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Mathematical Analysis/Calculus of Variations

Polyconvexity equals rank-one convexity for connected isotropic sets in $\mathbb{M}^{2\times 2}$

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Abstract

We give a short, self-contained argument showing that, for compact connected sets in $\mathbb{M}^{2\times 2}$ which are invariant under the left and right action of SO(2), polyconvexity is equivalent to rank-one convexity (and even to lamination convexity). As a corollary, the same holds for O(2)-invariant compact sets. These results were first proved by Cardaliaguet and Tahraoui. We also give an example showing that the assumption of connectedness is necessary in the SO(2) case. ***To cite this article: S. Conti et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

La polyconvexité est équivalente à la 1-rang convexité pour les ensembles isotropiques et connexes dans $M^{2\times 2}$. Nous donnons un argument simple montrant que pour les ensembles connexes et compacts dans $M^{2\times 2}$ qui sont invariants sous les actions à gauche et à droite de SO(2) la polyconvexité est équivalente à la 1-rang convéxité et même à la lamination-convexité. Comme corollaire la même chose est vraie pour les ensembles compacts O(2)-invariants. Ces résultats ont été démontrés par Cardaliaguet et Tahraoui pour la première fois. Nous donnons aussi un exemple montrant que l'hypothèse de connectivité est nécessaire pour le cas SO(2). ***Pour citer cet article : S. Conti et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

La notion de quasiconvexité a été introduite par Morrey, aussi que la caractérisation des densitées des énergies pour lesquelles les fonctionnels $I[u]=\int W(\nabla u)$ sont semi-continus inférieurement (ces fonctionnels sont définies sur l'ensemble des applications $u:\Omega\subset\mathbb{R}^m\to\mathbb{R}^n$; dans ce qui suit nous considérons $m=n=2$). Si K est un ensemble compact où W atteint son minimum, son enveloppe quasiconvexe K^{qc} est l'ensemble des gradients des applications affines où la relaxation de I atteint son minimum [6]. Physiquement K^{qc} représente l'ensemble des gradients des déformations macroscopiques à énergie nulle qui sont atteintes par un solide élastique.

On n'a pas une méthode directe pour caractériser explicitement K^{qc}. Pourtant, des bornes inférieures et supérieures peuvent être dérivées. D'un côté K^{qc} est contenu dans l'enveloppe polyconvexe K^{pc} définie comme

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l'ensemble des matrices qui ne peuvent pas être séparées de K par une fonction polyconvexe ($\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R}$ est polyconvexe si $\varphi(X) = \psi(X, \det X)$ pour une fonction convexe $\psi : \mathbb{R}^5 \to \mathbb{R}$). Par séparation nous entendons que pour tout $X \notin K^{pc}$ il y a une fonction polyconvexe φ telle que $\varphi(X) > \varphi(Y)$ pour tout $Y \in K$. D'autre part K^{qc} contient l'enveloppe 1-rang convexe K^{rc} définie comme l'ensemble des matrices qui ne peuvent pas être séparées de K par une fonction 1-rang convexe, c'est-à-dire une fonction qui est convexe sur les droites de rang 1, $t \to A + ta \otimes b$. L'ensemble K^{rc} contient en particulier l'enveloppe lamination-convexe K^{lc} définie comme l'intersection de tous les ensembles H contenants K tels que pour tout $X, Y \in H$ avec $\operatorname{rank}(X - Y) = 1$, le segment $[X, Y]$ entier appartient à H. Les exemples les plus connus des ensembles quasiconvexes sont obtenus en démontrant que dans certains cas spécifiques $K = K^{pc} = K^{lc}$. Alors c'est intéressant de chercher dans quels cas la dernière propriété est vraie, c'est-à-dire caractériser les classes des ensembles qui sont à la fois lamination-convexes et polyconvexes.

Ici nous nous concentrons sur les matériaux isotropiques en 2 dimensions. L'isotropie est appropriée, par exemple, pour les matériaux élastomériques ou polycristallins et mathématiquement est définie par l'identité $W(X) = W(QXQ')$ pour toutes les rotations $Q, Q' \in \mathrm{SO}(2)$. Aussi nous disons que l'ensemble $K \subset M^{2\times 2}$ est $\mathrm{SO}(2)$-invariant si $QKQ' = K$ pour tout $Q, Q' \in \mathrm{SO}(2)$. Nous disons qu'un ensemble K est polyconvexe (1-rang convexe, lamination-convexe) si $K = K^{pc}$ ($K = K^{rc}$, $K = K^{lc}$ respectivement).

Théorème 0.1. *Soit $K \subset M^{2\times 2}$ un ensemble compact, connexe et* $\mathrm{SO}(2)$*-invariant. Alors K est lamination-convexe si et seulment si il est polyconvexe.*

Ce résultat est un des plus interessants dans les travaux de Cardialaguet et Tahraoui [3–5] (avec l'hypothèse supplémentaire que toutes les matrices dans K ont le déterminant non négatif). Le but de notre article est de fournir une démonstration simple et autonome du Théorème 0.1. D'ailleurs, nous voudrions ajouter que [3–5] contiennent bien d'autres résultats interessants, y compris une caractérisation explicite des ensembles isotropiques convexes de rang un et une discussion détaillée du cas avec invariance sous l'action de $\mathrm{O}(2)$, qui conduit à une méthode de calcul de l'enveloppe quasi convexe de tout ensemble isotropique.

1. Introduction

Quasiconvexity was introduced by Morrey as a characterization of the energy densities W which give rise to lower semicontinuous energy functionals $I[u] = \int W(\nabla u)$ defined on maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ (in what follows we consider $n = m = 2$). If K is the compact set where W attains its minimum, then its quasiconvex envelope K^{qc} is the set of gradients of affine maps where the relaxation of I attains its minimum [6]. From a physical point of view, K^{qc} represents the set of macroscopic zero-energy deformation gradients which can be attained by an elastic solid.

A direct method to characterize K^{qc} explicitly is missing. Inner and outer bounds, however, can often be derived. On the one hand, K^{qc} is contained in the polyconvex hull K^{pc}, defined as the set of matrices which cannot be separated from K by a polyconvex function ($\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R}$ is polyconvex if $\varphi(X) = \psi(X, \det X)$ for some convex $\psi : \mathbb{R}^5 \to \mathbb{R}$). By separation we mean that for every $X \notin K^{pc}$ there exists a polyconvex φ such that $\varphi(X) > \varphi(Y)$ for any $Y \in K$. On the other hand, K^{qc} contains the rank-one convex hull K^{rc}, defined as the set of matrices which cannot be separated by a rank-one convex function, i.e., a function which is convex along rank-one lines $t \to A + ta \otimes b$. The set K^{rc} contains in particular the lamination-convex hull K^{lc}, defined as the intersection of all sets H containing K such that for every $X, Y \in H$ with $\operatorname{rank}(X - Y) = 1$ the whole segment $[X, Y]$ belongs to H. Most known examples of quasiconvex sets are obtained by proving that in specific cases $K = K^{pc} = K^{lc}$. It is therefore interesting to investigate in which cases the latter property holds, i.e., to characterize classes of sets which are both lamination convex and polyconvex.

2. Main result

We focus here on isotropic materials in two dimensions. Isotropy is appropriate for example for elastomeric or polycristalline materials, and mathematically means that the energy density W satisfies $W(X) = W(QXQ')$ for all rotations $Q, Q' \in \mathrm{SO}(2)$. Correspondingly, we say that the set $K \subset \mathbb{M}^{2\times 2}$ is SO(2)-invariant if $QKQ' = K$ for all Q, Q' in SO(2). We say that a set K is polyconvex (rank-one convex, lamination convex) if $K = K^{pc}$ ($K = K^{rc}$, $K = K^{lc}$ resp.).

Theorem 2.1. *Let $K \subset \mathbb{M}^{2\times 2}$ be compact, connected and* SO(2)*-invariant. Then K is lamination convex if and only if it is polyconvex.*

This is one of the main results in the papers [3–5] by Cardaliaguet and Tahraoui (with the additional assumption that all matrices in K have nonnegative determinant). The purpose of this note is to give a short, self-contained proof of Theorem 2.1. One should note, however, that [3–5] contain a number of other interesting results, including an explicit characterization of isotropic rank-one convex sets and a detailed discussion of the O(2)-invariant case, which leads to method to compute the quasiconvex envelope of any set.

Isotropic functions can be naturally written in terms of the scalar parameters $\lambda_1(X)$ and $\lambda_2(X)$, which we define as the only real numbers $|\lambda_1| \leqslant \lambda_2$ such that $Q\,\mathrm{diag}(\lambda_1, \lambda_2)Q' = X$ for some $Q, Q' \in \mathrm{SO}(2)$. We remark that $|\lambda_1|$ and λ_2 are the singular values of X, i.e., the eigenvalues of $(XX^{\mathrm{T}})^{1/2}$. Hence $\lambda_1^2(X) + \lambda_2^2(X) = |X|^2 := \mathrm{Tr}\, X^{\mathrm{T}} X$ and $\lambda_1(X)\lambda_2(X) = \det X$. If $X \in \mathbb{M}^{2\times 2}$, then $Y \in \mathrm{SO}(2)\, X\, \mathrm{SO}(2)$ if and only if $\lambda_i(X) = \lambda_i(Y)$. For an analysis of rank-one convexity for isotropic functions see [7] and references therein.

On key idea of both our proof and the one in [3–5] is a separation argument using suitable hyperbolae in the space of singular values. To this end, one uses the following lemma (see [5], see also [1,2])

Lemma 2.2. *Let $c \in \mathbb{R} \setminus \{0\}$. Then the functions*

$$\varphi_c^{\pm}(X) := \lambda_2(X) \pm \lambda_1(X) - (\det X)/c \tag{1}$$

are polyconvex. The same holds for $\varphi_0^{\pm}(X) := -\det(X)$.

Proof. The lemma follows from the convexity of the functions $\lambda_2 \pm \lambda_1$, which in turn is proved by the explicit computation $\lambda_2(X) \pm \lambda_1(X) = \sqrt{|X|^2 \pm 2\det X} = \sqrt{(X_{11} \pm X_{22})^2 + (X_{21} \mp X_{12})^2}$. □

Remark 1. We observe that any SO(2)-invariant polyconvex function can be written as supremum of linear combinations of the functions $\varphi_c^{\pm}$, as can be seen by writing it first as supremum of polyaffine functions and then exploiting SO(2)-invariance.

Before giving the detailed proof of Theorem 2.1, we illustrate how the argument can be visualized in the plane (λ_1, λ_2). Suppose that K is lamination convex. The level-sets of the functions $\varphi_c^{\pm}$ through a given matrix A form a one-parameter family of hyperbolic arcs. These arcs are at the same time images of rank-one lines, hence if A is not in K they cannot intersect K both 'before' and 'after' A. We divide each hyperbolic arc into two pieces, separated by the matrix A, and parametrize each ray with the vector $\mathbf{e} \in \mathbf{S}^1$ tangent to it in A (see Fig. 1). By continuity, the set $\gamma \subset \mathbf{S}^1$ of $\mathbf{e}$ for which the corresponding ray intersects the compact set K is closed. Since γ and $-\gamma$ are closed and disjoint subsets of the connected set $\mathbf{S}^1$, there is an $\mathbf{e} \in \mathbf{S}^1$ such that neither $\mathbf{e}$ nor $-\mathbf{e}$ lies in γ. The corresponding hyperbola $\{\bar\varphi = h\}$ does not intersect K, and since K is assumed to be connected it lies on one side of it. Using Remark 2 we show below that either K is a subset of $\{\bar\varphi < h\}$, or K lies also on one side of $\{\det = \det A\}$. This will conclude the proof.

Remark 2. Suppose that K is lamination convex. Then

$$\text{if } X \in K, \text{ then } \bigl\{Y\colon \det(Y) = \det(X),\ \lambda_2(Y) \leqslant \lambda_2(X)\bigr\} \subset K. \tag{2}$$

To see this, consider the rank-one segment joining the matrices

$$X_{\pm} := \begin{pmatrix} |\det X|^{1/2} & \pm\sqrt{|X|^2 - 2|\det X|} \\ 0 & (\det X)/|\det X|^{1/2}. \end{pmatrix} \tag{3}$$

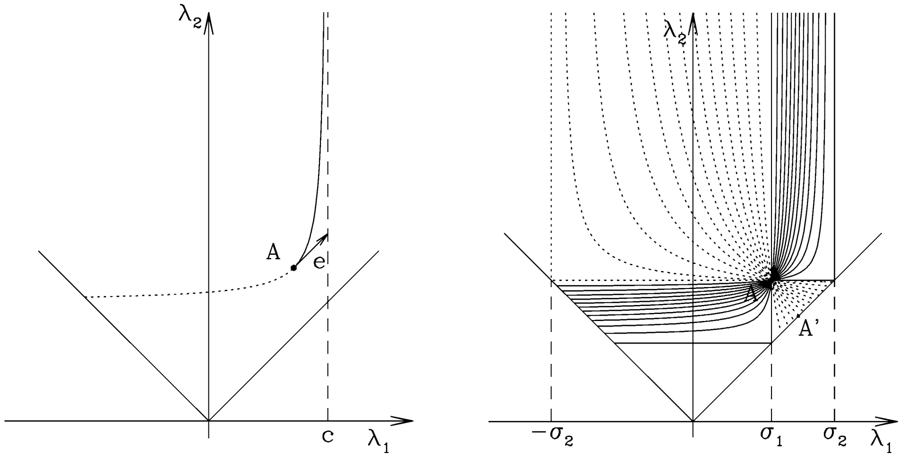

Fig. 1. Left panel: one of the level sets L_c. The dotted part is $\Lambda(c+2\sigma_2)$. Right panel: the family of closed sets $\Lambda(d)$. The dotted lines are the level sets of φ_c^-, the continuous are the level sets of φ_c^+. The point (x, y) represents the set $\mathrm{SO}(2)\operatorname{diag}(x, y)\,\mathrm{SO}(2)$.

Clearly $X_\pm \in \mathrm{SO}(2)\,X\,\mathrm{SO}(2) \subset K$ and thus the whole rank-one segment $[X^+, X^-]$ belongs to K. Along this segment the product of $|\lambda_1|$ and λ_2 is constant, and the sum of their squares is minimal when they are equal and maximal at the endpoints $X^\pm$. Therefore for any Y in the set (2) there exists $\widetilde{Y} \in [X^+, X^-]$ such that $\lambda_i(\widetilde{Y}) = \lambda_i(Y)$.

3. Proof of Theorem 2.1

Since det is affine on rank-one lines, a set of matrices which is polyconvex is also lamination convex, and one implication of the theorem follows. To prove the other implication we need to show that for any $A \notin K$, there is a polyconvex φ with $\varphi(A) > \max_K \varphi$. It is actually sufficient to show that, for any K which satisfies the assumptions above,

$$\text{if } A = \operatorname{diag}(\sigma_1, \sigma_2) \notin K, \text{ with } 0 \leqslant \sigma_1 \leqslant \sigma_2, \text{ there is } \varphi \text{ polyconvex s.t. } \varphi(A) > \max_K \varphi. \tag{4}$$

Indeed suppose that (4) holds, and take $B \in \mathbb{M}^{2\times 2} \setminus K$. Then there are $O_1, O_2 \in \mathrm{O}(2)$ such that $A := O_1 B O_2$ has the form above. The set $K' := O_1 K O_2$ is still compact, connected, lamination convex and SO(2)-invariant. By (4) there is a polyconvex function φ' which separates A from K'. Then $\varphi(X) := \varphi'(O_1 X O_2)$ is polyconvex and separates B from K.

We now start proving (4). If $\sigma_1 = \sigma_2$, we claim that $\{Y\colon \det Y = \det A\}$ does not intersect K. Otherwise there exists $X \in K$ with $\det X = \det A$ and thus $\lambda_2(X) \geqslant \sigma_2 = \sigma_1 \geqslant \lambda_1(X)$. Hence (2) yields that $A \in K$, a contradiction. Thus, the connectedness of K implies that either $\max_K \det < \det A$ or $\max_K - \det < -\det A$. Since $\pm\det$ are both polyconvex we are done.

If instead $\sigma_2 > \sigma_1$, we show below that at least one of the level sets

$$L_c := \begin{cases} \{X \mid \varphi_c^-(X) = \varphi_c^-(A)\} & \text{for } c \in [-\sigma_2, \sigma_1[, \\ \{X \mid \varphi_c^+(X) = \varphi_c^+(A)\} & \text{for } c \in [\sigma_1, \sigma_2[\end{cases} \tag{5}$$

does not intersect K (in the (λ_1, λ_2) plane with $|\lambda_1| \leqslant \lambda_2$, L_c is an arc of hyperbola with vertical asymptote $\lambda_1 = c$, see Fig. 1). This implies immediately the result. Indeed, let $\bar\varphi$ be the one of the $\varphi_c^\pm$ which generates this level set. Since K is compact and connected then either $\max_K \bar\varphi < \bar\varphi(A)$ or $\min_K \bar\varphi > \bar\varphi(A)$. In the first case our proof is finished. In the second case, consider $A' = (\sigma_1\sigma_2)^{1/2}\,\mathrm{Id}$. Since $\varphi_c^\pm(A) \geqslant \varphi_c^\pm(A')$ for all c, the matrix A' is not in K. Then by the argument above either det or $-\det$ separates A', and hence A, from K.

We now come to the core of the proof, which consists in showing that one of the level sets (5) does not intersect K. For $c \neq \pm\sigma_2$ we split any of the L_c into two pieces (see Fig. 1),

$$L_c^> := L_c \cap \{\lambda_2(X) \geqslant \sigma_2\} \quad \text{and} \quad L_c^< := L_c \cap \{\lambda_2(X) \leqslant \sigma_2\}. \tag{6}$$

One basic remark is that for any $c \in]-\sigma_2, \sigma_2[$,

$$\text{either } L_c^> \cap K = \emptyset \text{ or } L_c^< \cap K = \emptyset. \tag{7}$$

Indeed, if $B \in L_c^>$ and $C \in L_c^<$, we can find a rank-one segment between an element of $\mathrm{SO}(2)\,B\,\mathrm{SO}(2)$ and an element $\mathrm{SO}(2)\,C\,\mathrm{SO}(2)$ which contains A. To show this we distinguish two cases. If $\sigma_1 \leqslant c < \sigma_2$, we choose $s \in [0,1]$ such that $s\sigma_1 + (1-s)\sigma_2 = c$, and define

$$A(t) := A + t\begin{pmatrix} 1-s & \sqrt{s(1-s)} \\ \sqrt{s(1-s)} & s \end{pmatrix}, \qquad \begin{aligned} t_B &:= \lambda_2(B) + \lambda_1(B) - \sigma_2 - \sigma_1, \\ t_C &:= \lambda_2(C) + \lambda_1(C) - \sigma_2 - \sigma_1. \end{aligned}$$

It is easy to check that $(\lambda_2 + \lambda_1)^2(A(t)) = |A(t)|^2 + 2\det A(t) = (\sigma_2 + \sigma_1 + t)^2,\ t_B \geqslant 0,\ t_C \leqslant 0$, and $\varphi_c^+(A(t)) = \varphi_c^+(A) = \varphi_c^+(\mathrm{diag}(\sigma_1, \sigma_2))$. Hence $\lambda_i(A(t_B)) = \lambda_i(B),\ \lambda_i(A(t_C)) = \lambda_i(C)$. If instead $-\sigma_2 < c < \sigma_1$ we reason in the same way choosing s such that $-s\sigma_2 + (1-s)\sigma_1 = c$ and

$$A(t) := A + t\begin{pmatrix} -s & -\sqrt{s(1-s)} \\ \sqrt{s(1-s)} & 1-s \end{pmatrix}, \qquad \begin{aligned} t_B &:= \lambda_2(B) - \lambda_1(B) - \sigma_2 + \sigma_1, \\ t_C &:= \lambda_2(C) - \lambda_1(C) - \sigma_2 + \sigma_1 \end{aligned}$$

and considering that $(\lambda_2 - \lambda_1)^2(A(t)) = (\sigma_2 - \sigma_1 + t)^2$.

The second basic remark is that for $c \in]-\sigma_2, \sigma_2[$ and $\square \in \{>, <\}$, the following holds:

$$\text{if } X_n \in L_{c_n}^{\square},\ c_n \to c, \quad \text{and} \quad X_n \to X,\ \text{then } X \in L_c^{\square}. \tag{8}$$

We now extend and reparametrize $L_c^>$ and $L_c^<$ to obtain an $\mathbf{S}^1$-parameter family of sets $\Lambda(d)$, for $d \in [-\sigma_2, 3\sigma_2]$, such that the properties (7) and (8) still hold. For $|c| < \sigma_2$ we set $\Lambda(c) := L_c^>$ and $\Lambda(2\sigma_2 + c) := L_c^<$. The remaining sets are $\Lambda(\sigma_2) := L_{\sigma_2} \cap \{\lambda_1(X) \geqslant \sigma_1\}$ and $\Lambda(-\sigma_2) = \Lambda(3\sigma_2) := L_{-\sigma_2} \cap \{\lambda_1(X) \leqslant \sigma_1\}$. To better visualize the $\mathbf{S}^1$ family one could reparametrize it replacing d with the (oriented) tangent vector to the sets Λ in the point (σ_1, σ_2) in the (λ_1, λ_2) plane (see Fig. 1).

We call $\widetilde{K}$ the set of those $d \in [-\sigma_2, 3\sigma_2]$ such that $\Lambda(d)$ intersects K. In view of (8), and its obvious extension to the limits cases $c = \pm\sigma_2$, $\widetilde{K}$ is closed, and in view of (7), $\widetilde{K}$ and $(\widetilde{K} + 2\sigma_2)$ are disjoint. Hence their union cannot cover all of the connected set $[-\sigma_2, 3\sigma_2]$, and there is $d \in [-\sigma_2, \sigma_2]$ such that both $\Lambda(d)$ and $\Lambda(d + 2\sigma_2)$ do not intersect K. Since L_d is contained in $\Lambda(d) \cup \Lambda(d + 2\sigma_2)$ the proof is finished.

Corollary 3.1. *Let $K \subset \mathbb{M}^{2\times 2}$ be compact and* $\mathrm{O}(2)$*-invariant. Then K is lamination convex if and only if it is polyconvex.*

Proof. The claim follows from the fact that any $\mathrm{O}(2)$-invariant nonempty lamination convex set is connected and $\mathrm{SO}(2)$-invariant. To show that K is connected, we remark that if $X = \mathrm{diag}(\mu_1, \mu_2) \in K$, then the rank-one segment connecting X with $X' = \mathrm{diag}(-\mu_1, \mu_2)$ is in K, and hence also the one connecting $\mathrm{diag}(0, \mu_2)$ with $\mathrm{diag}(0, -\mu_2)$, which contains 0. □

4. Example for the disconnected case

We now show with an example that if K is not connected, with the remaining assumptions of Theorem 2.1 still holding, then rank-one convexity (and hence lamination convexity) does not imply polyconvexity. Consider the function

$$f(X) := \begin{cases} 4(\lambda_2 + \lambda_1 - 1 - \lambda_1\lambda_2)(X), & \lambda_1(X) \leqslant 1, \\ \frac{1}{4}(\lambda_2 + \lambda_1 - 1 - \lambda_1\lambda_2)(X), & \lambda_1(X) > 1, \end{cases}$$

which is rank-one convex by Lemma 3.1 of [8]. Define

$$K := \{X\colon\ f(X) \leqslant -1,\ \lambda_2(X) \leqslant 7\} \tag{9}$$

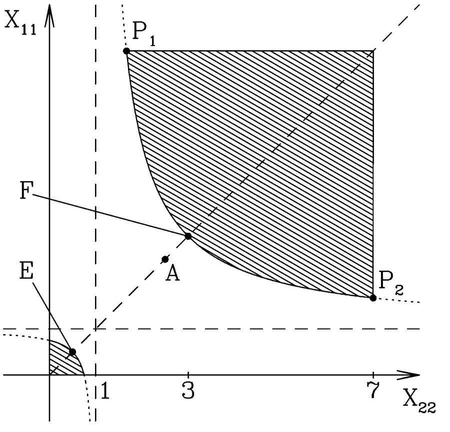

Fig. 2. Intersection of the set K defined in (9) with the diagonal matrices with nonnegative entries. The matrices A, E, F, P_1 and P_2 entering (10) are also shown.

and $A = \mathrm{diag}(5/2, 5/2)$. The set K is rank-one convex, compact, SO(2)-invariant, and does not contain A (see Fig. 2). We now show that there is no polyaffine function ψ, and hence no polyconvex function, which separates A from K. This implies that K is not polyconvex. Consider the four matrices $E = \mathrm{diag}(1/2, 1/2)$, $F = \mathrm{diag}(3, 3)$, $P_1 = \mathrm{diag}(7, 5/3)$, and $P_2 = \mathrm{diag}(5/3, 7)$ (all of which belong to K). It is a simple check that the inequalities

$$\psi(A) > \psi(E), \qquad \psi(A) > \psi(F), \qquad 2\psi(A) > \psi(P_1) + \psi(P_2) \tag{10}$$

are incompatible. To see this, write $\psi(X) = B : X - c \det X + d$, with $B \in \mathbb{M}^{2\times 2}$ and $c, d \in \mathbb{R}$ (and $B : X = \mathrm{Tr}\, B^{\mathrm{T}} X$). Setting $b := \mathrm{Tr}\, B/2$, the inequalities above become $5b - \frac{25}{4}c > b - \frac{1}{4}c$, $5b - \frac{25}{4}c > 6b - 9c$, $5b - \frac{25}{4}c > \frac{26}{3}b - \frac{35}{3}c$. It is easy to see that they are incompatible. Graphically, this corresponds to the fact that, if A is sufficiently close to F, any hyperbola which separates A from P_1, F and P_2 is very close to the one which contains the latter three points (see Fig. 2). Then, its second branch does not separate A from E.

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