rewrite in (B.14):

$$\begin{aligned} (X^T D h_{i+m,j+m}(X))_{ab} = \sum_{1 \le d \le n} x_{da} \partial_{db} h_{ij} &= \sum_{1 \le d \le n, 1 \le k \le m} \delta_{id} x_{da} S_{bk} x_{jk} + \sum_{1 \le d \le n, 1 \le l \le m} \delta_{jd} S_{lb} x_{il} x_{da} \\ &\quad - \sum_{1 \le l,k \le m} S_{lb} x_{il} x_{jk} (\delta_{ka} - S_{ka}) - \sum_{1 \le l,k \le n} S_{kb} x_{il} x_{jk} (\delta_{la} - S_{la}) \\ &= \sum_{1 \le d \le n, 1 \le k \le m} \delta_{id} x_{da} S_{bk} x_{jk} + \sum_{1 \le d \le n, 1 \le l \le m} \delta_{jd} S_{lb} x_{il} x_{da} \\ &\quad - \sum_{1 \le l,k \le m} S_{lb} x_{il} x_{jk} \delta_{ka} - \sum_{1 \le l,k \le m} S_{kb} x_{il} x_{jk} \delta_{la} \\ &\quad + \sum_{1 \le l,k \le m} S_{lb} x_{il} x_{jk} S_{ka} + \sum_{1 \le l,k \le m} S_{kb} x_{il} x_{jk} S_{la} \\ &= \sum_{1 \le k \le m} x_{ia} S_{bk} x_{jk} + \sum_{1 \le l \le m} S_{lb} x_{il} x_{ja} \\ &\quad - \sum_{1 \le l \le m} S_{lb} x_{il} x_{ja} - \sum_{1 \le k \le m} S_{kb} x_{ia} x_{jk} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{1\le l,k\le m} S_{lb}x_{il}x_{jk}S_{ka} + \sum_{1\le l,k\le m} S_{kb}x_{il}x_{jk}S_{la} \\
&= \sum_{1\le l,k\le m} S_{lb}x_{il}x_{jk}S_{ka} + \sum_{1\le l,k\le m} S_{kb}x_{il}x_{jk}S_{la}.
\end{aligned}
$$

Consider once again Y fulfilling (HP). Then:

$$
\begin{aligned}
\partial_{ab}h_{i+m,j+m}(Y) = &\sum_{1\le k\le m} \delta_{ia}S_{bk}y_{jk} + \sum_{1\le l\le m} \delta_{ja}S_{lb}y_{il} \\
&- \sum_{1\le l,k,c\le m} S_{lb}S_{kc}y_{ac}y_{il}y_{jk} - \sum_{1\le l,k,c\le m} x_{ac}S_{lc}S_{kb}y_{il}y_{jk}.
\end{aligned}
$$

Since Y^TY is diagonal, this expression simplifies as:

$$
\begin{aligned}
\partial_{ab}h_{i+m,j+m}(Y) = &\,\delta_{ia}S_{bb}y_{jb} + \delta_{ja}S_{bb}y_{ib} \\
&- \sum_{1\le c\le m} S_{bb}S_{cc}y_{ac}y_{ib}y_{jc} - \sum_{1\le c\le m} y_{ac}S_{cc}S_{bb}y_{ic}y_{jb}.
\end{aligned}
$$

For every $1 \le b \le m, 1 \le j \le n$,

$$
\mathcal{A}(Y)S_{bb}|y_{jb}| \le \begin{cases} 0 & \text{if } Y^b = 0, \\ \frac{\mathcal{A}(Y)}{\sqrt{1+\|Y^b\|^2}} & \text{otherwise.} \end{cases} \tag{B.15}
$$

This yields

$$
\mathcal{A}(Y)S_{bb}|y_{jb}| \lesssim 1 + \|Y\|^{\min\{m,n\}-1}.
$$

To prove that

$$
\mathcal{A}(Y)|\partial_{ab}h_{i+mj+m}(Y)| \lesssim 1 + \|Y\|^{\min\{m,n\}-1}, \tag{B.16}
$$

we still need to estimate terms of the form

$$
\mathcal{A}(Y)S_{bb}S_{cc}|y_{ac}y_{ib}y_{jc}|,
$$

for $1 \le b, c \le m$, $1 \le a, i, j \le n$. Anyway, observe that

$$
S_{cc}|y_{ac}||y_{jc}| \le 1, \forall 1 \le c \le m, 1 \le a, j \le n,
$$

hence

$$
\mathcal{A}(Y)S_{bb}S_{cc}|y_{ac}y_{ib}y_{jc}| \le \mathcal{A}(Y)S_{bb}|y_{ib}|
$$

and we can therefore apply again estimate (B.15) to deduce (B.16). To finish the proof of this case and of the present Lemma, we still need to show that

$$
|(Y^TDh_{i+m,j+m}(Y))_{ab}| \lesssim 1 + \|Y\|^{\min\{m,n\}-1}. \tag{B.17}
$$

To do so, recall that

$$
\begin{aligned}
(Y^TDh_{i+m,j+m}(Y))_{ab} &= \sum_{1\le l,k\le m} S_{lb}y_{il}y_{jk}S_{ka} + \sum_{1\le l,k\le m} S_{kb}y_{il}y_{jk}S_{la} \\
&= S_{bb}S_{aa}y_{ib}y_{ja} + S_{bb}S_{aa}y_{ia}y_{jb},
\end{aligned}
$$

but now, for every $1 \le a, b \le m, 1 \le i, j \le n$,

$$
\mathcal{A}(Y)S_{bb}S_{aa}|y_{ib}y_{ja}| \le \begin{cases} 0 & \text{if } Y^b = 0 \text{ or } Y^a = 0, \\ \frac{\mathcal{A}(Y)}{\sqrt{1+\|Y^b\|^2}\sqrt{1+\|Y^a\|^2}} & \text{otherwise.} \end{cases}
$$

The proof of (B.17) is now analogous to the one of (B.16).

C

APPENDIX TO "DIVERGENCE-FREE MATRIX FIELDS IN $\mathrm{Sym}^+(n)$"

In this appendix to Chapter ii, we use the family of functions of Lemma 5.3 to deduce optimal integrability of the *pointwise* determinant of the Hessian of convex functions.

Instead of *divergence-free* tensor fields A, here we consider *curl-free* tensor fields $A : \Omega \to \mathrm{Sym}^+(n)$, where $\Omega \subset \mathbb{R}^n$ is convex. The system $\mathrm{curl}(A) = 0$ and the convexity of Ω, together with the symmetry of A, defines the class of Hessians of functions. Once we also add the non-negativity of the eigenvalues, we are led to consider exactly Hessians of convex functions. We ask the following question: given $\Omega \subset \mathbb{R}^n$, an open and convex set, let $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $p \in [1, +\infty)$, be a convex function. What can be said about the integrability of $\det(H\varphi)$? For a Sobolev convex function the distributional Hessian $H\varphi$ can also be computed with the classical definition of the second derivatives almost everywhere, as proved in [31, Theorem 6.9]. On the other hand there are various definitions of determinant of the Hessian. The one we are interested in is the pointwise determinant of the Hessian matrix, and the results we are proving concern its integrability. Another notion of determinant of the Hessian we will be using is the Monge-Ampère measure associated to φ, defined in the introduction. The two notions are related by the fact that $\det(H\varphi)$ is the density of the absolutely continuous part of the Monge-Ampère measure μ_φ associated to φ (see for instance [70, Lemma 1.18]). In particular by the Radon-Nikodym Theorem we have

$$\det(H\varphi) \in L^1_{\mathrm{loc}}(\Omega) \qquad \forall\, \varphi \ \text{ convex.} \tag{C.1}$$

Let us start with the analysis of the optimal integrability of $x \mapsto \det(H\varphi)(x)$. The case $p = n$ has been covered (in a more general setting) by S. Müller in [59, 60] (see also [73] for the same result for mappings with determinants of arbitrary sign). More precisely, it is proved in [59, Theorem 1] that

$$\det(H\varphi) \in \mathcal{H}^1_{\mathrm{loc}}(\Omega),$$

where $\mathcal{H}^1_{\mathrm{loc}}(\Omega)$ is the local Hardy space, see Section A.3, and moreover that this is optimal in the following sense. In [60, Counterexample 7.2], Müller finds a sequence of maps $u_j \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ such that for every function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{\gamma(z)}{z\log(1+z)} \to +\infty, \text{ as } z \to \infty,$$

one has

$$\|\gamma(\det(\nabla u_j))\|_{L^1(B_1(0))} \to +\infty, \text{ as } j \to \infty.$$

It is immediate to see that this sequence u_j is actually $u_j = \nabla\varphi_j$, for some convex function $\varphi_j \in W^{2,n}_{\mathrm{loc}}(\mathbb{R}^n)$, hence Müller's results close the question in the case $p = n$. Theorem C.1 answers the question in the case $p \in [1, \infty) \setminus \{n\}$. Let us first introduce the following space: for any compact set $K \subset \Omega$, with $\mathrm{clos}(\mathrm{int}(K)) = K \neq \emptyset$,

$$H_{p,K} \doteq \{\varphi \in W^{2,p}(\Omega) : \varphi \text{ is convex, } H\varphi \equiv \bar{A} \text{ outside } K,$$
$$\text{for some fixed } \bar{A} \in \mathrm{Sym}^+(n)\}.$$

This is a complete metric space when endowed with the distance

$$d(\varphi_1, \varphi_2) \doteq \|\varphi_1 - \varphi_2\|_{W^{2,p}(\Omega)}.$$

Theorem C.1. *The following hold*

(i) If $p \in [1, n)$, then $\forall \varphi \in W^{2,p}(\Omega)$ convex, $\det(H\varphi) \in L^1_{\mathrm{loc}}(\Omega)$, but there exists a convex function $\bar{\varphi} \in W^{2,p}(\Omega)$ such that $\det(H\bar{\varphi}) \in L^1_{\mathrm{loc}}(\Omega) \setminus \mathcal{H}^1_{\mathrm{loc}}(\Omega)$;

(ii) If $p \in (n, +\infty)$, then $\forall \varphi \in W^{2,p}(\Omega)$ convex, $\det(H\varphi) \in L^{\frac{p}{n}}_{\mathrm{loc}}(\Omega)$, but there exists a convex function $\bar{\varphi} \in W^{2,p}(\Omega)$ such that $\det(H\bar{\varphi}) \in L^{\frac{p}{n}}_{\mathrm{loc}}(\Omega) \setminus L^{\frac{p}{n}+\varepsilon}_{\mathrm{loc}}(\Omega), \forall \varepsilon > 0$;

Proof. The *positive* part of the statements of the Theorem are clear: if $p < n$, then it was written in (C.1), while if $p > n$, $\det(H\varphi) \in L^{\frac{p}{n}}_{\mathrm{loc}}(\Omega)$ by Hölder inequality. Let us now show the optimality of these results. The optimality for the case (i) is the content of Proposition C.2. To find a convex function $\bar{\varphi} \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $p > n$, such that $\det(H\bar{\varphi}) \in L^{\frac{p}{n}}_{\mathrm{loc}}(\Omega) \setminus L^{\frac{p}{n}+\varepsilon}_{\mathrm{loc}}(\Omega)$ for every $\varepsilon > 0$, consider again the family of functions f_α defined in (5.1). As proved in Step 2 of Lemma 5.3 and Lemma 5.4, we find that if $\alpha > \frac{p-n}{p}$, then $f_\alpha \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and for every $\varepsilon > 0$, we find $\alpha = \alpha(\varepsilon) > 0$ such that $f_\alpha \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ but

$$\det(Hf_\alpha) \notin L^{\frac{p}{n}+\varepsilon}(B_r(0)), \text{ for any } r > 0.$$

With a construction analogous to the one of Lemma 5.3 and the same proof as in Theorem 5.1, it is possible to prove that the set

$$\{\varphi \in H_{p,K} : \det(H\varphi) \in L^{\frac{p}{n}}(\Omega) \setminus \bigcup_{\varepsilon > 0} L^{\frac{p}{n}+\varepsilon}(\Omega)\}$$

is residual in $H_{p,K}$. By Baire's Theorem A.4, we then deduce the existence of such a function $\bar{\varphi}$. $\square$

We will now prove the optimality of (i) of Theorem C.1, namely

Proposition C.2. *Let $p \in [1, n)$. The set*

$$U_{p,K} \doteq \{\varphi \in H_{p,K} : \det(H\varphi) \in L^1(\Omega) \setminus \mathcal{H}^1(\Omega)\}$$

is residual in $H_{p,K}$.

To prove Proposition C.2, we first need the following result.

Lemma C.3. *Let $\varphi : \Omega \to \mathbb{R}$ be a convex function such that its Monge-Ampère measure μ_φ has a non-trivial singular part with respect to the Lebesgue measure. Then, for every sequence of smooth and convex functions φ_j converging locally uniformly to φ and for every $z_0 \in \mathrm{spt}(\mu^s_\varphi)$, we have*

$$\|\det(H\varphi_j)\|_{\mathcal{H}^1(B_r(z_0))} \to +\infty, \text{ as } j \to \infty,$$

for every $B_r(z_0)$ compactly contained in Ω.

Proof. By contradiction, suppose there exists a sequence $(\varphi_j)_j$, a point $z_0 \in \mathrm{spt}(\mu^s_\varphi)$ and $r > 0$ as in the statement such that

$$\sup_j \|\det(H\varphi_j)\|_{\mathcal{H}^1(B_r(z_0))} < +\infty.$$

Equi-boundedness in $\mathcal{H}^1$ tells us that we can also assume, up to a non-relabeled subsequence, that $\det(H\varphi_j)$ converges weakly in L^1 to a function $F \in L^1(B_r(z_0))$ (see [60, Theorem 4.1] for a proof). By definition of the Monge-Ampère measure and the regularity of φ_j, we can write, $\forall f \in C^0_c(B_r(z_0))$,

$$\int_\Omega f(x) d\mu_{\varphi_j}(x) = \int_\Omega f(x) \det(H\varphi_j)(x)\, dx\,.$$

Now, the uniform convergence $\varphi_j \to \varphi$ implies that $\mu_{\varphi_n} \overset{*}{\rightharpoonup} \mu_\varphi$ (see [34, Proposition 2.9]), and the weak convergence of $\det(H\varphi_j)$ to F combined with the previous equality implies that, in the limit,

$$\int_\Omega f(x) d\mu_\varphi(x) = \int_\Omega F(x) f(x)\, dx\,, \quad \forall f \in C^0_c(B_r(z_0)).$$

The last equality implies $\mu_\varphi \llcorner B_r(z_0) = F\chi_{B_r(z_0)}\, dx$, contradicting the fact that $z_0 \in \mathrm{spt}(\mu^s_\varphi)$. $\square$

Proof of Proposition C.2. Fix $p \in [1, n)$. We consider the function f_0 constructed in the Steps 1 and 2 of Lemma 5.3. By Lemma 5.4, $f_0 \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ Analogously to Step 4 of the same lemma, for every $\beta, \delta, \varepsilon > 0$ and $x_0 \in \mathrm{int}(K)$, we consider $\varphi_{\beta,\delta,\varepsilon,x_0}$ defined as in (5.8). We choose x_0 arbitrarily and β such that $B_{2\beta}(x_0) \subset K$. For this proof we will not need ε, that we consider fixed. Therefore, we will write φ_δ instead of $\varphi_{\beta,\delta,\varepsilon,x_0}$ for the sake of readability. To prove the Proposition, we write $U^c_{p,K}$ as the countable union of closed sets:

$$U^c_{p,K} \doteq \bigcup_{m\in\mathbb{N}} \left\{ \varphi \in H_{p,K} : \| \det(H\varphi)\|_{\mathcal{H}^1(\Omega)} \leq m \right\}.$$

Each set $C_m \doteq \left\{ \varphi \in H_{p,K} : \| \det(H\varphi)\|_{\mathcal{H}^1(\Omega)} \leq m \right\}$ is closed. To prove that it has empty interior we reason by contradiction. Therefore, we find m, $\rho > 0$ and $\bar\varphi$ such that the ball $\mathcal{N}_\rho(\bar\varphi) \subset C_m$. Now choose $\delta > 0$ in such a way that $\|\varphi_\delta\|_{W^{2,p}(\Omega)} \leq \frac{\rho}{2}$. This can be done in view of (5.3) (in the case $\alpha = 0$). If we now mollify φ_δ, we get a sequence of smooth convex functions $\varphi_{\delta,j} \in H_{p,K}$ such that $\|\varphi_{\delta,j}\|_{W^{2,p}(\Omega)} \leq \frac{\rho}{2}, \forall j \in \mathbb{N}$. This sequence is also converging locally uniformly to φ_δ, since real-valued convex functions are locally Lipschitz. By the definition of φ_δ in (5.8) and the fact (see [34, Example 2.2(2)]) that

$$\mu_{f_0} \llcorner B_1(0) = \omega_n \delta_0,$$

we find that $x_0 \in \mathrm{spt}(\mu^s_{\varphi_\delta})$ and, by Lemma C.3, that

$$\| \det(H\varphi_{\delta,j})\|_{\mathcal{H}^1(\Omega)} \to +\infty, \text{ as } j \to +\infty. \tag{C.2}$$

Now, by our choice of δ, for every $j \in \mathbb{N}$, we have that

$$\bar\varphi + \varphi_{\delta,j} \in \mathcal{N}_\rho(\varphi),$$

hence

$$\| \det(H\bar\varphi + H\varphi_{\delta,j})\|_{\mathcal{H}^1(\Omega)} = \int_\Omega \det(H\bar\varphi + H\varphi_{\delta,j}) \log(1 + \det(H\bar\varphi + H\varphi_{\delta,j})) \leq m, \forall j \in \mathbb{N}. \tag{C.3}$$

By the monotonicity of the determinant on the cone of non-negative definite symmetric matrices, we have

$$\det(H\varphi_{\delta,j}) \leq \det(H\bar\varphi + H\varphi_{\delta,j}),$$

and since the function $x \mapsto x\log(1 + x)$ is increasing for $x \geq 0$, then

$$\| \det(H\varphi_{\delta,j})\|_{\mathcal{H}^1(\Omega)} \leq \| \det(H\bar\varphi + H\varphi_{\delta,j})\|_{\mathcal{H}^1(\Omega)} \overset{(C.3)}{\leq} m, \forall j \in \mathbb{N}.$$

The last inequality is in contradiction with (C.2). $\square$

D

APPENDIX TO "SHARP ENERGY REGULARITY FOR EULER EQUATIONS"

In this appendix to Part iii, we prove Proposition 7.2, a technical result concerning the improvement in the regularity in time for solutions of the Euler Equations (7.1) that is necessary for the proof of Theorem 7.5.

D.1 TIME ESTIMATES OF EULER EQUATIONS

Using the same technique introduced in [12] to prove the time regularity for Hölder solutions of Euler, we show Proposition 7.2:

Proposition D.1. *Let $u, v : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ be two weak solutions of (7.1) such that $u, v \in C^0(([0,T]; C^\theta(\mathbb{T}^3))$ for some $\theta \in (0,1)$. Then there exists a constant $C > 0$, depending only on θ, $\|u\|_\theta$ and $\|v\|_\theta$, such that*

$$\|u - v\|_{C^\theta_{x,t}} \le C\|u - v\|_\theta.$$

Proof. We define $w \doteq u - v$. We start by noticing that the Hölder norm, in the space-time variables, decouples as follows

$$\frac{|w(x,s) - w(y,t)|}{|(x,s) - (y,t)|^\theta} \le \frac{|w(x,s) - w(y,s)|}{|x - y|^\theta} + \frac{|w(y,s) - w(y,t)|}{|t - s|^\theta} \le \|w\|_\theta + \frac{|w(y,s) - w(y,t)|}{|t - s|^\theta}.$$

Thus it is enough to show that there exists a constant $C > 0$, independent of y, t, s, such that

$$\frac{|w(y,s) - w(y,t)|}{|t - s|^\theta} \le C\|w\|_\theta. \tag{D.1}$$

If p and q are the corresponding pressures associated to u and v respectively, one has that w solves

$$\partial_t w + \operatorname{div}(w \otimes u + v \otimes w) + \nabla(p - q) = 0. \tag{D.2}$$

By taking the divergence of (D.2), we get

$$-\Delta(p - q) = \operatorname{div}\operatorname{div}(w \otimes u + v \otimes w),$$

from which, by Schauder estimates, we get

$$\|p - q\|_\theta \le \|w\|_\theta \left(\|u\|_\theta + \|v\|_\theta\right) \le C\|w\|_\theta. \tag{D.3}$$

Let now $w_\delta = w * \varphi_\delta$ the space mollification of w, for some $\delta > 0$ that will be fixed at the end of the proof. Since $w \in C^0([0,T]; C^\theta(\mathbb{T}^3))$ we have

$$|w(y,t) - w_\delta(y,t)| \le C\|w\|_\theta \delta^\theta \qquad \forall t \in [0,T],$$

from which, by adding and subtracting $w_\delta(y,s)$ and $w_\delta(y,t)$, we can estimate

$$|w(y,s) - w(y,t)| \le C\|w\|_\theta \delta^\theta + |w_\delta(y,s) - w_\delta(y,t)|. \tag{D.4}$$

Moreover, since w solves (D.2), we get

$$|w_\delta(y,s) - w_\delta(y,t)| \le |t - s| \|\partial_t w_\delta\|_{C^0_{x,t}} \le |t - s| \left(\|(w \otimes u + v \otimes w)_\delta\|_1 + \|(p - q)_\delta\|_1\right). \tag{D.5}$$

By estimate (D.3) and (7.5), we have

$$\|(p - q)_\delta\|_1 \le C\|w\|_\theta \delta^{\theta - 1}, \quad \forall \delta > 0,$$

and also

$$\|(w \otimes u + v \otimes w)_\delta\|_1 \leq C\delta^{\theta-1}\|w \otimes u + v \otimes w\|_\theta \leq C\|w\|_\theta \delta^{\theta-1}, \quad \forall \delta > 0.$$

Thus, by plugging these two last inequalities in (D.5), we get

$$|w_\delta(y,s) - w_\delta(y,t)| \leq C|t-s|\delta^{\theta-1}\|w\|_\theta,\ \forall \delta > 0,$$

from which, by (D.4), we conclude

$$|w(y,s) - w(y,t)| \leq C(\delta^\theta + |t-s|\delta^{\theta-1})\|w\|_\theta, \quad \forall \delta > 0.$$

By choosing $\delta = |t-s|$ we finally achieve (D.1), and this concludes the proof. □

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