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The Chain Rule for the Divergence of BV–like Vector Fields

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ABSTRACT. We illustrate an approach to a conjecture of Bressan concerning compactness of solutions to ordinary differential equations. This approach combines classical results in the theory of BV functions with recent developments in the theory of renormalized solutions to transport equations. Moreover it leads naturally to some new questions in the theory of BV functions, which turn out to be connected with regularity issues for solutions to Hamilton–Jacobi PDEs and hyperbolic systems of conservation laws.

1. Introduction

In [9] the author advanced the following conjecture on compactness of ODEs

Conjecture 1.1 (Bressan's compactness conjecture). Let $b_n : \mathbf{R}_t \times \mathbf{R}^d_x \to \mathbf{R}^d$ be smooth maps and denote by Φ_n the solution of the ODEs:

$$\begin{cases} \frac{d}{dt}\Phi_n(t,x) \;=\; b_n(t,\Phi_n(t,x)), \\ \\ \Phi_n(0,x) \;=\; x\,. \end{cases} \tag{1}$$

Assume that the fluxes Φ_n are nearly incompressible, i.e. that for some constant C we have

$$C^{-1} \;\leq\; \det(\nabla_x \Phi_n(t,x)) \;\leq\; C\,, \tag{2}$$

and that $\|b_n\|_\infty + \|\nabla b_n\|_{L^1}$ is uniformly bounded. Then the sequence $\{\Phi_n\}$ is strongly precompact in $L^1_{\rm loc}$.

Thanks to the compactness of the space of BV functions, after possibly extracting a subsequence (not relabeled) we can assume the existence of a vector field $b \in BV_{loc} \cap L^\infty$ such that $b_n \to b$ strongly in L^1_{loc}. Therefore a positive answer to Conjecture 1.1 would provide the existence of a map Φ which solves (in a suitable weak sense) the ODE

$$\begin{cases} \frac{d}{dt}\Phi(t,x) \;=\; b(t,\Phi(t,x)), \\ \\ \Phi(0,x) \;=\; x\,. \end{cases} \tag{3}$$

The ODEs (1) are naturally connected with the transport equations

$$\begin{cases} \partial_t u_n + b_n \cdot \nabla_x u_n \;=\; 0, \\ \\ u(0,x) = \overline{u}(x)\,. \end{cases} \tag{4}$$

Indeed classical solutions of (4) are constant along the trajectories of (1). Next, denote by $J_n(t,\cdot)$ the Jacobian determinant of $\nabla_x \Phi_n(t,\cdot)$ and by $\Psi_n(t,\cdot)$ the inverse of the map $\Phi_n(t,\cdot)$. If we set $\rho_n(t,x) := J_n(t,\Psi_n(t,x))$, then a classical theorem of Liouville states that ρ_n solves

$$\begin{cases} \partial_t \rho_n + \mathrm{div}_x(\rho_n b_n) \;=\; 0, \\ \\ \rho_n(0,x) = 1\,. \end{cases} \tag{5}$$

Therefore the u_n solving (4) satisfy

$$\partial_t(\rho_n u_n) + \mathrm{div}_x(\rho_n u_n b_n) \;=\; 0\,.$$

The bounds (2) imply that, up to subsequences, we can assume the existence of a $\rho \in L^\infty$ with $\rho \geq C^{-1}$ such that $\rho_n \rightharpoonup^* \rho$ in L^∞. This ρ solves

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho b) \;=\; 0, \\ \\ \rho(0,x) = 1\,, \end{cases} \tag{6}$$

in the sense of distributions.

In [3] the authors suggested Conjecture 1.2 below (compare with Conjecture 4.1 of [3]). Using the DiPerna–Lions theory of renormalized solutions to transport equations (see [10]) they proved that a positive answer to Conjecture 1.2 would imply Bressan's Compactness Conjecture (see Proposition 4.4 of [3]).

Conjecture 1.2. Let $f \in L^\infty \cap BV_{loc}(\mathbf{R}\times\mathbf{R}^m, \mathbf{R}^m)$ and $\rho \in L^\infty(\mathbf{R}\times\mathbf{R}^m)$ be such that ρ is positive and bounded away from zero and $\partial_t \rho + \mathrm{div}_x(\rho f) = 0$ in the sense of distribution. If $u \in L^\infty$ satisfies $\partial_t(\rho u) + \mathrm{div}_x\,(\rho u f) = 0$ and $\beta \in C^1(\mathbf{R},\mathbf{R})$, then $\partial_t(\rho\beta(u)) + \mathrm{div}_x(\rho\beta(u) f) = 0$.

Actually a positive answer to this conjecture would imply even more: In particular it would show that the limit Φ of the fluxes Φ_n depends only on the limit b of the coefficients b_n. Therefore this flux would be a sort of "canonical" flow for the ODE (3). In this paper we want to illustrate some recent partial results on Conjecture 1.2, and related questions.

2. Chain Rule

First of all let us recast Conjecture 1.2 in a more general formulation. Consider the vector function $v = (v_1, v_2) := (\rho, \rho u)$, the vector field

$$B := (1, b) : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R} \times \mathbf{R}^m ,$$

and the coordinates $z := (t, x)$. Then we have $\mathrm{div}_z(v \otimes B) = 0$. To simplify the notation, from now on we will write $D \cdot (vB)$ instead of $\mathrm{div}\,(v \otimes B)$. If we define

$$\tilde{\beta} : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$$

as $\tilde{\beta}(x_1, x_2) := \beta(x_2/x_1)$, the goal of Conjecture 1.2 can be rewritten as $D \cdot (\tilde{\beta}(v)B) = 0$. Since $\rho = v_1$ is bounded away from zero, we can find a C^1 function h which coincides with $\tilde{\beta}$ on the range of v. Hence we might restate our problem in the following way:

Problem 2.1. Let $\Omega \subset \mathbf{R}^n$ be an open set, $B \in L^\infty \cap BV(\Omega, \mathbf{R}^N)$ and $v \in L^\infty(\Omega, \mathbf{R}^k)$. Assume that $D \cdot (vB) = 0$. Can we right a formula which allows to compute $D \cdot (h(v)B)$ for $h \in C^1(\mathbf{R}^k, \mathbf{R})$?

This time we cannot simply hope that $D \cdot (h(v)B) = 0$. Indeed, let us look at the following example. Consider an open set $A \subset \mathbf{R}^N$ with smooth boundary and two maps $B \in L^\infty \cap BV(\mathbf{R}^N, \mathbf{R}^N)$ and $v \in L^\infty(\mathbf{R}^N)$ such that

- $B|_A$, $u|_A$, $B|_{\mathbf{R}^N \setminus \overline{A}}$, and $B|_{\mathbf{R}^N \setminus \overline{A}}$ are C^1 and continuous up to the boundary ∂A;
- $D \cdot (vB) = 0$ on $\mathbf{R}^N$.

Denote by $\nabla \cdot B$ the bounded function which on $\mathbf{R}^N \setminus \partial A$ coincides with the pointwise divergence of B. Then it is easy to show that

$$\begin{aligned} D \cdot (h(v)B) &= \left[h(v) - \sum_i v_i \frac{\partial h}{\partial y_i}(v) \right] \nabla \cdot B \\ &\quad + \Big\{ h(v^+)\langle \nu, B^+ \rangle - h(v^-)\langle \nu, B^- \rangle \Big\} \mathcal{H}^{N-1} \llcorner \partial A \end{aligned} \tag{7}$$

where

- ν is the exterior unit normal to ∂A;
- $B^\pm$, $v^\pm$ denote the interior and exterior traces of B and v on ∂A;
- $\mathcal{H}^{n-1} \llcorner \partial A$ denotes the nonnegative Radon measure which to any Borel set E assigns the $n-1$–dimensional Hausdorff measure (that is the $n-1$–dimensional volume) of $E \cap \partial A$.

One can easily show examples where none of the terms in the right hand side of (7) vanish. However, when h takes the special form induced by the hypotheses of Conjecture 1.2, these terms do vanish (cp. with Lemma 7.6 of [5]).

3. Decomposition of Measures

The BV Structure Theorem states that a BV function is approximately continuous outside a rectifiable set J of codimension 1, and on this set it undergoes jump discontinuities (in a suitable measure theoretic sense). For the precise statement of this theorem we refer to Section 2.4 of [3] and to the book [7] (Section 3.7). Moreover, the total variation measure $|DB|$ of the (matrix–valued) Radon measure DB can be decomposed as the sum of three mutually orthogonal measures $|DB|^a$, $|DB|^c$, and $|DB|^j$, where

- $|DB|^a$ is absolutely continuous with respect to the Lebesgue measure;
- $|DB|^j$ is absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner J$;
- $|DB|^c$ is a singular measure such that $|DB|^c(A) = 0$ for every Borel set A with $\mathcal{H}^{N-1}(A) < \infty$.

Recall that $DB = M|DB|$ for some bounded Borel matrix–valued function M, and that $D \cdot B = \mathrm{tr}M\,|DB|$. Therefore we can write

$$\begin{array}{lclcl} DB & = & D^aB + D^cB + D^jB & := & M|D^aB| + M|D^cB| + M|D^jB| \\ D \cdot B & = & D^a \cdot B + D^c \cdot B + D^j \cdot B & := & \mathrm{tr}M|D^aB| + \mathrm{tr}M|D^cB| + \mathrm{tr}M|D^jB| \end{array}$$

(see Sections 2.1 and 2.4 of [5] and Section 3.9 of [7]). The measures of these decompositions are called *absolutely continuous part*, *Cantor part*, and *jump part*.

In the recent ground–breaking paper [2] the author extended the DiPerna–Lions theory of renormalized solutions to BV vector fields with bounded divergence. One consequence of his analysis is the following

Theorem 3.1. *Let B, v, and h be as in Problem 2.1. Then*

$$D \cdot (h(v)B) \;=\; \left[h(v) - \sum_i v_i \frac{\partial h}{\partial y_i}(v)\right] D^a \cdot B + \mu\,, \tag{8}$$

where $|\mu| \le C|D^c \cdot B| + C|D^j \cdot B|$, for some constant C.

A more general statement, proved with the techniques introduced in [2], can be found in Section 3 of [5].

4. Ambrosio–Crippa–Maniglia Trace Theorem

Consider a bounded vector field C such that $D \cdot C$ is a Radon measure. For every bounded open set Ω with C^1 boundary we can define the normal trace of C on Ω as the distribution $\mathrm{Tr}_{\partial\Omega}C$ given by

$$\langle \mathrm{Tr}_{\partial\Omega}C, \varphi\rangle \;:=\; \int_\Omega \nabla\varphi \cdot C + \int_\Omega \varphi d\,[D \cdot C] \qquad \text{for every } \varphi \in C^\infty_c(\mathbf{R}^N). \tag{9}$$

It was proved in [8] that there exists a unique $g \in L^\infty_{\mathrm{loc}}(\partial\Omega)$ such that

$$\langle \mathrm{Tr}_{\partial\Omega}C, \varphi\rangle \;=\; \int_{\partial\Omega} g\varphi\,.$$

The normal trace is local in the following sense:

(L) Let $E = \partial\Omega \cap \partial\Omega'$ and the exterior unit normals to Ω and Ω' coincide on E. Then, up to a set of zero $\mathcal{H}^{N-1}$ measure, $\mathrm{Tr}_{\partial\Omega} C = \mathrm{Tr}_{\partial\Omega} C'$ on E .

This locality property allows to define the left and right traces $\mathrm{Tr}^{\pm}_{A} C$ on any rectifiable set A of codimension 1, once we fix , a Borel unit vector field ν normal to A (see for instance Section 2.2 of [5]).

In [4], the authors proved the following Chain–Rule formula (see Theorem 4.2 therein):

Theorem 4.1. *Assume that $C = wB$, where w is a bounded function and B is a BV vector field. If β is a smooth function and J is a rectifiable set, then*

$$\mathrm{Tr}^{\pm}_{J}(\beta(w)B) \;=\; \beta\left(\frac{\mathrm{Tr}^{\pm}_{J}(wB)}{\mathrm{Tr}^{\pm}_{J}B}\right)\mathrm{Tr}^{\pm}_{J}B\,, \tag{10}$$

where we use the convention that the right hand side is zero whenever $\mathrm{Tr}^{\pm}_{J} C = 0$.

As a corollary of this chain–rule we have (cp. with Theorem 4.1 of [5]):

Theorem 4.2. *Let B, v, and h be as in Problem 2.1. Then*

$$\begin{aligned} D\cdot(h(v)B) \;&=\; \left\{h\left(\frac{\mathrm{Tr}^{+}_{J}(vB)}{\mathrm{Tr}^{+}_{J}B}\right)\mathrm{Tr}^{+}_{J}B - h\left(\frac{\mathrm{Tr}^{-}_{J}(vB)}{\mathrm{Tr}^{-}_{J}B}\right)\mathrm{Tr}^{-}_{J}B\beta\right\}\mathcal{H}^{N-1}\llcorner J \\ &\quad + \left[h(v) - \sum_i v_i\frac{\partial h}{\partial y_i}(v)\right] D^a\cdot B \;+\; \nu\,, \end{aligned} \tag{11}$$

where $|\nu| \leq C|D^c\cdot B|$.

5. A New Commutator Lemma

At this point it would be desirable to have a formula looking like

$$\nu \;=\; \left[h(v) - \sum_i v_i\frac{\partial h}{\partial y_i}(v)\right] D^c\cdot B\,. \tag{12}$$

However, recall that $D^c\cdot B$ is a singular measure, whereas v is a bounded function, and therefore it is defined only up to sets of zero Lebesgue measure. Hence, the right hand side of (12) does not have a meaning. It is tempting to conjecture that a similar formula holds at least where the approximate limit exists. For the reader's convenience we recall the definition of approximate limit.

Definition 5.1. *We say that v has approximate limit at x if there exists $\tilde v(x)$ such that*

$$\lim_{r\downarrow 0}\frac{1}{r^N}\int_{B_r(x)}|v(y)-\tilde v(x)|\,dy \;=\; 0\,.$$

We denote by S_v the set of points where v does not have approximate limit.

Indeed, using a new commutator lemma (see Lemma 5.1 and Theorem 5.2 of [5]), we have proved:

Theorem 5.2. *Let ν be the measure of Theorem 4.2, then*

$$\nu\llcorner(\Omega\setminus S_v) \;=\; \left[h(\tilde{v})-\sum_i \tilde{v}_i\frac{\partial h}{\partial y_i}(\tilde{v})\right] D^c\cdot B\llcorner(\Omega\setminus S_v)\,. \tag{13}$$

6. Divergence Problem

From Theorems 3.1, 4.2 and 5.2 we conclude that Problem 2.1 would be completely solved if we could show $\nu\llcorner S_v = 0$. Note that $|\nu\llcorner S_v|\leq C|D^c\cdot B|\llcorner S_v$. Therefore one could hope that $|D^c\cdot B|\llcorner S_v = 0$.

As before, we let M be a Borel matrix–valued function such that $D^cB = M|D^cB|$ and we observe that $|D^c\cdot B| = |\mathrm{tr}\, M||D^cB|$. In order to get some more insight about this measure, we recall that $|D^cB|$ can be *disintegrated* in the following way. There exists

- a (measurable) one–parameter family of rectifiable sets $\{P_t\}_{t\in[0,\infty[}$ of dimension $N-1$
- and a Borel function f

such that

$$|D^cB|(A) \;=\; \int_0^\infty\left[\int_{A\cap P_t} f(x)\,d\mathcal{H}^{N-1}(x)\right]dt \qquad \text{for any Borel set } A.$$

This decomposition is a consequence of the coarea formula for BV functions (see for instance Theorem 3.4 of [7]). Moreover, as a corollary of the BV Structure Theorem, we have that:

- For $\mathcal{H}^{N-1}$–a.e. $x\in P_t$, the approximate limit $\tilde{B}(x)$ of B at x exists.

For each t we denote by ν_t a Borel unit vector field normal to P_t . It is natural to call *characteristic* the set

$$E_t \;:=\; \left\{x\in S_t \;:\; \langle\nu_t(x),\tilde{B}(x)\rangle = 0\right\}.$$

Next, if we define $M|D^cB| = D^cB$, then a deep result of Alberti (see [1]) implies that

$$|D_cB|\left(E\setminus\bigcup_t E_t\right)+|D_cB|\left(\bigcup_t E_t\setminus E\right) \;=\; 0\,.$$

These considerations give an intuitive explanation for the following result of [4] (compare with Theorem 6.5 therein):

Theorem 6.1. $|D^cB|((\Omega\setminus E)\setminus S_v) = 0$.

Therefore we have $|\nu\llcorner S_v|\leq C|D^c\cdot B|\llcorner E$. Following [5] we call E *tangential set of* B. Our discussion leads naturally to the following:

Problem 6.2 (Divergence problem). Does $|D^c\cdot B|$ vanish on the tangential set of B?

A positive answer to this question would solve Problem 2.1 completely. Not only this would allow to answer positively to Bressan's compactness Conjecture, but it would also give a DiPerna–Lions theory for nearly incompressible BV coefficients. We refer the reader to Section 7 of [5].

7. SBV Regularity

It turns out that Problem 6.2 has connections with some regularity questions about solutions of 1–dimensional hyperbolic conservation laws and Hamilton–Jacobi equations. We start by introducing the space of SBV functions

Definition 7.1. *Let $\Omega \subset \mathbf{R}^N$ be an open set. Then $SBV(\Omega, \mathbf{R}^k)$ is given by the maps $u \in BV(\Omega, \mathbf{R}^k)$ such that $D^c u$ vanishes identically.*

The proof of the following theorem can be found in Section 8 of [5]:

Theorem 7.2 (SBV regularity)**.** *A positive answer to Question 6.2 would have the following corollaries:*

- *Let $H \in C^2(\mathbf{R}^d)$ be uniformly convex and let $u \in W^{1,\infty}(\Omega)$ be such that*
$$H(\nabla u) = 0 \qquad \mathcal{L}^d\text{–a.e. in } \Omega \ . \tag{14}$$
If $\nabla u \in BV(\Omega)$, then $\nabla u \in SBV(\Omega)$.
- *Let $f \in C^2(\mathbf{R})$ and assume that the set $\{f'' = 0\}$ is $\mathcal{L}^1$-negligible. If $\Omega \subset \mathbf{R}_t \times \mathbf{R}_x$ and $u \in L^\infty \cap BV(\Omega)$ is a weak solution of*
$$\partial_t u + \partial_x[f(u)] = 0 \quad \text{in } \Omega, \tag{15}$$
then $u \in SBV(\Omega)$.
- *Let $F \in C^2(\mathbf{R}^k, \mathbf{R}^k)$ be such that the system of conservation laws*
$$\partial_t U + \partial_x[F(U)] = 0 \tag{16}$$
is strictly hyperbolic and genuinely nonlinear. Then, if $\Omega \subset \mathbf{R}_t \times \mathbf{R}_x$, any weak solution $U \in L^\infty \cap BV(\Omega)$ is $SBV(\Omega)$.

In the following cases we gave in [6] a complete and independent proof of the regularity stated above:

- Entropy solutions to 1–d uniformly convex conservation laws;
- Viscosity solutions of planar Hamilton–Jacobi equations with a uniformly convex hamiltonian.

8. Interesting Cases of the Divergence Problem

We state here some cases of the divergence problem which still would have interesting consequences. First of all, we note that Bressan's compactness conjecture would follow from a positive answer to

Problem 8.1. Let $B \in BV_{\text{loc}} \cap L^\infty$ and $\rho \in L^\infty$ be such that $\rho \geq C > 0$ and $D \cdot (\rho B) = 0$. Is it true that $|D^c \cdot B|$ vanishes on the tangential set of B ?

The next variants are concerned with some important subclasses of bounded BV vector fields. We remark that a positive answer to Problem 8.4 would imply SBV regularity of gradients of viscosity solutions to Hamilton–Jacobi PDEs with uniformly convex hamiltonians, in any space dimension.

Problem 8.2. Let $\alpha \in W^{1,\infty}_{\rm loc}$ be such that $\nabla\alpha \in BV_{\rm loc}$. Is it true that $D^c \cdot \nabla\alpha$ (that is the Cantor part of the Laplacian $\Delta\alpha$) vanishes on the tangential set of $\nabla\alpha$?

Problem 8.3. Let B be as in Problem 6.2 and assume in addition that there is $\lambda \in \mathbf{R}$ such that

$$\langle B(y) - B(x), y - x\rangle \;\geq\; \lambda |x-y|^2 \qquad \forall x, y \in \Omega\,. \tag{17}$$

Does $|D^c \cdot B|$ vanish on the tangential set of B ?

Problem 8.4. We assume that $B = \nabla\alpha$ for α as in Problem 8.2 and that (17) holds for some $\lambda \in \mathbf{R}$. Does $|D^c \cdot B|$ vanish on the tangential set of B ?

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