# Boundary Regularity for the Plateau Problem 

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The Plateau Problem has been named after a Belgium Physicist, Joseph Pleateau (18011883), who has been experimenting with soap films and soap bubbles. Already in the 18 th century, Weierstrass observed that (when neglecting the gravitational force) such soap films are spanning surfaces of least area and calculated that they must have mean curvature zero. Such surfaces are called minimal surfaces and have been a topic of research ever since. The Plateau Problem can then be formulated as follows.

Question. For a fixed boundary (represented by a wire), is there a minimal surface (a soap film) spanning this boundary?

In the 1930's, T. Radó [31] and J. Douglas [22] proved the existence of 2-dimensional minimal surfaces in $\mathbb{R}^{3}$ and for this work, Douglas was awarded the Fields medal. His proof is based on the fact, that in three dimensions, minimizing the area functional is equivalent to minimizing the Dirichlet functional. In higher dimensions, this no longer holds and so his ideas do not allow to generalize his result. Instead, H. Federer and W. Fleming introduced in the 196o's more general objects than surfaces, the so called integral currents and proved in [24] existence of area-minimizing currents. The latter is then supported on a rectifiable set, thus a priori it can have many singularities. It took many mathematicians to prove that the singularities are rather rare. Indeed, in the interior of the support of an area minimizing current, we know thanks to the works of E. Bombieri, E. De Giorgi, E. Giusti [6], W. Allard [1-3] and J. Simons [34], that the set of singularities of an $n$-dimensional current in an ( $n+1$ )-dimensional ambient manifold is of dimension at most $n-7$. This result is sharp, as the so called Simons Cone

$$
\left\{x \in \mathbb{R}^{8}: x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2}=x_{5}^{2}+x_{6}^{2}+x_{7}^{2}+x_{8}^{2}\right\}
$$

is minimal and has the origin as a singular point. In the case of higher codimension (i.e. when the dimension of the ambient manifold is greater than $n+1$ ), the dimension bound is $n-2$ which was first proven in Almgren's Big regularity paper [5] and then revisited and shortened by De Lellis and Spadaro in [12-16]. The sharpness of this result is demonstrated by identifying $\mathbb{C}^{2}$ with $\mathbb{R}^{4}$ and looking at the two dimensional holomorphic subvariety

$$
\left\{(z, w) \in \mathbb{C}^{2}: z^{2}=w^{3}\right\}
$$

In this thesis, we focus on the regularity of an area-minimizing current near its boundary. The two parts are separately submitted for publication and their notation is independent of each other.

### 1.1 THE CODIMENSION ONE CASE

In his Ph.D. thesis [1], Allard proved that, in case the boundary is contained in the boundary of a uniformly convex set and the ambient manifold is the euclidean space, then all boundary points are regular (we explain more about this in Section 1.2). This means, in a neighborhood of a boundary point, the support of the current is a regular submanifold with boundary. Later, R. Hardt and L. Simon came to the same conclusion in [27] when having replaced the assumption of the uniform convexity by the fact that the current is of codimension 1. However, the result of Hardt and Simon is stated and proved only in the euclidean ambient space. In [35] and the first part of this thesis, we provide an adaptation of the arguments to the case of general Riemannian manifolds. We show the following theorem.

Theorem 1.1. Let $U \subset \mathbb{R}^{n+k}$ be open and $T$ an n-dimensional locally rectifiable current in $U$ that is area minimizing in some smooth $(n+1)$-manifold $\mathcal{M}$ and such that $\partial T$ is an oriented $\mathcal{C}^{2}$ submanifold of $U$. Then for any point $a \in \operatorname{spt}(\partial T)$, there is a neighborhood $V$ of a in $U$ satisfying that $V \cap \operatorname{spt}(T)$ is an embedded $\mathcal{C}^{1, \frac{1}{4}}$ submanifold with boundary.

The theorem of Hardt and Simon is then a case of the one stated above, however we follow their strategy of proof with a few modifications in order to deal with additional error terms coming from the ambient manifold. The main difference to [27] is that the blow-up procedure depends on the ambient manifold. On a technical level, even though the current has codimension one compared to the ambient manifold, we embed both in some higher dimensional euclidean space, and thus every point has many more components which have to be estimated (compared to the arguments in [27]).

Notice that the complete absence of singular points only happens at the boundary and only in codimension one. Indeed, in 2018, C. De Lellis, G. De Philippis, J. Hirsch and A. Massaccesi showed in [21] that in the case of higher codimension and on a general Riemannian manifold, there can be singular boundary points, but regardless, the set of regular boundary points is dense.

### 1.1.1 Outline of the proof of part I

We would like to measure how flat a current $T$ is. Therefore we introduce its excess in a cylinder of radius $r$ and denote it by $\mathbf{E}_{C}(T, r)$. It is the scaled version of the difference between the mass of the current in a cylinder and the mass of its projection onto an n-plane. The main ingredient to deduce the boundary regularity is the fact that this excess scales (up to a small rotation) like $r$ assuming that the curvature of both the boundary of the current $\kappa_{T}$ and the ambient manifold $\mathbf{A}$ are small.

Theorem 1.2 (Excess decay). Let $\mathcal{M}$ be a smooth manifold and let $T$ be area minimizing in $\mathcal{M}$ such that $\max \left\{\boldsymbol{E}_{C}(T, 1), \mathbf{A}, \kappa_{T}\right\} \leq \frac{1}{C}$. Then there is a real number $\eta$ such that for all $0<r<R$ the following holds

$$
E_{C}\left(\gamma_{\eta \#} T, r\right) \leq C r .
$$

The precise statement can be found in Theorem 4.2. In order to prove it, we first analyze in Section 3 the current away from the boundary. There we can use results from the interior regularity theory in order to find that the current is supported on the union of graphs of functions fulfilling the minimal surface equation. When zooming in (up to rescaling), the boundary (and the ambient manifold) become more flat and therefore, we can find the interior graphs closer to the boundary. The point is then to study what happens in the limit when the graphs on both sides of the boundary grow together. These limiting rescaled functions we call the harmonic blow-ups and they are introduced in section 4.

After proving the uniform convergence of the harmonic blow-ups also at boundary points, we show in a first step that in case the harmonic blow-ups are linear, they coincide on both sides of the boundary, see the Collapsing Lemma 5.4. Having proven some technical estimates on the excess (Theorem 6.3), the assumption of linearity then is dropped in Theorem 7.2. This follows by blowing up the harmonic blow-ups a second time. To guarantee the existence of this second blow-up, we need first to prove some a priori estimates (Lemma 7.1).

Once we know that the harmonic blow-ups coincide and in fact merge together in a smooth way, we prove the excess decay via a compactness argument: if the excess decay did not hold, there would be a sequence of currents whose blow-ups cannot coincide. Then this decay leads to a $\mathcal{C}^{1, \frac{1}{4}}$-continuation up to the boundary of the functions whose graphs describe the current (Corollary 4.3) assuming that the excess and the curvatures are sufficiently small. In section 9 we then collect everything together and deduce that either the current lies only on one side of the boundary or both sides merge together smoothly. In case of a one-sided boundary, Allard's boundary regularity theory [3] covers the result.

### 1.2 THE HIGHER MULTIPLICITY CASE

In the second part of this thesis, we consider an area-minimizing integral current $T$ of dimension $m \geq 2$ in $\mathbb{R}^{m+n}$ and assume that $\partial T$ is a smooth submanifold, namely $\partial T=\sum_{i} Q_{i} \llbracket \Gamma_{i} \rrbracket$, where $Q_{i}$ are (positive) integer multiplicites and $\Gamma_{i}$ finitely many pairwise disjoint oriented smooth and connected submanifolds of dimension $m-1$. We are focused on understanding how regular $T$ can be at points $p \in \cup_{i} \Gamma_{i}$ and our primary interest is that the integer multiplicities are allowed to be larger than 1 and the codimension $n$ is at least 2 . This has been done in the joint work [10] with C. De Lellis and S. Nardulli. When the codimension $m$ is 1 , the situation is completely understood (cf. [8, Problem 4.19]): first of all the coarea formula for functions of bounded variation allows to decompose, locally, the current $T$ into a sum of area minimizing integral currents which take the boundary with multiplicity 1 ; hence we can apply the main Theorem 9.1 of the first part of this thesis to each piece of the decomposition, which guarantees the absence of any singularity.

A quite general boundary regularity theory was developed by Allard in the pioneering fundamental work [3], which covers any dimension and codimension and is valid for more general objects than currents, namely stationary varifolds. In [3] Allard restricts his attention to boundary points where the density, namely the limit of the mass ratio

$$
\Theta(T, q):=\lim _{r \downarrow 0} \frac{\|T\|\left(\mathbf{B}_{\rho}(q)\right)}{\rho^{m}}
$$

is sufficiently close to $\frac{1}{2}$. His Boundary Regularity Theorem guarantees then that, under such assumption, $q$ is always a regular point. Indeed this generalizes a similar statement in his PhD thesis [1], which covered the case of area minimizing currents in codimension 1.

In the introduction to [1], Allard points out that when the multiplicity of the boundary $\Gamma$ is allowed to be an arbitrary natural number $Q>1$, the assumption $\Theta(T, q)<\frac{1}{2}+\varepsilon$ is empty and should be replaced by $\Theta(T, q)<\frac{Q}{2}+\varepsilon$. However, he quotes a possible extension of his theorem as a very challenging problem. This basic question was raised again by B. White in the collection of open problems [8], cf. Problem 4.19, where he also explains that the nontrivial situation is in higher codimension, given the decomposition through the coarea formula already explained a few paragraphs above. Our work gives the very first result in that direction and solves Allard's "higher multiplicity" question for 2-dimensional integral currents. Before stating it, we wish to discuss what we mean by "regularity at the boundary".

Definition 1.3. Assume $T$ is an area minimizing 2-dimensional integral current in $U \subset \mathbb{R}^{2+n}$ such that $\partial T\left\llcorner U=Q \llbracket \Gamma \rrbracket\right.$ for some integer $Q \geq 1$ and some $C^{1}$ embedded arc $\Gamma$. $p$ is called a regular boundary point if $T$ consists, in a neighborhood of $p$, of the union
of finitely many smooth submanifolds with boundary $\Gamma$, counted with appropriated integer multiplicities, which meet at $\Gamma$ transversally. More precisely, if there are:
(i) a neighborhood $U$ of $p$;
(ii) a finite number $\Lambda_{1}, \ldots, \Lambda_{J}$ of $C^{1}$ oriented embedded 2-dimensional surfaces in $U$;
(iii) and a finite number of positive integers $k_{1}, \ldots, k_{J}$
such that:
(a) $\partial \Lambda_{j} \cap U=\Gamma \cap U=\Gamma_{i} \cap U$ (in the sense of differential topology) for every $j$;
(b) $\Lambda_{j} \cap \Lambda_{l}=\Gamma \cap U$ for every $j \neq l$;
(c) for all $j \neq l$ and at each $q \in \Gamma$ the tangent planes to $\Lambda_{j}$ and $\Lambda_{l}$ are distinct;
(d) $T\left\llcorner U=\sum_{j} k_{j} \llbracket \Lambda_{j} \rrbracket\right.$ (hence $\sum_{j} k_{j}=Q_{i}$ ).

The set $\operatorname{Reg}_{b}(T)$ of boundary regular points is a relatively open subset of $\Gamma$ and its complement in $\Gamma$ will be denoted by $\operatorname{Sing}_{b}(T)$.

Our main theorem reads as follows.
Theorem 1.4. Let $U \subset \mathbb{R}^{n+2}$ be an open set, $\Gamma \subset U$ be a $C^{3, \alpha_{0}}$ embedded arc for some $\alpha_{0}>0$, and $T$ be a 2 -dimensional area-minimizing integral current such that $\partial T=Q \llbracket \Gamma \rrbracket$. If $q \in \Gamma$ and $\Theta(T, q)<\frac{Q+1}{2}$, then $T$ is regular at $q$ in the sense of Definition 1.3.

Note that it is well known that there are smooth curves (counted with multiplicity 1) in the Euclidean space, even in $\mathbb{R}^{3}$, which span more than one area-minimizing current. In particular, if $\Gamma \subset \mathbb{R}^{3}$ is such a curve and $T_{1}, T_{2}$ two area minimizing currents with $\partial T_{i}=\llbracket \Gamma \rrbracket, i=1,2$, then $T:=T_{1}+T_{2}$ is an area minimizing current with $\partial T=2 \llbracket \Gamma \rrbracket$ (this follows because any area-minimizing current $S$ with boundary $\partial S=2 \llbracket \Gamma \rrbracket$ must have mass which doubles that of $T_{i}$, and hence equals that of $T$ ). Let us analyze the above example more accurately. In view of the interior and boundary regularity theory, both $T_{1}$ and $T_{2}$ are smooth submanifolds up to the boundary, i.e. a standard argument using Allard's boundary regularity theorem [3] (cf. [4, Section 5.23]) implies that $T_{i}=\llbracket \Lambda_{i} \rrbracket$ for two connected smooth submanifolds such that $\partial \Lambda_{i}=\Gamma$ in the classical sense of differential topology. Since any integral area-minimizing 2-dimensional current in $\mathbb{R}^{3}$ is an embedded submanifold (with integer multiplicity) away from the boundary, we also conclude that $\Lambda_{1}$ and $\Lambda_{2}$ do not intersect except at their common boundary $\Gamma$. The Hopf boundary lemma then implies that at every point $p \in \Gamma$ the two currents have distinct tangents, i.e. $\Lambda_{1}$ and $\Lambda_{2}$ meet at their common boundary transversally.

In view of the above observation we cannot expect, in general, a "better" conclusion than the one of Theorem 1.4 or, in other words, we cannot expect that the number $J$ in Definition 1.3 to be 1 . However, an obvious corollary of Theorem 1.4 is the following.

Theorem 1.5. Let $U, T, \Gamma$ and $q$ be as in Theorem 1.4. Then there is a neighborhood $U^{\prime}$ of $q$ in which $T=Q \llbracket \Lambda \rrbracket$ for some smooth minimal surface $\Lambda$ if and only if one tangent cone to $T$ at $q$ is "flat", i.e. contained in a 2-dimensional linear subspace of $\mathbb{R}^{2+n}$.

Even though the assumption that $\Theta(T, q)$ is sufficiently close to $\frac{Q}{2}$ seems, at a first glance, very restrictive, we can either follow a lemma of Allard in [3] (valid in any dimension and codimension) or a simple classificaton of the boundary tangent cones (cf. [10]) to show that it holds when $\mathrm{spt}(\partial T)$ is contained in the boundary of a bounded $C^{2}$ uniformly convex set $\Omega$. For this reason, complete regularity can be achieved when there is a "convex barrier". Since this is an assumption which will be used often in some sections of the work, we wish to isolate its statement.
Assumptions 1.6. $\Omega \subset \mathbb{R}^{2+n}$ is a bounded $C^{3, \alpha_{0}}$ uniformly convex set for some $\alpha_{0}>0$, $\Gamma \subset \partial \Omega$ is the disjoint union of finitely many $C^{3, \alpha_{0}}$ simple closed curves $\left\{\Gamma_{i}\right\}_{i=1, \ldots, N}$. $T$ is a 2-dimensional area-minimizing integral current in $\mathbb{R}^{2+n}$ such that $\partial T=\sum_{i} Q_{i} \llbracket \Gamma_{i} \rrbracket$.
Theorem 1.7. Let $\Gamma, \Omega$ and $T$ be as in Assumption 1.6. Then $\operatorname{Sing}_{b}(T)$ is empty.
In fact we can give a suitable local version of the above statement from which Theorem 1.7 can be easily concluded, cf. Theorem 10.5.

In the next section we will outline the arguments to prove Theorem 1.4, 1.5, and 1.7. Before coming to it we wish to point out two things. We are confident that the methods used in this work generalize to cover the same statement as in Theorem 1.4 in an arbitrary smooth (i.e. $C^{3, \alpha_{0}}$ ) complete Riemannian manifold, but in order to keep the technicalities at bay we have decided to restrict our attention to Euclidean ambient spaces. Even though the basic ideas behind this work are quite simple, the overall proof of the theorems is quite lengthy. For instance before the recent paper [21], not even the existence of a single boundary regular point was known, without some convex barrier assumption and in a general Riemannian manifold. Part of the challenge is that several crucial PDE ingredients are absent in codimension higher than 1. Let us in particular mention three facts:
(a) There is no "soft" decomposition theorem which allows to reduce the general case to that of multiplicity 1 boundaries;
(b) Boundary singularities occur even in the case of multipliciy 1 smooth boundaries;
(c) There is no maximum principle (and in particular no Hopf boundary lemma) available even if we knew apriori that the minimizing currents are completely smooth.

### 1.2.1 Outline of the proof of part II

In the first step (cf. Section 10), we use the classical convex hull property to reduce the statement of Theorem 1.7 to a local version, cf. Theorem 10.5. The latter statement
will then focus only on a portion of the boundary, but under the assumption that the support of the current is contained in a suitable convex region, cf. Assumption 10.4. The crucial point is that this convex region forms a "wedge" at each point of the boundary, cf. Definition 10.2.

In the second step (cf. Section 11) we recall the classical Allard's monotonicity formula and we appeal to a classification result for 2-dimensional area-minimizing integral cones with a straight boundary (see [10]) to conclude that, in all the cases we are dealing we can assume, without loss of generality, that all the tangent cones to $T$ at every boundary point $p$ consist of a finite number of halfplanes with common boundary $T_{p} \Gamma$, counted with a positive integer multiplicity, cf. Theorem 11.5.

At this point, taking advantage of pioneering ideas of White, cf. [37], and of a recent paper by Hirsch and Marini, cf. [29], the tangent cone can be shown to be unique at each point $p \in \Gamma$. We need, strictly speaking, a suitable generalization of [29], but the simple technical details are given in the shorter paper [11]. This uniqueness result has two important outcomes:
(a) At any point $p \in \Gamma$ where the tangent cone is not flat (i.e. it is not contained in a single half-plane) we can decompose the current into simpler pieces, cf. Theorem 12.3;
(b) the convergence rate of the current to the cone is polynomial (cf. also Corollary 23.1.

Point (a) reduces all our regularity statement to Theorem 1.5. In fact we will focus on a slightly more technical version of it, cf. Theorem 12.6 Point (b) gives one crucial piece of information which will allow us to conclude Theorem 12.6. The remaining part of this work will in fact be spent to argue for Theorem 12.6 by contradiction: if a flat boundary point $p$ is singular, then the convergence rate to the flat tangent cone at $p$ must be slower than polynomial, contradicting thus (b).

We first address a suitable linearized version of Theorem 12.6: we introduce multivalued functions and define the counterpart of flat boundary points in that context, which are called contact points. In Theorem 13.5, we then prove an analog of Theorem 12.6 in the case of multivalued functions minimizing the Dirichlet energy using a version of the frequency function (see Definition 13.6) first introduced by Almgren. However, while the proof of Theorem 13.5 might be instructive to the reader because it illustrates, in a very simplified setting, the idea behind the "slow decay" at singular points, the crucial fact which will be used to show Theorem 12.6 is contained in Theorem 13.3: the latter states that, if a multi-function vanishes identically at a straight line and it is $I$-homogeneous, either it is a multiple copy of a single classical harmonic function, or the homogeneity equals 1 .

The overall idea is that, if $p$ is a singular flat point, then it can be efficiently approximated at small scales by an homogeneous harmonic (i.e. Dirichlet minimizing)
multivalued function as above (not necessarily unique), which however cannot be a multiple copy of a single classical harmonic function. Since the homogeneity of the latter will be forced to be 1, we will infer from it the slow decay of the "cylindrical excess" (cf. Definition 14.1). However, the work to accomplish the latter approximation proves to be quite laborious and it will pass through a series of more and more refined approximations.

First of all, in the Sections 14, 15, 16, and 17 we prove that the current can be efficiently approximated by multivalued Lipschitz functions when sufficiently flat (cf. Theorem 17.1) and that the latter approximation almost minimizes the Dirichlet energy (cf. Theorem 15.3). These sections take heavily advantage of the tools introduced in [13, 14] and of some ideas in [21]. However these approximations are not sufficient to carry on our program.

A new refined approximation is then devised in Section 18. At every sufficiently small scale we can construct a center manifold (i.e. a classical $C^{3}$ surface with boundary $\Gamma$ ) and a multivalued Lipschitz approximation over its normal bundle (called normal approximation), which approximates the current as efficiently as the "straight" approximation in Theorem 17.1, cf. Theorem 18.16 and Theorem 18.21 for the relevant statements. This new normal approximation has however two important features:
(i) It approximates the current well not only at the "starting scale" but also across smaller scales as long as certain decay conditions are ensured.
(ii) At all such scales the normal approximation has average close to 0 (namely it is never close to a multiple copy of a single harmonic function, compared to its own Dirichlet energy).

The Sections 19, 20, and 21 provide a proof of Theorem 18.16 and Theorem 18.21. While the first center manifold was introduced in the monograph [4] by Almgren, our constructions borrows from the ideas and tools introduced in [15] and [21].

Our proof would be at this point much easier if the validity of (ii) above would hold, around the given singular flat point $p$, at all scales smaller than the one where we start the construction of the center manifold. Unfortunately we do not know how to achieve this. We are therefore forced to construct a sequence of center manifolds which cover different sets of scales, cf. again Section 23.1. At certain particular scales we need therefore to change approximating maps, i.e. to pass from one center manifold to the next. Section 22 provides then important information about the latter "exchange scales". Both sections are heavily influenced by similar considerations made in the papers [15, 16].

The remaining parts of the thesis are thus focused to show that, at a sufficiently small scale around the flat point $p$, all these normal approximations are close to some homogeneous Dir-minimizing function (not necessarily the same across all scales), which by Theorem 13.3 will then result to be 1-homogeneous. The key ingredient to
show this homogeneity is the almost monotonicity of the frequency function of the normal approximation (a celebrated quantity introduced by Almgren in his pioneering work [4]). In order to deal with the boundary we resort to an important variant introduced in [21]. The key point is to show that, as $r \downarrow 0$, the frequency function $I(r)$ of the approximation at scale $r$ converges to a limit. However, since our approximation might change at some particular scales, the function I undergoes a possibly infinite number of jump discontinuities, while it is almost monotone in the complement of these discontinuities. In order to show that the limit exists we thus need:
(1) a suitable quantification of the monotonicity on each interval delimited by two consecutive discontinuities;
(2) a suitable bound on the series of the absolute values of such jumps.

The relevant estimates, namely (23.13) and (23.14), are contained in Theorem 23.5. While the proof of (23.13) takes advantage of similar cases handled in [16] and [21], (23.14) is entirely new and we expect that the underlying ideas behind it will prove useful in other contexts. The Sections 24 and 25 are dedicated to prove the respective estimates.
Finally, in Section 26 we carry on the (relatively simple) argument which, building upon all the work of the previous sections, shows that the rate of convergence to the tangent cone at a singular flat point must to be slower than any polynomial rate. As already mentioned, since the convergence rate has to be polynomial at every point, this shows that a singular flat point cannot exist.

## Part I

THE CODIMENSION ONE CASE

### 2.1 NOTATION

In this part of the thesis, $k, m$ and $n$ denote fixed natural numbers with $m \geq 1$ and $n, k \geq 2 . C_{1}, \ldots, C_{80}$ are positive constants depending only on $n, k$ and $m$.

### 2.1.1 Notation associated to the domain

We define the following sets for $y \in \mathbb{R}^{n}, j \in\{1, \ldots, n\}$ and any real numbers $r>0$ and $0<\sigma<1$

$$
\begin{aligned}
\mathbf{B}_{r}^{n}(y) & =\left\{x \in \mathbb{R}^{n}:|x-y|<r\right\}, \\
\overline{\mathbf{B}}_{r}^{n}(y) & =\left\{x \in \mathbb{R}^{n}:|x-y| \leq r\right\}, \\
\boldsymbol{\omega}_{n} & =\mathcal{L}^{n}\left(\mathbf{B}_{1}^{n}(0)\right), \\
\mathbf{L} & =\left\{x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbf{B}_{1}^{n}(0): x_{n}=0\right\}, \\
\mathbf{V} & =\left\{x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbf{B}_{1}^{n}(0): x_{n}>0\right\}, \\
\mathbf{W} & =\left\{x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbf{B}_{1}^{n}(0): x_{n}<0\right\}, \\
\mathbf{V}_{\sigma} & =\{x \in \mathbf{V}: \operatorname{dist}(x, \partial \mathbf{V})>\sigma\}, \\
\mathbf{W}_{\sigma} & =\{x \in \mathbf{W}: \operatorname{dist}(x, \partial \mathbf{W})>\sigma\}, \\
Y_{j} & : \mathbb{R}^{n} \rightarrow \mathbb{R}, Y_{j}(y)=y_{j} .
\end{aligned}
$$

### 2.1.2 Notation associated to the ambient space

We define the following sets for $a \in \mathbb{R}^{n+k}, j \in\{1, \ldots, n+k\}$ and any real numbers $\omega$ and $r>0$

$$
\begin{aligned}
\mathbf{B}_{r} & =\left\{x \in \mathbb{R}^{n+k}:|x|<r\right\}, \\
\overline{\mathbf{B}}_{r} & =\left\{x \in \mathbb{R}^{n+k}:|x| \leq r\right\}, \\
\mathbf{C}_{r} & =\left\{x \in \mathbb{R}^{n+k}:|\mathbf{p}(x)| \leq r\right\} \text { where } \mathbf{p}: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n}, \mathbf{p}\left(x_{1}, \ldots, x_{n+k}\right)=\left(x_{1}, \ldots, x_{n}\right), \\
\mathbf{e}_{j} & =(0, \ldots, 0,1,0, \ldots, 0) \text { where the } 1 \text { is at the } j \text {-th component, } \\
X_{j} & : \mathbb{R}^{n+k} \rightarrow \mathbb{R}, X_{j}(x)=x_{j}, \\
X & :=\left(X_{1}, \ldots, X_{n+k}\right),
\end{aligned}
$$

For the following maps, we identify $\mathbb{R}^{n+k}$ with $\mathbb{R}^{n+1} \times \mathbb{R}^{k-1}$.

$$
\begin{aligned}
\boldsymbol{\tau}_{a} & : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}, \boldsymbol{\tau}_{a}(x, y)=(x, y)+a \\
\boldsymbol{\mu}_{r} & : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}, \boldsymbol{\mu}_{r}(x, y)=r(x, y) \\
\gamma_{\omega} & : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}, \\
& \gamma_{\omega}(x, y)=\left(x_{1}, \ldots, x_{n-1}, x_{n} \cos (\omega)-x_{n+1} \sin (\omega), x_{n} \sin (\omega)+x_{n+1} \cos (\omega), y\right)
\end{aligned}
$$

### 2.1.3 Notation associated to the current $T$

For any real number $r>0$, we define the cylindrical excess as

$$
\mathbf{E}_{C}(T, r)=r^{-n} \mathbb{M}\left(T\left\llcorner\mathbf{C}_{r}\right)-r^{-n} \mathbb{M}\left(\mathbf{p}_{\#}\left(T\left\llcorner\mathbf{C}_{r}\right)\right)\right.\right.
$$

and the spherical excess as

$$
\mathbf{E}_{S}(T, r)=r^{-n} \mathbb{M}\left(T\left\llcorner\overline{\mathbf{B}}_{r}\right)-\boldsymbol{\omega}_{n} \Theta^{n}(\|T\|, 0)\right.
$$

whenever $\Theta^{n}(\|T\|, 0)=\lim _{r \downarrow 0} \frac{\|T\|\left(\mathbf{B}_{r}\right)}{\boldsymbol{\omega}_{n} r^{n}}$ exists. Notice that this differs from

$$
\frac{1}{2 r^{n}} \int_{B_{r}}\left|\vec{T}-\mathbf{e}_{1} \wedge \cdots \wedge \mathbf{e}_{n}\right|^{2} \mathrm{~d}\|T\|
$$

which is in the literature also called the spherical excess.
In Chapter 9, we will see that it suffices to consider only currents with compact support and whose boundary lies on a $(n-1)$-dimensional $\mathcal{C}^{2}$-graph going through the origin. Namely, we define $\mathcal{T}$ to be the collection of pairs $(T, \mathcal{M})$ where $\mathcal{M}$ is an embedded $(n+1)$-manifold and $T \in \mathcal{R}_{n}\left(\mathbb{R}^{n+k}\right)$ is an absolutely area minimizing integer rectifiable current for which there exist a positive integer $m, \varphi_{T}, \psi_{T} \in \mathcal{C}^{2}(\{z \in$ $\left.\mathbb{R}^{n-1}:|z| \leq 2\right\}$ ) and a smooth $\operatorname{map} \boldsymbol{\Phi}_{\mathcal{M}}: \mathbf{B}_{4}^{n+1}(0) \rightarrow \mathbb{R}^{k-1}$, such that

- $\left\{z \in \mathbf{C}_{3}: z \in \mathcal{M}\right\}=\left\{\left(x, \boldsymbol{\Phi}_{\mathcal{M}}(x)\right): x \in \mathbf{B}_{3}^{n+1}(0)\right\}$,
- $\boldsymbol{\Phi}_{\mathcal{M}}(0)=0$ and $D \boldsymbol{\Phi}_{\mathcal{M}}(0)=0$,
- $\mathbf{A} \leq 1$,
- $\operatorname{spt}(T) \subset \overline{\mathbf{B}}_{3} \cap \mathcal{M}$,
- $\mathbb{M}(T) \leq 3^{n}\left(1+m \boldsymbol{\omega}_{n}\right)$,
- $\Theta^{n}(\|T\|, 0)=m-1 / 2$,
- $\begin{aligned} \mathbf{p}_{\#}\left(T\left\llcorner\mathbf{C}_{2}\right)=\right. & m\left(\mathbf{E}^{n}\left\llcorner\left\{y \in \mathbf{B}_{2}^{n}(0): y_{n}>\varphi_{T}\left(y_{1}, \ldots, y_{n-1}\right)\right\}\right)\right. \\ & +(m-1)\left(\mathbf{E}^{n}\left\llcorner\left\{y \in \mathbf{B}_{2}^{n}(0): y_{n}<\varphi_{T}\left(y_{1}, \ldots, y_{n-1}\right)\right\}\right)\right.\end{aligned}$
- $\varphi_{T}(0)=0=\psi_{T}(0)$,
- $\varphi_{T}(0)=0=\psi_{T}(0)$,
- $D \varphi_{T}(0)=0=D \psi_{T}(0)$,
- $(\partial T)\left\llcorner\left\{x \in \mathbb{R}^{n+1}:\left|\left(x_{1}, \ldots, x_{n-1}\right)\right|<2,\left|x_{n}\right|<2\right\}\right.$

$$
=(-1)^{n+k}\left(F_{T}\right)_{\#}\left(\mathbf{E}^{n-1}\left\llcorner\left\{z \in \mathbb{R}^{n-1}:|z|<2\right\}\right)\right.
$$

- $\kappa_{T} \leq 1$,
where

$$
\begin{aligned}
\mathbf{A} & :=\left\|D^{2} \boldsymbol{\Phi}_{\mathcal{M}}\right\|_{\mathcal{C}^{1}\left(\overline{\mathbf{B}}_{2}\right)}, \\
\mathbf{E}^{j} & :=\llbracket \mathbb{R}^{j} \times\{0\} \rrbracket \in \mathcal{R}_{j}\left(\mathbb{R}^{n+k}\right) \text { for all } j \leq n, \\
F_{T}(z) & :=\left(z, \varphi_{T}(z), \psi_{T}(z), \boldsymbol{\Phi}_{\mathcal{M}}\left(z, \varphi_{T}(z), \psi_{T}(z)\right)\right), \\
\kappa_{T} & :=\left\|D^{2}\left(\varphi_{T}, \psi_{T}\right)\right\|_{\mathcal{C}^{0}} .
\end{aligned}
$$

Notice that $\mathbf{A}$ and $\kappa_{T}$ are comparable to the second fundamental forms of $\mathcal{M}$ and $\operatorname{spt}(\partial T)$ respectively.

### 2.2 FIRST VARIATION AND MONOTONICITY

We start this section with the following monotonicity estimates. The first two can be read in [21, Theorem 3.2] and the third one, we prove in the Appendix a.1.

Lemma 2.1 (Monotonicity Formula). For $(T, \mathcal{M}) \in \mathcal{T}$ and $0<r<s<2$, the following holds

$$
\begin{aligned}
\frac{\|T\|\left(\overline{\boldsymbol{B}}_{s}\right)}{s^{n}}-\frac{\|T\|\left(\overline{\boldsymbol{B}}_{r}\right)}{r^{n}} & -\int_{\overline{\boldsymbol{B}}_{s} \backslash \overline{\boldsymbol{B}}_{r}}\left|X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| \\
& =\int_{r}^{s} \rho^{-n-1}\left(\int_{\boldsymbol{B}_{\rho}} X^{\perp} \cdot \vec{H} \mathrm{~d}\|T\|+\int_{\operatorname{spt}(\partial T) \cap \boldsymbol{B}_{\rho}} X \cdot \vec{n} \mathrm{~d} \mathcal{H}^{n-1}\right) \mathrm{d} \rho
\end{aligned}
$$

where $X^{\perp}$ denotes the component orthogonal to the tangent plane of $T$ and $\vec{H}$ the curvature vector of $\mathcal{M}$.

Remark 2.2. There exists $C_{1}$ such that $|\vec{H}| \leq C_{1} \mathbf{A}_{\mathcal{M}}$.
Lemma 2.3. There is a dimensional constant $C_{2}>0$ such that for $(T, \mathcal{M}) \in \mathcal{T}$ and $0<r<2$, the map

$$
r \mapsto \exp \left(C_{2}\left(\boldsymbol{A}_{\mathcal{M}}+\kappa_{T}\right) r\right) \frac{\|T\|\left(\overline{\boldsymbol{B}}_{r}\right)}{r^{n} \boldsymbol{\omega}_{n}}
$$

is monotonously increasing.

Corollary 2.4. For $(T, \mathcal{M}) \in \mathcal{T}$ and $0<r<s<2$, the following holds

$$
\left.\left.\left|\frac{\|T\|\left(\overline{\boldsymbol{B}}_{s}\right)}{s^{n}}-\frac{\|T\|\left(\overline{\boldsymbol{B}}_{r}\right)}{r^{n}}-\int_{\overline{\boldsymbol{B}}_{s} \backslash \overline{\boldsymbol{B}}_{r}}\right| X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| \right\rvert\, \leq C_{3}\left(A_{\mathcal{M}}+\kappa_{T}\right)(s-r) .
$$

Letting $r \downarrow 0$, we deduce the following corollary.
Corollary 2.5. For $(T, \mathcal{M}) \in \mathcal{T}$ and $0<r<2$, we have

$$
\left.\left|E_{S}(T, r)-\int_{\bar{B}_{r}}\right| X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| \mid \leq C_{4}\left(A_{\mathcal{M}}+\kappa_{T}\right) .
$$

In this chapter we prove that the minimizing current is, away from the boundary, supported on graphs.
Definition 3.1. Let $u: U \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$. Then we define

$$
\operatorname{graph}(u, \boldsymbol{\Phi}):=\{(x, u(x), \boldsymbol{\Phi}(x, u(x))): x \in U\} .
$$

Away from the boundary, the interior regularity theory gives us functions whose graphs describe the current. Moreover they fulfill the Riemannian minimal surface equation (see Definition 3.7) that is elliptic and therefore, we can deduce estimates on the gradient of these functions. These estimates are crucial as they guarantee the existence of the
 harmonic blow-ups introduced in section 4.

Theorem 3.2. Let $(T, \mathcal{M}) \in \mathcal{T}$ and assume $A \leq 1 / 4$. Then there are constants $C_{5} \geq 12$, $C_{6} \geq 1$ such that if

$$
\boldsymbol{E}_{C}(T, 1)+\kappa_{T}+\boldsymbol{A} \leq\left(4 C_{5}\right)^{-2 n-3}
$$

and we denote $\sigma_{T}:=C_{5}\left(E_{C}(T, 1)+\kappa_{T}+A\right)^{1 /(2 n+3)}, V_{T}:=V_{\sigma_{T}}$ and $W_{T}:=W_{\sigma_{T}}$, then for $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ and $k \in\{1,2,3\}$ there are smooth functions $v_{i}^{T}: V_{T} \rightarrow \mathbb{R}$ and $w_{j}^{T}: \boldsymbol{W}_{T} \rightarrow \mathbb{R}$ satisfying the Riemannian minimal surface equation and such that
(i.) $v_{1}^{T} \leq v_{2}^{T} \leq \cdots \leq v_{m}^{T} \quad$ and $\quad w_{1}^{T} \leq w_{2}^{T} \leq \cdots \leq w_{m-1}^{T}$,
(ii.) $p^{-1}\left(\boldsymbol{V}_{T}\right) \cap \operatorname{spt}(T)=\bigcup_{i=1}^{m} \operatorname{graph}\left(v_{i}^{T}, \boldsymbol{\Phi}\right)$ and

$$
\boldsymbol{p}^{-1}\left(\boldsymbol{W}_{T}\right) \cap \operatorname{spt}(T)=\bigcup_{i=1}^{m-1} \operatorname{graph}\left(w_{i}^{T}, \boldsymbol{\Phi}\right),
$$

(iii.) $\left|D^{k} v_{i}^{T}(y)\right| \leq C_{7} \sqrt{E_{C}(T, 1)+\kappa_{T}+A} \operatorname{dist}(y, \partial V)^{-k-n-1 / 2}$ for all $y \in V_{T}$,
(iv.) $\left|D^{k} w_{j}^{T}(y)\right| \leq C_{7} \sqrt{E_{C}(T, 1)+\kappa_{T}+\boldsymbol{A}} \operatorname{dist}(y, \partial W)^{-k-n-1 / 2}$ for all $y \in \boldsymbol{W}_{T}$,

$$
\begin{aligned}
& \text { (v.) } \int_{\boldsymbol{V}_{T}}\left(\frac{\partial}{\partial r} \frac{v_{i}^{T}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)+\int_{\boldsymbol{W}_{T}}\left(\frac{\partial}{\partial r} \frac{w_{j}^{T}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \\
& \leq 2^{n+7} E_{S}(T, 1)+C_{8}\left(A+\kappa_{T}\right) \\
& \leq 2^{n+7} E_{C}(T, 1)+C_{9}\left(A+\kappa_{T}\right), \quad \quad \text { where } \frac{\partial}{\partial r} f(y):=\frac{y}{|y|} \cdot D f(y) \text {. }
\end{aligned}
$$

For the existence of these graphs, we need to split the current into pieces and show that each piece is supported on a graph. Then, once we have these graphs, we show the estimates by using the regularity theory of elliptic PDEs. This will be done in detail in section 3.3.

### 3.1 COMPARISON BETWEEN EXCESS AND HEIGHT

To prove the estimate in Theorem 3.2(iii.), (iv.), we will deduce from the PDE theory an estimate on the values of the functions $v_{i}^{T},\left(w_{j}^{T}\right.$ respectively). This can be translated into the height of the current in the $(n+1)$-component. We wish to estimate the latter quantity with the excess of $T$ and hence, we need the following lemmata comparing the (cylindrical) excess with the height. The proofs are given in chapter a.

First notice that as in the original paper [27, 1.4(1)], we infer that for $0<r \leq s \leq 2$, the following holds

$$
\begin{equation*}
\mathbf{E}_{C}(T, r) \leq\left(\frac{s}{r}\right)^{n} \mathbf{E}_{C}(T, s) \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{E}_{S}(T, r) \leq \mathbf{E}_{C}(T, r)+m r \kappa_{T} \tag{3.2}
\end{equation*}
$$

Lemma 3.3. There are positive constants $C_{10}$ and $C_{11}$ such that for all $0<\sigma<1$ and $(T, \mathcal{M}) \in \mathcal{T}$, the following holds

$$
\frac{\sigma^{2}}{C_{10}} E_{C}(T, 1)-\kappa_{T}-A \leq \int_{C_{1+\sigma}} X_{n+1}^{2} \mathrm{~d}\|T\| \leq C_{11} \sup _{C_{1+\sigma} \cap \operatorname{spt}(T)} X_{n+1}^{2}
$$

Not only it is true, that the height bounds the excess, but also the other way around. The following estimates rely on an area comparison lemma (Lemma a.1). Its proof will give us a constant $C_{12}$ which we will use to prove the following

Lemma 3.4. If $0<\sigma<1, A^{2} \leq \sigma / 8$ and $A \leq\left(7 C_{1}+C_{12}+1\right)^{-1}$ then there are positive constants $C_{13}$ and $C_{14} \geq 2$ such that for $(T, \mathcal{M}) \in \mathcal{T}$, the following holds
(i.) $\frac{\sigma^{n}}{C_{13}} \sup _{C_{1-\sigma} \cap \operatorname{spt}(T)} X_{n+1}^{2} \leq \int_{C_{1-\sigma / 2}} X_{n+1}^{2} \mathrm{~d}\|T\|+\kappa_{T}$.
(ii.) $\int_{C_{1-\sigma / 2}} X_{n+1}^{2} \mathrm{~d}\|T\| \leq \frac{C_{14}-1}{\sigma^{n+1}}\left(\boldsymbol{E}_{C}(T, 1)+\kappa_{T}+\boldsymbol{A}\right)$.

In particular, we have

$$
\sup _{C_{1-\sigma} \cap \operatorname{spt}(T)} X_{n+1}^{2} \leq \frac{C_{13} C_{14}}{\sigma^{2 n+1}}\left(E_{C}(T, 1)+\kappa_{T}+A\right) .
$$

### 3.2 SPLITTING OF THE MINIMIZING CURRENT $T$

Here we prove the fact, that if a current has no boundary, its excess is not too large and the projection has multiplicity $j$, then it consists of $j$ many layers whose projection are of multiplicity 1 .
Lemma 3.5. Let $j \in \mathbb{N}_{+}, V \subset \mathbb{R}^{n}$ be open and consider the cylinder $\Gamma:=\left\{x \in \mathbb{R}^{n+1}\right.$ : $\left.\left(x_{1}, \ldots, x_{n}\right) \in V\right\}$ and the modified version $\tilde{\Gamma}:=\{(x, \boldsymbol{\Phi}(x)) \in \mathcal{M}: \boldsymbol{p}(x) \in V\}$. If $S \in \mathcal{R}_{n}(\tilde{\Gamma})$ satisfies

- $(\partial S)\llcorner\tilde{\Gamma}=0$
- $\boldsymbol{p}_{\#} S=j\left(\boldsymbol{E}^{n}\llcorner V)\right.$
- $\mathbb{M}(S)-\mathbb{M}\left(\boldsymbol{p}_{\#} S\right)<\mathcal{H}^{n}(V)$,
then for all $i \in\{1, \ldots, j\}$ there exists $S_{i} \in \mathcal{R}_{n}(\tilde{\Gamma})$ such that

$$
\begin{aligned}
\tilde{\Gamma} \cap \operatorname{spt}\left(\partial S_{i}\right)=\varnothing, & \boldsymbol{p}_{\#} S_{i}=\boldsymbol{E}^{n}\llcorner V, \\
S=\sum_{i=1}^{j} S_{i}, & \|S\|=\sum_{i=1}^{j}\left\|S_{i}\right\| .
\end{aligned}
$$

Proof. Denote by $\tilde{p}$ the projection to $\mathbb{R}^{n+1}$ and consider $\tilde{S}:=\tilde{p} \# S$. Then we have

- $(\partial \tilde{S})\left\llcorner\Gamma=\left(\tilde{p}_{\#}(\partial S)\right)\left\llcorner\Gamma=\tilde{p}_{\#}((\partial S)\llcorner\tilde{\Gamma})=0\right.\right.$
- $\mathbf{p}_{\#} \tilde{S}=\mathbf{p}_{\#} S=j\left(\mathbf{E}^{n}\llcorner V)\right.$
- $\mathbb{M}(\tilde{S})-\mathbb{M}\left(\mathbf{p}_{\#} \tilde{S}\right) \leq \mathbb{M}(S)-\mathbb{M}\left(\mathbf{p}_{\#} S\right) \leq \mathcal{H}^{n}(V)$.

Therefore, we can argue as in the original paper [27, Lemma 5.1] to deduce a decomposition for $\tilde{S}$ : There are $\tilde{S}_{i} \in \mathcal{R}_{n}\left(\mathbb{R}^{n+1}\right)$ such that

$$
\begin{aligned}
\Gamma \cap \operatorname{spt}\left(\partial \tilde{S}_{i}\right)=\varnothing, & \mathbf{p}_{\#} S_{i}=\mathbf{E}^{n}\llcorner V, \\
\tilde{S}=\sum_{i=1}^{j} \tilde{S}_{i}, & \|\tilde{S}\|=\sum_{i=1}^{j}\left\|\tilde{S}_{i}\right\| .
\end{aligned}
$$



We conclude by putting $S_{i}:=(\mathrm{id}, \boldsymbol{\Phi})_{\#} \tilde{S}_{i}$.

In the situation of Theorem 3.2, each of these $S_{i}$ is area minimizing in $\mathcal{M}$ and so the smallness of the excess implies that locally the function, whose graph describe $\operatorname{spt}\left(S_{i}\right)$, fulfills an elliptic equation. Thus, we can deduce the following Schauder estimate:

Lemma 3.6. Let $U$ be an open neighborhood of $0 \in \mathbb{R}^{n}$ and $u: U \rightarrow \mathbb{R}$ such that $u(0)=0$, $D u(0)=0$ and $\operatorname{graph}(u, \boldsymbol{\Phi}) \subset \mathcal{M}$ is a minimal surface in $\mathcal{M}$. Then there is $r_{0}>0$ such that for all $0<r<r_{0}$ we have

$$
r\|D u\|_{\mathcal{C}^{0}\left(B_{r / 2}\right)}+r^{2}\left\|D^{2} u\right\|_{\mathcal{C}^{0}\left(B_{r / 2}\right)}+r^{2+\alpha}\left[D^{2} u\right]_{\mathcal{C}^{\alpha}\left(B_{r / 2}\right)} \leq C_{15}\left(\|u\|_{\mathcal{C}^{0}\left(B_{r}\right)}+\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{\alpha}\left(B_{r}\right)}^{*}\right)
$$

where

$$
\|f\|_{\mathcal{C}^{\alpha}(\Omega)}^{*}:=\sup _{x \in \Omega} \operatorname{dist}(x, \partial \Omega)^{2}|f(x)|+\sup _{\substack{x, y \in \Omega \\ x \neq y}} \max \{\operatorname{dist}(x, \partial \Omega), \operatorname{dist}(y, \partial \Omega)\}^{2+\alpha} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}
$$

Proof. We use the Euler-Lagrange equation in the form of Schoen-Simon in [32, Chapter 1]. Then we use Gilbarg-Trudinger [25, Theorem 6.2] to deduce the estimate. Indeed, we define

$$
\begin{aligned}
F(z, v(z)) & :=\sqrt{\operatorname{det}\left((\mathrm{id}-v \otimes v) g_{\mathcal{M}}(\mathrm{id}-v \otimes v)\right)} \\
& =\sqrt{\operatorname{det}\left(\left\langle e_{i}+\partial_{i} u, e_{j}+\partial_{j} u\right\rangle_{g_{\mathcal{M}}}\right)}=J_{g_{\mathcal{M}}}\left(\mathrm{id}_{\mathbb{R}^{n}}, u\right)
\end{aligned}
$$

where $g_{\mathcal{M}}=\left(\operatorname{id}_{\mathbb{R}^{n+1}}, \Phi\right)^{\#} g_{\mathbb{R}^{n+1+k}}=\left(\delta_{i j}+\left\langle\partial_{i} \Phi, \partial_{j} \Phi\right\rangle\right)_{i j}$ is the pullback metric on $\mathcal{M}$. Then

$$
\int_{\operatorname{graph}(u) \cap \mathbf{C}_{r_{0}}} F(z, v(z)) \mathrm{d} \mathcal{H}^{n}(z)=\int_{\mathbf{B}_{r_{0}}} J_{g_{\mathcal{M}}}\left(\operatorname{id}_{\mathbb{R}^{n}}, u\right)=\operatorname{Vol}(\operatorname{graph}(u, \boldsymbol{\Phi}))
$$

and we can apply [32] (in particular, [32, Remark 1] describes our situation perfectly). The Euler-Lagrange equation then reads

$$
\operatorname{div}\left(\frac{D u}{\sqrt{1+|D u|^{2}}}\right)=\sum_{i, j=1}^{n} a_{i j}(x) \partial_{i j} u(x)+b(x)
$$

where

$$
\begin{aligned}
a_{i j}(x) & =\int_{0}^{1} \sum_{k=1}^{n+1} z_{k} \partial_{z_{k} p_{i} p_{j}} F(t z,-D u, 1) \mathrm{d} t \\
b(x) & =\sum_{i=1}^{n+1} \partial_{z_{i}, p_{i}} F(z, p)
\end{aligned}
$$

are evaluated in $z=(x, u(x)), p=(-D u, 1)$. In order to use elliptic estimates, we define

$$
A_{i j}:=\frac{\delta_{i j}\left(1+|D u|^{2}\right)-\partial_{i} u \partial_{j} u}{\left(1+|D u|^{2}\right)^{3 / 2}}-a_{i j}
$$

and notice that for $r_{0}>0$ small enough, we have $|D u|+\max _{i j}\left|a_{i j}\right| \leq 1 / 12$ in $\mathbf{B}_{r_{0}}$ and therefore $\frac{1}{2} \mathrm{id} \leq A \leq 2 \mathrm{id}$ as a quadratic form. The only thing left to do is to notice that

$$
\|b\|_{\mathcal{C}^{\alpha}}^{*} \leq C_{16}\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{\alpha}}^{*}
$$

Definition 3.7. We define (3.3) to be the Riemannian minimal surface equation.

### 3.3 PROOF OF THEOREM 3.2

## Proof of Theorem 3.2.

Let $\varepsilon>0$ be as in [23, Theorem 5.3.14] with $\lambda, \kappa, m, n$ replaced by $1,1, n, n+1$ respectively and we choose the parametric integrand to be the one associated to $\mathcal{M}$ :

$$
\begin{aligned}
& \Psi: \mathbb{R}^{n} \times \mathbb{R} \times \Lambda_{n}\left(\mathbb{R}^{n+k}\right) \longrightarrow \mathbb{R} \\
& \quad((x, y), \zeta) \longmapsto|\zeta| h\left(\binom{\text { id }}{D \boldsymbol{\Phi}(x, y)}\left(v_{1} \cdots v_{n}\right)\right),
\end{aligned}
$$


where $h$ is the map from Lemma 3.6 and $\left\{v_{1}, \ldots, v_{n}\right\}$ are orthonormal and such that

$$
v_{1} \wedge \cdots \wedge v_{n}=\frac{\zeta}{|\zeta|}
$$

We require $C_{5}$ to fulfil $\left(4 C_{5}\right)^{-2 n-3} \leq \mathcal{L}^{n}\left(\mathbf{V}_{T}\right)$ implying that $\operatorname{spt}(\partial T) \cap \mathbf{p}^{-1}\left(\mathbf{V}_{\sigma_{T} / 3} \cup\right.$ $\left.\mathbf{W}_{\sigma_{T} / 3}\right)=\varnothing$, because $\kappa_{T}<9^{-n} \sigma_{T}$. Indeed,

$$
\kappa_{T} \leq\left(\frac{\sigma_{T}}{C_{5}}\right)^{2 n+3} \leq \frac{\sigma_{T}}{9^{n}} \frac{9^{n}}{4^{2 n+2} C_{5}^{2 n+3}} \leq \frac{\sigma_{T}}{9^{n}} .
$$

Then, we have
$\mathbf{p}_{\#}\left(T\left\llcorner\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma_{T} / 3}\right)\right)=m\left(\mathbf{E}^{n}\left\llcorner\mathbf{V}_{\sigma_{T} / 3}\right) \quad\right.\right.$ and $\quad \mathbf{p}_{\#}\left(T\left\llcorner\mathbf{p}^{-1}\left(\mathbf{W}_{\sigma_{T} / 3}\right)\right)=(m-1)\left(\mathbf{E}^{n}\left\llcorner\mathbf{W}_{\sigma_{T} / 3}\right)\right.\right.$ and we can apply Lemma 3.5. We obtain for $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ on $\mathcal{M}$-area minimizing currents $S_{i}$ and $\tilde{S}_{j}$ satisfying

$$
\begin{gathered}
\sum_{i=1}^{m} S_{i}=T\left\llcorner\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma_{T} / 3}\right) \quad \text { and } \quad \sum_{j=1}^{m-1} \tilde{S}_{j}=T\left\llcorner\mathbf{p}^{-1}\left(\mathbf{W}_{\sigma_{T} / 3}\right),\right.\right. \\
\mathbf{p}_{\#} S_{i}=\mathbf{E}^{n}\left\llcorner\mathbf{V}_{\sigma_{T} / 3} \quad \text { and } \quad \mathbf{p}_{\#} \tilde{S}_{j}=\mathbf{E}^{n}\left\llcorner\mathbf{W}_{\sigma_{T} / 3} .\right.\right.
\end{gathered}
$$

Denote again by $\tilde{p}$ the projection to $\mathbb{R}^{n+1}$. Then $\tilde{p}_{\#} S_{i}$ and $\tilde{p}_{\#} \tilde{S}_{j}$ are absolutely $\Psi-$ minimizing. Now, we cover $\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma_{T} / 3}\right)$ with cylinders $\mathbf{C}_{\sigma_{T} / 3}(x)$ for all $x \in \mathbf{V}_{2 \sigma_{T} / 3} \cap$
$\operatorname{spt}\left(\mathbf{p}_{\#}\left(S_{i}\right)\right)$. In each of these cylinders, we want to use [23, Corollary 5.3.15] replacing $\lambda, \kappa, m, n, r, S$ by $1,1, n, n+1, \sigma_{T} / 3, \tau_{-x \#} \tilde{p}_{\#} S_{i}$ respectively. To do so, we must have $\left(4 C_{5}\right)^{-2 n-3} \leq(\varepsilon / 2)^{n+1}$. As a result, we get in each cylinder $C$ a solution $u_{C}$ of the Riemannian minimal surface equation whose graph forms $\operatorname{spt}\left(\tilde{p}_{\#} S_{i}\right) \cap \overline{\mathbf{B}}_{\sigma_{T} / 3}(x)$ and hence, $\operatorname{graph}\left(u_{C}, \boldsymbol{\Phi}\right) \cap \overline{\mathbf{B}}_{\sigma_{T} / 3}(x)=\operatorname{spt}\left(S_{i}\right) \cap \overline{\mathbf{B}}_{\sigma_{T} / 3}(x)$. These solutions yield a unique function $v_{i}^{T}$ whose graph on $\mathcal{M}$ is $\operatorname{spt}\left(S_{i}\right) \cap \mathbf{p}^{-1}\left(\mathbf{V}_{2 \sigma_{T} / 3}\right)$. As the integrand is smooth in $(x, y)$, so are the solutions. In $\mathbf{p}^{-1}\left(\mathbf{W}_{\sigma_{T} / 3}\right)$ we argue analogously. By construction of the splitting $\left\{S_{i}\right\}_{i}$, there is a numbering such that (i.) holds.
Now, we prove (iii.). We want to use Lemma 3.6 and Lemma 3.4 with $\sigma=2 \sigma_{T} / 3$. To do so, we notice that as $C_{5} \geq 12$, we have

$$
\mathbf{A}^{2} \leq \frac{C_{5}}{12}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right)^{1 /(2 n+3)}=\frac{\sigma_{T}}{12}=\frac{1}{8}\left(\frac{2}{3} \sigma_{T}\right) .
$$

Thus,

$$
\begin{equation*}
\sup _{\mathrm{C}_{1-2 \sigma_{T} / 3} n \mathrm{spt}(T)} X_{n+1}^{2} \leq \frac{C_{13} C_{14}}{\left(\frac{2}{3}\right)^{2 n+1} \sigma_{T}^{2 n+1}}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right) \leq\left(\frac{3}{2}\right)^{2 n+1} \frac{C_{13} C_{14}}{C_{5}^{2 n+3}} \sigma_{T}^{2} . \tag{3.4}
\end{equation*}
$$

Let $y \in \mathbf{V}_{T}$. We differ between two cases. Either $y$ is near the boundary having distance to $\partial \mathbf{V}$ which is comparable with $\sigma_{T}$, or $y$ lies more in the inner of $\mathbf{V}$, then $\sigma_{T}$ is much smaller than the distance, but on the other hand, we can choose larger balls. More formally:
Case 1: $\sigma_{T}<\operatorname{dist}(y, \partial \mathbf{V})<2 \sigma_{T}$.
We define $\delta:=\operatorname{dist}\left(y, \partial \mathbf{V}_{2 \sigma_{T} / 3}\right)$. Notice that

$$
\overline{\mathbf{B}}_{\delta}^{n}(y) \subset \mathbf{V}_{2 \sigma_{T} / 3} \quad \text { and } \quad \delta=\operatorname{dist}(y, \partial \mathbf{V})-\frac{2}{3} \sigma_{T} \geq \frac{1}{3} \operatorname{dist}(y, \partial \mathbf{V}) \geq \frac{1}{3} \sigma_{T} .
$$

Lemma 3.6, (3.4) and Lemma 3.4 (with $\sigma$ replaced by $2 \sigma_{T} / 3$ ) then yield for $k \in\{1,2,3\}$

$$
\begin{aligned}
\left|D^{k} v_{i}^{T}(y)\right| & \leq \frac{2^{k} C_{15}}{\delta^{k}}\left(\sup _{\overline{\mathbf{B}}_{\delta}^{n}(y)}\left|v_{i}^{T}\right|+\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right) \\
& \leq 24 C_{15}\left(\frac{1}{\sigma_{T}^{k}} \sup _{\mathbf{C}_{1-2 \sigma_{T} / 3} \cap \operatorname{spt}(T)}\left|X_{n+1}\right|+\frac{1}{\delta^{k}}\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right) \\
& \leq 24 C_{15}\left(\sqrt{\frac{C_{13} C_{14}}{\left(\frac{2}{3}\right)^{2 n+2 k+1} \sigma_{T}^{2 n+1}}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right)}+\frac{1}{\delta^{3}}\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right) \\
& \leq \frac{C_{7}}{\operatorname{dist}(y, \partial \mathbf{V})^{n+k+1 / 2}} \sqrt{\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}} .
\end{aligned}
$$

Case 2: $\operatorname{dist}(y, \partial \mathbf{V}) \geq 2 \sigma_{T}$.
We choose $\delta:=\operatorname{dist}(y, \partial \mathbf{V}) / 2 \geq \sigma_{T}$. Notice that also in this case $\overline{\mathbf{B}}_{\delta}^{n}(y) \subset \mathbf{V}_{2 \sigma_{T} / 3}$. Indeed, the following holds

$$
\operatorname{dist}\left(y, \partial \mathbf{V}_{2 \sigma_{T} / 3}\right)=\operatorname{dist}(y, \partial \mathbf{V})-\frac{2}{3} \sigma_{T} \geq 2 \delta-\frac{2}{3} \delta>\delta
$$

also $\overline{\mathbf{B}}_{\delta}^{n}(y) \subset \overline{\mathbf{V}}_{\delta}$. Therefore, Lemma 3.6 and Lemma 3.4 (with $\sigma$ replaced by $\delta$ ) imply

$$
\begin{aligned}
\left|D^{k} v_{i}^{T}(y)\right| & \leq \frac{2^{k} C_{15}}{\delta^{k}}\left(\sup _{\overline{\mathbf{B}}_{\delta}^{n}(y)}\left|v_{i}^{T}\right|+\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right) \\
& \leq 16 C_{15}\left(\frac{1}{\sigma_{T}^{k}} \sup _{\mathbf{C}_{1-2 \sigma_{T} / 3} \cap \operatorname{spt}(T)}\left|X_{n+1}\right|+\frac{1}{\delta^{k}}\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right) \\
& \leq 16 C_{15}\left(\sqrt{\frac{C_{13} C_{14}}{\left(\frac{2}{3}\right)^{2 n+2 k+1} \sigma_{T}^{2 n+1}}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right)}+\frac{1}{\delta^{3}}\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right) \\
& \leq \frac{\boldsymbol{C}_{7}}{\operatorname{dist}(y, \partial \mathbf{V})^{n+k+1 / 2}} \sqrt{\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}} .
\end{aligned}
$$

This shows (iii.).
(iv.) is done as (iii.).

For (v), we fix $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ and abbreviate $v:=v_{i}^{T}, w:=$ $w_{j}^{T}$. Additionally to the conditions before, we now require for $C_{5}$ to fulfil $C_{5}^{2 n+3} \geq$ $C_{13} C_{14}\left(2^{2 /(n+2)}-1\right)^{-1}$. Then Lemma 3.4 implies that

$$
\sup _{\mathrm{C}_{1-\sigma_{T}} \cap \operatorname{spt}(T)} X_{n+1}^{2} \leq \sigma_{T}^{2} \frac{C_{13} C_{14}}{C_{5}^{2 n+3}} \leq \sigma_{T}^{2}\left(2^{2 /(n+2)}-1\right)
$$

In the following, let $y \in \mathbf{V}_{T}$ (thus $\left.|y| \geq y_{n}>\sigma_{T}\right)$ and $\delta:=\operatorname{dist}\left(y, \mathbf{V}_{2 \sigma_{T} / 3}\right)$. Then we have

$$
\begin{align*}
|(y, v(y), \boldsymbol{\Phi}(y, v(y)))|^{2} & =|y|^{2}+v(y)^{2}+|\boldsymbol{\Phi}(y, v(y))|^{2} \\
& \leq\left(1-\sigma_{T}\right)^{2}+\sigma_{T}^{2}+|D \boldsymbol{\Phi}|^{2} \leq 1+|D \boldsymbol{\Phi}|^{2} . \tag{3.5}
\end{align*}
$$

Denote by $K:=\|D \boldsymbol{\Phi}\|_{\mathcal{C}^{0}\left(\overline{\mathbf{B}}_{1}\right)}$. Then $\mathbf{p}^{-1}\left(\mathbf{V}_{T}\right) \cap \operatorname{spt}(T) \subset \mathbf{B}_{1+K}$.
Last, we let $C_{5}$ also fulfil $C_{5}^{2 n+3} \geq 144\left(\frac{3}{2}\right)^{2 n+1} C_{15}^{2} C_{13} C_{14}$. By Lemma 3.6, the following holds

$$
\begin{align*}
|(y, v(y), \boldsymbol{\Phi}(y, v(y)))|^{n+2} & =\left(|y|^{2}+v(y)^{2}+|\boldsymbol{\Phi}(y, v(y))|^{2}\right)^{(n+2) / 2} \\
& \leq\left(|y|^{2}+\left(2^{2 /(n+2)}-1\right)|y|^{2}|D \boldsymbol{\Phi}|^{2}\left(1+|D v|^{2}\right)|y|^{2}\right)^{(n+2) / 2} \\
& \leq 2^{2+n / 2}|y|^{n+2}, \tag{3.6}
\end{align*}
$$

$$
\begin{align*}
|D v(y)|^{2} & \left.\leq\left(\frac{2 C_{15}}{\delta}\right)^{2}\left(\sup _{\overline{\mathbf{B}}_{\delta}^{n}(y)}\left|v_{i}^{T}\right|+\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right)\right)^{2} \\
& \leq \frac{8 C_{15}^{2}}{\left(\frac{\sigma_{T}}{3}\right)^{2}} \frac{C_{13} C_{14}}{\left(\frac{2}{3}\right)^{2 n+1} \sigma_{T}^{2 n+1}}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right)+\frac{8 C_{15}^{2}}{\delta^{2}}\left(\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{1}\left(B_{\delta}\right)}^{*}\right)^{2} \\
& \leq 72 C_{15}^{2}\left(\left(\frac{3}{2}\right)^{2 n+1} \frac{C_{13} C_{14}}{C_{5}^{2 n+3}}+\left\|D^{2} \boldsymbol{\Phi}\right\|_{\mathcal{C}^{\alpha}}\right) \leq 1 . \tag{3.7}
\end{align*}
$$

Now, we compute

$$
\begin{equation*}
\frac{\partial}{\partial r} \frac{v(y)}{|y|}=\frac{y}{|y|}\left(\frac{D v(y)}{|y|}-v(y) \frac{y}{|y|^{3}}\right)=\frac{y D v(y)-v(y)}{|y|^{2}} . \tag{3.8}
\end{equation*}
$$

We notice that this is similar to the projection on $\operatorname{span}\{(\operatorname{Dv}(y),-1,0)\}$. Let $\zeta(y):=$ $\frac{1}{\sqrt{1+|D v(y)|^{2}}}(D v(y),-1,0) \in \mathbb{R}^{n+k}$. Then

$$
\begin{equation*}
\langle(y, v(y), \boldsymbol{\Phi}(y, v(y))), \zeta(y)\rangle=\frac{\langle y, D v(y)\rangle-v(y)}{\sqrt{1+|D v(y)|^{2}}} . \tag{3.9}
\end{equation*}
$$

Moreover, the approximate tangent space of $\operatorname{spt}(T)$ at $(y, v(y), \boldsymbol{\Phi}(y, v(y)))$ is spanned by the vectors $\partial_{i} G(y)$ for $i \leq n$ and $G(y)=(y, v(y), \boldsymbol{\Phi}(y, v(y)))$. As $(D v(y),-1,0)$ is normal to all of the $\partial_{i} G(y)$, we have that $\zeta(y)$ is normal to the approximate tangent space of $\operatorname{spt}(T)$ at $(y, v(y), \boldsymbol{\Phi}(y, v(y)))$. In particular,

$$
\begin{equation*}
|\langle(y, v(y), \boldsymbol{\Phi}(y, v(y))), \zeta(y)\rangle| \leq\left|(y, v(y), \boldsymbol{\Phi}(y, v(y)))^{\perp}\right|, \tag{3.10}
\end{equation*}
$$

where $X^{\perp}$ denotes the component orthogonal to the approximate tangent space of $T$. Therefore, we deduce by using (3.8), (3.9), (3.7), (3.6), (3.10) and (3.5)

$$
\begin{aligned}
& \int_{\mathbf{V}_{T}}\left(\frac{\partial}{\partial r} \frac{v(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \\
& =\int_{\mathbf{V}_{T}}\langle(y, v(y), \boldsymbol{\Phi}(y, v(y))), \zeta(y)\rangle^{2} \frac{1+|D v(y)|^{2}}{|y|^{n+2}} \mathrm{~d} \mathcal{L}^{n}(y) \\
& \leq \int_{\mathbf{V}_{T}}\left|(y, v(y), \boldsymbol{\Phi}(y, v(y)))^{\perp}\right|^{2} \frac{2^{2+n / 2} \sqrt{2}}{\mid\left(y, v(y),\left.\boldsymbol{\Phi}(y, v(y))\right|^{n+2}\right.} \sqrt{1+|D v(y)|^{2}} \mathrm{~d} \mathcal{L}^{n}(y) \\
& \leq 2^{(n+5) / 2} \int_{\overline{\mathbf{B}}_{1+K} \cap \mathbf{p}^{-1}\left(\mathbf{V}_{T}\right)}\left|X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| .
\end{aligned}
$$

We argue in the same manner to extract

$$
\int_{\mathbf{W}_{T}}\left(\frac{\partial}{\partial r} \frac{w(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \leq 2^{(n+5) / 2} \int_{\overline{\mathbf{B}}_{1+K} \cap \mathbf{p}^{-1}\left(\mathbf{W}_{T}\right)}\left|X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| .
$$

By Corollary 2.5 and (3.2), we could conclude here the desired estimate but with radius of the excess being $1+K$ instead of 1 . However, we use (a.1) to see

$$
\begin{aligned}
\int_{\overline{\mathbf{B}}_{1+K} \backslash \overline{\mathbf{B}}_{1}}\left|X \cdot \zeta^{T}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| & \leq 4\|T\|\left(\overline{\mathbf{B}}_{1+K} \backslash \overline{\mathbf{B}}_{1}\right) \\
& \leq 4\left(C_{43}(1+K)^{n}-\frac{1}{C_{43}}\right) \leq C_{17} K \leq C_{17} \mathbf{A} .
\end{aligned}
$$

In total, we deduce

$$
\begin{aligned}
\int_{\mathbf{V}_{T}}\left(\frac{\partial}{\partial r} \frac{v_{i}^{T}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) & +\int_{\mathbf{W}_{T}}\left(\frac{\partial}{\partial r} \frac{w_{j}^{T}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \\
& \leq 2^{(n+5) / 2}\left(\mathbf{E}_{S}(T, 1)+C_{4}\left(\mathbf{A}+\kappa_{T}\right)+C_{17} \mathbf{A}\right) \\
& \leq 2^{(n+5) / 2}\left(\mathbf{E}_{C}(T, 1)+\left(C_{4}+m+C_{17}\right)\left(\mathbf{A}+\kappa_{T}\right)\right) .
\end{aligned}
$$

We now know that, away from the boundary, our minimizing current $T$ is supported on graphs. We would like to extend that fact up to the boundary. To do so, we use that the functions describing the current are bounded by the square root of the excess such that we can introduce a blow-up procedure by rescaling by the latter quantity. Notice that the domain of the functions converges to the half ball as the excess tends to zero.

We aim to extend the graphs up to the boundary of $T$ and such that they are merging together smoothly. To do so, we will show that the harmonic blow-ups on $\mathbf{V}$ (or $\mathbf{W}$ respectively) are all identical (see Theorem 7.2), which will lead to an excess decay (Theorem 4.2) which will then lead to the extension of the graphs (Corollary 4.3).

First we describe the blow-up procedure.
Definition 4.1. For $v \in \mathbb{N}, v \geq 1, i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ and $\left(T_{v}, \mathcal{M}_{v}\right) \in \mathcal{T}$, we define $\mathbf{A}_{v}:=\mathbf{A}_{\mathcal{M}_{v}}, \varepsilon_{v}:=\sqrt{\mathbf{E}_{C}\left(T_{v}, 1\right)}, \kappa_{v}:=\kappa_{T_{v}}, v_{i}^{(v)}:=v_{i}^{T_{v}} \mathbb{1}_{\mathbf{V}_{T_{v}}}: \mathbf{V} \rightarrow \mathbb{R}$ and $w_{j}^{(v)}:=w_{j}^{T_{v}} \mathbb{1}_{\mathbf{W}_{T_{v}}}: \mathbf{W} \rightarrow \mathbb{R}$. We call $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1}$ a blowup sequence with associated harmonic blow-ups $f_{i}, g_{j}$ if the following holds as $v \rightarrow \infty$,
(i.) $\varepsilon_{v} \rightarrow 0$,
(ii.) $\varepsilon_{v}^{-2} \kappa_{v} \rightarrow 0$,
(iii.) $\mathbf{A}_{v} \rightarrow 0$,
(iv.) $\frac{v_{i}^{(v)}}{\max \left\{\varepsilon_{v}, \mathbf{A}_{v}^{1 / 4}\right\}} \longrightarrow f_{i}$ uniformly on compact subsets of $\mathbf{V}$,
(v.) $\frac{w_{j}^{(v)}}{\max \left\{\varepsilon_{v}, \mathbf{A}_{v}^{1 / 4}\right\}} \longrightarrow g_{i}$ uniformly on compact subsets of $\mathbf{W}$.

Notice that by the estimates of Theorem 3.2, the Riemannian minimal surface equation and [23, Lemma 5.3.7], it follows that $f_{i}, g_{j}$ are harmonic. Furthermore, by Lemma 3.4, we have for $0<\rho<1$

$$
\begin{align*}
\sup _{\mathbf{V} \cap \overline{\mathbf{B}}_{\rho}^{n}(0)}\left|f_{i}\right|^{2}+\sup _{\mathbf{W} \cap \overline{\mathbf{B}}_{\rho}^{n}(0)}\left|g_{j}\right|^{2} & \leq \limsup _{v \rightarrow \infty}\left(\frac{2}{\max \left\{\varepsilon_{v}, \mathbf{A}_{v}^{1 / 4}\right\}^{2}} \sup _{\mathbf{C}_{\rho} \cap \operatorname{spt}\left(T_{v}\right)} X_{n+1}^{2}\right) \\
& \leq \frac{4 C_{13} C_{14}}{(1-\rho)^{2 n+1}} \tag{4.1}
\end{align*}
$$

Notice that by the Arzelà-Ascoli Theorem and Theorem 3.2, every sequence $\left\{\left(S_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset$ $\mathcal{T}$ satisfying

$$
\begin{equation*}
\lim _{v \rightarrow \infty}\left(\mathbf{E}_{C}\left(S_{v}, 1\right)+\frac{\kappa_{S_{v}}}{\mathbf{E}_{C}\left(S_{v}, 1\right)}+\mathbf{A}_{v}\right)=0 \tag{4.2}
\end{equation*}
$$

contains a blow-up subsequence.
The main result of this section is the following excess decay: We define $C_{18}, C_{19}$ and $\theta$ later (in Remark 7.3, Remark 8.1 and Theorem 8.1) and claim
Theorem 4.2. Let $(T, \mathcal{M}) \in \mathcal{T}$ and assume that $\max \left\{\boldsymbol{E}_{C}(T, 1), \mathbf{A}, C_{19} \kappa_{T}\right\} \leq \frac{\theta}{C_{19}}$. Then there is a real number $|\eta| \leq 2 C_{18} \sqrt{\frac{\theta}{C_{19}}}$ such that for all $0<r<\theta / 4$ the following holds

$$
E_{C}\left(\gamma_{\eta \sharp} T, r\right) \leq \frac{\theta^{-n-1}}{C_{19}} r .
$$

A direct consequence of the Theorem 4.2 is the following
Corollary 4.3. Let $T, \mathcal{M}, \eta, C_{5}$ and $\theta$ be as in Theorem 4.2 and Theorem 3.2. If we define the real numbers $\beta:=\frac{1}{4 n+10}$ and $\delta:=\theta^{2(1+n)}\left(4 C_{5}\right)^{-(4 n+6)}$ and the sets

$$
\tilde{V}:=\left\{y \in \boldsymbol{B}_{\delta}^{n}(0): y_{n}>|y|^{1+\beta}\right\} \quad \text { and } \quad \tilde{W}:=\left\{y \in \boldsymbol{B}_{\delta}^{n}(0): y_{n}<-|y|^{1+\beta}\right\},
$$

then there are functions $\tilde{v}_{i} \in \mathcal{C}^{1, \frac{1}{4}}(\overline{\tilde{V}}), \tilde{w}_{j} \in \mathcal{C}^{1, \frac{1}{4}}(\bar{W})$ such that
(i.) $\boldsymbol{p}^{-1}(\tilde{V}) \cap \operatorname{spt}\left(\gamma_{\eta \#} T\right)=\bigcup_{i=1}^{m} \operatorname{graph}\left(\tilde{v}_{i}, \gamma_{\eta} \circ \boldsymbol{\Phi}\right) \quad$ and

$$
\boldsymbol{p}^{-1}(\tilde{W}) \cap \operatorname{spt}\left(\gamma_{\eta \#} T\right)=\bigcup_{j=1}^{m-1} \operatorname{graph}\left(\tilde{w}_{j}, \gamma_{\eta} \circ \boldsymbol{\Phi}\right)
$$

(ii.) $\left.\tilde{v}_{i}\right|_{\tilde{v}},\left.\tilde{w}_{j}\right|_{\tilde{w}}$ satisfy the Riemannian minimal surface equation.
(iii.) $D \tilde{v}_{i}(0)=0=D \tilde{w}_{j}(0)$.
(iv.) $\tilde{v}_{1} \leq \tilde{v}_{2} \leq \cdots \leq \tilde{v}_{m}$ and $\tilde{w}_{1} \leq \tilde{w}_{2} \leq \cdots \leq \tilde{w}_{m-1}$.

In order to handle the rotations and scalings of $T$, we state the following computations that we will prove in chapter a.
Remark 4.4. For $C_{20}:=C_{3}+6^{n}\left(1+m \omega_{n}\right),(T, \mathcal{M}) \in \mathcal{T}$ and $r \geq 3$ the following holds: if $\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A} \leq \frac{1}{C_{3}}$, then

$$
\left(\left(\boldsymbol{\mu}_{r \#} T\right)\left\llcorner\mathbf{B}_{3}, \boldsymbol{\mu}_{r}(\mathcal{M})\right) \in \mathcal{T}, \quad \mathbf{A}_{\boldsymbol{\mu}_{r}(\mathcal{M})} \leq \frac{\mathbf{A}_{\mathcal{M}}}{r} \quad \text { and } \quad \kappa_{\left(\mu_{\not r \sharp} T\right)\left\llcorner\mathbf{B}_{3}\right.} \leq \frac{\kappa_{T}}{r} .\right.
$$

Indeed, we apply Corollary 2.4 with $r, s$ replaced by $3 / r, 1$ :

$$
\left(\frac{r}{3}\right)^{n}\|T\|\left(\overline{\mathbf{B}}_{3 / r}\right)+\int_{\overline{\mathbf{B}}_{1} \backslash \overline{\mathbf{B}}_{3 / r}}\left|X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| \leq\|T\|\left(\overline{\mathbf{B}}_{1}\right)+C_{3}\left(\mathbf{A}+\kappa_{T}\right)\left(1-\frac{3}{r}\right) .
$$

Therefore, we have

$$
\begin{aligned}
\mathbb{M}\left(\left(\boldsymbol{\mu}_{r \#} T\right)\left\llcorner\overline{\mathbf{B}}_{3}\right)\right. & \leq r^{n} \mathbb{M}\left(T\left\llcorner\overline{\mathbf{B}}_{3 / r}\right)\right. \\
& \leq 3^{n}\left(\|T\|\left(\overline{\mathbf{B}}_{1}\right)+C_{3}\left(\mathbf{A}+\kappa_{T}\right)\right) \\
& \leq 3^{n}\left(\mathbf{E}_{C}(T, 1)+m \boldsymbol{\omega}_{n}+C_{3}\left(\mathbf{A}+\kappa_{T}\right)\right) \\
& \leq 3^{n}\left(1+m \boldsymbol{\omega}_{n}\right) .
\end{aligned}
$$

The estimate on $\kappa_{\left(\mu_{t} \# T\right)\left\llcorner\mathbf{B}_{3}\right.}$ follows from classical differential geometry.
Remark 4.5. Let $(T, \mathcal{M}) \in \mathcal{T}$ and $|\omega| \leq 1 / 8$ and assume that

$$
\mathbf{A} \leq \max \left\{\frac{1}{8},\left(7 C_{1}+C_{12}+1\right)^{-1}\right\}
$$

Then, we have
(i.) if $\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A} \leq\left(C_{13} C_{14} 4^{2 n+4}\right)^{-1}$, then $\sup _{\mathrm{C}_{3 / 4} \cap \mathrm{spt}(T)}\left|X_{n+1}\right| \leq \frac{1}{8}$.
(ii.) if $\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A} \leq \min \left\{C_{20}^{-1},\left(C_{13} C_{14} 4^{2 n+4}\right)^{-1}\right\}$, then

$$
\left(\boldsymbol{\mu}_{4 \#} \gamma_{\omega \#} T\right)\left\llcorner\mathbf{B}_{3}=\left(\gamma_{\omega \sharp} \boldsymbol{\mu}_{4 \#} T\right)\left\llcorner\mathbf{B}_{3} \in \mathcal{T} .\right.\right.
$$

(iii.) if $12 \leq r<\infty$ and $\omega^{2}+\mathbf{E}_{C}(T, 1)+\kappa_{T}+|D \boldsymbol{\Phi}|^{2}+\left|D^{2} \boldsymbol{\Phi}\right| \leq C_{21}{ }^{-1}$, where $C_{21}=$ $4^{2 n+4} C_{20} C_{10}\left(1+C_{11}\right)\left(1+C_{13}\right) C_{14}$, then $\left(\mu_{r \#} \gamma_{\omega \#} T\right)\left\llcorner\mathbf{B}_{3} \in \mathcal{T}\right.$ and

$$
\kappa_{\left(\mu_{\| t *} \gamma_{\omega *} T\right)\left\llcorner\mathbf{B}_{3}\right.} \leq \frac{\kappa_{T}}{r} .
$$

Proof of Corollary 4.3. We only show it for $\tilde{v}_{i}$, the argument for $\tilde{w}_{j}$ goes analogously.

Let $0<\rho<\delta$ and define $S_{\rho}:=\left(\boldsymbol{\mu}_{1 / \rho \#} \gamma_{\eta \#} T\right)\left\llcorner\mathbf{B}_{3}, \mathcal{M}_{\rho}:=\mu_{1 / \rho}(\mathcal{M})\right.$. By Remark 4.5 $\left(S_{\rho}, \mathcal{M}_{\rho}\right) \in \mathcal{T}$. Moreover, notice that by Theorem 3.2, Theorem 4.2 and Remark 4.5 the following holds

$$
\begin{aligned}
\sigma_{S_{\rho}} & =C_{5}\left(\mathbf{E}_{C}\left(S_{\rho}, 1\right)+\kappa_{S_{\rho}}+\mathbf{A}_{\mathcal{M}_{\rho}}\right)^{1 /(2 n+3)} \\
& \leq C_{5}\left(\mathbf{E}_{C}\left(\gamma_{\eta \#} T, \rho\right)+\rho\left(\kappa_{T}+\mathbf{A}\right)\right)^{1 /(2 n+3)} \\
& \leq C_{5}\left(\theta^{-n-1} \frac{\rho}{C_{19}}+\rho \frac{2 \theta}{C_{19}}\right)^{1 /(2 n+3)} \\
& =C_{5} \rho^{1 /(4 n+6)}\left(\rho^{1 / 2} \frac{3 \theta^{-n-1}}{C_{19}}\right)^{1 /(2 n+3)} \\
& \leq C_{5} \rho^{\beta}\left(\delta^{1 / 2} \frac{4}{C_{19}} \theta^{-n-1}\right)^{1 /(2 n+3)} \\
& =C_{5} \rho^{\beta}\left(\frac{4^{2 n+5}}{C_{19} C_{5}^{2 n+3}}\right)^{1 /(2 n+3)} \\
& \leq \frac{\rho^{\beta}}{4} .
\end{aligned}
$$

Now, we look for functions whose graph contain $\operatorname{spt}\left(\gamma_{\eta \#} T\right)$. For a fixed $\rho$, we apply Theorem 3.2 to $\left(S_{\rho}, \mathcal{M}_{\rho}\right)$ and get $v_{1}^{S_{\rho}} \leq v_{2}^{S_{\rho}} \leq \cdots \leq v_{m}^{S_{\rho}}$ whose $\boldsymbol{\Phi}_{\mathcal{M}_{\rho}}$-graph form the $\operatorname{spt}\left(S_{\rho}\right)$. Using Theorem 3.2(iii.) (with $T, \mathcal{M}, k$ replaced by $S_{\rho}, \mathcal{M}_{\rho}, 1$ and 2) for all $i \in\{1, \ldots, m\}$

$$
\begin{align*}
\sup _{\mathbf{V}_{\frac{1}{4} \rho^{\beta}}}\left|D v_{i}^{S_{\rho}}\right| & \leq C_{7} \sqrt{\mathbf{E}_{C}\left(S_{\rho}, 1\right)+\kappa_{S_{\rho}}+\mathbf{A}_{\mathcal{M}_{\rho}}} \sup _{y \in \mathbf{V}_{\frac{1}{4} \rho^{\beta}}} \operatorname{dist}(y, \partial \mathbf{V})^{-1-n-1 / 2} \\
& \leq C_{7} \sqrt{3 \rho \theta^{-n-1}}\left(\frac{4}{\rho^{\beta}}\right)^{n+3 / 2} \\
& \leq C_{22} \rho^{1 / 4},  \tag{4.3}\\
\sup _{\mathbf{V}_{\frac{1}{4} \rho^{\beta}}}\left|D^{2} v_{i}^{S_{\rho}}\right| & \leq C_{7} \sqrt{\mathbf{E}_{C}\left(S_{\rho}, 1\right)+\kappa_{S_{\rho}}+\mathbf{A}_{\mathcal{M}_{\rho}}} \sup _{y \in \mathbf{V}_{\frac{1}{4} \rho^{\beta}}} \operatorname{dist}(y, \partial \mathbf{V})^{-2-n-1 / 2} \\
& \leq C_{7} \sqrt{3 \rho \theta^{-n-1}}\left(\frac{4}{\rho^{\beta}}\right)^{n+5 / 2} \\
& \leq C_{22} \rho^{1 / 4} .
\end{align*}
$$

Define $\rho_{k}:=\left(\frac{7}{8}\right)^{k}, k \in \mathbb{Z}$, and look at the annuli

$$
A_{k}:=\left\{y \in \tilde{V}: \frac{1}{2} \rho_{k} \leq|y| \leq \frac{2}{3} \rho_{k}\right\} .
$$

These annuli are overlapping as $\frac{1}{2} \rho_{k}<\frac{2}{3} \rho_{k+1}$ and moreover their union covers all of $\tilde{V}$. Notice that for $y \in A_{k}$ the following holds

$$
\frac{y_{n}}{\rho_{k}}>\frac{|y|^{1+\beta}}{\rho_{k}} \geq\left(\frac{\rho_{k}}{2}\right)^{1+\beta} \frac{1}{\rho_{k}} \geq \frac{\rho_{k}^{\beta}}{4} \geq \sigma_{S_{\rho_{k}}} .
$$

Therefore, $y / \rho_{k} \in \mathbf{V}_{S_{\rho_{k}}}$ and we can define for $y \in A_{k}$

$$
\tilde{v}_{i}(y)=\rho_{k} v_{i}^{S_{\rho_{k}}}\left(\frac{y}{\rho_{k}}\right) .
$$

Then

$$
\mathbf{p}^{-1}(\tilde{V}) \cap \operatorname{spt}\left(\gamma_{\eta \#} T\right)=\bigcup_{i=1}^{m} \operatorname{graph}\left(\tilde{v}_{i}, \gamma_{\eta} \circ \boldsymbol{\Phi}\right),
$$

because $S_{\rho}:=\left(\mu_{1 / \rho \#} \gamma_{\eta \#} T\right)\left\llcorner\mathbf{B}_{3}\right.$. Moreover, all $\tilde{v}_{i}$ fulfil the Riemannian minimal surface equation on $\tilde{V}$ and $\tilde{v}_{1} \leq \tilde{v}_{2} \leq \cdots \leq \tilde{v}_{m}$. The only thing we still have to prove is the $\mathcal{C}^{1, \frac{1}{4}-r e g u l a r i t y . ~ B y ~ u s i n g ~ t h e ~ b o u n d s ~ i n ~(4.3), ~ w e ~ e s t i m a t e ~ f o r ~ e a c h ~} y \in \tilde{V}$

$$
\begin{align*}
\left|D \tilde{v}_{i}(y)\right| & \leq C_{22} \rho_{k}^{1 / 4} \leq 2 C_{22}|y|^{1 / 4}  \tag{4.4}\\
\left|D^{2} \tilde{v}_{i}(y)\right| & \leq \frac{1}{\rho_{k}} C_{22} \rho_{k}^{1 / 4} \leq C_{22}|y|^{-3 / 4} \tag{4.5}
\end{align*}
$$

Let $y, z \in \tilde{V}$ be arbitrary. We want to deduce that $\left|D \tilde{v}_{i}(y)-D \tilde{v}_{i}(z)\right| \leq 4 C_{22}|y-z|^{1 / 4}$. We differ between the following cases:
Case 1: $\max \{|y|,|z|\} \leq 2|y-z|$.
Then the following holds by (4.4)

$$
\begin{aligned}
\left|D \tilde{v}_{i}(y)-D \tilde{v}_{i}(z)\right| & \leq\left|D \tilde{v}_{i}(y)\right|+\left|D \tilde{v}_{i}(z)\right| \\
& \leq 2 C_{22}|y|^{1 / 4}+2 C_{22}|z|^{1 / 4} \\
& \leq 4 C_{22}|y-z|^{1 / 4} .
\end{aligned}
$$

Case 2: $\max \{|y|,|z|\}>2|y-z|$.
Wlog $\max \{|y|,|z|\}=|y|$. We claim that also the path between these two points fulfils this inequality. Indeed, for every $t \in[0,1]$ we have

$$
|y+t(y-z)| \geq||y|-t| z-y| | \geq 2|y-z|-t|y-z| \geq|y-z|
$$

and

$$
|y+t(y-z)|^{-3 / 4} \leq|y-z|^{-3 / 4}
$$

We use this together with (4.5) to infer

$$
\left|D \tilde{v}_{i}(y)-D \tilde{v}_{i}(z)\right| \leq|y-z| \int_{0}^{1}\left|D^{2} \tilde{v}_{i}(y+t(y-z))\right| \mathrm{d} t \leq C_{22}|y-z|^{1 / 4} .
$$

Thus the Hölder regularity holds up to the boundary and by (4.4) we conclude (iii.).

## GLUEING OF HARMONIC BLOW-UPS AND FIRST COLLAPSING LEMMA

We aim to prove that under certain conditions, the harmonic blow-ups agree in order to deduce later that the graphs are equal on $\mathbf{V}$ and $\mathbf{W}$ respectively. The first step in this direction is to show that if we glue them together, the result is weakly differentiable.

Lemma 5.1. Let $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ be a blow-up sequence with associated harmonic blowups $f_{i}, g_{j}$. Define $h, \mu: B_{1}^{n}(0) \rightarrow \mathbb{R}$ by

$$
h(x)= \begin{cases}\sum_{i=1}^{m} f_{i}(x), & \text { if } x \in \boldsymbol{V} \\ \sum_{j=1}^{m-1} g_{j}(x), & \text { if } x \in \boldsymbol{W} \\ 0, & \text { if } x \in \boldsymbol{L}\end{cases}
$$

and

$$
\mu(x)= \begin{cases}\min \left\{\left|f_{1}(x)\right|, \ldots,\left|f_{m}(x)\right|\right\}, & \text { if } x \in V \\ 0, & \text { if } x \in W \cup L .\end{cases}
$$

Then $h$ and $\mu$ are in $W_{\text {loc }}^{1,2}\left(B_{1}^{n}(0)\right)$.
Remark 5.2. Consider the notion of trace as in [36, Chapter 26]. The previous lemma implies that $\left.\mu\right|_{\mathbf{V}}$ has zero trace on $\mathbf{L}$.

Proof of Leamm 5.1. Let $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ be a blow-up sequence with associated harmonic blow-ups $f_{i}, g_{j}$ and denote $\mathbf{A}_{v}, \boldsymbol{\varepsilon}_{v}, \kappa_{v}$ as in the Definition 4.1 and $\boldsymbol{\Phi}_{v}:=\boldsymbol{\Phi}_{\mathcal{M}_{v}}$. Observe that $\sqrt{1+t} \geq 1+\frac{1}{2} t-\frac{1}{9} t^{2}$ for all $0 \leq t \leq 1$. We use Theorem 3.2 (iii.) to estimate for any $i \in\{1, \ldots, m\}$

$$
\begin{aligned}
\varepsilon_{v}^{2} & =\mathbb{M}\left(T_{v}\left\llcorner\mathbf{C}_{1}\right)-\mathbb{M}\left(\mathbf{p}_{\#}\left(T_{v}\left\llcorner\mathbf{C}_{1}\right)\right)\right.\right. \\
& \geq \mathbb{M}\left(T_{v}\left\llcorner\mathbf{p}^{-1}\left(\mathbf{V}_{\sqrt{\sigma_{v}}}\right)\right)-\mathbb{M}\left(\mathbf{p}_{\#}\left(T_{v}\left\llcorner\mathbf{p}^{-1}\left(\mathbf{V}_{\sqrt{\sigma_{v}}}\right)\right)\right)\right.\right. \\
& \geq \int_{\mathbf{V}_{\sqrt{\sigma_{v}}}}\left(\sqrt{1+\left|D v_{i}^{(v)}\right|^{2}}-1\right) \mathrm{d} \mathcal{L}^{n} \\
& \geq \int_{\mathbf{V}_{\sqrt{\sigma_{v}}}}\left(\frac{1}{2}\left|D v_{i}^{(\nu)}\right|^{2}-\frac{1}{9}\left|D v_{i}^{(v)}\right|^{4}\right) \mathrm{d} \mathcal{L}^{n} \\
& \geq \frac{1}{2} \int_{\mathbf{V}_{\sqrt{\sigma_{v}}}}\left|D v_{i}^{(v)}\right|^{2}\left(1-\frac{2}{9} C_{7}^{2}\left(\varepsilon_{v}^{2}+\kappa_{v}+\mathbf{A}_{v}\right) \sigma_{v}{ }^{-n-3 / 2}\right) \mathrm{d} \mathcal{L}^{n} \\
& =\frac{1}{2} \int_{\mathbf{V}_{\sqrt{\sigma_{v}}}}\left|D v_{i}^{(\nu)}\right|^{2}\left(1-\frac{2}{9} C_{5}^{-n-3 / 2} C_{7}^{2} \sqrt{\varepsilon_{v}^{2}+\kappa_{v}+\mathbf{A}_{v}}\right) \mathrm{d} \mathcal{L}^{n} .
\end{aligned}
$$

Hence, for $v$ large enough, it follows that

$$
\int_{\mathbf{v}_{\sqrt{\sigma v}}}\left|D v_{i}^{(\nu)}\right|^{2} \mathrm{~d} \mathcal{L}^{n} \leq 3 \varepsilon_{v}^{2} .
$$

Moreover, fix $\delta>0$. For all $v$ such that $\sigma_{v} \leq \delta^{2}$ the following holds

$$
\int_{\mathbf{V}_{\delta}} \frac{\left|D v_{i}^{(v)}\right|^{2}}{\max \left\{\varepsilon_{v}^{2}, \mathbf{A}_{v}^{1 / 2}\right\}} \mathrm{d} \mathcal{L}^{n} \leq \int_{\mathbf{v}_{\delta}} \frac{\left|D v_{i}^{(v)}\right|^{2}}{\varepsilon_{v}^{2}} \mathrm{~d} \mathcal{L}^{n} \leq 3
$$

and by locally uniform convergence, we deduce

$$
\int_{\mathbf{V}_{\delta}}\left|D f_{i}\right|^{2} \mathrm{~d} \mathcal{L}^{n} \leq 3 .
$$

As $\delta$ was arbitrary, we can conclude the integrability of the weak derivative of $f_{i}$ in all of $\mathbf{V}$ and analogously for $g_{j}$ in $\mathbf{W}$. The fact that the trace is zero, we deduce in the same manner as in [27, Lemma 6.2] (which is based on [23, Lemma 5.3.7]). Thus, we also conclude that $h$ and $\mu$ are in $W_{l o c}^{1,2}$.

As a next step, we see that also around boundary points, we have local uniform convergence. In fact, the proof of the original paper [27, Lemma 6.3] carries over and thus, we omit the details here.

Lemma 5.3. Let $0<\sigma<1 / 2, a \in \boldsymbol{L} \cap \boldsymbol{B}_{1-2 \sigma}^{n}(0), U:=\boldsymbol{B}_{\sigma}^{n}(a), B:=\partial U, C \subset \boldsymbol{p}^{-1}(U)$ compact and $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ a blowup sequence with associated harmonic blowups $f_{i}$ and $g_{j}$. Denote $\varepsilon_{v}:=\sqrt{E_{\mathcal{C}}\left(T_{v}, 1\right)}$ and $\mathfrak{m}_{v}:=\max \left\{\varepsilon_{v}, \boldsymbol{A}_{v}^{1 / 4}\right\}$. Then, the following holds

$$
\begin{aligned}
& \limsup _{v \rightarrow \infty} \sup _{C \cap \operatorname{spt}\left(T_{v}\right)} \frac{X_{n+1}}{\mathfrak{m}_{v}} \leq \max \left\{\sup _{B \cap V} f_{m}, \sup _{B \cap W} g_{m-1}, 0\right\} \\
& \liminf _{v \rightarrow \infty} \inf _{C \cap \operatorname{spt}\left(T_{v}\right)} \frac{X_{n+1}}{\mathbf{m}_{v}} \geq \min \left\{\inf _{B \cap V} f_{1}, \inf _{B \cap W} g_{1}, 0\right\}
\end{aligned}
$$

As a first step to the fact, that the harmonic blow-ups coincide, we prove it under the strong assumptions that they are linear. This will be useful, as for the excess decay we will use a blow-up argument in which the inequality of Theorem $3.2(v$.$) forces them to$ be linear. The argument for the equality of the blow-ups relies on the fact, that in case they are not equal, we find a better competitor for the minimization problem.

Lemma 5 -4 (Collapsing lemma). Let $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ be a blowup sequence and denote $\varepsilon_{v}:=\sqrt{\boldsymbol{E}_{C}\left(T_{v}, 1\right)}$ and $\mathfrak{m}_{v}:=\max \left\{\varepsilon_{v}, \boldsymbol{A}_{v}^{1 / 4}\right\}$.
Assume the harmonic blowups are of the form

$$
f_{i}=\left.\beta_{i} Y_{n}\right|_{V^{\prime}} \quad g_{j}=\left.\gamma_{j} Y_{n}\right|_{W^{\prime}}
$$

for some real numbers $\beta_{1} \leq \cdots \leq \beta_{m}$ and $\gamma_{1} \geq \cdots \geq \gamma_{m-1}$. Then the following holds

$$
\beta_{1}=\cdots=\beta_{m}=\gamma_{1}=\cdots=\gamma_{m-1}
$$

and for every $0<\rho<1$

$$
\lim _{v \rightarrow \infty} \sup _{\mathcal{C}_{\rho} \cap \operatorname{spt}\left(T_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta_{1} X_{n}\right|=0 .
$$



Proof. Let $v_{i}^{(\nu)}$ and $w_{j}^{(\nu)}$ be as in Definition 4.1, define $\zeta:=\max \left\{\left|\beta_{1}\right|,\left|\beta_{m}\right|,\left|\gamma_{1}\right|,\left|\gamma_{m-1}\right|\right\}$, $\delta:=\min \left\{\{1\} \cup\left\{\beta_{i+1}-\beta_{i}: \beta_{i+1} \neq \beta_{i}\right\} \cup\left\{\gamma_{i}-\gamma_{i+1}: \gamma_{i} \neq \gamma_{i+1}\right\}\right\}$ and let $0<\sigma<$ $\min \{\delta / 2,1 / 16\}$. By Theorem 3.2(iii.), (iv.), Definition $4.1(i)-.(v$.$) and the previous$ Lemma 5.3, we can choose $N_{\sigma}>0$ such that for all $v \geq N_{\sigma}$ the following holds for all $0 \leq i \leq m$ and $0 \leq j \leq m-1$


$$
\begin{align*}
\sigma_{T_{v}}<\frac{\sigma}{4}, \quad \mathfrak{m}_{v}^{2}<\sigma, \quad \kappa_{T_{v}} & <\sigma^{3} \mathfrak{m}_{v}^{2}  \tag{5.1}\\
\sup _{\mathbf{V}_{\sigma / 2}}\left|v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right|^{2} & \leq \sigma^{n+4} \mathfrak{m}_{v}^{2}  \tag{5.2}\\
\sup _{\mathbf{W}_{\sigma / 2}}\left|w_{j}^{(v)}-\mathfrak{m}_{v} \gamma_{j} Y_{n}\right|^{2} & \leq \sigma^{n+4} \mathfrak{m}_{v}^{2}  \tag{5.3}\\
\sup \left|X_{n+1}\right| & \leq 2 \zeta \sigma \mathfrak{m}_{v}+\sigma \mathfrak{m}_{v} \tag{5.4}
\end{align*}
$$

The grey area in the sketch stands for the set where the supremum in (5.4) is taken. We divide the proof into several steps.
Step 1: For all $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ the following holds

$$
\begin{equation*}
\sup _{\mathbf{V}_{\sigma}}\left|D\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)\right|^{2}, \sup _{\mathbf{W}_{\sigma}}\left|D\left(w_{j}^{(v)}-\mathfrak{m}_{v} \gamma_{j} Y_{n}\right)\right|^{2} \leq C_{23} \sigma^{2} \mathfrak{m}_{v}^{2} . \tag{5.5}
\end{equation*}
$$

Step 2: There is a Lipschitzian map $F_{v}^{\sigma}$ such that

$$
\mathbb{M}\left(F_{v \#}^{\sigma} T_{v}\right)-\mathbb{M}\left(T_{v}\right) \leq C_{24}(1+\zeta)^{2} \sigma \mathfrak{m}_{v}^{2}
$$

The maps $F_{v}^{\sigma}$ are constructed by performing the blowup process backwards: we multiply the harmonic blowups with $\varepsilon_{v}$ and move it by $\sigma$ to the origin. These compressed sheets then almost recreate the original currents.

Step 3: With the help of $F_{v}^{\sigma}$, we show that

$$
\eta: \overline{\mathbf{B}}_{1 / 2}^{n}(0) \rightarrow \mathbb{R}, \eta(y)= \begin{cases}\beta_{m} Y_{n}(y), & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \cap \overline{\mathbf{V}} \\ \gamma_{m-1} Y_{n}(y), & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \cap \overline{\mathbf{W}}\end{cases}
$$

is harmonic in $\mathbf{B}_{1 / 2}^{n}(0)$. In particular, $\eta$ is differentiable in 0 and hence, $\beta_{m}=\gamma_{m-1}$. We argue similarly to deduce that also $\beta_{1}=\gamma_{1}$.
Step 4: $\lim _{v \rightarrow \infty} \sup _{\operatorname{sen}_{\rho} \cap \operatorname{spt}\left(T_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta_{1} X_{n}\right|=0$.
Proof of step 1:
Away from the boundary, we want to use [25, Corollary 6.3] on the function $u:=$ $v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}$. Recall the coefficients $a_{i j}$ and $b$ of (3.3) and define $a_{i j}^{(v)}, b^{(v)}$ accordingly. Then for

$$
A_{k l}:=\frac{\delta_{k, l}}{\sqrt{1+\left|D v_{i}^{(v)}\right|^{2}}}-\frac{D_{k} v_{i}^{(v)} D_{l} v_{i}^{(v)}}{\left(1+\left|D v_{i}^{(v)}\right|^{2}\right)^{3 / 2}}-a_{k l} .
$$

we have $\sum_{k, l=1}^{n} A_{k l} \partial_{k l} u=\sum_{k, l=1}^{n} A_{k l} \partial_{k l} v_{i}^{(v)}=b^{(v)}$ and for $v$ large enough, $A_{k l}$ are elliptic in $\mathbf{V}_{\sigma / 3}$. Hence, we have

$$
\begin{aligned}
\sup _{\mathbf{V}_{\sigma}}\left|D\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)\right|^{2} & \leq \frac{C_{25}}{\sigma^{2}}\left(\sup _{\mathbf{v}_{\sigma / 3}}\left|v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right|^{2}+\left\|b^{(v)}\right\|_{\mathcal{C}^{1}\left(\mathbf{V}_{\sigma / 3}\right)}\right) \\
& \leq \frac{C_{23}}{2}\left(\sigma^{2} \mathfrak{m}_{v}^{2}+\mathfrak{m}_{v}^{8}\right) \\
& \leq C_{23} \sigma^{2} \mathfrak{m}_{v}^{2} .
\end{aligned}
$$

In the same manner we show that

$$
\sup _{\mathbf{W}_{\sigma}}\left|D\left(w_{j}^{(v)}-\mathfrak{m}_{v} \gamma_{j} Y_{n}\right)\right|^{2} \leq C_{23} \sigma^{2} \mathfrak{m}_{v}^{2} .
$$

Proof of step 2: Fix $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ and define the following subsets of $\mathbb{R}^{n+1}$ :

$$
\begin{aligned}
H^{\sigma} & :=\left\{x \in \mathbb{R}^{n+1}:\left|x_{n}\right| \leq \sigma\right\}, \\
I_{i}^{\sigma} & :=\left\{x \in \mathbb{R}^{n+1}:\left(x_{1}, \ldots, x_{n}\right) \in \mathbf{V}_{\sigma} \text { and }\left|x_{n+1}-\beta_{i} x_{n}\right|<\frac{\delta \sigma}{2}\right\}, \\
J_{j}^{\sigma} & :=\left\{x \in \mathbb{R}^{n+1}:\left(x_{1}, \ldots, x_{n}\right) \in \mathbf{W}_{\sigma} \text { and }\left|x_{n+1}-\gamma_{j} x_{n}\right|<\frac{\delta \sigma}{2}\right\} .
\end{aligned}
$$

Notice that $I_{i}^{\sigma} \cap I_{k}^{\sigma}=\varnothing$ for all $\beta_{i} \neq \beta_{k}$ and $J_{j}^{\sigma} \cap$ $J_{l}^{\sigma}=\varnothing$ for all $\gamma_{j} \neq \gamma_{l}$ by the definition of $\delta$. Additionally, define the maps $\boldsymbol{\beta}_{r}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1},(x, y) \mapsto$ $\left(x_{1}, \ldots, x_{n}, r x_{n+1}\right)$ for $r>0$. We define

$$
\begin{aligned}
G_{v}^{\sigma}:=H^{\sigma} \cup \beta_{\mathfrak{m}_{v}}\left(\bigcup_{i=1}^{m} I_{i}^{\sigma}\right) \cup \beta_{\mathfrak{m}_{v}}\left(\bigcup_{j=1}^{m-1} J_{j}^{\sigma}\right), & \sigma-1 \\
\lambda_{v}^{\sigma}: G_{v}^{\sigma} & \rightarrow \mathbb{R}^{n+1} \\
& x \mapsto \begin{cases}\left(x_{1}, \ldots, x_{n}, 0\right), & \text { if } x \in H^{\sigma} \\
\left(x_{1}, \ldots, x_{n}, \mathfrak{m}_{v} \beta_{i}\left(x_{n}-\sigma\right)\right), & \text { if } x \in \beta_{\mathfrak{m}_{v}}\left(I_{i}^{\sigma}\right) \\
\left(x_{1}, \ldots, x_{n}, \mathfrak{m}_{v} \gamma_{j}\left(x_{n}-\sigma\right)\right), & \text { if } x \in \beta_{\mathfrak{m}_{v}}\left(J_{j}^{\sigma}\right),\end{cases}
\end{aligned}
$$




Now, we want to construct a homotopy between $\lambda_{\nu}^{\sigma}$ and the identity map. For this we take a $\mathcal{C}^{1}$ function $\mu: \overline{\mathbf{B}}_{1}^{n}(0) \rightarrow[0,1]$ satisfying $\left.\mu\right|_{\overline{\mathbf{B}}_{1 / 2}^{n}(0)} \equiv 0,\left.\mu\right|_{\overline{\mathbf{B}}_{1}^{n}(0) \backslash \mathbf{B}_{3 / 4}^{n}(0)} \equiv 1$ and $\sup |D \mu| \leq 5$. Then, we define $\overline{\mathbf{B}}_{1}^{n}(0)$

$$
\begin{array}{rlr}
\Lambda_{v}^{\sigma}:=G_{v}^{\sigma} \cup\left(\mathbb{R}^{n+1} \backslash \mathbf{C}_{3 / 4}\right) \longrightarrow \mathbb{R}^{n+1} \\
& x \mapsto \begin{cases}x, & \text { if } x \in \mathbb{R}^{n+1} \backslash \mathbf{C}_{3 / 4} \\
(1-\mu \circ \mathbf{p}(x)) \lambda_{v}^{\sigma}(x)+(\mu \circ \mathbf{p}(x)) x, & \text { if } x \in G_{v}^{\sigma}\end{cases}
\end{array}
$$

and finally map everything to $\mathcal{M}_{v}$ with

$$
\begin{aligned}
F_{v}^{\sigma}:\left(G_{v}^{\sigma} \times \mathbb{R}^{k-1}\right) \cup\left(\mathbb{R}^{n+k} \backslash \mathbf{C}_{3 / 4}\right) & \longrightarrow \mathcal{M}_{v} \subset \mathbb{R}^{n+k} \\
(x, y) & \mapsto\left(\Lambda_{v}^{\sigma}(x), \boldsymbol{\Phi}_{v}\left(\Lambda_{v}^{\sigma}(x)\right)\right) .
\end{aligned}
$$

We know that in $\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma}\right)$, $\operatorname{spt}\left(T_{v}\right)$ lives on the $\boldsymbol{\Phi}_{v}$-graphs of $v_{i}^{(\nu)}$. As $v_{i}^{(\nu)} \mathfrak{m}_{v}^{-1}$ converges to $\beta_{i} Y_{n}$, for $v$ big enough, $\operatorname{graph}\left(v_{i}^{(v)}, \boldsymbol{\Phi}_{v}\right) \subset\left(\mathrm{id}, \boldsymbol{\Phi}_{v}\right) \circ \boldsymbol{\beta}_{\mathfrak{m}_{v}}\left(I_{i}^{\sigma}\right)$. Therefore

$$
\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma}\right) \cap \operatorname{spt}\left(T_{v}\right)=\bigcup_{i=1}^{m} \operatorname{graph}\left(v_{i}^{(v)}, \boldsymbol{\Phi}_{v}\right) \subset\left(\mathrm{id}, \boldsymbol{\Phi}_{v}\right)\left(G_{v}^{\sigma}\right) .
$$

Now, we compute the functions whose $\boldsymbol{\Phi}_{v}$-graph describes $\operatorname{spt}\left(F_{v \#}^{\sigma} T_{v}\right) \cap \mathbf{p}^{-1}\left(\mathbf{V}_{\sigma}\right)$ :

$$
\begin{aligned}
u_{i}^{(v)} & =(1-\mu) \mathfrak{m}_{v} \beta_{i}\left(Y_{n}-\sigma\right)+\mu v_{i}^{(v)} \\
& =(1-\mu) \mathfrak{m}_{v} \beta_{i} Y_{n}+\mu v_{i}^{(v)}-(1-\mu) \mathfrak{m}_{v} \beta_{i} \sigma \\
& =\mu\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)+\mathfrak{m}_{v} \beta_{i} Y_{n}-(1-\mu) \mathfrak{m}_{v} \beta_{i} \sigma .
\end{aligned}
$$

Then the following holds

$$
u_{i}^{(v)}-v_{i}^{(v)}=\mu\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)-\left(v_{i}^{(v)}-\mathfrak{m}_{\nu} \beta_{i} Y_{n}\right)-(1-\mu) \mathfrak{m}_{\nu} \beta_{i} \sigma .
$$

Recall $\zeta:=\max \left\{\left|\beta_{1}\right|,\left|\beta_{m}\right|,\left|\gamma_{1}\right|,\left|\gamma_{m-1}\right|\right\}$. We bound by step 1 and (5.2)

$$
\begin{aligned}
& \sup _{\mathbf{V}_{\sigma}}\left|D u_{i}^{(v)}\right| \leq \sup _{\mathbf{V}_{\sigma}}\left(|D \mu|\left|v-\mathfrak{m}_{v} \beta_{i} Y_{n}\right|+\left|D\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)\right|+\mathfrak{m}_{v}\left|\beta_{i}\right|+\mathfrak{m}_{v} \sigma\left|\beta_{i} D \mu\right|\right) \\
& \leq 5 \sigma \mathfrak{m}_{v}+\sqrt{C_{23}} \sigma \mathfrak{m}_{v}+\mathfrak{m}_{v} \zeta+5 \mathfrak{m}_{\nu} \zeta \sigma \\
& \leq C_{26} \mathfrak{m}_{v}(1+\zeta), \\
& \sup _{\mathbf{V}_{\sigma}}\left|D v_{i}^{(v)}\right| \leq \sup _{\mathbf{V}_{\sigma}}\left(\left|D\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)\right|+\left|D\left(\mathfrak{m}_{v} \beta_{i} Y_{n}\right)\right|\right) \\
& \leq C_{26} \mathfrak{m}_{v}(1+\zeta), \\
& \sup _{\mathbf{V}_{\sigma}}\left|D u_{i}^{(v)}-D v_{i}^{(v)}\right| \\
& \leq \sup _{\mathbf{V}_{\sigma}}\left(\left|D \mu \|\left|v-\mathfrak{m}_{v} \beta_{i} Y_{n}\right|+|1+\mu|\right| D\left(v_{i}^{(v)}-\mathfrak{m}_{v} \beta_{i} Y_{n}\right)\left|+\mathfrak{m}_{v} \sigma\right| \beta_{i} D \mu \mid\right) \\
& \leq 5 \sigma+2 \sqrt{C_{23} \sigma \mathfrak{m}_{v}+5 \mathfrak{m}_{v} \zeta \sigma} \\
& \leq C_{26} \sigma \mathfrak{m}_{v}(1+\zeta) .
\end{aligned}
$$

With this we can estimate

$$
\begin{align*}
& \mathbb{M}\left(F_{v \#}^{\sigma}\right.\left(T_{v}\left\llcorner\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma}\right)\right)\right)-\mathbb{M}\left(T_{v}\left\llcorner\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma}\right)\right)\right. \\
& \quad \leq \sum_{i=1}^{m} \sqrt{1+\left|D \boldsymbol{\Phi}_{v}\right|^{2}} \int_{\mathbf{V}_{\sigma}}\left(\sqrt{1+\left|D u_{i}^{(v)}\right|^{2}}-\sqrt{1+\left|D v_{i}^{(v)}\right|^{2}}\right) \mathrm{d} \mathcal{L}^{n} \\
& \quad \leq 2 \sum_{i=1}^{m} \int_{\mathbf{V}_{\sigma}}\left(1+\left|D u_{i}^{(v)}\right|^{2}-1-\left|D v_{i}^{(v)}\right|^{2}\right) \mathrm{d} \mathcal{L}^{n}  \tag{5.6}\\
& \quad \leq 2 \sum_{i=1}^{m} \int_{\mathbf{V}_{\sigma}}\left|D u_{i}^{(v)}-D v_{i}^{(v)}\right|\left(\left|D u_{i}^{(v)}\right|+\left|D v_{i}^{(v)}\right|\right) \mathrm{d} \mathcal{L}^{n} \\
& \quad \leq C_{27}(1+\zeta)^{2} \mathfrak{m}_{v}^{2} \sigma .
\end{align*}
$$

In the same manner, we deduce

$$
\begin{equation*}
\mathbb{M}\left(F_{v \#}^{\sigma}\left(T_{v}\left\llcorner\mathbf{p}^{-1}\left(\mathbf{W}_{\sigma}\right)\right)\right)-\mathbb{M}\left(T_{v}\left\llcorner\mathbf{p}^{-1}\left(\mathbf{W}_{\sigma}\right)\right) \leq C_{27}(1+\zeta)^{2} \mathfrak{m}_{v}^{2} \sigma .\right.\right. \tag{5.7}
\end{equation*}
$$

Outside of $\mathbf{p}^{-1}\left(\mathbf{V}_{\sigma} \cup \mathbf{W}_{\sigma}\right)$ we notice that $F_{v}^{\sigma}$ is the identity in $\mathcal{M}_{v} \cap\left(\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \backslash\right.$ $\mathrm{C}_{3 / 4}$ ) and hence

$$
\begin{equation*}
\mathbb{M}\left(F_{v \#}^{\sigma}\left(T_{v}\left\llcorner\left(\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \backslash \mathbf{C}_{3 / 4}\right)\right)\right)=\mathbb{M}\left(T_{v}\left\llcorner\left(\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \backslash \mathbf{C}_{3 / 4}\right)\right) .\right.\right. \tag{5.8}
\end{equation*}
$$

In $\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4}$, the following holds

$$
F_{v}^{\sigma}(x, y)=\left(x_{1}, \ldots, x_{n}, \mu\left(x_{1}, \ldots, x_{n}\right) x_{n+1}, \boldsymbol{\Phi}_{v}\left(x_{1}, \ldots, x_{n}, \mu\left(x_{1}, \ldots, x_{n}\right) x_{n+1}\right)\right)
$$

Hence, we can use Lemma a.1 (with $\left.A=\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4}, \tau=\sigma, \rho=5 \sigma\right)$ to bound

$$
\begin{aligned}
& \mathbb{M}\left(F_{v \#}^{\sigma}\right.\left(T\left\llcorner\left(H^{\sigma} \times \mathbb{R}^{k-1}\right)\right)\right)-\mathbb{M}\left(T\left\llcorner\left(H^{\sigma} \times \mathbb{R}^{k-1}\right)\right)\right. \\
& \quad \stackrel{(5.8)}{=} \mathbb{M}\left(F_{v \#}^{\sigma}\left(T\left\llcorner\left(\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4}\right)\right)\right)-\mathbb{M}\left(T\left\llcorner\left(\left(H^{\sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4}\right)\right)\right.\right. \\
& \leq \frac{C_{44}}{\sigma^{2}}\left(\kappa_{T_{v}}^{2}+2 \int_{\left(H^{2 \sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4+\sigma}} X_{n+1}^{2} \mathrm{~d}\left\|T_{v}\right\|+27 \mathbf{A}_{v}\right) \\
& \stackrel{(5.4)}{\leq} \frac{C_{44}}{\sigma^{2}}\left(\kappa_{T_{v}}^{2}+27 \mathbf{A}_{v}+2\left\|T_{v}\right\|\left(\left(H^{2 \sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4+\sigma}\right)\left(2 \zeta \sigma \mathfrak{m}_{v}+\sigma \mathfrak{m}_{v}\right)^{2}\right) .
\end{aligned}
$$

Further, we see that by the monotonicity property (3.1) and the projection property of currents in $\mathcal{T}$, the following holds

$$
\begin{aligned}
\left\|T_{v}\right\| & \left(\left(H^{2 \sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4+\sigma}\right) \\
& =\left(\frac{3}{4}+\sigma\right)^{n} \mathbf{E}_{C}\left(T_{v}, \frac{3}{4}+\sigma\right)+\mathbb{M}\left(\mathbf{p}_{\#}\left(T_{v}\left\llcorner\left(\left(H^{2 \sigma} \times \mathbb{R}^{k-1}\right) \cap \mathbf{C}_{3 / 4+\sigma}\right)\right)\right)\right. \\
& \leq \varepsilon_{v}^{2}+m \sigma\left(\frac{3}{4}+\sigma\right)^{n-1} \\
& \leq C_{28} \sigma
\end{aligned}
$$

where we used (5.1) in the last inequality.
Therefore,

$$
\begin{aligned}
& \mathbb{M}\left(F_{v \#}^{\sigma}\left(T\left\llcorner\left(H^{2 \sigma} \times \mathbb{R}^{k-1}\right)\right)\right)-\mathbb{M}\left(T\left\llcorner\left(H^{2 \sigma} \times \mathbb{R}^{k-1}\right)\right)\right.\right. \\
& \quad \leq \frac{C_{44}}{\sigma^{2}}\left(\kappa_{T_{v}}^{2}+27 \mathbf{A}_{v}+2 C_{28} \sigma\left(2 \zeta \sigma \mathfrak{m}_{v}+\sigma \mathfrak{m}_{v}\right)^{2}\right) \\
& \quad \stackrel{(5.1)}{\leq} C_{29}(1+\zeta)^{2} \mathfrak{m}_{v}^{2} \sigma .
\end{aligned}
$$

Putting this toghether with (5.6) and (5.7) yields

$$
\mathbb{M}\left(F_{v \#}^{\sigma} T_{v}\right)-\mathbb{M}\left(T_{v}\right) \leq C_{24}(1+\zeta)^{2} \mathfrak{m}_{v}^{2} \sigma
$$

for all $v \geq N_{\sigma}$.


Proof of step 3: We define

$$
\eta: \overline{\mathbf{B}}_{1 / 2}^{n}(0) \rightarrow \mathbb{R}, \eta(y)= \begin{cases}\beta_{m} Y_{n}(y), & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \cap \overline{\mathbf{V}} \\ \gamma_{m-1} Y_{n}(y), & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \cap \overline{\mathbf{W}} .\end{cases}
$$

To show that $\eta$ is harmonic, we prove that it minimizes the Dirichlet integral. To do so, we take some arbitrary Lipschitz function $\theta: \overline{\mathbf{B}}_{1 / 2}^{n}(0) \rightarrow \mathbb{R}$ satisfying $\left.\theta\right|_{\partial \overline{\mathbf{B}}_{1 / 2}^{n}(0)}=$ $\left.\eta\right|_{\partial \overline{\mathbf{B}}_{1 / 2}^{n}(0)}$. Then we notice that $\int|D \eta|^{2}-\int|D \theta|^{2}$ is comparable to the difference of the Hausdorff measure of the graphs of $\eta$ and $\theta$. These graphs, we express as currents and use the minimality of $T_{\nu}$ to deduce that $\int|D \eta|^{2}-\int|D \theta|^{2} \leq 0$. To make this precise, we approximate both of these functions. Indeed, let $\left\{\sigma_{k}\right\}_{k \geq 1}$ be a monotonously decreasing null sequence with $\sigma_{1}<\min \{\delta / 2,1 / 16\}$. For each $k \geq 1$, let $v_{k}=N_{\sigma_{k}}$,

$$
\eta_{k}: \overline{\mathbf{B}}_{1 / 2}^{n}(0) \rightarrow \mathbb{R}, \eta_{k}(y)= \begin{cases}\beta_{m}\left(Y_{n}(y)-\sigma_{k}\right), & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \cap \mathbf{V}_{\sigma_{k}} \\ \gamma_{m-1}\left(Y_{n}(y)+\sigma_{k}\right), & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \cap \mathbf{W}_{\sigma_{k}} \\ 0, & \text { if } y \in \overline{\mathbf{B}}_{1 / 2}^{n}(0) \backslash\left(\mathbf{V}_{\sigma_{k}} \cup \mathbf{W}_{\sigma_{k}}\right),\end{cases}
$$

and choose some $\mathcal{C}^{1}$ function $\theta_{k}: \overline{\mathbf{B}}_{1 / 2}^{n}(0) \rightarrow \mathbb{R}$ with $\left.\theta_{k}\right|_{\partial_{\overline{\mathbf{B}}}^{1 / 2}} ^{n}(0)=\left.\eta_{k}\right|_{\partial \overline{\mathbf{B}}_{1 / 2}^{n}(0)}$,
$\limsup _{k \rightarrow \infty} \sup _{\overline{\mathbf{B}}_{1 / 2}^{n}(0)}\left|D \theta_{k}\right|<\infty$ and $\lim _{k \rightarrow \infty} \int_{\overline{\mathbf{B}}_{1 / 2}^{n}(0)}\left|D \theta_{k}-D \theta\right|^{2} \mathrm{~d} \mathcal{L}^{n}=0$.
With this, we define two auxiliary currents associated to the $\boldsymbol{\Phi}_{\nu}$-graphs of $\mathfrak{m}_{\nu} \eta_{k}$ and $\mathfrak{m}_{v} \theta_{k}$ respectively:

$$
\begin{aligned}
R_{k} & :=\left(( \operatorname { i d } _ { n } , \mathfrak { m } _ { v _ { k } } \theta _ { k } , \boldsymbol { \Phi } _ { v _ { k } } ( \mathrm { id } _ { n } , \mathfrak { m } _ { v _ { k } } \eta _ { k } ) ) _ { \# } ( \mathbf { E } ^ { n } \llcorner \overline { \mathbf { B } } _ { 1 / 2 } ^ { n } ) ) \left\llcorner\stackrel{\circ}{\mathbf{C}}_{1 / 2},\right.\right. \\
S_{k} & :=\left(( \operatorname { i d d } _ { n } , \mathfrak { m } _ { v _ { k } } \theta _ { k } , \boldsymbol { \Phi } _ { v _ { k } } ( \mathrm { id } _ { n } , \mathfrak { m } _ { v _ { k } } \theta _ { k } ) ) _ { \# } ( \mathbf { E } ^ { n } \llcorner \overline { \mathbf { B } } _ { 1 / 2 } ^ { n } ) ) \left\llcorner\stackrel{\circ}{\mathbf{C}}_{1 / 2} .\right.\right.
\end{aligned}
$$

Notice that $R_{k}, S_{k}$ are supported in $\mathcal{M}_{v_{k}}$ and moreover, in $\mathbf{C}_{1 / 2} \cap G_{v}^{\sigma}$ the following holds $F_{v_{k}}^{\sigma_{k}}=\left(\mathrm{id}, \boldsymbol{\Phi}_{v}\right) \circ \Lambda_{v_{k}}^{\sigma_{k}}=\left(\mathrm{id}, \boldsymbol{\Phi}_{v}\right) \circ \lambda_{v_{k}}^{\sigma_{k}}$ and hence,

$$
\begin{equation*}
\mathbb{M}\left(F_{v_{k} \#}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)\right)=\mathbb{M}\left(F_{v_{k} \#}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)-R_{k}\right)+\mathbb{M}\left(R_{k}\right) .\right.\right. \tag{5.9}
\end{equation*}
$$

In addition, we define $q(t, x)=(\mathrm{id}, \boldsymbol{\Phi})\left(x_{1}, \ldots, x_{n-1}, t x_{n}, t x_{n+1}\right)$ and $Q_{v_{k}}:=q_{\#}([0,1] \times$ $\left(\left(\partial T_{v_{k}}\right)\left\llcorner\mathbf{C}_{2}\right)\right)\left\llcorner\mathbf{C}_{1}\right.$. This is the filling between $\overline{\mathbf{B}}_{1}^{n-1} \times\{0\}$ and $\operatorname{spt}\left(\partial T_{v}\right) \cap \mathbf{C}_{1}$ mapped onto $\mathcal{M}_{\nu_{k}}$. Then we consider $P_{k}:=Q_{v_{k}}-\left(F_{\nu_{k}}^{\sigma_{k}}\right) \# Q_{v_{k}}$. Because $\left.F_{\nu_{k}}^{\sigma_{k}}\right|_{\partial \mathrm{c}_{1}}=\left.(\mathrm{id}, \boldsymbol{\Phi})\right|_{\partial \mathrm{c}_{1}}$,
$\left.\theta_{k}\right|_{\partial \overline{\mathbf{B}}_{1 / 2}^{n}(0)}=\left.\eta_{k}\right|_{\partial \overline{\mathbf{B}}_{1 / 2}^{n}(0)}$ and the homotopy formula [23, Section 4.1.9], the following holds

$$
\begin{aligned}
\partial R_{k}= & \partial S_{k,} \\
\partial P_{k}= & \partial Q_{v_{k}}-\partial\left(F_{v_{k} \#}^{\sigma_{k}} Q_{v_{k}}\right) \\
= & \left(\partial T_{v_{k}}\right)\left\llcorner\mathbf{C}_{1}-\left(\mathrm{id}, \boldsymbol{\Phi}_{v_{k}}\right) \#\left(\left(\mathbf{E}^{n-1} \times\{0\}\right)\left\llcorner\mathbf{C}_{1}\right)\right.\right. \\
& \quad-F_{v_{k} \#}^{\sigma_{k}}\left(\left(\partial T_{v_{k}}\right)\left\llcorner\mathbf{C}_{1}\right)+\left(\mathrm{id}, \boldsymbol{\Phi}_{v_{k}}\right) \#\left(\left(\mathbf{E}^{n-1} \times\{0\}\right)\left\llcorner\mathbf{C}_{1}\right)\right.\right. \\
= & \left(\partial T_{v_{k}}\right)\left\llcorner\mathbf{C}_{1}-F_{v_{k} \#}^{\sigma_{k}}\left(\left(\partial T_{v_{k}}\right)\left\llcorner\mathbf{C}_{1}\right)\right.\right. \\
= & \partial\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)-\partial\left(F_{v_{k} \#}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)\right) .\right.\right.
\end{aligned}
$$

Moreover, the area minimality of $T_{v_{k}}$ in $\mathcal{M}_{v_{k}}$ implies

$$
\begin{aligned}
\mathbb{M}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)\right. & \leq \mathbb{M}\left(F_{v_{k} \#}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)+P_{k}-R_{k}+S_{k}\right)\right. \\
& \leq \mathbb{M}\left(F_{v_{k} \#}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)-R_{k}\right)+\mathbb{M}\left(P_{k}\right)+\mathbb{M}\left(S_{k}\right) .\right.
\end{aligned}
$$

Together with step 2 and (5.9), we deduce

$$
\begin{aligned}
\mathbb{M}\left(R_{k}\right)-\mathbb{M}\left(S_{k}\right) & =\mathbb{M}\left(F_{v_{k_{\#}}}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)\right)-\mathbb{M}\left(F_{v_{k} \#}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)-R_{k}\right)-\mathbb{M}\left(S_{k}\right)\right.\right. \\
& \leq \mathbb{M}\left(F_{v_{k} \sigma_{k}}^{\sigma_{k}}\left(T_{v_{k}}\left\llcorner\mathbf{C}_{1}\right)\right)-\mathbb{M}\left(T_{v_{v_{k}}}\left\llcorner\mathbf{C}_{1}\right)+\mathbb{M}\left(P_{k}\right)\right.\right. \\
& \leq \mathbb{M}\left(P_{k}\right)+C_{24}(1+\zeta)^{2} \mathfrak{m}_{v_{k}}^{2} \sigma_{v_{k}} .
\end{aligned}
$$

Notice that again by the homotopy formula [23, Section 4.1.9], $\mathbb{M}\left(Q_{v_{k}}\right) \leq C_{30}\left(\kappa_{T_{v_{k}}}+\mathfrak{m}_{v_{k}}^{4}\right)$. Then the condition (ii.) in Definition 4.1 yields

$$
\limsup _{k \rightarrow \infty} \frac{\mathbb{M}\left(P_{k}\right)}{\mathfrak{m}_{v_{k}}^{2}} \leq \limsup _{k \rightarrow \infty}\left(1+\operatorname{Lip}\left(F_{v_{k}}^{\sigma_{k}}\right)^{n}\right) \frac{\mathbb{M}\left(Q_{v_{k}}\right)}{\mathfrak{m}_{v_{k}}^{2}}=0 .
$$

Thus,

$$
\begin{aligned}
& 0 \geq \limsup _{k \rightarrow \infty} \frac{\mathbb{M}\left(R_{k}\right)-\mathbb{M}\left(S_{k}\right)}{\mathfrak{m}_{v_{k}}^{2}} \\
& =\limsup _{k \rightarrow \infty}\left(\int_{\overline{\mathbf{B}}_{1 / 2}^{n}(0)} \frac{\sqrt{1+\mathfrak{m}_{v_{k}}^{2}\left|D \eta_{k}\right|^{2}}}{\mathfrak{m}_{v_{k}}^{2}} \mathrm{~d} \mathcal{L}^{n}-\int_{\overline{\mathbf{B}}_{1 / 2}^{n}(0)} \frac{\sqrt{1+\mathfrak{m}_{v_{k}}^{2}\left|D \theta_{k}\right|^{2}}}{\mathfrak{m}_{v_{k}}^{2}} \mathrm{~d} \mathcal{L}^{n}-C_{31} \frac{\left|D \boldsymbol{\Phi}_{v_{k}}\right|}{\mathfrak{m}_{v_{k}}^{2}}\right) \\
& =\limsup _{k \rightarrow \infty} \int_{\overline{\mathbf{B}}_{1 / 2}^{n}(0)} \frac{\left(1+\mathfrak{m}_{v_{k}}^{2}\left|D \eta_{k}\right|^{2}\right)-\left(1+\mathfrak{m}_{v_{k}}^{2}\left|D \theta_{k}\right|^{2}\right)}{\mathfrak{m}_{v_{k}}^{2}\left(\sqrt{1+\mathfrak{m}_{v_{k}}^{2}\left|D \eta_{k}\right|^{2}}+\sqrt{1+\mathfrak{m}_{v_{k}}^{2}\left|D \theta_{k}\right|^{2}}\right)} \mathrm{d} \mathcal{L}^{n} \\
& =\frac{1}{2} \int_{\overline{\mathbf{B}}_{1 / 2}^{n}(0)}\left(|D \eta|^{2}-|D \theta|^{2}\right) \mathrm{d} \mathcal{L}^{n} .
\end{aligned}
$$

As $\theta$ was arbitrary, $\eta$ minimizes the Dirichlet integral and hence, is a harmonic function. In particular, $\eta$ is differentiable in 0 and thus, $\beta_{m}=\gamma_{m-1}$. We argue similarly to deduce that also $\beta_{1}=\gamma_{1}$.

Step 4: Let $0<\rho<1$ and assume $0<\sigma<(1-\rho) / 2$. Then by Definition 4.1(iii.),(iv.), it follows that

$$
\limsup _{v \rightarrow \infty} \sup _{\operatorname{spt}\left(T_{v}\right) \backslash H^{\sigma / 2}}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta_{1} X_{n}\right|=0
$$

and by Lemma $5 \cdot 3$

$$
\begin{aligned}
\limsup _{v \rightarrow \infty} \sup _{\operatorname{spt}\left(T_{v}\right) \cap H^{\sigma / 2} \cap \mathbf{C}_{\rho}}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta_{1} X_{n}\right| & \leq \limsup _{v \rightarrow \infty} \sup _{\operatorname{spt}\left(T_{v}\right) \cap H^{\sigma / 2} \cap \mathbf{C}_{\rho}}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}\right|+\left|\beta_{1}\right| \frac{\sigma}{2} \\
& \leq\left|\beta_{1}\right| \sigma
\end{aligned}
$$

Letting $\sigma \downarrow 0$ concludes the proof.

## COMPARISON BETWEEN SPHERICAL AND CYLINDRICAL EXCESS

In some situations it is more convenient to work with the spherical excess rather than with the cylindrical one. However, in the context of blow-ups, we see that they are in fact comparable.
Lemma 6.1. There exist positive constants $C_{32}, C_{33}, C_{34}$ such that if $(T, \mathcal{M}) \in \mathcal{T}$ satisfies

$$
E_{C}(T, 1)+\kappa_{T}+A \leq \frac{1}{C_{32}} \quad \text { and } \quad \sup _{C_{1 / 4} \cap \operatorname{spt}(T)} X_{n+1}^{2} \leq \frac{E_{C}\left(T, \frac{1}{3}\right)}{C_{33}}
$$

then

$$
E_{C}\left(T, \frac{1}{3}\right) \leq C_{34}\left(E_{S}(T, 1)+\kappa_{T}+\boldsymbol{A}\right)
$$

We will give the very technical proof for this in chapter a. It follows by computing the first variation of a suitable vectorfield.

Instead of asking $X_{n+1}^{2}$ to be small, we now only assume that $T$ is optimal with respect to rotations. We will argue by contradiction, finding a suitable blow-up sequence and then we will reduce it to the case when the harmonic blow-ups are linear (in order to use Lemma 5.4). Here, we give a sufficient condition for this to happen.
Remark 6.2. Let $h: \mathbf{V} \rightarrow \mathbb{R}$ be a harmonic function such that for all $y \in \mathbf{V}$ and $0<\rho<1$ the following holds $h(\rho y)=\rho h(y)$. Then it follows
(i.) If $h \geq 0$, then $h$ has zero trace on $\mathbf{L}$.
(ii.) If $h$ has zero trace on $\mathbf{L}$, then there is some $\beta \in \mathbb{R}$ satisfying $h=\left.\beta Y_{n}\right|_{\mathbf{V}}$.

The proof of this fact can be read in the original paper [27, Remark 7.2].
Theorem 6.3. Let $(T, \mathcal{M}) \in \mathcal{T}$ and recall $C_{32}$ and $C_{34}$ from Lemma 6.1. Then there is a positive constant $C_{35}$ such that if for all real numbers $|\eta|<1 / 8$ the following holds

- $E_{C}(T, 1)+\kappa_{T}+A \leq \frac{1}{2 C_{32}}$,
- $E_{C}\left(T, \frac{1}{3}\right)+\frac{\kappa_{T}}{E_{C}\left(T, \frac{1}{3}\right)} \leq \frac{1}{C_{35}}$,
- $E_{C}\left(T, \frac{1}{4}\right) \leq 2 E_{C}\left(\gamma_{\eta \#} T, \frac{1}{4}\right)$,
then

$$
E_{C}\left(T, \frac{1}{4}\right) \leq C_{35}\left(\boldsymbol{E}_{S}(T, 1)+\kappa_{T}+\boldsymbol{A}\right)
$$

Proof. We argue by contradiction. Assume that no matter how large $C_{35}$ is, there is a current satisfying the four conditions but not the fifth one. This means, there is a sequence $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ such that for every $v \geq 1$ and $|\eta|<1 / 8$ the following holds

$$
\begin{align*}
\mathbf{E}_{C}\left(T_{v}, 1\right)+\kappa_{T_{v}}+\mathbf{A}_{v} & \leq \frac{1}{2 C_{32}}, \\
\mathbf{E}_{C}\left(T_{v}, \frac{1}{4}\right) & \leq 2 \mathbf{E}_{C}\left(\gamma_{\eta \#} T_{v}, \frac{1}{4}\right),  \tag{6.1}\\
\lim _{v \rightarrow \infty}\left(\mathbf{E}_{C}\left(T_{v}, \frac{1}{3}\right)+\frac{\kappa_{T_{v}}}{\mathbf{E}_{C}\left(T_{v}, \frac{1}{3}\right)}\right) & =0,  \tag{6.2}\\
\lim _{v \rightarrow \infty}\left(\frac{\mathbf{E}_{S}\left(T_{v}, 1\right)+\kappa_{T_{v}}+\mathbf{A}_{v}}{\mathbf{E}_{C}\left(T_{v}, \frac{1}{4}\right)}\right) & =0 . \tag{6.3}
\end{align*}
$$

We define $S_{v}:=\left(\boldsymbol{\mu}_{3 \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}, \varepsilon_{v}:=\sqrt{\mathbf{E}_{C}\left(S_{v}, 1\right)}, \kappa_{v}:=\kappa_{S_{v}}\right.$ and $\mathfrak{m}_{v}:=\max \left\{\varepsilon_{v},\left(\frac{1}{3} \mathbf{A}_{v}\right)^{1 / 4}\right\}$. By Remark $4.4\left(S_{v}, \mu_{3}\left(\mathcal{M}_{v}\right)\right) \in \mathcal{T}$ and moreover,

$$
\varepsilon_{v}=\sqrt{\mathbf{E}_{C}\left(T_{v}, \frac{1}{3}\right)} \quad \text { and } \quad \kappa_{v} \leq \kappa_{T_{v}} .
$$

Up to subsequence (which we do not relabel) is $\left\{\left(S_{v}, \boldsymbol{\mu}_{3}\left(\mathcal{M}_{v}\right)\right)\right\}_{v \geq 1}$ a blowup sequence (see (4.2)) with harmonic blowups $f_{i}$ and $g_{j}$. We want to show that they are of the form $\beta Y_{n}$. Then we will be able to deduce that $\beta \neq 0$ which will make it impossible for $\mathrm{E}_{C}\left(T_{v}, \frac{1}{4}\right) \varepsilon_{v}^{-2}$ to converge to zero. This then leads to a contradiction to (6.1). Notice that by Lemma 2.3, the following holds

$$
e^{\frac{C_{2}}{3}\left(\mathbf{A}_{v}+\kappa_{T_{v}}\right)} 3^{n}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{1 / 3}\right) \leq e^{\mathcal{C}_{2}\left(\mathbf{A}_{v}+\kappa_{T_{v}}\right)}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{1}\right) .
$$

From this, it follows

$$
\begin{aligned}
\mathbf{E}_{S}\left(S_{v}, 1\right) & =\mathbf{E}_{S}\left(T_{v}, \frac{1}{3}\right) \\
& \leq 3^{n}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{1 / 3}\right)-\omega_{v}\left(m-\frac{1}{2}\right) \\
& \leq e^{\frac{2}{3} C_{2}\left(\mathbf{A}_{v}+\kappa_{T_{v}}\right)}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{1}\right)-\boldsymbol{\omega}_{v}\left(m-\frac{1}{2}\right) \\
& \leq e^{\frac{2}{3} C_{2}\left(\mathbf{A}_{v}+\kappa_{T_{v}}\right)} \mathbf{E}_{S}\left(T_{v}, 1\right)+\left(e^{\frac{2}{3} C_{2}\left(\mathbf{A}_{v}+\kappa_{T_{v}}\right)}-1+\kappa_{T_{v}}\right) \boldsymbol{\omega}_{v}\left(m-\frac{1}{2}\right) \\
& \leq\left(e^{C_{2} / C_{32}}+2 \frac{C_{2}}{C_{32}}\right)\left(\mathbf{E}_{S}\left(T_{v}, 1\right)+\kappa_{T_{v}}\right)
\end{aligned}
$$

and hence,

$$
\begin{equation*}
\limsup _{v \rightarrow \infty} \frac{\mathbf{E}_{S}\left(S_{v}, 1\right)}{\varepsilon_{v}^{2}} \leq\left(\frac{4}{3}\right)^{n}\left(e^{C_{2} / C_{32}}+2 \frac{C_{2}}{C_{32}}\right) \limsup _{v \rightarrow \infty} \frac{\mathbf{E}_{S}\left(T_{v}, 1\right)+\kappa_{T_{v}}}{\mathbf{E}_{C}\left(T_{v}, \frac{1}{4}\right)}=0, \tag{6.4}
\end{equation*}
$$

where we used (6.3).

We can apply Theorem 3.2(v.) (with $T$ replaced by $S_{v}$ ) combined with Definition $4.1(\mathrm{iv}),.(v$.$) (with T_{v}$ replaced by $S_{v}$ ), (6.2) and (6.4) to infer

$$
\begin{aligned}
\int_{\mathbf{V}_{T}}\left(\frac{\partial}{\partial r} \frac{f_{i}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)+\int_{\mathbf{W}_{T}}\left(\frac{\partial}{\partial r} \frac{g_{j}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \\
\leq 2^{n+7} \limsup _{v \rightarrow \infty} \frac{\mathbf{E}_{S}\left(S_{v}, 1\right)+C_{8}\left(\mathbf{A}_{v}+\kappa_{T_{v}}\right)}{\mathfrak{m}_{v}^{2}}=0
\end{aligned}
$$

Hence, both terms must vanish and therefore the following holds for all $0<\rho<1$

$$
f_{i}(\rho y)=\rho f_{i}(y) \quad \text { for } y \in \mathbf{V} \quad \text { and } \quad g_{j}(\rho y)=\rho g_{j}(y) \quad \text { for } y \in \mathbf{W} .
$$

This allows us to use Remark $6.2(i$.$) to the nonnegative functions f_{m}-f_{1}, g_{m-1}-g_{1}$ having vanishing trace on $\mathbf{L}$. We notice that

$$
\begin{aligned}
\left|f_{i}\right| & =\left(\left|f_{i}\right|-\min \left\{\left|f_{1}\right|, \cdots,\left|f_{m}\right|\right\}\right)+\min \left\{\left|f_{1}\right|, \cdots,\left|f_{m}\right|\right\} \\
& \leq\left(f_{m}-f_{1}\right)+\min \left\{\left|f_{1}\right|, \cdots,\left|f_{m}\right|\right\}
\end{aligned}
$$

and so, also each $f_{i}$ has zero trace on $\mathbf{L}$ by Lemma 5.1. Remark 6.2(ii.) gives that $f_{i}=\left.\beta_{i} Y_{n}\right|_{\mathbf{V}}$ for some $\beta_{i} \in \mathbb{R}$. The analogues statement holds for $g_{j}$ because Lemma 5.1 implies that also $\sum_{l=1}^{m-1} g_{l}$ has zero trace on $\mathbf{L}$ and we can bound

$$
(m-1)\left|g_{j}\right|=\left|\sum_{l=1}^{m-1}\left(g_{j}-g_{l}\right)+\sum_{l=1}^{m-1} g_{l}\right| \leq(m-1)\left(g_{m-1}-g_{1}\right)+\left|\sum_{l=1}^{m-1} g_{l}\right| .
$$

Then we can apply Lemma 5.4 to deduce

$$
\begin{align*}
& \beta_{1}=\cdots=\beta_{m}=\gamma_{1}=\cdots=\gamma_{m-1} \quad \text { and } \\
& \lim _{v \rightarrow \infty} \sup _{C_{7 / 8} \cap \operatorname{spt}\left(S_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta_{1} X_{n}\right|=0 . \tag{6.5}
\end{align*}
$$

Next, we infer $\beta_{1} \neq 0$. Indeed, if this were not the case, then Lemma 6.1 would imply that

$$
\begin{aligned}
0=\limsup _{v \rightarrow \infty}\left(\frac{\mathbf{E}_{S}\left(T_{v}, 1\right)+\kappa_{T_{v}}}{\mathbf{E}_{C}\left(T_{v}, \frac{1}{4}\right)}\right) & \geq \underset{v \rightarrow \infty}{\limsup }\left(\frac{\frac{1}{C_{34}} \mathbf{E}_{C}\left(T_{v}, \frac{1}{3}\right)-\mathbf{A}_{v}}{\mathbf{E}_{C}\left(T_{v}, \frac{1}{4}\right)}\right) \\
& \geq \frac{3^{n}}{4^{n} C_{34}}>0,
\end{aligned}
$$

where we used (6.3) for the last inequality.

Now, we rotate $T_{v}$ such that the new blowup sequence has a vanishing harmonic blowups. To do so, let $\eta_{v}:=\arctan \left(\beta_{1} \mathfrak{m}_{v}\right)$ and consider $R_{v}:=\left(\boldsymbol{\mu}_{4 \#} \gamma_{\eta_{v} \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}\right.$. From Remark 4.5(ii.), we know that $\left(R_{v},\left(\boldsymbol{\mu}_{4} \circ \gamma_{\eta_{v}}\right)\left(\mathcal{M}_{v}\right)\right) \in \mathcal{T}$ for $v$ large enough. We use again Lemma 3.3 (with $T, \sigma$ replaced by $R_{v}, 1 / 6$ ) and Lemma 5.3 to obtain

$$
\begin{align*}
\underset{v \rightarrow \infty}{\limsup } \frac{\mathbf{E}_{C}\left(\gamma_{\eta_{v}} \not T_{v}, \frac{1}{4}\right)}{\mathfrak{m}_{v}^{2}} & =\underset{v \rightarrow \infty}{\limsup } \frac{\mathbf{E}_{C}\left(R_{v}, 1\right)}{\mathfrak{m}_{v}^{2}} \\
& \leq \limsup _{v \rightarrow \infty} 36 C_{10}\left(C_{11} \sup _{\mathrm{C}_{7 / 6} \cap \operatorname{spt}\left(R_{v}\right)} \frac{X_{n+1}^{2}}{\mathfrak{m}_{v}^{2}}+\frac{\kappa_{T_{v}}+\mathbf{A}_{v}}{\mathfrak{m}_{v}^{2}}\right)  \tag{6.6}\\
& =0 .
\end{align*}
$$

But by Lemma 3.4 (with $T, \sigma$ replaced by $R_{v}, 7 / 8$ )

$$
\begin{aligned}
\liminf _{v \rightarrow \infty} \frac{\mathbf{E}_{C}\left(T_{v}, \frac{1}{4}\right)}{\mathfrak{m}_{v}^{2}} & =\liminf _{v \rightarrow \infty} \frac{\mathbf{E}_{C}\left(\mu_{4 \#} T_{v}, 1\right)}{\mathfrak{m}_{v}^{2}} \\
& \geq \liminf _{v \rightarrow \infty}\left(\frac{7}{8}\right)^{2 n+1} \frac{1}{C_{13} C_{14}}\left(\sup _{C_{1 / 8} \cap \operatorname{spt}\left(\mu_{4 \#} T_{v}\right)} \frac{X_{n+1}^{2}}{\mathfrak{m}_{v}^{2}}-\frac{\kappa_{T_{v}}+\mathbf{A}_{v}}{\mathfrak{m}_{v}^{2}}\right) \\
& =\frac{7^{2 n+1}}{8^{2 n+1} C_{13} C_{14}}\left(\frac{\beta_{1}}{8}\right)^{2}>0 .
\end{aligned}
$$

For $v$ large enough, together with (6.6), this contradicts (6.1).

As mentioned before, the excess decay will follow from the fact, that the harmonic blow-ups coincide on $\mathbf{V}$ and $\mathbf{W}$ respectively. To see this, we want to blow-up the harmonic blow-ups in a homogeneous way. Thus, we need to make sure that the limit exists, i.e. we prove that the harmonic blow-ups are $\mathcal{C}^{0,1}$ up to the boundary. The proof uses suitable rotations of $T_{v}$ and the uniform convergence of the blow-up sequence at the boundary.
Lemma 7.1. Let $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \in \mathcal{T}$ be a blow-up sequence with harmonic blow-ups $f_{i}$ and $g_{j}$. Then for all $0<\rho<1, i \in\{1, \ldots, m\}$ and $j \in\{1, \ldots, m-1\}$ the following holds

Proof. For $v \in \mathbb{N}$ with $v \geq 1$, we define $\varepsilon_{v}:=\sqrt{\mathbf{E}_{C}\left(T_{v}, 1\right)}$ and $\kappa_{v}:=\kappa_{T_{v}}$. Let $0<\sigma \leq$ $1 / 12$ and $\omega(\nu, \sigma) \in \mathbb{R}$ such that for all $|\eta| \leq 1 / 8$

$$
\begin{equation*}
\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{4}\right) \leq 2 \mathbf{E}_{C}\left(\gamma_{\eta \#} T_{v}, \frac{\sigma}{4}\right) \tag{7.1}
\end{equation*}
$$

Notice that by the monotonicity of the excess (3.1) and Definition 4.1 (i.), it follows $\lim _{v \rightarrow \infty} \mathbf{E}_{C}\left(T_{v}, \sigma\right)=0$. As (7.1) also must hold for $\eta=0$, it follows by Lemma 3.3 that also

$$
\begin{equation*}
\lim _{v \rightarrow \infty} \omega(v, \sigma)=0 \tag{7.2}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\lim _{v \rightarrow \infty} \mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \sigma\right)=0 . \tag{7.3}
\end{equation*}
$$

In a first step, we show that there is a constant $C_{36}$ such that for infinitely many $v$ the following holds

$$
\sup _{\mathrm{C}_{\sigma / 5} \cap \operatorname{spt}\left(\gamma_{\omega(v, \sigma) \neq \pm} T_{v}\right)}\left|X_{n+1}\right| \leq C_{36} \mathfrak{m}_{v} \sigma .
$$

To do so, we first bound $\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{4}\right)$ by looking at two different cases:
Case 1: $\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{3}\right)<\varepsilon_{v}^{2}$ for infintely many $v$.
We use the monotonicity of the excess (3.1) to deduce

$$
\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{4}\right) \leq\left(\frac{4}{3}\right)^{n} \mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{3}\right) \leq\left(\frac{4}{3}\right)^{n} \varepsilon_{v}^{2}
$$

for infinitely many $v$.

Case 2: $\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{3}\right) \geq \varepsilon_{v}^{2}$ for all $v \geq N$ for some $N$ large enough.
We define $S_{v}:=\left(\gamma_{\omega(v, \sigma) \#} \boldsymbol{\mu}_{\frac{1}{\sigma} \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}\right.$ and $\tilde{M}_{v}:=\gamma_{\omega(v, \sigma)} \circ \boldsymbol{\mu}_{\frac{1}{\sigma}}\left(\mathcal{M}_{v}\right)$. By Remark 4.5(iii.) is $\left(S_{v}, \tilde{M}_{v}\right) \in \mathcal{T}$. Recall the constants $C_{32}$ and $C_{35}$ of Theorem 6.3. By (3.1), (a.28), (7.3), (7.1) and Definition 4.1, there is an integer $N_{\sigma}$ such that for all $v \geq N_{\sigma}$ the following holds

- $\kappa_{v} \leq \varepsilon_{v}^{2}$,
- $\mathbf{E}_{C}\left(S_{v}, 1\right)+\kappa_{S_{v}}+\mathbf{A}_{\tilde{M}_{v}} \leq \mathbf{E}_{C}\left(\left(\gamma_{\omega(v, \sigma) \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}, \sigma\right)+\sigma\left(\kappa_{v}+\mathbf{A}_{v}\right) \leq \frac{1}{2 C_{32}}\right.$,
- $\mathbf{E}_{C}\left(S_{v}, \frac{1}{3}\right)+\frac{\kappa_{S_{v}}}{\mathbf{E}_{C}\left(S_{v}, \frac{1}{3}\right)} \leq 3^{n} \mathbf{E}_{C}\left(\left(\gamma_{\omega(v, \sigma) \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}, \sigma\right)+\sigma \frac{\kappa_{v}}{\varepsilon_{v}^{2}} \leq \frac{1}{C_{35}}\right.$,
- $\mathbf{E}_{C}\left(S_{v}, \frac{1}{4}\right) \leq 2 \mathbf{E}_{C}\left(\gamma_{\eta \#} S_{v}, \frac{1}{4}\right) \quad$ for all $|\eta| \leq \frac{1}{8}$.

Therefore, we can apply Theorem 6.3 (with $T$ replaced by $S_{v}$ for $v \geq N_{\sigma}$ ) to deduce

$$
\begin{align*}
\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{4}\right) & =\mathbf{E}_{C}\left(S_{v}, \frac{1}{4}\right) \leq C_{35}\left(\mathbf{E}_{S}\left(S_{v}, 1\right)+\kappa_{S_{v}}+\mathbf{A}_{\tilde{M}_{v}}\right)  \tag{7.4}\\
& \leq C_{35}\left(\mathbf{E}_{S}\left(T_{v}, \sigma\right)+\sigma\left(\kappa_{v}+\mathbf{A}_{v}\right)\right) .
\end{align*}
$$

Notice that by Lemma 2.3, the following holds

$$
e^{\mathcal{C}_{2}\left(\mathbf{A}_{v}+\kappa_{v}\right) \sigma} \sigma^{-n}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{\sigma}\right) \leq e^{\mathcal{C}_{2}\left(\mathbf{A}_{v}+\kappa_{v}\right)}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{1}\right) .
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}_{S}\left(T_{v}, \sigma\right) & =\sigma^{-n}\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{\sigma}\right)-\left(m-\frac{1}{2}\right) \boldsymbol{\alpha}(n) \\
& \leq e^{\mathcal{C}_{2}\left(\mathbf{A}_{v}+\kappa_{v}\right)}\left(\left\|T_{v}\right\|\left(\overline{\mathbf{B}}_{1}\right)-\left(m-\frac{1}{2}\right) \boldsymbol{\omega}_{n}\right)+\left(e^{\mathcal{C}_{2}\left(\mathbf{A}_{v}+\kappa_{v}\right)}-1\right)\left(m-\frac{1}{2}\right) \boldsymbol{\omega}_{n} .
\end{aligned}
$$

With this and (3.2), we can continue to estimate (7.4) with

$$
C_{35}\left(\mathbf{E}_{S}\left(T_{v}, \sigma\right)+\sigma\left(\kappa_{v}+\mathbf{A}_{v}\right)\right) \leq C_{35}\left(\mathbf{E}_{C}\left(T_{v}, 1\right)+\kappa_{v}+\mathbf{A}_{v}\right) \leq C_{37} \mathfrak{m}_{v}^{2} .
$$

Hence, in both cases we have infinitely many $v$ satisfying

$$
\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{4}\right) \leq C_{38} \mathbf{m}_{v}^{2} .
$$

For these $v$ we apply Lemma 3.4 (with $\sigma, T$ replaced by $1 / 5,\left(\gamma_{\omega(v, \sigma) \#} \boldsymbol{\mu}_{4 / \sigma \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}\right)$ and infer

$$
\begin{aligned}
& \sup _{\mathrm{C}_{\sigma / 5} \cap \operatorname{spt}\left(\gamma_{\left.\omega(v, \sigma) \neq T_{v}\right)}\right.}\left|X_{n+1}\right|=\sup _{\mathrm{C}_{4 / 5} \cap \operatorname{spt}\left(\gamma_{\left.\omega(v, \sigma) \neq \mu_{4 / \sigma \not / \pi} T_{v}\right)}\right.} \frac{\sigma}{4}\left|X_{n+1}\right| \\
& \leq \frac{\sigma}{4} \sqrt{C_{13} C_{14} 5^{2 n+1}\left(\mathbf{E}_{C}\left(\gamma_{\omega(v, \sigma) \#} T_{v}, \frac{\sigma}{4}\right)+\frac{\sigma}{4}\left(\kappa_{v}+\mathbf{A}_{v}\right)\right)} \\
& \leq C_{39} \mathfrak{m}_{v} \sigma .
\end{aligned}
$$

With this, we now prove the bound on $f_{i}$ and $g_{j}$. To be able to jump between $\mathbf{V}$ and $\mathbf{W}$, we define for $y \in \mathbb{R}^{n}$ the map $y \mapsto \bar{y}:=$ $\left(y_{1}, \ldots, y_{n-1},-y_{n}\right)$. Denote by $v_{i}^{(\nu)}$ and $w_{j}^{(\nu)}$ the maps whose $\boldsymbol{\Phi}_{\nu}$-graphs form the $\operatorname{spt}\left(T_{v}\right)$ as in Definition 4.1. By the previous inequality and (7.2), we can bound for infintely many
 $v$, arbitrary $0<\tau<1, i \in\{1, \ldots, m\}$ and $j \in\{1, \ldots, m-1\}$

$$
\left|v_{i}^{(v)}(y)+w_{j}^{(v)}(\bar{y})\right| \leq 2 C_{39} \mathbf{m}_{v} \sigma \quad \text { for } y \in \mathbf{V}_{\tau} \cap \overline{\mathbf{B}}_{\sigma / 5}^{n}(0)
$$

Consider now any $0 \neq y \in \mathbf{V} \cap \overline{\mathbf{B}}_{1 / 60}^{n}(0)$. Then let $\sigma:=5|y| \leq 1 / 12$. The previous bounds imply that

$$
\left|\frac{v_{i}^{(v)}(y)}{\boldsymbol{m}_{v}}+\frac{w_{j}^{(v)}(\bar{y})}{\mathfrak{m}_{v}}\right| \leq 2 C_{39} \sigma=10 C_{39}|y|
$$

for infintely many $v$. Hence, by local uniform convergence,

$$
\begin{equation*}
\left|f_{i}(y)+g_{j}(\bar{y})\right| \leq 10 C_{39}|y| \quad \text { for } y \in \mathbf{V} \cap \overline{\mathbf{B}}_{1 / 60}^{n}(0) \tag{7.5}
\end{equation*}
$$

Moreover, by (4.1), for $y \in \mathbf{V} \cap\left(\overline{\mathbf{B}}_{\rho}^{n}(0) \backslash \overline{\mathbf{B}}_{1 / 60}^{n}(0)\right), i \in\{1, \ldots, m\}$ and $j \in\{1, \ldots, m-$ $1\}$, the following holds

$$
\begin{equation*}
\left|f_{i}(y)\right|^{2}+\left|g_{j}(\bar{y})\right|^{2} \leq \frac{4 C_{13} C_{14}}{(1-\rho)^{2 n+1}}(60|y|)^{2} \tag{7.6}
\end{equation*}
$$

Now, we define the following auxiliary functions

$$
\begin{gathered}
h: \mathbf{B}_{1}^{n}(0) \rightarrow \mathbb{R}, h(y)= \begin{cases}\sum_{i=1}^{m} f_{i}(y), & \text { for } y \in \mathbf{V} \\
\sum_{j=1}^{m-1} g_{j}(y), & \text { for } y \in \mathbf{W}, \\
0, & \text { for } y \in \mathbf{L}\end{cases} \\
H: \mathbf{B}_{1}^{n}(0) \rightarrow \mathbb{R}, H(y)=h(y)-h(\bar{y}) .
\end{gathered}
$$

By Lemma 5.1, these two functions have locally square integrable weak gradients. Moreover, $H$ is odd in the $n$-th variable and $\left.H\right|_{\mathbf{V} \cup \mathbf{W}}$ is harmonic. The weak version of the Schwarz reflection principle implies that $H$ is harmonic on all $\mathbf{B}_{1}^{n}(0)$. Therefore, the following holds for all $0<\rho<1$

$$
\begin{equation*}
\sup _{\overline{\mathbf{B}}_{\rho}^{n}(0)} \frac{|H(y)|}{|y|}<\infty . \tag{7.7}
\end{equation*}
$$

Notice that for $y \in \mathbf{V}$, we can write

$$
\begin{aligned}
& f_{i}(y)=H(y)-\sum_{k=1}^{i-1}\left(f_{k}(y)+g_{k}(\bar{y})\right)-\sum_{k=i+1}^{m}\left(f_{k}(y)+g_{k-1}(\bar{y})\right), \\
& g_{j}(\bar{y})=\left(f_{1}(y)+g_{j}(\bar{y})\right)-f_{1}(y) .
\end{aligned}
$$

(7.5), (7.6) and (7.7) then imply the lemma.

Now, we are ready to prove that all harmonic blowups coincide even if they are not linear. The definition of the homogeneous blow-up of the harmonic blow-ups and the estimate in Theorem 3.2(v.) will imply that they are linear, and hence, coincide with each other. Then we will use the E.Hopf boundary point Lemma for harmonic functions to deduce that also the harmonic blow-ups need to coincide themselves.

Theorem 7.2. Let $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ be a blowup sequence with harmonic blowups $f_{i}, g_{j}$. Then
(i.) $f_{1}=\cdots=f_{m}$ and $g_{1}=\cdots=g_{m-1}$.
(ii.) The functions

$$
\begin{aligned}
& f: V \cup L \rightarrow \mathbb{R}, y \mapsto \begin{cases}f_{1}(y), & \text { for } y \in V \\
0, & \text { for } y \in L\end{cases} \\
& g: W \cup L \rightarrow \mathbb{R}, y \mapsto \begin{cases}g_{1}(y), & \text { for } y \in W \\
0, & \text { for } y \in L\end{cases}
\end{aligned}
$$

are $\mathcal{C}^{2}$.
(iii.) $D f(0)=D g(0)$.

Proof. We first blow $f_{i}, g_{j}$ up and show the equality of these limiting functions. Then we deduce that also the $f_{i}, g_{j}$ coincide.

Let $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}, 4 \leq \rho<\infty$ and define the functions $f_{i}^{(\rho)}:=$ $\rho f_{i}(\dot{\bar{\rho}})$ and $g_{j}^{(\rho)}:=\rho g_{j}(\dot{\bar{\rho}})$. Then $f_{i}^{(\rho)}$ and $g_{j}^{(\rho)}$ are harmonic and by Lemma 7.1 uniformly bounded.

Indeed, for all $4 \leq \rho<\infty$

$$
\sup _{\mathbf{V}}\left|f_{i}^{(\rho)}\right|=\rho \sup _{\mathbf{V}}\left|f_{i}\left(\frac{y}{\rho}\right)\right|=\rho \sup _{\mathbf{V} \cap \overline{\mathbf{B}}_{1 / \rho}^{n}(0)}\left|f_{i}\right| \leq \sup _{\mathbf{V} \cap \overline{\mathbf{B}}_{1 / \rho}^{n}(0)} \frac{\left|f_{i}(y)\right|}{|y|} \leq \sup _{\mathbf{V} \cap \overline{\mathbf{B}}_{1 / 4}^{n}(0)} \frac{\left|f_{i}(y)\right|}{|y|}<\infty .
$$

Then [25, Theorem 2.11] implies that, up to subsequence, they converge pointwise to a harmonic function. This means, there exist a strictly increasing sequence $\rho_{k} \rightarrow \infty$ as $k \rightarrow \infty$ and harmonic functions $f_{1}^{*}, \ldots, f_{m}^{*}$ on $\mathbf{V}, g_{1}^{*}, \ldots, g_{m-1}^{*}$ on $\mathbf{W}$ such that for all $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$

$$
\begin{array}{llll}
\lim _{k \rightarrow \infty} f_{i}^{\left(\rho_{k}\right)}(y)=f_{i}^{*}(y) & \text { and } & \lim _{k \rightarrow \infty} D f_{i}^{\left(\rho_{k}\right)}(y)=D f_{i}^{*}(y) & \text { for } y \in \mathbf{V}, \\
\lim _{k \rightarrow \infty} g_{j}^{\left(\rho_{k}\right)}(y)=g_{j}^{*}(y) & \text { and } & \lim _{k \rightarrow \infty} D g_{j}^{\left(\rho_{k}\right)}(y)=D g_{j}^{*}(y) & \text { for } y \in \mathbf{W} .
\end{array}
$$

We want to deduce their equality by using Lemma 5.4. To do so, we first must show that $f_{i}^{*}, g_{j}^{*}$ are of the form $\beta Y_{n}$ for some $\beta \in \mathbb{R}$. A sufficient condition for this is the following identity $\frac{\partial}{\partial r} \frac{f_{i}^{*}(y)}{|y|}=0=\frac{\partial}{\partial r} \frac{g_{i}^{*}(\bar{y})}{|\bar{y}|}$, as we have seen in the proof of Theorem 6.3. By Theorem 3.2(v.), we have

$$
\int_{\mathbf{V}}\left(\frac{\partial}{\partial r} \frac{f_{i}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)+\int_{\mathbf{W}}\left(\frac{\partial}{\partial r} \frac{g_{j}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \leq 2^{n+5} C_{40}<\infty,
$$

and hence, Fatou's Lemma implies that

$$
\begin{aligned}
& \int_{\mathbf{V}}\left(\frac{\partial}{\partial r} \frac{f_{i}^{*}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)+\int_{\mathbf{W}}\left(\frac{\partial}{\partial r} \frac{g_{j}^{*}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y) \\
& \leq \liminf _{k \rightarrow \infty}\left(\int_{\mathbf{V}}\left(\frac{\partial}{\partial r} \frac{f_{k}^{\left(\rho_{k}\right)}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)+\int_{\mathbf{W}}\left(\frac{\partial}{\partial r} \frac{g_{j}^{\left(\rho_{k}\right)}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)\right) \\
& \leq \liminf _{k \rightarrow \infty}\left(\int_{\mathbf{V} \cap \overline{\mathbf{B}}_{1 / \rho_{k}}^{n}(0)}\left(\frac{\partial}{\partial r} \frac{f_{i}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)\right. \\
&\left.+\int_{\mathbf{W} \cap \overline{\mathbf{B}}_{1 / \rho_{k}}^{n}(0)}\left(\frac{\partial}{\partial r} \frac{g_{j}(y)}{|y|}\right)^{2}|y|^{2-n} \mathrm{~d} \mathcal{L}^{n}(y)\right)
\end{aligned}
$$

$$
=0
$$

Therefore, there exist real numbers $\beta_{1} \leq \cdots \leq \beta_{m}, \gamma_{1} \geq \cdots \geq \gamma_{m-1}$ such that $f_{i}^{*}=\left.\beta_{i} Y_{n}\right|_{\mathbf{V}}, g_{j}^{*}=\left.\gamma_{j} Y_{n}\right|_{\mathbf{W}}$. Now, we show that all these numbers coincide.
This must hold by Lemma 5.4, if we find a blowup sequence whose associated harmonic blowups are exactly $f_{i}^{*}, g_{j}^{*}$. For $k \in \mathbb{N}, k \geq 1$, we define

$$
S_{v}^{k}:=\left(\boldsymbol{\mu}_{\rho_{k}} T_{v}\right)\left\llcorner\mathbf{B}_{3} .\right.
$$

Then there is an $N>0$ such that for $v \geq N$ the following holds $\mathbf{E}_{C}\left(T_{v}, 1\right)+\kappa_{v}+\mathbf{A}_{v} \leq$ $\frac{1}{C_{20}}$ and hence, by Remark $4.4,\left(S_{v}^{k}, \boldsymbol{\mu}_{\rho_{k}}\left(\mathcal{M}_{v}\right)\right) \in \mathcal{T}$. Moreover, by Definition 4.1(iv.), (v.) for all $i \in\{1, \ldots, m\}, j \in\{1, \ldots, m-1\}$ we have

$$
\begin{array}{ll}
\lim _{v \rightarrow \infty} \frac{v_{i}^{S_{v}^{k}}}{\mathfrak{m}_{v}}=f_{i}^{\left(\rho_{k}\right)} \quad \text { on compact subsets of } \mathbf{V}, \\
\lim _{v \rightarrow \infty} \frac{w_{j}^{s_{v}^{k}}}{\mathfrak{m}_{v}}=g_{j}^{\left(\rho_{k}\right)} \quad \text { on compact subsets of } \mathbf{W} .
\end{array}
$$

We choose now for every $k$ an $v_{k} \geq \max \{N, k\}$ satisfying the following three properties:

1. $\quad \max \left\{\sup _{\mathbf{V} \cap \overline{\mathbf{B}}_{1 / 2}}\left|f_{1}^{\left(\rho_{k}\right)}\right|, \sup _{\mathbf{V} \cap \overline{\mathbf{B}}_{1 / 2}}\left|f_{m}^{\left(\rho_{k}\right)}\right|, \sup _{\mathbf{W} \cap \overline{\mathbf{B}}_{1 / 2}}\left|g_{1}^{\left(\rho_{k}\right)}\right|, \sup _{\mathbf{W} \cap \overline{\mathbf{B}}_{1 / 2}}\left|g_{m-1}^{\left(\rho_{k}\right)}\right|\right\}$

$$
\leq \sup _{\mathrm{C}_{1 / 2} \cap \operatorname{spt}_{v_{v_{k}}}^{k}} \frac{\left|X_{n+1}\right|}{\mathfrak{m}_{v_{k}}}+\frac{1}{k}
$$

2. $\sup _{\mathbf{C}_{3 / 2} \cap \operatorname{spt}\left(S_{v_{k}}^{k}\right)} \frac{\left|X_{n+1}\right|}{\mathfrak{m}_{v_{k}}}$

$$
\leq 3 \max \left\{\sup _{\mathbf{V}}\left|f_{1}^{\left(\rho_{k} / 3\right)}\right|, \sup _{\mathbf{V}}\left|f_{m}^{\left(\rho_{k} / 3\right)}\right|, \sup _{\mathbf{W}}\left|g_{1}^{\left(\rho_{k} / 3\right)}\right|, \sup _{\mathbf{W}}\left|g_{m-1}^{\left(\rho_{k} / 3\right)}\right|\right\}+\frac{1}{k}
$$

This is possible by Lemma 5•3, where $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1}, a, \sigma$ are replaced by $\left\{\left(\boldsymbol{\mu}_{\rho_{k} / 3 \#} T_{v_{k}}, \boldsymbol{\mu}_{\rho_{k} / 3}\left(\mathcal{M}_{v_{k}}\right)\right\}_{k \geq 1}, 0,1 / 2\right)$ and because

$$
\sup _{\mathbf{C}_{3 / 2} \cap \operatorname{spt}\left(S_{v_{k}}^{k}\right)} \frac{\left|X_{n+1}\right|}{\mathfrak{m}_{v_{k}}}=\sup _{\mathbf{C}_{1 / 2} \cap \operatorname{spt}\left(\boldsymbol{\mu}_{1 / 3 \#} S_{v_{k}}^{k}\right)} 3 \frac{\left|X_{n+1}\right|}{\mathfrak{m}_{v_{k}}}=3 \sup _{\mathbf{C}_{1 / 2} \cap \operatorname{spt}\left(\boldsymbol{\mu}_{\rho_{k} / 3 \# \#^{T} T_{v_{k}}^{k}}\right)} \frac{\left|X_{n+1}\right|}{\mathfrak{m}_{v_{k}}} .
$$

3. We define the (blowup) sequence $\left\{\left(S_{k}^{*}, \mathcal{M}_{k}^{*}\right)\right\}_{k \geq 1}$ by $S_{k}^{*}:=S_{v_{k}}^{k}$ and $\mathcal{M}_{k}^{*}:=$ $\boldsymbol{\mu}_{\rho_{k}}\left(\mathcal{M}_{v_{k}}\right)$ and notice

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{v_{i}^{S_{k}^{*}}}{\mathfrak{m}_{v_{k}}}=f_{i}^{*} \quad \text { and } \quad \lim _{k \rightarrow \infty} \frac{w_{j}^{S_{k}^{*}}}{\mathfrak{m}_{v_{k}}}=g_{j}^{*} \tag{7.8}
\end{equation*}
$$

If all $f_{i}^{*}, g_{j}^{*}$ vanish, then also $0=\beta_{1}=\cdots=\beta_{m}=\gamma_{1}=\cdots=\gamma_{m-1}$. If not, we want to see whether $\left\{S_{k}^{*}\right\}_{k \geq 1}$ is a blowup sequence to $f_{i}^{*}, g_{j}^{*}$. Hence, we aim for (7.8) with $\mathfrak{m}_{v_{k}}$ replaced by $\mathfrak{m}_{S_{k}^{*}}$. Therefore, we shall compare these two quantities. First, we notice that by Remark 4.4,

$$
0 \leq \frac{\kappa_{S_{k}^{*}}+\mathbf{A}_{\mathcal{M}_{k}^{*}}}{\mathfrak{m}_{v_{k}}^{2}} \leq \frac{\kappa_{v_{k}}+\mathbf{A}_{v_{k}}}{\rho_{k} \mathfrak{m}_{v_{k}}^{2}} \rightarrow 0 \quad \text { as } k \rightarrow \infty
$$

Then by Lemmas 3.3 and 3.4 (with $T, \mathcal{M}, \sigma$ replaced by $S_{k}^{*}, \mathcal{M}_{k}^{*} 1 / 2$ ) and the conditions 1. and 2., it follows that

$$
\begin{aligned}
\limsup _{k \rightarrow \infty} \frac{\mathbf{E}_{C}\left(S_{k}^{*}, 1\right)}{\mathfrak{m}_{v_{k}}^{2}} & \leq \limsup _{k \rightarrow \infty} 4 C_{10}\left(\frac{C_{11}}{\mathfrak{m}_{v_{k}}^{2}} \sup _{\mathbf{C}_{3 / 2} \cap \operatorname{spt}\left(S_{k}^{*}\right)} X_{n+1}^{2}+\frac{\kappa_{S_{k}^{*}}+\mathbf{A}_{\mathcal{M}_{k}^{*}}}{\mathfrak{m}_{v_{k}}^{2}}\right) \\
& \leq 36 C_{10} C_{11} \max \left\{\sup _{\mathbf{V}}\left(f_{i}^{*}\right)^{2}, \sup _{\mathbf{W}}\left(g_{j}^{*}\right)^{2}: i, j\right\}, \\
\liminf _{k \rightarrow \infty} \frac{\mathbf{E}_{C}\left(S_{k^{\prime}}^{*}, 1\right)}{\mathfrak{m}_{v_{k}}^{2}} & \geq \liminf _{k \rightarrow \infty}\left(\frac{1}{2^{2 n+1} C_{13} C_{14}} \sup _{C_{1 / 2} \cap \operatorname{spt}\left(S_{k}^{*}\right)} \frac{X_{n+1}^{2}}{\mathfrak{m}_{v_{k}}^{2}}-\frac{\kappa_{S_{k}^{*}}+\mathbf{A}_{\mathcal{M}_{k}^{*}}}{\mathfrak{m}_{v_{k}}^{2}}\right) \\
& \geq \frac{1}{2^{2 n+1} C_{13} C_{14}} \max \left\{\sup _{\mathbf{V}}\left(f_{i}^{*}\right)^{2}, \sup _{\mathbf{W}}\left(g_{j}^{*}\right)^{2}: i, j\right\} .
\end{aligned}
$$

Hence,

$$
0<\liminf _{k \rightarrow \infty} \frac{\max \left\{\mathbf{E}_{C}\left(S_{k}^{*}, 1\right), \mathbf{A}_{S_{k}^{*}}^{1 / 2}\right\}}{\mathfrak{m}_{v_{k}}^{2}} \leq \limsup _{k \rightarrow \infty} \frac{\max \left\{\mathbf{E}_{C}\left(S_{k}^{*}, 1\right), \mathbf{A}_{S_{k}^{*}}^{1 / 2}\right\}}{\mathfrak{m}_{v_{k}}^{2}}<\infty
$$

and we can find a subsequence $\left\{\left(S_{k_{l}}^{*}, \mathcal{M}_{k_{l}}^{*}\right)\right\}_{l \geq 1}$ which is a blowup sequence and whose associated harmonic blowups are $\gamma f_{i}^{*}, \gamma g_{j}^{*}$ for some fixed $\gamma \in \mathbb{R}$ by (7.8). As they are of the form as in Lemma 5.4 it follows that there is a $\beta \in \mathbb{R}$ satisfying

$$
f_{1}^{*}=\cdots=f_{m}^{*}=\left.\beta Y_{n}\right|_{\mathbf{V}} \quad \text { and } \quad g_{1}^{*}=\cdots=g_{m-1}^{*}=\left.\beta Y_{n}\right|_{\mathbf{W}} .
$$

From this, we want to deduce that also $f_{1}=\cdots=f_{m}$ and $g_{1}=\cdots=g_{m-1}$. Notice that $f_{1}-f_{m}$ and $g_{1}-g_{m-1}$ are nonpositive and harmonic functions. By Lemma 5.1, $f_{i}$ and $g_{j}$ have zero trace on $\mathbf{L}$. Hence,

$$
\sup _{\mathbf{V}}\left(f_{1}-f_{m}\right)=0=\sup _{\mathbf{W}}\left(g_{1}-g_{m-1}\right) .
$$

Moreover, the E. Hopf boundary point Lemma [25, Lemma 3.4] implies that if $y_{0} \in \mathbf{L}$ is a strict maximum point, then the outer normal derivative at $y_{0}$ (if it exists) must be positive. But at zero, the following holds

$$
\begin{gathered}
\frac{\partial\left(f_{1}-f_{m}\right)}{\partial v}(0)=\lim _{t \downarrow 0} \frac{\left(f_{1}-f_{m}\right)(0, \ldots, 0, t)}{t}=\left(f_{1}^{*}-f_{m}^{*}\right)(0, \ldots, 0,1)=0, \\
\frac{\partial\left(g_{1}-g_{m-1}\right)}{\partial v}(0)=\lim _{t \downarrow 0} \frac{\left(g_{1}-g_{m-1}\right)(0, \ldots, 0,-t)}{t}=\left(g_{1}^{*}-g_{m-1}^{*}\right)(0, \ldots, 0,-1)=0 .
\end{gathered}
$$

Hence, 0 is not a strict maximum point and there must be a point in $\mathbf{V}$ ( $\mathbf{W}$ respectively) reaching 0 (i.e. the maximum) as well. Then [25, Theorem 3.5] implies that $f_{1}-f_{m}$, and $g_{1}-g_{m-1}$ must be constant. In fact, by the vanishing trace, $f_{1}-f_{m}=0=g_{1}-g_{m-1}$. Therefore, (i.) must hold. Also by the vanishing trace and weak version of the Schwarz reflection principle, there are harmonic functions $f \in \mathcal{C}^{2}(\mathbf{V} \cap \mathbf{L}), g \in \mathcal{C}^{2}(\mathbf{W} \cup \mathbf{L})$ satisfying (ii.) and (iii.).

Remark 7.3. Let $f, g$ denote harmonic blow-ups as in Theorem 7.2(ii.). Then there are constants $C_{41}, C_{18}$ such that
(i.) $|D f(0)|=|D g(0)| \leq C_{41} \min \left\{\sqrt{\int_{\mathbf{V} \cap \mathbf{B}_{1 / 2}^{n}(0)}|f|^{2} \mathrm{~d} \mathcal{L}^{n}}, \sqrt{\int_{\mathbf{W} \cap \mathbf{B}_{1 / 2}^{n}(0)}|g|^{2} \mathrm{~d} \mathcal{L}^{n}}\right\} \leq C_{18}$.
(ii.) For all $y \in \overline{\mathbf{B}}_{1 / 4}^{n}(0)$ the following holds

$$
|f(y)-y \cdot D f(0)| \leq C_{41}|y|^{2} \sqrt{\int_{\mathbf{V} \cap \mathbf{B}_{1 / 2}^{n}(0)}|f|^{2} \mathrm{~d} \mathcal{L}^{n}} \leq C_{18}|y|^{2}
$$

(iii.) For all $y \in \overline{\mathbf{B}}_{1 / 4}^{n}(0)$ the following holds

$$
|g(y)-y \cdot D g(0)| \leq C_{41}|y|^{2} \sqrt{\int_{\mathbf{W} \cap \mathbf{B}_{1 / 2}^{n}(0)}|g|^{2} \mathrm{~d} \mathcal{L}^{n}} \leq C_{18}|y|^{2}
$$

Proof. (i.) By the Schwarz reflection principle, we can extend $f$ to an harmonic function $\tilde{f}$ defined on $\overline{\mathbf{B}}_{1 / 2}^{n}(0)$. Then by the interior estimates for harmonic functions [25, Theorem 2.10], the mean value property and Hölder's inequality, it follows that

$$
|D f(0)| \leq 8 n \sup _{\overline{\mathbf{B}}_{1 / 4}^{n}}|\tilde{f}| \leq 8 n \frac{2^{n}}{\omega_{n}} \int_{\overline{\mathbf{B}}_{1 / 2}^{n}}|\tilde{f}| \mathrm{d} \mathcal{L}^{n} \leq 8 n\left(\frac{2^{n}}{\omega_{n}}\right)^{2} \sqrt{\int_{\overline{\mathbf{B}}_{1 / 2}^{n}}|f|^{2} \mathrm{~d} \mathcal{L}^{n}} .
$$

Moreover, by Lemma 3.4(ii.) (with $\sigma$ replaced by $1 / 2$ ) and Definition 4.1 (iii.), this integral is bounded by $2^{n+1} C_{14}$. The same holds for $g$.
(ii.) By the Taylor formula, $|f(y)-y \cdot D f(0)| \leq C\left|D^{2} f(0)\right||y|^{2}$. Also by [25, Theorem 2.10], the following holds

$$
\left|D^{2} f(0)\right| \leq \frac{n^{2}}{16} \sup _{\overline{\mathbf{B}}_{1 / 4}^{n}}|\tilde{f}| .
$$

The inequalities follow then as in (i.).
(iii.) Similar to (ii.).

With the $\mathcal{C}^{2}$ functions from Theorem 7.2, we prove the following inequalities of the excess. We will use them to prove Theorem 4.2 by constructing inductively a sequence of currents which will show that the excess of the (slightly rotated) original current decays at most proportional to the radius.

Theorem 8.1. Let $(T, \mathcal{M}) \in \mathcal{T}$ and define $\theta:=\left(C_{21}\left(1+C_{18}\right)\right)^{-2}$ (see Remarks 4.5 (iii.) and 7.3). There is a constant $C_{19} \geq 1$ such that if $T$ fulfils $\max \left\{E_{C}(T, 1), C_{19} \kappa_{T}, \sqrt{A}\right\} \leq \frac{1}{C_{19}}$, then there is a real number $\omega$ satisfying

$$
|\omega|^{2} \leq C_{18}^{2} \max \left\{E_{C}(T, 1), \sqrt{A}\right\} \quad \text { and } \quad E_{C}\left(\gamma_{\omega \#} T, \theta\right) \leq \theta \max \left\{E_{C}(T, 1), C_{19} \kappa_{T}, \sqrt{A}\right\} .
$$

Proof. We argue by contradiction. If the theorem did not hold, then there would be a sequence $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1} \subset \mathcal{T}$ such that for all $|\omega| \leq \mathcal{C}_{18} \boldsymbol{m}_{v}$ the following holds

$$
\begin{align*}
\max \left\{\varepsilon_{v}^{2}, \sqrt{\mathbf{A}_{v}}, v \kappa_{v}\right\} & \leq \frac{1}{v}  \tag{8.1}\\
\mathbf{E}_{C}\left(\gamma_{\omega \#} T_{v}, \theta\right) & >\theta \max \left\{\varepsilon_{v}^{2}, \sqrt{\mathbf{A}_{v}}, \nu \kappa_{v}\right\} \tag{8.2}
\end{align*}
$$

where $\varepsilon_{v}:=\sqrt{\mathbf{E}_{C}\left(T_{v}, 1\right)}, \kappa_{v}:=\kappa_{T_{v}}$ and $\mathbf{A}_{v}:=\mathbf{A}_{\mathcal{M}_{v}}$. Notice that by the monotonicity of the excess (3.1), the condition (8.2) (with $\omega=0$ ) implies

$$
\theta v \kappa_{v} \leq \theta \max \left\{\mathbf{E}_{C}\left(T_{v}, 1\right), \sqrt{\mathbf{A}_{v}}, \nu \kappa_{v}\right\}<\mathbf{E}_{C}\left(T_{v}, \theta\right) \leq \frac{\varepsilon_{v}^{2}}{\theta^{n}}
$$

Hence, by (8.1), we can assume that

$$
\varepsilon_{v}^{2}+\frac{\kappa_{v}}{\varepsilon_{v}^{2}}+\mathbf{A}_{v}<\frac{2}{v}+\frac{1}{v \theta^{n+1}} .
$$

Therefore, we notice that as in (4.2), $\left\{\left(T_{v}, \mathcal{M}_{v}\right)\right\}_{v \geq 1}$ is, up to subsequence, a blowup sequence with associated harmonic blowups $f_{i}, g_{j}$. Let $f, g$ denote the $\mathcal{C}^{2}$-functions as in Theorem 7.2(ii.). As they vanish on $\mathbf{L}$, for every $0<\sigma<1$ the functions $\varepsilon_{v}^{-1} v_{i}^{(v)}$, $\varepsilon_{v}^{-1} w_{j}^{(v)}$ converge uniformly on $\mathbf{V}_{\sigma}, \mathbf{W}_{\sigma}$. Thus, we derive from Lemma 5.3 that

$$
\begin{align*}
& \underset{v \rightarrow \infty}{\limsup \sup ^{2}} \sup _{\mathbf{C}_{1 / 2} \cap \mathbf{p}^{-1}(\mathbf{V}) \cap \operatorname{spt}\left(T_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-f \circ \mathbf{p}\right|=0,  \tag{8.3}\\
& \underset{v \rightarrow \infty}{\limsup } \sup _{\mathbf{C}_{1 / 2} \cap \mathbf{p}^{-1}(\mathbf{W}) \cap \operatorname{spt}\left(T_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-g \circ \mathbf{p}\right|=0 .
\end{align*}
$$

From Remark 7.3 and the proof of Theorem 7.2, we deduce the existence of some $\beta \in\left[-C_{18}, C_{18}\right]$ satisfying $D f(0)=(0, \ldots, 0, \beta)=D g(0)$. Therefore, by applying Remark 7.3(ii.), (iii.), it follows

$$
\begin{align*}
\left|f(x)-\beta x_{n}\right|=|f(x)-x D f(0)| \leq C_{18}|x|^{2} & \text { for } x \in \mathbf{V} \cap \overline{\mathbf{B}}_{1 / 4}^{n}(0) \\
\left|g(x)-\beta x_{n}\right|=|g(x)-x D g(0)| \leq C_{18}|x|^{2} & \text { for } x \in \mathbf{W} \cap \overline{\mathbf{B}}_{1 / 4}^{n}(0) \tag{8.4}
\end{align*}
$$

Then we rotate the currents such that the new differential vanishes. Indeed, let $\omega_{v}:=$ $\arctan \left(\beta \mathfrak{m}_{v}\right)$. Then

$$
\begin{equation*}
\left|\omega_{v}\right| \leq|\beta| \mathfrak{m}_{v} \leq C_{18} \mathfrak{m}_{v} \tag{8.5}
\end{equation*}
$$

Consider now $S_{v}:=\left(\boldsymbol{\mu}_{1 / \theta \#} \gamma_{\omega_{v} \#} T_{v}\right)\left\llcorner\mathbf{B}_{3}\right.$ and $\left.\tilde{\mathcal{M}}_{v}:=\boldsymbol{\mu}_{1 / \theta} / \mathcal{M}_{v}\right)$. By (8.1), the assumptions of Remark 4.5 (iii.) are fulfilled for $v$ large enough, and hence, $\left(S_{v}, \tilde{\mathcal{M}}_{v}\right) \in \mathcal{T}$ and

$$
\begin{equation*}
\kappa_{S_{v}} \leq \theta \kappa_{v}, \quad \mathbf{A}_{\tilde{\mathcal{M}}_{v}} \leq \theta \mathbf{A}_{v} \tag{8.6}
\end{equation*}
$$

By (8.3), (8.4) and the Remark 7.3(ii.), (iii.), it follows

$$
\begin{aligned}
& \limsup \sup _{v \rightarrow \infty}\left|\frac{X_{n+1}}{\operatorname{m}_{2} \cap \operatorname{spt}\left(S_{v}\right)}\right| \leq \limsup _{v \rightarrow \infty} \sup _{\mathrm{C}_{3} \cap \operatorname{spt}\left(\mu_{1 / \theta \#} T_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta X_{n}\right| \\
& \leq \frac{1}{\theta} \limsup _{v \rightarrow \infty} \sup _{\mathrm{Cup}_{3 \theta} \cap \operatorname{spt}\left(T_{v}\right)}\left|\frac{X_{n+1}}{\mathfrak{m}_{v}}-\beta X_{n}\right| \\
& \leq \frac{1}{\theta} \limsup _{v \rightarrow \infty}\left(\sup _{\substack{\sup _{3 \theta} \cap \mathbf{V} \\
\mathbf{C s p t}^{2 p}\left(T_{v}\right)}}\left|f \circ \mathbf{p}-\beta Y_{n}\right|+\sup _{\substack{\mathbf{C u}_{3 \theta} \cap \mathbf{W} \\
\cap \operatorname{spt}\left(T_{v}\right)}}\left|g \circ \mathbf{p}-\beta Y_{n}\right|\right) \\
& \leq \frac{1}{\theta} C_{18}\left((3 \theta)^{2}+(3 \theta)^{2}\right) \\
& =18 C_{18} \theta \text {. }
\end{aligned}
$$

Together with Lemma 3.3 (with $\sigma \uparrow 1$ and $T$ replaced by $S_{v}$ ), (8.6) and Definition 4.1 (iii.), we yield

$$
\begin{aligned}
\limsup _{v \rightarrow \infty} \frac{\mathbf{E}_{C}\left(\gamma_{\omega_{v} \#} \# T_{v}, \theta\right)}{\mathfrak{m}_{v}^{2}} & =\limsup _{v \rightarrow \infty} \frac{\mathbf{E}_{C}\left(S_{v}, 1\right)}{\mathfrak{m}_{v}^{2}} \\
& \leq \limsup _{v \rightarrow \infty} C_{10}\left(\frac{C_{11} \sup _{\mathbf{C}_{2} \cap \operatorname{spt}\left(S_{v}\right)} X_{n+1}^{2}}{\mathfrak{m}_{v}^{2}}+\frac{\kappa_{S_{v}}+\mathbf{A}_{\tilde{\mathcal{M}}_{v}}}{\mathfrak{m}_{v}^{2}}\right) \\
& \leq C_{10}\left(C_{11} \limsup _{v \rightarrow \infty} \sup _{\mathrm{C}_{2} \cap \operatorname{spt}\left(S_{v}\right)} \frac{X_{n+1}^{2}}{\mathfrak{m}_{v}^{2}}+\theta \limsup _{v \rightarrow \infty} \frac{\kappa_{v}+\mathbf{A}_{v}}{\mathfrak{m}_{v}^{2}}\right) \\
& \leq(18)^{2} C_{10} C_{11} C_{18}^{2} \theta^{2} \\
& <\theta
\end{aligned}
$$

As $\omega_{v}$ is bounded (see (8.5)), the latter inequality contradicts (8.2) for $v$ large enough.

### 8.1 PROOF OF THEOREM 4.2

Proof. We construct a sequence of currents $\left\{\left(T_{\nu}, \mathcal{M}_{\nu}\right)\right\}_{v \in \mathbb{N}} \subset \mathcal{T}$ and real numbers $\left\{\omega_{v}\right\}_{v \geq 1}$ inductively. We start with $\left(T_{0}, \mathcal{M}_{0}\right):=(T, \mathcal{M})$. Assume that for some fixed $j \in$ $\mathbb{N}$, we already have $\left(T_{j}, \mathcal{M}_{j}\right) \in \mathcal{T}$. Denote by $\mathbf{A}_{j} ;=\mathbf{A}_{\mathcal{M}_{j}}$ and $\mathfrak{m}_{j}:=\max \left\{\sqrt{\mathbf{E}_{\mathcal{C}}\left(T_{j}, 1\right)}, \mathbf{A}_{j}^{1 / 4}\right\}$. By Theorem 8.1, there is a real number $\left|\omega_{j+1}\right| \leq C_{18} \mathfrak{m}_{j}$ such that if we define

$$
T_{j+1}:=\left(\boldsymbol{\mu}_{1 / \theta \#} \gamma_{\omega_{j+1} \#} \not T_{j}\right)\left\llcorner\mathbf{B}_{3} \quad \text { and } \quad \mathcal{M}_{j+1}:=\boldsymbol{\mu}_{1 / \theta}\left(\mathcal{M}_{j}\right)\right.
$$

then $\left(T_{j+1}, \mathcal{M}_{j+1}\right) \in \mathcal{T}$ and by Remark 4.5 (iii.)

$$
\max \left\{\mathbf{E}_{C}\left(T_{j+1}, 1\right), \mathbf{A}_{j+1}, C_{19} \kappa_{T_{j+1}}\right\} \leq \theta \max \left\{\mathbf{E}_{C}\left(T_{j}, 1\right), \mathbf{A}_{j}, C_{19} \kappa_{T_{j}}\right\} .
$$

Using this inequality $j$ times, we deduce

$$
\max \left\{\mathbf{E}_{C}\left(T_{j+1}, 1\right), \mathbf{A}_{j}, C_{19} \kappa_{T_{j+1}}\right\} \leq \theta^{j+1} \max \left\{\mathbf{E}_{C}(T, 1), \mathbf{A}, C_{19} \kappa_{T}\right\} \leq \frac{\theta^{j+2}}{C_{19}}
$$

Moreover, the following holds

$$
\begin{gather*}
\left|\omega_{j+1}\right| \leq C_{18} \sqrt{\frac{\theta^{j+1}}{C_{19}}}  \tag{8.7}\\
\mathbf{E}_{C}\left(T_{j}, 1\right)+\kappa_{T_{j}}+\mathbf{A}_{j} \leq 3 \max \left\{\mathbf{E}_{C}\left(T_{j}, 1\right), \mathbf{A}_{j}, \kappa_{T_{j}}\right\} \leq 3 \frac{\theta^{j+1}}{C_{19}} \tag{8.8}
\end{gather*}
$$

Then we define $\eta_{j}:=\sum_{k=1}^{j} \omega_{k}$ and $\eta:=\lim _{j \rightarrow \infty} \eta_{j}$. This is a valid choice for $\eta$ as (8.7) and the fact that $\theta^{1 / 2} \leq 1 / 2$ implies

$$
|\eta| \leq C_{18} \sum_{k=1}^{\infty} \sqrt{\frac{\theta^{k}}{C_{19}}}=\frac{C_{18}}{\sqrt{C_{19}}} \sum_{k=1}^{\infty}\left(\theta^{1 / 2}\right)^{k}=\frac{C_{18}}{\sqrt{C_{19}}} \frac{\theta^{1 / 2}}{1-\theta^{1 / 2}} \leq 2 \frac{C_{18}}{\sqrt{C_{19}}} \theta^{1 / 2} .
$$

Fix $0<r<\theta / 4$ and choose an appropriate $j \in \mathbb{N}$ such that $\theta^{j+1} \leq 4 r<\theta^{j}$. Then we use the inequalities (8.7), (8.8) together with (a.28) from the proof of Remark 4.5 (iii.) (with $T, \mathcal{M}, \omega$ replaced by $T_{j}, \mathcal{M}_{j}, \eta-\eta_{j}$ ) and the excess monotonicity (3.1) to derive

$$
\begin{aligned}
\mathbf{E}_{C}\left(\gamma_{\eta \#} T, r\right) & \leq\left(\frac{\theta^{j}}{4 r}\right)^{n} \mathbf{E}_{C}\left(\gamma_{\eta \#} T, \frac{\theta^{j}}{4}\right) \leq \theta^{-n} \mathbf{E}_{C}\left(\gamma_{\eta \#} T, \frac{\theta^{j}}{4}\right) \\
& =\theta^{-n} \mathbf{E}_{C}\left(\mu_{4 \#} \gamma_{\eta^{\#}} T, \theta^{j}\right) \\
& =\theta^{-n} \mathbf{E}_{C}\left(\gamma_{\gamma_{j} \#} \mu_{4 \#} \gamma_{\eta-\eta_{j} \#} T, \theta^{j}\right) \\
& =\theta^{-n} \mathbf{E}_{C}\left(\boldsymbol{\mu}_{(1 / \theta) \ddagger \#} \gamma_{\eta_{j} \#} \mu_{4 \#} \gamma_{\eta-\eta_{j} \# T} T, 1\right) \\
& =\theta^{-n} \mathbf{E}_{C}\left(\mu_{4 \#} \gamma_{\eta-\eta_{j} \#} T_{j}, 1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\theta^{-n} \mathbf{E}_{C}\left(\mu_{4 \#} \gamma_{\eta-\eta_{j} \#} T_{j}, 1\right) & \leq \theta^{-n} \frac{C_{21}}{C_{20}}\left(\left(\sum_{k=j+1}^{\infty} \omega_{k}\right)^{2}+\mathbf{E}_{C}\left(T_{j}, 1\right)+\kappa_{T_{j}}+\mathbf{A}_{j}\right) \\
& \leq \theta^{-n} \frac{C_{21}}{C_{20}}\left(\sum_{k=j+1}^{\infty} \omega_{k}^{2}+3 \frac{\theta^{j+1}}{C_{19}}\right) \\
& \leq \theta^{-n} \frac{C_{21}}{C_{20}}\left(\frac{C_{18}^{2}}{C_{19}} \frac{\theta^{j+1}}{1-\theta}+3 \frac{\theta^{j+1}}{C_{19}}\right) \\
& \leq \theta^{-n} \frac{C_{21}}{C_{20}} \frac{3\left(C_{18}^{2}+1\right)}{C_{19}} \theta^{j+1} \\
& \leq \theta^{-n} \frac{C_{21}}{C_{20}} \frac{3\left(C_{18}^{2}+1\right)}{C_{19}}(4 r) \\
& \leq \frac{\theta^{-n-1}}{C_{19}} r .
\end{aligned}
$$

## THE BOUNDARY REGULARITY THEOREM

Theorem 9.1. Let $U \subset \mathbb{R}^{n+k}$ be open and $T$ an $n$-dimensional locally rectifiable current in $U$ that is area minimizing in some smooth $(n+1)$-manifold $\mathcal{M}$ and such that $\partial T$ is an oriented $\mathcal{C}^{2}$ submanifold of $U$. Then for any point $a \in \operatorname{spt}(\partial T)$, there is a neighborhood $V$ of a in $U$ satisfying that $V \cap \operatorname{spt}(T)$ is an embedded $\mathcal{C}^{1, \frac{1}{4}}$ submanifold with boundary.

Hardt and Simon found out, that it is enough to consider currents whose tangent cones at boundary are in fact a tangent planes. Once we have this tangent plane, we can parametrize the support of the current with graphs over the plane.

Lemma 9.2. Let $Q \in \mathcal{R}_{n}^{\text {loc }}\left(\mathbb{R}^{n+1}\right)$ be an absolutely area minimizing cone with $\partial Q=E^{n-1} \times$ $\delta_{0} \times \delta_{0}$. Then, the support of $Q$ is contained in a hyperplane.

Proof. This can be read in the original paper [27, Theorem 11.1, Step II].
Lemma 9.3. Let $U, T$ and $\mathcal{M}$ be as in Theorem 9.1 and assume further that for every $a \in \operatorname{spt}(\partial T)$, there is a tangent cone $C$ at a such that $\operatorname{spt}(C)$ is contained in a hyperplane. Then for any point $a \in \operatorname{spt}(\partial T)$, there is a neighborhood $V$ of a in $U$ satisfying that $V \cap \operatorname{spt}(T)$ is an embedded $\mathcal{C}^{1, \frac{1}{4}}$ submanifold with boundary.

Proof. After some translation, reflection and rotation, we can assume wlog that $a=0$ and the hyperplane is $\left\{(y, 0): y \in \mathbb{R}^{n}\right\} \subset \mathbb{R}^{n+k}$. Hence, for $m=\Theta^{n}(\|T\|, 0)+\frac{1}{2} \in \mathbb{N}$,

$$
\left(m \left(\mathbf{E}^{n}\left\llcorner\left\{y \in \mathbb{R}^{n}: y_{n}>0\right\}\right)+(m-1)\left(\mathbf{E}^{n}\left\llcorner\left\{y \in \mathbb{R}^{n}: y_{n}<0\right\}\right)\right) \times \delta_{0}\right.\right.
$$

is an oriented tangent cone of $T$ at 0 by [23, 4.1.31(2)]. Therefore, we find a nullsequence $\left\{r_{k}\right\}_{k \geq 1} \subset \mathbb{R}_{+}$such that $\mu_{1 / r_{k} \#} T$ converges in $\mathcal{R}_{n}^{\text {loc }}\left(\mathbb{R}^{n+k}\right)$ to this cone as $k \rightarrow \infty$. Moreover, we assume that for every $k$ we have $3 r_{k}<\operatorname{dist}(0, \partial U)$. Then it follows that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sup _{\overline{\mathbf{B}}_{r_{k}} \cap \operatorname{spt}(T)} \frac{X_{n+1}}{r_{k}}=\lim _{k \rightarrow \infty} \sup _{\overline{\mathbf{B}}_{1} \cap \operatorname{spt}\left(\mu_{1} / r_{k}{ }^{\#} T\right)} X_{n+1}=0 . \tag{9.1}
\end{equation*}
$$

By [23, Section 5.4.2], also the associated measures converge weakly and hence,

$$
\begin{aligned}
\lim _{k \rightarrow \infty} r_{k}^{-n} \mathbb{M}\left(T\left\llcorner\left(\mathbf{B}_{3 r_{k}} \cap \mathbf{C}_{r_{k}}\right)\right)\right. & =\lim _{k \rightarrow \infty} \mathbb{M}\left(\left(\boldsymbol{\mu}_{1 / r_{k} \#} T\right)\left\llcorner\left(\mathbf{B}_{3} \cap \mathbf{C}_{1}\right)\right)\right. \\
& =m \mathcal{L}^{n}(\mathbf{V})+(m-1) \mathcal{L}^{n}(\mathbf{W})=\left(m-\frac{1}{2}\right) \boldsymbol{\omega}_{n}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \lim _{k \rightarrow \infty} \left\lvert\, r_{k}^{-n} \mathbb{M}\left(\left.\mathbf{p}_{\#}\left(T\left\llcorner\left(\mathbf{B}_{3 r_{k}} \cap \mathbf{C}_{r_{k}}\right)\right)\right)-\left(m-\frac{1}{2}\right) \boldsymbol{\omega}_{n} \right\rvert\,\right.\right. \\
& \quad \leq \lim _{k \rightarrow \infty} \mid \mathbb{M}\left(\mathbf{p}_{\#}\left(\left(\boldsymbol{\mu}_{1 / r_{k} \#} T\right)\left\llcorner\left(\mathbf{B}_{3} \cap \mathbf{C}_{1}\right)\right)\right)-\mathbb{M}\left(\left(\boldsymbol{\mu}_{1 / r_{k} \#} T\right)\left\llcorner\left(\mathbf{B}_{3} \cap \mathbf{C}_{1}\right)\right) \mid\right.\right. \\
& \quad=0,
\end{aligned}
$$

where we also have used (9.1).
Thus, if we define $T_{k}:=\left(\mu_{1 / r_{k} \sharp T} T\right) \mathbf{B}_{3}$ and $\mathcal{M}_{k}:=\mu_{1 / r_{k}}(\mathcal{M})$, then for $k$ large enough, we have $\left(T_{k}, \mathcal{M}_{k}\right) \in \mathcal{T}$ and

$$
\max \left\{\mathbf{E}_{C}\left(T_{k}, 1\right), C_{19} \kappa_{T_{k}}, \mathbf{A}_{k}\right\} \leq \frac{\theta}{C_{19}} .
$$

Then we can apply Theorem 4.2 (with $T$ replaced by $T_{k}$ ) and notice that we can choose $\eta$ to be zero, to find the decay

$$
\mathbf{E}_{C}\left(T_{k}, r\right) \leq \frac{\theta^{-n-1}}{C_{19}} r .
$$

Now, we differ between two cases.
Case 1: $m=1$. This is a corollary of Allard's interior regularity theorem. However, a self-contained proof could be given from the results of the previous chapters. Observe first that, by Corollary 4.3, in a sufficiently small neighborhood of $x$, the current $T$ is supported in the $\boldsymbol{\Phi}$-graph of $\tilde{v}_{1}$ and so we can assume, wlog, that $\operatorname{spt}(T) \backslash \operatorname{spt}(\partial T)$ is connected. By the Constancy Lemma, it follows that the density $\Theta$ is an an integer constant $k$ at every interior point of such neighborhood. So the current is actually $k$ times the one induced by the $\boldsymbol{\Phi}$-graph of $\tilde{v}_{1}$. However, since the boundary of $T$ is a current with multiplicity 1 we easily conclude that $k$ is actually 1 . The current $T$ is thus the current induced by the $\boldsymbol{\Phi}$-graph of the $C^{1, \frac{1}{4}}$ function $\tilde{v}_{1}$. Notice that there is a neighborhood $U$ of 0 such that $\Theta^{n}(\|T\|, y)=\frac{1}{2}$ for all $y \in U \cap \operatorname{spt}(\partial T)$.

Case 2: $m>1$. We fix $k$ and use Corollary 4.3 with $\gamma_{\eta \#} T$ replaced by $T_{k}$. Hence, we get functions $\tilde{v}_{i}, \tilde{w}_{j}$ whose $\boldsymbol{\Phi}$-graphs around zero form $\operatorname{spt}\left(T_{k}\right)$. Moreover, we know that $D \tilde{v}_{i}(0)=0=D \tilde{w}_{j}$. Hence, similar to the proof of Theorem 7.2, by the E. Hopf boundary point Lemma for quasilinear equations [30, Theorem 2.7.1], we deduce that $\tilde{v}_{m}-\tilde{v}_{1} \equiv 0 \equiv \tilde{w}_{m-1}-\tilde{w}_{1}$. Therefore, they all coincide.
Notice that the regular points of

$$
\mathbf{B}_{r_{k}} \cap(\operatorname{spt}(T) \backslash \operatorname{spt}(\partial T))=\boldsymbol{\mu}_{r_{k}}\left(\mathbf{B}_{1} \cap\left(\operatorname{spt}\left(T_{k}\right) \backslash \operatorname{spt}\left(\partial T_{k}\right)\right)\right) \supseteq \boldsymbol{\mu}_{r_{k}}\left(\operatorname{graph}\left(\tilde{v}_{1}\right) \cup\left(\operatorname{graph}\left(\tilde{w}_{1}\right)\right)\right.
$$

consist of at least two connected components. Let $G$ denote that component of the regular points containing $\boldsymbol{\mu}_{r_{k}}\left(\operatorname{graph}\left(\tilde{v}_{1}\right)\right)$ and consider

$$
S:=\frac{1}{m}(T\llcorner G) .
$$

Notice that by [23, 4.1.31(2)], the density $\Theta(\|T\|, x)$ is constantly $m$ for all $x \in G$. We will show later that on some open neighborhood $V$ of 0 in $U$, we have that $\operatorname{spt}(T)=\operatorname{spt}(T-S), T-S$ has no boundary in $W$ and then, we apply interior regularity theory.
First notice that as $T, S$ are area minimizing in $\mathcal{M}$ and $\|T\|=\|S\|+\|T-S\|$ holds, is follows that $T-S$ is also area minimizing $\mathcal{M}$.
Then, we denote $W:=\mathbf{B}_{r_{k}} \cap \mathbf{C}_{\delta r_{k}}$, where $\delta$ is as in Corollary 4.3, and aim to show that

$$
\begin{equation*}
(\partial S)\llcorner W=(\partial T)\llcorner W \tag{9.2}
\end{equation*}
$$

Notice that

$$
\operatorname{spt}(\partial S) \subset \operatorname{spt}((\partial T)\llcorner G) \cup \operatorname{spt}(T\llcorner(\partial G))
$$

and hence,

$$
\operatorname{spt}((\partial S)\llcorner W) \subset \operatorname{spt}((\partial T)\llcorner W) \cup \operatorname{spt}(T\llcorner(\partial G \cap W))=\operatorname{spt}((\partial T)\llcorner W)
$$

Moreover, we can use the Constancy Theorem [23, Section 4.1.7] to derive

$$
\begin{aligned}
\mathbf{p}_{\#}((\partial S)\llcorner W) & =\left(\partial\left(\frac{1}{m} \mathbf{p}_{\#}(T\llcorner(G \cap W)))\right)\llcorner\mathbf{p}(W)\right. \\
& =\left(\partial \left(\mathbf{E}^{n}\left\llcorner\left\{r_{k} y \in \mathbf{p}(W): y_{n}>\varphi_{T_{k}}\left(y_{1}, \ldots, y_{n-1}\right)\right\}\right)\llcorner\mathbf{p}(W)\right.\right. \\
& =\left(\partial\left(\mathbf{p}_{\#}(T\llcorner W))\right)\llcorner\mathbf{p}(W)\right. \\
& =\mathbf{p}_{\#}((\partial T)\llcorner W) .
\end{aligned}
$$

As the map $\left.\mathbf{p}\right|_{\mathrm{spt}((\partial T)\llcorner W)}$ is a $\mathcal{C}^{2}$-diffeomorphism, (9.2) must hold. Then $T-S$ has no boundary in $W$ and by (9.1), a tangent cone of $T-S$ at 0 is contained in $X_{n+1}^{-1}(0)$. Therefore, we can apply [23, Theorem 5.3.18] to $p_{\#}(T-S)$ and deduce that there is an open neighborhood $V$ of 0 in $U$ such that

$$
V \cap \operatorname{spt}(T)=V \cap \operatorname{spt}(T-S)
$$

is a smooth embedded submanifold of $\mathcal{M}$.
Putting the previous two lemmas together, we deduce the boundary regularity theorem:

Proof of Theorem 9.1. Let $a \in \operatorname{spt}(\partial T)$. Then by [9, Theorem 3.6], $T$ has an absolutely area minimizing tangent cone $Q \in \mathcal{R}_{n}^{l o c}\left(T_{a} \mathcal{M}\right)$ at $a$. After some rotation, we can assume that $\partial Q=(-1)^{n} \mathbf{E}^{n-1} \times \delta_{0} \times \delta_{0}$. By Lemma 9.2, the cone is contained in some hyperplane and by Lemma 9.3, we conclude that $T$ is regular at $a$.

Part II
THE HIGHER MULTiPLICITY CASE

We start recalling the following well known fact:
Proposition 10.1. Assume $T$ is an area minimizing m-dimensional current in $\mathbb{R}^{m+n}$ with $\mathrm{spt}(\partial T)$ compact. Then $\mathrm{spt}(T)$ is contained in the convex hull of $\mathrm{spt}(\partial T)$.

Proof. The statement can be concluded from much stronger ones, for instance we can use that $\|T\|$ is an integral stationary varifold in $\mathbb{R}^{m+n} \backslash \operatorname{spt}(T)$ and invoke [33, Theorem 19.2].

We then take advantage of a simple and elementary fact which combines the regularity of $\Gamma$ with the uniform convexity of the barrier $\Omega$. We will state this fact in higher generality than we actually need in this manuscript.

Definition 10.2. First of all, given an ( $m-1$ )-dimensional plane $V \subset \mathbb{R}^{m+n}$ we denote by $\mathbf{p}_{V}$ the orthonogonal projection onto $V$. Given additionally a unit vector $v$ normal to $V$ and an angle $\vartheta \in\left(0, \frac{\pi}{2}\right)$ we then define the wedge with spine $V$, axis $v$ and opening angle $\vartheta$ as the set

$$
\begin{equation*}
W(V, v, \vartheta):=\left\{y:\left|y-\mathbf{p}_{V}(y)-(y \cdot v) v\right| \leq(\tan \vartheta) y \cdot v\right\} . \tag{10.1}
\end{equation*}
$$



Figure 1: An illustration of the wedge where $V$ is the tangent $T_{q} \Gamma$ to $\Gamma$ at some boundary point $q$, whereas $v$ the interior unit normal $v(q)$ to the convex barrier $\Omega$ at $q$.

In particular we have the following lemma.

Lemma 10.3. Let $\Omega \subset \mathbb{R}^{m+n}$ be a $C^{2}$ bounded open set with uniformly convex boundary and $\Gamma a C^{2}(m-1)$-dimensional submanifold of $\Omega$ without boundary. Then there is $a<\vartheta<\frac{\pi}{2}$ (which depends only on $\Gamma$ and $\Omega$ ) such that the convex hull of $\Gamma$ satisfies

$$
\operatorname{ch}(\Gamma) \subset \bigcap_{q \in \Gamma}\left(q+W\left(T_{q} \Gamma, v(q), \vartheta\right)\right) .
$$

We postpone the proof of the lemma to the end of the section Using Proposition 10.1 and Lemma 10.3 we can reduce Theorem 1.7 to a suitable local statement. In particular we will replace Assumption 1.6 with the following one:

Assumptions 10.4. $Q \geq 1$ is an arbitrary integer and $\vartheta$ a given positive real number smaller than $\frac{\pi}{2}$. $\Gamma$ is a $C^{3, \alpha}$ arc in $U=\mathbf{B}_{1}(0) \subset \mathbb{R}^{2+n}$ with endpoints lying in $\partial \mathbf{B}_{1}(0)^{1}$. Moreover $v: \Gamma \rightarrow \mathrm{S}^{n+1}$ is a $\mathrm{C}^{2, \alpha}$ map such that $v(q) \perp T_{q} \Gamma$. $T$ is a 2 -dimensional area-minimizing integral current in $U$ such that:

$$
\begin{align*}
& (\partial T)\llcorner U=Q \llbracket \Gamma \rrbracket  \tag{10.2}\\
& \operatorname{spt}(T) \subset \bigcap_{q \in \Gamma}\left(q+W\left(T_{q} \Gamma, v(q), \vartheta\right)\right) . \tag{10.3}
\end{align*}
$$

Moreover,

$$
\begin{equation*}
\mathbf{A}:=\|\kappa\|_{L^{\infty}}+\|\dot{v}\|_{L^{\infty}} \leq 1, \tag{10.4}
\end{equation*}
$$

where $\kappa$ denotes the curvature of $\Gamma$ and $\dot{v}$ is the derivative, in the arclength parametrization, of $v$.

Theorem 10.5. Let $\Gamma$ and $T$ be as in Assumption 10.4. Then $\operatorname{Sing}_{b}(T)$ is empty.
Proof of Lemma 10.3. Since $q+W(V, v, \vartheta)$ is a convex set, we just need to show the existence of a $0<\vartheta<\frac{\pi}{2}$ such that $\Gamma \subset\left(q+W\left(T_{q} \Gamma, v, \vartheta\right)\right)$ for every $q \in \Gamma$. The latter is equivalent to show the existence of a constant $C>0$ such that

$$
\left|(p-q)-((p-q) \cdot v(q)) v(q)-\mathbf{p}_{V}(p-q)\right| \leq C((p-q) \cdot v(q)) \quad \forall p, q \in \Gamma . \text { (10.5) }
$$

The strict convexity of $\partial \Omega$ ensures that for every $\varepsilon>0$ there is a constant $C$ such that (10.5) holds if additionally $|p-q| \geq \varepsilon$. Thus we just have to show the inequality for a sufficiently small $\varepsilon$. In order to do that, fix $q$ and assume w.l.o.g. that it is the origin, while at the same time we assume that $T_{q} \Gamma=T_{0} \Gamma=\left\{x_{m}=\ldots=x_{m+n}=0\right\}$ and $v=\frac{\partial}{\partial x_{m+n}}$. We will use accordingly the coordinates $(y, z, w)$, with $y \in \mathbb{R}^{m-1}, z \in \mathbb{R}^{n}$, and $w \in \mathbb{R}$. By the $C^{2}$ regularity of $\Omega$ and $\Gamma$, in a sufficiently small ball $\mathbf{B}_{\varepsilon}(q)=\mathbf{B}_{\varepsilon}(0)$ the points $p$ in $\Gamma$ are described by

$$
\begin{equation*}
p=(y, z, w)=(y, f(y), g(y, f(y))) \tag{10.6}
\end{equation*}
$$

1 I.e. $\Gamma=\hat{\gamma}([0,1])$ where $\hat{\gamma}:[0,1] \rightarrow \overline{\mathbf{B}_{1}(0)}$ is a $C^{3, \alpha}$ diffeomorphism onto its image.
for some $f$ and $g$ which are $C^{2}$ functions. Observe that $f(0)=0, D f(0)=0, g(0)=0$, and $D g(0)=0$. Moreover $\left\|D^{2} f\right\|_{C^{0}} \leq C_{0}$ and $D^{2} g \geq c_{0} \mathrm{I} d$ for constants $c_{0}>0$ and $C_{0}$, which depend only on $\Gamma$ and $\Omega$. Similarly, the size of the radius $\varepsilon$ in which the formula (10.6) and the estimates are valid depends only on $\Omega$ and $\Gamma$ and not on the choice of the point $q$. Next, compute

$$
((p-q) \cdot v(q))=g(y, f(y)) \geq c_{0}\left(|y|^{2}+|f(y)|^{2}\right) \geq c_{0}|y|^{2}
$$

and

$$
\left|(p-q)-((p-q) \cdot v(q)) v(q)-\mathbf{p}_{V}(p-q)\right|=|f(y)| \leq C_{0}|y|^{2} .
$$

The desired inequality is then valid for $C:=\frac{C_{0}}{c_{0}}$.

We start recalling Allard's boundary monotonicity formula. More specifically, we first define

Definition 11.1. For every point $p \in \mathbf{B}_{1}$, we define the density of $T$ at the point $p$

$$
\Theta(T, p):=\lim _{r \downarrow 0} \frac{\|T\|\left(\mathbf{B}_{r}(p)\right)}{\pi r^{2}}
$$

whenever the latter limit exists.
Next, we introduce the notation $\kappa$ for the curvature of $\Gamma$ and we consider the functions $\Theta_{\mathrm{i}}(T, p, r)$ and $\Theta_{\mathrm{b}}(T, p, r)$ given by

$$
\begin{align*}
\Theta_{\mathrm{i}}(T, p, r) & :=\frac{\|T\|\left(\mathbf{B}_{r}(p)\right)}{\pi r^{2}}  \tag{11.1}\\
\Theta_{\mathrm{b}}(T, p, r) & :=\exp \left(C_{0}\|\kappa\|_{0} r\right) \frac{\|T\|\left(\mathbf{B}_{r}(p)\right)}{\pi r^{2}} \tag{11.2}
\end{align*}
$$

where $C_{0}=C_{0}(n)$ is a suitably large constant.
Theorem 11.2. Let $T$ be as in Assumption 10.4.
(a) If $p \in \mathbf{B}_{1} \backslash \Gamma$, then $r \mapsto \Theta_{\mathrm{i}}(T, p, r)$ is monotone on $(0, \min \{\operatorname{dist}(p, \Gamma), 1-|p|\})$,
(b) if $p \in \mathbf{B}_{1} \cap \Gamma$, then $r \mapsto \Theta_{\mathrm{b}}(T, p, r)$ is monotone on $(0,1-|p|)$.

Thus the density exists at every point of $\mathbf{B}_{1}$. Moreover, the restrictions of the map $p \mapsto \Theta(T, p)$ to $\Gamma \cap \mathbf{B}_{1}$ and to $\mathbf{B}_{1} \backslash \Gamma$ are both upper semicontinuous.

If $X \in C_{c}^{1}\left(\mathbf{B}_{1}, \mathbb{R}^{2+n}\right)$, then the first variation of $T$ with respect to $X$ satisfies

$$
\begin{equation*}
\delta T(X)=Q \int_{\Gamma} X \cdot \vec{n}(x) d \mathcal{H}^{1}(x) \tag{11.3}
\end{equation*}
$$

where $\vec{n}$ is a Borel vector field with $|\vec{n}| \leq 1$.
Moreover, if $p \in \Gamma$ and $0<s<r<1-|p|$, we then have the following precise monotonicity identity

$$
\begin{align*}
& r^{-2}\|T\|\left(\mathbf{B}_{r}(p)\right)-s^{-2}\|T\|\left(\mathbf{B}_{s}(p)\right)-\int_{\mathbf{B}_{r}(p) \backslash \mathbf{B}_{s}(p)} \frac{\left|(x-p)^{\perp}\right|^{2}}{|x-p|^{4}} d\|T\|(x) \\
= & Q \int_{s}^{r} \int_{\Gamma \cap \mathbf{B}_{\rho}(p)}(x-p) \cdot \vec{n}(x) d \mathcal{H}^{1}(x) d \rho \tag{11.4}
\end{align*}
$$

where $Y^{\perp}(x)$ denotes the component of the vector $Y(x)$ orthogonal to the tangent plane of $T$ at $x$ (which is oriented by $\vec{T}(x)$ ).

Note that $\delta T(X)=0$ for $X \in C_{c}^{1}\left(\mathbf{B}_{1} \backslash \Gamma\right)$ follows in a straightforward way from the minimality property of $T$. In particular $\|T\|$ is a stationary integral varifold in $\mathbf{B}_{1} \backslash \Gamma$ and (a) and (b) are consequences of the celebrated works of Allard, cf. [2] and [3]. Next note that (11.3) follows from (11.4) arguing, for instance, as in [7] for [7, Eq. (31)] (see [2, 3] as well). Coming to (11.3), note first that the derivation of [21, (3.8)] is valid under our assumptions, with the additional information $\delta T=\delta T_{s}$ (following the terminology and notation of [21, Section 3]). We then just need to show that $\left\|\delta T_{s}\right\| \leq Q \cdot \mathcal{H}^{1}\llcorner\Gamma$. The latter follows easily arguing as in [21, Section 3.4] once we have shown that $\Theta(T, p)=\frac{Q}{2}$ at every $p \in \Gamma$, see below.

As in [21, Section 3] we introduce the following notation and terminology.
Definition 11.3. Fix a point $p \in \operatorname{spt}(T)$ and define for all $r>0$

$$
\iota_{p, r}(q):=\frac{q-p}{r}
$$

We denote by $T_{p, r}$ the currents

$$
T_{p, r}:=\left(\iota_{p, r}\right)_{\sharp} T .
$$

We call the current $T_{p, r}$ the blow up at the point $p$ and scale $r$ of $T$. Let $T_{0}$ be a current such that there exists a sequence $r_{k} \rightarrow 0$ of radii such that $T_{p, r_{k}} \rightarrow T_{0}$, we say that $T_{0}$ is a tangent cone to $T$ at $p$.

We recall the following consequence of the Allard's monotonicity formula, cf. [3].
Theorem 11.4. Let $T$ be as in Assumption 10.4 or as in Theorem 1.4. Fix $p \in \operatorname{spt}(T)$ and take any sequence $r_{k} \downarrow 0$. Up to subsequences $T_{p, r_{k}}$ is converging locally in the sense of currents to an area-minimizing integral current $T_{0}$
(a) $T_{0}$ is a cone with vertex 0 and $\left\|T_{0}\right\|\left(\mathbf{B}_{1}(0)\right)=\pi \Theta(T, p)$;
(b) if $p \in \operatorname{spt}(T) \backslash \Gamma$, then $\partial T_{0}=0$;
(c) if $p \in \Gamma$, then $\partial T_{0}=Q \llbracket T_{p} \Gamma \rrbracket$.

Moreover $\left\|T_{p, r_{k}}\right\|$ converges, in the sense of measures, to $\left\|T_{0}\right\|$.
We next show the following elementary fact:
Theorem 11.5. Let $T$ be as in Assumption 10.4 and $p \in \Gamma$. Any tangent cone $T_{0}$ at $p \in \Gamma$ has then the following properties:
(a) $\operatorname{spt}\left(T_{0}\right)$ is contained in $W\left(T_{p} \Gamma, v(p), \vartheta\right)$ (where $v(p)$ and $\vartheta$ are the vector and the constant given in Assumption 10.4);
(b) There are $k_{1}, \ldots k_{N} \in \mathbb{N} \backslash\{0\}$ and 2-dimensional distinct oriented half-planes $V_{1}, \ldots, V_{N}$ with $\partial \llbracket V_{i} \rrbracket=\llbracket T_{p} \Gamma \rrbracket$ such that

$$
\begin{equation*}
T_{0}=\sum_{i} k_{i} \llbracket V_{i} \rrbracket . \tag{11.5}
\end{equation*}
$$

Note in particular that $2 \Theta(T, p)=Q=\sum_{i} k_{i}$, and thus $1 \leq N \leq Q$.
Conclusion (b) holds under the assumptions of Theorem 1.4 provided we choose $p$ sufficiently close to $q$.

The first part of the theorem is in fact at the same time a particular case of a more general theorem of Allard in higher dimensions (under Assumption 10.3) and of a general classification of all 2-dimensional area-minimizing cones with $\partial T_{0}=Q \llbracket \ell \rrbracket$, where $\ell$ is a straight line, given [10]. In particular since point (a) is obvious, point (b) is a direct corollary of [10, Proposition 4.1] and of (a). As for the second part of the statement, observe that, by [10, Proposition 4.1], $2 \Theta(T, p)$ is always an integer no smaller than $Q$. Recalling that $\Gamma \ni p \mapsto \Theta(T, p)$ is upper semicontinuous, under the assumptions of Theorem 1.4 we must necessarily have $\Theta(T, P)=\frac{Q}{2}$ for every $p$ sufficiently close to $q$. Then conclusion (b) follows again from [10, Proposition 4.1]. Since it will be useful later, we introduce a notation for the cones as in (11.5).

Definition 11.6. Let $\ell \subset \mathbb{R}^{2+n}$ be a 1 - dimensional line passing through the origin and let $Q \in \mathbb{N} \backslash\{0\}$. We denote by $\mathscr{B}_{Q}(\ell)$ the set of area minimizing cones of the form $T=\sum_{i=1}^{N} k_{i} \llbracket V_{i} \rrbracket$, for any finite collection of distinct half-planes $V_{i}$ such that $\partial \llbracket V_{i} \rrbracket=\llbracket \ell \rrbracket$ and any finite collection of positive integers $\left\{k_{i}\right\}_{i=1}^{N}$ such that $\sum_{i=1}^{N} k_{i}=Q$. Moreover we will call such cones open books.

In this section we appeal to [10, Theorem 1.1], which follows the ideas of Hirsch and Marini in [29], in order to claim that the tangent cone to $T$ at $p \in \Gamma$ is unique.

Theorem 12.1. Let $T$ and $\Gamma$ be as in Assumption 10.4. Then the tangent cone at each $p \in \Gamma$ is unique and from now on will be denoted by $T_{p, 0}$. The same conclusion holds under the assumptions of Theorem 1.4 provided $q$ is sufficiently close to $p$.

In fact such a uniqueness theorem comes with a power-law decay (cf. [10, Theorem 2.1]), which in turn allows us to decompose the current at any point $p \in \Gamma$ where the tangent cone is not contained in a single half-plane. Before coming to its statement, we introduce the following terminology.

Definition 12.2. Let $T$ and $\Gamma$ be as in Assumption 10.4. If the tangent cone $T_{p, 0}$ to $T$ at $p \in \Gamma$ is of the form $Q \llbracket V \rrbracket$ for some 2-dimensional half-plane $V$, then $p$ is called a flat boundary point.

Theorem 12.3 (Decomposition). Let $T$ and $\Gamma$ be:

- either as in Assumption 10.4,
- or either as in Theorem 1.4.

Assume that $p \in \Gamma$ is not a flat boundary point and in the second case assume further that $p$ is sufficiently close to $q$. Then there is $\rho>0$ with the following property. There are two positive integers $Q_{1}$ and $Q_{2}$ and two area-minimizing currents $T_{1}$ and $T_{2}$ in $\mathbf{B}_{\rho}(p)$ such that:
(a) $T_{1}+T_{2}=T\left\llcorner\mathbf{B}_{\rho}(p)\right.$ (thus $Q_{1}+Q_{2}=Q$ ),
(b) $\partial T_{i}\left\llcorner\mathbf{B}_{\rho}(p)=Q_{i} \llbracket \Gamma \cap \mathbf{B}_{\rho}(p) \rrbracket\right.$,
(c) $\operatorname{spt}\left(T_{1}\right) \cap \operatorname{spt}\left(T_{2}\right)=\Gamma \cap \mathbf{B}_{\rho}(p)$,
(d) at each point $q \in \mathbf{B}_{\rho}(p)$ the tangent cones to $T_{1}$ and $T_{2}$ have only the line $T_{q} \Gamma$ in common, i.e., $\left(T_{1}\right)_{q, 0} \in \mathscr{C}_{\text {min }, \mathrm{Q}_{1}}\left(T_{q} \Gamma\right)$ and to $\left(T_{2}\right)_{q, 0} \in \mathscr{C}_{\text {min, } \mathrm{Q}_{2}}\left(T_{q} \Gamma\right)$.

At flat points we are not able to decompose the current further and in fact the final byproduct of the regularity theory of this paper is that in a neighborhood of each flat point, the current is supported in a single smooth minimal sheet. For the moment the uniqueness of the tangent cones (and the corresponding decay from which we derive it) allows us to draw the following conclusion.

Theorem 12.4. Let $T$ and $\Gamma$ be as in Assumption 10.4 or as in Theorem 1.4. Assume that $p \in \Gamma$ is a flat boundary point, that $Q \llbracket V \rrbracket$ is the unique tangent cone of $T$ at $p$, and, in the case of Theorem 1.4 that $p$ is sufficiently close to $q$. Let $n(p) \in V$ be the unit normal to $\Gamma$ at $p$ and define in a neighborhood of $p$

$$
\begin{equation*}
n(q)=\frac{n(p)-n(p) \cdot \tau(q) \tau(q)}{|n(p)-n(p) \cdot \tau(q) \tau(q)|} \tag{12.1}
\end{equation*}
$$

where $\tau$ is the unit tangent vector to $\Gamma$ orienting it.
Then, for every $\theta>0$ there is a $\rho>0$ such that

$$
\begin{equation*}
\mathrm{spt}(T) \cap \mathbf{B}_{\rho}(p) \subset \bigcap_{q \in \mathbf{B}_{\rho}(p) \cap \Gamma}\left(q+W\left(T_{q} \Gamma, n(q), \theta\right)\right) \tag{12.2}
\end{equation*}
$$

The previous two theorems allow us to reduce both Theorem 10.5 and Theorem 1.4 to the following simpler statement. We postpone the proof to Section 12.3.

Assumptions 12.5. $Q \geq 1$ is an arbitrary integer and $\vartheta$ a given positive real number smaller than $\frac{\pi}{2}$. $\Gamma$ is a $C^{3, \alpha}$ arc in $\mathbf{B}_{1}(0) \subset \mathbb{R}^{2+n}$ with endpoints lying in $\partial \mathbf{B}_{1}(0)$. $T$ is a 2 -dimensional area-minimizing integral current in $U$ such that $(\partial T)\llcorner U=Q \llbracket \Gamma \rrbracket .0 \in \Gamma$ is a flat point, $Q \llbracket V \rrbracket$ is the unique tangent cone to $T$ at 0 and we let $n$ be as in (12.1). Moreover

$$
\begin{equation*}
\operatorname{spt}(T) \subset \bigcap_{q \in \mathbf{B}_{1}(0) \cap \Gamma}\left(q+W\left(T_{q} \Gamma, n(q), \vartheta\right)\right) \tag{12.3}
\end{equation*}
$$

where $\vartheta$ is a small constant.
Theorem 12.6. Let $T$ and $\Gamma$ be as in Assumption 12.5. Then there is a neighborhood $U$ of 0 and a smooth minimal surface $\Sigma$ in $U$ with boundary $\Gamma$ such that $T\llcorner U=Q \llbracket \Sigma \rrbracket$.

Obviously the latter theorem implies as well Theorem 1.5.

### 12.1 DECAY TOWARDS THE CONE

We first state a more precise version of Theorem 12.1. To that end we recall the flat norm $\mathcal{F}$ and the definition of spherical excess. Given an integral 2-dimensional current $S$ we set

$$
\mathcal{F}(S):=\inf \left\{\mathbf{M}(P)+\mathbf{M}(R): S=\partial P+R, R \in \mathbf{I}_{2}, P \in \mathbf{I}_{3}\right\} .
$$

Moreover, for $T$ as in Assumption 10.4 and $p \in \Gamma$ we define the spherical excess $e(p, r)$ at the point $p$ and with radius $r$ by

$$
\begin{equation*}
e(p, r):=\frac{\|T\|\left(\mathbf{B}_{r}(p)\right)}{\pi r^{2}}-\Theta(T, p)=\frac{\|T\|\left(\mathbf{B}_{r}(p)\right)}{\pi r^{2}}-\frac{Q}{2} . \tag{12.4}
\end{equation*}
$$

We are now ready to state the main decay theorem. Its proof follows the ideas of [29], but it is in fact a consequence of a more general result, which is proved separately in our work [10], cf. [10, Theorem 2.1].

Theorem 12.7. Let $T$ and $\Gamma$ be as in Theorem 12.1. Then there are positive constants $\varepsilon_{0}, C$ and $\alpha$ with the following property. If $p \in \Gamma$ and $e(p, r) \leq \varepsilon_{0}^{2}$ for some $r \leq \operatorname{dist}\left(p, \partial \mathbf{B}_{1}\right)$, then:
(a) $|e(p, \rho)| \leq C|e(p, r)|\left(\frac{\rho}{r}\right)^{2 \alpha}+C \rho^{2 \alpha}$ for every $\rho \leq r$,
(b) There is a unique tangent cone $T_{p, 0}$ to $T$ at $p$,
(c) The following estimates hold for every $\rho \leq r$

$$
\begin{align*}
& \mathcal{F}\left(T _ { p , \rho } \left\llcorner\mathbf{B}_{1}, T_{p, 0}\left\llcorner\mathbf{B}_{1}\right) \leq C(r)|e(p, r)|^{1 / 2}\left(\frac{\rho}{r}\right)^{\alpha}+C \rho^{\alpha},\right.\right.  \tag{12.5}\\
& \operatorname{dist}_{H}\left(\operatorname{spt}\left(T_{p, \rho}\right) \cap \overline{\mathbf{B}}_{1}, \operatorname{spt}\left(T_{p, 0}\right) \cap \overline{\mathbf{B}}_{1}\right) \leq C\left(\frac{\rho}{r}\right)^{\alpha} . \tag{12.6}
\end{align*}
$$

### 12.2 FROM THEOREM 12.7 TO THEOREM 12.3

We fix a point $p$ as in the statement of Theorem 12.3, we choose a radius $r_{0}$ so that $\mathbf{B}_{2 r_{0}}(p) \subset \mathbf{B}_{1}(0)$. We fix thus $\varepsilon_{0}, \alpha$ and $C$ given by Theorem 12.7. Moreover, in order to simplify the notation, we write $T_{p}$ rather than $T_{p, 0}$ for the unique tangent cone to $T$ and $p$.

First of all we observe that

$$
\begin{aligned}
e\left(q, r_{0}\right) & =\frac{\|T\|\left(B_{r_{0}}(q)\right)}{\pi r_{0}^{2}}-\frac{Q}{2} \leq \frac{\|T\|\left(B_{r_{0}+|p-q|}(p)\right)}{\pi r_{0}^{2}}-\frac{Q}{2} \\
& =\left(\frac{r_{0}+|p-q|}{r_{0}}\right)^{2} e\left(p, r_{0}+|p-q|\right)+\left(\left(\frac{r_{0}+|p-q|}{r_{0}}\right)^{2}-1\right) \frac{Q}{2}
\end{aligned}
$$

In particular, if $r_{0}$ is chosen sufficiently small, we can assume that $e\left(q, r_{0}\right) \leq 5 \varepsilon_{0}^{2}$ for every point $q \in \Gamma \cap \mathbf{B}_{r_{0}}(p)$. The rest of the proof is divided into three steps

In a first step we compare tangent cones between different points and prove

$$
\begin{equation*}
\mathcal{F}\left(T _ { q } \left\llcorner\mathbf{B}_{1}, T_{p}\left\llcorner\mathbf{B}_{1}\right) \leq C|q-p|^{\alpha} \quad \forall q \in \mathbf{B}_{r_{0}}(p) .\right.\right. \tag{12.7}
\end{equation*}
$$

Next, since $T_{p}$ is not flat by assumption and because of the classification of tangent cones, we can find half-planes $V$ and $V_{1}, \ldots V_{N}$ all distinct, such that

$$
\begin{equation*}
T_{p}=Q_{1} \llbracket V \rrbracket+\sum_{i} \bar{Q}_{i} \llbracket V_{i} \rrbracket, \tag{12.8}
\end{equation*}
$$

where $Q_{1}<Q$ and $Q_{2}:=Q-Q_{1}=\sum_{i} \bar{Q}_{i}>0$. Let $n$ be the unit vector in $V$ which is orthogonal to $T_{p} \Gamma$. We then infer the existence of a positive $\vartheta_{0}$ with the property that

$$
\begin{equation*}
\bigcup_{i} V_{i} \subset \overline{\mathbb{R}^{2+n} \backslash W\left(T_{p} \Gamma, n, 8 \vartheta_{0}\right)}=: W^{c}\left(T_{p} \Gamma, n, 8 \vartheta_{0}\right) . \tag{12.9}
\end{equation*}
$$

For every point $q \in \Gamma$ sufficiently close to $p$ we project $n$ onto the orthogonal complement of $T_{q} \Gamma$ and normalize it to a unit vector $n(q)$. (12.7) will then be used to show the existence of $r>0$ such that

$$
\begin{equation*}
\operatorname{spt}\left(T_{q}\right) \subset W\left(T_{q} \Gamma, n(q), 2 \vartheta_{0}\right) \cup W^{c}\left(T_{q} \Gamma, n(q), 7 \vartheta_{0}\right) \quad \forall q \in \Gamma \cap \mathbf{B}_{r}(p) . \tag{12.10}
\end{equation*}
$$

Hence we use (12.5) to show the existence of $\bar{r}>0$ such that

$$
\begin{equation*}
\operatorname{spt}(T) \cap \mathbf{B}_{\bar{r}}(q) \subset\left(q+W\left(T_{q} \Gamma, n(q), 3 \vartheta_{0}\right)\right) \cup\left(q+W^{c}\left(T_{q} \Gamma, n(q), 6 \vartheta_{0}\right)\right) \tag{12.11}
\end{equation*}
$$

(12.11) allows us to define

$$
\begin{align*}
& T_{1}:=T\left\llcorner\left(\mathbf{B}_{\bar{r}}(p) \cap \bigcap_{q}\left(q+W\left(T_{q} \Gamma, n(q), 3 \vartheta_{0}\right)\right)\right),\right.  \tag{12.12}\\
& T_{2}:=T\left\llcorner\left(\mathbf{B}_{\bar{r}}(p) \cap \bigcap_{q}\left(q+W^{c}\left(T_{q} \Gamma, n(q), 6 \vartheta_{0}\right)\right)\right),\right. \tag{12.13}
\end{align*}
$$

and to show that $T_{1}+T_{2}=T\left\llcorner\mathbf{B}_{\bar{r}}(p)\right.$ and that each of the $T_{i}$ is area-minimizing. The final step is then to prove that

$$
\begin{equation*}
\partial T_{1}\left\llcorner\mathbf{B}_{\bar{r}}(p)=Q_{1} \llbracket \Gamma \cap \mathbf{B}_{\bar{r}}(p) \rrbracket .\right. \tag{12.14}
\end{equation*}
$$

Step 1. Proof of (12.7) In order to prove (12.7) set $\rho_{0}:=|p-q|$ and observe that, it suffices to show the estimate

$$
\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}, T_{q, \rho}\left\llcorner\mathbf{B}_{1}\right) \leq C \rho^{\alpha}\right.\right.
$$

for some $\rho \in\left[\rho_{0}, 2 \rho_{0}\right]$, whose choice will be specified later. For $v \in \mathbb{R}^{2+n}$, denote by $\tau_{v}$ the translation by the vector $v$. If we choose $v:=(q-p) / \rho$ it is easy to see that $T_{q, \rho}\left\llcorner\mathbf{B}_{1}=\left(\tau_{-v}\right) \sharp\left(T_{p, \rho}\left\llcorner\mathbf{B}_{1}(v)\right)\right.\right.$ and since the flat norm is invariant under translations, we get

$$
\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}, T_{q, \rho}\left\llcorner\mathbf{B}_{1}\right)=\mathcal{F}\left(( \tau _ { v } ) _ { \sharp } \left(T_{p}\left\llcorner\mathbf{B}_{1}(0)\right), T_{p, \rho}\left\llcorner\mathbf{B}_{1}(v)\right) .\right.\right.\right.\right.
$$

On the other hand, observe that $T_{p}$ is invariant by translation along $T_{p} \Gamma$ and that, if we write $v=w+\mathbf{p}_{T_{p} \Gamma}(v)=: w+z$, then $|w| \leq C \rho$. Hence we have

$$
\begin{aligned}
& \mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}, T_{q, \rho}\left\llcorner\mathbf{B}_{1}\right)=\right.\right. \mathcal{F}\left(( \tau _ { w } ) _ { \sharp } \left(T_{p}\left\llcorner\mathbf{B}_{1}(z)\right), T_{p, \rho}\left\llcorner\mathbf{B}_{1}(v)\right)\right.\right. \\
& \leq \mathcal{F}\left(( \tau _ { w } ) _ { \sharp } \left(T_{p}\left\llcorner\mathbf{B}_{1}(z)\right), T_{p}\left\llcorner\mathbf{B}_{1}(z)\right)+\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}(z), T_{p}\left\llcorner\mathbf{B}_{1}(v)\right)\right.\right.\right.\right. \\
&+\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}(v), T_{p, \rho}\left\llcorner\mathbf{B}_{1}(v)\right) .\right.\right.
\end{aligned}
$$

The first two summands can be easily estimated with $C \rho$. Indeed for the first term we write

$$
\left(\tau_{w}\right)_{\sharp}\left(T_{p}\left\llcorner\mathbf{B}_{1}(z)\right)-T_{p}\left\llcorner\mathbf{B}_{1}(z)=\partial\left(\left(T_{p}\left\llcorner\mathbf{B}_{1}(z)\right) \times \llbracket[0, w] \rrbracket\right)=: \partial Z\right.\right.\right.
$$

and we estimate $\mathbf{M}(Z) \leq C|w| \leq C \rho$, whereas for the second term we can estimate directly

$$
\mathbf{M}\left(T _ { p } \left\llcorner\mathbf{B}_{1}(z)-T_{p}\left\llcorner\mathbf{B}_{1}(v)\right) \leq C|w| .\right.\right.
$$

It remains to bound the third summand. To that end we employ the fact that we are free to choose $\rho \in\left[\rho_{0}, 2 \rho_{0}\right]$ appropriately. Note that the point $v$ depends on $\rho$ : we will therefore write $v(\rho)$ from now on and use $v_{0}$ for $v\left(\rho_{0}\right)$, while we define $\sigma:=\frac{\rho}{\rho_{0}}$. By a simple rescaling argument we observe that

$$
\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}(v(\rho)), T_{p, \rho}\left\llcorner\mathbf{B}_{1}(v(\rho)) \leq \mathcal{C}\left(T _ { p } \left\llcorner\mathbf{B}_{\sigma}\left(v_{0}\right), T_{p, \rho_{0}}\left\llcorner\mathbf{B}_{\sigma}\left(v_{0}\right)\right) \quad \text { for all } \sigma \in[1,2] .\right.\right.\right.\right.\right.
$$

We complete the proof by showing that, if $\sigma$ is chosen appropriately, then

$$
\begin{equation*}
\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{\sigma}\left(v_{0}\right), T_{p, p_{0}}\left\llcorner\mathbf{B}_{\sigma}\left(v_{0}\right)\right) \leq C \mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{3}(0), T_{p, p_{0}}\left\llcorner\mathbf{B}_{3}(0)\right),\right.\right.\right.\right. \tag{12.15}
\end{equation*}
$$

since, again using a simple scaling argument, we can estimate

$$
\mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{3}(0), T_{p, \rho_{0}}\left\llcorner\mathbf{B}_{3}(0)\right) \leq C \mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{1}(0), T_{p, 3 \rho_{0}}\left\llcorner\mathbf{B}_{1}(0)\right)\right.\right.\right.\right.
$$

and take advantage of (12.5). In order to show (12.15), fix currents $R$ and $S$ such that $\left(T_{p}-T_{p, \rho_{0}}\right)\left\llcorner\mathbf{B}_{3}(0)=R+\partial S\right.$ with

$$
\mathbf{M}(R)+\mathbf{M}(S) \leq 2 \mathcal{F}\left(T _ { p } \left\llcorner\mathbf{B}_{3}(0), T_{p, p_{0}}\left\llcorner\mathbf{B}_{3}(0)\right) .\right.\right.
$$

Let now $d(x):=\left|x-v_{0}\right|$ and for every $\sigma$ we can then use the slicing formula [33, Lemma 28.5] to write

$$
\left(T_{p}-T_{p, p_{0}}\right)\left\llcorner\mathbf{B}_{\sigma}\left(v_{0}\right)=R\left\llcorner\mathbf{B}_{\sigma}(v)+\partial\left(S\left\llcorner\mathbf{B}_{\sigma}\left(v_{0}\right)\right)-\langle S, d, \sigma\rangle .\right.\right.\right.
$$

Since

$$
\int_{1}^{2} \mathbf{M}(\langle S, d, \sigma\rangle) d \sigma \leq \mathbf{M}\left(S\left\llcorner\mathbf{B}_{2}\left(v_{0}\right)\right) \leq \mathbf{M}(S),\right.
$$

it suffices to choose a $\sigma$ for which $\mathbf{M}(\langle S, d, \sigma\rangle) \leq 2 \mathbf{M}(S)$.
Step 2. Proof of (12.11) The latter is a simple consequence of the estimates proved in the previous two steps and of (12.6) and is left to the reader.

Step 3. Proof of (12.14) Observe that $\partial T_{1}\left\llcorner\mathbf{B}_{\bar{r}}(p)\right.$ is supported in $\Gamma \cap \mathbf{B}_{\bar{r}}(p)$ and is a flat chain without boundary in $\mathbf{B}_{\bar{r}}(p)$. By the Constancy Lemma of Federer [23, Section 4.1.7], it follows that $\partial T_{1}\left\llcorner\mathbf{B}_{\bar{r}}(p)=\Theta \llbracket \Gamma \cap \mathbf{B}_{\bar{r}}(p) \rrbracket\right.$ for some constant $\Theta$. In particular $T_{1}$ is integral and thus $\Theta$ is an integer. Since it is area minimizing, it follows from our analysis that $T_{1}$ has a unique tangent cone $\left(T_{1}\right)_{p}$ at $p$ and that $\pi \Theta$ equals twice the mass of $\left(T_{1}\right)_{p}$ in $\mathbf{B}_{1}(0)$. On the other hand the latter cone is the restricion of $T_{p}$ to $W\left(T_{p} \Gamma, n(p), 3 \vartheta_{0}\right)$, which by assumption is $Q_{1} \llbracket V \rrbracket$ for a fixed half-plane $V$ with boundary $T_{p} \Gamma$. Thus $\Theta=Q_{1}$, which completes the proof.

### 12.3 FROM THEOREM 12.6 TO THEOREM 10.5

In this subsection we show how to conclude Theorem 10.5 from Theorem 12.6 and Theorem 12.3. We argue by induction on $Q$. We start observing that for $Q=1$ there are no boundary singular points, as it can be concluded by [3]. Assume therefore that Theorem 10.5 holds for all $Q$ strictly smaller than some fixed positive integer $\bar{Q}$ : our aim is to show that it holds for $Q=\bar{Q}$. First of all observe that by Theorem 12.3 we know that the set $F:=\{p \in \Gamma: p$ is a flat boundary point $\}$ is closed in $\Gamma$. If $F=\Gamma$, then $T$ has no boundary singularities. Otherwise, by Theorem 12.6(a), it suffices to show that the dimension of $\operatorname{Sing}_{b}(T) \backslash F$ is 0 . It then suffices to show that for every $p \in \Gamma \backslash F$ there is a radius $\rho$ such that $\operatorname{Sing}_{b}(T) \cap \mathbf{B}_{\rho}(p)$ has dimension 0 . Fix $\rho$ as in Theorem 12.3 and let $T_{1}$ and $T_{2}$ satisfy the conclusion of that theorem. We claim that

$$
\begin{equation*}
\operatorname{Sing}_{b}(T) \cap \mathbf{B}_{\rho}(p) \subset \operatorname{Sing}_{b}\left(T_{1}\right) \cup \operatorname{Sing}_{b}\left(T_{2}\right) \tag{12.16}
\end{equation*}
$$

Since by the induction hypothesis each $\operatorname{Sing}_{b}\left(T_{i}\right)$ has dimension 0 , the latter claim would conclude the proof. In order to show (12.16), consider a point $q$ which is a boundary regular point for both $T_{1}$ and $T_{2}$ : we aim to prove that $q$ is a regular point for $T$ as well. By the very definition of boundary regular point, for each $i$ there is a neighborhood $U_{i} \subset \mathbf{B}_{\rho}(p)$ of $p$, minimal surfaces $\Lambda_{j}^{i}$, and integer coefficients $k_{j}^{i}$ such that:

- $T_{i}\left\llcorner U_{i}=\sum_{j} k_{j}^{i} \llbracket \Lambda_{j}^{i} \rrbracket ;\right.$
- $\Lambda_{j}^{i} \cap \Lambda_{k}^{i} \subset \Gamma$ for every $j \neq k$;
- the tangents of $\Lambda_{j}^{i}$ at every point $\bar{q} \in \Gamma \cap U$ are all distinct.

Now, in $U:=U_{1} \cap U_{2}$ we clearly have

$$
T\left\llcorner U=\sum_{i=1}^{2} \sum_{j} k_{j}^{i} \llbracket \Lambda_{j}^{i} \cap U \rrbracket .\right.
$$

Note that, by Theorem 12.3(c) $\Lambda_{j}^{1} \cap \Lambda_{k}^{2} \subset \operatorname{spt}\left(T_{1}\right) \cap \operatorname{spt}\left(T_{2}\right) \subset \Gamma$ for every $j \neq k$. Moreover, if $\bar{q} \in \Gamma \cap U$, then $\left(T_{1}\right)_{\bar{q}, 0}=\sum_{j} k_{j}^{1} \llbracket T_{\bar{q}} \Lambda_{j}^{1} \rrbracket$ and $\left(T_{2}\right)_{\bar{q}, 0}=\sum_{k} k_{k}^{2} \llbracket T_{\bar{q}} \Lambda_{k}^{2} \rrbracket$. We conclude from Theorem 12.3(d) that for every $j$ and $k$ the half planes $T_{\bar{q}} \Lambda_{j}^{1}$ and $T_{\bar{q}} \Lambda_{k}^{2}$ are distinct, i.e. intersect only in $T_{\bar{q}} \Gamma$. This shows that $q$ is then a boundary regular point of $T$.

## 13

The next step of our proof is a detailed study of the boundary behaviour of Dirminimizing multi-valued functions. In this section we consider maps $u: B_{\rho}(x) \cap D \rightarrow$ $\mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ where $D \subset \mathbb{R}^{2}$ is a planar domain such that $\partial D$ is $C^{2}$. We will be interested in maps which take a preassigned value $Q \llbracket f \rrbracket$ at $\partial D \cap B_{\rho}(x)$. Since by subtracting the average $\eta \circ u$ we still get a Dir-minimizer, we can without loss of generality, assume that $f$ vanishes identically. We summarize the relevant assumptions in the following

Assumptions 13.1. $D \subset \mathbb{R}^{2}$ is a $C^{2}$ open set, $U$ is a bounded open set and $u \in W^{1,2}(D \cap$ $U, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ ) a multivalued function such that $\left.u\right|_{\partial D \cap U} \equiv Q \llbracket 0 \rrbracket$ and $\eta \circ u \equiv 0$. u is Dir minimizing in the sense that, for every $K \subset U$ compact and for every $v \in W^{1,2}\left(D \cap U, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ which coincides with $u$ on $(U \backslash K) \cap D$ and vanishes on $\partial D \cap U$, we have

$$
\operatorname{Dir}(u) \leq \operatorname{Dir}(v)
$$

Observe that under our assumptions, we can apply the regularity theory of [12] and [28] to conclude that $u$ is Hölder continuous in $K \cap \bar{D}$ for every compact set $K \subset U$. More precisely we have the following

Theorem 13.2. There is a geometric constant $\alpha(Q)>0$ and a constant $C$ which depends only on $Q$ and $D$ such that, if $u$ and $D$ are as in Assumption 13.1, then

$$
[u]_{0, \alpha, B_{\rho}(x) \cap D} \leq C \rho^{-\alpha}\left(\operatorname{Dir}\left(u, B_{2 \rho}(x) \cap D\right)\right)^{\frac{1}{2}}
$$

for every $B_{2 \rho}(x) \subset U$.
In the final blow-up in Section 26, we will prove that the limit of a suitable approximating sequence is a homogeneous Dir-minimizer. The following theorem will then exclude the existence of singular boundary points. It is a consequence of the classification of tangent functions (Theorem 13.9).

Theorem 13.3. Assume $D=\left\{x_{2}>0\right\}, U=B_{1}(0)$ and $u: D \cap U \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ is a Dirminimizing I-homogeneous map such that $\left.u\right|_{\partial D}=Q \llbracket 0 \rrbracket$. Either $u$ is a single harmonic function with multiplicity $Q$ (i.e. $u=Q \llbracket \eta \circ u \rrbracket$ ) or $I=1$.

Observe that under the additional information that $\eta \circ u \equiv 0$, the first alternative would imply that $u$ vanishes identically.

In case that the approximating sequence consisted of Dir-minimizers (which it does not in our case), we mention for completeness here the analouge definition of singular
boundary points for Dir-minimizers (i.e. points at the boundary where the order of "vanishing" of the Dir-minimizer is larger than 1) and prove its absence. Even though we will not need Definition 13.4 nor Theorem 13.5 for our analysis, it illustrates the ideas of our argument.

Definition 13.4. Let $D, u$ and $U$ be as in Assumption 13.1. $x \in \partial D$ will be called a contact point if there is a positive $\delta>0$ such that

$$
\begin{equation*}
\underset{\rho \downarrow 0}{\liminf } \frac{1}{\rho^{2+\delta}} \int_{B_{\rho}(x) \cap D}|D u|^{2}=0 . \tag{13.1}
\end{equation*}
$$

In section 13.3 we will show the following multi-valued counterpart of Theorem 12.6.
Theorem 13.5. Let $D, u$ and $U$ be as in Assumption 13.1. If $x \in \partial D$ is a contact point, then $u$ vanishes identically on the connected component of $D \cap U$ whose boundary contains $x$.

### 13.1 MONOTONICITY OF THE FREQUENCY FUNCTION

We introduce here the basic tool of our analysis, the frequency function, pioneered by Almgren. The version of the Almgren's frequency function used here is an extension introduced for the first time in the literature in [21] to deal with boundary regularity. One of the outcomes of our analysis is that the limit of the frequency function exists at every boundary point $x$ unless $u$ vanishes identically in a neighborhood of it.

We recall the definition of the frequency function as in [21, Definition 4.13].
Definition 13.6. Consider $u \in W_{l o c}^{1,2}\left(D, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ and fix any cut-off $\phi:[0, \infty[\rightarrow$ $[0, \infty]$ which equals 1 in a neighborhood of 0 , it is non increasing and equals 0 on $\left[1, \infty\left[\right.\right.$. We next fix a function $d: \mathbb{R}^{2} \rightarrow \mathbb{R}^{+}$which is $C^{2}$ on the punctured space $\mathbb{R}^{2} \backslash\{0\}$ and satisfies the following properties:
(i) $d(x)=|x|+O\left(|x|^{2}\right)$,
(ii) $\nabla d(x)=\frac{x}{|x|}+O(|x|)$,
(iii) $D^{2} d(x)=|x|^{-1}\left(\mathrm{I} d-|x|^{-2} x \otimes x\right)+O(1)$.

By [21, Lemma 4.25], we deduce the existence of such a $d$ satisfying also that $\nabla d$ is tangent to $\partial D$. We define the following quantities:

$$
\begin{aligned}
D_{\phi, d}(u, r) & :=\int_{D} \phi\left(\frac{d(x)}{r}\right)|D u|^{2}(x) d x \\
H_{\phi, d}(u, r) & :=-\int_{D} \phi^{\prime}\left(\frac{d(x)}{r}\right)|\nabla d(x)|^{2} \frac{|u(x)|^{2}}{d(x)} d x .
\end{aligned}
$$

The frequency function is then the ratio

$$
I_{\phi, d}(u, r):=\frac{r D_{\phi, d}(u, r)}{H_{\phi, d}(u, r)}
$$

This quantity is essentially monotone.
Theorem 13.7. Let $D, U$ and $u$ be as in Assumption 13.1. Then there is a function $d$ satisfying the requirements of Definition 13.6 such that the following holds for every $\phi$ as in the same definition. Either $u \equiv Q \llbracket 0 \rrbracket$ in a neighborhood of 0 , or $D_{\phi, d}(u, r)$ is positive for every $r$ (hence $I_{\phi, d}(u, r)$ is well defined) and the limit

$$
0<\lim _{r \downarrow 0} I_{\phi, d}(u, r)<+\infty
$$

exists and it is a positive finite number. In fact, there is an $r_{0}>0$ and $C$ such that $r \mapsto$ $e^{C r} I_{\phi, d}(u, r)$ is monotone for all $0<r<r_{0}$.

We first recall the following identities (compare [21, Proposition 4.18]).
Proposition 13.8. Let $\phi$ and $d$ be as in Definition 13.6 and assume in addition that $\phi$ is Lipschitz. Let $\Omega, D, U$ and $u$ be as in Assumption 13.1. Then, for every $0<r<1$, we have

$$
\begin{align*}
& D^{\prime}(r)=-\int_{D} \phi^{\prime}\left(\frac{|d(x)|}{r}\right) \frac{|d(x)|}{r^{2}}|D u|^{2} d x,  \tag{13.2}\\
& H^{\prime}(r)=\left(\frac{1}{r}+O(1)\right) H(r)+2 E(r), \tag{13.3}
\end{align*}
$$

where

$$
\begin{equation*}
E(r):=-\frac{1}{r} \int_{D} \phi^{\prime}\left(\frac{d(x)}{r}\right) \sum_{i} u_{i}(x) \cdot\left(D u_{i}(x) \cdot \nabla d(x)\right) d x \tag{13.4}
\end{equation*}
$$

and the constant $O(1)$ appearing in (13.3) depends on the function $d$ but not on $\phi$.
Theorem 13.7 follows as in [21], as soon as we can show the validity of the above identities. In turn the latter can be proved following also the computations in [21], provided we prove that both the outer variations $g_{\varepsilon}(x):=\sum_{i} \llbracket u_{i}(x)+\varepsilon \varphi\left(\frac{d(x)}{r}\right) u_{i}(x) \rrbracket$ and the inner variations $u \circ \psi_{t}$, with $\psi_{t}$ being the flow of $Y(x):=\varphi\left(\frac{d(x)}{r}\right) \frac{d(x) \nabla d(x)}{\left.\nabla d(x)\right|^{2}}$, are competitors to our problem. This is however obvious. Clearly the outer variations are well defined and preserve the condition that $\left.u\right|_{\partial D \cap u} \equiv Q \llbracket 0 \rrbracket$. As for the inner variations note that, since $\nabla d$ is tangent to $\partial D$, so is $Y$ and thus its flow maps $\partial D$ onto itself and $D$ into itself. This shows that the inner variations are well defined and provide admissible competitors too.

### 13.2 CLASSIFICATION OF TANGENT FUNCTIONS

Following a common path which started with Almgren's monumental work (see [21], but also [12-15, 17-20]) we use the monotonocity of the frequency function to define
tangent functions to $u$. Let $D, u, U$ and $f$ be as in Assumption 13.1. Let $x \in \partial D$ and denote by $n(x)$ the interior unit normal to $\partial D$. If we denote by $V^{+}$the half space $\{y: n(x) \cdot y>0\}$, the tangent functions to $u$ at $x$ are multivalued functions defined on $V^{+}$, which turn out to be locally Dir-minimizing and in fact satisfy Assumption 13.1 with $D=V^{+}$for any bounded open set $U$.

The central result is the following theorem of which Theorem 13.3 is a direct corollary.
Theorem 13.9. Let $D, U$ and $u$ be as in Assumption 13.1. Let $x \in \partial D$ and assume that, for some $\rho>0, D \cap B_{\rho}(x)$ is connected and u does not vanish identically on $B_{\rho}(x) \cap D$. Define

$$
u_{x, \rho}(y):=\sum_{i} \llbracket \frac{u_{i}(x+\rho y)}{\operatorname{Dir}\left(u, B_{\rho}(x)\right)^{1 / 2}} \rrbracket .
$$

Then $I_{0}(x):=\lim _{r \rightarrow 0} I(u(\cdot-x), r)=1$ and, for every sequence $\rho_{k} \downarrow 0$, there is a subsequence (not relabeled) such that $u_{x, \rho_{k}}$ converges locally uniformly on $V^{+}$to a Dir-minimizer $u_{x, 0}=$ $\sum_{i} \llbracket v_{i} \rrbracket$ satisfying the following properties:
(a) each $v_{i}: V^{+} \rightarrow \mathbb{R}^{n}$ is a linear function that vanishes at $\partial V^{+}$;
(b) for every $i \neq j$, either $v_{i} \equiv v_{j}$, or $v_{i}(y) \neq v_{j}(y)$ for every $y \in V^{+}$;
(c) $\operatorname{Dir}\left(u_{x, 0}, B_{1}\right)=1$ and $\eta \circ u_{x, 0}=0$.

Proof. First of all we let $I:=I_{0}(x)$. It follows from the same arguments of [21, Lemma 4.28] that a subsequence, not relabeled, of $u_{x, \rho_{k}}$ converges to a Dir-minimizer $u_{x, 0}=$ $\sum_{i} \llbracket v_{i} \rrbracket$ which has the property (c) and which is $I$-homogeneous. Up to a rotation of the system of coordinates we can assume that $V^{+}=\left\{x_{1}>0\right\}$ (and hence $\partial V^{+}$is the $x_{2}$-axis). From now on we use polar coordinates on $V^{+}$and in particular we identify $\partial B_{1} \cap V^{+}$ with $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let $g=\sum_{i} \llbracket g_{i} \rrbracket$ be the restriction of $u_{x, 0}$ on $\partial B_{1} \cap V^{+}$. We can then use [12, Proposition 1.2] to conclude the existence of Hölder maps $g_{1}, \ldots, g_{Q}:(-\pi, \pi) \rightarrow \mathbb{R}^{n}$ such that

$$
g(\theta)=\sum_{i} \llbracket g_{i}(\theta) \rrbracket .
$$

In particular

$$
u_{x, 0}(\theta, r)=\sum_{i} \llbracket r^{I} g_{i}(\theta) \rrbracket,
$$

and each $u_{i}(\theta, r)=r^{I} g_{i}(\theta)$ is an harmonic polynomial. In particular $I$ must be an integer. Since however $u_{x .0} \equiv Q \llbracket 0 \rrbracket$ on $\left\{x_{1}=0\right\}$ and $\operatorname{Dir}\left(u_{x, 0}, B_{1}\right)>0$, it must be a positive integer.

Observe that, if $i \neq j$ and $\theta_{0} \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a point where $g_{i}\left(\theta_{0}\right)=g_{j}\left(\theta_{0}\right)$, then $g_{i}$ and $g_{j}$ must coincide in a neighborhood of $\theta_{0}$, otherwise the whole halfline $\left\{\left(r \cos \theta_{0}, r \sin \theta_{0}\right)\right\}$ consists of singularities of $u_{x, 0}$, contradicting [12, Theorem o.11]. In particular by the unique continuation principle for harmonic functions we have
(Alt)' either $u_{i}(r, \theta) \neq u_{j}(r, \theta)$ for every $\left.(r, \theta) \in\right] 0,+\infty\left[\times\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$, or $u_{i}(r, \theta)=u_{j}(r, \theta)$ for every $(r, \theta) \in] 0,+\infty\left[\times\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$,
so
(Alt) either $g_{i}(\theta) \neq g_{j}(\theta)$ for every $\theta \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, or $g_{i}(\theta)=g_{j}(\theta)$ for every $\theta \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Next, using the classification of 2-dimensional harmonic polynomials, we know that there are coefficients $a_{i}, b_{i} \in \mathbb{R}^{n}$ such that

$$
g_{i}(\theta)=a_{i} \cos (I \theta)+b_{i} \sin (I \theta)
$$

If $I$ were even, since $g_{i}\left(\frac{\pi}{2}\right)=g_{i}\left(-\frac{\pi}{2}\right)=0$, we conclude that $a_{i}=0$. But then all the $g_{i}$ 's would vanish at $\theta=0$ and (Alt) would imply that they all coincide everywhere. This would however contradict (c). Likewise, if $I$ were odd and larger than 1 , then we would have $b_{i}=0$ and all the $g_{i}{ }^{\prime}$ s would vanish at $\theta=\frac{\pi}{2 I}$. We thus conclude that $I$ is necessarily equal to 1 . This proves then (a), while (Alt) shows (b).

### 13.3 PROOF OF THEOREM 13.5

Fix a point $x \in \partial D$ and assume that $u$ does not vanish in any neighborhood of $x$. Then Theorem 13.9 implies that the frequency function $I_{0}(x)$ is 1 . Arguing as in [21, Corollary 4.27] we conclude however that, for every $\delta>0$, there is a radius $\rho>0$ such that

$$
\frac{D(r)}{r^{2+\delta}} \geq(1-\delta) \frac{D(\rho)}{\rho^{2+\delta}}>0 \quad \forall r<\rho
$$

This shows that $x$ cannot be a contact point.

## 14

In this section we consider a neighborhood of a flat point and we introduce the cylindrical excess $\mathbf{E}\left(T, \mathbf{C}_{r}(p, V)\right)$ as in [21, Definition 5.1]. Then, under the assumption that $\mathbf{E}\left(T, \mathbf{C}_{r}(p, V)\right)$ is sufficiently small, we produce an efficient approximation of the current with a multivalued graph. One important point is that the graph of such approximation, considered as an integral current, will also have boundary $Q \llbracket \Gamma \rrbracket$. From now on, given a point $p$ and a plane $V$ through the origin, $B_{r}(p, V)$ will denote the disk $\mathbf{B}_{r}(p) \cap(p+V), V^{\perp}$ the orthogonal complement of $V$ and $\mathbf{C}_{r}(p, V)$ the cylinder $B_{r}(p, V)+V^{\perp}$. We then denote by $\mathbf{p}_{V}$ and $\mathbf{p}_{V}^{\perp}$ the orthogonal projections respectively on $V$ and its orthogonal complement.

Definition 14.1. For a current $T$ in a cylinder $\mathbf{C}_{r}(p, V)$ we define the cylindrical excess $\mathbf{E}\left(T, \mathbf{C}_{r}(p, V)\right)$ and the excess measure $\mathbf{e}_{T}$ of a set $F \subset B_{4 r}\left(\mathbf{p}_{V}(p), V\right)$ as

$$
\begin{aligned}
\mathbf{E}\left(T, \mathbf{C}_{r}(p, V)\right) & :=\frac{1}{2 \pi r^{2}} \int_{\mathbf{C}_{r}(p, V)}|\vec{T}-\vec{V}|^{2} d\|T\|, \\
\mathbf{e}_{T}(F) & :=\frac{1}{2} \int_{F+V^{\perp}}|\vec{T}-\vec{V}|^{2} d\|T\|
\end{aligned}
$$

The height in a set $G \subset \mathbb{R}^{2+n}$ with respect to a plane $V$ is defined as

$$
\begin{equation*}
\mathbf{h}(T, G, V):=\sup \left\{\left|\mathbf{p}_{V}^{\frac{1}{V}}(q-p)\right|: q, p \in \operatorname{spt}(T) \cap G\right\} . \tag{14.1}
\end{equation*}
$$

If $p$ and $V$ are omitted, then we understand that $\mathbf{C}_{r}=\mathbf{C}_{r}\left(0, \mathbb{R}^{2} \times\{0\}\right)$ and $V=$ $\mathbb{R}^{2} \times\{0\}$.

Assumptions 14.2. Let $\Gamma$ and $T$ be as in Assumption 12.5. 9 is a fixed point, which without loss of generality we assume to be the origin, $r$ an arbitrary radius such that $(\partial T)\left\llcorner\mathbf{C}_{4 r}=\right.$ $Q \llbracket \Gamma \rrbracket \perp \mathbf{C}_{4 r}$ and
(i) $q=(0,0) \in \Gamma$ and $T_{q} \Gamma=\mathbb{R} \times\{0\} \subset V_{0}=\mathbb{R}^{2} \times\{0\}$;
(ii) $\gamma=\mathbf{p}(\Gamma)$ divides $B_{4 r}$ in two disjoint open sets $D$ and $B_{4 r} \backslash \bar{D}$;
(iii) $\mathbf{p}_{\#} T\left\llcorner\mathbf{C}_{4 r}=Q \llbracket D \rrbracket\right.$.

Observe that thanks to (iii) we have the identities

$$
\begin{align*}
\mathbf{E}\left(T, \mathbf{C}_{4 r}\right) & =\frac{1}{2 \pi(4 r)^{2}}\left(\|T\|\left(\mathbf{C}_{4 r}\right)-Q|D|\right),  \tag{14.2}\\
\mathbf{e}_{T}(F) & =\|T\|\left(F \times \mathbb{R}^{n}\right)-Q|D \cap F| . \tag{14.3}
\end{align*}
$$

Following a classical terminology we define noncentered maximal functions for Radon measures $\mu$ and (Lebesgue) integrable functions $f: U \rightarrow \mathbb{R}_{+}$by setting

$$
\begin{aligned}
\mathbf{m} f(z) & :=\sup _{z \in B_{s}(y) \subset U} \frac{1}{\pi s^{2}} \int_{B_{s}(y)} f \\
\mathbf{m} \mu(z) & :=\sup _{z \in B_{s}(y) \subset U} \frac{\mu\left(B_{s}(y)\right)}{\pi s^{2}}
\end{aligned}
$$

Remark 14.3. Observe that by our assumptions there is an interval $I \subset \mathbb{R}$ containing $(-5 r, 5 r)$ and function $\psi: I \rightarrow \mathbb{R}^{n+1}$ with the property that $\mathbf{C}_{5 r} \cap \Gamma=\{(t, \psi(t)): t \in I\}$. Moreover $\psi(0)=0, \dot{\psi}(0)=0$ and $\|\ddot{\psi}\|_{C^{0}} \leq C A$ for a geometric constant $C(n)$. In particular $|\psi(t)| \leq C A t^{2}$ and $|\dot{\psi}(t)| \leq C A t$. Finally observe that, if we write $\psi=\left(\psi_{1}, \bar{\psi}\right)$, then $\partial D=\left(t, \psi_{1}(t)\right)$ and $\Gamma$ can be written as the graph of a function $g$ on $\partial D$ defined by $g\left(t, \psi_{1}(t)\right)=\bar{\psi}(t)$.


Figure 2: An illustration of the maps describing the boundary.

Proposition 14.4 (First Lipschitz approximation). There are positive constants $C$ and $c_{0}$ (depending only on $Q$ and $n$ ) with the following properties. Assume $T$ satisfies Assumption 14.2, $E:=\mathbf{E}\left(T, \mathbf{C}_{4 r}\right) \leq c_{0}$. Then, for any $\delta_{*} \in(0,1)$, there are a closed set $K \subset D \cap B_{3 r}$ and a $Q$-valued function $u$ on $D \cap B_{3 r}$ with the following properties:

$$
\begin{align*}
\left.u\right|_{\partial D \cap B_{3 r}} & =Q \llbracket g \rrbracket  \tag{14.4}\\
\operatorname{Lip}(u) & \leq C\left(\delta_{*}^{1 / 2}+r \mathbf{A}\right)  \tag{14.5}\\
\operatorname{osc}(u) & \leq C \mathbf{h}\left(T, \mathbf{C}_{4 r}\right)+C r E^{1 / 2}+C r^{2} \mathbf{A}  \tag{14.6}\\
K & \subset B_{3 r} \cap\left\{\mathbf{m e}_{T} \leq \delta_{*}\right\}  \tag{14.7}\\
\mathbf{G}_{u}\left\llcorner\left[K \times \mathbb{R}^{n}\right]\right. & =T\left\llcorner\left[K \times \mathbb{R}^{n}\right]\right.  \tag{14.8}\\
\left|\left(D \cap B_{s}\right) \backslash K\right| & \leq \frac{C}{\delta_{*}} \mathbf{e}_{T}\left(\left\{\mathbf{m e}_{T}>4^{-1} \delta_{*}\right\} \cap B_{s+r_{1} r}\right)+C \frac{\mathbf{A}^{2}}{\delta_{*}} s^{2} \quad \forall s \leq 3 r+r_{1} r  \tag{14.9}\\
\frac{\left\|T-\mathbf{G}_{u}\right\|\left(\mathbf{C}_{2 r}\right)}{r^{2}} & \leq \frac{C}{\delta_{*}}\left(E+\mathbf{A}^{2} r^{2}\right) \tag{14.10}
\end{align*}
$$

where $r_{1}=c \sqrt{\frac{E+\mathbf{A}^{2} r^{2}}{\delta_{*}}}$ and $c$ is a geometric constant.
Proof. Since the statement is invariant under dilations we assume w.l.o.g. that $r=1$. Consider the extension $\hat{g}$ of the function $g$ defined in Remark 14.3 which is simply given by $\hat{g}\left(x_{1}, x_{2}\right)=\bar{\psi}\left(x_{1}\right)$. In order to simplify our notation, we drop the hat symbol and denote the extension by $g$ as well. Consider next the current $\hat{T} \in \mathbf{I}_{2}\left(\mathbf{C}_{4}\right)$ which consists of $\hat{T}=T\left\llcorner\mathbf{C}_{4}+Q \mathbf{G}_{g}\left\llcorner\left(\left(B_{4} \backslash D\right) \times \mathbb{R}^{n}\right)\right.\right.$, where we use notation $\mathbf{G}_{g}$ for the integer rectifiable current naturally associated to the graph of a function $g: B_{4} \rightarrow \mathbb{R}^{n}$. More formally, if $\bar{g}(x)=(x, g(x))$, then

$$
\begin{equation*}
\mathbf{G}_{g}\left\llcorner\left(\left(B_{4} \backslash D\right) \times \mathbb{R}^{n}\right)=\bar{g}_{\sharp}\left(\llbracket B_{4} \backslash D \rrbracket\right) .\right. \tag{14.11}
\end{equation*}
$$

In particular from (14.11) and the classical theory of currents we see that

$$
\begin{align*}
(\partial \hat{T})\left\llcorner\mathbf{C}_{4}\right. & =Q \llbracket \Gamma \rrbracket\left\llcorner\mathbf{C}_{4}-Q \bar{g}_{\sharp}\left(\llbracket \partial D \cap B_{4} \rrbracket\right)=Q \llbracket \Gamma \rrbracket\left\llcorner\mathbf{C}_{4}-Q \llbracket \Gamma \rrbracket\left\llcorner\mathbf{C}_{4}=0,\right.\right.\right.  \tag{14.12}\\
\mathbf{p}_{\sharp} \hat{T} & =Q \llbracket D \rrbracket+Q \llbracket B_{4} \backslash D \rrbracket=Q \llbracket B_{4} \rrbracket . \tag{14.13}
\end{align*}
$$

Moreover, we can use [14, Corollary 3.3] to estimate

$$
\begin{align*}
\|\hat{T}\|\left(\mathbf{C}_{4}\right)-Q \pi 4^{2} & =\mathbf{E}\left(T, \mathbf{C}_{4}\right)+Q\left(\left\|\mathbf{G}_{g}\right\|\left(\left(B_{4} \backslash D\right) \times \mathbb{R}^{n}\right)-|D|\right) \\
& \leq \mathbf{E}\left(T, \mathbf{C}_{4}\right)+Q \int_{B_{4} \backslash D}|D g|^{2} \leq E+\mathbf{C A}^{2} \tag{14.14}
\end{align*}
$$

Similarly, we can define for $F \subset B_{4}$

$$
\mathbf{e}_{\hat{T}}(F)=\|\hat{T}\|\left(F \times \mathbb{R}^{n}\right)-Q|F|
$$

and the same considerations give

$$
\mathbf{e}_{\hat{T}}(F) \leq \mathbf{e}_{T}(F \cap D)+C \mathbf{A}^{2}|F \backslash D| .
$$

Moreover, we can apply [13, Proposition 3.2] to $\hat{T}$ to obtain a closed set $\hat{K} \subset B_{3}$ and $\hat{u} \in \operatorname{Lip}\left(B_{3}, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ which satisfy all the estimates (14.5)-(14.10), with the only relevant differences in (14.9), which becomes

$$
\left|B_{s} \backslash \hat{K}\right| \leq \frac{C}{\delta_{*}} \mathbf{e}_{T}\left(\left\{\mathbf{m e}_{T}>4^{-1} \delta_{*}\right\} \cap B_{s+r_{1} r}(x)\right)+C \frac{\mathbf{A}^{2}}{\delta_{*}} s^{2} \quad \text { for every } s \leq 3 r . \text { (14.15) }
$$

In order to show (14.4), we define an "almost reflection" $h$ on the boundary $\partial D$ in the following way:

$$
h\left(x_{1}, x_{2}\right)=\left(x_{1}, 2 \psi_{1}\left(x_{1}\right)-x_{2}\right)
$$

and set $K:=h(\hat{K}) \cap \hat{K}$. We now take the map $\hat{u}$, restrict it to $\hat{K}$ and then extend it again to a Lipschitz map $u$ with the additional property that (14.4) holds. In fact we first define $u: K \cup\left(\partial D \cap B_{2}\right) \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ as

$$
u(y)= \begin{cases}Q \llbracket g(y) \rrbracket & , \text { if } y \in \partial D \\ \hat{u}(y) & , \text { else. }\end{cases}
$$

Note that in principle a point $y$ could belong to both $K$ and $\partial D$ : in that case we are ignoring the value given by $\hat{u}$ and force such value to be the one given by $Q \llbracket g \rrbracket$. However a byproduct of the next elementary argument is that in fact $\hat{u}(y)=Q \llbracket g(y) \rrbracket$ for every $y \in \partial D$.

We now wish to show that the bound on $\operatorname{Lip}(u)$ and $\operatorname{osc}(u)$ becomes worse only by a geometric factor. In fact, since the oscillation of $Q \llbracket g \rrbracket$ is controlled by $\mathbf{A}$, we just need to focus on the Lipschitz bound. Consider $p \in \partial D, q \in K$. By construction of $h$, let $\sigma$ be the vertical segment joining $q$ and $h(q)$ and let $\tilde{q}$ be the only intersection of $\sigma$ with $\partial D$. Thus

$$
\begin{aligned}
\mathcal{G}(u(p), u(q)) & \leq \mathcal{G}(u(q), u(h(q)))+\mathcal{G}(u(h(q)), u(p)) \\
& \leq \mathcal{G}(u(q), u(h(q)))+\operatorname{C\mathcal {G}}(u(\tilde{q}), u(p)) \\
& \leq \mathcal{G}(u(q), u(h(q)))+\operatorname{CQ}|g(p)-g(\tilde{q})| \\
& \leq 2|q-p| \operatorname{Lip}(\hat{u})+\operatorname{CQA}|p-q| .
\end{aligned}
$$

Now we can use the Lipschitz Extension Theorem [12, Theorem 1.7] to extend $u$ to the whole domain $B_{2}$, while enlarging the Lipschitz constant and the oscillation by a geometric factor.

So far our map satisfies (14.4), (14.5), and (14.6). However, (14.7) and (14.8) are obvious because $K \subset \hat{K}$.

Next we show (14.9) holds with a slightly larger constant. First of all notice that, provided A is sufficiently small, $h$ is a diffeomorphism and that $h^{-1}\left(B_{s}\right) \subset B_{s+\text { CAs }^{2}}$, because $h(0)=0$ and $\|D h-I d\|_{C\left(B_{s}\right)}=\|D h-D h(0)\|_{C\left(B_{s}\right)} \leq C A s$. In particular we can estimate

$$
\begin{aligned}
\left|\left(B_{s} \cap D\right) \backslash K\right| & \leq\left|B_{s} \backslash \hat{K}\right|+\left|B_{s} \backslash h(\hat{K})\right| \\
& \leq\left|B_{s} \backslash \hat{K}\right|+C\left|h^{-1}\left(B_{s}\right) \backslash \hat{K}\right| \leq C\left|h\left(B_{s+C A s^{2}} \backslash \hat{K}\right)\right| .
\end{aligned}
$$

Finally we conclude

$$
\left\|T-\mathbf{G}_{u}\right\|\left(\mathbf{C}_{2}\right) \leq\left\|T-\mathbf{G}_{i}\right\|\left(\mathbf{C}_{2}\right)+\left\|\mathbf{G}_{u}-\mathbf{G}_{i}\right\|\left(\mathbf{C}_{2}\right) .
$$

For the first summand, we already have the desired estimate from [13, Proposition 3.2]. For the second we observe

$$
\left\|\mathbf{G}_{u}-\mathbf{G}_{\hat{u}}\right\|\left(\mathbf{C}_{2}\right)=\left\|\mathbf{G}_{u}-\mathbf{G}_{\hat{u}}\right\|\left(\left(B_{2} \backslash K\right) \times \mathbb{R}^{n}\right) \leq C\left|B_{2} \backslash K\right|,
$$

and we then use (14.9). This shows (14.10).
The proof would be complete, except that our approximation and estimates hold on slightly smaller balls than claimed. It can however easily be checked that in [13, Proposition 3.2], we just need to reduce slightly the size of the radius from 4 to a fixed smaller one, while the argument is literally the same: the price to pay are just worse constants in the estimates.

Definition 15.1 ( $E^{\beta}$-Lipschitz approximation). Let $\beta \in(0,1)$ and $T$ be as in Proposition 14.4. After setting $\delta_{*}=\left(E+\mathbf{A}^{2}\right)^{2 \beta}$, the corresponding map $u$ given by the proposition will be called the $E^{\beta}$-Lipschitz approximation of $T$ in $\mathbf{C}_{3 r}$ and will be denoted by $f$.

In this section we use the minimimizing assumption on $T$ to show that the $E^{\beta}$ Lipschitz approximation is close to a Dir-minimizing function $w$. We first introduce some notation.

Assumptions 15.2. $D \subset \mathbb{R}^{2}$ is a $C^{2}$ open set, $U$ is a bounded open set and $u \in W^{1,2}(D \cap$ $U, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ ) a multivalued function such that $\left.u\right|_{\partial D \cap u} \equiv Q \llbracket g \rrbracket$, where $g$ is as in Remark 14.3. $u$ is Dir-minimizing in the sense that, for every $K \subset U$ compact and for every $v \in$ $W^{1,2}\left(D \cap U, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ which coincides with $u$ on $(U \backslash K) \cap D$ and $\left.v\right|_{\partial D \cap u} \equiv Q \llbracket g \rrbracket$ we have

$$
\operatorname{Dir}(u) \leq \operatorname{Dir}(v)
$$

Theorem 15.3 (First harmonic approximation). For every $\eta>0$ and every $\beta \in(0,1)$, there exist a constant $\varepsilon=\varepsilon(\eta, \beta)>0$ with the following property. Let $T$ and $\Gamma$ be as in Assumption 14.2 in $\mathbf{C}_{4 r}$ (in particular $T$ is area minimizing in $\mathbf{C}_{4 r}$ ). If $E=\mathbf{E}\left(T, \mathbf{C}_{4 r}\right) \leq \varepsilon$ and $r \mathbf{A} \leq \varepsilon E^{\frac{1}{2}}$, then the $E^{\beta}$-Lipschitz approximation $f$ in $\mathbf{C}_{3 r}$ satisfies

$$
\begin{equation*}
\int_{B_{2} \sqcap D \backslash K}|D f|^{2} \leq \eta E \pi(4 r)^{2}=\eta \mathbf{e}_{T}\left(B_{4 r}\right) . \tag{15.1}
\end{equation*}
$$

Moreover, there exists a Dir-minimizing function $w$ such that $\left.w\right|_{\partial D \cap B_{2 r}}=Q \llbracket g \rrbracket$ and

$$
\begin{align*}
& r^{-2} \int_{B_{2 r \cap D}} \mathcal{G}(f, w)^{2}+\int_{B_{2 r} \cap D} \mathcal{G}(D f, D w)^{2} \leq \eta E \pi(4 r)^{2}=\eta \mathbf{e}_{T}\left(B_{4 r}\right),  \tag{15.2}\\
& \int_{B_{2} \cap D}|D(\eta \circ f)-D(\eta \circ w)|^{2} \leq \eta E \pi(4 r)^{2}=\eta \mathbf{e}_{T}\left(B_{4 r}\right) . \tag{15.3}
\end{align*}
$$

The following proposition provides a Taylor expansion of the mass of the current associated to the graph of a $Q$-valued function. It is proven in [14, Corollary 3.3] (although the corollary is stated for $V$ open, the proof works obviously when $V$ is merely measurable).

Proposition 15.4. (Taylor expansion of the mass, see [14, Corollary 3.3]). There are dimensional constants $c, C>0$ such that the following holds. Let $V \subset \mathbb{R}^{2}$ be a bounded measurable set and let $u: V \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ be a Lipschitz function with $\operatorname{Lip}(u) \leq c$. Denote by $\mathbf{G}_{u}$ the integer
rectifiable current associated to the graph of $u$ as in [14, Definition 1.10]. Then, the following Taylor expansion of the mass of $\mathbf{G}_{u}$ holds:

$$
\mathbf{M}\left(\mathbf{G}_{u}\right)=Q|V|+\frac{1}{2} \int_{V}|D u|^{2}+\int_{V} \sum_{i} R\left(D u_{i}\right),
$$

where $R: \mathbb{R}^{n \times 2} \rightarrow \mathbb{R}$ is a $C^{1}$ function satisfying $|R(D)|=|D|^{3} L(D)$ for some positive function $L$ such that $L(0)=0$ and $\operatorname{Lip}(L) \leq C$.

Remark 15.5. We write here the analog of ([13, Remark 5.5]). There exists a dimensional constant $c>0$ such that, if $E \leq c$, then the $E^{\beta}$-Lipschitz approximation satisfies the following estimates:

$$
\begin{array}{r}
\operatorname{Lip}(f) \leq C\left(E+C \mathbf{A}^{2}\right)^{\beta} \\
\int_{B_{3 s}(x) \cap D}|D f|^{2} \stackrel{(14.9)}{\leq} C\left(E+\mathbf{A}^{2}\right) s^{2} . \tag{15.5}
\end{array}
$$

Indeed (15.4) follows from Proposition 14.4, by the choice of $\beta$ and the scaling of $\mathbf{A}$. While (15.5) follows from Proposition 15.4 since for $E$ sufficiently small

$$
\int_{B_{3 s}(x) \cap D} \sum_{i} R\left(D f_{i}\right) \leq C E^{2 \beta} \int_{B_{3 s}(x) \cap D}|D f|^{2}<\frac{1}{4} \int_{B_{3 s}(x) \cap D}|D f|^{2},
$$

and therefore

$$
\begin{aligned}
\int_{B_{3 s}(x) \cap D}|D f|^{2} & \leq C\left(\mathbf { M } \left(\mathbf{G}_{f}\left\llcorner\mathbf{C}_{3 s}(x) \cap\left(D \times \mathbb{R}^{n}\right)-Q|D|\right)\right.\right. \\
& \leq C\left(\mathbf{M}\left(T\left\llcorner\mathbf{C}_{3 s}(x)\right)-Q|D|\right)+C \mathbf{M}\left(\mathbf{G}_{f}\left\llcorner\left(B_{3 s}(x) \cap D \backslash K\right) \times \mathbb{R}^{n}\right)\right.\right. \\
& \leq C E s^{2}+C\left(E+\mathbf{A}^{2}\right)^{2 \beta}\left|B_{3 s}(x) \cap D \backslash K\right| \leq C\left(E+\mathbf{A}^{2}\right) s^{2} .
\end{aligned}
$$

Proof of Theorem 15.3. By rescaling, it is not restrictive to assume that $r=1$. The proof of (15.1) is by contradiction. Assume there exist a constant $c_{1}>0$, a sequence of currents $\left(T_{k}\right)_{k \in \mathbb{N}}$ satisfying Assumption 14.2 and corresponding $E_{k}^{\beta}$-Lipschitz approximations $\left(f_{k}\right)_{k \in \mathbb{N}}$ which violate (15.1) for $\eta=c_{1}>0$. At the same time $\partial T\left\llcorner\mathbf{C}_{4}(0)=Q \llbracket \Gamma_{k} \rrbracket\right.$, where $\Gamma_{k}$ is a sequence of $C^{2}$ curves. For the latter we have $T_{0} \Gamma_{k}=\mathbb{R} \times\{0\}$ and a parametrization $\psi^{k}: \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ of the form

$$
\psi^{k}(t)=\left(\psi_{1}^{k}(t), \bar{\psi}^{k}(t)\right) .
$$

Moreover we assume $\left\|\psi^{k}\right\|_{C^{2}} \leq C \mathbf{A}_{k} \leq C \varepsilon_{k} E_{k}^{\frac{1}{2}}$. The domain of definition of the map $f_{k}$ is a set $D_{k}$ which can be explicitly written as

$$
D_{k}=\left\{\left(x_{1}, x_{2}\right) \in B_{3}: x_{2}>\psi_{1}^{k}\left(x_{1}\right)\right\} .
$$

Summarizing, our currents satisfy the following:

$$
\begin{equation*}
\mathbf{E}\left(T_{k}, \mathbf{C}_{4}\right) \leq \varepsilon_{k} \rightarrow 0, \quad \mathbf{A}_{k} \leq \varepsilon_{k} E_{k}^{\frac{1}{2}} \quad \text { and } \quad \int_{D_{k} \backslash K_{k}}\left|D f_{k}\right|^{2} \geq c_{1} E_{k} \tag{15.6}
\end{equation*}
$$

where $K_{k}:=\left\{x \in B_{3}: \mathbf{m e}_{T_{k}}(x)<E_{k}^{2 \beta}\right\}$. Set $\Lambda_{k}:=\left\{x \in D_{k}: \mathbf{m e}_{T_{k}}(x) \leq 2^{-2} E_{k}^{2 \beta}\right\}$ and observe that $\Lambda_{k} \cap B_{3} \subset K_{k}$. From Proposition 14.4 it follows that for every $r \leq 3$

$$
\begin{align*}
\operatorname{Lip}\left(f_{k}\right) & \leq C E_{k}^{\beta}  \tag{15.7}\\
\left|B_{r} \cap D_{k} \backslash K_{k}\right| & \leq C E_{k}^{-2 \beta} \mathbf{e}_{T_{k}}\left(B_{r+r_{0}(k)} \backslash \Lambda_{k}\right)+C \varepsilon_{k}^{2} E_{k}^{2(1-\beta)} \tag{15.8}
\end{align*}
$$

where $r_{0}(k)=16 E_{k}^{(1-2 \beta) / 2}<\frac{1}{2}$. Then, (15.6), (15.7), and (15.8) give

$$
\begin{equation*}
c_{1} E_{k} \leq \int_{B_{2} \cap D_{k} \backslash K_{k}}\left|D f_{k}\right|^{2} \leq C \mathbf{e}_{T_{k}}\left(B_{s} \backslash \Lambda_{k}\right)+C \varepsilon_{k}^{2} E_{k}^{2} \quad \text { for every } s \in\left[\frac{5}{2}, 3\right] \tag{15.9}
\end{equation*}
$$

Setting $c_{2}:=c_{1} /(2 C)$, we have

$$
2 c_{2} E_{k} \leq \mathbf{e}_{T_{k}}\left(B_{s} \cap D_{k} \backslash \Lambda_{k}\right)=\mathbf{e}_{T_{k}}\left(B_{s} \cap D_{k}\right)-\mathbf{e}_{T_{k}}\left(B_{s} \cap \Lambda_{k}\right)
$$

implying

$$
\begin{equation*}
\mathbf{e}_{T_{k}}\left(\Lambda_{k} \cap B_{s}\right) \leq \mathbf{e}_{T_{k}}\left(D_{k} \cap B_{s}\right)-2 c_{2} E_{k} \tag{15.10}
\end{equation*}
$$

Next observe that $2 \pi 4^{2} E_{k}=\mathbf{e}_{T_{k}}\left(B_{4} \cap D_{k}\right) \geq \mathbf{e}_{T_{k}}\left(B_{s} \cap D_{k}\right)$. Therefore, by the Taylor expansion in [13, Remark 5.4], (15.10) and the fact that $E_{k} \downarrow 0$, it follows that for every $s \in[5 / 2,3]$ and $k$ large enough so that $C E^{2 \beta_{k}} \leq c_{2}$, we have

$$
\begin{align*}
\frac{1}{2} \int_{\Lambda_{k} \cap B_{s}}\left|D f_{k}\right|^{2} & \stackrel{\text { Taylor }}{\leq}\left(1+C E_{k}^{2 \beta}\right) \mathbf{e}_{T_{k}}\left(\Lambda_{k} \cap B_{s}\right) \\
& \stackrel{(15 \cdot 10)}{\leq}\left(1+C E_{k}^{2 \beta}\right)\left(\mathbf{e}_{T_{k}}\left(B_{s} \cap D_{k}\right)-2 c_{2} E_{k}\right) \\
& \leq \mathbf{e}_{T_{k}}\left(B_{s} \cap D_{k}\right)-c_{2} E_{k} \tag{15.11}
\end{align*}
$$

Our aim is to show that (15.11) contradicts the minimality of $T_{k}$. To construct a competitor, we write $f_{k}(x)=\sum_{i} \llbracket f_{k}^{i}(x) \rrbracket \in \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$. We consider $h_{k}:=E_{k}^{-1 / 2} f_{k}$. Observe that $\left.h_{k}\right|_{\partial D_{k}}=Q \llbracket E_{k}^{-1 / 2} \bar{\psi}^{k} \rrbracket$ and that in turn $\left\|\bar{\psi}^{k}\right\|_{C^{2}} \leq C \varepsilon_{k} E_{k}^{\frac{1}{2}}$. In particular $E_{k}^{-1 / 2} \bar{\psi}^{k}$ converges strongly to 0 in $C^{2}$. Extend $\bar{\psi}^{k}$ to $B_{3} \cap D_{k}$ by keeping it constant in the variable $x_{2}$. Thus $\mathcal{G}\left(h_{k}, Q \llbracket E_{k}^{-1 / 2} \bar{\psi}^{k} \rrbracket\right)$ is a classical $W^{1,2}$ function that vanishes on $\partial D_{k}$. Since by [13, Remark 5.5(5.5)] we have $\sup _{k} \operatorname{Dir}\left(h_{k}, B_{3} \cap D\right)<\infty$, the Poincaré inequality gives

$$
\left\|\mathcal{G}\left(h_{k}, Q \llbracket E_{k}^{-1 / 2} \bar{\psi}^{k} \rrbracket\right)\right\|_{L^{2}\left(D_{k} \cap B_{3}\right)} \leq C
$$

which in turn implies $\left\|\mathcal{G}\left(h_{k}, Q \llbracket 0 \rrbracket\right)\right\|_{L^{2}\left(D_{k} \cap B_{3}\right)} \leq C$. Hence $\left\{h_{k}\right\}$ is bounded in $W^{1,2}$. Even though the domains of the $h_{k}$ depend on $k$, we can extend the maps identically equal to
$Q\left\{\bar{\psi}^{k}\right\}$ on their complement, and thus treat them as maps on $B_{3}$. Up to a subsequence, not relabeled, we can thus assume that the maps converge to some $h \in W^{1,2}$. Observe that $h$ vanishes identically on the lower half disk $B_{3}^{-}:=\left\{\left(x_{1}, x_{2}\right) \in B_{3}: x_{2}<0\right\}$ and thus we will also consider it as a map defined on the upper half disk $B_{3}^{+}$, taking the value $Q \llbracket 0 \rrbracket$ on the $x_{1}$-axis.
Since

$$
\begin{equation*}
\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}\left(B_{3}\right)} \rightarrow 0 \tag{15.12}
\end{equation*}
$$

and the following inequalities hold for every open $\Omega^{\prime} \subset B_{3}$ and any sequence of measurable sets $J_{k}$ with $\left|J_{k}\right| \rightarrow 0$,

$$
\begin{align*}
\liminf _{k \rightarrow+\infty}\left(\int_{\Omega^{\prime} \backslash J_{k}}\left|D h_{k}\right|^{2}-\int_{\Omega^{\prime}}|D h|^{2}\right) & \geq 0  \tag{15.13}\\
\limsup _{k \rightarrow+\infty} \int_{\Omega}\left(\left|D h_{k}\right|-|D h|\right)^{2} & \leq \limsup _{k \rightarrow+\infty} \int_{\Omega}\left(\left|D h_{k}\right|^{2}-|D h|^{2}\right) \tag{15.14}
\end{align*}
$$

Applying the first inequality with $J_{k}$ being the complement of $\Lambda_{k}$ we reach the following inequality

$$
\begin{equation*}
\frac{1}{2} \int_{B_{s}^{+}}|D h|^{2} \leq \liminf _{k \rightarrow \infty} E_{k}^{-1} \mathbf{e}_{T_{k}}\left(B_{s} \cap D_{k}\right)-c_{2} \quad \text { for every } s<3 . \tag{15.15}
\end{equation*}
$$

Now we wish to find a radius $r \in\left[\frac{5}{2}, 3\right]$ and a competitor function $H_{k}$ such that

- $\left.H_{k}\right|_{\left(B_{3} \backslash B_{r}\right) \cap D_{k}}=\left.h_{k}\right|_{\left(B_{3} \backslash B_{r}\right) \cap D_{k}} ;$
- $\left.H_{k}\right|_{\partial D_{k} \cap B_{3}}=\left.h_{k}\right|_{\partial D_{k} \cap B_{3}} ;$
- The following estimates hold for a subsequence (not relabeled)

$$
\begin{align*}
\lim _{k \rightarrow \infty} \operatorname{Dir}\left(H_{k}, B_{r}\right) & \leq \operatorname{Dir}\left(h, B_{r}\right)+\frac{c_{2}}{4},  \tag{15.16}\\
\operatorname{Lip}\left(H_{k}\right) & \leq C^{*} E_{k}^{\beta-1 / 2},  \tag{15.17}\\
\left\|\mathcal{G}\left(H_{k}, h_{k}\right)\right\|_{L^{2}\left(B_{r}^{+}\right)} & \leq C \operatorname{Dir}\left(h_{k}, B_{r}^{+}\right)+C \operatorname{Dir}\left(H_{k}, B_{r}^{+}\right) \leq M<+\infty, \tag{15.18}
\end{align*}
$$

where $C^{*}$ is a constant independent of $k$.
After proving that such a function exists, we can then follow the proof of [13, Theorem 5.2] mutatis mutandis.

In order to show our claim we will use (15.12), the Lipschitz bound $\operatorname{Lip}\left(h_{k}\right) \leq$ $C E_{k}^{\beta-1 / 2}$, the bound $\sup _{k} \operatorname{Dir}\left(h_{k}, B_{3}\right) \leq C$, and (15.15). Note next that, since $\left\|\bar{\psi}^{k} / E^{1 / 2}\right\|_{C^{2}} \downarrow$ 0 , all these facts remain true if we replace $h_{k}$ with the map

$$
\bar{h}_{k}(x):=\sum_{i} \llbracket\left(h_{k}\right)_{i}-\bar{\psi}^{k} \rrbracket .
$$

The advantage of the latter is that $\left.\bar{h}_{k}\right|_{\partial D_{k}}=Q \llbracket 0 \rrbracket$. Assuming that we find corresponding maps $\bar{H}_{k}$ satisfying all the properties above, we can then simply get $H_{k}$ by adding back $\bar{\psi}^{k}$ :

$$
H_{k}(x)=\sum_{i} \llbracket\left(\bar{H}_{k}\right)_{i}+\bar{\psi}^{k} \rrbracket
$$

(because the difference in the Dirichlet energies of $H_{k}$ and $\bar{H}_{k}$ and the difference in the Lipschitz constants are both infinitesimal).

The next issue is that the domains $D_{k} \cap B_{s}$ are curved compared to $B_{s}^{+}$. To resolve this, we invoke Lemma 15.6 below. For each $k$ we apply the lemma to $\psi_{1}^{k}$ and get a corresponding diffeomorphism $\Phi_{k}$ which maps each $B_{s} \cap D_{k}$ diffeomorphically onto $B_{s}^{+}$. Observe that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left(\left\|\Phi_{k}-\mathrm{I} d\right\|_{C^{1}}+\left\|\Phi_{k}^{-1}-\mathrm{I} d\right\|_{C^{1}}\right)=0 \tag{15.19}
\end{equation*}
$$

because $\left\|\psi_{1}^{k}\right\|_{C^{1}} \rightarrow 0$. For this reason the maps $\tilde{h}_{k}:=\bar{h}_{k} \circ \Phi_{k}^{-1}$ satisfy the same assumptions as $\bar{h}_{k}$ (and hence as $h_{k}$ ). Indeed, after having built the corresponding competitors $\tilde{H}_{k}$, we can then define $\bar{H}_{k}:=\tilde{H}_{k} \circ \Phi_{k}$. Again the desired conclusion follows because the difference of the Lipschitz constants and Dirichlet energies are infinitesimal.
Summarizing, we have reduced the proof of the proposition to showing that the competitor $H_{k}$ can be constructed, without loss of generality, under the additional assumptions that all $h_{k}$ 's are defined on the same domain $B_{3}^{+}$and that they all vanish on $\left\{\left(x_{1}, x_{2}\right) \in B_{3}^{+}: x_{2}=0\right\}$. This is accomplished in Proposition 15.7 below. Now that we have illustrated how to construct suitable competitors we can proceed with the proof of the theorem. We restart observing that, when $k$ is large enough, (15.13) implies the following inequalities

$$
\begin{equation*}
\operatorname{Dir}\left(h, B_{r}\right) \leq \operatorname{Dir}\left(h_{k}, B_{r} \cap \Gamma_{k}\right)+\frac{c_{2}}{4} \stackrel{(5.11)}{\leq} \frac{\mathbf{e}_{T_{k}}\left(B_{r}\right)}{E_{k}}-\frac{3 c_{2}}{4} E_{k} \tag{15.20}
\end{equation*}
$$

Note that (15.17) follows from (15.27) as $E_{k}^{\beta-1 / 2} \uparrow \infty$. Thus $C^{*}$ depends on $c_{2}$ and on the choice of the two sequences, but not on $k$. From now on, although this and similar constants are not dimensional, we will keep denoting them by $C$, with the understanding that they do not depend on $k$. Note that, from (15.7) and (15.8), one gets

$$
\begin{aligned}
\left\|T_{k}-\mathbf{G}_{f_{k}}\right\|\left(\mathbf{C}_{3}\right) & \leq\left\|T_{k}\right\|\left(\left(B_{3} \backslash K_{k}\right) \times \mathbb{R}^{n}\right)+\left\|\mathbf{G}_{f_{k}}\right\|\left(\left(B_{3} \backslash K_{k}\right) \times \mathbb{R}^{n}\right) \\
& \leq Q\left|B_{3} \backslash K_{k}\right|+E_{k}+Q\left|B_{3} \backslash K_{k}\right|+C\left|B_{3} \backslash K_{k}\right| \operatorname{Lip}\left(f_{k}\right) \\
& \leq E_{k}+C E_{k}^{1-2 \beta} \leq C E_{k}^{1-2 \beta} .
\end{aligned}
$$

Let $(z, y)$ denote the coordinates on $\mathbb{R}^{2} \times \mathbb{R}^{n}$ and consider the function $\varphi(z, y)=|z|$ and the slice $\left\langle T_{k}-\mathbf{G}_{f_{k}} \varphi, r\right\rangle$. Observe that, by the coarea formula and Fatou's lemma,

$$
\int_{r}^{3} \liminf _{k} E_{k}^{2 \beta-1} \mathbf{M}\left(\left\langle T_{k}-\mathbf{G}_{f_{k}}, \varphi, s\right\rangle\right) d s \leq \liminf _{k} E_{k}^{2 \beta-1}\left\|T_{k}-\mathbf{G}_{f_{k}}\right\|\left(\mathbf{C}_{3}\right) \leq C
$$

Therefore, for some $\bar{r} \in(r, 3)$, up to subsequences (not relabeled) $\mathbf{M}\left(\left\langle T_{k}-\mathbf{G}_{f_{k}}, \varphi, \bar{r}\right\rangle\right)$ $\leq C E_{k}^{1-2 \beta}$. Let now $v_{k}:=\left.E_{k}^{1 / 2} H_{k}\right|_{B_{\bar{F}}}$ and consider the current $Z_{k}:=\mathbf{G}_{v_{k}}\left\llcorner\mathbf{C}_{\bar{r}}\right.$. Since $\left.\left(v_{k}\right)\right|_{\partial B_{F}}=\left.f_{k}\right|_{\partial B_{\bar{r}}}$, one gets $\partial Z_{k}=\left\langle\mathbf{G}_{f_{k^{\prime}}} \varphi, \bar{r}\right\rangle$ and hence, $\mathbf{M}\left(\partial\left(T_{k}\left\llcorner\mathbf{C}_{\bar{r}}-Z_{k}\right)\right) \leq C E_{k}^{1-2 \beta}\right.$. We define

$$
\begin{equation*}
S_{k}=T_{k}\left\llcorner\left(\mathbf{C}_{4} \backslash \mathbf{C}_{\bar{r}}\right)+Z_{k}+R_{k},\right. \tag{15.21}
\end{equation*}
$$

where (cp. [13, Remark 5.3]) $R_{k}$ is an integral current such that

$$
\partial R_{k}=\partial\left(T_{k}\left\llcorner\mathbf{C}_{\bar{r}}-Z_{k}\right) \quad \text { and } \quad \mathbf{M}\left(R_{k}\right) \leq C E_{k}^{(1-2 \beta) 2}\right.
$$

In particular, we have $\partial S_{k}=\partial\left(T_{k}\left\llcorner\mathbf{C}_{4}\right)\right.$. We now show that, since $\beta<\frac{1}{4}$, for $k$ large enough, the mass of $S_{k}$ is strictly smaller than the one of $T_{k}$. To this aim we write

$$
\operatorname{Dir}\left(v_{k}, B_{\bar{r}}\right)-\operatorname{Dir}\left(f_{k}, B_{\bar{F}} \cap \Lambda_{k}\right)=\int_{B_{\bar{r}}}\left|D v_{k}\right|^{2}-\int_{B_{\bar{F}} \cap \Lambda_{k}}\left|D f_{k}\right|^{2}=: I_{1} .
$$

The first term is estimated by (15.16) and (15.13). Indeed, recall that $v_{k}=E_{k}^{1 / 2} H_{k}$ and $f_{k}=E_{k}^{1 / 2} h_{k}$ (but also that the two functions coincide on $B_{\bar{r}} \backslash B_{r}$ ). We thus deduce that $I_{1} \leq \frac{c_{2}}{2} E_{k}$ for $k$ large enough. Hence, by using (15.11) we observe

$$
\begin{align*}
\mathbf{M}\left(S_{k}\right)-\mathbf{M}\left(T_{k}\right) & \leq \mathbf{M}\left(Z_{k}\right)+C \mathbf{M}\left(R_{k}\right)-\mathbf{M}\left(T_{k}\left\llcorner\mathbf{C}_{\bar{r}}\right)\right. \\
& \leq Q\left|B_{\bar{r}}\right|+\int_{B_{\bar{F}}} \frac{\left|D v_{k}\right|^{2}}{2}+C E_{k}^{1+2 \beta}+C E_{k}^{(1-2 \beta) 2}-Q\left|B_{\bar{r}}\right|-\mathbf{e}_{T_{k}}\left(B_{\bar{r}}\right) \\
& \leq \int_{B_{\digamma} \cap \Lambda_{k}} \frac{\left|D f_{k}\right|^{2}}{2}+\frac{1}{2} c_{2} E_{k}+C E_{k}^{1+2 \beta}+C E_{k}^{(1-2 \beta) 2}-\mathbf{e}_{T_{k}}\left(B_{\bar{r}}\right) \\
& \leq-\frac{C_{2} E_{k}}{2}+C E_{k}^{1+\beta}+C E_{k}^{(1-2 \beta) 2}<0, \tag{15.22}
\end{align*}
$$

as soon as $E_{k}$ is small enough, i.e., $k$ large enough. This gives the desired contradiction and proves (15.1).

Now, we come to the proof of (15.2) and (15.3). To this aim, we argue again by contradiction using similar constructions of competitors. Without loss of generality, we assume $x=0$ and $s=1$. Suppose $\left(T_{k}\right)_{k}$ is a sequence with $E_{k}:=\mathbf{E}\left(T_{k}, \mathbf{C}_{4}\right)$ satisfying

$$
\begin{equation*}
\mathbf{E}\left(T_{k}, \mathbf{C}_{4}\right) \leq \varepsilon_{k} \rightarrow 0, \quad \mathbf{A}_{k} \leq \varepsilon_{k} E_{k}^{\frac{1}{2}}, \tag{15.23}
\end{equation*}
$$

but contradicting (15.2) or (15.3). Let us denote by $f_{k}$ the $E_{k}^{\beta}$-Lipschitz approximation of $T_{k}$. We know that, for any sequence of Dir-minimizing functions $\bar{u}_{k}$ which we might choose, we will have by the contradiction assumption that

$$
\begin{equation*}
\liminf _{k} \underbrace{E_{k}^{-1} \int_{B_{2}}\left(\mathcal{G}\left(f_{k}, \bar{u}_{k}\right)^{2}+\left(\left|D f_{k}\right|-\left|D \bar{u}_{k}\right|\right)^{2}+\left|D\left(\boldsymbol{\eta} \circ f_{k}-\boldsymbol{\eta} \circ \bar{u}_{k}\right)\right|^{2}\right)}_{=: I(k)}>0 \tag{15.24}
\end{equation*}
$$

As in the previous argument, we introduce the auxiliary normalized functions $h_{k}=$ $E_{k}^{-1 / 2} f_{k}$ and, after extraction of a subsequence, the function $h$ satisfies (15.13) and (15.14). Moreover $\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}\left(B_{3}\right)} \rightarrow 0$. We next claim (and prove)
(i) $\lim _{k} \int_{B_{2}}\left|D h_{k}\right|^{2}=\int_{B_{2}}|D h|^{2}$,
(ii) $h$ is Dir-minimizing in $B_{2}$.

Indeed, if $(i)$ were false, then there is a positive constant $c_{2}$ such that, for any $r \in[5 / 2,3]$,

$$
\begin{equation*}
\int_{B_{r}} \frac{|D h|^{2}}{2} \leq \int_{B_{r}} \frac{\left|D h_{k}\right|^{2}}{2}-c_{2} \leq \frac{\mathbf{e}_{T_{k}}\left(B_{r}\right)}{E_{k}}-\frac{c_{2}}{2} \tag{15.25}
\end{equation*}
$$

provided $k$ is large enough (where the last inequality is again an effect of the Taylor expansion of [13, Remark 5.4]). We next define the competitor currents $S_{k}$ as in the argument leading to (15.22). Replacing in the argument above (15.11) and (15.20) by (15.25), we deduce again (15.22). On the other hand (15.22) contradicts the minimality of $T_{k}$. So we conclude that $(i)$ is true.

If (ii) were false, then $h$ is not Dir-minimizing in $B_{2}$. Thus, we can find a competitor $\tilde{h} \in W^{1,2}\left(B_{3}, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ with less energy in the ball $B_{2}$ than $h$ and such that $\tilde{h}=h$ on $B_{3} \backslash B_{5 / 2}$. So for any $r \in[5 / 2,3]$, the function $\tilde{h}$ satisfies

$$
\begin{equation*}
\int_{B_{r}} \frac{|D \tilde{h}|^{2}}{2} \leq \int_{B_{r}} \frac{|D h|^{2}}{2}-c_{2}=\lim _{k \rightarrow \infty} \int_{B_{r}} \frac{\left|D h_{k}\right|^{2}}{2}-c_{2} \leq \frac{\mathbf{e}_{T}\left(B_{r}\right)}{E_{k}}-\frac{c_{2}}{2} \tag{15.26}
\end{equation*}
$$

provided $k$ is large enough (here $c_{2}>0$ is some constant independent of $r$ and $k$ ). On the other hand, $\tilde{h}=h$ on $B_{3} \backslash B_{5 / 2}$ and therefore $\left\|\mathcal{G}\left(\tilde{h}, h_{k}\right)\right\|_{L^{2}\left(B_{3} \backslash B_{5 / 2}\right)} \rightarrow 0$. We then construct the competitor current $S_{k}$ of (15.21). This time however, we use the map $\tilde{h}$ in place of $h$ to construct $H_{k}$ via Proposition 15.7 and we reach the contradiction (15.22) using (15.26) in place of (15.11) and (15.20). We next set $\bar{u}_{k}:=E_{k}^{1 / 2} h$ and we will show that $I(k) \rightarrow 0$, violating (15.24). Observe first that as $\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}} \rightarrow 0$, we have $D\left(\boldsymbol{\xi} \circ h_{k}\right)-D(\boldsymbol{\xi} \circ h) \rightarrow 0$ weakly in $L^{2}$ (recall the definition of $\boldsymbol{\xi}=\boldsymbol{\xi}_{B W}$ in [13, Section 2.5]). So, (i) and the identities $\left|D\left(\boldsymbol{\xi} \circ h_{k}\right)\right|=\left|D h_{k}\right|,|D(\boldsymbol{\xi} \circ h)|=|D h|$ imply that $D\left(\boldsymbol{\xi} \circ h_{k}\right)-D(\boldsymbol{\xi} \circ h)$ converges strongly to 0 in $L^{2}$. If we next set $\hat{h}=\sum_{i} \llbracket h^{i}-\boldsymbol{\eta} \circ h \rrbracket$ and $\hat{h}_{k}=\sum_{i} \llbracket h_{k}^{i}-\boldsymbol{\eta} \circ h_{k} \rrbracket$, we obviously have $\left\|\mathcal{G}\left(\hat{h}, \hat{h}_{k}\right)\right\|_{L^{2}}+\left\|\boldsymbol{\eta} \circ h-\boldsymbol{\eta} \circ h_{k}\right\|_{L^{2}} \rightarrow 0$. Recall however that the Dirichlet energy enjoys the splitting

$$
\operatorname{Dir}\left(h_{k}\right)=Q \int\left|D\left(\boldsymbol{\eta} \circ h_{k}\right)\right|^{2}+\operatorname{Dir}\left(\hat{h}_{k}\right), \quad \operatorname{Dir}(h)=Q \int|D(\boldsymbol{\eta} \circ h)|^{2}+\operatorname{Dir}(\hat{h})
$$

So (i) implies that the Dirichlet energy of $\eta \circ h_{k}$ and $\hat{h}_{k}$ converge, respectively, to the one of $\eta \circ h$ and $\hat{h}$ (which, we recall again, are independent of $k$ because the $h_{k}$ 's are
translating sheets). We thus infer that $D(\boldsymbol{\eta} \circ h)-D\left(\boldsymbol{\eta} \circ h_{k}\right)$ converges to o strongly in $L^{2}$. Coming back to $\bar{u}_{k}$ we observe that $\bar{u}_{k}$ is Dir-minimizing and

$$
E_{k}^{-1} \int_{B_{2}} \mathcal{G}\left(\bar{u}_{k}, f_{k}\right)^{2}=\int_{B_{2}} \mathcal{G}\left(h, h_{k}\right)^{2} \rightarrow 0 .
$$

So,

$$
\limsup I(k) \leq \limsup \int_{k}\left(\left|D h_{k}\right|-|D h|\right)^{2}+\left|D\left(\boldsymbol{\eta} \circ h_{k}-\boldsymbol{\eta} \circ h\right)\right|^{2}
$$

Thus $I(k) \rightarrow 0$, which contradicts (15.24).

### 15.1 TECHNICAL LEMMAS

Lemma 15.6. There is a positive geometric constant $c>0$ with the following property. Consider a $C^{1}$ function $\psi_{1}:[0,4] \rightarrow \mathbb{R}$ such that $\psi_{1}(0)=\psi_{1}^{\prime}(0)=0$ and $\left\|\psi_{1}\right\|_{C^{1}} \leq c$. Then there is a map $\Phi: B_{4} \rightarrow B_{4}$ such that

- $\Phi$ maps $B_{s}$ diffeomorphically onto itself for every $s \in(0,4]$;
- if we set $D:=\left\{\left(x_{1}, x_{2}\right):\left|x_{1}\right| \leq 4, x_{2}>\psi_{1}\left(x_{1}\right)\right\}$ then $\Phi$ maps $D \cap B_{s}$ diffeomorphically onto $B_{s}^{+}$for every $s \in(0,4]$;
- $\left\|\Phi^{-1}-\mathrm{I} d\right\|_{C^{1}}+\|\Phi-\mathrm{I} d\|_{C^{1}} \leq C\left\|\psi_{1}\right\|_{C^{1}}$.

Proof. We use polar coordinates $(\theta, r)$ and let the angle $\theta$ vary from $-\frac{\pi}{2}$ (included) to $\frac{3 \pi}{2}$ (excluded). It is in fact easier to define the map $\Phi^{-1}$. If $c$ is sufficiently small, each circle $\partial B_{s}$ intersects the graph of $\psi_{1}$ in exactly two points, given in polar coordinates by $\left(\theta_{r}(s), s\right)$ and $\left(\theta_{l}(s), s\right)$, with $\theta_{l}(s)>\theta_{r}(s)$. Furthermore, again assuming $c$ is sufficiently small, $\left|\theta_{r}(s)\right| \leq \frac{\pi}{4}$ and $\left|\theta_{l}(s)-\pi\right| \leq \frac{\pi}{4}$. In polar coordinates the map $\Phi^{-1}$ is then defined on $B_{4}^{+}$by the formula

$$
\Phi^{-1}(\theta, s)=\left(\frac{\theta_{r}(s)(\pi-\theta)+\theta_{l}(s) \theta}{\pi}, s\right)
$$

The verification that $\left\|\Phi^{-1}-\mathrm{I} d\right\|_{C^{1}} \leq C\left\|\psi_{1}\right\|_{C^{1}}$ is left to the reader.
We then need to extend the map to the lower half disk keeping the same estimate. This could be reached for instance by the formula

$$
\Phi^{-1}(\theta, s)=\left(\frac{2 \pi-\left(\theta_{l}-\theta_{r}\right)}{\pi} \theta e^{a(\theta-\pi)(\theta-2 \pi)}+2 \theta_{l}-\theta_{r}, s\right) \quad \text { for } \pi<\theta<2 \pi
$$

where $a=a(s):=\pi^{-2}\left(1-\frac{\theta_{l}(s)-\theta_{r}(s)}{2 \pi-\left(\theta_{l}(s)-\theta_{r}(s)\right)}\right)$.
In the next proposition we want to "patch" functions defined on the upper half disk $B_{s}^{+}$which vanish on the $x_{1}$-axis. For convenience we introduce the notation $\mathcal{H}_{s}$ horizontal boundary for $\mathcal{H}_{s}=\left\{\left(x_{1}, 0\right):\left|x_{1}\right|<s\right\}$.

Proposition 15.7. Consider two radii $1 \leq r_{0}<r_{1}<4$ and maps $h_{k}, h \in W^{1,2}\left(B_{r_{1}}^{+}, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ satisfying

$$
\sup _{k} \operatorname{Dir}\left(h_{k}, B_{r_{1}}^{+}\right)<+\infty \quad \text { and } \quad\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}\left(B_{r_{1}}^{+} \backslash B_{r_{0}}\right)} \rightarrow 0
$$

and $\left.h_{k}\right|_{\mathcal{H}_{r_{1}}}=\left.h\right|_{\mathcal{H}_{r_{1}}}=Q \llbracket 0 \rrbracket$. Then for every $\eta>0$, there exist $\left.r \in\right] r_{0}, r_{1}[$, a subsequence of $\left\{h_{k}\right\}_{k}$ (not relabeled) and functions $H_{k} \in W^{1,2}\left(B_{r_{1}}^{+}, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ such that:

- $\left.H_{k}\right|_{B_{r_{1}}^{+} \backslash B_{r}^{+}}=\left.h_{k}\right|_{B_{r_{1}}^{+} \backslash B_{r}^{+}} ;$
- $\left.H_{k}\right|_{\mathcal{H}_{s}}=Q \llbracket 0 \rrbracket$ and
- $\operatorname{Dir}\left(H_{k}, B_{r_{1}}^{+}\right) \leq \operatorname{Dir}\left(h, B_{r_{1}}^{+}\right)+\eta$.

Moreover, there is a dimensional constant $C$ and a constant $C^{*}$ (depending on $\eta$ and the two sequences, but not on $k$ ) such that

$$
\begin{align*}
\operatorname{Lip}\left(H_{k}\right) & \leq C^{*}\left(\operatorname{Lip}\left(h_{k}\right)+1\right),  \tag{15.27}\\
\left\|\mathcal{G}\left(H_{k}, h_{k}\right)\right\|_{L^{2}\left(B_{r}^{+}\right)} & \leq C \operatorname{Dir}\left(h_{k}, B_{r}^{+}\right)+C \operatorname{Dir}\left(H_{k}, B_{r}^{+}\right),  \tag{15.28}\\
\left\|\boldsymbol{\eta} \circ H_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+}\right)} & \leq C^{*}\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+}\right)}+C\|\boldsymbol{\eta} \circ h\|_{L^{1}\left(B_{r_{1}}^{+}\right)} . \tag{15.29}
\end{align*}
$$

Before coming to the proof of the proposition we state the following variant of the Lipschitz approximation in [13, Lemma 4.5]. Observe that the only difference is that our functions are defined on the upper half disks and vanish on the horizontal boundary. We need the Lipschitz approximation $f_{\varepsilon}$ to satisfy the same requirement.

Lemma 15.8 (Lusin type Lipschitz approximation). Let $f \in W^{1,2}\left(B_{r}^{+}, \mathcal{A}_{Q}\right)$ be such that $\left.f\right|_{\mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$. Then for every $\varepsilon>0$ there exists $f_{\varepsilon} \in \operatorname{Lip}\left(B_{r}^{+}, \mathcal{A}_{Q}\right)$ satisfying $\left.f_{\varepsilon}\right|_{\mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$ and

$$
\int_{B_{r}^{+}} \mathcal{G}\left(f, f_{\varepsilon}\right)^{2}+\int_{B_{r}^{+}}\left(|D f|-\left|D f_{\varepsilon}\right|\right)^{2}+\int_{B_{r}^{+}}\left(|D(\boldsymbol{\eta} \circ f)|-\left|D\left(\boldsymbol{\eta} \circ f_{\varepsilon}\right)\right|\right)^{2} \leq \varepsilon
$$

If in addition $\left.f\right|_{\partial B_{r}^{+} \backslash \mathcal{H}_{r}} \in W^{1,2}\left(\partial B_{r}, \mathcal{A}_{Q}\right)$, then $f_{\varepsilon}$ can be chosen to satisfy also

$$
\begin{equation*}
\int_{\partial B_{r}^{+} \backslash \mathcal{H}_{r}} \mathcal{G}\left(f, f_{\varepsilon}\right)^{2}+\int_{\partial B_{r}^{+} \backslash \mathcal{H}_{r}}\left(|D f|-\left|D f_{\mathcal{\varepsilon}}\right|\right)^{2} \leq \varepsilon \tag{15.31}
\end{equation*}
$$

Now we need the following interpolation lemma.
Lemma 15.9 (Interpolation). There exists a constant $C_{0}=C_{0}(n, Q)>0$ with the following property. Assume $r \in] 1,3\left[, f \in W^{1,2}\left(B_{r}, \mathcal{A}_{Q}\right)\right.$ satisfies $\left.f\right|_{\mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$ and $\left.f\right|_{\partial B_{r}} \in$ $W^{1,2}\left(\partial B_{r}, \mathcal{A}_{Q}\right)$, and $g \in W^{1,2}\left(\partial B_{r}^{+}, \mathcal{A}_{Q}\right)$ is such that $\left.g\right|_{\mathcal{H}_{r} \cap \partial B_{r}^{+}}=Q \llbracket 0 \rrbracket$.

Then, for every $\varepsilon \in] 0, r\left[\right.$, there exists a function $h_{\varepsilon} \in W^{1,2}\left(B_{r}, \mathcal{A}_{Q}\right)$ such that $\left.h_{\varepsilon}\right|_{\partial B_{r}}=g$, $\left.h_{\varepsilon}\right|_{\mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$ and

$$
\begin{align*}
\int_{B_{r}^{+}}\left|D h_{\varepsilon}\right|^{2} & \leq \int_{B_{r}^{+}}|D f|^{2}+\varepsilon \int_{\partial B_{r}^{+}}\left(\left|D_{\tau} f\right|^{2}+\left|D_{\tau} g\right|^{2}\right)+\frac{C_{0}}{\varepsilon} \int_{\partial B_{r}^{+}} \mathcal{G}(f, g)^{2},  \tag{15.32}\\
\operatorname{Lip}\left(h_{\varepsilon}\right) & \leq C_{0}\left\{\operatorname{Lip}(f)+\operatorname{Lip}(g)+\varepsilon^{-1} \sup _{\partial B_{r}^{+}} \mathcal{G}(f, g)\right\},  \tag{15.33}\\
\int_{B_{r}^{+}}\left|\eta \circ h_{\varepsilon}\right| & \leq C_{0} \int_{\partial B_{r}^{+}}|\eta \circ g|+C_{0} \int_{B_{r}^{+}}|\eta \circ f|, \tag{15.34}
\end{align*}
$$

where $D_{\tau}$ denotes the tangential derivative.
Proof. The proof is the same as in [13, Lemma 4.6], because the map constructed there by the linear interpolation on the annulus and taking $f$ in the interior disk vanishes on $\mathcal{H}_{r_{1}}$.

Proof of Lemma 15.8. We can apply directly [21, Lemma 5.5] to obtain a Lipschitz function $\tilde{f}_{\varepsilon}$ satisfying $\left(\tilde{f}_{\varepsilon}\right)_{\mid \mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$ and (15.30).

Proof of Proposition 15.7. The proof goes along the same lines as the proof of [13, Proposition 4.4] using Lemmas 15.8 and 15.9 instead of [13, Lemma 4.5, Lemma 4.6], taking into account that the situation here is simpler because we do not have translating sheets. For the sake of completeness we report here the details. Set for simplicity $A_{k}:=\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}\left(B_{r_{1}} \backslash B_{r_{0}}^{+}\right)}$and $B_{k}:=\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+}\right)}$. If for any $k$ large enough $A_{k} \equiv 0$, then there is nothing to prove and so we can assume that, for a subsequence (not relabeled) $A_{k}>0$. In case that for yet another subsequence (not relabeled) $B_{k}>0$, we consider the function

$$
\begin{equation*}
\psi_{k}(r):=\int_{\partial B_{r}}\left(\left|D h_{k}\right|^{2}+|D h|^{2}\right)+A_{k}^{-2} \int_{\partial B_{r}} \mathcal{G}\left(h_{k}, h\right)^{2}+B_{k}^{-1} \int_{\partial B_{r}}\left|\eta \circ h_{k}\right| . \tag{15.35}
\end{equation*}
$$

By assumption $\lim \inf _{k} \int_{r_{0}}^{r_{1}} \psi_{k}(r) d r<\infty$. Hence by Fatou's Lemma, there is an $\left.r \in\right] r_{0}, r_{1}[$ and a subsequence (not relabeled) such that $\lim _{k} \psi_{k}(r)<\infty$. Thus, for some $M>0$ we have

$$
\begin{align*}
\int_{\partial B_{r}^{+}} \mathcal{G}\left(h_{k}, h\right)^{2} & \rightarrow 0,  \tag{15.36}\\
\operatorname{Dir}\left(h, \partial B_{r}^{+}\right)+\operatorname{Dir}\left(h_{k}, \partial B_{r}^{+}\right) & \leq M,  \tag{15.37}\\
\int_{\partial B_{r}^{+}}\left|\boldsymbol{\eta} \circ h_{k}\right| & \leq M\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}\right)} . \tag{15.38}
\end{align*}
$$

In case $B_{k}=0$ for all $k$ large enough, we define $\psi_{k}$ by dropping the last summand in (15.35) and reach the same conclusion. We apply Lemma 15.8 with $f=h, r=r_{1}$ and find
a Lipschitz function $h_{\bar{\varepsilon}_{1}}$ satisfying the conclusion of the lemma with $\bar{\varepsilon}_{1}=\bar{\varepsilon}_{1}(\eta, M)>0$ (which will be chosen later). In particular we have

$$
\begin{aligned}
\left\|\mathcal{G}\left(h_{k}, h_{\bar{\varepsilon}_{1}}\right)\right\|_{L^{2}\left(B_{r_{1}}^{+} \backslash B_{r_{0}}^{+}\right)} \leq\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}\left(B_{r_{1}}^{+} \backslash B_{r_{0}}^{+}\right)}+\left\|\mathcal{G}\left(h, h_{\bar{\varepsilon}_{1}}\right)\right\|_{L^{2}\left(B_{r_{1}}^{+} \backslash B_{r_{0}}^{+}\right)} \leq o(1)+\bar{\varepsilon}_{1}, \\
\operatorname{Dir}\left(h_{\bar{\varepsilon}_{1}}, \partial B_{r}^{+}\right) \leq \operatorname{Dir}\left(h, \partial B_{r}^{+}\right) \leq M+\bar{\varepsilon}_{1}
\end{aligned}
$$

To obtain also the estimate (15.29), which will be required in the construction of the center manifold, we argue along the same lines of [13, Proposition 4.4]. For $h_{\bar{\varepsilon}_{1}}=$ $\sum_{i=1}^{Q} \llbracket\left(h_{\bar{\varepsilon}_{1}}\right)_{i} \rrbracket$ we set $\bar{h}_{\bar{\varepsilon}_{1}}:=\sum_{i=1}^{Q} \llbracket\left(h_{\bar{\varepsilon}_{1}}\right)_{i}-\boldsymbol{\eta} \circ h_{\bar{\varepsilon}_{1}}+(\boldsymbol{\eta} \circ h) * \varphi_{\rho} \rrbracket$, where $\varphi_{\rho}(x):=\frac{1}{\rho^{n}} \varphi\left(\frac{x}{\rho}\right)$, and $\varphi(x)=\bar{\varphi}\left(x-z_{0}\right)$ with $\bar{\varphi}$ being the standard bump function with support in $B_{1}(0)$, $z_{0}:=(0,-2)$ and $\rho$ will be chosen small enough later. Observe that $\operatorname{spt}\left(\varphi_{\rho}\right)=B_{\rho}\left(\rho z_{0}\right) \subseteq$ $B_{r}^{-}$for every $\rho$ small enough and $\operatorname{spt}(\varphi)=B_{1}\left(z_{0}\right)$. The reason to introduce this convolution kernel $\varphi_{\rho}$ with support contained in $B_{r}^{-}$is that we need to preserve the zero boundary condition on $\mathcal{H}_{r}$. Indeed, we claim that such an $\bar{h}_{\bar{\varepsilon}_{1}}$ satisfies $\left.\left(\bar{h}_{\bar{\varepsilon}_{1}}\right)\right|_{\mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$ in addition to all the other conclusion of the proposition. The fact that $\left.\left(\bar{h}_{\varepsilon}\right)\right|_{\mathcal{H}_{r}}=Q \llbracket 0 \rrbracket$ is a simple consequence of the definitions and we leave it to the reader. Observe that the standard approximation properties of mollifiers reinterpreted suitably extends to this new kind of kernel. In particular, we can choose $\rho$ small enough to have

$$
\begin{align*}
Q^{2}\left\|\boldsymbol{\eta} \circ h-(\boldsymbol{\eta} \circ h) * \varphi_{\rho}\right\|_{L^{2}}^{2} & \leq \bar{\varepsilon}_{1},  \tag{15.39}\\
\left\|D(\boldsymbol{\eta} \circ h)-D\left((\boldsymbol{\eta} \circ h) * \varphi_{\rho}\right)\right\|_{L^{2}}^{2} & \leq \bar{\varepsilon}_{1}, \tag{15.40}
\end{align*}
$$

for some small $\bar{\varepsilon}_{1}$. These last two inequalities combined with (15.36), (15.37), (15.38) imply

- $\left\|\mathcal{G}\left(h_{k}, \bar{h}_{\bar{\varepsilon}_{1}}\right)\right\|_{L^{2}} \stackrel{(15 \cdot 39)}{\leq}\left\|\mathcal{G}\left(h_{k}, h\right)\right\|_{L^{2}}+2\left\|\mathcal{G}\left(h, \bar{h}_{\bar{\varepsilon}_{1}}\right)\right\|_{L^{2}}+\bar{\varepsilon}_{1} \leq o(1)+3 \bar{\varepsilon}_{1}$,
- $\operatorname{Dir}\left(\bar{h}_{\bar{\varepsilon}_{1}}, \partial B_{r}\right) \leq 2 M+2 \bar{\varepsilon}_{1}$,
- $\operatorname{Dir}\left(\bar{h}_{\bar{\varepsilon}_{1}}, B_{r}\right)=\sum_{i} \int_{B_{r}}\left|D\left(\bar{h}_{\bar{\varepsilon}_{1}}\right)_{i}-D\left(\boldsymbol{\eta} \circ \bar{h}_{\bar{\varepsilon}_{1}}\right)+D\left((\boldsymbol{\eta} \circ h) * \varphi_{\bar{\rho}}\right)\right|^{2}$
$=\int_{B_{r}}\left(\left|D \bar{h}_{\bar{\varepsilon}_{1}}\right|^{2}-Q\left|D\left(\boldsymbol{\eta} \circ \bar{h}_{\bar{\varepsilon}_{1}}\right)\right|^{2}+Q\left|D\left((\boldsymbol{\eta} \circ h) * \varphi_{\bar{\rho}}\right)\right|^{2}\right)$
$=Q \int_{B_{r}}\left(|D(\boldsymbol{\eta} \circ h)|^{2}-\left|D\left(\boldsymbol{\eta} \circ \bar{h}_{\bar{\varepsilon}_{1}}\right)\right|^{2}+\left|D\left(\boldsymbol{\eta} \circ h * \varphi_{\bar{\rho}}\right)\right|^{2}-|D(\boldsymbol{\eta} \circ h)|^{2}\right)$ $+\operatorname{Dir}\left(\bar{h}_{\bar{\varepsilon}_{1}}, B_{r}\right)$
$\leq \operatorname{Dir}\left(h_{\bar{\varepsilon}_{1}}, B_{r}\right)+2 Q \bar{\varepsilon}_{1}$,
where we used (15.30),(15.40) in the last inequality. We can then apply the interpolation Lemma 15.9 with $f=\bar{h}_{\bar{\varepsilon}_{1}}$ and $g=h_{k \mid \partial B_{r}^{+}}$, and $\varepsilon=\bar{\varepsilon}_{2}=\bar{\varepsilon}_{2}(\eta, M)>0$ to get maps $H_{k}$
satisfying $\left.H_{k}\right|_{\partial B_{r}^{+}}=\left.h_{k}\right|_{\partial B_{r}^{+}},\left.H_{k}\right|_{B_{r_{1}}^{+} \backslash B_{r}^{+}}=\left.h_{k}\right|_{B_{r_{1}}^{+} \backslash B_{r}^{+}}$. Now, we use (15.36), (15.37), (15.38) (15.30) and (15.31) to deduce

$$
\begin{array}{rll}
\operatorname{Dir}\left(H_{k}, B_{r}^{+}\right) & \stackrel{(15.32)}{\leq} & \operatorname{Dir}\left(\bar{h}_{\bar{\varepsilon}_{1}}, B_{r}^{+}\right)+\bar{\varepsilon}_{2} \operatorname{Dir}\left(\bar{h}_{\bar{\varepsilon}_{1}}, \partial B_{r}^{+}\right)+\bar{\varepsilon}_{2} \operatorname{Dir}\left(h_{k}, \partial B_{r}^{+}\right) \\
& +\frac{C_{0}}{\bar{\varepsilon}_{2}} \int_{\partial B_{r}^{+}} \mathcal{G}\left(\bar{h}_{\bar{\varepsilon}_{1}}, h_{k}\right)^{2} \\
& \stackrel{(15.31)}{\leq} & \operatorname{Dir}\left(h, B_{r}^{+}\right)+\bar{\varepsilon}_{1}+2 Q \bar{\varepsilon}_{1}+3 \bar{\varepsilon}_{2}\left[\operatorname{Dir}\left(h, \partial B_{r}^{+}\right)+\bar{\varepsilon}_{1}\right]+\bar{\varepsilon}_{2} M \\
& +\frac{C_{0}}{\bar{\varepsilon}_{2}}\left[\int_{\partial B_{r}^{+}} \mathcal{G}\left(h, h_{k}\right)^{2}+\int_{\partial B_{r}^{+}} \mathcal{G}\left(h_{\bar{\varepsilon}_{1}}, h\right)^{2}\right] \\
& \leq \quad \operatorname{Dir}\left(h, B_{r}^{+}\right)+\bar{\varepsilon}_{1}(1+2 Q)+\bar{\varepsilon}_{2}\left(4 M+3 \bar{\varepsilon}_{1}\right)+C_{0} \bar{\varepsilon}_{2}^{-1}\left[o(1)+\bar{\varepsilon}_{1}\right] .
\end{array}
$$

An appropriate choice of the parameters $\bar{\varepsilon}_{1}$ and $\bar{\varepsilon}_{2}$ gives the desired bound $\operatorname{Dir}\left(H_{k}, B_{r}\right) \leq$ $\operatorname{Dir}\left(h, B_{r}\right)+\eta$ for $k$ large enough. Observe next that, by construction, Lip $\left(\bar{h}_{\bar{\varepsilon}_{1}}\right)$ depends on $\eta$ and $h$, but not on $k$. Moreover, we have

$$
\left\|\mathcal{G}\left(\bar{h}_{\bar{\varepsilon}_{1}}, h_{k}\right)\right\|_{L^{\infty}\left(\partial B_{r}\right)} \leq C\left\|\mathcal{G}\left(\bar{h}_{\bar{\varepsilon}_{1}}, h_{k}\right)\right\|_{L^{2}\left(\partial B_{r}\right)}+C \operatorname{Lip}\left(h_{k}\right)+C \operatorname{Lip}\left(\bar{h}_{\bar{\varepsilon}_{1}}\right) .
$$

To prove the last inequality put $F(x):=\mathcal{G}\left(\bar{h}_{\bar{\varepsilon}_{1}}(x), h_{k}(x)\right)$ and observe that $F(x) \leq$ $F(y)+\operatorname{Lip}(F)|x-y|$, then integrate in $y$ and use the Cauchy-Schwarz inequality combined with the fact that $\operatorname{Lip}(F) \leq C\left(\operatorname{Lip}\left(\bar{h}_{\bar{\varepsilon}_{1}}\right)+\operatorname{Lip}\left(h_{k}\right)\right)$. Thus (15.27) follows from (15.33). Finally, (15.28) follows from the Poincaré inequality applied to $\mathcal{G}\left(H_{k}, h_{k}\right)$ (which vanishes identically on $\partial B_{r}^{+}$), in fact we have

$$
\left\|\mathcal{G}\left(H_{k}, h_{k}\right)\right\|_{L^{2}\left(B_{r_{1}}^{+}\right)}^{2} \leq C\left\|\nabla \mathcal{G}\left(H_{k}, h_{k}\right)\right\|_{L^{2}\left(B_{r_{1}}^{+}\right)}^{2} \leq C \operatorname{Dir}\left(h_{k}, B_{r_{1}}^{+}\right)+C \operatorname{Dir}\left(H_{k}, B_{r_{1}}^{+}\right) .
$$

(15.29) follows from (15.34), because of (15.38) and $\left\|\boldsymbol{\eta} \circ \bar{h}_{\bar{\varepsilon}_{1}}\right\|_{L^{1}\left(B_{r}\right)}=\left\|(\boldsymbol{\eta} \circ h) * \varphi_{\bar{\rho}}\right\|_{L^{1}\left(B_{r}\right)} \leq$ $\|\boldsymbol{\eta} \circ h\|_{L^{1}\left(B_{r_{1}}\right)}$ if $\bar{\rho}$ is also chosen small enough such that $r+\bar{\rho}<r_{1}$. Indeed, observe that

$$
\begin{aligned}
\left\|\boldsymbol{\eta} \circ H_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+}\right)} & =\left\|\boldsymbol{\eta} \circ H_{k}\right\|_{L^{1}\left(B_{r}^{+}\right)}+\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+} \backslash B_{r}^{+}\right)} \\
& \stackrel{(15.34)}{\leq} C_{0} \int_{\partial B_{r}^{+}}\left|\boldsymbol{\eta} \circ h_{k}\right|+C_{0} \int_{B_{r}^{+}}\left|\boldsymbol{\eta} \circ \bar{h}_{\bar{\varepsilon}_{1}}\right|+\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+} \backslash B_{r}^{+}\right)} \\
& \stackrel{(15.38)}{\leq} C_{0}\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r}^{+}\right)}+C_{0} \int_{B_{r}^{+}}\left|(\boldsymbol{\eta} \circ h) * \varphi_{\rho}\right|+\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+} \backslash B_{r}^{+}\right)} \\
& \stackrel{(15 \cdot 39)}{\leq} C_{0}\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r}^{+}\right)}+C\|\boldsymbol{\eta} \circ h\|_{L^{1}\left(B_{r}^{+}\right)}+\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+} \backslash B_{r}^{+}\right)} \\
& \leq C\left\|\boldsymbol{\eta} \circ h_{k}\right\|_{L^{1}\left(B_{r_{1}}^{+}\right)}+C\|\boldsymbol{\eta} \circ h\|_{L^{1}\left(B_{r_{1}}^{+}\right)}
\end{aligned}
$$

provided $\rho$ is chosen so small that $\bar{r}+\rho<r$.

We consider the density $\mathbf{d}_{T}$ of the measure $\mathbf{e}_{T}$ with respect to the Lebesgue measure $|\cdot|$, i.e.

$$
\mathbf{d}_{T}(y)=\underset{s \rightarrow 0}{\limsup } \frac{\mathbf{e}_{T}\left(B_{s}(y)\right)}{\pi s^{2}} .
$$

We will drop the subscript $T$ when the current in question is clear from the context. Clearly, under the assumptions of Proposition 14.4, $\left\|\mathbf{d}_{T}\right\|_{L^{1}} \leq C E$. Now, following the approach of [13], we wish to prove an $L^{p}$ estimate for a $p>1$, which is just a geometric constant.
Theorem 16.1. There exist constants $p>1, C$, and $\varepsilon>0$ (depending on $n$ and $Q$ ) such that, if $T$ is as in Proposition 14.4, then

$$
\begin{equation*}
\int_{\{\mathbf{d} \leq 1\} \cap B_{2}} \mathbf{d}^{p} \leq C\left(E+\mathbf{A}^{2}\right)^{p} . \tag{16.1}
\end{equation*}
$$

### 16.1 HIGHER INTEGRABILITY FOR DIR-MINIMIZERS

We start with an analogous estimate for the gradient of Dir-minimizers.
Proposition 16.2. There are constants $q>1, \delta>0$ and $C$ (depending only on $Q$ and n) with the following property. Consider a connected domain $D$ in $\mathbb{R}^{2}$ such that:

- the curvature $\kappa$ of $\partial D$ enjoys the bound $\|\kappa\|_{\infty} \leq \delta$;
- $\partial D \cap B_{16}(x)$ is connected for every $x$.

Let $0<\rho \leq 1$ and $u: B_{8 \rho}(x) \cap D \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ be a Dir-minimizing function such that $\left.u\right|_{\partial D \cap B_{p}(x)}=Q \llbracket g \rrbracket$ for some $C^{1}$ function $g$. Then

$$
\begin{equation*}
\left(f_{B_{\rho}(x) \cap D}|D u|^{2 q}\right)^{\frac{1}{q}} \leq C f_{B_{B_{\rho}(x) \cap D}}|D u|^{2}+C\|D g\|_{\infty}^{2} \tag{16.2}
\end{equation*}
$$

Proof. First of all, the claim follows from [13, Theorem 6.1] when $B_{2 \rho}(x) \subset D$, while it is trivial if $B_{2 \rho}(x) \subset \operatorname{int}\left(D^{c}\right)$. We can thus assume, without loss of generality, that $B_{2 \rho}(x)$ intersects $\partial D$. Let $y$ be a point in such intersection and observe that $B_{\rho}(x) \subset B_{4 \rho}(y)$. The claim thus follows if we can show

$$
\begin{equation*}
\left(f_{B_{r}(y) \cap D}|D u|^{2 q}\right)^{\frac{1}{q}} \leq C f_{B_{2 r}(y) \cap D}|D u|^{2}+C\|D g\|_{\infty}^{2} \tag{16.3}
\end{equation*}
$$

for every $y \in \partial D$ and every $r \leq 4$. We now define

$$
\bar{u}(z)=\sum_{i} \llbracket u_{i}(z)-\eta \circ u(z) \rrbracket,
$$

and observe that $|D u| \leq|D \bar{u}|+Q|D \eta \circ u|$, while $\eta \circ u$ is a classical harmonic function such that $\left.\eta \circ u\right|_{\partial D \cap B_{2}}=g$, and $\bar{u}$ is a Dir-minimizing function such that $\left.\bar{u}\right|_{\partial D \cap B_{2}}=Q \llbracket 0 \rrbracket$. Observe that

$$
\left(f_{B_{r}(y) \cap D}|D \eta \circ u|^{2 q}\right)^{\frac{1}{q}} \leq C f_{B_{2 r}(y) \cap D}|D \eta \circ u|^{2}+C\|D g\|_{\infty}^{2}
$$

is a classical estimate for (single-valued) harmonic functions and that $|D \eta \circ u| \leq|D u|$. Hence, it suffices to prove (16.3) when $g=Q \llbracket 0 \rrbracket$. Moreover without loss of generality we can assume that $y=0$ and $r=1$. Our goal is thus to show

$$
\left\|\left|D u\left\|_{L^{2 q}\left(B_{1} \cap D\right)} \leq C\right\|\right| D u\right\|_{L^{2}\left(B_{2} \cap D\right)},
$$

under the assumption that $\left.u\right|_{\partial D \cap B_{2}}=Q \llbracket 0 \rrbracket$. If we extend $|D u|$ trivially to the complement of $D$, by setting it identically equal to 0 , the inequality is just an higher integrability estimate for the function $|D u|$ on $B_{1}$. By Gehring's lemma, it suffices to prove the existence of a constant $C$ such that

$$
\begin{equation*}
\left\|\left|D u\left\|_{L^{2}\left(B_{\rho}(x)\right)} \leq C\right\|\right| D u\right\|_{L^{1}\left(B_{8 \rho}(x)\right)} \tag{16.4}
\end{equation*}
$$

whenever $B_{8 \rho}(x) \subset B_{2}$. However, in the "interior case" $B_{2 \rho}(x) \subset D$, the stronger

$$
\left\|\left|D u\left\|_{L^{2}\left(B_{\rho}(x)\right)} \leq C\right\|\right| D u\right\|_{L^{1}\left(B_{2 \rho}(x)\right)}
$$

is already proved in [13, Proposition 6.2]. Hence, arguing as above, it suffices to prove (16.4), with the ball $B_{4 \rho}(x)$ replacing $B_{\rho}(x)$ in the left hand side, under the additional assumption $x \in \partial D$. Again by scaling, we are reduced to prove the following estimate

$$
\begin{equation*}
\left\|\left|D u\left\|_{L^{2}\left(B_{1} \cap D\right)} \leq C\right\|\right| D u\right\|_{L^{1}\left(B_{2} \cap D\right)} \quad \text { if } 0 \in \partial D . \tag{16.5}
\end{equation*}
$$

First of all observe that, by our assumptions, if $\delta$ is sufficiently small, for every $r \in(1,2)$ the domain $D \cap B_{r}$ is biLipschitz equivalent to the half disk $B_{r} \cap\left\{\left(x_{1}, x_{2}\right): x_{2}>0\right\}$, with uniform bounds on the Lipschitz constants of the homeomorphism and its inverse. In particular, we recall that, by classical Sobolev space theory, we have

$$
\min _{c \in \mathbb{R}}\|f-c\|_{H^{1 / 2}\left(\partial\left(B_{r} \cap D\right)\right)} \leq C\|D f\|_{L^{1}\left(\partial\left(B_{r} \cap D\right)\right)}
$$

for every classical function $f \in W^{1,1}\left(\partial B_{r}, \mathbb{R}\right)$. Moreover there is an extension $F \in$ $W^{1,2}\left(B_{r} \cap D\right)$ of $f$ such that

$$
\begin{equation*}
\|D F\|_{L^{2}\left(B_{r} \cap D\right)} \leq C\|f-c\|_{H^{1 / 2}\left(\partial\left(B_{r} \cap D\right)\right)} \leq C\|D f\|_{L^{1}\left(\partial\left(B_{r} \cap D\right)\right)} . \tag{16.6}
\end{equation*}
$$

Thus, using Fubini and (16.6), under our assumptions on $u$, we find a radius $r \in(1,2)$ and an extension $v$ of the classical function $\left.\xi \circ u\right|_{\partial\left(B_{r} \cap D\right)}$ to $B_{r} \cap D$ such that

$$
\|D \xi \circ u\|_{L^{2}\left(B_{r} \cap D\right)} \leq C\|D \xi \circ u\|_{L^{1}\left(\partial\left(B_{r} \cap D\right)\right)} \leq C\|D \xi \circ u\|_{L^{1}\left(B_{2} \cap D\right)} \leq C \mid D u \|_{L^{1}\left(D \cap B_{2}\right)} .
$$

If we consider the multivalued function $\boldsymbol{\xi}^{-1} \circ \boldsymbol{\rho} \circ v$, the latter has trace $w:=\boldsymbol{\xi}^{-1} \circ \boldsymbol{\zeta} \circ u$ on $\partial\left(B_{r} \cap D\right)$. Therefore, by minimality of $u$,

$$
\|\mid D u\|_{L^{2}\left(B_{r} \cap D\right)} \leq\|D w\|_{L^{2}\left(B_{r} \cap D\right)} \leq C\|D v\|_{L^{2}\left(B_{r} \cap D\right)} .
$$

Combining the latter inequality with (16.7) we achieve (16.5).

### 16.2 IMPROVED EXCESS ESTIMATES

Proposition 16.3 (Weak excess estimate). For every $\eta>0$, there exists $\varepsilon>o$ with the following property. Let $T$ be area minimizing and assume it satisfies Assumption 14.2 in $\mathbf{C}_{4 s}(x)$. If $E=\mathbf{E}\left(T, \mathbf{C}_{4 s}(x)\right) \leq \varepsilon$, then

$$
\begin{equation*}
\mathbf{e}_{T}(A) \leq \eta_{10} E s^{2}+C \mathbf{A}^{2} s^{4} \tag{16.8}
\end{equation*}
$$

for every $A \subset B_{s}(x) \cap D$ Borel with $|A| \leq \varepsilon\left|B_{s}(x)\right|$.
Proof. Without loss of generality, we can assume $s=1$ and $x=0$. We distinguish the two regimes: $E \leq \mathbf{A}^{2}$ and $\mathbf{A}^{2} \leq E$. In the former, clearly $\mathbf{e}_{T}(A) \leq C E \leq C \mathbf{A}^{2}$. In the latter, we let $f$ be the $E^{\frac{1}{8}}$-Lipschitz approximation of $T$ in $\mathbf{C}_{3}$ and, arguing as for the proof of [13, Theorem 5.2] we find a radius $r \in(1,2)$ and a current $R$ such that

$$
\partial R=\left\langle T-\mathbf{G}_{f}, \varphi, r\right\rangle
$$

and

$$
\mathbf{M}(R) \leq\left(\frac{C}{\delta_{*}}\left(E+\mathbf{A}^{2} r^{2}\right)\right)^{2} \leq C E^{2-\frac{1}{2}}
$$

Therefore, by the Taylor expansion in Remark 5.4 and the minimality of $T$, we observe

$$
\begin{align*}
\|T\|\left(\mathbf{C}_{r}\right) & \leq \mathbf{M}\left(\mathbf{G}_{f}\left\llcorner\mathbf{C}_{r}+R\right) \leq\left\|\mathbf{G}_{f}\right\|\left(\mathbf{C}_{r}\right)+C E^{\frac{3}{2}}\right. \\
& \leq Q\left|B_{r}\right|+\int_{B_{r}} \frac{|D f|^{2}}{2}+C E^{\frac{5}{4}} . \tag{16.9}
\end{align*}
$$

On the other hand, using again the Taylor expansion for the part of the current which coincides with the graph of $f$, we deduce as well that

$$
\begin{equation*}
\|T\|\left(\left(B_{r} \cap K\right) \times \mathbb{R}^{n}\right) \geq Q\left|B_{r} \cap K\right|+\frac{1}{2} \int_{B_{r} \cap K}|D f|^{2}-C E^{\frac{5}{4}} . \tag{16.10}
\end{equation*}
$$

Subtracting (16.10) from (16.9), we deduce

$$
\begin{equation*}
\mathbf{e}_{T}\left(B_{r} \cap D \backslash K\right) \leq \frac{1}{2} \int_{B_{r} \cap D \backslash K}|D f|^{2}+C E^{\frac{5}{4}} \tag{16.11}
\end{equation*}
$$

If $\varepsilon$ is chosen small enough, we infer from (16.11) and (15.1) in Theorem 15.3 that

$$
\mathbf{e}_{T}\left(B_{r} \cap D \backslash K\right) \leq \bar{\eta} E+C E^{1+\gamma}
$$

for a suitable $\bar{\eta}>0$ to be chosen. Let now $A \subset B_{1}$ be such that $|A| \leq \varepsilon \pi$. If $\varepsilon$ is small enough, we can again apply Theorem 15.3 and so by (16.2) there is a Dir-minimizing $w$ such that $|D f|$ is close in $L^{2}$ (with an error $\bar{\eta} E$ ) to $|D w|$ and by [13, Remark 5.5] $\operatorname{Dir}(w) \leq C E$. By Proposition 16.2 we have $\|\mid D w\|_{L^{q}\left(B_{1}\right)} \leq C E^{\frac{1}{2}}$. Therefore, using (15.1) and (15.2), we can deduce

$$
\begin{aligned}
\mathbf{e}_{T}(A) & \leq \int_{A}|D w|^{2}+3 \eta E+C E^{1+\gamma} \\
& \leq C\|D g\|_{\infty}^{2}|A|^{1-2 / q}+C\left(|A|^{1-2 / q}+\bar{\eta}\right) E+C E^{1+\gamma} \\
& \leq C\left(|A|^{1-2 / q}+\bar{\eta}\right) E+C E^{\frac{5}{4}}
\end{aligned}
$$

Hence, for a suitable choice of $\varepsilon$ and $\eta$, (16.8) follows.
16.3 PROOF OF THEOREM 16.1

The proof follows from Proposition 16.3 arguing exactly as in [13, Section 6.3].

## 17

In this section we show how Theorem 16.1 gives a simple proof of the following approximation result analogous to [13, Theorem 2.4].

Theorem 17.1 (Boundary Almgren Strong Approximation). There are geometric constants $\gamma_{1}>0, \varepsilon_{A}>0$, and $C>0$ with the following properties. Let $T$ and $\Gamma$ be as in Assumption 14.2 with $\varepsilon=\varepsilon_{A}$, let $f$ be the $E^{\gamma}$-Lipschitz approximation and $K \subset B_{3 r}$ the corresponding set where $\mathbf{G}_{f}$ and $T$ coincide. Then:

$$
\begin{align*}
\operatorname{Lip}(f) & \leq C\left(E+r^{2} \mathbf{A}^{2}\right)^{\gamma_{1}}  \tag{17.1}\\
\operatorname{osc}(f) & \leq C \mathbf{C}\left(T, \mathbf{C}_{4 r}\right)+C r\left(E+r^{2} \mathbf{A}^{2}\right)^{\frac{1}{2}}  \tag{17.2}\\
\left|B_{r} \backslash K\right|+\mathbf{e}_{T}\left(B_{r} \backslash K\right) & \leq C r^{2}\left(E+r^{2} \mathbf{A}^{2}\right)^{1+\gamma_{1}}  \tag{17.3}\\
\left.\left.\left|\|T\|\left(A \times \mathbb{R}^{n}\right)-Q\right| A \cap D\left|-\frac{1}{2} \int_{A}\right| D f\right|^{2} \right\rvert\, & \leq C r^{2}\left(E+r^{2} \mathbf{A}^{2}\right)^{1+\gamma_{1}} \tag{17.4}
\end{align*}
$$

for every closed set $A \subset B_{r}$.
We postpone the proof till the end of this section however we anticipate that it goes along the same line of [13, Theorem 2.4] using Theorems 17.2 and 17.4 below instead of [13, Theorem 7.1] and [13, Theorem 7.3] respectively. The substantial changes necessary to adapt the argument of the interior case, i.e., [13, Theorem 2.4] concerns mainly the proof of Theorem 17.4 while the proof of Theorem 17.2 is essentially the same as that of [13, Theorem 7.1]. So we start by stating the Almgren's boundary strong excess estimate.

Theorem 17.2 (Almgren's boundary strong excess estimate). There are constants $\varepsilon_{11}, \gamma_{11}>$ 0 and $C>0$ (depending on $n, Q$ ) with the following property. Assume $T$ satisfies Assumption 14.2 in $\mathbf{C}_{4}$ and is area minimizing. If $E=\mathbf{E}\left(T, \mathbf{C}_{4}\right)<\varepsilon_{11}$, then

$$
\begin{equation*}
\mathbf{e}_{T}(A) \leq C\left(\left(E+\mathbf{A}^{2}\right)^{\gamma_{11}}+|A|^{\gamma_{11}}\right)\left(E+\mathbf{A}^{2}\right), \tag{17.5}
\end{equation*}
$$

for every Borel set $A \subset B_{\frac{9}{8}}$.
This estimate complements (16.1) enabling to control the excess also in the region where $\mathbf{d}>1$. We call it boundary strong Almgren's estimate because a similar formula in the interior case can be found in the big regularity paper (cf. [5, Sections 3.24-3.26 and $3.30(8)]$ ) and is a strengthened version of Proposition 16.3 that we called weak excess estimate. To prove (17.5) we construct a suitable competitor to estimate the size of the set $\tilde{K}$ where the graph of the $E^{\beta}$ Lipschitz approximation $f$ differs from $T$. Following

Almgren, we embed $\mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ in a large Euclidean space, via a bilipschitz embedding $\xi$. We then regularize $\boldsymbol{\xi} \circ f$ by convolution and project it back onto $\mathcal{Q}=\boldsymbol{\xi}\left(\mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$. To avoid loss of energy we need a rather special "almost projection" $\rho_{\delta}^{\star}$ that preserves zero boundary data, i.e., $\boldsymbol{\rho}_{\delta}^{\star}(0)=0$.
Proposition 17.3. (Analogue to [13, Proposition 7.2]) For every $n, Q \in \mathbb{N} \backslash\{0\}$ there are geometric constants $\delta_{0}, C>0$ with the following property. For every $\left.\delta \in\right] 0, \delta_{0}[$ there is $\boldsymbol{\rho}_{\delta}^{\star}: \mathbb{R}^{N(Q, n)} \rightarrow \mathcal{Q}=\boldsymbol{\xi}\left(\mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ such that $\boldsymbol{\rho}_{\delta}^{\star}(0)=0,\left|\boldsymbol{\rho}_{\delta}^{\star}(P)-P\right| \leq C \delta^{8^{-n Q}}$ for all $P \in \mathcal{Q}$ and, for every $u \in W^{1,2}\left(\Omega, \mathbb{R}^{N}\right)$, the following holds

$$
\begin{equation*}
\int\left|D\left(\rho_{\delta}^{\star} \circ u\right)\right|^{2} \leq\left(1+C \delta^{8^{-n Q-1}}\right) \int_{\left\{\operatorname{dist}(u, Q) \leq \delta^{n Q+1}\right\}}|D u|^{2}+C \int_{\left\{\operatorname{dist}(u, Q)>\delta^{n Q+1}\right\}}|D u|^{2} . \tag{17.6}
\end{equation*}
$$

Proof. $\rho_{\delta}^{\star}$ is the projection obtained in [13, Proposition 7.2].
Here we show the Strong Excess Approximation of Almgren in our version that takes into account the non-homogeneous boundary value problem, concluding in this way the proof of Theorem 17.1. Theorem 16.1 enters crucially in the argument when estimating the second summand of (17.6) for the regularization of $\xi \circ f$.

### 17.1 REGULARIZATION BY CONVOLUTION WITH A NON CENTERED KERNEL

Here we construct the competitor preserving the boundary conditions.
Proposition 17.4. Let $\beta_{1} \in\left(0, \frac{1}{4}\right)$ and $T$ be an area minimizing current satisfying Assumption 14.2 in $\mathbf{C}_{4}$. Let $f$ be its $E^{\beta_{1}}$-Lipschitz approximation. Then, there exist constants $\bar{\varepsilon}_{12}, \gamma_{12}, C>0$ and a subset of radii $B \subset[9 / 8,2]$ with $|B|>1 / 2$ with the following properties. If $\mathbf{E}\left(T, \mathbf{C}_{4}\right) \leq$ $\bar{\varepsilon}_{12}$, for every $\sigma \in B$, there exists a Q-valued function $h \in \operatorname{Lip}\left(B_{\sigma} \cap D, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ such that

$$
\begin{align*}
\left.h\right|_{B_{\sigma} \cap \partial D} & =g \\
\left.h\right|_{\partial B_{\sigma} \cap D} & =\left.f\right|_{\partial B_{\sigma} \cap D} \\
\operatorname{Lip}(h) & \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}} \\
\int_{B_{\sigma} \cap D}|D h|^{2} & \leq \int_{B_{\sigma} \cap K \cap D}|D f|^{2}+C\left(E+\mathbf{A}^{2}\right)^{1+\gamma_{12}} \tag{17.7}
\end{align*}
$$

Proof. Since $|D f|^{2} \leq C \mathbf{d}_{T} \leq C E^{2 \beta_{1}} \leq 1$ on $K$, by Theorem 16.1 there is $q_{1}=2 p_{1}>2$ such that

$$
\begin{equation*}
\||D f|\|_{L^{q_{1}}\left(K \cap B_{2}\right)}^{2} \leq C\left(E+\mathbf{A}^{2}\right) \tag{17.8}
\end{equation*}
$$

Given two (vector-valued) functions $h_{1}$ and $h_{2}$ and two radii $0<\bar{r}<r$, we denote by lin $\left(h_{1}, h_{2}\right)$ the linear interpolation in $B_{r} \backslash \bar{B}_{\bar{r}}$ between $\left.h_{1}\right|_{\partial B_{r}}$ and $\left.h_{2}\right|_{\partial B_{\bar{r}}}$. More precisely, if $(\theta, t) \in S_{+}^{m-1} \times[0, \infty)$ are spherical coordinates, then

$$
\operatorname{lin}\left(h_{1}, h_{2}\right)(\theta, t)=\frac{r-t}{r-\bar{r}} h_{2}(\theta, t)+\frac{t-\bar{r}}{r-\bar{r}} h_{1}(\theta, t) .
$$

Next, let $\delta>0$ and $\varepsilon>0$ be two parameters and let $1<r_{1}<r_{2}<r_{3}<2$ be three radii, all to be chosen suitably later. First of all extend the function $g$ to the whole disk $B_{3}$ by making it coinstant in the direction $x_{2}$, i.e. $g\left(x_{1}, x_{2}\right)=g\left(x_{1}, \psi_{1}\left(x_{1}\right)\right)$. We then extend the $E^{\beta_{1}}$-Lipschitz approximation to a function $f^{*}$ defined on the entire $B_{3}$ by setting

$$
f^{*}(x)= \begin{cases}f(x) & \text { if } x \in B_{3} \cap D \\ Q \llbracket g(x) \rrbracket & \text { if } x \in B_{3} \cap D^{-}\end{cases}
$$

From now to keep our notation simpler we denote $f^{*}$ as well by $f$. Observe moreover that

$$
\left.(\boldsymbol{\eta} \circ f)\right|_{D^{-}}=g
$$

We next define a translation operator $\oplus: A_{Q}\left(\mathbb{R}^{N}\right) \times \mathbb{R}^{N} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{N}\right)$ setting

$$
T \oplus t=\sum_{i=1}^{Q} \llbracket t_{i}+t \rrbracket \quad \text { for } \quad T=\sum_{i=1}^{Q} \llbracket t_{i} \rrbracket .
$$

We then introduce $\tilde{f}:=f \oplus(-\eta \circ f)$, so that $\left.\tilde{f}\right|_{D^{-}}=Q \llbracket 0 \rrbracket$ and $\eta \circ \tilde{f}=0$.
Next we define, as in the proof of Proposition 15.7, $\varphi_{\varepsilon}(x):=\frac{1}{\varepsilon^{n}} \varphi\left(\frac{x}{\varepsilon}\right)$, and $\varphi(x)=$ $\bar{\varphi}\left(x-z_{0}\right)$ with $\bar{\varphi}$ being the standard bump function with support in $B_{1}(0)$ and $z_{0}:=$ $(0,-2)$. We therefore set

$$
\tilde{h}_{\varepsilon}:=(\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}-g * \varphi_{\varepsilon}+g .
$$

We easily see that $\left.\left(\tilde{h}_{\varepsilon}\right)\right|_{\partial D \cap B_{r_{3}}}=\left.g\right|_{\partial D \cap B_{r_{3}}}$, and

$$
\operatorname{Lip}\left(\tilde{h}_{\varepsilon}\right) \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}}
$$

Recall the maps $\rho_{\delta}^{\star}$ and $\xi$ of [LS11 b, Theorem 2.1] and observe that $\xi(Q \llbracket 0 \rrbracket)=0$ and $\rho_{\delta}^{\star}\left(0_{\mathbb{R}^{n}}\right)=0_{\mathbb{R}^{n}}$. We then set $\tilde{f}_{1}^{\prime}:=\tilde{\xi} \circ \tilde{f}$

$$
\tilde{g}_{\delta, \varepsilon, s}^{\prime}:= \begin{cases}\sqrt{E+\mathbf{A}^{2}} \boldsymbol{\rho} \circ \Phi \circ \operatorname{lin}\left(\frac{\tilde{f}_{\prime}^{\prime} \circ \Phi^{-1}}{\sqrt{E+\mathbf{A}^{2}}} \boldsymbol{\rho}_{\delta}^{\star}\left(\frac{\tilde{f}_{1}^{\prime} \circ \Phi^{-1}}{\sqrt{E+\mathbf{A}^{2}}}\right)\right), & \text { in }\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D, \\ \sqrt{E+\mathbf{A}^{2}} \boldsymbol{\rho} \circ \Phi \circ \operatorname{lin}\left(\boldsymbol{\rho}_{\delta}^{\star}\left(\frac{\tilde{f}_{1}^{\prime} \circ \Phi^{-1}}{\sqrt{E+\mathbf{A}^{2}}}\right), \boldsymbol{\rho}_{\delta}^{\star}\left(\left(\frac{\left.\tilde{f}_{1}^{\prime} * \varphi_{\varepsilon}\right) \Phi^{-1}}{\sqrt{E+\mathbf{A}^{2}}}\right)\right),\right. & \text { in }\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D, \\ \sqrt{E+\mathbf{A}^{2}} \boldsymbol{\rho}_{\delta}^{\star}\left(\frac{\tilde{f}_{1}^{\prime} \times \varphi_{\varepsilon}}{\sqrt{E+\mathbf{A}^{2}}}\right), & \text { in } B_{r_{1}} \cap D,\end{cases}
$$

where $\Phi$ is the diffeomorphism constructed in Proposition 15.6. Now, we define

$$
\begin{equation*}
\hat{h}_{\delta, \varepsilon, s}:=\sum_{i=1}^{Q} \llbracket\left(\xi^{-1} \circ \tilde{g}_{\delta, \varepsilon, S}^{\prime}\right)_{i}-\eta \circ\left(\xi^{-1} \circ \tilde{g}_{\delta, \varepsilon, S}^{\prime}\right) \rrbracket, \text { in } B_{r_{3}} \cap D, \tag{17.9}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{\delta, \varepsilon, s}:=\sum_{i=1}^{Q} \llbracket\left(\xi^{-1} \circ \tilde{g}_{\delta, \varepsilon, s}^{\prime}\right)_{i}-\eta \circ\left(\xi^{-1} \circ \tilde{g}_{\delta, \varepsilon, s}^{\prime}\right)+\tilde{h}_{\varepsilon} \rrbracket, \text { in } B_{r_{3}} \cap D . \tag{17.10}
\end{equation*}
$$

Notice that the convolution of any function $u$ satisfying $u_{\mid B_{3} \backslash D} \equiv 0$ with $\varphi_{\varepsilon}$ for $\varepsilon$ small enough always produces smooth function $u * \varphi_{\varepsilon}$ satisfying $\left(u * \varphi_{\varepsilon}\right)_{\mid B_{3} \backslash D} \equiv 0$, because we have assumed that $\partial D$ is the graph of a Lipschitz function and so it stays inside a cone with fixed angles. With this last fact in mind it is easy to see that $\left(\tilde{g}_{\delta}^{\prime}\right)_{\mid \partial D}=0$, and $\left(h_{\delta}\right)_{\mid \partial D}=g, \eta \circ \hat{h}_{\delta, \varepsilon, s}=0$. We will prove that, for $\sigma:=r_{3}$ in a suitable set $B \subset[9 / 8,2]$ with $|B|>1 / 2$, we can choose $r_{2}=r_{3}-s$ and $r_{1}=r_{2}-s$ so that $h$ satisfies the conclusion of the proposition. Our choice of the parameters will imply the following inequalities:

$$
\begin{equation*}
\delta^{2 \cdot 8^{-n Q}} \leq s, \quad \varepsilon \leq s, \quad \text { and } \quad E^{1-2 \beta_{1}} \leq \varepsilon^{2} \tag{17.11}
\end{equation*}
$$

We estimate the Lipschitz constant of $\tilde{g}_{\delta}^{\prime}$. This can be easily done observing that

- in $B_{r_{1}} \cap D$, we have

$$
\operatorname{Lip}\left(\tilde{g}_{\delta}^{\prime}\right) \leq C \operatorname{Lip}\left(\tilde{f}_{1}^{\prime} * \varphi_{\varepsilon}\right) \leq C \operatorname{Lip}\left(\tilde{f}_{1}^{\prime}\right) \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}}
$$

- in $\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D$, we have

$$
\operatorname{Lip}\left(\tilde{g}_{\delta}^{\prime}\right) \leq C \operatorname{Lip}\left(\tilde{f}_{1}^{\prime}\right)+C \frac{\left\|\tilde{f}_{1}^{\prime}-\tilde{f}_{1}^{\prime} * \varphi_{\varepsilon}\right\|_{L^{\infty}}}{s} \leq C\left(1+\frac{\varepsilon}{s}\right) \operatorname{Lip}\left(\tilde{f}_{1}^{\prime}\right) \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}}
$$

- in $\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D$, we have

$$
\begin{align*}
\operatorname{Lip}\left(\tilde{g}_{\delta}^{\prime}\right) & \leq C \operatorname{Lip}\left(\tilde{f}_{1}^{\prime}\right)+C\left(E+\mathbf{A}^{2}\right)^{1 / 2} \frac{\delta^{8^{-n Q}}}{s} \\
& \leq C E^{\beta_{1}}+C\left(E+\mathbf{A}^{2}\right)^{1 / 2} \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}} \tag{17.12}
\end{align*}
$$

In the first inequality of the last line we have used that, since $\mathcal{Q}$ is a cone, $(E+$ $\left.\mathbf{A}^{2}\right)^{-1 / 2} \tilde{f}_{1}^{\prime}(x) \in \mathcal{Q}$ for every $x$, hence

$$
\left|\boldsymbol{\rho}_{\delta}^{\star}\left(\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E+\mathbf{A}^{2}}}\right)-\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E+\mathbf{A}^{2}}}\right| \leq C \delta^{8^{-n Q}}
$$

From (17.12) and (17.11) we deduce easily that $\tilde{g}_{\delta}^{\prime}$ is continuous and piecewise Lipschitz and so globally Lipschitz and furthermore that

$$
\begin{equation*}
\operatorname{Lip}\left(h_{\delta, \varepsilon, s}\right) \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}} \tag{17.13}
\end{equation*}
$$

In the following Steps 1-3 we estimate the Dirichlet energy of $h_{\delta, \varepsilon, s}$ and finally in Step 4 we obtain the desired estimate (17.7) of Theorem 17.4 for a suitable choice of $\delta, \varepsilon, s$ depending on some powers of the infinitesimal quantity $E$ (see (17.39) below). Before we realize this program, we recall that for every $f \in W^{1,2}\left(\Omega, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ we have

$$
\begin{equation*}
0 \leq \operatorname{Dir}(f \oplus(-\boldsymbol{\eta} \circ f))=\operatorname{Dir}(f)-Q \operatorname{Dir}(\boldsymbol{\eta} \circ f) \tag{17.14}
\end{equation*}
$$

We write here the estimate of the Dirichlet energy of $\tilde{h}_{\varepsilon}$ which will be useful in combination with (17.14).

$$
\begin{align*}
\int\left|D g * \varphi_{\varepsilon}-D g\right|^{2} & \leq C \mathbf{A}^{2} \varepsilon^{2},  \tag{17.15}\\
\left\|D g * \varphi_{\varepsilon}-D g\right\|_{\infty} & \leq C\left\|D^{2} g\right\|_{\infty} \leq C \mathbf{A} \varepsilon, \\
\left|\int\left(D g * \varphi_{\varepsilon}-D g\right)\left(D(\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right)\right| & \leq C \mathbf{A} \varepsilon \int\left|D(\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right| \\
& \leq C \mathbf{A} \varepsilon\left(E+\mathbf{A}^{2}\right)^{\frac{1}{2}} \\
& \leq C \varepsilon\left(E+\mathbf{A}^{2}\right) . \tag{17.16}
\end{align*}
$$

where we used Young's inequality and Remark 15.5. Summing (17.16), (17.15), we obtain

$$
\begin{array}{rl}
\int\left|D \tilde{h}_{\varepsilon}\right|^{2}= & \int\left|D(\eta \circ f) * \varphi_{\varepsilon}\right|^{2}+\int\left|D g * \varphi_{\varepsilon}-D g\right|^{2} \\
& -2 \int\left(D g * \varphi_{\varepsilon}-D g\right)\left(D(\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right) \\
\leq & \int|D(\boldsymbol{\eta} \circ f)|^{2}+C \mathbf{A}^{2} \varepsilon^{2}+C \varepsilon\left(E+\mathbf{A}^{2}\right) \\
\leq C & C|D f|^{2}+C \varepsilon\left(E+\mathbf{A}^{2}\right) .
\end{array}
$$

Step 1. Energy in $B_{r_{3}} \backslash B_{r_{2}}$. By Proposition 17.3, we have $\left|\boldsymbol{\rho}_{\delta}^{\star}(P)-P\right| \leq C \delta^{8-n \ell}$ for all $P \in \mathcal{Q}:=\xi\left(\mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$. Thus, elementary estimates on the linear interpolation give

$$
\begin{align*}
\int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}\left|D \tilde{\delta}_{\delta}^{\prime}\right|^{2} \leq & \left.\frac{C\left(E+\mathbf{A}^{2}\right)}{\left(r_{3}-r_{2}\right)^{2}} \int_{\left(B_{\left.r_{3} \backslash B_{r_{2}}\right) \cap D}\left|\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E+\mathbf{A}^{2}}}-\boldsymbol{\rho}_{\delta}^{\star}\left(\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E+\mathbf{A}^{2}}}\right)\right|^{2}\right.} \quad+C \int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}\left|D \tilde{f}_{1}^{\prime}\right|^{2}+C \int_{\left(B_{\left.r_{3} \backslash B_{r_{2}}\right) \cap D}\left|D\left(\boldsymbol{\rho}_{\delta}^{\star} \circ \tilde{f}_{1}^{\prime}\right)\right|^{2}\right.}^{\leq} \right\rvert\, \\
& C \int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}\left|D \tilde{f}_{1}^{\prime}\right|^{2}+C\left(E+\mathbf{A}^{2}\right) s^{-1} \delta^{2 \cdot 8^{-n \ell}} .
\end{align*}
$$

Hence, using that $\operatorname{Lip}(\mathcal{\xi}) \leq 1$ and (17.14), we estimate

$$
\begin{align*}
\int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}\left|D h_{\delta, \varepsilon, S}\right|^{2} & =\int_{\left(B_{\left.r_{3} \backslash B_{r_{2}}\right) \cap D}\left|D \hat{h}_{\delta, \varepsilon, S}\right|^{2}+Q \int_{\left(B_{r_{3}} \backslash{r_{r}}_{2}\right) \cap D}\left|D \tilde{h}_{\varepsilon}\right|^{2}\right.} \\
& \leq \int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}\left|D \tilde{g}_{\delta}^{\prime}\right|^{2}-Q \int \eta+C \int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}\left|D \tilde{h}_{\varepsilon}\right|^{2} \\
& \leq C \int_{\left(B_{r_{3}} \backslash B_{r_{2}}\right) \cap D}|D f|^{2}+C\left(E+\mathbf{A}^{2}\right)\left(\varepsilon+s^{-1} \delta^{2 \cdot 88^{-n Q}}\right) . \tag{17.18}
\end{align*}
$$

Step 2. Energy in $B_{r_{2}} \backslash B_{r_{1}}$. Here, using the same interpolation inequality and a standard estimate on convolutions of $W^{1,2}$ functions, we get

$$
\begin{aligned}
\int_{\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D}\left|D \tilde{g}_{\delta}^{\prime}\right|^{2} & \leq C \int_{\left(B_{r_{2}+\varepsilon} \backslash B_{r_{1}-\varepsilon}\right) \cap D}\left|D \tilde{f}_{1}^{\prime}\right|^{2}+\frac{C C_{\Phi}}{\left(r_{2}-r_{1}\right)^{2}} \int_{B_{r_{2}} \backslash B_{r_{1}}}\left|\tilde{f}_{1}^{\prime}-\varphi_{\varepsilon} * \tilde{f}_{1}^{\prime}\right|^{2} \\
& \leq C C_{\Phi} \int_{\left(B_{r_{2}+\varepsilon} \backslash B_{r_{1}-\varepsilon}\right) \cap D}\left|D \tilde{f}_{1}^{\prime}\right|^{2}+C C_{\Phi} \varepsilon^{2} s^{-2} \int_{B_{3} \cap D}\left|D \tilde{f}_{1}^{\prime}\right|^{2} \\
& \leq C \int_{\left(B_{r_{2}+\varepsilon} \backslash B_{r_{1}-\varepsilon}\right) \cap D}\left|D \tilde{f}_{1}^{\prime}\right|^{2}+C \varepsilon^{2}\left(E+\mathbf{A}^{2}\right) s^{-2} \\
& \leq C \int_{\left(B_{r_{2}+\varepsilon} \backslash B_{r_{1}-\varepsilon}\right) \cap D}|D f|^{2}+C \varepsilon^{2}\left(E+\mathbf{A}^{2}\right) s^{-2}
\end{aligned}
$$

So coming back to the energy estimate on $h_{\delta, \varepsilon, s}$ we get

$$
\begin{align*}
\int_{\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D}\left|D h_{\delta, \varepsilon, S}\right|^{2} & =\int_{\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D}\left|D \hat{h}_{\delta, \varepsilon, S}\right|^{2}+Q \int_{\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D}\left|D \tilde{h}_{\varepsilon}\right|^{2} \\
& \leq \int_{\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D}\left|D \tilde{g}_{\delta}^{\prime}\right|^{2}+C \int_{\left(B_{r_{2}} \backslash B_{r_{1}}\right) \cap D}\left|D \tilde{h}_{\varepsilon}\right|^{2} \\
& \leq C \int_{\left(B_{r_{2}+\varepsilon} \backslash B_{r_{1}-\varepsilon}\right) \cap D}|D f|^{2}+C \varepsilon^{2}\left(E+\mathbf{A}^{2}\right) s^{-2}+C \varepsilon\left(E+\mathbf{A}^{2}\right) \tag{17.19}
\end{align*}
$$

Step 3. Energy in $B_{r_{1}}$. Define $Z:=\left\{\operatorname{dist}\left(\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E}} * \varphi_{\varepsilon}, \mathcal{Q}\right)>\delta^{n Q+1}\right\} \subseteq D$ and use (17.6) to get

$$
\begin{align*}
& \int_{B_{r_{1} \cap D}}\left|D \tilde{g}_{\delta}^{\prime}\right|^{2} \\
& \leq\left(1+C \delta^{8^{-\bar{n} Q-1}}\right) \int_{\left(B_{\left.r_{1} \cap D\right) \backslash Z}\right.}\left|D\left(\tilde{f}_{1}^{\prime} * \varphi_{\varepsilon}\right)\right|^{2}+C \int_{Z}\left|D\left(\tilde{f}_{1}^{\prime} * \varphi_{\varepsilon}\right)\right|^{2}  \tag{17.20}\\
& =: I_{1}+I_{2}
\end{align*}
$$

We consider $I_{1}$ and $I_{2}$ separately. For $I_{1}$ we first observe the elementary inequality

$$
\begin{align*}
\left\|D\left(\tilde{f}_{1}^{\prime} * \varphi_{\varepsilon}\right)\right\|_{L^{2}}^{2} \leq & \left\|\left(D \tilde{f}_{1}^{\prime}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
\leq & \left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}^{2}+\left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K^{c}}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
& +2\left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}\left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K^{c}}\right) * \varphi_{\varepsilon}\right\|_{L^{2}} \tag{17.21}
\end{align*}
$$

where $K^{c}$ is the complement of $K$ in $D$. Recalling $r_{1}+\varepsilon \leq r_{1}+s=r_{2}$ we estimate the first summand in (17.21) as follows

$$
\begin{equation*}
\left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{2}\left(B_{r_{1}} \cap D\right)}^{2} \leq \int_{B_{r_{1}+\varepsilon} \cap D}\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right)^{2} \leq \int_{B_{r_{2} \cap K}}\left|D \tilde{f}_{1}^{\prime}\right|^{2} \tag{17.22}
\end{equation*}
$$

In order to treat the other terms, recall that $\operatorname{Lip}\left(\tilde{f}_{1}^{\prime}\right) \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}}$ and $\left|K^{c}\right| \leq C(E+$ $\left.A^{2}\right)^{1-2 \beta_{1}}$. Thus, we have

$$
\begin{align*}
\left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K^{c}}\right) * \varphi_{\varepsilon}\right\|_{L^{2}\left(B_{r_{1}} \cap D\right)}^{2} & \leq C\left(E+\mathbf{A}^{2}\right)^{2 \beta_{1}}\left\|\mathbf{1}_{K^{c}} * \varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
& \leq C\left(E+\mathbf{A}^{2}\right)^{2 \beta_{1}}\left\|\mathbf{1}_{K^{c}}\right\|_{L^{1}}^{2}\left\|\varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
& \leq \frac{C\left(E+\mathbf{A}^{2}\right)^{2-2 \beta_{1}}}{\varepsilon^{2}} . \tag{17.23}
\end{align*}
$$

Putting (17.22) and (17.23) in (17.21) and recalling $\left(E+\mathbf{A}^{2}\right)^{1-2 \beta_{1}} \leq \varepsilon^{2}$ and $\int\left|D \tilde{f}_{1}^{\prime}\right|^{2} \leq$ $C\left(E+\mathbf{A}^{2}\right)$, we get

$$
\begin{equation*}
I_{1} \leq \int_{B_{r_{2}} \cap K}\left|D \tilde{f}_{\prime}^{\prime}\right|^{2}+C \delta^{8^{-n Q-1}}\left(E+\mathbf{A}^{2}\right)+C \varepsilon^{-1}\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}} \tag{17.24}
\end{equation*}
$$

For what concerns $I_{2}$, first we argue as for $I_{1}$, splitting in $K$ and $K^{c}$, to deduce that

$$
\begin{equation*}
I_{2} \leq C \int_{Z}\left(\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right)^{2}+C \varepsilon^{-1}\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}} . \tag{17.25}
\end{equation*}
$$

Then, regarding the first summand in (17.25), we note that

$$
\begin{equation*}
|Z| \delta^{2 n Q+2} \leq \int_{B_{r_{1}} \cap D}\left|\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E+\mathbf{A}^{2}}} * \varphi_{\varepsilon}-\frac{\tilde{f}_{1}^{\prime}}{\sqrt{E+\mathbf{A}^{2}}}\right|^{2} \leq C \varepsilon^{2} \tag{17.26}
\end{equation*}
$$

Next, we recall that $q_{1}=2 p_{1}>2$ and use (17.8) to obtain

$$
\begin{align*}
\int_{Z}\left(\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right)^{2} & \leq|Z|^{\frac{p_{1}-1}{p_{1}}}\left\|\left(\left|D \tilde{f}_{1}^{\prime}\right| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{4}}^{2} \\
& \leq C\left(\frac{\varepsilon}{\delta^{n \ell+1}}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}\left\|\left|D \tilde{f}_{1}^{\prime}\right|\right\|_{L^{1_{1}}(K)}^{2} \\
& \leq C\left(\frac{\varepsilon}{\delta^{n Q+1}}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}\left(E+\mathbf{A}^{2}\right) . \tag{17.27}
\end{align*}
$$

Gathering all the estimates together (17.20), (17.24), (17.25) and (17.27) gives

$$
\begin{align*}
\int_{B_{r_{1} \cap D} \cap}\left|D \tilde{g}_{\delta}^{\prime}\right|^{2} \leq & \int_{B_{r_{1} \cap K}}\left|D \tilde{f}_{1}^{\prime}\right|^{2}+C\left(E+\mathbf{A}^{2}\right) \delta^{8^{-n Q-1}}+C \frac{\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}}}{\varepsilon} \\
& +C\left(E+\mathbf{A}^{2}\right)\left(\frac{\varepsilon}{\delta^{n Q+1}}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}} \\
= & \int_{B_{r_{1} \cap K}}|D f|^{2}-Q \int_{B_{r_{1} \cap K} \cap}|D(\boldsymbol{\eta} \circ f)|^{2}+C\left(E+\mathbf{A}^{2}\right) \delta^{8^{-n Q-1}} \\
& +C \frac{\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}}}{\varepsilon}+C\left(E+\mathbf{A}^{2}\right)\left(\frac{\varepsilon}{\delta^{n} Q+1}\right)^{\frac{2\left(p_{\left.p_{1}-1\right)}^{p_{1}}\right.}{}} \tag{17.28}
\end{align*}
$$

Define $Z:=\left\{\operatorname{dist}\left((\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}, \mathcal{Q}\right)>\delta^{n Q+1}\right\}$ to get

$$
\begin{align*}
\int_{B_{r_{1}} \cap D}\left|D(\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right|^{2} & \leq \int_{B_{r_{1}} \cap D \backslash Z}\left|D\left((\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right)\right|^{2}+\int_{Z}\left|D\left((\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right)\right|^{2}  \tag{17.29}\\
& =: \hat{I}_{1}+\hat{I}_{2} .
\end{align*}
$$

We consider $\hat{I}_{1}$ and $\hat{I}_{2}$ separately. For $\hat{I}_{1}$ we first observe the elementary inequality

$$
\begin{align*}
\left\|D\left((\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right)\right\|_{L^{2}}^{2} \leq & \left\|(D(\boldsymbol{\eta} \circ f)) * \varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
\leq & \left\|\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}^{2}+\left\|\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K^{c}}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
& +2\left\|\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{2}}\left\|\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K^{c}}\right) * \varphi_{\varepsilon}\right\|_{L^{2}} . \tag{17.30}
\end{align*}
$$

Recalling $r_{1}+\varepsilon \leq r_{1}+s=r_{2}$, we estimate the first summand in (17.30) as follows

$$
\left\|\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{2}\left(B_{r_{1}} \cap D\right)}^{2} \leq \int_{B_{r_{1}+\varepsilon} \cap D}\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right)^{2} \leq \int_{B_{r_{2}} \cap K}|D(\boldsymbol{\eta} \circ f)|^{2} .(17 \cdot 31)
$$

In order to treat the other terms, recall that $\operatorname{Lip}(\eta \circ f) \leq C\left(E+\mathbf{A}^{2}\right)^{\beta_{1}}$ and $\left|K^{c}\right| \leq$ $C\left(E+\mathbf{A}^{2}\right)^{1-2 \beta_{1}}$. We thus have

$$
\begin{align*}
\left\|\left(|D(\eta \circ f)| \mathbf{1}_{K^{c}}\right) * \varphi_{\varepsilon}\right\|_{L^{2}\left(B_{r_{1}} \cap D\right)}^{2} & \leq C\left(E+\mathbf{A}^{2}\right)^{2 \beta_{1}}\left\|\mathbf{1}_{K^{c}} * \varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
& \leq C\left(E+\mathbf{A}^{2}\right)^{2 \beta_{1}}\left\|\mathbf{1}_{K^{c}}\right\|_{L^{1}}^{2}\left\|\varphi_{\varepsilon}\right\|_{L^{2}}^{2} \\
& \leq \frac{C\left(E+\mathbf{A}^{2}\right)^{2-2 \beta_{1}}}{\varepsilon} . \tag{17.32}
\end{align*}
$$

Putting (17.31) and (17.32) in (17.30), and recalling $E^{1-2 \beta_{1}} \leq \varepsilon^{2}$ and $\int|D(\eta \circ f)|^{2} \leq C E$ we get

$$
\hat{I}_{1} \leq \int_{B_{r_{2}} \cap D \cap K}|D(\boldsymbol{\eta} \circ f)|^{2}+C \varepsilon^{-1}\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}} .
$$

For what concerns $\hat{I}_{2}$, first we argue as for $\hat{I}_{1}$ (splitting in $K$ and $K^{c}$ ) to deduce that

$$
\begin{equation*}
\hat{I}_{2} \leq C \int_{Z}\left(\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right)^{2}+C \varepsilon^{-1}\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}} . \tag{17.34}
\end{equation*}
$$

Then, regarding the first summand in (17.34), we note that

$$
\begin{equation*}
|Z| \delta^{2 n Q+2} \leq \int_{B_{r_{1}} \cap D}\left|\frac{(\eta \circ f)}{\sqrt{E+\mathbf{A}^{2}}} * \varphi_{\varepsilon}-\frac{(\eta \circ f)}{\sqrt{E+\mathbf{A}^{2}}}\right|^{2} \leq C \varepsilon^{2} \tag{17.35}
\end{equation*}
$$

Recalling that $q_{1}=2 p_{1}>2$, we use (17.8) to obtain

$$
\begin{align*}
\int_{Z}\left(\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right)^{2} & \leq|Z|^{\frac{p_{1}-1}{p_{1}}}\left\|\left(|D(\boldsymbol{\eta} \circ f)| \mathbf{1}_{K}\right) * \varphi_{\varepsilon}\right\|_{L^{4}}^{2} \\
& \leq C\left(\frac{\varepsilon}{\delta^{n} Q+1}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}\||D(\eta \circ f)|\|_{L^{q_{1}}(K)}^{2} \\
& \leq C\left(\frac{\varepsilon}{\delta n Q+1}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}\left(E+\mathbf{A}^{2}\right) . \tag{17.36}
\end{align*}
$$

Gathering all the estimates together, (17.29), (17.33), (17.34) and (17.36) gives

$$
\begin{align*}
\int_{B_{r_{1}} \cap D}\left|D(\boldsymbol{\eta} \circ f) * \varphi_{\varepsilon}\right|^{2} \leq & \int_{B_{r_{1} \cap K}}|D(\boldsymbol{\eta} \circ f)|^{2}+C \frac{\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}}}{\varepsilon} \\
& +C\left(E+\mathbf{A}^{2}\right)\left(\frac{\varepsilon}{\delta^{n} Q+1}\right)^{2-\frac{1}{p_{1}}} \tag{17.37}
\end{align*}
$$

So combining (17.28) and (17.37) yields

$$
\begin{align*}
& \int_{B_{r_{1}} \cap D}\left|D h_{\delta, \varepsilon, s}\right|^{2}=\int_{B_{r_{1}} \cap D}\left|D \hat{h}_{\delta, \varepsilon, s}\right|^{2}+Q \int_{B_{r_{1} \cap D}}\left|D \tilde{h}_{\varepsilon}\right|^{2} \\
& \leq \int_{B_{r_{1} \cap D} \cap}\left|D \tilde{g}_{\delta}^{\prime}\right|^{2}+Q \int_{B_{r_{1} \cap D} \cap D}\left|D \tilde{h}_{\varepsilon}\right|^{2} \\
& \leq \int_{B_{r_{1} \cap K} \cap K}|D f|^{2}-Q \int_{B_{r_{1} \cap K} \cap}|D(\eta \circ f)|^{2}+Q \int_{B_{r_{1}} \cap K}|D \eta \circ f|^{2}+C \varepsilon\left(E+\mathbf{A}^{2}\right) \\
&+C\left(\left(E+\mathbf{A}^{2}\right) \delta^{8^{-n Q-1}}+\frac{\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}}}{\varepsilon}+\left(E+\mathbf{A}^{2}\right)\left(\frac{\varepsilon}{\delta^{n Q+1}}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}\right) \\
& \leq \int_{B_{r_{1} \cap K} \cap}|D f|^{2}+C\left(E+\mathbf{A}^{2}\right) \delta^{8^{-n Q-1}}+C \frac{\left(E+\mathbf{A}^{2}\right)^{3 / 2-\beta_{1}}}{\varepsilon} \\
&+C\left(E+\mathbf{A}^{2}\right)\left(\frac{\varepsilon}{\delta^{n Q+1}}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}+C \varepsilon\left(E+\mathbf{A}^{2}\right) . \tag{17.38}
\end{align*}
$$

Step 4. Final estimate. This part is analogue to [13, Step 4 of Proposition 7.3]. Summing (17.18), (17.19), (17.38), and recalling that $\varepsilon<s$, we conclude

$$
\begin{aligned}
\int_{B_{r_{3}} \cap D}\left|D h_{\delta, \varepsilon, s}\right|^{2} \leq & \int_{B_{r_{1} \cap K} \cap}|D f|^{2}+C \int_{\left(B_{r_{1}+3 s} \backslash B_{r_{1}-s}\right) \cap D}\left|D f^{\prime}\right|^{2}+C\left(E+\mathbf{A}^{2}\right)\left(\varepsilon+\delta^{8^{-n \ell-1}}\right) \\
& +C\left(E+\mathbf{A}^{2}\right)\left(\frac{\varepsilon^{2}}{s^{2}}+\frac{\delta^{2 \cdot 8^{-n \ell}}}{s}+\frac{\left(E+\mathbf{A}^{2}\right)^{1 / 2-\beta_{1}}}{\varepsilon}+\left(\frac{\varepsilon}{\delta^{n \ell+1}}\right)^{\frac{2\left(p_{1}-1\right)}{p_{1}}}\right) .
\end{aligned}
$$

We set $\varepsilon=\left(E+\mathbf{A}^{2}\right)^{a}, \delta=\left(E+\mathbf{A}^{2}\right)^{b}$ and $s=\left(E+\mathbf{A}^{2}\right)^{c}$, where

$$
\begin{equation*}
a=\frac{1-2 \beta_{1}}{4}, \quad b=\frac{1-2 \beta_{1}}{8(n Q+1)}, \quad \text { and } \quad c=\frac{1-2 \beta_{1}}{8^{n} Q 8(n Q+1)} \tag{17.39}
\end{equation*}
$$

and we finally let $h$ be the corresponding function $h_{\delta, \varepsilon, s}$. This choice respects (17.11). Assume $\left(E+\mathbf{A}^{2}\right)$ is small enough so that $s \leq \frac{1}{16}$. Now, if $C>0$ is a sufficiently large constant, there is a set $B^{\prime} \subset\left[\frac{9}{8}, \frac{29}{16}\right]$ with $\left|B^{\prime}\right|>1 / 2$ such that,

$$
\int_{\left(B_{\left.r_{1}+3 s \backslash B_{r_{1}-s}\right) \cap D}\left|D f^{\prime}\right|^{2} \leq C s\right.} \int_{B_{2} \cap D}\left|D f^{\prime}\right|^{2} \leq C\left(E+\mathbf{A}^{2}\right)^{1+c} \quad \text { for every } r_{1} \in B^{\prime} .
$$

For $\sigma=r_{3} \in B=2 s+B^{\prime}$ we then conclude the existence of a $\bar{\gamma}\left(\beta_{1}, n, Q\right)>0$ such that

$$
\int_{B_{\sigma} \cap D}|D h|^{2} \leq \int_{B_{\sigma} \cap K}|D f|^{2}+C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}} .
$$

Proof of Theorem 17.2. Here we proceed as in the proof of [13, Theorem 7.1]. Choose $\beta_{1}=\frac{1}{8}$ and consider the set $B \subset[9 / 8,2]$ given in Proposition 17.4. Using the coarea formula and the isoperimetric inequality (the argument and the map $\varphi$ are the same in the proof of Theorem 15.3 and that of Proposition 16.3), we find $s \in B$ and an integer rectifiable current $R$ such that

$$
\partial R=\left\langle T-\mathbf{G}_{f}, \varphi, s\right\rangle \quad \text { and } \quad \mathbf{M}(R) \leq C E^{\frac{3}{2}} .
$$

Since $\left.h\right|_{\partial\left(D \cap B_{s}\right)}=\left.f\right|_{\partial\left(D \cap B_{s}\right)}$ we can use $h$ in place of $f$ in the estimates and, arguing as before (see e.g. the proof of Proposition 16.3), we get, for a suitable $\gamma>0$

$$
\begin{align*}
\|T\|\left(\mathbf{C}_{s}\right) & \leq Q\left|B_{s} \cap D\right|+\frac{1}{2} \int_{B_{s} \cap D}|D g|^{2}+C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}} \\
& \stackrel{(17.7)}{\leq} Q\left|B_{s} \cap D\right|+\frac{1}{2} \int_{B_{s} \cap K}|D f|^{2}+C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}} . \tag{17.40}
\end{align*}
$$

On the other hand, by Taylor's expansion in [13, Remark 5.4],

$$
\begin{align*}
\|T\|\left(\mathbf{C}_{s}\right)= & \|T\|\left(\left(B_{s} \cap D \backslash K\right) \times \mathbb{R}^{n}\right)+\left\|\mathbf{G}_{f}\right\|\left(\left(B_{s} \cap K\right) \times \mathbb{R}^{n}\right) \\
\geq & \|T\|\left(\left(B_{s} \cap D \backslash K\right) \times \mathbb{R}^{n}\right)+Q\left|K \cap B_{s}\right| \\
& +\frac{1}{2} \int_{K \cap B_{s}}|D f|^{2}-C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}} . \tag{17.41}
\end{align*}
$$

Hence, from (17.40) and (17.41), we get $\mathbf{e}_{T}\left(B_{s} \cap D \backslash K\right) \leq C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}}$. This is enough to conclude the proof. Indeed, let $A \subset B_{9 / 8} \cap D$ be a Borel set. Using the higher integrability of $|D f|$ in $K$ (see (17.8)) and possibly selecting a smaller $\bar{\gamma}>0$, we get

$$
\begin{aligned}
\mathbf{e}_{T}(A) & \leq \mathbf{e}_{T}(A \cap K)+\mathbf{e}_{T}(A \backslash K) \\
& \leq \frac{1}{2} \int_{A \cap K}|D f|^{2}+C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}} \\
& \leq C|A \cap K|^{\frac{p_{1}-1}{p_{1}}}\left(\int_{A \cap K}|D f|^{q_{1}}\right)^{2 / q_{1}}+C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}}+ \\
& \leq C|A|^{\frac{p_{1}-1}{p_{1}}}\left(E+\mathbf{A}^{2}\right)+C\left(E+\mathbf{A}^{2}\right)^{1+\bar{\gamma}} .
\end{aligned}
$$

Proof of Theorem 17.1. Here we proceed exactly as in the proof of [13, Theorem 2.4]. Assume $r=1$ and $x=0$. Choose $\beta_{11}<\min \left\{\frac{1}{4}, \frac{\gamma_{11}}{2\left(1+\gamma_{11}\right)}\right\}$, where $\gamma_{11}$ is the constant in Theorem 17.4. Let $f$ be the $E^{\beta_{11}}$-Lipschitz approximation of $T$. Clearly (17.1) and (17.2) follow directly from Proposition 14.4, if $\gamma<\beta_{11}$. Set next $A:=$ $\left\{\mathbf{m e}_{T}>2^{-m}\left(E+\mathbf{A}^{2}\right)^{2 \beta_{11}}\right\} \cap B_{9 / 8}$. By Proposition 14.4 we have $|A| \leq C\left(E+\mathbf{A}^{2}\right)^{1-2 \beta_{11}}$. If $\varepsilon_{A}>0$ is sufficiently small, apply (14.9) and the estimate (17.5) to $A$ in order to conclude

$$
\left|B_{1} \cap D \backslash K\right| \leq C\left(E+\mathbf{A}^{2}\right)^{-2 \beta_{11}} \mathbf{e}_{T}(A) \leq C\left(E+\mathbf{A}^{2}\right)^{\gamma_{11}-2 \beta_{11}\left(1+\gamma_{11}\right)}\left(E+\mathbf{A}^{2}\right) .
$$

By our choice of $\gamma_{11}$ and $\beta_{11}$, this last inequality gives (17.3) for some positive $\gamma_{1}$. Finally, set $S=\mathbf{G}_{f}$. Recalling the strong Almgren estimate (17.5) and the Taylor expansion in [13, Remark 5.4] we conclude for every $0<\sigma \leq 1$

$$
\begin{align*}
& \left|\|T\|\left(\mathbf{C}_{\sigma}\right)-Q\right| D\left|-\int_{B_{\sigma} \cap D} \frac{|D f|^{2}}{2}\right|  \tag{17.42}\\
& \leq \mathbf{e}_{T}\left(B_{\sigma} \cap D \backslash K\right)+\mathbf{e}_{S}\left(B_{\sigma} \cap D \backslash K\right)+\left|\mathbf{e}_{S}\left(B_{\sigma} \cap D\right)-\int_{B_{\sigma} \cap D} \frac{|D f|^{2}}{2}\right|  \tag{17.43}\\
& \leq C\left(E+\mathbf{A}^{2}\right)^{1+\gamma_{11}}+C\left|B_{\sigma} \cap D \backslash K\right|+C \operatorname{Lip}(f)^{2} \int_{B_{\sigma} \cap D}|D f|^{2}  \tag{17.44}\\
& \leq C\left(E+\mathbf{A}^{2}\right)^{1+\gamma_{11}} . \tag{17.45}
\end{align*}
$$

We conclude the proof by noticing that the $L^{\infty}$ bound follows from Proposition 14.4.

## 18

This section is devoted to prove an analog of [21, Theorem 8.13], namely to construct, in a neighborhood of a flat point $p$, a smooth $C^{3, \alpha}$ submanifold with boundary $\Gamma$ and a normal multivalued map $N$ on it. The first is, roughly, an approximation of the average of the sheets lying over the unique tangent plane $V$ to $T$ at $p$. The second is a more accurate approximation of the current $T$, which compared to the one in Section 14 has the additional property of having (almost) zero average.

We start by introducing the spherical excess and the cylindrical excess with respect to a general plane.

Definition 18.1. Given a current $T$ as in Assumption 12.5 and 2-dimensional planes $V, V^{\prime}$, we define the excess of $T$ in balls and cylinders with respect to planes $V, V^{\prime}$ as

$$
\begin{aligned}
\mathbf{E}\left(T, \mathbf{B}_{r}(x), V\right) & :=\left(2 \pi r^{2}\right)^{-1} \int_{\mathbf{B}_{r}(x)}|\vec{T}-\vec{V}|^{2} d\|T\|, \\
\mathbf{E}\left(T, \mathbf{C}_{r}(x, V), V^{\prime}\right): & =\left(2 \pi r^{2}\right)^{-1} \int_{\mathbf{C}_{r}(x, V)}\left|\vec{T}-\vec{V}^{\prime}\right|^{2} d\|T\| .
\end{aligned}
$$

Definition 18.2 (Optimal planes). For the case of balls we define the spherical excess as follows. The optimal spherical excess at some $x \in \operatorname{spt}(T) \backslash \Gamma$ is given by

$$
\begin{equation*}
\mathbf{E}\left(T, \mathbf{B}_{r}(x)\right):=\min _{V} \mathbf{E}\left(T, \mathbf{B}_{r}(x), V\right), \tag{18.1}
\end{equation*}
$$

but in the case of $x \in \Gamma$ we define the optimal boundary spherical excess as

$$
\mathbf{E}^{b}\left(T, \mathbf{B}_{r}(x)\right):=\min \left\{\mathbf{E}\left(T, \mathbf{B}_{r}(x), V\right): V \supset T_{x} \Gamma\right\} .
$$

The plane $V$ which minimizes $\mathbf{E}$, resp. $\mathbf{E}^{b}$, is not unique but since for notational purposes it is convenient to define a unique "height" $\mathbf{h}\left(T, \mathbf{B}_{r}(x)\right)$ we set

$$
\begin{equation*}
\mathbf{h}\left(T, \mathbf{B}_{r}(x)\right):=\min \left\{\mathbf{h}\left(T, \mathbf{B}_{r}(x), V\right): V \text { optimizes } \mathbf{E}\left(\text { resp. } \mathbf{E}^{b}\right)\right\} . \tag{18.2}
\end{equation*}
$$

In the case of cylinders we denote by $\mathbf{E}\left(T, \mathbf{C}_{r}(x, V)\right)=\mathbf{E}\left(T, \mathbf{C}_{r}(x, V), V\right)$ and $\mathbf{h}\left(T, \mathbf{C}_{r}(x, V)\right)=\mathbf{h}\left(T, \mathbf{C}_{r}(x, V), V\right)$.

We recall that under the above assumptions $\mathbf{C}_{5 R_{0}}=\mathbf{C}_{5 R_{0}}\left(0, V_{0}\right)$ and $\mathbf{p}_{\sharp} T\left\llcorner\mathbf{C}_{5 R_{0}}=\right.$ $Q \llbracket D \rrbracket$, where $D \subset B_{5 R_{0}}$ is one of the two connected components in which $B_{5 R_{0}}$ is subdivided by the curve $\gamma=\mathbf{p}(\Gamma)$. Moreover $T_{0} \Gamma=\mathbb{R} \times\{0\}$ and in particular $\Gamma \cap \mathbf{C}_{5 R_{0}}=\{(t, \psi(t))\}=\left\{\left(t, \psi_{1}(t), \bar{\psi}(t)\right)\right\}$, where $\psi_{1}:\left(-5 R_{0}, 5 R_{0}\right) \rightarrow \mathbb{R}$ and $\bar{\psi}:\left(-5 R_{0}, 5 R_{0}\right) \rightarrow \mathbb{R}^{n}$. In particular $\gamma$ is the graph of $\psi_{1}$ and without loss of generality
we assume that $D=\left\{\left(x_{1}, x_{2}\right) \in B_{5 R_{0}}: x_{2}>\psi_{1}\left(x_{1}\right)\right\}$, namely it is the upper half of $B_{5 R_{0}} \backslash \gamma$.

In this section we will then work under the following assumptions.
Assumptions 18.3. $p=q=(0,0), V=V_{0}=\mathbb{R}^{2} \times\{0\}, Q, T$, and $\Gamma$ are as in Assumption 14.2 in the cylinder $C_{5 R_{0}}$, where $R_{0} \geq 1+\sqrt{2}$ is a sufficiently large geometric constant which will be specified later. Moreover $Q \llbracket V_{0} \rrbracket$ is the (unique) tangent cone to $T$ at 0 . We moreover assume in the sequel that

$$
\mathbf{E}\left(T, \mathbf{C}_{5 R_{0}}\left(0, V_{0}\right)\right)+\mathbf{A}^{2} \leq \varepsilon_{C M},
$$

for some small positive parameter $\varepsilon_{C M}=\varepsilon_{C M}\left(n, Q, R_{0}\right)$.
Under the above assumptions we show now that the height of $T$ in $\mathrm{C}_{4 R_{0}}$ is also under control.

Lemma 18.4. There are constants $\varepsilon_{C M}, C$ depending on $Q, n$ and $R_{0}$ such that, if Assumption 18.3 holds, then for all $p \in \Gamma$ and $r>0$ such that $\mathbf{C}_{5 r}\left(p, V_{0}\right) \subset \mathbf{C}_{5 R_{0}}$, we have

$$
\begin{equation*}
\mathbf{h}\left(T, \mathbf{C}_{4 r}\left(p, V_{0}\right)\right) \leq \operatorname{Cr}\left(\mathbf{E}\left(T, \mathbf{C}_{5 r}\left(p, V_{0}\right)\right)+r \mathbf{A}\right)^{\frac{1}{2}} . \tag{18.3}
\end{equation*}
$$

Proof. We divide the proof into two steps.
Step 1: $\sup _{z \in \operatorname{spt}(T) \cap \mathrm{C}_{4 r}\left(p, V_{0}\right)}\left|\mathbf{p}_{V_{0}}^{\perp}(z-p)\right|^{2} \leq \mathrm{Cr}^{-2} \int_{\mathbf{C}_{9 r / 2}\left(p, V_{0}\right)}\left|\mathbf{p}_{V_{0}}^{\perp}(z-p)\right|^{2} d\|T\|(z)+C_{0} \mathbf{A}^{2} r^{4}$.
This is shown in [21, Lemma 6.6] and carries over word by word to our setting as the only part where the stationarity of the associated integral varifold is needed, is for the harmonicity of the coordinate functions. This however is true, as we test with functions which are supported away from the boundary of $T$. We use this to apply a Moser iteration scheme and estimate the $L^{\infty}$ norm by the limsup of the $L^{p}$ norms as $p \rightarrow \infty$.
Step 2: $r^{-2} \int_{\mathbf{C}_{9 r / 2}\left(p, V_{0}\right)}\left|\mathbf{p}_{V_{0}}^{\perp}(z-p)\right|^{2} d\|T\|(z) \leq C \mathbf{E}\left(T, \mathbf{C}_{5 r}\left(p, V_{0}\right)\right) r^{2}+C \mathbf{A} r^{3}$.
Also for this, the proof of [21, Lemma 6.7] carries over as the difference to our situation is a factor $Q$ in the monotonicity formula (Theorem 11.2). From there, we estimate the remainder term by $r^{2}\left(\mathbf{E}\left(T, \mathbf{C}_{5 r}\left(0, V_{0}\right)\right)+\mathbf{A}\right)$.

### 18.1 WHITNEY DECOMPOSITION

We specify next some notation which will be recurrent when dealing with squares inside $V_{0}$. For each $j \in \mathbb{N}, \mathscr{C}_{j}$ denotes the family of closed squares $L$ of $V_{0}$ of the form

$$
\begin{equation*}
\left[a_{1}, a_{1}+2 \ell\right] \times\left[a_{2}, a_{2}+2 \ell\right] \times\{0\} \subset V_{0} \tag{18.4}
\end{equation*}
$$

which intersect $D$, where $2 \ell=2^{1-j}=: 2 \ell(L)$ is the side-length of the square, $a_{i} \in 2^{1-j} \mathbb{Z}$ $\forall i$ and we require in addition $-4 \leq a_{i} \leq a_{i}+2 \ell \leq 4$. To avoid cumbersome notation, we will usually drop the factor $\{0\}$ in (18.4) and treat each squares, its subsets and its points as subsets and elements of $\mathbb{R}^{2}$. Thus, for the center $x_{L}$ of $L$ we will use the notation $x_{L}=\left(a_{1}+\ell, a_{2}+\ell\right)$, although the precise one is $\left(a_{1}+\ell, a_{2}+\ell, 0, \ldots, 0\right)$. Next we set $\mathscr{C}:=\bigcup_{j \in \mathbb{N}} \mathscr{C}_{j}$. If $H$ and $L$ are two squares in $\mathscr{C}$ with $H \subset L$, then we call $L$ an ancestor of $H$ and $H$ a descendant of $L$. When in addition $\ell(L)=2 \ell(H), H$ is a child of $L$ and $L$ the parent of $H$. Moreover, if $H \cap L \neq \varnothing$ but they are not contained in each other, we call them neighbours.

Definition 18.5. A Whitney decomposition of $\bar{D} \cap[-4,4]^{2} \subset V_{0}$ consists of a closed set $\Delta \subset[-4,4]^{2} \cap \bar{D}$ and a family $\mathscr{W} \subset \mathscr{C}$ satisfying the following properties:
(W1) $\Delta \cup \bigcup_{L \in \mathscr{W}} L \cap \bar{D}=[-4,4]^{2} \cap \bar{D}$ and $\Delta$ does not intersect any element of $\mathscr{W}$;
(w2) the interiors of any pair of distinct squares $L_{1}, L_{2} \in \mathscr{W}$ are disjoint;
(w3) if $L_{1}, L_{2} \in \mathscr{W}$ have nonempty intersection, then $\frac{1}{2} \ell\left(L_{1}\right) \leq \ell\left(L_{2}\right) \leq 2 \ell\left(L_{1}\right)$.
Remark 18.6. Because of (WI) we will assume that any $L \in \mathscr{W}$ intersects $\bar{D}$.
Observe that (w1) - (w3) imply

$$
\begin{equation*}
\operatorname{sep}(\Delta, L):=\inf \{|x-y|: x \in L, y \in \Delta\} \geq 2 \ell(L) \quad \text { for every } L \in \mathscr{W} \tag{18.5}
\end{equation*}
$$

since there is an infinite chain of neighbouring squares $\left\{L_{i}\right\}_{i \in \mathbb{N}}$ with $L_{0}=L, \operatorname{dist}\left(\boldsymbol{\Delta}, L_{i}\right) \rightarrow$ 0 and $\ell\left(L_{i}\right) \geq 2 \ell\left(L_{i+1}\right)$ for all $i$. However, we do not require any inequality of the form sep $(\Delta, L) \leq C \ell(L)$, although this would be customary for what is commonly called a Whitney decomposition in the literature.

Assumptions 18.7. In the rest of this section we will use several different parameters:
(a) $\delta_{1}$ and $\beta_{1}$ are two small geometric constants which depends only on $Q, n$, the constant $\gamma_{1}$ of Theorem 17.1, in fact they will be chosen smaller than $\frac{\gamma_{1}}{8}$ and $\delta_{1} \leq \frac{\beta_{1}}{2}$;
(b) $M_{0}$ is a large geometric constant which depends only on $\delta_{1}$, while $N_{0} \geq \frac{\ln (132 \sqrt{2})}{\ln (2)}$ is a large natural number which will be chosen depending on $\beta_{1}, \delta_{1}$, and $M_{0}$;
(c) $C_{e}^{b}$ is a large constant $C_{e}^{b}\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}\right)$, while $C_{e}^{\natural}$ is larger and depends also on $C_{e}^{b}$;
(d) $C_{h}$ is large and depends on $\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{b}$ and $C_{e}^{\natural}$;
(e) the small threshold $\varepsilon_{C M}$ is the last to be chosen, it depends on all the previous parameters and also on the constant $\varepsilon_{A}$ of Theorem 17.1.

Definition 18.8. For each square $L \in \mathscr{C}$ we set $r_{L}:=\sqrt{2} M_{0} \ell(L)$ and we say that $L$ is an interior square if $\operatorname{dist}\left(x_{L}, \gamma\right) \geq 64 r_{L}$, otherwise we say that $L$ is a boundary square and we use, respectively, the notation $\mathscr{C}^{\natural}$ for the interior squares contained in $D$ and $\mathscr{C}^{b}$ for the boundary squares. Next, we define a corresponding $(n+2)$-dimensional balls $\mathbf{B}_{L}$, resp. $\mathbf{B}_{L}^{b}$, for such $L^{\prime}$ s:
(a) If $L \in \mathscr{C}^{\natural}$, we pick a point $p_{L}=\left(x_{L}, y_{L}\right) \in \operatorname{spt}(T) \cap\left(\left\{x_{L}\right\} \times \mathbb{R}^{n}\right)$ and we set $\mathbf{B}_{L}:=\mathbf{B}_{64 r_{L}}\left(p_{L}\right)$;
(b) If $L \in \mathscr{C}^{b}$, we pick $x_{L}^{b}=\left(t, \psi_{1}(t)\right) \in \gamma$ such that $\operatorname{dist}\left(x_{L}, \gamma\right)=\left|x_{L}^{b}-x_{L}\right|$, define $p_{L}^{b}=(t, \psi(t)) \in \Gamma \cap\left(\left\{x_{L}^{b}\right\} \times \mathbb{R}^{n}\right)$ and $\operatorname{set} \mathbf{B}_{L}^{b}=\mathbf{B}_{2^{7} 64 r_{L}}\left(p_{L}^{b}\right)$.

We are now ready to prescribe $N_{0}$ : we require the inequality

$$
\begin{equation*}
2^{7} 64 r_{L} \leq 2^{7} 64 \sqrt{2} M_{0} 2^{-N_{0}} \leq 1 \tag{18.6}
\end{equation*}
$$

so that, in particular, all the balls $\mathbf{B}_{L}$ and $\mathbf{B}_{L}^{b}$ considered above are contained in the cylinder $\mathrm{C}_{4 R_{0}}$.

The following remark will be useful in the sequel.
Remark 18.9. If $L \in \mathscr{C}^{b}$ and $J$ is the parent of $L$, then $J \in \mathscr{C}^{b}$, while if $L \in \mathscr{C}^{\natural}$, then every child of $L$ is an element of $\mathscr{C}^{\natural}$. In fact, if $H$ and $L$ are two squares with nonempty intersection, $\ell(H)<\ell(L)$ and $H$ is a boundary cube, then necessarily $L$ is a boundary cube too.
Remark 18.10. Fix $L \in \mathscr{C}^{b}$ and subdivide it into the canonical four squares $M$ with half the sidelength. For $M$ any of the following three cases can occur: $M$ might be a boundary square, an interior square, or might simply not belong to $\mathscr{C}^{b} \cup \mathscr{C}^{\natural}$ (i.e. $M \cap \bar{D}=\varnothing$ ). However, because of the enlarged radius for boundary squares, it still holds that the ball of a child is contained in the ball of its parent (compare to Proposition 19.1 (i)). Moreover, $\mathbf{B}_{L}^{b} \supset L$ for any boundary square $L$.

We are now ready to define the refining procedure leading to the desired Whitney decomposition.

Definition 18.11. First of all we set $m_{0}:=\mathbf{E}\left(T, \mathbf{C}_{5 R_{0}}\right)+\|\psi\|_{C^{3, \alpha}(]-5 R_{0}, 5 R_{0}[)}^{2}$. We start with all $L \in \mathscr{C}^{b} \cup \mathscr{C}^{\sharp}$ with $\ell(L)=2^{-N_{0}}$ and we assign all of them to $\mathscr{S}$. Next, inductively, for each $j>N_{0}$ and each $L \in \mathscr{C}_{j}^{b} \cup \mathscr{C}_{j}^{\natural}$ such that its parent belongs to $\mathscr{S}$ we assign to $\mathscr{S}$ or to $\mathscr{W}=\mathscr{W}^{e} \cup \mathscr{W}^{h} \cup \mathscr{W}^{n}$ in the following way:
(EX) $L \in \mathscr{W}^{e}$ if $\mathbf{E}\left(T, \mathbf{B}_{L}\right)>C_{e}^{\natural} \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}}$, resp. if $\mathbf{E}^{b}\left(T, \mathbf{B}_{L}^{b}\right)>C_{e}^{b} \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}}$;
(HT) $L \in \mathscr{W}^{h}$ if $L \notin \mathscr{W}^{e}$ and $\mathbf{h}\left(T, \mathbf{B}_{L}\right) \geq C_{h} \boldsymbol{m}_{0}^{\frac{1}{4}} \ell(L)^{1+\beta_{1}}, \operatorname{resp} . \mathbf{h}\left(T, \mathbf{B}_{L}^{b}\right) \geq C_{h} \boldsymbol{m}_{0}^{\frac{1}{4}} \ell(L)^{1+\beta_{1}}$;
(NN) $L \in \mathscr{W}^{n}$ if $L \notin \mathscr{W}^{h} \cup \mathscr{W}^{e}$ but there is a $J \in \mathscr{W}$ such that $\ell(J)=2 \ell(L)$ and $L \cap J \neq \varnothing ;$
(S) $L \in \mathscr{S}$ if none of three conditions above are satisfied.

We denote by $\mathscr{C}_{j}^{b}:=\mathscr{C}^{b} \cap \mathscr{C}_{j}, \mathscr{C}_{j}^{\sharp}:=\mathscr{C}^{\sharp} \cap \mathscr{C}_{j}, \mathscr{S}_{j}:=\mathscr{S} \cap \mathscr{C}_{j}, \mathscr{W}_{j}:=\mathscr{W} \cap \mathscr{C}_{j}, \mathscr{W}_{j}^{e}:=\mathscr{W}^{e} \cap \mathscr{C}_{j}$, $\mathscr{W}_{j}^{h}:=\mathscr{W}^{h} \cap \mathscr{C}_{j}$ and $\mathscr{W}_{j}^{n}:=\mathscr{W}^{n} \cap \mathscr{C}_{j}$. Finally, we set

$$
\begin{equation*}
\Delta:=\left([-4,4]^{2} \cap \bar{D}\right) \backslash \bigcup_{L \in \mathscr{W}} L=\bigcap_{j \geq N_{0}} \bigcup_{L \in \mathscr{S}_{j}} L \tag{18.7}
\end{equation*}
$$

A simple consequence of our refining procedure is the following proposition which we will prove in the next section.

Proposition 18.12. Let $V_{0}, Q, T$, and $\Gamma$ be as in Assumption 18.3 and assume the parameter $N_{0}$ satisfies (18.6). Then $(\boldsymbol{\Delta}, \mathscr{W})$ is a Whitney decomposition of $\bar{D} \cap[-4,4]^{2}$. Moreover, for any choice of $M_{0}$ and $N_{0}$, there is $C^{\star}\left(M_{0}, N_{0}\right)$ such that, if $C_{e}^{b}$, and $C_{e}^{\natural} / C_{e}^{b}, C_{h} / C_{e}^{\natural}$, are larger than $C^{\star}$, then
(a) $\mathscr{W}_{N_{0}}=\varnothing$;
(b) if $L \in \mathscr{C}^{\natural} \cap \mathscr{W}^{e}$ then the parent of $L$ belongs to $\mathscr{C}^{\natural}$.

Moreover, the following estimates hold for some geometric constant $C$ depending on $\beta_{1}$ and $\delta_{1}$, provided $\varepsilon_{C M}$ is sufficiently small (depending on all the previous parameters as detailed in Assumption 18.7)

$$
\begin{align*}
& \mathbf{E}^{b}\left(T, \mathbf{B}_{L}^{b}\right) \leq C C_{e}^{b} \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}} \text {, and } \mathbf{h}\left(T, \mathbf{B}_{L}^{b}\right) \leq C C_{h} \boldsymbol{m}_{0}^{\frac{1}{4}} \ell(L)^{1+\beta_{1}}, \forall L \in \mathscr{W} \cap \mathscr{C}^{b},  \tag{18.8}\\
& \mathbf{E}\left(T, \mathbf{B}_{L}\right) \leq C C_{e}^{\natural} \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}} \text { and } \mathbf{h}\left(T, \mathbf{B}_{L}\right) \leq C C_{h} \boldsymbol{m}_{0}^{\frac{1}{4}} \ell(L)^{1+\beta_{1}}, \forall L \in \mathscr{W} \cap \mathscr{C}^{\natural} . \tag{18.9}
\end{align*}
$$

### 18.2 CONSTRUCTION OF THE CENTER MANIFOLD

First of all for each $\mathbf{B}_{L}$ and $\mathbf{B}_{L}^{b}$, we let $V_{L}$ be the choice of optimal plane for the excess and the height in the sense of Definition 18.2: note that for boundary squares, namely in $\mathbf{B}_{L}^{b}$, the plane $V_{L}$ optimizes the excess $\mathbf{E}^{b}$, and thus it is constrained to contain the line $T_{p_{L}^{b}} \Gamma$. The following key lemma allows us to apply Theorem 17.1 (and its interior version [13, Theorem 2.4]) to corresponding cylinders.

Lemma 18.13. For any choice of the other parameters, if $\varepsilon_{C M}$ is sufficiently small, the following holds for every $L \in \mathscr{S} \cup \mathscr{W}$.
(a) If $L \in \mathscr{C} \mathscr{C}^{\natural}$, then $T$ satisfies the assumptions of [13, Theorem 2.4] in $\mathbf{C}_{32 r_{L}}\left(p_{L}, V_{L}\right)$.
(b) If $L \in \mathscr{C}^{b}$, then $T$ satisfies the assumptions of Theorem 17.1 in $\mathbf{C}_{2^{7} 32 r_{L}}\left(p_{L}^{b}, V_{L}\right)$.

The corresponding $Q$-valued strong Lipschitz approximations will be denoted by $f_{L}$ and will be called $V_{L}$-approximations.

Given a square $L \in \mathscr{C}^{b}$ which belongs to $\mathscr{S} \cup \mathscr{W}$, we denote by $D_{L} \subset B_{2^{7} 24 r_{L}}\left(p_{L}^{b}, V_{L}^{b}\right)$ the domain of the function $f_{L}$, which coincides with the orthogonal projection on $p_{L}^{b}+V_{L}^{b}$ of $\operatorname{spt}(T) \cap \mathbf{C}_{2^{7} 24 r_{L}}\left(p_{L}^{b}, V_{L}^{b}\right)$. Note in particular that $\partial D_{L} \cap B_{2^{7} 24 r_{L}}\left(p_{L^{\prime}}^{b}, V_{L}^{b}\right)$ is the projection of $\Gamma \cap \mathbf{C}_{2^{7} 24 r_{L}}\left(p_{L}^{b}, V_{L}^{b}\right)$ onto $p_{L}^{b}+V_{L}^{b}$, which we will denote by $\gamma_{L}$. Likewise, we denote by $g_{L}$ the function over $\gamma_{L}$ whose graph gives $\Gamma \cap \mathbf{C}_{2^{7} 24 r_{L}}\left(p_{L}^{b}, V_{L}^{b}\right)$. In particular, Theorem 17.1 implies that $\left.f_{L}\right|_{\gamma_{L}}=Q \llbracket g_{L} \rrbracket$. We now regularize the averages $\eta \circ f_{L}$ to suitable harmonic functions $h_{L}$ in the following fashion.
Definition 18.14. We denote by $h_{L}$ the harmonic function on $B_{16 r_{L}}\left(p_{L}, V_{L}\right)$, resp. $D_{L} \cap$ $B_{2^{7} 16 r_{L}}\left(p_{L}^{b}, V_{L}\right)$, for $L \in \mathscr{C}^{\natural}$, resp. $L \in \mathscr{C}^{b}$, such that the boundary value of $h_{L}$ on the respective domain is given by $\eta \circ f_{L}$ (in particular it coincides with $g_{L}$ on $\gamma_{L}$ ). $h_{L}$ will be called tilted harmonic interpolating function.

In order to complete the description of our algorithm we need a second important technical lemma.

Lemma 18.15. Consider $L \in \mathscr{S} \cup \mathscr{W}$. For every $L \in \mathscr{C}^{b}$, resp. $L \in \mathscr{C}^{\natural}$, there is a smooth function $u_{L}: D \cap B_{2^{78} r_{L}}\left(\mathbf{p}_{0}\left(p_{L}^{\mathrm{b}}\right), V_{0}\right) \rightarrow V_{0}^{\perp}$, resp. $u_{L}: B_{8 r_{L}}\left(\mathbf{p}_{0}\left(p_{L}\right), V_{0}\right) \rightarrow V_{0}^{\perp}$, such that

$$
\begin{align*}
& \mathbf{G}_{u_{L}}\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}^{b}, V_{0}\right)=\mathbf{G}_{h_{L}}\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}^{b}, V_{0}\right), \quad\right. \text { resp. }\right.  \tag{18.10}\\
& \mathbf{G}_{u_{L}}\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}, V_{0}\right)=\mathbf{G}_{h_{L}}\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}, V_{0}\right) .\right.\right. \tag{18.11}
\end{align*}
$$

The function $u_{L}$ will be called interpolating function.
The center manifold is the result of gluing the interpolating functions appropriately. To that we fix a bump function $\vartheta \in C_{c}^{\infty}\left(\left(-\frac{3}{2}, \frac{3}{2}\right)^{2}\right)$ which is identically 1 on $[-1,1]^{2}$ and define

$$
\vartheta_{L}(x):=\vartheta\left(\frac{x-x_{L}}{\ell(L)}\right) .
$$

Hence, for any fixed $j \geq N_{0}$ we define

$$
\begin{equation*}
\mathscr{P}^{j}:=\mathscr{S}_{j} \cup \bigcup_{i \leq j} \mathscr{W}_{i} \tag{18.12}
\end{equation*}
$$

and the following function $\boldsymbol{\varphi}_{j}$, defined over $D \cap[-4,4]^{2} \subset V_{0}$ and taking values in $V_{0}^{\perp}$

$$
\begin{equation*}
\boldsymbol{\varphi}_{j}(x):=\frac{\sum_{L \in \mathscr{P j}} \vartheta_{L}(x) u_{L}(x)}{\sum_{H \in \mathscr{P} j} \vartheta_{H}(x)} . \tag{18.13}
\end{equation*}
$$

The center manifold is the graph of the function $\boldsymbol{\varphi}$ which is the limit of $\boldsymbol{\varphi}_{j}$ as explained in the statement of the next theorem.

Theorem 18.16 (Center manifold). Let $T$ be as in Assumption 18.3 and assume that the parameters satisfy the conditions of Assumption 18.7. Then there is a positive $\omega$ (depending only on $\delta_{1}$ and $\beta_{1}$ ), with the following properties:
(a) $\left.\boldsymbol{\varphi}_{j}\right|_{\gamma}=g$ for every $j$;
(b) $\left\|\boldsymbol{\varphi}_{j}\right\|_{C^{3, \omega}} \leq$ Cm $_{0}^{\frac{1}{2}}$ for some constant $C$ which depends on $\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{\natural}, C_{e}^{b}$, and $C_{h}$, but not on $\varepsilon_{C M}$;
(c) For every $k, k^{\prime} \geq j+2, \boldsymbol{\varphi}_{k}=\boldsymbol{\varphi}_{k^{\prime}}$ on every cube $L \in \mathscr{W}_{j}$;
(d) $\boldsymbol{\varphi}_{j}$ converges uniformly to a $C^{3, \omega}$ function $\boldsymbol{\varphi}$.

Definition 18.17. The graph of the function $\varphi$ will be called center manifold and denoted by $\mathcal{M}$. We will define $\boldsymbol{\Phi}(x):=(x, \boldsymbol{\varphi}(x))$ as the graphical parametrization of $\mathcal{M}$ over $[-4,4]^{2} \cap \bar{D}$. The set $\boldsymbol{\Phi}(\boldsymbol{\Delta})$ will be called the contact set, while for every $L \in \mathscr{W}$ the corresponding $\mathcal{L}:=\boldsymbol{\Phi}(L \cap D)$ will be called Whitney region.

### 18.3 THE $\mathcal{M}$-NORMAL approximation and Related estimates

In what follows we assume that the conclusions of Theorem 18.16 apply. For any Borel set $\mathcal{V} \subset \mathcal{M}$ we will denote by $|\mathcal{V}|$ its $\mathcal{H}^{2}$-measure and will write $\int_{\mathcal{V}} f$ for the integral of $f$ with respect to $\mathcal{H}^{2} . \mathcal{B}_{r}(q)$ denotes the geodesic balls in $\mathcal{M}$. Moreover, we refer to [14] for all the relevant notation pertaining to the differentiation of (multiple valued) maps defined on $\mathcal{M}$, induced currents, differential geometric tensors and so on.

Assumptions 18.18. We fix the following notation and assumptions.
(U) $\mathbf{U}:=\{x+y: x \in \mathcal{M},|y|<1$, and $y \perp \mathcal{M}\}$.
(P) $\mathbf{p}: \mathbf{U} \rightarrow \mathcal{M}$ is the map defined by $(x+y) \mapsto x$.
(R) For any choice of the other parameters, we assume $\varepsilon_{C M}$ to be so small that $\mathbf{p}$ extends to $C^{2, \kappa}(\overline{\mathbf{U}})$ and $\mathbf{p}^{-1}(y)=y+\overline{B_{1}\left(0,\left(T_{y} \mathcal{M}\right)^{\perp}\right)}$ for every $y \in \mathcal{M}$.
(L) We denote by $\partial_{l} \mathbf{U}:=\mathbf{p}^{-1}(\partial \mathcal{M})$ the lateral boundary of $\mathbf{U}$.

The following is then a corollary of Theorem 18.16 and the construction algorithm.
Corollary 18.19. Under the hypotheses of Theorem 18.16 and of Assumption 18.18 we have:
(i) $\operatorname{spt}\left(\partial(T\llcorner\mathbf{U})) \subset \partial_{l} \mathbf{U}, \operatorname{spt}\left(T\left\llcorner\left[-\frac{7}{2}, \frac{7}{2}\right]^{2} \times \mathbb{R}^{n}\right) \subset \mathbf{U}\right.\right.$ and $\mathbf{p}_{\sharp}(T\llcorner\mathbf{U})=Q \llbracket \mathcal{M} \rrbracket$;
(ii) $\operatorname{spt}(\langle T, \mathbf{p}, \boldsymbol{\Phi}(q)\rangle) \subset\left\{y:|\boldsymbol{\Phi}(q)-y| \leq \operatorname{Cm}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}}\right\}$ for every $q \in L \in \mathscr{W}$, where $C$ depends on all the parameters except $\varepsilon_{C M}$;
(iii) $\langle T, \mathbf{p}, p\rangle=Q \llbracket p \rrbracket$ for every $p \in \boldsymbol{\Phi}(\Delta) \cup(\Gamma \cap \partial \mathcal{M})$.

The main reason for introducing the center manifold of Theorem 18.16 is that we are able to pair it with a good approximating map defined on it.

Definition 18.20 ( $\mathcal{M}$-normal approximation). An $\mathcal{M}$-normal approximation of $T$ is given by a pair $(\mathcal{K}, F)$ such that
(A1) $F: \mathcal{M} \rightarrow \mathcal{A}_{Q}(\mathbf{U})$ is Lipschitz (with respect to the geodesic distance on $\mathcal{M}$ ) and takes the special form $F(x)=\sum_{i} \llbracket x+N_{i}(x) \rrbracket$, with $N_{i}(x) \perp T_{x} \mathcal{M}$.
(A2) $\mathcal{K} \subset \mathcal{M}$ is closed and $\mathbf{T}_{F}\left\llcorner\mathbf{p}^{-1}(\mathcal{K})=T\left\llcorner\mathbf{p}^{-1}(\mathcal{K})\right.\right.$.
(A3) $\mathcal{K}$ contains $\boldsymbol{\Phi}\left(\Delta \cap\left[-\frac{7}{2}, \frac{7}{2}\right]^{2}\right)$ and $\Gamma \cap \boldsymbol{\Phi}\left(\bar{D} \cap\left[-\frac{7}{2}, \frac{7}{2}\right]^{2}\right)$, and on the latter two sets the map $N$ equals $Q \llbracket 0 \rrbracket$.
The map $N=\sum_{i} \llbracket N_{i} \rrbracket: \mathcal{M} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{2+n}\right)$ is the normal part of $F$.
Theorem 18.21 (Existence and local estimates for the $\mathcal{M}$-normal approximation). Let $\gamma_{2}:=\frac{\gamma_{1}}{4}$, with $\gamma_{1}$ the constant of Theorem 17.1. Under the hypotheses of Theorem 18.16 and Assumption 18.18, if $\varepsilon_{C M}$ is suitably small (depending upon all other parameters but not the current $T$ ), then there is an $\mathcal{M}$-normal approximation $(\mathcal{K}, F)$ such that the following estimates hold on every Whitney region $\mathcal{L}$ associated to a cube $L \in \mathscr{W}$, with constants $C=C\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{\natural}, C_{e}^{b}, C_{h}\right)>0:$

$$
\begin{align*}
& \operatorname{Lip}\left(\left.N\right|_{\mathcal{L}}\right) \leq \mathrm{Cm}_{0}^{\gamma_{2}} \ell(L)^{\gamma_{2}} \quad \text { and } \quad\left\|\left.N\right|_{\mathcal{L}}\right\|_{C^{0}} \leq \operatorname{Cm}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}},  \tag{18.14}\\
& |\mathcal{L} \backslash \mathcal{K}|+\left\|\mathbf{T}_{F}-T\right\|\left(\mathbf{p}^{-1}(\mathcal{L})\right) \leq \operatorname{Cm}_{0}^{1+\gamma_{2}} \ell(L)^{4+\gamma_{2}},  \tag{18.15}\\
& \int_{\mathcal{L}}|D N|^{2} \leq \operatorname{Cm}_{0} \ell(L)^{4-2 \delta_{1}} . \tag{18.16}
\end{align*}
$$

Moreover, for any $a>0$ and any Borel $\mathcal{V} \subset \mathcal{L}$, we have $\left(\right.$ for $\left.C=C\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{b}, C_{e}^{\natural}, C_{h}\right)\right)$

$$
\begin{equation*}
\int_{\mathcal{V}}|\boldsymbol{\eta} \circ N| \leq C m_{0}\left(\ell(L)^{5+\beta_{1} / 3}+a \ell(L)^{2+\gamma_{2} / 2}|\mathcal{V}|\right)+\frac{C}{a} \int_{\mathcal{V}} \mathcal{G}(N, Q \llbracket \boldsymbol{\eta} \circ N \rrbracket)^{2+\gamma_{2}} \tag{18.17}
\end{equation*}
$$

From (18.14) - (18.16) it is not difficult to infer analogous "global versions" of the estimates.

Corollary $\mathbf{1 8 . 2 2}$ (Global estimates for the $\mathcal{M}$-normal approximation). Let $\mathcal{M}^{\prime}$ be the domain $\boldsymbol{\Phi}\left(D \cap\left[-\frac{7}{2}, \frac{7}{2}\right]^{2}\right)$ and $N$ the map of Theorem 18.21. Then, there is a constant $C=$ $C\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{\natural}, C_{e}^{b}, C_{h}\right)$ such that

$$
\begin{align*}
& \operatorname{Lip}\left(\left.N\right|_{\mathcal{M}^{\prime}}\right) \leq \mathrm{Cm}_{0}^{\gamma_{2}} \quad \text { and } \quad\left\|\left.N\right|_{\mathcal{M}^{\prime}}\right\|_{\mathrm{C}^{0}} \leq \mathrm{Cm}_{0}^{1 / 4},  \tag{18.18}\\
& \left|\mathcal{M}^{\prime} \backslash \mathcal{K}\right|+\left\|\mathbf{T}_{F}-T\right\|\left(\mathbf{p}^{-1}\left(\mathcal{M}^{\prime}\right)\right) \leq \mathrm{Cm}_{0}^{1+\gamma_{2}},  \tag{18.19}\\
& \int_{\mathcal{M}^{\prime}}|D N|^{2} \leq \boldsymbol{C m}_{0} . \tag{18.20}
\end{align*}
$$

In addition, since $N=Q \llbracket 0 \rrbracket$ on $\boldsymbol{\Gamma} \cap \mathcal{M}^{\prime}$, we also get

$$
\begin{equation*}
\int_{\mathcal{M}^{\prime}}|N|^{2} \leq C m_{0} \tag{18.21}
\end{equation*}
$$

### 18.4 ADDITIONAL $L^{1}$ ESTIMATE

While the estimates claimed so far have all appropriate counterparts in the papers [15] and [21], we will need an additional important estimate which is noticed here for the first time, even though it is still a consequence of the same arguments leading to Theorem 18.16 and Theorem 18.21.

Proposition 18.23. Consider the function $f: B_{3} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ with the property that $\mathbf{G}_{f}=$ $\mathbf{T}_{F}\left\llcorner\mathbf{C}_{3}\right.$. For every $L \in \mathscr{W}^{e}$ we then have the estimate

$$
\begin{equation*}
\|\boldsymbol{\varphi}-\boldsymbol{\eta} \circ f\|_{L^{1}(L)} \leq C m_{0}^{3 / 4} \ell(L)^{4} \tag{18.22}
\end{equation*}
$$

and in particular, as long as $r \leq 3$ is a radius such that $\ell(L) \leq r$ for every $L \in \mathscr{W}$ with $L \cap B_{r} \neq \varnothing$, we have the estimate

$$
\begin{equation*}
\|\boldsymbol{\varphi}-\boldsymbol{\eta} \circ f\|_{L^{1}\left(B_{r}\right)} \leq C m_{0}^{3 / 4} r^{4} \tag{18.23}
\end{equation*}
$$

We estimate the changes of excess and height when tilting the reference planes of nearby squares.

Proposition 19.1 (Tilting of optimal planes). Let $Q, T$ and $\Gamma$ be as in Assumption 18.3 and recall the parameters of Assumption 18.7. There are constants $\bar{C}=\bar{C}\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{\natural}, C_{e}^{b}\right)>$ 0 and $C=C\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{\natural}, C_{e}^{b}, C_{h}\right)>0$ such that, if $\varepsilon_{C M}=\varepsilon_{C M}\left(Q, n, R_{0}, C_{h}\right)>0$ is small enough, for any $H, L \in \mathscr{S} \cup \mathscr{W}$ with $H$ being equal or a descendant of $L$ we have
(i) $\mathbf{B}_{H}^{\square} \subset \mathbf{B}_{L}^{\square} \subset \mathbf{B}_{4 R_{0}}$,
(ii) $\left|V_{H}-V_{L}\right| \leq \bar{C} m_{0}^{1 / 2} \ell(L)^{1-\delta_{1}}$,
(iii) $\left|V_{H}-V_{0}\right| \leq \bar{C} m_{0}^{1 / 2}$,
(iv) ${ }^{\natural}$ if $H \in \mathscr{C}^{\natural}$, then
$\mathbf{h}\left(T, \mathbf{C}_{36 r_{H}}\left(p_{H}, V_{0}\right)\right) \leq C m_{0}^{1 / 4} \ell(H)$ and $\operatorname{spt}(T) \cap \mathbf{C}_{36 r_{H}}\left(p_{H}, V_{0}\right) \subset \mathbf{B}_{H}$,
(iv) ${ }^{b}$ if $H \in \mathscr{C}^{b}$, then

$$
\mathbf{h}\left(T, \mathbf{C}_{2^{7} 36 r_{H}}\left(p_{H}^{b}, V_{0}\right)\right) \leq \mathrm{Cm}_{0}^{1 / 4} \ell(H) \text { and } \operatorname{spt}(T) \cap \mathbf{C}_{2^{7} 36 r_{H}}\left(p_{H}^{b}, V_{0}\right) \subset \mathbf{B}_{H},
$$

$(v)^{\natural}$ if $H, L \in \mathscr{C}^{\natural}$, then

$$
\mathbf{h}\left(T, \mathbf{C}_{36 r_{L}}\left(p_{L}, V_{H}\right)\right) \leq \mathrm{Cm}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}} \text { and } \operatorname{spt}(T) \cap \mathbf{C}_{36 r_{L}}\left(p, V_{H}\right) \subset \mathbf{B}_{L}
$$

$(v)^{b}$ if $L \in \mathscr{C}^{b}$, then
$\mathbf{h}\left(T, \mathbf{C}_{2^{7} 36 r_{L}}\left(p_{L}^{b}, V_{H}\right)\right) \leq \operatorname{Cm}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}}$ and $\operatorname{spt}(T) \cap \mathbf{C}_{2^{7} 36 r_{L}}\left(p_{L}^{b}, V_{H}\right) \subset \mathbf{B}_{L}$.
where $\square=$ or $\square=b$ depending on whether the square is a boundary square or not. Moreover, (ii) $-(v)$ also hold if $H$ and $L$ are neighbours with $\frac{1}{2} \ell(L) \leq \ell(H) \leq \ell(L)$.

Proof. We argue by induction on $i=-\log _{2}(\ell(H))$. The base step is when $i=N_{0}$ and $H=L$ while we pass to children squares in the induction step. By the choice of $M_{0}$ and $N_{0}$, we notice that there are no squares with side length $2^{-N_{0}}$ in $\mathscr{W}$.

The second inclusion of (i), we already observed in (18.6) while the first inclusion of (i) and the inequality in (ii) is redundant for $H=L$. Thus, we show now (iii). We use (i), the optimality of $V_{H}$, the monotonicity formula of Theorem 11.2 and the definition of $m_{0}$ to deduce

$$
\begin{align*}
\left|V_{H}-V_{0}\right|^{2} & \leq \bar{C} r_{H}^{-2} \int_{\mathbf{B}_{H}^{\square}}\left|\vec{T}-\vec{V}_{H}\right|^{2} d\|T\|(x)+\bar{C} r_{H}^{-2} \int_{\mathbf{B}_{H}^{\square}}\left|\vec{T}-\vec{V}_{0}\right|^{2} d\|T\|(x) \\
& \leq 2 \bar{C} \mathbf{E}\left(T, \mathbf{B}_{H}^{\square}, V_{0}\right) \leq \bar{C} \mathbf{E}\left(T, \mathbf{B}_{5 R_{0}}, V_{0}\right) \leq \bar{C} m_{0} . \tag{19.1}
\end{align*}
$$

For (iv) we use the height estimate (18.3) of Lemma 18.4. Notice that $\mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{0}\right) \subset$ $\mathrm{C}_{4 R_{0}}\left(0, V_{0}\right)$ and hence,

$$
\mathbf{h}\left(T, \mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{0}\right)\right) \leq \mathbf{h}\left(T, \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)\right) \leq \overline{\mathrm{C}} \boldsymbol{m}_{0}^{1 / 4}=\overline{\mathrm{C}} \boldsymbol{m}_{0}^{1 / 4} \ell(H)
$$

Then also the inclusion $\operatorname{spt}(T) \cap \mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{0}\right) \subset \mathbf{B}_{H}^{\square}$ holds, as long as $\varepsilon_{C M}$ is small enough. For $(v)$ we observe that as $\mathbf{B}_{H}^{\square} \subset \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)$ we can estimate

$$
\left|p_{H}^{\square}\right|^{2} \leq 9 R_{0}^{2}+\mathbf{h}\left(T, \mathbf{C}\left(4 R_{0}, V_{0}\right)\right)^{2} \leq 9 R_{0}^{2}+\mathbf{C m}_{0} .
$$

Thus if $\varepsilon_{C M}$ (and thus $m_{0}$ ) is small enough, then $\mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{H}\right) \cap \mathbf{B}_{4 R_{0}} \subset \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)$. Hence, also spt $(T) \cap \mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{H}\right) \subset \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)$ and we can estimate

$$
\begin{aligned}
\mathbf{h}\left(T, \mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{H}\right)\right) & \leq \mathbf{h}\left(T, \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)\right)+\bar{C}\left|V_{H}-V_{0}\right| \\
& \leq \bar{C} m_{0}^{1 / 4}=\bar{C} m_{0}^{1 / 4} \ell(H)^{1+\beta_{1}},
\end{aligned}
$$

where we used (iii) and (iv).


Figure 3: An illustration of the various relevant points in the Whitney square.
Induction step: Let $H \in \mathscr{S}_{i+1} \cup \mathscr{W}_{i+1}$ for some $i \geq N_{0}$. Thus there is a chain of squares such that $H_{i+1}:=H \subset H_{i} \subset \cdots \subset H_{N_{0}}$ with $H_{j} \in \mathscr{S}_{j}$ for each $j \leq i$. Assume the validity of $(i)-(v)$ for $H_{l}$ and $H_{k}$ with $N_{0} \leq l \leq k \leq i$. We want to show $(i)-(v)$ for $H=H_{i+1}$ and $L=H_{j}$ with $N_{0} \leq j \leq i$. For $(i)$, we notice that it is enough to show
the inclusion for $j=i$. Then we have $\left|x_{H_{i}}-x_{H}\right| \leq \sqrt{2} \ell\left(H_{i}\right)$ and hence, if $\varepsilon_{C M}$ is small enough, we use the induction hypothesis for (iv) to estimate

$$
\begin{aligned}
\left|p_{H_{i}}^{\square}-p_{H}^{\square}\right|^{2} & \leq\left(\sqrt{2} \ell\left(H_{i}\right)+96 r_{H_{i}}\right)^{2}+\mathbf{h}\left(T, \mathbf{C}_{2 r_{H_{i}}}\left(p_{H_{i}}^{\square} V_{0}\right)\right)^{2} \\
& \leq \ell\left(H_{i}\right)^{2}\left(\sqrt{2}\left(1+96 M_{0}\right)\right)^{2}+\text { Cm }_{0}^{1 / 2} \ell\left(H_{i}\right)^{2} \leq 2^{16} M_{0}^{2} \ell\left(H_{i}\right)^{2} .
\end{aligned}
$$

Now we check that $\mathbf{B}_{H}^{\square} \subset \mathbf{B}_{H_{i}}^{\square}$. Indeed, we have

$$
\begin{aligned}
2^{7} 64 r_{H}+\left|p_{H_{i}}^{\square}-p_{H}^{\square}\right| & \leq 2^{7} 32 \sqrt{2} M_{0} \ell\left(H_{i}\right)+2^{8} M_{0} \ell\left(H_{i}\right) \\
& \leq 2^{7} 32 \sqrt{2} M_{0} \ell\left(H_{i}\right)+2^{7} 32 \sqrt{2} M_{0} \ell\left(H_{i}\right)=2^{7} 64 r_{H_{i}} .
\end{aligned}
$$

For (ii), we first show the special case where $j=i$. We notice that by $(i)$, the fact that $2 r_{H}=r_{H_{i}}$ and $H_{i} \in \mathscr{S}_{i}$, we have by the monotonicity formula

$$
\begin{aligned}
\left|V_{H}-V_{H_{i}}\right|^{2} & \leq \bar{C} \frac{r_{H}^{2}}{\|T\|\left(\mathbf{B}_{H}^{\square}\right)}\left(\mathbf{E}^{\square}\left(T, \mathbf{B}_{H}^{\square}\right)+\mathbf{E}^{\square}\left(T, \mathbf{B}_{H_{i}}^{\square}\right)\right) \\
& \leq \bar{C}\left(\mathbf{E}\left(T, \mathbf{B}_{H}^{\square}, V_{H_{i}}\right)+\mathbf{E}^{\square}\left(T, \mathbf{B}_{H_{i}}^{\square}\right)\right) \\
& \leq 2 \bar{C} \mathbf{E}^{\square}\left(T, \mathbf{B}_{H_{i}}^{\square}\right) \\
& \leq \bar{C} C_{e}^{\square} \boldsymbol{m}_{0} \ell(H)^{2-2 \delta_{1}} .
\end{aligned}
$$

Now for a general $j \in\left\{N_{0}, \ldots, i\right\}$, we use the geometric series to conclude

$$
\begin{aligned}
\left|V_{H}-V_{H_{j}}\right| \leq \sum_{l=j}^{i}\left|V_{H_{l+1}}-V_{H_{l}}\right| & \leq \bar{C} C_{e}^{\square} m_{0} \sum_{l=j}^{i} \ell\left(H_{l}\right)^{1-\delta_{1}} \\
& \leq \bar{C} C_{e}^{\square} m_{0} \sum_{l=j}^{\infty}\left(2^{-l+j} \ell\left(H_{j}\right)\right)^{1-\delta_{1}} \leq \bar{C} C_{e}^{\square} m_{0} \ell\left(H_{j}\right)^{1-\delta_{1}} .
\end{aligned}
$$

(iii) follows by (ii) and (19.1). To prove (iv) ${ }^{\natural}$, we observe that by the induction hypothesis, we already know $\operatorname{spt}(T) \cap \mathbf{C}_{36 r_{H_{i}}}\left(p_{H_{i}}^{\square}, V_{H_{i}}\right) \subset \mathbf{B}_{H_{i}}^{\square}$. Now we want to see that $\mathbf{C}_{36 r_{H}}\left(p_{H}^{\square}, V_{0}\right) \subset \mathbf{C}_{36 r_{H_{i}}}\left(p_{H_{i}}^{\square}, V_{0}\right)$. In case where $H_{i} \in \mathscr{C}$, we have $\left|x_{H}-x_{H_{i}}\right| \leq \sqrt{2} \ell\left(H_{i}\right)$, hence

$$
36 r_{H}+\left|x_{H}-x_{H_{i}}\right| \leq 36 r_{H_{i}} .
$$

On the other hand, if $H_{i} \in \mathscr{C}^{b}$, then we recall $\left|p_{H}-p_{H_{i}}^{b}\right| \leq 2^{8} M_{0} \ell\left(H_{i}\right)$ which implies

$$
36 r_{H}+\left|x_{H}-x_{H_{i}}^{b}\right| \leq 36 r_{H}+\left|p_{H}-p_{H_{i}}^{b}\right| \leq 2^{7} 36 r_{H_{i}} .
$$

Thus the desired inclusion of the cylinders holds. We deduce

$$
\begin{aligned}
\mathbf{h}\left(T, \mathbf{C}_{36 r_{H}}\left(p_{H}, V_{0}\right)\right) \leq \mathbf{h}\left(T, \mathbf{B}_{H_{i}}^{\square}, V_{0}\right) & \leq \mathbf{h}\left(T, \mathbf{B}_{H_{i}}^{\square}\right)+\overline{\mathrm{C}} r_{H_{i}}\left|V_{H_{i}}-V_{0}\right| \\
& \leq C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell\left(H_{i}\right)^{1+\beta_{1}}+\overline{\mathrm{C}} \ell\left(H_{i}\right) \boldsymbol{m}_{0}^{1 / 2} \\
& \leq \overline{\mathrm{C}} C_{h} \ell\left(H_{i}\right) \boldsymbol{m}_{0}^{1 / 4},
\end{aligned}
$$

where we used the induction hypothesis and that $H_{i} \in \mathscr{S}_{i}$.

The previous estimate shows also that $\operatorname{spt}(T) \cap \mathbf{C}_{36 r_{H}}\left(p_{H}, V_{0}\right) \subset \mathbf{B}_{H}$ assuming that $\varepsilon_{C M}$ is small enough. The proof of $(i v)^{b}$ is analogous because if $H \in \mathscr{C}^{b}$, then also $H_{i} \in \mathscr{C}^{b}$ and so as before

$$
2^{7} 36 r_{H}+\left|x_{H}^{b}-x_{H_{i}}^{b}\right| \leq 2^{7} 36 r_{H}+\left|p_{H}^{b}-p_{H_{i}}^{b}\right| \leq 2^{7} 36 r_{H_{i}} .
$$

Now we show $(v)^{\natural},(v)^{b}$ for $H=H_{i+1}$ and $L=H_{j}$ for some $j \in\left\{N_{0}, \ldots, i\right\}$ by induction on $j$. For $j=N_{0}$, we use the estimate on $\left|V_{H}-V_{H_{N_{0}}}\right|$ to deduce

$$
\left(\mathbf{C}_{2^{7} 36 r_{H_{N_{0}}}}\left(p_{H_{N_{0}}}^{\square}, V_{H}\right) \cap \mathbf{B}_{4 R_{0}}\right) \subset\left(\mathbf{C}_{2^{7} 36 r_{H_{N_{0}}}}\left(p_{H_{N_{0}}}^{\square}, V_{H_{N_{0}}}\right) \cap \mathbf{B}_{5 R_{0}}\right) \subset \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)
$$

provided that $\varepsilon_{C M}$ is small enough. Therefore, we have

$$
\mathbf{h}\left(T, \mathbf{C}_{2^{7} 36 r_{H_{N_{0}}}}\left(p_{H_{N_{0}}}^{\square}, V_{H}\right)\right) \leq \mathbf{h}\left(T, \mathbf{C}_{4 R_{0}}\left(0, V_{0}\right)\right)+C\left|V_{H}-V_{0}\right| \leq \bar{C} m_{0}^{1 / 2} .
$$

Again if $\varepsilon_{C M}$ is small, this also implies that $\left.\operatorname{spt}(T) \cap \mathbf{C}_{2^{7} 36 r_{H_{N_{0}}}}\left(p_{H_{N_{0}}}^{\square}, V_{H}\right)\right) \subset \mathbf{B}_{H_{N_{0}}}^{\square}$. Now assume that $(v)^{\text {b }},(v)^{b}$ hold for some $j \geq N_{0}$ and denote $L=H_{j+1}$. We first consider the case where $L \in \mathscr{C}^{\natural}$. Then its parent $H_{j}$ is still unknown, but in any case, $\mathbf{B}_{L} \subset \mathbf{B}_{H_{j}}^{\square}$ and thus, $\mathbf{C}_{36 r_{L}}\left(p_{L}, V_{H}\right) \subset \mathbf{C}_{36 r_{H_{j}}}\left(p_{H_{j}}, V_{H}\right)$ or $\mathbf{C}_{36 r_{L}}\left(p_{L}, V_{H}\right) \subset \mathbf{C}_{2^{7} 36 r_{H_{j}}}\left(p_{H_{j^{\prime}}}^{b}, V_{H}\right)$ respectively. Using the induction hypothesis, we find $\mathbf{h}\left(T, \mathbf{C}_{36 r_{H_{j}}}\left(p_{H_{j}}, V_{H}\right)\right) \leq \mathbf{h}\left(T, \mathbf{B}_{H_{j}}, V_{H}\right)$ or $\mathbf{h}\left(T, \mathbf{C}_{2^{7} 36 r_{H_{j}}}\left(p_{H_{j^{\prime}}}^{\mathrm{b}}, V_{H}\right)\right) \leq \mathbf{h}\left(T, \mathbf{B}_{H_{j}}^{b}, V_{H}\right)$ respectively. Moreover, using (ii), we deduce

$$
\begin{aligned}
\mathbf{h}\left(T, \mathbf{B}_{H_{j}}^{\square}, V_{H}\right) & \leq \mathbf{h}\left(T, \mathbf{B}_{H_{j}}^{\square}\right)+\overline{\mathrm{C}} r_{H_{j}}\left|V_{H}-V_{H_{j}}\right| \\
& \leq \bar{C} C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell\left(H_{j}\right)^{1+\beta_{1}}+\overline{\mathrm{C}} \boldsymbol{m}_{0}^{1 / 2} \ell\left(H_{j}\right)^{2-\delta_{1}} \leq \bar{C} C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell\left(H_{j}\right) .
\end{aligned}
$$

Thus, we have also $\left.\operatorname{spt}(T) \cap \mathbf{C}_{36 r_{L}}\left(p_{L}, V_{H}\right)\right) \subset \mathbf{B}_{L}$ and finally

$$
\mathbf{h}\left(T, \mathbf{C}_{36 r_{L}}\left(p, V_{H}\right)\right) \leq \mathbf{h}\left(T, \mathbf{B}_{L}\right)+\bar{C} r_{L}\left|V_{H}-V_{L}\right| \leq \bar{C} C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}} .
$$

On the other hand, if $L \in \mathscr{C}^{b}$, then also $H_{j} \in \mathscr{C}^{b}$ and we can perform the same argument since $\mathbf{B}_{L}^{b} \subset \mathbf{B}_{H_{j}}^{b}$ and $\mathbf{C}_{2^{7} 36 r_{L}}\left(p_{L}^{b}, V_{H}\right) \subset \mathbf{C}_{2^{7} 36 r_{H_{j}}}\left(p_{H_{j}}^{b}, V_{H}\right)$. This shows both $(v)^{\natural}$ and $(v)^{b}$.

For neighbor squares, the argument works exactly the same as everything follows from the smallness of $\left|p_{L}^{\square}-p_{H}^{\square}\right|$ and the fact that $\mathbf{B}_{L}^{\square} \cup \mathbf{B}_{H}^{\square} \subset \mathbf{B}_{J}^{\square}$, where $J$ is the parent of $L$.

Very similarly we now prove the excess estimates using the fact, that the parent of any square belongs to $\mathscr{S}$.

Proof of Proposition 18.12. For squares $L$ of side length $2^{-N_{0}}$, we know by Proposition 19.1 (i) that $\mathbf{B}_{L}^{\square} \subset \mathbf{B}_{4 R_{0}}$ and so we can choose $C_{e}^{\natural}$ and $C_{e}^{b}$ large enough such that

$$
\mathbf{E}^{\square}\left(T, \mathbf{B}_{L}\right) \leq C\left(R_{0}, N_{0}\right) \mathbf{E}\left(T, \mathbf{B}_{4 R_{0}}, V_{0}\right) \leq C\left(R_{0}, N_{0}\right) m_{0} \leq C_{e}^{\square} m_{0} \ell(L)^{2-2 \delta_{1}} .
$$

Hence, $L \notin \mathscr{W}^{e}$. Similarly we see that $L \notin \mathscr{W}^{h}$. Indeed, we use Proposition 19.1 (ii) and the height estimate of Lemma 18.4

$$
\mathbf{h}\left(T, \mathbf{B}_{L}^{\square}\right) \leq \mathbf{h}\left(T, \mathbf{B}_{4 R_{0}}, V_{0}\right)+C\left(R_{0}, n, Q\right)\left|V_{L}^{\square}-V_{0}\right| \leq C\left(R_{0}, n, Q\right) m_{0}^{1 / 4}
$$

Thus, we can choose $C_{h}$ large enough such that $\mathbf{h}\left(T, \mathbf{B}_{L}^{\square}\right) \leq C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}}$. This shows (a).

We claim that (b) holds as long as $C_{e}^{\natural} \geq 16 C_{e}^{b}$. Let $L \in \mathscr{C}^{\natural}$ and assume its parent $H \in \mathscr{C}^{b}$. We want to show that $L \notin \mathscr{W}^{e}$. Recall that $\left|p_{L}-p_{H}^{b}\right| \leq 2^{8} M_{0} \ell(L)$ and thus $\mathbf{B}_{L} \subset \mathbf{B}_{H}^{b}$. Moreover, as $H$ is a parent, it belongs to $\mathscr{S}$, thus

$$
\mathbf{E}^{b}\left(T, \mathbf{B}_{H}^{b}\right) \leq C_{e}^{b} \boldsymbol{m}_{0} \ell(H)^{2-2 \delta_{1}}
$$

This then implies

$$
\mathbf{E}\left(T, \mathbf{B}_{L}\right) \leq \mathbf{E}\left(T, \mathbf{B}_{L}, V_{H}\right) \leq 4 \mathbf{E}^{b}\left(T, \mathbf{B}_{H}^{b}\right) \leq 16 C_{e}^{b} \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}}
$$

Now let $L \in \mathscr{W} \cap \mathscr{C}^{b}$ and denote by $H \in \mathscr{S}$ the parent of $L$. As $L$ is a boundary square, so is $H$. By Proposition 19.1 (i) and (ii), we know that $\mathbf{B}_{L}^{b} \subset \mathbf{B}_{H}^{b}$ and

$$
\begin{aligned}
\mathbf{E}^{b}\left(T, \mathbf{B}_{L}^{b}\right) & \leq 4 \mathbf{E}^{b}\left(T, \mathbf{B}_{H}^{b}\right) \leq C C_{e}^{b} \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}} \\
\mathbf{h}\left(T, \mathbf{B}_{L}^{b}\right) & \leq \mathbf{h}\left(T, \mathbf{B}_{H}^{b}\right)+\bar{C} r_{L}\left|V_{L}-V_{H}\right| \leq C C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}}
\end{aligned}
$$

On the other hand, for $L \in \mathscr{W} \cap \mathscr{C}^{\natural}$, the parent $H$ of $L$ could be either a boundary square or an interior square. So we estimate

$$
\begin{aligned}
& \mathbf{E}\left(T, \mathbf{B}_{L}\right) \leq 4 \mathbf{E}^{\square}\left(T, \mathbf{B}_{H}^{\square}\right) \leq C\left(C_{e}^{b}+C_{e}^{\natural}\right) \boldsymbol{m}_{0} \ell(L)^{2-2 \delta_{1}}, \\
& \mathbf{h}\left(T, \mathbf{B}_{L}\right) \leq \mathbf{h}\left(T, \mathbf{B}_{H}^{\square}\right)+\bar{C} r_{L}\left|V_{L}-V_{H}^{\square}\right| \leq C C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}} .
\end{aligned}
$$

We notice that our construction fulfills the estimates needed for the strong Lipschitz approximation.
Proposition 20.1. Suppose that Assumption 18.3 holds true, recall the constants from Assumption 18.7 and assume that $\varepsilon_{C M}$ is small enough. Let either $H, L \in \mathscr{S} \cup \mathscr{W}$ be neighbors with $\frac{1}{2} \ell(L) \leq \ell(H) \leq \ell(L)$ or let $H$ be a descendant of $L$. Then we have

$$
\begin{align*}
\operatorname{spt}(T) \cap \mathbf{C}_{32 r_{L}}\left(p_{L}, V_{H}\right) \subset \mathbf{B}_{L}, & \text { if } L \in \mathscr{C}^{\natural},  \tag{20.1}\\
\operatorname{spt}(T) \cap \mathbf{C}_{2^{7} 32 r_{L}}\left(p_{L}^{b}, V_{H}\right) \subset \mathbf{B}_{L}^{b}, & \text { if } L \in \mathscr{C}^{b}, \tag{20.2}
\end{align*}
$$

and [13, Theorem 2.4] can be applied to $T$ in the cylinder $\mathbf{C}_{32 r_{L}}\left(p_{L}, V_{H}\right)$ and Theorem 17.1 in $\mathrm{C}_{2^{7} 32 r_{L}}\left(p_{L}^{b}, V_{H}\right)$ respectively. The resulting strong Lipschitz approximation we call $f_{H L}$.
Proof. The proof of Proposition 20.1 is completely analogous to [15, Proposition 4.2] for interior squares and to [21, Proposition 8.25] for boundary squares.

Remark 20.2. Observe that if $\ell(H)<\ell(L)$ and $H$ is a boundary square, then $L$ is necessarily also a boundary square, since either $H$ and $L$ are neighbors or $H \subset L$. When $\ell(H)=\ell(L)$, in case $H$ is a boundary square and $L$ is an interior square, we can simply swap their roles. In particular, without loss of generality, we will in the sequel ignore the case in which $H$ is a boundary square and $L$ is an interior square.
Definition 20.3. We denote by $f_{H L}$ the strong Lipschitz approximation produced by Proposition 20.1. We will however consider the domain of the function $f_{H L}$ a subset of $p_{H}+V_{H}$, resp. $p_{H}^{b}+V_{H}$. More precisely, for interior squares the domain is $\mathrm{C}_{24 r_{L}}\left(p_{L}, V_{H}\right) \cap\left(p_{H}+V_{H}\right)$, while for boundary squares it is $D_{H L}:=D_{H} \cap \mathbf{C}_{2^{7} 24 r_{L}}\left(p_{L}^{b}, V_{H}\right)$, where we recall that $D_{H}$ is the projection on $p_{H}^{b}+V_{H}$ of $\operatorname{spt}(T)$. Observe next that $\mathbf{C}_{24 r_{L}}\left(p_{L}, V_{H}\right) \cap\left(p_{H}+V_{H}\right)$ and $\mathbf{C}_{2^{7} 24 r_{L}}\left(p_{L}^{b}, V_{H}\right) \cap\left(p_{H}^{\square}+V_{H}\right)$ are discs, whose centers are given by

$$
\begin{aligned}
& p_{H L}:=p_{H}+\mathbf{p}_{V_{H}}\left(p_{L}\right), \quad \text { resp. } \\
& p_{H L}^{b}:=p_{H}^{\square}+\mathbf{p}_{V_{H}}\left(p_{L}^{b}\right) .
\end{aligned}
$$

(Note that, when $L$ is a boundary square, $H$ might be a boundary square but it might also be an interior square).
Definition 20.4. We then let $h_{H L}$ be the harmonic function on $B_{16 r_{L}}\left(p_{H L}, V_{H}\right)$, resp. $D_{H} \cap \mathbf{C}_{2^{7} 16 r_{L}}\left(p_{H L}^{b}, V_{H}\right)$, such that the boundary value of $h_{H L}$ on the respective domain is given by $\eta \circ f_{H L}$, in particular it coincides with $g_{H}$ on $\gamma_{H}$. $h_{H L}$ will be called the ( $H, L$ )-tilted harmonic interpolating function.

Lemma 18.15 will then be a particular case of the following more general lemma.
Lemma 20.5. Consider $H$ and $L$ as in Proposition 20.1. Then there is a smooth function $u_{H L}: D \cap B_{2^{7} 8 r_{L}}\left(\mathbf{p}_{0}\left(p_{L}^{b}\right), V_{0}\right) \rightarrow V_{0}^{\perp}$, resp. $u_{H L}: B_{8 r_{L}}\left(\mathbf{p}_{0}\left(p_{L}\right), V_{0}\right) \rightarrow V_{0}^{\perp}$, such that

$$
\begin{aligned}
\mathbf{G}_{u_{H L}}\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}, V_{0}\right)\right. & =\mathbf{G}_{h_{H L}}\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}, V_{0}\right),\right. \\
\mathbf{G}_{u_{H L}}\left\llcorner\mathbf{C}_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{0}\right)\right. & =\mathbf{G}_{h_{H L}}\left\llcorner\mathbf{C}_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{0}\right), \quad\right. \text { respectively. }
\end{aligned}
$$

The function $u_{H L}$ will be called interpolating function.
20.1 LINEARIZATION AND FIRST ESTIMATES ON $h_{H L}$

Proposition 20.6. Under the Assumptions of Proposition 20.1 the following estimates hold for every pair of squares $H$ and $L$ as in Proposition 20.1. First of all

$$
\begin{equation*}
\int D\left(\boldsymbol{\eta} \circ f_{H L}\right): D \zeta \leq C m_{0} r_{L}^{4+\beta_{1}}\|D \zeta\|_{0} \tag{20.3}
\end{equation*}
$$

for every function $\zeta$ in $C_{c}^{\infty}\left(B_{8 r_{L}}\left(p_{H L}, V_{H}\right), V_{H}^{\perp}\right)$, resp. $C_{c}^{\infty}\left(D_{H} \cap B_{2^{7} 8 r_{L}}\left(p_{H L}^{b}, V_{H}\right), V_{H}^{\perp}\right)$, depending on whether $L \in \mathscr{C}^{\natural}$ or $L \in \mathscr{C}^{b}$. Moreover,

$$
\begin{align*}
\left\|h_{H L}-\eta \circ f_{H L}\right\|_{L^{1}\left(B_{8 r_{L}}\left(p_{H L}, V_{H}\right)\right)} & \leq \operatorname{Cm}_{0} r_{L}^{5+\beta_{1}}, & & \text { if } L \in \mathscr{C}^{\natural} ; \\
\left\|h_{H L}-\eta \circ f_{H L}\right\|_{L^{1}\left(D_{H} \cap B_{2^{7} 8 r_{L}}\left(p_{H L}^{b}, V_{H}\right)\right)} & \leq \operatorname{Cm}_{0} r_{L}^{5+\beta_{1}}, & & \text { if } L \in \mathscr{C}^{b} ;  \tag{20.5}\\
\left\|D h_{H L}\right\|_{L^{\infty}\left(B_{7 r_{L}}\left(p_{H L}, V_{H}\right)\right)} & \leq \operatorname{Cm}_{0}^{\frac{1}{2}} r_{L}^{1-\delta_{1}}, & & \text { if } L \in \mathscr{C}^{\natural} ;  \tag{20.6}\\
\left\|D h_{H L}\right\|_{L^{\infty}\left(D_{H} \cap B_{2^{7} 7_{r_{L}}}\left(p_{H L}^{b}, V_{H}\right)\right)} & \leq \operatorname{Cm}_{0}^{\frac{1}{2}} r_{L}^{1-\delta_{1}}, & & \text { if } L \in \mathscr{C}^{b} . \tag{20.7}
\end{align*}
$$

Proof. Proof of (20.3). Without loss of generality consider a system of coordinates $(x, y)$ with the property that $p_{H L}^{\square}$ is the origin, $(x, 0) \in V_{H}$ and $(0, y) \in V_{H}^{\perp}$. Fix $\zeta$ as in the statement of the proposition and in the cylinder $\mathbf{C} \in\left\{\mathbf{C}_{32 r_{L}}\left(p_{H L}, V_{H}\right), \mathbf{C}_{2^{7} 32 r_{L}}\left(p_{H L}^{b}, V_{H}\right)\right\}$ we consider the vector field $\chi(x, y)=(0, \zeta(x))$. Observe that, by assumption, the vector field vanishes on $\Gamma$. Observe that, though $\chi$ is not compactly supported, since the height of the current in the cylinder $\mathbf{C}$ is bounded, we can multiply $\chi$ by a cut-off function in the variable $y$ but keeping its values the same on $\operatorname{spt}(T)$. The latter vector field is a valid first variation for the area-minimizing current $T$ and thus we have $\delta T(\chi)=0$. Thus we can use Theorem 17.1 and Proposition 19.1 to estimate

$$
\begin{aligned}
\left|\delta \mathbf{G}_{f_{H L}}(\chi)\right| & =\left|\delta\left(T-\mathbf{G}_{f_{H L}}\right)(\chi)\right| \leq\|D \zeta\|_{0}\left\|T-\mathbf{G}_{f_{H L}}\right\|(\mathbf{C}) \\
& \leq C\|D \zeta\|_{0} r_{L}^{2}\left(\mathbf{E}^{\square}\left(T, \mathbf{C}, V_{H}\right)+\mathbf{A}^{2} r_{L}^{2}\right)^{1+\gamma_{1}} \\
& \leq C\|D \zeta\|_{0} r_{L}^{2}\left(\mathbf{E}^{\square}\left(T, \mathbf{B}_{L}^{\square}\right)+\left|V_{H}-V_{L}\right|^{2}+\mathbf{A}^{2} r_{L}^{2}\right)^{1+\gamma_{1}} \\
& \leq C\|D \zeta\|_{0} r_{L}^{2}\left(\boldsymbol{m}_{0} r_{L}^{2-2 \delta_{1}}\right)^{1+\gamma_{1}} \leq C\|D \zeta\|_{0} \boldsymbol{m}_{0} r_{L}^{4+\beta_{1}}
\end{aligned}
$$

provided $\delta_{1}$ and $\beta_{1}$ are chosen small enough to satisfy $\left(2-2 \delta_{1}\right)\left(1+\gamma_{1}\right) \geq 2+\beta_{1}$.
Next we use the Taylor expansion [14, Theorem 4.1] to estimate

$$
\begin{aligned}
& \left|\delta \mathbf{G}_{f_{H L}}(\chi)-Q \int \eta \circ D f_{H L}: D \zeta\right| \leq C\|D \zeta\|_{0} \int\left|D f_{H L}\right|^{3} \\
& \leq C\|D \zeta\|_{0} \operatorname{Lip}\left(f_{H L}\right) \int\left|D f_{H L}\right|^{2} \\
& \leq C\|D \zeta\|_{0}\left(\mathbf{E}^{\square}\left(T, \mathbf{C}, V_{H}\right)+\mathbf{A}^{2} r_{L}^{2}\right)^{\gamma_{1}} r_{L}^{2}\left(\mathbf{E}^{\square}\left(T, \mathbf{C}, V_{H}\right)+\mathbf{A}^{2} r_{L}^{2}\right) \\
& \leq C\|D \zeta\|_{0} r_{L}^{2}\left(\boldsymbol{m}_{0} r_{L}^{2-2 \delta_{1}}\right)^{1+\gamma_{1}} .
\end{aligned}
$$

Proof of (20.4)-(20.5). Consider $v:=h_{H L}-\eta \circ f_{H L}$ on its respective domain $\Omega$ which equals either $B_{8 r_{L}}\left(p_{H L}, V_{H}\right)$ or $D_{H} \cap B_{2^{7} 8 r_{L}}\left(p_{H L}^{b}, V_{H}\right)$. Observe that $v$ vanishes on the boundary of $\Omega$. For every $w \in L^{2}$ we denote by $\zeta=P(w)$ the unique solution of $\Delta \zeta=w$ in $\Omega$ with $\left.\zeta\right|_{\partial \Omega}=0$, which is an element of the Sobolev space $W_{0}^{1,2}(\Omega)$. Next notice that by a simple density argument, the estimate (20.3) remains valid for any test function $\zeta \in W_{0}^{1,2}$ and recall also the standard estimate

$$
\|D(P(w))\|_{0} \leq C r\|w\|_{0}
$$

Therefore we can write

$$
\begin{aligned}
\|v\|_{L^{1}} & =\sup _{w:\|w\|_{0} \leq 1} \int_{\Omega} v \cdot w=\sup _{w:\|w\|_{0} \leq 1} \int_{\Omega} v \cdot \Delta(P(w)) \\
& =\sup _{w:\|w\|_{0} \leq 1}\left(-\int_{\Omega} D v: D(P(w))\right)=\sup _{w:\|w\|_{0} \leq 1} \int_{\Omega} D \eta \circ f_{H L}: D(P(w)) \\
& \leq C \sup _{w:\|w\|_{0} \leq 1} m_{0} r_{L}^{4+\beta_{1}}\|D P(w)\|_{0} \leq C m_{0} r_{L}^{5+\beta_{1}} .
\end{aligned}
$$

Proof of (20.6). Using the mean-value inequality for harmonic functions we simply get

$$
\begin{aligned}
\left\|D h_{H L}\right\|_{L^{\infty}\left(B_{7_{L}}\left(p_{H L}, V_{H}\right)\right)} & \leq \frac{C}{r_{L}^{2}} \int_{B_{8_{r_{L}}\left(p_{H L}, V_{H}\right)}}\left|D h_{H L}\right| \\
& \leq \frac{C}{r_{L}}\left(\int_{B_{8_{r_{L}}\left(p_{H L}, V_{H}\right)}}\left|D h_{H L}\right|^{2}\right)^{1 / 2} \\
& \leq \frac{C}{r_{L}}\left(\int_{B_{8_{r_{L}}\left(p_{H L}, V_{H}\right)}}\left|D \eta \circ f_{H L}\right|^{2}\right)^{1 / 2} \\
& \leq \frac{C}{r_{L}}\left(r_{L}^{2}\left(\mathbf{E}\left(T, \mathbf{C}, V_{H}\right)+\mathbf{A}^{2} r_{L}^{2}\right)\right)^{\frac{1}{2}} \leq \mathrm{Cm}_{0}^{\frac{1}{2}} r_{L}^{1-\delta_{1}} .
\end{aligned}
$$

Proof of (20.7). Using standard Schauder estimates for harmonic functions, we get

$$
\begin{aligned}
\left.\left\|D h_{H L}\right\|_{L^{\infty}\left(D_{H} \cap B_{2^{77_{L}}}\right.}\left(p_{H L}^{b}, V_{H}\right)\right) \leq & \frac{C}{r_{L}^{2}} \int_{D_{H} \cap B_{278 r_{L}}\left(p_{H L}^{b}, V_{H}\right)}\left|D h_{H L}\right| \\
& +C\left(\left\|D g_{H}\right\|_{0}+r_{L}^{-\alpha}\left[g_{H}\right]_{\alpha}\right)
\end{aligned}
$$

where we recall that $g_{H}: \partial D_{H} \cap B_{2^{7} 8 r_{L}}\left(p_{H L}^{b}, V_{H}\right)$ is the graphical parametrization of our boundary curve $\Gamma$ and $\alpha$ is a positive number smaller than 1 , to be chosen later. The first summand on the right hand side is estimated as in the proof above of (20.6). As for the second summand, recall that $T_{p_{L}^{b}} \Gamma$ is contained in the plane $V_{L}$ and that $\left|V_{L}-V_{H}\right| \leq C m_{0}^{1 / 2} r_{L}^{1-\delta_{1}}$. This implies that

$$
\left|D g_{H}\left(p_{H L}^{b}\right)\right| \leq C m_{0}^{1 / 2} r_{L}^{1-\delta_{1}}
$$

In particular we have

$$
\left\|D g_{H}\right\|_{L^{\infty}\left(\partial D_{H} \cap B_{2^{7} 8 r_{L}}\left(p_{H L}^{b}, V_{H}\right)\right)} \leq\left|D g_{H}\left(p_{H L}^{b}\right)\right|+C \mathbf{A} r_{L} \leq C m_{0}^{1 / 2} r_{L}^{1-\delta_{1}}
$$

On the other hand,

$$
r_{L}^{-\alpha}\left[g_{H}\right]_{\alpha} \leq C r_{L}^{1-2 \alpha} \mathbf{A} \leq C m_{0}^{1 / 2} r_{L}^{1-2 \alpha}
$$

and thus it suffices to choose $2 \alpha<\delta_{1}$.

### 20.2 TILTED ESTIMATE

We follow here [21, Section 8.5] almost verbatim to establish a suitable comparison between tilted interpolating functions which are defined in different system of coordinates.

Definition 20.7. Four cubes $H, J, L, M \in \mathscr{C}$ make a distant relation between $H$ and $L$ if $J, M$ are neighbors (possibly the same cube) with same side length and $H$ and $L$ are descendants respectively of $J$ and $M$.

Lemma 20.8 (Tilted $L^{1}$ estimate). Under the Assumptions of Theorem 18.16 the following holds for every quadruple $H, J, L$ and $M$ in $\mathscr{S} \cup \mathscr{W}$ which makes a distant relation between $H$ and $L$.

- If $J \in \mathscr{C}^{\natural}$, then there is a map $\hat{h}_{L M}: B_{4 r_{J}}\left(p_{H J}, V_{H}\right) \rightarrow V_{H}^{\perp}$ such that

$$
\mathbf{G}_{\hat{h}_{L M}}=\mathbf{G}_{h_{L M}}\left\llcorner\mathbf{C}_{4 r_{J}}\left(p_{H J}, V_{H}\right)\right.
$$

and

$$
\begin{equation*}
\left\|h_{H J}-\hat{h}_{L M}\right\|_{L^{1}\left(B_{2 r_{J}}\left(p_{H J}, V_{H}\right)\right)} \leq \operatorname{Cm}_{0} \ell(J)^{5+\beta_{1} / 2} \tag{20.8}
\end{equation*}
$$

- If both $J$ and $M$ belong to $\mathscr{C}^{b}$, then there is a map $\hat{h}_{L M}: D_{H J} \cap B_{2^{7} 4 r_{J}}\left(p_{H J}^{b}, V_{H}\right) \rightarrow V_{H}^{\perp}$ such that

$$
\mathbf{G}_{\hat{h}_{L M}}=\mathbf{G}_{h_{L M}} L \mathbf{C}_{2^{7} 4 r_{J}}\left(p_{H J}^{b}, V_{H}\right)
$$

and

$$
\begin{equation*}
\left\|h_{H J}-\hat{h}_{L M}\right\|_{L^{1}\left(D_{H J} \cap B_{2^{7} 2 r_{J}}\left(p_{H J}^{b}, V_{H}\right)\right)} \leq \operatorname{Cm}_{0} \ell(J)^{5+\beta_{1} / 2} \tag{20.9}
\end{equation*}
$$

The proof follows verbatim the arguments given in [21, Section 8.5]. The only difference is the absence of the "ambient Riemannian" manifold which in [21, Lemma 8.31] is the graph of a function $\Psi$. The case needed for our arguments is the clearly simpler situation in which the linear subspaces $\omega$ and $\bar{\omega}$ in [21, Lemma 8.31] are given by the trivial subspace $\{0\}$. The proof of this version of the lemma (which is in fact [ 15, Lemma 5.6]) is even less complicated. However there is a direct way to conclude it directly from the more general statement of [21, Lemma 8.31]: we can consider $\mathbb{R}^{2+n}$ as a subspace of $\mathbb{R}^{2+n+1}$ and apply [21, Lemma 8.31] to a generic choice of $\varkappa, \bar{x}, \pi, \bar{\pi}$ and the specific choice of $\omega=\bar{\omega}=\{0\} \times \mathbb{R}$ and $\Psi=\bar{\Psi}: \pi \times \varkappa=\bar{\pi} \times \bar{\varkappa} \rightarrow \omega=\bar{\omega}$ given by the trivial map $\Psi \equiv 0$.

Proposition 21.1. There is a constant $\omega$ depending upon $\delta_{1}$ and $\beta_{1}$ such that, under the assumptions of Theorem 18.16, the following holds for every pair of squares $H, L \in \mathscr{P}^{j}$ (cf. (18.12)).
(a) $\left\|u_{H}\right\|_{C^{3}, \omega\left(B_{4 r_{H}}\left(x_{H}\right)\right.} \leq \operatorname{Cm}_{0}^{1 / 2}$, resp. $\left.\left\|u_{H}\right\|_{C^{3, \omega}\left(D \cap B_{2^{7} t_{r}}\right.}\left(x_{H}^{b}\right)\right) \leq \operatorname{Cm}_{0}^{1 / 2}$, for $H \in \mathscr{C}^{\natural}$, resp. $H \in \mathscr{C}^{\text {b }}$;
(b) If $H$ and $L$ are neighbors, then we have for every $i \in\{0,1,2,3\}$

$$
\begin{array}{rlr}
\left\|u_{H}-u_{L}\right\|_{C^{i}\left(B_{r_{H}}\left(x_{H}\right)\right)} \leq \text { Cm }_{0}^{1 / 2} \ell(H)^{3+\omega-i} & & \text { when } H \in \mathscr{C}^{\natural}, \\
\left\|u_{H}-u_{L}\right\|_{C^{i}\left(D \cap B_{2^{7} r_{H}}\left(x_{H}^{b}\right)\right)} \leq \text { Cm }_{0}^{1 / 2} \ell(H)^{3+\omega-i} & \text { when } H, L \in \mathscr{C}^{b} ; \tag{21.2}
\end{array}
$$

(c) $\left|D^{3} u_{H}\left(x_{H}^{\square}\right)-D^{3} u_{L}\left(x_{L}^{\square}\right)\right| \leq \operatorname{Cm}_{0}^{1 / 2}\left|x_{H}^{\square}-x_{L}^{\square}\right|^{\omega}$, where $\square=$ if the corresponding square is a non-boundary square and $\square=b$ if it is a boundary square;
(d) if $H \in \mathscr{C}^{\natural}$, then $\left\|u_{H}-\mathbf{p}_{V_{0}}^{\perp}\left(p_{H}\right)\right\|_{C^{0}\left(B_{4 r_{H}}\left(x_{H}\right)\right)} \leq \operatorname{Cm}_{0}^{1 / 2} \ell(H)$ and if $H \in \mathscr{C}^{b}$, then $\left.\left.u_{H}\right|_{\partial D \cap B_{2} 7_{4 r_{H}}}\left(x_{H}^{b}\right)\right)=g$;
(e) $\left|V_{H}-T_{\left(x, u_{H}(x)\right)} \mathbf{G}_{u_{H}}\right| \leq \operatorname{Cm}_{0}^{1 / 2} \ell(H)^{1-\delta_{1}}$ for every $x \in B_{4 r_{H}}\left(x_{H}\right)$, resp. $x \in D \cap$ $B_{2^{7} 4 r_{H}}\left(x_{H}^{b}\right)$;
(f) If $H^{\prime}$ is the square concentric to $H \in \mathscr{W}_{j}$ with $\ell\left(H^{\prime}\right)=\frac{9}{8} \ell(H)$, then we have for every $i \geq j+1$

$$
\begin{equation*}
\left\|\boldsymbol{\varphi}_{i}-u_{H}\right\|_{L^{1}\left(H^{\prime}\right)} \leq \operatorname{Cm}_{0} \ell(H)^{5+\beta_{1} / 2} \tag{21.3}
\end{equation*}
$$

### 21.1 PROOF OF PROPOSITION 21.1

Proof. We follow the proof of [21, Proposition 8.32] and often we drop here for simplicity the domains where we estimate the norm in.
(a) By [13, Lemma B.1], it is enough to make the estimates on $h_{H}$ instead of $u_{H}$. Fix any square $H \in \mathscr{P}^{j}$ and consider the family tree $H=H_{i} \subset H_{i-1} \subset \cdots \subset H_{N_{0}}$. We estimate

$$
\left\|h_{H}\right\|_{C^{3, \omega}} \leq \sum_{j=N_{0}+1}^{i}\left\|h_{H H_{j}}-h_{H H_{j-1}}\right\|_{C^{3, \omega}}+\left\|h_{H H_{N_{0}}}\right\|_{C^{3, \omega}} .
$$

As these are all harmonic functions, by the mean value property, it is enough to estimate the $L^{1}$ norms. Again using the harmonicity we see that

$$
\left\|h_{H H_{j}}-h_{H H_{j-1}}\right\|_{L^{1}\left(\Omega_{j}\right)} \leq\left\|\boldsymbol{\eta} \circ f_{H H_{j}}-\boldsymbol{\eta} \circ f_{H H_{j-1}}\right\|_{L^{1}\left(\Omega_{j}\right)}+\operatorname{Cm}_{0} r_{H_{j-1}}^{5+\beta_{1}},
$$

where $\Omega_{j}$ either is $B_{7 r_{H_{j}}}\left(p_{H_{j}}, V_{H}\right)$ if $H_{j} \in \mathscr{C}^{\natural}$ or $D_{H} \cap B_{2^{7} 7 r_{H_{j}}}\left(p_{H_{j}}^{b}, V_{H}\right)$ if $H_{j} \in \mathscr{C}^{b}$. Using Theorem 17.1, we see that both $f_{H H_{j}}$ and $f_{H H_{j-1}} \operatorname{describe} \operatorname{spt}(T)$ on a large set $K$, thus their average agree on $K$. Together with the oscillation estimate we then deduce

$$
\begin{aligned}
\left\|\boldsymbol{\eta} \circ f_{H H_{j}}-\boldsymbol{\eta} \circ f_{H H_{j-1}}\right\|_{L^{1}\left(\Omega_{j}\right)} & \leq C \ell\left(H_{j-1}\right)^{2}\left(m_{0} \ell\left(H_{j-1}\right)^{2-2 \delta_{1}}\right)^{1+\gamma_{1}} m_{0}^{1 / 4} \ell\left(H_{j-1}\right)^{1+\beta_{1}} \\
& \leq C m_{0} \ell\left(H_{j-1}\right)^{5+\beta_{1}} .
\end{aligned}
$$

For $\left\|h_{H H_{N_{0}}}\right\|_{C^{3, \omega}}$ we argue similarly and use Proposition 20.6.
(b) By [13, Lemma C.2], we have

$$
\left\|D^{j}\left(u_{H}-u_{L}\right)\right\|_{C^{0}} \leq C C r_{L}^{-2-j}\left\|u_{H}-u_{L}\right\|_{L^{1}}+C r_{L}^{3+\omega-j}\left\|D^{3}\left(u_{H}-u_{L}\right)\right\|_{C^{\omega}} .
$$

The second term is already bounded in (a), thus we are left with showing the $L^{1}$ estimate. To do so, we again use [13, Lemma B.1] to replace $u_{L}$ and $u_{H}$ with functions which have the same graph. It is enough to notice that, by Lemma 20.8

$$
\left\|h_{H}-\hat{h}_{L}\right\|_{L^{1}} \leq C m_{0}^{1 / 2} \ell(H)^{5+\delta_{1} / 2} .
$$

(c) Let $H, L \in \mathscr{P}^{j}$. In case that $\left|x_{H}-x_{L}\right| \geq 2^{-N_{0}}$, the statement follows from (a). Otherwise, we can find ancestors $J, M$ such that $H, L$ are in a distant relation where $\ell(J)=\ell(M)$ is comparable to $\left|x_{H}^{\square}-x_{L}^{\square}\right|$. Then we estimate

$$
\begin{aligned}
\left|D^{3} u_{H}\left(x_{H}^{\square}\right)-D^{3} u_{L}\left(x_{L}^{\square}\right)\right| \leq & \left|D^{3} u_{H}\left(x_{H}^{\square}\right)-D^{3} u_{H J}\left(x_{J}^{\square}\right)\right|+\left|D^{3} u_{L M}\left(x_{M}^{\square}\right)-D^{3} u_{L}\left(x_{L}^{\square}\right)\right| \\
& +\left|D^{3} u_{H J}\left(x_{J}^{\square}\right)-D^{3} u_{L M}\left(x_{M}^{\square}\right)\right| .
\end{aligned}
$$

The bound on the last term is already shown in (b), while for the first two we argue similarly as before. Consider the family tree $H \subset H_{i-1} \subset \cdots \subset J$. By the previous arguments, we deduce

$$
\left\|u_{H H_{i}}-u_{H H_{i-1}}\right\|_{C^{3}} \leq \mathrm{Cm}_{0}^{1 / 2} \ell\left(H_{i-1}\right)^{\omega} .
$$

(d) The claim is obvious by construction for boundary cubes. For non-boundary cubes, consider that the height bound for $T$ and the Lipschitz regularity for $f_{H}$ give that

$$
\left\|\mathbf{p}_{V_{H}}^{\perp}\left(p_{H}\right)-\boldsymbol{\eta} \circ f_{H}\right\|_{\infty} \leq C m_{0}^{1 / 4} \ell(H) .
$$

We also get $\left\|\mathbf{p}_{V_{H}}^{\perp}\left(p_{H}\right)-\eta \circ f_{H}\right\|_{\infty} \leq \mathrm{Cm}_{0}^{1 / 4} \ell(H)$. On the other hand the Lipschitz regularity of the tilted $H$-interpolating function $h_{H}$ and the $L^{1}$ estimate on $h_{H}-\boldsymbol{\eta} \circ f_{H}$ easily gives $\left\|\mathbf{p}_{V_{H}}^{\perp}\left(p_{H}\right)-h_{H}\right\|_{\infty} \leq \mathrm{Cm}_{0}^{1 / 4} \ell(H)$. The estimate claimed in (d) follows then from [15, Lemma B.1].
(e) follows from the estimates on $D \bar{h}_{H L}$ of Lemma 20.8.
(f) By definition of $\boldsymbol{\varphi}_{j}$, it is enough to estimate that for $L$ a neighbour square of $H$, we have

$$
\left\|u_{H}-u_{L}\right\|_{L^{1}} \leq C m_{0} \ell(H)^{5+\delta_{1} / 2} .
$$

### 21.2 PROOF OF THEOREM 18.16

Proof. (a) is an immediate consequence of the definition of $\boldsymbol{\varphi}_{j}$ and the fact that $u_{L}$ satisfies the correct boundary condition (for $L \in \mathscr{C}^{b}$ ). (b) follows exactly as in the proof of [13, Theorem 1.17] and from Proposition 21.1. In fact, we are in the simpler situation where our "ambient manifold" is just $\mathbb{R}^{n+2}$ and thus, we can choose $\Psi \equiv 0$. (c) and (d) are consequences of (b).

### 21.3 Proof of corollary 18.19 and theorem 18.21

Proof. We extend $\boldsymbol{\varphi}$ to all of $[-4,4]^{2}$ changing the $C^{3, \omega}$-norm only by geometric constant and call this extension $\tilde{\boldsymbol{\varphi}}$. Then consider

$$
\tilde{T}:=T+Q \cdot \mathbf{G}_{\left.\tilde{\varphi}\right|_{\mid-4,4)^{2} \backslash D}} .
$$

Then as $\partial \mathcal{M}=\Gamma$, so $\partial \tilde{T}=0$. We cannot directly apply the corresponding interior paper, [15, Corollary 2.2], to $\tilde{T}$ because the latter is not area-minimizing. However, the argument given in [15, Proof of Corollary 2.2] does not use the area-minimizing assumption. It uses only the height estimates of Proposition 19.1 (which can be trivially extended to $\tilde{T}$ since the portion added to $T$ is regular) and the constancy theorem (which is valid in our case, since $\tilde{T}$ has no boundary).

As for the existence and estimates on the normal approximation, we also can follow the same argument as in [15, Section 6.2] substituting the current $\tilde{T}$ to the current $T$ in there and the $\operatorname{map} \tilde{\varphi}$ to the $\operatorname{map} \varphi$ in there. First of all notice that the extension is done locally on each square and the ones surrounding it, and thus, even though the union of the squares in our $\mathscr{W}$ and the set $\square$ does not cover $[-4,4]^{2}$, this does not prevent us from applying the same procedure. Next, the construction algorithm and the estimates performed in [15, Section 6.2] depend only on the following two facts:
(a) The map $\varphi$ in $\left[15\right.$, Section 6.2] has, on every $L \in \mathscr{W}$, the same control on the $C^{3, \omega}$ norm that we have for the map $\tilde{\boldsymbol{\varphi}}$ (up to a constant).
(b) For each square $L \in \mathscr{W}$ (which in the case of [15, Section 6.2] corresponds to an interior square for us) we have a Lipschitz approximation $f_{L}$ of the current
$T\left\llcorner\mathbf{C}_{8 r_{L}}\left(p_{L}, V_{L}\right)\right.$, which in turn coincides with the current $T$ on a set $K_{L} \times V_{L}^{\perp}$, where $\left|B_{8 r_{L}} \backslash K_{L}\right|$ is small and the Lipschitz constant and the height of $f_{L}$ are both suitably small too. This is literally the case with the very same estimates for our interior squares, because $\tilde{T} L \mathbf{C}_{8 r_{L}}\left(p_{L}, V_{L}\right)=T L \mathbf{C}_{8 r_{L}}\left(p_{L}, V_{L}\right)$. In the case of boundary squares, we apply Theorem 17.1 and we extend the corresponding $f_{L}$ to a map $\tilde{F}_{L}$ on the whole disk $B_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{L}\right)$ by setting it equal to $Q$ copies of the graph of $\tilde{\boldsymbol{\varphi}}$ outside of the domain $D_{L} \cap B_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{L}\right)$. We then notice that such extension satisfies the same estimates on the Lipschitz constant and the height. Moreover, over the new region, by construction the extension coincides with the current $T$. Hence, if we denote by $\tilde{K}_{L}$ the complement of the projection on $V_{L}$ of the difference set $\operatorname{spt}(\tilde{T}) \Delta \mathrm{spt}\left(\mathbf{G}_{L}\left(f_{L}\right)\right)$, then

$$
B_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{L}\right) \backslash \tilde{K}_{L}=\left(B_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{L}\right) \cap D_{L}\right) \backslash K_{L} .
$$

In particular $\left|B_{2^{7} 8 r_{L}}\left(p_{L}^{b}, V_{L}\right) \backslash \tilde{K}_{L}\right|$ has the desired estimate.
Finally, observe the following. By the construction of [15, Section 6.2] we have a specific description of the set $\mathcal{K}$ consistsing of those points $p$ in the center manifold for which we know that the slice $\langle T, \mathbf{p}, p\rangle$ coincides with the slice of the multivalued approximation, namely $\sum_{i} \llbracket F_{i}(p) \rrbracket$. First of all, $\mathcal{K}$ contains $\boldsymbol{\Phi}(\mathbf{■})$. Secondly, for every Whitney region $\mathcal{L}$ corresponding to some square $L \in \mathscr{W}, \mathcal{K} \cap \mathcal{L}$ is defined in the following fashion. First of all, we denote by $\mathscr{D}(L)$ the family of squares $M \in \mathscr{W}$ which have nonempty intersection with $L$ (i.e. its neighbors), hence we consider in each $\mathbf{C}_{M}:=\mathbf{C}_{8 r_{M}}\left(p_{M}, V_{M}\right)$, resp. $\mathbf{C}_{M}:=\mathbf{C}_{2^{78} r_{M}}\left(p_{M}, V_{M}\right)$, the corresponding Lipschitz approximation $f_{L}$ and define

$$
\mathcal{K} \cap \mathcal{L}:=\bigcap_{M \in \mathscr{D}(L)} \mathbf{p}\left(\operatorname{spt}(T) \cap \mathrm{g} r\left(f_{M}\right)\right) .
$$

Since for boundary cubes $\Gamma \cap \mathbf{C}_{M} \subset \operatorname{spt}(T) \cap \operatorname{gr}\left(f_{M}\right)$, we conclude that $\Gamma \cap \mathcal{L} \subset \mathcal{K}$. On the other hand every point of $\Gamma \cap \mathcal{M}$ which does not belong to some Whitney region is necessarily contained in the contact set $\boldsymbol{\Phi}(\mathbf{\square})$. Thus we conclude that $\Gamma \subset \mathcal{K}$. Observe, moreover, that by construction the map $N$ vanishes identically on the contact set, while we also know that for each $f_{M}$ as above $f_{M}$ coincides with the function $g_{M}$ on $\mathbf{p}_{V_{M}}(\Gamma)$. In particular this implies that $N$ vanishes identically on the intersection of $\Gamma$ with any Whitney region.

### 21.4 PROOF OF PROPOSITION 18.23

(18.23) is an ovious consequence of (18.22) since on the complement of the squares $L \in \mathscr{W}^{e}$ the two functions $\varphi$ and $f$ coincide.

We now turn to (18.22) Observe next that, by Proposition 21.1(f), it suffices to show the claim for the function $u_{H}$ in place of $\varphi$. Observe also that we already know from the above argument that, if we replace $u_{H}$ with the tilted interpolating function $h_{H}$ and
$f$ with the Lipschitz approximation $f_{H}=f_{H H}$, the estimate holds, as it is in fact just a special case of (20.5) and (20.4). Fix now a point $x \in H$ and the corresponding point let $y(x):=\mathbf{p}_{V_{H}}\left(u_{H}(x)\right)$ be the corresponding projection on the plane $V_{H}$. We can use [15, (5.4)] (where we identify the manifold $\mathcal{M}$ in there with the affine plane $V_{H}+\boldsymbol{\varphi}(p)$ ) to compute

$$
\left|\boldsymbol{\eta} \circ f(x)-u_{H}(x)\right| \leq C\left|\boldsymbol{\eta} \circ f_{H}(y)-h_{H}(y)\right|+C\left|V_{H}-V_{0}\right| \operatorname{Lip}(f) \mathbf{h}\left(T, \mathbf{B}_{H}\right) .
$$

In particular we conclude

$$
\left|\boldsymbol{\eta} \circ f(x)-u_{H}(x)\right| \leq C\left|\boldsymbol{\eta} \circ f_{H}(y)-h_{H}(y)\right|+C \boldsymbol{m}_{0}^{1 / 2} \ell(H)^{1-\delta_{1}} \boldsymbol{m}_{0}^{\gamma_{2}} \ell(L)^{\gamma_{2}} \boldsymbol{m}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}} .
$$

Observing that $x \mapsto y(x)$ is a Lipschitz function with Lipschitz constant bounded by $|D \varphi|$, i.e. by $C m_{0}^{1 / 2}$ and integrating in $x$, we easily conclude the claimed estimate.

## LOCAL LOWER BOUNDS FOR THE DIRICHLET ENERGY AND THE $L^{2}$ NORM OF $N$

As in [15, Section 3] the aim of this section is to conclude suitable lower bounds for $\int|D N|^{2}$ and $|N|$ over regions of the center manifold which are close (and sizable) enough to some Whitney region $\mathcal{L}$. Depending on the reason why the refinement was stopped, we will either bound $|N|$ from below in terms of $\ell(L)^{1+\beta_{1}}$ or we will bound $\int|D N|^{2}$ from below in terms of the excess of the current in $\mathbf{B}_{L}$

### 22.1 LOWER BOUND ON $N$

We start with the following conclusion.
Proposition 22.1 (Separation because of the height). If $L \in \mathscr{W}^{h}$ then $L$ is necessarily an interior square. Moreover, there is constant $\tilde{C}>0$ depending on $M_{0}$ such that whenever $\left(C_{h}\right)^{4} \geq \tilde{C} C_{e}^{\natural}$ and $\varepsilon_{C M}>0$ is small enough, then every $L \in \mathscr{W}^{h}$ fulfills
(SI) $\Theta(T, p) \leq Q-\frac{1}{2} \quad$ for all $p \in \mathbf{B}_{16 r_{L}}\left(p_{L}\right)$,
(S2) $L \cap H=\varnothing \quad$ for all $H \in \mathscr{W}^{n}$ with $\ell(H) \leq \frac{1}{2} \ell(L)$,
(S3) $\mathcal{G}(N(x), Q \llbracket \boldsymbol{\eta} \circ N(x) \rrbracket) \geq \frac{1}{4} C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(L)^{1+\beta_{1}} \quad$ for all $x \in \boldsymbol{\Phi}\left(B_{2 \sqrt{2} \ell(L)}\left(x_{L}\right)\right)$.
Proof. We only have to prove that $L \in \mathscr{C}^{\natural}$ as the rest follows from the interior theory in [15, Section 3]. We show that any boundary square $H$ which did not stop because of the excess, also did not stop because of the height. Fix such an $H \in \mathscr{C}^{b} \backslash \mathscr{W}^{e}$. Then we know that its parent $M \in \mathscr{C}^{b} \cap \mathscr{S}$ satisfies

$$
\mathbf{E}\left(T, \mathbf{B}_{M}^{b}\right) \leq C_{e}^{b} \boldsymbol{m}_{0} \ell(H)^{2-2 \delta_{1}}
$$

and we want to show that

$$
\mathbf{h}\left(T, \mathbf{B}_{H}^{b}\right) \leq C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(H)^{1+\beta_{1}}
$$

To do so, we apply the height bound of Lemma 18.4 to a suitable rotated current $\tilde{T}:=O_{\sharp} T$, where $O$ is a rotation which maps $V_{0}$ onto $V_{H}$. Notice that the proof of this
lemma is based on the first variation and thus on the minimality of $T$. As $\tilde{T}$ is area minimizing (with respect to the tilted boundary $O(\Gamma)$ ), we can directly deduce

$$
\begin{aligned}
\mathbf{h}\left(T, \mathbf{B}_{H}^{b}\right) \leq \mathbf{h}\left(T, \mathbf{C}_{2^{7} 64 r_{H}}\left(p_{H}^{b}, V_{H}\right)\right) & \leq C r_{H}\left(\mathbf{E}\left(T, \mathbf{C}_{2^{7} 80 r_{H}}\left(p_{H}^{b}, V_{H}\right)+\mathbf{A} r_{H}\right)^{1 / 2}\right. \\
& \leq C r_{H}\left(\mathbf{E}\left(T, \mathbf{B}_{M}^{b}\right)+C\left|V_{M}-V_{H}\right|^{2}+\mathbf{A} r_{H}\right)^{1 / 2} \\
& \leq C m_{0}^{1 / 2} r_{H}^{3 / 2} \\
& \leq C C_{h} \boldsymbol{m}_{0}^{1 / 4} \ell(H)^{1+\beta_{1}},
\end{aligned}
$$

where we also used Proposition 19.1 and the sufficient small choice of $\varepsilon_{C M}$.
A simple corollary of the above proposition is that if a square stopped because of the neighbor condition, then this originated from a larger nearby square which stopped because of the excess.

Corollary 22.2. For every $H \in \mathscr{W}^{n}$, there is a chain of squares $L_{0}, L_{1}, \ldots, L_{j}=H$ such that
(a) $L_{i} \in \mathscr{W}^{n}$ for all $i>0$ and $L_{0} \in \mathscr{W}^{e}$,
(b) they are all neighbors, i.e. $L_{i} \cap L_{i-1} \neq \varnothing$ and $\ell\left(L_{i}\right)=\frac{1}{2} \ell\left(L_{i-1}\right)$.

In particular, $H \subset B_{3 \sqrt{2} \ell\left(L_{0}\right)}\left(x_{L_{0}}, V_{0}\right)$.
Accordingly, we can collect all the squares $H$ which have such a chain relating $H$ to a specific square $L \in \mathscr{W}^{e}$. The latter square is not necessarily unique, but it will be convenient to fix a consistent choice of $L$.

Definition 22.3 (Domains of influence). First, let us fix an ordering $\left\{J_{i}\right\}_{i \in \mathbb{N}}$ of $\mathscr{W}^{e}$ such that the side length is non-increasing. For $J_{0}$, we define its domain of influence by

$$
\mathscr{W}^{n}\left(J_{0}\right):=\left\{H \in \mathscr{W}^{n}: \text { there is a chain as in Corollary } 22.2 \text { with } L_{0}=J_{0} \text { and } L_{j}=H\right\} .
$$

Inductively, we define for $k>0$ the domain of influence $\mathscr{W}^{n}\left(J_{k}\right)$ of $J_{k}$ by all $H \in$ $\mathscr{W}^{n} \backslash \bigcup_{i<k} \mathscr{W}^{n}\left(J_{i}\right)$ which have a chain as in Corollary 22.2 with $L_{0}=J_{k}$ and $L_{j}=H$. As it is easy to check using Corollary 22.2 we have $\mathscr{W}^{n}=\bigcup_{k \in \mathbb{N}}^{\circ} \mathscr{W}^{n}\left(J_{k}\right)$.

### 22.2 LOWER BOUND ON THE DIRICHLET ENERGY

Having handled the case of "height stopped" squares we turn to squares which were stopped because they exceed the excess bound.

Proposition 22.4. (Splitting) There are constants $C_{1}\left(\delta_{1}\right), C_{2}\left(M_{0}, \delta_{1}\right), C_{3}\left(M_{0}, \delta_{1}\right)$ such that, if $M_{0} \geq C_{1}\left(\delta_{1}\right), C_{e}^{\natural} \geq C_{2}\left(M_{0}, \delta_{1}\right), C_{e}^{b} \geq C_{3}\left(M_{0}, \delta_{1}\right)$, if the hypotheses of Theorem 18.21 hold and if $\varepsilon_{C M}$ is chosen sufficiently small, then the following holds. If $L \in \mathscr{W}^{e}, q \in V_{0}$
with $\operatorname{dist}(L, q) \leq 4 \sqrt{2} \ell(L), B_{\ell(L) / 4}\left(q, V_{0}\right) \subset D$ and $\Omega=\boldsymbol{\Phi}\left(B_{\ell(L) / 4}\left(q, V_{0}\right)\right)$, then (with $\left.C, C_{4}=C\left(\beta_{1}, \delta_{1}, M_{0}, N_{0}, C_{e}^{\natural}, C_{e}^{b}, C_{h}\right)\right):$

$$
\begin{align*}
& C_{e}^{\square} \boldsymbol{m}_{0} \ell(L)^{4-2 \delta_{1}} \leq \ell(L)^{2} \mathbf{E}\left(T, \mathbf{B}_{L}^{\square}\right) \leq C \int_{\Omega}|D N|^{2},  \tag{22.1}\\
& \int_{\mathcal{L}}|D N|^{2} \leq C \ell(L)^{2} \mathbf{E}\left(T, \mathbf{B}_{L}^{\square}\right) \leq C_{4} \ell(L)^{-2} \int_{\Omega}|N|^{2} . \tag{22.2}
\end{align*}
$$

Before coming to the proof of the Proposition, let us first observe an important point. Fix $L$ as in the statement of the Proposition and consider its parent $H$ and its ancestor $J$ 6 generations before. If $L$ is a boundary square, then $H$ and $J$ are both boundary squares. On the other hand, if $L$ is an interior square, since $C_{e}^{\natural}$ is chosen much larger than $C_{e}^{b}$, we can ensure that both $L$ and $J$ are also interior squares. Indeed, when $\mathbf{B}_{L} \subset \mathbf{B}_{J}^{b}$ and $J \notin \mathscr{W}^{e}$, we have the obvious estimate

$$
\mathbf{E}\left(T, \mathbf{B}_{L}\right) \leq 2^{26} \mathbf{E}\left(T, \mathbf{B}_{J}^{b}\right) \leq 2^{26} C_{e}^{b} m_{0} \ell(J)^{2-2 \delta_{1}} \leq 2^{38} C_{e}^{b} m_{0} \ell(L)^{2-2 \delta_{1}}
$$

which therefore, by choosing $C_{e}^{\natural} \geq 2^{38} C_{e}^{b}$ implies that $L$ does not satisfy the excess stopping condition.

Hence we can invoke [15, Proposition 3.4] to cover the case in which $L \in \mathscr{W}^{e} \cap \mathscr{C}^{\natural}$, since the proof given in [15, Section $7 \cdot 3$ ] just uses the fact that all squares $L, H$ and $J$ are interior squares (i.e. the repsective balls $\mathbf{B}_{L}, \mathbf{B}_{H}$, and $\mathbf{B}_{J}$ do not intersect the boundary $\Gamma$ ). We are thus left to handle the case in which $L$ (and therefore also $H$ and $J$ ) are boundary squares.

To do so, we need analogues of three lemmas from [15].
Lemma 22.5. Let $B^{+} \subset \mathbb{R}^{2}$ be a half ball centered at the origin and $w \in W^{1,2}\left(B^{+}, \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)\right)$ be Dir-minimizing with $w=Q \llbracket 0 \rrbracket$ on $B^{+} \cap(\mathbb{R} \times\{0\})$. Denoting $\bar{w}:=w \oplus(-\eta \circ w)=$ $\sum_{i} \llbracket w_{i}-\boldsymbol{\eta} \circ w \rrbracket$ and $u:=\boldsymbol{\eta} \circ w$, we have

$$
Q \int_{B^{+}}|D u-D u(0)|^{2}=\int_{B^{+}} \mathcal{G}(D w, Q \llbracket D u(0) \rrbracket)^{2}-\operatorname{Dir}\left(\bar{w}, B^{+}\right)
$$

Proof. We extend $w$ in an odd way to all of the ball $B$. Notice that then also the extension of $u$ is harmonic in all of $B$. Now the proof is the same as in [15, Lemma $7 \cdot 3$ ], but we repeat it here anyway. First notice, that $u$ is a classical harmonic function and in particular, fulfills the mean value property. We use it to deduce

$$
\begin{align*}
Q \int_{B}|D u-D u(0)|^{2} & =Q \int_{B}\left(|D u|^{2}+|D u(0)|^{2}-2 D u \cdot D u(0)\right) \\
& =Q \int_{B}|D u|^{2}+Q|B||D u(0)|^{2}-2 Q\left(\int_{B} D u\right) \cdot D u(0)  \tag{22.3}\\
& =Q \int_{B}|D u|^{2}-Q|B||D u(0)|^{2}
\end{align*}
$$

Similarly we compute

$$
\begin{align*}
Q \int_{B}|D w|^{2} & =\sum_{i} \int_{B}\left|D w_{i}\right|^{2}=\sum_{i} \int_{B}\left(\left|D w_{i}-D u(0)\right|^{2}-|D u(0)|^{2}+2 D w_{i} \cdot D u(0)\right) \\
& =\int_{B} \mathcal{G}(D w, Q \llbracket D u(0) \rrbracket)^{2}-Q|B||D u(0)|^{2}+2 Q\left(\int_{B} \frac{1}{Q} \sum_{i} D w_{i}\right) \cdot D u(0) \\
& =\int_{B} \mathcal{G}(D w, Q \llbracket D u(0) \rrbracket)^{2}+Q|B \| D u(0)|^{2} . \tag{22.4}
\end{align*}
$$

Last we split the Dirichlet energy of $w$ into the average and the average-free part (as already observed in (17.14)).

$$
\begin{align*}
\int_{B}|D \bar{w}|^{2} & =\sum_{i} \int_{B}\left|D w_{i}-D u\right|^{2}=\sum_{i} \int_{B}\left(\left|D w_{i}\right|^{2}+|D u|^{2}-2 D w_{i} \cdot D u\right) \\
& =\int_{B}|D w|^{2}+Q \int_{B}|D u|^{2}-2 Q \int_{B}\left(\frac{1}{Q} D w_{i}\right) \cdot D u  \tag{22.5}\\
& =\int_{B}|D w|^{2}-Q \int_{B}|D u|^{2} .
\end{align*}
$$

The three identities (22.3), (22.4), (22.5) and dividing everything by 2 conclude the lemma.

An other important ingredient is the unique continuation for Dir-minimizers (compare to [15, Lemma 7.1]).

Lemma 22.6 (Unique Continuation for Dir-minimizers). For every $0<\eta<1$ and $c>0$, there is a $\delta>0$ such that whenever $B_{2 r}^{+} \subset V_{0}$ is the half ball and $w: B_{2 r}^{+} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ is Dir-minimizing with $w=Q \llbracket 0 \rrbracket$ on $B_{2 r}^{+} \cap(\mathbb{R} \times\{0\}), \operatorname{Dir}\left(w, B_{2 r}^{+}\right)=1$, and $\operatorname{Dir}\left(w, B_{r}^{+}\right) \geq c$, then

$$
\operatorname{Dir}\left(w, B_{s}(q)\right) \geq \delta \quad \text { for every } B_{s}(q) \subset B_{2 r}^{+} \text {with } s \geq \eta r
$$

Proof. The qualitative statement (UC) of the proof of [15, Lemma 7.1] applies directly to our situation while the quantitative statement follows from a blow-up argument that goes analogously for us as $B_{s}(q) \subset B_{2 r}^{+}$.

The previous two lemmas imply the following energy decay for Dir-minimizers (compare to [15, Proposition 7.2]) which itself implies the Proposition 22.4. First fix a number $\lambda>0$ such that

$$
(1+\lambda)^{4}<2^{\delta_{1}} .
$$

Proposition 22.7 (Decay estimate for Dir-minimizers). For any $\eta>0$ there is a $\delta>0$ such that whenever $B_{2 r}^{+} \subset V_{0}$ is the half ball and $w: B_{2 r}^{+} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ is Dir-minimizing with $w=Q \llbracket 0 \rrbracket$ on $B_{2 r}^{+} \cap(\mathbb{R} \times\{0\})$ and satisfies

$$
\int_{B_{(1+\lambda) r}^{+}} \mathcal{G}(D w, Q \llbracket D(\eta \circ w)(0) \rrbracket)^{2} \geq 2^{\delta_{1}-4} \operatorname{Dir}\left(w, B_{2 r}^{+}\right),
$$

then we have for any $B_{s}(q) \subset B_{2 r}^{+}$with $s \geq \eta r$

$$
\delta \operatorname{Dir}\left(w, B_{(1+\lambda) r}^{+}\right) \leq \operatorname{Dir}\left(\bar{w}, B_{(1+\lambda) r}^{+}\right) \leq \frac{1}{\delta r^{2}} \int_{B_{s}(q)}|\bar{w}|^{2} .
$$

Here we used again the notation $\bar{w}:=w \oplus(-\boldsymbol{\eta} \circ w)=\sum_{i} \llbracket w_{i}-\boldsymbol{\eta} \circ w \rrbracket$.
Proof. We follow word by word the proof of [ 15 , Proposition 7.2] using Lemma 22.6 and Lemma 22.5 instead of [ 15, Lemma 7.1] and [15, Lemma 7.3]. We reach the contradicting inequality

$$
\int_{B_{1+\lambda}^{+}}|D u-D u(0)|^{2} \geq 2^{\delta_{1}-4} \int_{B_{2}^{+}}|D u|^{2}
$$

which is false as one can see by reflecting such that $u$ stays harmonic and then using the classical decay for harmonic functions.

## FREQUENCY FUNCTION AND MONOTONICITY

In this section we take a further crucial step towards the proof of Theorem 12.6. We recall our key Assumption 12.5 and we add a further one on the smallness of the excess. Before doing that, we observe a corollary of the decay estimate in Theorem 12.7.

Corollary 23.1. Let $T$ and $\Gamma$ be as in Assumption 10.4 and assume that $0 \in \Gamma$ is a flat point and that $Q \llbracket V \rrbracket$ is the unique tangent cone to $T$ at 0 . Then there is a geometric constant $\kappa>0$ and constants $C$ and $r_{0}>0$ (depending on $\Gamma$ and $T$ ) such that

$$
\begin{equation*}
\mathbf{E}\left(T, \mathbf{C}_{r}\right) \leq C r^{4 \kappa} \quad \forall r \leq r_{0} . \tag{23.1}
\end{equation*}
$$

Thus, upon rescaling the current appropriately, if 0 is a flat point we can assume, without loss of generality, the following.

Assumptions 23.2. Let $T$ and $\Gamma$ be as in Assumption 10.4. $0 \in \Gamma$ is a flat point, $Q \llbracket V \rrbracket$ is the unique tangent cone to $T$ at 0 , we let $n$ be as in (12.1) and assume that (12.3) holds. In addition we assume to have fixed a choice of the parameters so that Theorem 18.16 and Theorem 18.21 hold and that

$$
\begin{equation*}
\mathbf{E}\left(T, \mathbf{C}_{4 R_{0} \rho}\right)+\mathbf{A}^{2} \rho^{2} \leq \varepsilon_{C M} \rho^{2 \kappa} \quad \text { for all } \rho \leq 1 \tag{23.2}
\end{equation*}
$$

Observe that, by (23.2), we conclude that both Theorem 18.16 and Theorem 18.21 can be applied to the current $T_{0, \rho}$ whenever $\rho \leq 1$.

### 23.1 INTERVALS OF FLATTENING

We start defining a decreasing set of radii $\left\{t_{1}>t_{2}>\ldots\right\} \subset(0,1]$, which at the moment can be both finite and infinite: in the first case one $t_{N}$ will be equal to 0 , while in the second case all $t_{k}$ 's are positive and $t_{k} \downarrow 0$.
$t_{1}$ is defined to be equal to 2 . We then let $\overline{\mathcal{M}}_{1}=\mathcal{M}_{1}$ be the center manifold and $\bar{N}_{1}=N_{1}$ the corresponding normal approximation which results after we apply Theorem 18.16 and Theorem 18.21 to the current $T$. Moreover we let $\mathscr{W}^{(1)}$ be the squares of the Whitney decomposition described in Definition 18.11. We then distinguish two cases:
(Stop) There is a square $H \in \mathscr{W}^{(1)}$ such that

$$
\begin{equation*}
\operatorname{dist}(0, H) \leq 64 \sqrt{2} \ell(H) \tag{23.3}
\end{equation*}
$$

(Go) There is no such square.

Notice that every such square $H$ satisfying (23.3) is a boundary square. In the first case we select an $H$ as in (Stop) which has maximal sidelength and we define $\bar{t}_{2}:=66 \sqrt{2} \ell(H)$ and $t_{2}:=t_{1} \bar{t}_{2}=132 \sqrt{2} \ell(H)$. Otherwise we define $t_{2}=0$. Observe that

$$
\begin{equation*}
\frac{t_{2}}{t_{1}} \leq 66 \sqrt{2} 2^{-N_{0}} \tag{23.4}
\end{equation*}
$$

Before proceeding further, we record an important consequence of the Whitney decomposition.

Corollary 23.3. If (Stop) holds, then the square $H$ of maximal sidelength that satisfies (23.3) must be an element of $\mathscr{W}^{e}$, i.e. it violates the excess condition.

Proof. Observe that if $H$ is an (NN) square, then there is a neighboring square $H^{\prime}$ of double sidelength which also belongs to $\mathscr{W}$ and it is easy to see that the latter satisfies (23.3) too, violating the maximilaity of $H$. Note next that (23.3) implies that $H$ is a boundary square, and as such it cannot belong to $\mathscr{W}^{h}$.

In case $t_{2}>0$ we then apply Theorem 18.16 and Theorem 18.21 to $T_{0, t_{2}}$ and let $\overline{\mathcal{M}}_{2}$ and $\bar{N}_{2}$ be the corresponding objects. The pair $\left(\mathcal{M}_{2}, N_{2}\right)$ will be derived by scaling back the objects at scale $t_{2}$, namely

$$
\begin{align*}
\mathcal{M}_{2} & =\left\{t_{2} q: q \in \overline{\mathcal{M}}_{2}\right\}, \\
N_{2}(q) & =t_{2} \bar{N}_{2}\left(\frac{q}{t_{2}}\right) \tag{23.6}
\end{align*}
$$

We then apply the procedure above to $\overline{\mathcal{M}}_{2}$ in place of $\overline{\mathcal{M}}_{1}$ and determine $\bar{t}_{3}$ analogously, while we set $t_{3}:=t_{2} \bar{t}_{3}$.

We proceed inductively and define $\overline{\mathcal{M}}_{k}, \mathcal{M}_{k}, \bar{N}_{k}, N_{k}, \bar{t}_{k}$, and $t_{k}:=t_{k-1} \bar{t}_{k}$ : the procedure stops when one $t_{k}$ equals 0 , otherwise goes indefinitely. Observe that for every $k$ we have the estimate

$$
\begin{equation*}
\frac{t_{k}}{t_{k-1}} \leq 66 \sqrt{2} 2^{-N_{0}} \tag{23.7}
\end{equation*}
$$

### 23.2 FREQUENCY FUNCTION

Observe that the conclusion of Theorem 12.6 is equivalent to $T$ coinciding with $Q \llbracket \mathcal{M}_{k} \rrbracket$ for some $k$ in a neighborhood of the origin. A simple corollary of the interior regularity is in fact the following

Corollary 23.4. If $N_{k} \equiv Q \llbracket 0 \rrbracket$ on some nontrivial open subset of $\mathcal{M}_{k}$, then $T=Q \llbracket \mathcal{M}_{k} \rrbracket$ in a neighborhood of 0 and in particular Theorem 12.6 holds.

We next consider a function $d$ which is $C^{2}$ in the punctured ball $\mathbf{B}_{1}(0)$, whose gradient $\nabla d$ is tangent to $\Gamma$ and such that (i)-(ii)-(iii) of Definition 13.6 hold. Likewise we fix the function $\phi:[0, \infty) \rightarrow[0, \infty)$ given by

$$
\phi(t):= \begin{cases}1, & \text { if } t \in\left[0, \frac{1}{2}\right] \\ (1-2 t), & \text { if } t \in\left[\frac{1}{2}, 1\right] \\ 0, & \text { if } t \geq 1\end{cases}
$$

From now on we denote by $D$ the classical Euclidean differentiation of functions, tensors, and vector fields, which for objects defined on the manifold $\mathcal{M}_{k}$ will mean that we compute derivatives along the tangents to the manifold. On the other hand we use the notation $\nabla_{\mathcal{M}_{k}}, D^{\mathcal{M}_{k}}$, and div $\mathcal{M}_{\mathcal{M}_{k}}$, respectively for the gradient, Levi-Civita connection, and divergence of (respectively), functions, tensors, and vector fields on $\mathcal{M}_{k}$ understood as a Riemannian submanifold of the Euclidean space $\mathbb{R}^{2+n}$.

We then define

$$
\begin{align*}
D(r) & :=\int_{\mathcal{M}_{k}} \phi\left(\frac{d(x)}{r}\right)\left|D N_{k}\right|^{2}(x) d \mathcal{H}^{2}(x), \quad \text { if } r \in\left(t_{k+1}, t_{k}\right],  \tag{23.8}\\
H(r) & :=-\int_{\mathcal{M}_{k}} \phi^{\prime}\left(\frac{d(x)}{r}\right)\left|\nabla_{\mathcal{M}_{k}} d(x)\right|^{2} \frac{\left|N_{k}(x)\right|^{2}}{d(x)} d \mathcal{H}^{2}(x), \quad \text { if } r \in\left(t_{k+1}, t_{k}\right] .  \tag{23.9}\\
S(r) & :=\int_{\mathcal{M}_{k}} \phi\left(\frac{d(x)}{r}\right)\left|N_{k}(x)\right|^{2} d \mathcal{H}^{2}(x) . \tag{23.10}
\end{align*}
$$

We are then ready to state our main estimate.
Theorem 23.5. Let $T$ be as in Assumption 23.2. Either $T=Q \llbracket \mathcal{M}_{k} \rrbracket$ in a neighborhood of the origin for some $k$ (and in that case note that $t_{k+1}=0$ ), or else $H(r)>0$ and $D(r)>0$ for every $r$. In the latter case the function $I(r):=\frac{r D(r)}{H(r)}$ satisfies the following properties for some constants $C$ and $\tau>0$ :
(a) For all $r>0$, we have

$$
\begin{equation*}
I(r) \geq C^{-1} \tag{23.11}
\end{equation*}
$$

and

$$
\begin{equation*}
D(r) \leq C r^{2+\tau} \tag{23.12}
\end{equation*}
$$

(b) I is continuous and differentiable on each open interval $\left(t_{k+1}, t_{k}\right)$ and moreover

$$
\begin{equation*}
\left.\frac{d}{d r}\left(\log I(r)+C D(r)^{\tau}-C t_{k}^{2 \tau-2} \frac{S(r)}{D(r)}\right) \geq-C r^{\tau-1} \quad \text { for a.e. } r \in\right] t_{k+1}, t_{k}[\text {. } \tag{23.13}
\end{equation*}
$$

(c) At each $t_{k}$ the function I has one-sided limits

$$
\begin{aligned}
& I\left(t_{k}^{+}\right)=\lim _{t \downarrow t_{k}} I(t), \\
& I\left(t_{k}^{-}\right)=\lim _{t \uparrow t_{k}} I(t),
\end{aligned}
$$

and moreover

$$
\begin{equation*}
\sum_{k}\left|I\left(t_{k}^{+}\right)-I\left(t_{k}^{-}\right)\right|<\infty \tag{23.14}
\end{equation*}
$$

We will prove (a) and (b) in Section 24 while we devote Section 25 to show (c). An obvious corollary of Theorem 23.5 is the following

Corollary 23.6. Let $T$ be as in Assumption 23.2. Either 0 is a regular point, or else $I(r)$ is well defined for every $r$ and the limit

$$
I_{0}:=\lim _{r \downarrow 0} I(r)
$$

exists, is finite and positive.
Proof. First of all observe that, since $I(r) \geq C^{-1}$,

$$
f(r):=\log I(r)-C t_{k}^{2 \tau-2} \frac{S(r)}{D(r)}+C D(r)^{\tau}+C r^{\tau} \geq-\log C .
$$

We will also see below in Lemma 24.1 that $S(r) \leq C r^{2} D(r)$. Hence, since the Lipschitz constant of $\log$ is bounded on [ $C^{-1}, \infty[$, we infer

$$
\begin{equation*}
\left|f\left(t_{j}^{+}\right)-f\left(t_{j}^{-}\right)\right| \leq C\left|I\left(t_{j}^{+}\right)-I\left(t_{j}^{-}\right)\right|+C\left(t_{j}^{+}\right)^{\tau} . \tag{23.15}
\end{equation*}
$$

Next we show that the two bounds (23.14) and (23.13) imply that $f$ is bounded from above: considering $\rho \in] 0,1\left[\right.$, we let $k$ the largest number such that $\rho<t_{k}$ and we can estimate

$$
f(1)-f(\rho)=\int_{\rho}^{t_{k}} f^{\prime}+\sum_{j=1}^{k-1} \int_{t_{j+1}}^{t_{j}} f^{\prime}+\sum_{j=2}^{k}\left(f\left(t_{j}^{+}\right)-f\left(t_{j}^{-}\right)\right)
$$

which turns into

$$
\begin{aligned}
f(\rho) & \leq f(1)-\int_{\rho}^{t_{k}} f^{\prime}-\sum_{j=1}^{k-1} \int_{t_{j+1}}^{t_{j}} f^{\prime}-\sum_{j}\left|f\left(t_{j}^{+}\right)-f\left(t_{j}^{-}\right)\right| \\
& \leq f(1)+C \int_{0}^{1} r^{\tau-1} d r+C \sum_{j}\left|I\left(t_{j}^{+}\right)-I\left(t_{j}^{-}\right)\right|<\infty
\end{aligned}
$$

(note that in the last line we have used (23.15)).

Next observe that the distributional derivative of $f$ consists of a nonnegative measure (on the union of the open intervals $\left(t_{k+1}, t_{k}\right)$ and a purely atomic Radon measure which has finite mass by (23.14). We thus conclude that the distributional derivative of $f$ is a Radon measure. Next fix any $\rho \leq 1$ and let $t_{k}$ be such that $2 t_{k+1}<\rho<2 t_{k}$. We then have the bound

$$
\begin{aligned}
|D f|(] \rho, 1[) & \leq D f\left(\rho, t_{k}^{-}\right)+\sum_{1 \leq j \leq k-1} D f(] t_{j+1}^{+}, t_{j}^{-}[)+\sum_{2 \leq j \leq k}\left|f\left(t_{j}^{+}\right)-f\left(t_{j}^{-}\right)\right| \\
& \leq 2 \sum_{j=1}^{\infty}\left|f\left(t_{j}^{+}\right)-f\left(t_{j}^{-}\right)\right|+\|f\|_{\infty}<\infty .
\end{aligned}
$$

Hence, letting $\rho$ go to 0 we discover that $|D f|(] 0,1[)<\infty$, that is $f \in B V(] 0,1[)$. This in turn implies that $f$ is a function of bounded variation and hence that $\lim _{r \downarrow 0} f(r)$ exists and is finite. Observe, moreover that by (24.11) we infer that $f(r)-\log (I(r))$ converges to 0 as $r \downarrow 0$. We thus conclude that

$$
\lim _{r \downarrow 0} f^{f(r)}=\lim _{r \downarrow 0} I(r)
$$

exists, it is finite, and it is positive.

### 24.1 PROOF OF (23.11)

The claim is simply equivalent to the existence of a constant $C$ such that $H(r) \leq C r D(r)$. The latter is a consequence of a Poincaré-type inequality which uses the fact that $N_{k}$ vanishes identically on the boundary curve $\Gamma$. The proof will be reduced to [21, Proposition 9.4]. However, in order to make the latter reduction, we employ a device which will be used in several subsequent computations. Having fixed a positive $r$ different from any $t_{j}$ we let $k$ be such that $t_{k+1}<r<t_{k}$ and we define the corresponding rescaled quantities $\bar{D}_{k}\left(t_{k}^{-1} r\right), \bar{H}_{k}\left(t_{k}^{-1} r\right), \bar{S}_{k}\left(t_{k}^{-1} r\right)$, and $\bar{I}_{k}\left(t_{k}^{-1} r\right)$. More precisely we define the function $d_{k}(x):=t_{k}^{-1} d\left(t_{k} x\right)$ and set

$$
\begin{align*}
\bar{D}_{k}(\rho) & :=\int_{\overline{\mathcal{M}}_{k}} \phi\left(\frac{d_{k}(x)}{\rho}\right)\left|D \bar{N}_{k}\right|^{2}(x) d \mathcal{H}^{2}(x),  \tag{24.1}\\
\bar{H}_{k}(\rho) & :=-\int_{\overline{\mathcal{M}}_{k}} \phi^{\prime}\left(\frac{d_{k}(x)}{\rho}\right)\left|\nabla_{\overline{\mathcal{M}}_{k}} d_{k}(x)\right|^{2} \frac{\left|\bar{N}_{k}(x)\right|^{2}}{d_{k}(x)} d \mathcal{H}^{2}(x),  \tag{24.2}\\
\bar{S}_{k}(\rho) & :=\int_{\overline{\mathcal{M}}_{k}} \phi\left(\frac{d_{k}(x)}{\rho}\right)\left|\bar{N}_{k}(x)\right|^{2} d \mathcal{H}^{2}(x) .
\end{align*}
$$

We then can immediately check the relations

$$
\begin{align*}
\bar{D}_{k}\left(t_{k}^{-1} r\right) & =t_{k}^{-2} D(r)  \tag{24.4}\\
\bar{H}_{k}\left(t_{k}^{-1} r\right) & =t_{k}^{-3} H(r) \\
\bar{S}_{k}\left(t_{k}^{-1} r\right) & =t_{k}^{-4} S(r)  \tag{24.6}\\
\bar{S}_{k}^{\prime}\left(t_{k}^{-1} r\right) & =t_{k}^{-3} S^{\prime}(r)  \tag{24.7}\\
\bar{D}_{k}^{\prime}\left(t_{k}^{-1} r\right) & =t_{k}^{-1} D^{\prime}(r) \tag{24.8}
\end{align*}
$$

Lemma 24.1. There is a constant $C$ such that

$$
\begin{align*}
H(r) & \leq C r D(r)  \tag{24.9}\\
S^{\prime}(r) & \leq C r D(r)  \tag{24.10}\\
S(r) & \leq C r^{2} D(r) \tag{24.11}
\end{align*}
$$

Proof. We observe that the corresponding inequalities for $\bar{D}_{k}, \bar{H}_{k}, \bar{S}_{k}$, and $\bar{S}_{k}^{\prime}$ follow from [21, Proposition 9.4], since the center manifold $\overline{\mathcal{M}}_{k}$, the functions $d_{k}$, and $N_{k}$ satisfy the assumptions of the Proposition.

### 24.2 Derivatives of $H$ and $D$

In order to prove (23.13) the first step consists in computing the derivatives of $H$ and $D$. In what follows we will use the usual convention of denoting by $O(g)$ any function $f$ of the real variable $r>0$ with the property that $|f(r)| \leq C g(r)$. Moreover, in order to avoid cumbersome notation, for $r \in\left(t_{k+1}, t_{k}\right]$ we will drop the subscript $\mathcal{M}_{k}$ from the gradient $\nabla_{\mathcal{M}_{k}}$ on the manifold.

Proposition 24.2. Under the assumptions of Theorem 23.5 we have, for every $r \in\left(t_{k+1}, t_{k}\right]$,

$$
\begin{align*}
& D^{\prime}(r)=-\int \phi^{\prime}\left(\frac{d(x)}{r}\right) \frac{d(x)}{r^{2}}|D N|^{2},  \tag{24.12}\\
& H^{\prime}(r)=\left(\frac{1}{r}+O(1)\right) H(r)+2 E(r), \tag{24.13}
\end{align*}
$$

and

$$
\begin{equation*}
E(r)=-\frac{1}{r} \int \phi^{\prime}\left(\frac{d(x)}{r}\right) \sum_{i} N_{i}(x) \cdot\left(D N_{i}(x) \nabla d(x)\right) . \tag{24.14}
\end{equation*}
$$

Proof. The first derivative is a straightforward computation. For the second, we can follow the computations of [21, Proof of Proposition 9.5] to conclude that

$$
H^{\prime}(r)=2 E(r)-\frac{1}{r} \int \phi^{\prime}\left(\frac{d(x)}{r}\right) \Delta_{\mathcal{M}_{k}} d(x)|N|^{2}(x),
$$

where $\Delta_{\mathcal{M}_{k}}$ is the Laplace-Beltrami operator on the manifold $\mathcal{M}_{k}$. Noticing that $\phi^{\prime}\left(\frac{d(x)}{r}\right)$ vanishes unless $C^{-1} r \leq|x| \leq C r$, our claim will follow once we show that

$$
\Delta_{\mathcal{M}_{k}} d(x)=\frac{1}{d(x)}+O(1)=\frac{1}{d(x)}(1+O(d(x))) .
$$

In order to show the latter estimate, we fix a point $x \in \mathcal{M}_{k}$ and observe first that the second fundamental form of the center manifold $\overline{\mathcal{M}}_{k}$ is bounded by $C\left(\mathbf{E}\left(T_{0, t_{k}}, 4 R_{0}\right)^{1 / 2}+\right.$ $\mathbf{A} t_{k}$ ), which in turn is bounded by $C t_{k}^{\kappa}$ for some positive $\kappa$. By rescaling, the second fundamental form $A_{\mathcal{M}_{k}}$ of $\mathcal{M}_{k}$ enjoys the bound $\left\|A_{\mathcal{M}_{k}}\right\|_{\infty} \leq C t_{k}^{\kappa-1}$. On the other hand, recalling that $|x|^{-1}|D d-D| x| |+\left|D^{2} d-D^{2}\right| x| | \leq C$ it is easy to see that

$$
\begin{aligned}
\left|\Delta_{\mathcal{M}_{k}} d(x)-\frac{1}{d(x)}\right| & \leq\left|\Delta_{\mathcal{M}_{k}}\right| x\left|-\frac{1}{|x|}\right|+C+C|x|\left\|A_{\mathcal{M}_{k}}\right\|_{\infty} \\
& \leq C+C\left\|A_{\mathcal{M}_{k}}\right\|_{\infty} \leq C t_{k}^{\kappa}+C \leq C .
\end{aligned}
$$

### 24.3 FIRST VARIATIONS AND APPROXIMATE IDENTITIES

We start by recalling that, since $T_{0, t_{k}}$ is area-minimizing and $\partial T_{0,2 t_{k}}\left\llcorner\mathbf{C}_{4 R_{0}}=Q \llbracket \Gamma_{k} \rrbracket\left\llcorner\mathbf{C}_{4 R_{0}}\right.\right.$, then $\delta T_{0, t_{k}}(X)=0$ for every $X$ which is tangent to $\Gamma$. In what follows we fix a $C^{3}$ extension $\tilde{\boldsymbol{\varphi}}_{k}$ of the function $\overline{\boldsymbol{\varphi}}_{k}$ to $[-4,4]^{2} \subset V$ (by increasing the $C^{3, \omega}$ estimate on $\boldsymbol{\varphi}_{k}$ by a constant factor) whose graph is the center manifold $\overline{\mathcal{M}}_{k}$ and we denote by $\mathbf{p}_{k}$ the orthogonal projection onto the graph of $\tilde{\boldsymbol{\varphi}}_{k}$ (which is of course defined only in a suitable normal neighborhood of it). We then fix the two relevant vector fields with which we will test the stationarity condition:

$$
\begin{aligned}
X_{o}(p) & :=\phi\left(\frac{d_{k}\left(\mathbf{p}_{k}(p)\right)}{r}\right)\left(p-\mathbf{p}_{k}(p)\right), \\
X_{i}(p) & :=-Y\left(\mathbf{p}_{k}(p)\right):=-\frac{1}{2} \phi\left(\frac{\left.d_{k}\left(\mathbf{p}_{k}(p)\right)\right)}{r}\right) \frac{\nabla d_{k}^{2}}{\left|\nabla d_{k}\right|^{2}}\left(\mathbf{p}_{k}(p)\right)
\end{aligned}
$$

(note that $\nabla$ means the gradient $\nabla_{\overline{\mathcal{M}}_{k}}$ here).
Note that $X_{i}$ is tangent to both $\mathcal{M}_{k}$ and $\Gamma_{k}$. Moreover, in [21, Sections 9.4 and 9.5], the estimates are done separately on both sides of $\Gamma_{k}$. Thus, it applies to our situation directly with $\mathcal{M}^{+}=\overline{\mathcal{M}}_{k}$. Note also that the fifth error terms vanish for us as our "ambient manifold" is $\mathbb{R}^{n+2}$. We summarize the statements here and first define the following function

$$
\varphi_{k}(p):=\phi\left(\frac{d_{k}\left(\mathbf{p}_{k}(p)\right)}{r}\right) .
$$

We also introduce the rescaled quantity

$$
\bar{E}_{k}(\rho):=-\frac{1}{\rho} \int_{\overline{\mathcal{M}}_{k}} \phi^{\prime}\left(\frac{d_{k}(x)}{\rho}\right) \sum_{i}\left(\bar{N}_{k}\right)_{i}(x) \cdot\left(D\left(\bar{N}_{k}\right)_{i}(x) \nabla d_{k}(x)\right)
$$

and record the corresponding relation with $E$, namely

$$
\begin{equation*}
\bar{E}_{k}\left(t_{k}^{-1} r\right)=t_{k}^{-2} E(r) . \tag{24.15}
\end{equation*}
$$

Proposition 24.3 (Outer variations). Let $A_{k}$ and $H_{\overline{\mathcal{M}}_{k}}$ denote the second fundamental form and the man curvature of $\overline{\mathcal{M}}_{k}$ respectively. Assume $\frac{t_{k+1}}{t_{k}}<r<1$. Then we have

$$
\begin{equation*}
\left|\bar{D}_{k}(r)-\bar{E}_{k}(r)\right|=\left|\int_{\overline{\mathcal{M}}_{k}}\left(\varphi_{k}\left|D \bar{N}_{k}\right|^{2}+\sum_{i}\left(\left(\bar{N}_{k}\right)_{i} \otimes D \varphi_{k}\right): D\left(\bar{N}_{k}\right)_{i}\right)\right| \leq \sum_{j=1}^{4}\left|\operatorname{Err}_{j}^{o}\right| \tag{24.16}
\end{equation*}
$$

with

$$
\begin{aligned}
\operatorname{Err}_{1}^{o}:= & -Q \int_{\overline{\mathcal{M}}_{k}} \varphi\left\langle H_{\overline{\mathcal{M}}_{k}} \eta \circ \bar{N}_{k}\right\rangle, \\
\left|\operatorname{Err}_{2}^{o}\right| \leq & C \int_{\overline{\mathcal{M}}_{k}}\left|\varphi_{k}\right|\left|A_{k}\right|^{2}\left|\bar{N}_{k}\right|^{2}, \\
\left|\operatorname{Err}_{3}^{o}\right| \leq & C \int_{\mathcal{M}}\left|\varphi_{k}\right|\left(\left|D \bar{N}_{k}\right|^{2}\left|\bar{N}_{k}\right|\left|A_{k}\right|+\left|D \bar{N}_{k}\right|^{4}\right) \\
& +C \int_{\mathcal{M}}\left|D \varphi_{k}\right|\left(\left|D \bar{N}_{k}\right|^{3}\left|\bar{N}_{k}\right|+\left|D \bar{N}_{k}\right|\left|\bar{N}_{k}\right|^{2}\left|A_{k}\right|\right), \\
\operatorname{Err}_{4}^{o}:= & \delta \mathbf{T}_{\bar{F}_{k}}\left(X_{o}\right)-\delta T_{0, t_{k}}\left(X_{o}\right)=\delta \mathbf{T}_{\bar{F}_{k}}\left(X_{o}\right) .
\end{aligned}
$$

For the inner variation, we introduce first a bit more of notation. First of all, we see $D\left(\bar{N}_{k}\right)_{j}$ as a map from $T \overline{\mathcal{M}}_{k}$ to $\mathbb{R}^{n+2}$. Denoting the components of $\left(\bar{N}_{k}\right)_{j}$ by $\left(\bar{N}_{k}\right)_{j}=$ $\left(\left(\bar{N}_{k}\right)_{j}^{1}, \ldots,\left(\bar{N}_{k}\right)_{j}^{n+2}\right)$ and choosing a vector field $Z$ tangent to $\overline{\mathcal{M}}_{k}$, we write

$$
D\left(\bar{N}_{k}\right)_{j}(Z)=\left(D_{Z}\left(\bar{N}_{k}\right)_{j}^{1}, \ldots, D_{Z}\left(\bar{N}_{k}\right)_{j}^{n+2}\right) .
$$

Similarly, we have

$$
D\left(\bar{N}_{k}\right)_{j} D^{\overline{\mathcal{M}}_{k}} Y(Z)=D\left(\bar{N}_{k}\right)_{j}\left(D^{\overline{\mathcal{M}}_{k}} Y(Z)\right)=\left(D_{D^{\bar{M}_{k} Y(Z)}}\left(\bar{N}_{k}\right)_{j}^{1}, \ldots, D_{D^{\bar{M}_{k} Y(Z)}}\left(\bar{N}_{k}\right)_{j}^{n+2}\right) .
$$

Thus, for the scalar product $D\left(\bar{N}_{k}\right)_{j}: D\left(\bar{N}_{k}\right)_{j} D^{\overline{\mathcal{M}}_{k}} Y$, we choose an orthonormal frame $e_{1}, e_{2}$ of $T \overline{\mathcal{M}}_{k}$ and express

$$
D\left(\bar{N}_{k}\right)_{j}: D\left(\bar{N}_{k}\right)_{j} D^{\overline{\mathcal{M}}_{k}} Y=\sum_{\ell}\left\langle D_{e_{\ell}}\left(\bar{N}_{k}\right)_{j}, D_{D^{\tilde{M}_{k} Y\left(e_{\ell}\right)}}\left(\bar{N}_{k}\right)_{j}\right\rangle=\sum_{\ell, i} D_{e_{\ell}}\left(\bar{N}_{k}\right)_{j}^{i} D_{D^{\tilde{M}_{k} Y\left(e_{\ell}\right)}}\left(\bar{N}_{k}\right)_{j}^{i} .
$$

We further introduce the quantity

$$
G(r):=-r^{-2} \int_{\mathcal{M}_{k}} \phi\left(\frac{d}{r}\right) \frac{d}{|\nabla d|^{2}} \sum_{j}\left|D\left(N_{k}\right)_{j} \cdot \nabla d\right|^{2}
$$

and its correspoding rescaled version

$$
\bar{G}_{k}(\rho)=-\rho^{-2} \int_{\overline{\mathcal{M}}_{k}} \phi\left(\frac{d_{k}}{\rho}\right) \frac{d_{k}}{\left|\nabla d_{k}\right|^{2}} \sum_{j}\left|D\left(\bar{N}_{k}\right)_{j} \cdot \nabla d_{k}\right|^{2},
$$

while we record the corresponding relation as in (24.4)-(24.8)

$$
\begin{equation*}
\bar{G}_{k}\left(t_{k}^{-1} r\right)=t_{k}^{-1} G(r) . \tag{24.17}
\end{equation*}
$$

Proposition 24.4 (Inner variations). Under the above assumptions we have

$$
\begin{align*}
& \left|\bar{D}_{k}^{\prime}(r)-O\left(t_{k}^{k}\right) \bar{D}_{k}(r)-2 \bar{G}_{k}(r)\right| \\
& =\frac{2}{r}\left|\int_{\overline{\mathcal{M}}_{k}}\left(\sum_{j} D\left(\bar{N}_{k}\right)_{j}: D\left(\bar{N}_{k}\right)_{j} D^{\overline{\mathcal{M}}_{k}} Y-\frac{1}{2}\left|D \bar{N}_{k}\right|^{2} \mathrm{~d} i v_{\mathcal{M}_{k}} Y\right)\right| \\
& \leq \frac{2}{r} \sum_{j=1}^{4}\left|\operatorname{Err}_{j}^{i}\right| \tag{24.18}
\end{align*}
$$

with

$$
\begin{aligned}
\operatorname{Err}_{1}^{i} & :=Q \int_{\overline{\mathcal{M}}_{k}}\left(\left\langle H_{\overline{\mathcal{M}}_{k^{\prime}}} \eta \circ \bar{N}_{k}\right\rangle d i v_{\mathcal{M}_{k}} Y+\left\langle D_{Y} H_{\mathcal{M}_{k^{\prime}}} \eta \circ \bar{N}_{k}\right\rangle\right), \\
\left|\operatorname{Err}_{2}^{i}\right| & \leq C \int_{\overline{\mathcal{M}}_{k}}\left|A_{k}\right|^{2}\left(|D Y|\left|\bar{N}_{k}\right|^{2}+|Y|\left|\bar{N}_{k}\right|\left|D \bar{N}_{k}\right|\right), \\
\left|\operatorname{Err}_{3}^{i}\right| & \leq C \int_{\overline{\mathcal{M}}_{k}}\left(\left|D \bar{N}_{k}\right|^{2}|Y|\left|A_{k}\right|\left(\left|\bar{N}_{k}\right|+\left|D \bar{N}_{k}\right|\right)+|D Y|\left(|A|\left|\bar{N}_{k}\right|^{2}\left|D \bar{N}_{k}\right|+\left|D \bar{N}_{k}\right|^{4}\right)\right) \\
\operatorname{Err}_{4}^{i} & :=\delta \mathbf{T}_{\bar{F}_{k}}\left(X_{i}\right)-\delta T_{0, t_{k}}\left(X_{i}\right)=\delta \mathbf{T}_{\bar{F}_{k}}\left(X_{i}\right) .
\end{aligned}
$$

Proof. The arguments for the proposition are the same as in [21, Proposition 9.10] and indeed they are based on the Taylor expansions of [14, Theorems 4.2 \& 4.3]. However some more care is required because the term $O\left(t_{k}^{\kappa}\right) D(r)$ appears in the corresponding inequality (namely [21, $(9.28)$ ] as $O(1) D(r)$. The reason for the improvement is based on the computations [21, (9.29)] and [21, Lemma 9.2]: the improvement follows easily from the fact that:

- The curvature of the rescaled boundary $\Gamma_{k}$ is bounded by $t_{k}$;
- The $C^{3}$ norm of the function $\overline{\boldsymbol{\varphi}}_{k}$ (whose graph is the center manifold $\overline{\mathcal{M}}_{k}$ ) is bounded by $\left(\mathbf{E}\left(T_{0, t_{k}}, \mathbf{C}_{4 R_{0}}\right)+\left\|\psi_{k}\right\|_{C^{3, \alpha_{0}}}\right)^{1 / 2}$, where $\psi_{k}$ is the function whose graph describes $\Gamma_{k}$; we thus have $\left\|\overline{\boldsymbol{\varphi}}_{k}\right\|_{\mathcal{C}^{3}} \leq C t_{k}^{\tau}$.


### 24.4 FAMILIES OF SUBREGIONS FOR ESTIMATING THE ERROR TERMS

We want to estimate the error terms over the Whitney regions in order to use the separation estimate (Proposition 22.1) and the splitting before tilting estimates (Proposition 22.4). To achieve this goal we goes along the same lines of [21, Section 9.6] and apply the arguments of [21, Section 9.6] to the current $T_{0, t_{k}}$ that gives rise to the center manifold $\overline{\mathscr{M}}_{k}$. Notice that in each error term, there is the cut-off $\phi\left(d_{k} / r\right)$, thus it is enough to consider squares which intersect $\mathscr{B}_{r}^{+}:=\left\{x \in V_{0} \cap D: d_{k}\left(\overline{\boldsymbol{\varphi}}_{k}(x)\right)<r\right\}$. However, to sum the estimates over all squares, we prefer the regions over which we integrate to be disjoint. For this purpose, we define a Besicovitch-type covering.

From now on we fix all the constants from Assumption 18.7 and treat them as geometric constants. We are going to consider the Whitney decomposition and the corresponding family $\mathscr{W}^{e}, \mathscr{W}^{h}, \mathscr{W}^{n}$ of squares whose definition is detailed in Section 18. Note that the construction is not applied to the current $T$ and the boundary $\Gamma$, but rather to the rescaled current $T_{0, t_{k}}$ and the rescaled boundary $\Gamma_{k}$. Note that the assumptions for the construction apply for each $k$. For our notation to be more precise we should add the dependence on $k$ of the various families $\mathscr{W}$, however, since $k$ is fixed at this stage, in order to make our formulas simpler we drop such dependence.

First we consider all squares which stopped for the excess or the height and which influence some square intersecting $\mathscr{B}_{r}^{+}$.
Definition 24.5. We define the family $\mathcal{T}$ to be

$$
\begin{aligned}
\mathcal{T}:=\{ & \left\{L \in \mathscr{W}^{e} \cup \mathscr{W}^{h}: L \cap \mathscr{B}_{r}^{+} \neq \varnothing\right\} \\
& \cup\left\{L \in \mathscr{W}^{e}: \text { there is an } L^{\prime} \in \mathscr{W}^{n}(L) \text { such that } L^{\prime} \cap \mathscr{B}_{r}^{+} \neq \varnothing\right\} .
\end{aligned}
$$

Notice that because in a chain of squares in $\mathscr{W}^{n}$, the sidelengths always double, we have for each $L \in \mathcal{T}$

$$
\operatorname{sep}\left(L, \mathscr{B}_{r}^{+}\right):=\inf \left\{|x-y|: x \in L, y \in \mathscr{B}_{r}^{+}\right\} \leq 3 \sqrt{2} \ell(L) .
$$

To each such square $L \in \mathcal{T}$, we associate a ball $B(L)$ which we call satellite ball. Preferably this ball is contained in the square and with radius comparable to the sidelength. However, as not every square in $\mathcal{T}$ is contained in $D$, we choose instead a nearby ball. Moreover we want that the concentric ball with twice the radius to be contained in $\mathscr{B}_{r}^{+}$. Notice that because of the intervals of flattening (23.3), the largest square $L$ contributing to the center manifold and intersecting $\mathscr{B}_{r}^{+}$satisfies $\ell(L) \leq \frac{1}{64 \sqrt{2}} r$.

- If $B_{\ell(L) / 2}\left(x_{L}\right) \subset \mathscr{B}_{r}^{+}$, we define $B(L):=B_{\ell(L) / 4}\left(x_{L}\right)$.
- If $B_{\ell(L) / 2}\left(x_{L}\right) \nsubseteq \mathscr{B}_{r}^{+}$, we choose a point $y \in \partial \mathscr{B}_{r}^{+}$minimizing the distance to $L$. Notice that the size length of the squares in the domain of influence of $L$ vary by a factor 2, we have $\left|x_{L}-y\right| \leq 4 \sqrt{2} \ell(L)$. The center of the satellite ball we want to be a point inside $\mathscr{B}_{r}^{+}$and close to $y$ (and thus close to $x_{L}$ ). Indeed, first notice that by the regularity assumption on $\Gamma_{k}, \overline{\boldsymbol{\varphi}}_{k}$ (Theorem 18.16) and $d_{k}$ (Definition 13.6) there is a $C^{1}$-diffeomorphism $\Psi_{r}: \bar{B}_{r}^{+} \rightarrow \overline{\mathscr{B}}_{r}^{+}$with $\left\|\Psi_{r}-\mathrm{Id}\right\| \leq \mathrm{Cm}_{0}^{1 / 2}$. Moreover, we define for any $\ell<\frac{r}{2}$ the vectorfield $n_{\ell}: \partial B_{r}^{+} \rightarrow B_{r}^{+}$describing $\partial\left\{y \in B_{r}^{+}: \operatorname{dist}\left(y, \partial B_{r}^{+}\right)>\ell\right\}$ by

$$
n_{\ell}\left(x_{1}, x_{2}\right):= \begin{cases}\left(x_{1}, \ell\right), & \text { if }\left|x_{1}\right|<r-\ell, x_{2}=0, \\ (r-\ell)\left(x_{1}, x_{2}\right), & \text { if } x_{2}>\ell, \\ (r-\ell, \ell), & \text { if } \ell-r<x_{1}<r, x_{2} \leq \ell \\ (-r+\ell, \ell), & \text { if }-r<x_{1}<-r+\ell, x_{2} \leq \ell\end{cases}
$$

Notice that if $\varepsilon_{C M}$ is small enough, we have for any $\ell<\frac{r}{2}$

$$
B_{\ell / 2}\left(\Psi_{r}\left(n_{\ell}(x)\right)\right) \subset \Psi_{r}\left(B_{\ell}\left(n_{\ell}(x)\right)\right) \subset \mathscr{B}_{r}^{+} .
$$

Thus for the $y \in B_{\ell(L) / 2}\left(x_{L}\right) \cap \partial \mathscr{B}_{r}^{+}$, we define

$$
q_{L}:=\Psi_{r}\left(n_{\ell(L) / 2}\left(\Psi_{r}^{-1}(y)\right)\right)
$$

and observe that

$$
B(L):=B_{\ell(L) / 4}\left(q_{L}\right) \subset \mathscr{B}_{r}^{+} .
$$

By construction and the estimates on $d_{k}$, we have if $\varepsilon_{C M}$ is small enough,

$$
\left|q_{L}-x_{L}\right| \leq 5 \sqrt{2} \ell(L) \quad \text { and thus } \quad \operatorname{dist}\left(q_{L}, L\right) \leq 4 \sqrt{2} \ell(L) .
$$

From this family $\mathcal{T}$, we now choose a maximal subfamily $\mathscr{T}$ for which the satellite balls are disjoint. Denote by $S:=\sup \{\ell(L): L \in \mathcal{T}\}$. We define $\mathscr{T}_{1} \subset\{L \in \mathcal{T}$ : $\left.\frac{1}{2} S \leq \ell(L) \leq S\right\}$ to be a maximal subfamily for which the associated satellite balls are pairwise disjoint. We inductively define $\mathscr{T}_{k+1} \subset\left\{L \in \mathcal{T}: 2^{-k-1} S \leq \ell(L) \leq 2^{-k} S\right\}$ to be a maximal subfamily such that all the satellite balls $B\left(L^{\prime}\right)$ with $L^{\prime} \in \mathscr{T}_{1} \cup \cdots \cup \mathscr{T}_{k}$ are pairwise disjoint. Finally we define $\mathscr{T}$ to be the union of all the $\mathscr{T}_{k}$. As we want to cover all of $\mathscr{B}_{r}^{+}$, we associate to each square in $L \in \mathscr{T}$ the nearby squares of $\mathcal{T}$ whose satellite balls intersect $B(L)$ and the domain of influence $\mathscr{W}^{n}(L)$. Indeed, by a standard covering argument, notice that if $H \in \mathcal{T}$, then there is at least one square $L \in \mathscr{T}$ such that $\operatorname{dist}(H, L) \leq 20 \sqrt{2} \ell(L)$. We fix an arbitrary choice to partition $\mathcal{T}$ into families $\mathcal{T}(L)$ such that $L \in \mathscr{T}$, for any $H \in \mathcal{T}(L)$ we have $\ell(H) \leq 2 \ell(L)$ and $\operatorname{dist}(H, L) \leq 20 \sqrt{2} \ell(L)$. Now we add the rest of $\mathscr{B}_{r}^{+}$and define

$$
\mathscr{W}(L):=\bigcup_{H \in \mathcal{T}(L)} \mathscr{W}^{n}(H) \cup\{H\}
$$

The associated Whitney regions will be called $\mathcal{U}(L) \subset \mathcal{M}$,

$$
\mathcal{U}(L):=\bigcup_{H \in \mathscr{W}(L)} \overline{\boldsymbol{\Phi}}_{k}(H)
$$

where the map $\overline{\boldsymbol{\Phi}}_{k}$ is the parametrization of the center manifold induced by $\overline{\boldsymbol{\varphi}}_{k}$, namely $\overline{\boldsymbol{\Phi}}_{k}(x)=\left(x, \overline{\boldsymbol{\varphi}}_{k}(x)\right)$.

For simplicity of notation, we enumerate $\mathscr{T}=\left\{L_{i}\right\}_{i}$ and denote

$$
\begin{aligned}
\mathcal{B}_{r}^{+} & :=\overline{\boldsymbol{\Phi}}_{k}\left(\mathscr{B}_{r}^{+}\right)=\overline{\mathcal{M}}_{k} \cap\left\{d_{k}<r\right\}, \\
\mathcal{U}_{i} & :=\mathcal{U}\left(L_{i}\right) \cap \mathcal{B}_{r}^{+}, \\
\mathcal{B}^{i} & :=\overline{\boldsymbol{\Phi}}_{k}\left(B\left(L_{i}\right)\right), \\
\ell_{i} & :=\ell\left(L_{i}\right) .
\end{aligned}
$$

Notice that by construction, every satellite ball $B\left(L_{i}\right)$ has distance at least $\ell_{i} / 4$ to $\partial \mathscr{B}_{r}^{+}$. In particular, there is a geometric constant $c>0$ such that

$$
c \frac{\ell_{i}}{r} \leq \inf _{\mathbf{p}_{k}^{-1}\left(\mathcal{B}^{i}\right)} \varphi_{k}=\inf _{\mathcal{B}^{i}} \varphi_{k} .
$$

As in [21, Section 9.6.2], we conclude that there is a geometric constant $C>0$ such that

$$
\begin{gather*}
\sup _{\mathbf{p}_{k}^{-1}\left(\mathcal{U}_{i}\right)} \varphi_{k}=\sup _{\mathcal{U}_{i}} \varphi_{k} \leq C \inf _{\mathbf{p}_{k}^{-1}\left(\mathcal{U}_{i}\right)} \varphi_{k}=C \inf _{\mathcal{U}_{i}} \varphi_{k}  \tag{24.19}\\
\sum_{H \in \mathscr{W}\left(L_{i}\right)} \ell(H)^{2} \leq C \ell_{i}^{2} . \tag{24.20}
\end{gather*}
$$

Applying the estimates of Theorem 18.21 and Corollary 18.19(ii) in each square of $\mathscr{W}\left(L_{i}\right)$ and summing over them yields

$$
\begin{align*}
\operatorname{Lip}\left(\bar{N}_{k} \mid \mathcal{U}_{i}\right) & \leq \mathrm{Cm}_{0}^{\gamma_{2}} \ell_{i}^{\gamma_{2}},  \tag{24.21}\\
\left\|\bar{N}_{k}\right\|_{C^{0}\left(\mathcal{U}_{i}\right)}+\sup _{\operatorname{spt(T)\cap \mathbf {p}^{-1}(\mathcal {U}_{i})} \mid}\left|\mathbf{p}^{\perp}\right| & \leq \mathrm{Cm}_{0}^{1 / 4} \ell_{i}^{1+\beta_{1}},  \tag{24.22}\\
\left\|\mathbf{T}_{\bar{F}_{k}}-T_{0, t_{k}}\right\|\left(\mathbf{p}_{k}^{-1}\left(\mathcal{U}_{i}\right)\right) & \leq \mathrm{Cm}_{0}^{1+\gamma_{2}} \ell_{i}^{4+\gamma_{2}},  \tag{24.23}\\
\int_{\mathcal{U}_{i}}\left|D \bar{N}_{k}\right|^{2} & \leq \mathrm{Cm} m_{0} \ell_{i}^{4-2 \delta_{1}},  \tag{24.24}\\
\int_{\mathcal{U}_{i}}\left|\eta \circ \bar{N}_{k}\right| & \leq \mathrm{Cm} m_{0} \ell_{i}^{4+\gamma_{2} / 2}+C \int_{\mathcal{U}_{i}}\left|\bar{N}_{k}\right|^{2+\gamma_{2}} . \tag{24.25}
\end{align*}
$$

On the other hand, we can use the the Separation Proposition 22.1, the Splitting Proposition 22.4 and the estimates (24.19), (24.20) to deduce estimates on the normal approximation as stated in the next lemma.

Lemma 24.6. Assume the assumption 18.18 holds. Then there is a geometric constant $C_{0}{ }^{1}$ such that

$$
\begin{align*}
m_{0} \sum_{i}\left(\ell_{i}^{4+2 \beta_{1}} \inf _{\mathcal{B}^{i}} \varphi_{k}\right) & \leq C_{0} \bar{D}_{k}(r),  \tag{24.26}\\
m_{0} \sum_{i} \ell_{i}^{4+\beta_{1}} & \leq C_{0} \int_{\mathcal{B}_{r}^{+}}\left|D \bar{N}_{k}\right|^{2} \leq C_{0}\left(\bar{D}_{k}(r)+r \bar{D}_{k}^{\prime}(r)\right) . \tag{24.27}
\end{align*}
$$

Moreover, we have

$$
\begin{equation*}
m_{0} \sup _{i} \ell_{i} \leq C_{0}\left(r \bar{D}_{k}(r)\right)^{1 /\left(5+\beta_{1}\right)} \quad \text { and } \quad m_{0} \sup _{i}\left(\ell_{i} \inf _{\mathcal{B}^{i}} \varphi_{k}\right) \leq C_{0} \bar{D}_{k}(r)^{1 /\left(4+\beta_{1}\right)}, \tag{24.28}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{D}_{k}(r) \leq C_{0} m_{0} r^{4-2 \delta_{1}} \leq C_{0} t_{k}^{2 \kappa} r^{4-2 \delta_{1}} . \tag{24.29}
\end{equation*}
$$

Proof. The proof goes completely analogous to the one of [21, Lemma 9.13] and we summarize it here. Fix an $L_{i} \in \mathscr{T}$. If $L_{i} \in \mathscr{W}^{h}$, it is an interior square and we can use Proposition 22.1 to deduce

$$
\begin{equation*}
\int_{\mathcal{B}^{i}}\left|\bar{N}_{k}\right|^{2} \geq c_{0} m_{0}^{1 / 2} \ell_{i}^{4+2 \beta_{1}} \tag{24.30}
\end{equation*}
$$

[^0]On the other hand, if $L_{i} \in \mathscr{W}^{e}$, then $L_{i}$ can be either a boundary square or an interior square. However the satellite ball does not intersect the boundary and also we can apply Proposition 22.4 in both situations. Thus, we have

$$
\begin{align*}
& \int_{\mathcal{B}^{i}}\left|D \bar{N}_{k}\right|^{2} \geq c_{0} m_{0} \ell_{i}^{4-2 \delta_{1}}, \\
& \int_{\mathcal{B}^{i}} \varphi\left|D \bar{N}_{k}\right|^{2} \geq c_{0} m_{0} \ell_{i}^{4-2 \delta_{1}} \inf _{\mathcal{B}^{i}} \varphi_{k} .
\end{align*}
$$

Summing over all squares and using (24.30), (24.31) and (24.32), we conclude

$$
\begin{gathered}
m_{0} \sum_{i} \ell_{i}^{4+2 \beta_{1}} \inf _{\mathcal{B}^{i}} \varphi_{k} \leq C_{0} \int_{\mathcal{B}_{r}^{+}}\left(\left|\bar{N}_{k}\right|^{2}+\varphi_{k}\left|D \bar{N}_{k}\right|^{2}\right), \\
m_{0} \sum_{i} \ell_{i}^{4+2 \beta_{1}} \leq C_{0} \int_{\mathcal{B}_{r}^{+}}\left(\left|\bar{N}_{k}\right|^{2}+\left|D \bar{N}_{k}\right|^{2}\right) \leq C_{0} \int_{\mathcal{B}_{r}^{+}}\left|D \bar{N}_{k}\right|^{2},
\end{gathered}
$$

where we used the Poincaré inequality and the fact that $\bar{N}_{k}$ vanishes on $\Gamma_{k}$. We conclude by noticing that, as $\phi^{\prime}=-2$ in $\left[\frac{1}{2}, 1\right]$, we have

$$
\begin{gathered}
\int_{\left\{r / 2<d_{k}<r\right\} \cap \overline{\mathcal{M}}_{k}}\left|D \bar{N}_{k}\right|^{2} \leq r \bar{D}_{k}^{\prime}(r), \\
\int_{\left\{d_{k}<r / 2\right\} \cap \overline{\mathcal{M}}_{k}}\left|D \bar{N}_{k}\right|^{2} \leq \bar{D}_{k}(r) .
\end{gathered}
$$

(24.29) is a consequence of (24.24).

We end this section with estimating the error terms (compare with [21, Proposition 9.14]).

Proposition 24.7. There are constants $C, \tau>0$ such that

$$
\begin{align*}
\left|\operatorname{Err}_{1}^{o}\right|+\left|\operatorname{Err}_{3}^{o}\right|+ & \left|\operatorname{Err}_{4}^{o}\right| \leq C \bar{D}_{k}(r)^{1+\tau},  \tag{24.33}\\
\left|\operatorname{Err}_{2}^{o}\right| & \leq C t_{k}^{2 \kappa} \bar{S}_{k}(r) \leq C t_{k}^{2 \kappa} r^{2} \bar{D}_{k}(r) \tag{24.34}
\end{align*}
$$

and

$$
\begin{align*}
\left|\operatorname{Err}_{1}^{i}\right|+\left|\operatorname{Err}_{3}^{i}\right|+\left|\operatorname{Err}_{4}^{i}\right| & \leq C \bar{D}_{k}(r)^{\tau}\left(\bar{D}_{k}(r)+r \bar{D}_{k}^{\prime}(r)\right),  \tag{24.35}\\
\left|\operatorname{Err}_{2}^{i}\right| & \leq C t_{k}^{2 \kappa} r \bar{D}_{k}(r) . \tag{24.36}
\end{align*}
$$

Proof. The detailed estimates can be found in the proof of [21, Proposition 9.14]. Notice that as there it is done for either side of the boundary separately, and as we have the same estimates on $N$, it applies directly to our situation. The idea is as follows. First we notice that

$$
|Y(p)| \leq \varphi(p) d_{k}\left(\mathbf{p}_{k}(p)\right) \quad \text { and } \quad|D Y(p)| \leq C \mathbf{1}_{\mathcal{B}_{r}^{+}}\left(\mathbf{p}_{k}(p)\right) .
$$

Then because of the Theorem 18.16, both the second fundamental form and the mean curvature of $\overline{\mathcal{M}}_{k}$ are bounded (and their derivatives) are bouned by $C t_{k}^{\kappa}$. The remaining terms in the errors can be split into the regions $\mathcal{U}_{j}$ and then be estimated by powers of $m_{0}$ and $\ell_{j}$ using (24.21) - (24.25). Choosing $\tau \ll \delta_{1}$ and recalling that $\delta_{1} \leq \beta_{1} \leq \gamma_{1} / 8$, we see that the powers are higher than what we need for (24.26) and (24.27). Thus with (24.28) we gain the additional $\bar{D}_{k}(r)^{\tau}$.

The only relevant difference in the estimates of [21, Proposition 9.14] is in the terms $\mathrm{Err}_{2}^{i}$ and $\mathrm{E}_{2}^{0}$, where our estimates have an improved factor $\mathrm{Ct}{ }_{k}^{2 \kappa}$ in the right hand side. But this follows easily from the fact that in our case we take advantage of $\left\|A_{k}\right\|_{\infty} \leq C t_{k}^{\kappa}$, while in [21, Proposition 9.14] the second fundamental form of the center manifold is only known to be bounded by a constant.

### 24.5 Proof of (23.12) and (23.13)

In order to prove (23.12) we exploit (24.4) and (24.29): we assume $t_{k+1}<r<t_{k}$ and estimate

$$
D(r)=t_{k}^{2} \bar{D}_{k}\left(t_{k}^{-1} r\right) \leq C t_{k}^{2+2 \kappa}\left(t_{k}^{-1} r\right)^{4-2 \delta_{1}} \leq C r^{2+2 \kappa} .
$$

In order to prove (23.13) we follow the computations of [21, Section 9.1], but in our setting some additional complications are created by the fact that we need to scale back our estimates for the rescaled quantities $\bar{D}_{k}, \bar{H}_{k}, \bar{S}_{k}, \bar{G}_{k}$, and $\bar{S}_{k}$. First of all we recall (24.13):

$$
\begin{equation*}
H^{\prime}(r)=r^{-1} H(r)+2 E(r)+O(1) H(r) . \tag{24.37}
\end{equation*}
$$

Next we combine (24.16), (24.33), and (24.34) to get

$$
\begin{equation*}
\left|\bar{D}_{k}\left(t_{k}^{-1} r\right)-\bar{E}_{k}\left(t_{k}^{-1} r\right)\right| \leq C \bar{D}_{k}\left(t_{k}^{-1} r\right)^{1+\tau}+C t_{k}^{2 \tau} \bar{S}_{k}\left(t_{k}^{-1} r\right) . \tag{24.38}
\end{equation*}
$$

We next can use (24.4), (24.6), and (24.15) to conclude

$$
\begin{equation*}
|D(r)-E(r)| \leq C D(r)\left(t_{k}^{-2} D(r)\right)^{\tau}+C t_{k}^{2 \tau-2} S(r) \tag{24.39}
\end{equation*}
$$

Next recall that $D(r) \leq C r^{2+2 \kappa}$. Since $r \leq t_{k}$ we can write

$$
t_{k}^{-2} D(r) \leq C t_{k}^{-2} r^{2} D(r)^{1-2 /(2+2 k)} \leq C D(r)^{1-1 /(1+k)} .
$$

Thus, at the prize of choosing $\tau$ smaller, we can translate (24.39) into

$$
\begin{equation*}
|D(r)-E(r)| \leq C D(r)^{1+\tau}+C t_{k}^{2 \tau-2} S(r) \tag{24.40}
\end{equation*}
$$

The final ingredient is derived by first combining (24.18), (24.35), and (24.36) to get

$$
\begin{align*}
& \left|\bar{D}_{k}^{\prime}\left(t_{k}^{-1} r\right)+O\left(t_{k}^{2 k}\right) \bar{D}_{k}\left(t_{k}^{-1} r\right)-\bar{G}_{k}\left(t_{k}^{-1} r\right)\right| \\
& \leq \frac{C}{t_{k}^{-1} r} \bar{D}_{k}\left(t_{k}^{-1} r\right)^{\tau}\left(\bar{D}_{k}\left(t_{k}^{-1} r\right)+t_{k}^{-1} r \bar{D}_{k}^{\prime}\left(t_{k}^{-1} r\right)\right)+C t_{k}^{2 \kappa} \bar{D}_{k}\left(t_{k}^{-1} r\right), \tag{24.41}
\end{align*}
$$

which in turn, using (24.4), (24.17), and (24.15) becomes

$$
\left|D^{\prime}(r)+O\left(t_{k}^{2 \kappa-1}\right) D(r)-2 G(r)\right| \leq C\left(t_{k}^{-2} D(r)\right)^{\tau}\left(r^{-1} D(r)+D^{\prime}(r)\right)+C t_{k}^{2 \kappa-1} D(r)
$$

But then, arguing as for (24.40) we can achieve

$$
\begin{equation*}
\left|D^{\prime}(r)-2 G(r)\right| \leq C t_{k}^{2 \kappa-1} D(r)+C D(r)^{\tau}\left(r^{-1} D(r)+D^{\prime}(r)\right) . \tag{24.42}
\end{equation*}
$$

We are now ready to estimate $\frac{d}{d r} \log I(r)$. We start by writing

$$
\frac{d}{d r} \log I(r)=\frac{1}{r}+\frac{D^{\prime}(r)}{D(r)}-\frac{H^{\prime}(r)}{H(r)} .
$$

Hence, using (24.37) we write

$$
\begin{equation*}
\frac{d}{d r} \log I(r) \geq-C+\frac{D^{\prime}(r)}{D(r)}-\frac{2 E(r)}{H(r)} \tag{24.43}
\end{equation*}
$$

Next recall (23.12) while Lemma 24.1 implies that for $\sigma \in] 0,1[$ we have

$$
t_{k}^{2 \sigma-2} S(r) \leq \mathrm{Cr}^{2} t_{k}^{2 \sigma-2} D(r) \leq \mathrm{Cr}^{2 \sigma} D(r) .
$$

In combination with the last two bounds, (24.40) becomes (after possibly choosing a new positive $\tau$ )

$$
|D(r)-E(r)| \leq C r^{\tau} D(r),
$$

which in turn implies

$$
\begin{equation*}
\frac{D(r)}{2} \leq E(r) \leq 2 D(r), \tag{24.44}
\end{equation*}
$$

provided $r \leq r_{0}$ is sufficiently small with $r_{0}>0$ depending only on $C$ and $\tau$.
By (24.44) we can turn (24.40) into

$$
\left|E(r)^{-1}-D(r)^{-1}\right| \leq C D(r)^{\tau-1}+C t_{k}^{2 \tau-2} \frac{S(r)}{D(r)^{2}}
$$

Inserting the latter into (24.43) (and considering that $D^{\prime}(r) \geq 0$ ) we then get

$$
\frac{d}{d r} \log (I(r)) \geq \frac{D^{\prime}(r)}{E(r)}-\frac{2 E(r)}{H(r)}-C \frac{D^{\prime}(r)}{D(r)^{1-\tau}}-C t_{k}^{2 \kappa-2} \frac{S(r) D^{\prime}(r)}{D(r)^{2}}-C .
$$

We can finally insert (24.42) to achieve

$$
\begin{aligned}
\frac{d}{d r} \log (I(r)) \geq & \frac{2 G(r)}{E(r)}-\frac{2 E(r)}{H(r)}-C \frac{D(r)}{E(r)}\left(\frac{D(r)^{\tau}}{r}+\frac{D^{\prime}(r)}{D(r)^{1-\tau}}+t_{k}^{2 \tau-2}\right) \\
& -C \frac{D^{\prime}(r)}{D(r)^{1-\tau}}-C t_{k}^{2 \kappa-2} \frac{S(r) D^{\prime}(r)}{D(r)^{2}}-C .
\end{aligned}
$$

Next note that:

- $G(r) H(r) \geq E(r)^{2}$, by Cauchy-Schwarz;
- $\frac{D(r)}{E(r)} \leq C$;
- $D(r) \leq C r^{2+2 \kappa}$.
- We can rewrite $-\frac{S(r) D^{\prime}(r)}{D(r)^{2}}=\frac{d}{d r} \frac{S(r)}{D(r)}-\frac{S^{\prime}(r)}{D(r)}$, and it is easy to see that $S^{\prime}$ is positive. So, after possibly choosing $\tau$ smaller, yet positive, we achieve

$$
\frac{d}{d r}\left(\log I(r)+C D(r)^{\tau}-C t_{k}^{2 \tau-2} \frac{S(r)}{D(r)}\right) \geq-C r^{\tau-1}
$$

This section is devoted to prove (23.14). We observe that, by the continuity of the functions

$$
t \mapsto H\left(N_{k}, t\right) \quad \text { and } \quad t \mapsto D\left(N_{k}, t\right)
$$

we have

$$
I\left(t_{k}^{+}\right)=\frac{t_{k} D\left(N_{k-1}, t_{k}\right)}{H\left(N_{k-1}, t_{k}\right)} \quad \text { and } \quad I\left(t_{k}^{-}\right)=\frac{t_{k} D\left(N_{k}, t_{k}\right)}{H\left(N_{k}, t_{k}\right)} .
$$

In order to simplify our notation we use the shortcut $\mathbf{E}(T, r)$ for $\mathbf{E}\left(T, \mathbf{B}_{r}\right)$. We will show the following two propositions

Proposition 25.1. There is a constant $C$ independent of $k$ such that, if $\varepsilon_{C M}$ is small enough then

$$
\begin{align*}
& C^{-1} t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right) \leq D\left(N_{k-1}, t_{k}\right) \leq C t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right)  \tag{25.1}\\
& C^{-1} t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right) \leq D\left(N_{k}, t_{k}\right) \leq C t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right)  \tag{25.2}\\
& C^{-1} t_{k}^{3} \mathbf{E}\left(T, 6 t_{k}\right) \leq H\left(N_{k-1}, t_{k}\right) \leq C t_{k}^{3} \mathbf{E}\left(T, 6 t_{k}\right) \\
& C^{-1} t_{k}^{3} \mathbf{E}\left(T, 6 t_{k}\right) \leq H\left(N_{k}, t_{k}\right) \leq C t_{k}^{3} \mathbf{E}\left(T, 6 t_{k}\right) . \tag{25.4}
\end{align*}
$$

Proposition 25.2. There is a positive exponent $\tau_{1}$ independent of $k$ such that, if $\varepsilon_{C M}$ is small enough then

$$
\begin{align*}
& \left|D\left(N_{k-1}, t_{k}\right)-D\left(N_{k}, t_{k}\right)\right| \leq C t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right)^{1+\tau_{1}},  \tag{25.5}\\
& \left|H\left(N_{k-1}, t_{k}\right)-H\left(N_{k}, t_{k}\right)\right| \leq C t_{k}^{3} \mathbf{E}\left(T, 6 t_{k}\right)^{1+\tau_{1}} \tag{25.6}
\end{align*}
$$

Observe that the estimates (25.2) (the second one), (25.3) (the first one), (25.4) (the first one), (25.5), and (25.6) imply

$$
\begin{equation*}
\left|I\left(t_{k}^{+}\right)-I\left(t_{k}^{-}\right)\right| \leq C \mathbf{E}\left(T, 6 t_{k}\right)^{\tau_{1}} \leq C t_{k}^{2 \kappa \tau_{1}} \tag{25.7}
\end{equation*}
$$

On the other hand, by the choice of $N_{0}$ in Assumption 18.7, by (23.7), we get $\frac{t_{k}}{t_{k-1}} \leq \frac{1}{2}$, which iterated implies $t_{k} \leq 2^{-k}$. We therefore get

$$
\begin{equation*}
\left|I\left(t_{k}^{+}\right)-I\left(t_{k}^{-}\right)\right| \leq C 2^{-2 \kappa \tau_{1} k} \tag{25.8}
\end{equation*}
$$

which clearly implies (23.14).

Proof of Proposition 25.1. As the center manifold $\overline{\mathcal{M}}_{k-1}$ stopped, and we are close to the boundary, it must have stopped for the excess and thus, there is a square $L \in \mathscr{W}^{e}$ such that $c \frac{t_{k}}{t_{k-1}} \leq \ell(L) \leq C \frac{t_{k}}{t_{k-1}}$ (recall section 23.1). Looking at its ancestors (as we did in Proposition 18.12), we notice

$$
\begin{equation*}
\mathbf{E}\left(T, \rho t_{k}\right)=\mathbf{E}\left(T_{0, t_{k-1}}, \rho t_{k} / t_{k-1}\right) \leq \operatorname{Cm}_{0}(k-1)\left(\rho \frac{t_{k}}{t_{k-1}}\right)^{2-2 \delta_{1}} \tag{25.9}
\end{equation*}
$$

for every $1 \leq \rho \leq 5 R_{0} \frac{t_{k-1}}{t_{k}}$ and some geometric constant $C$. Here we denote by $\boldsymbol{m}_{0}(k-1)$ and $m_{0}(k)$ the two quantities

$$
\begin{aligned}
\boldsymbol{m}_{0}(k-1) & =\mathbf{E}\left(T_{0, t_{k-1}}, \mathbf{C}_{5 R_{0}}\right)+\left\|\psi_{k-1}\right\|_{C^{3, \alpha}(]-5 R_{0}, 5 R_{0}[)^{\prime}}^{2} \\
\boldsymbol{m}_{0}(k) & =\mathbf{E}\left(T_{0, t_{k}}, \mathbf{C}_{5 R_{0}}\right)+\left\|\psi_{k}\right\|_{C^{3, \alpha}(]-5 R_{0}, 5 R_{0}[)}^{2}
\end{aligned}
$$

where $\psi_{k}$ and $\psi_{k-1}$ are the functions describing the rescaled boundaries $\Gamma_{k}$ and $\Gamma_{k-1}$. Observe that, since $\psi_{k}(0)=\psi_{k-1}(0)=0$ and $\psi_{k}^{\prime}(0)=\psi_{k-1}^{\prime}(0)=0$, it can be readily checked that

$$
\left\|\psi_{k}\right\|_{C^{3, \alpha}(]-5 R_{0}, 5 R_{0}[)}^{2} \leq \frac{t_{k}^{2}}{t_{k-1}^{2}}\left\|\psi_{k-1}\right\|_{C^{3, \alpha}(]-5 R_{0}, 5 R_{0}[)}^{2}
$$

so that we have

$$
\begin{equation*}
\boldsymbol{m}_{0}(k) \leq \mathbf{E}\left(T, \mathbf{C}_{5 R_{0} t_{k}}\right)+\frac{t_{k}^{2}}{t_{k-1}^{2}} \boldsymbol{m}_{0}(k-1) \leq \operatorname{Cm}_{0}(k-1)\left(\frac{t_{k}}{t_{k-1}}\right)^{2-2 \delta_{1}} \tag{25.10}
\end{equation*}
$$

where we also used (25.9). On the other hand, because of the stopping condition we also know that

$$
\begin{equation*}
\mathbf{E}\left(T, 6 t_{k}\right)=\mathbf{E}\left(T_{0, t_{k-1}}, 6 t_{k} / t_{k-1}\right) \geq C^{-1} \boldsymbol{m}_{0}(k-1)\left(\frac{t_{k}}{t_{k-1}}\right)^{2-2 \delta_{1}} \tag{25.11}
\end{equation*}
$$

In particular, we infer by (25.10) that

$$
\begin{equation*}
\mathbf{E}\left(T, 6 t_{k}\right) \geq C^{-1} m_{0}(k) . \tag{25.12}
\end{equation*}
$$

Observe now that for $D\left(\bar{N}_{k}, 1\right)$ we have the inequality

$$
D\left(\bar{N}_{k}, 1\right) \leq C m_{0}(k)
$$

by construction of the center manifold (i.e. (18.20)). In turn, by rescaling, we can conclude

$$
D\left(N_{k}, t_{k}\right)=t_{k}^{2} D\left(\bar{N}_{k}, 1\right) \leq C t_{k}^{2} m_{0}(k) \leq C t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right)
$$

namely the first of the two inequalities in (25.13). Then we observe that (25.1) and (25.3) follow from the Splitting Proposition 22.4 applied to to the current $T_{0, t_{k}}$ which in turn
produces the center manifold $\overline{\mathcal{M}}_{k-1}$ and the normal approximation $\bar{N}_{k-1}$ as we are in the situation where the center manifold stopped. Moreover, we recall that by the Poincaré inequality (as already observed in (23.11) and proved in Section 24), we have for any $r>0$

$$
H\left(N_{k}, r\right) \leq \operatorname{CrD}\left(N_{k}, r\right) .
$$

Thus (25.4) and (25.2) follow once we have shown the following inequalities

$$
\begin{equation*}
D\left(N_{k}, t_{k}\right) \leq C t_{k}^{2} \mathbf{E}\left(T, 6 t_{k}\right) \leq C t_{k}^{-1} H\left(N_{k}, 6 t_{k}\right) . \tag{25.13}
\end{equation*}
$$

For the second inequality in (25.13) we adapt the proof of [ 15 , Proposition 3.7 ] as the only difference to our situation is the cut-off function. We describe here the idea of the argument, the details can be read in [ 15, Section 9]. Again recall the square $L \in \mathscr{W}^{e}$ which stopped in the construction of $\overline{\mathcal{M}}_{k-1}$ according to the argument above. By the splitting Proposition 22.4, we then have a nearby ball $B_{\ell / 4}(z)$ not intersecting $\Gamma_{0, t_{k-1}}$ such that

$$
m_{0}(k-1)\left(\frac{t_{k}}{t_{k-1}}\right)^{6-2 \delta_{1}} \leq C \int_{\bar{\Phi}_{k-1}\left(B_{\ell / 4}(z)\right)}\left|\bar{N}_{k-1}\right|^{2} .
$$

The argument of [ 15 , Section 9] provides now a similar bound for the ball $B^{\prime}=$ $2 \frac{t_{k-1}}{t_{k}} B_{\ell / 4}(z)$, which has radius comparable to 1 , in the center manifold $\overline{\mathcal{M}}_{k}$. More precisely, since $\left(\frac{t_{k-1}}{t_{k}}\right)^{4}$ is exactly the scaling relating the $L^{2}$ norm on $B^{\prime}$ and $B_{\ell / 4}(z)$, while $\left(\frac{t_{k-1}}{t_{k}}\right)^{2-2 \delta_{1}}$ is the scaling factor which makes $m_{0}(k)$ and $m_{0}(k-1)$ comparable, the corresponding estimate is given by

$$
m_{0}(k) \leq C \int_{\overline{\boldsymbol{\Phi}}_{k}\left(B^{\prime}\right)}\left|\bar{N}_{k}\right|^{2} .
$$

Applying the rescaling which relates $\overline{\mathcal{M}}_{k}$ and $\mathcal{M}_{k}$, we find a corresponding rescaled ball $B^{\prime \prime}$ (of radius comparable to $t_{k}$ )

$$
\boldsymbol{m}_{0}(k) t_{k}^{4} \leq C \int_{B^{\prime \prime} \cap \mathcal{M}_{k}}\left|N_{k}\right|^{2} .
$$

Using that the center $z$ of the ball can be chosen arbitrarily as long as it is at a distance from $L$ compared to its diameter, we can ensure that $-d(p)^{-1} \phi^{\prime}\left(t_{k}^{-1} d(p)\right) \geq c t_{k}^{-1}$ on $B^{\prime \prime}$ (for some positive geometric constant $c$ ). We thus get

$$
\boldsymbol{m}_{0}(k) t_{k}^{3} \leq-\int_{B^{\prime \prime} \cap \mathcal{M}_{k}}\left|N_{k}\right|^{2} \frac{\phi^{\prime}\left(t_{k}^{-1} d(p)\right)}{d(p)} \leq C H\left(N_{k}, t_{k}\right) .
$$

However $\mathbf{E}_{k}\left(T, 6 t_{k}\right) \leq C m_{0}(k)$, and we have thus completed the proof of the second inequality in (25.13).

Proof of Proposition 25.2. Define for $p \in \mathcal{M}_{k}$ the map $F_{k}(p)=\sum_{i} \llbracket p+\left(N_{k}\right)_{i}(p) \rrbracket$ and for $q \in \mathcal{M}_{k-1}$ the map $F_{k-1}(q)=\sum_{i} \llbracket q+\left(N_{k-1}\right)_{i}(q) \rrbracket$. Moreover denote by $\mathbf{E}_{k}:=\mathbf{E}\left(T, 6 t_{k}\right)$ and $\mathbf{C}_{k}:=\mathbf{C}_{2 t_{k}}\left(0, V_{0}\right)$. In order to compare $N_{k}$ and $N_{k-1}$, we first apply Theorem 18.21 to the rescaled currents $T_{0, t_{k}}$ and $T_{0, t_{k-1}}$ to derive corresponding estimates for the normal approximations $\bar{N}_{k}$ and $\bar{N}_{k-1}$ of the currents on $\overline{\mathcal{M}}_{k}$ and $\overline{\mathcal{M}}_{k-1}$. We then scale them back to find corresponding estimates for $N_{k}$ and $N_{k-1}$. During this process we also observe that, by (25.9) and (25.10), we have

$$
\begin{equation*}
m_{0}(k)+m_{0}(k-1)\left(\frac{t_{k}}{t_{k-1}}\right)^{2-2 \delta_{1}} \leq C \mathrm{E}_{k} \tag{25.14}
\end{equation*}
$$

Moreover, we will prove later

$$
\begin{align*}
\left\|\boldsymbol{\varphi}_{k-1}\right\|_{C^{0}\left(B_{2 t_{k}}\right)} & \leq C t_{k} \mathbf{E}_{k}^{1 / 2},  \tag{25.15}\\
\left\|D \boldsymbol{\varphi}_{k-1}\right\|_{C^{0}\left(B_{2 t_{k}}\right)} & \leq C \mathbf{E}_{k}^{1 / 2}  \tag{25.16}\\
\left\|D^{2} \boldsymbol{\varphi}_{k-1}\right\|_{C^{0}\left(B_{4 t_{k}}\right)} & \leq C t_{k-1}^{-1} m_{0}(k-1)^{1 / 2} \leq C t_{k}^{-1} \mathbf{E}_{k}^{1 / 2}  \tag{25.17}\\
\left\|\boldsymbol{\varphi}_{k}\right\|_{C^{0}\left(B_{2 t_{k}}\right)} & \leq C t_{k} \mathbf{E}_{k}^{1 / 2},  \tag{25.18}\\
\left\|D \boldsymbol{\varphi}_{k}\right\|_{C^{0}\left(B_{5 t_{k}}\right)} & \leq C m_{0}(k)^{1 / 2} \leq C \mathbf{E}_{k}^{1 / 2},  \tag{25.19}\\
\left\|D^{2} \boldsymbol{\varphi}_{k}\right\|_{C^{0}\left(B_{5 t_{k}}\right)} & \leq C t_{k}^{-1} m_{0}(k)^{1 / 2} \leq C t_{k}^{-1} \mathbf{E}_{k}^{1 / 2},  \tag{25.20}\\
\left\|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right\|_{L^{2}\left(B_{2 t_{k}}\right)}^{2} & \leq C t_{k}^{2} \mathbf{E}^{1+2 \gamma_{2}} . \tag{25.21}
\end{align*}
$$

In particular we get by (25.14), (18.18), and (18.20) after rescaling back

$$
\begin{align*}
\operatorname{Lip}\left(N_{k}\right)+\operatorname{Lip}\left(N_{k-1}\right) & \leq C \mathbf{E}_{k}^{\gamma_{2}},  \tag{25.22}\\
\mathbf{M}\left(\mathbf { T } _ { F _ { k } } \left\llcorner\mathbf{C}_{k}-\mathbf{T}_{F_{k-1}}\left\llcorner\mathbf{C}_{k}\right)\right.\right. & \leq \mathbf{M}\left(\mathbf { T } _ { F _ { k } } \left\llcorner\mathbf{C}_{k}-T\left\llcorner\mathbf{C}_{k}\right)+\mathbf{M}\left(T \left\llcorner\mathbf{C}_{k}-\mathbf{T}_{F_{k-1}}\left\llcorner\mathbf{C}_{k}\right)\right.\right.\right.\right. \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}} . \tag{25.23}
\end{align*}
$$

Thus, we set $\hat{N}_{k}$ to be the $Q$-valued function defined on $\mathcal{M}_{k-1}$ satisfying

$$
\mathbf{G}_{\hat{N}_{k}}\left\llcorner\mathbf{C}_{k}=\mathbf{T}_{F_{k}}\left\llcorner\mathbf{C}_{k}=\mathbf{G}_{N_{k}}\left\llcorner\mathbf{C}_{k}=: S,\right.\right.\right.
$$

where with $\mathbf{G}_{\hat{N}_{k}}$ we mean the current associated to the function $p \mapsto p+\hat{N}_{k}(p)$. By comparing $D\left(N_{k}, t_{k}\right)$ with $D\left(\hat{N}_{k}, t_{k}\right)$ and $H\left(N_{k}, t_{k}\right)$ with $H\left(\hat{N}_{k}, t_{k}\right)$ we make an additional error of size $t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}$ and size $t_{k}^{3} \mathbf{E}_{k}^{1+\gamma_{2}}$ respectively. We will prove this later. With this
aim in mind we change coordinates in the integrals of $D$ and $H$ to flat ones. Denote by $\boldsymbol{\Phi}_{k}(x):=\left(x, \boldsymbol{\varphi}_{k}(x)\right)$ and $\boldsymbol{\Phi}_{k-1}(x):=\left(x, \boldsymbol{\varphi}_{k-1}(x)\right)$. We then estimate

$$
\begin{aligned}
\mid D\left(N_{k}, t_{k}\right)- & \int\left|D N_{k}\right|^{2}\left(\boldsymbol{\Phi}_{k}(x)\right) \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right) d x \mid \\
& \leq C \int_{B_{2 t_{k}}}\left|D N_{k}\right|^{2}\left(\boldsymbol{\Phi}_{k}(x)\right) \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right)\left|D \boldsymbol{\Phi}_{k}(x)-(\mathrm{I} d, 0)\right| d x \\
& \leq C\left\|D \boldsymbol{\varphi}_{k}\right\|_{C^{0}\left(B_{2 t_{k}}\right)} \int\left|D N_{k}\right|^{2}\left(\boldsymbol{\Phi}_{k}(x)\right) \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right) J \boldsymbol{\Phi}_{k}(x) d x \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{3 / 2},
\end{aligned}
$$

where we used (25.2) and (25.19) for the last inequality. Analogous estimates can be employed for $D\left(\hat{N}_{k}, t_{k}\right), H\left(N_{k}, t_{k}\right)$, and $H\left(\hat{N}_{k}, t_{k}\right)$.

Therefore, it is enough to prove

$$
\begin{align*}
& \left.\left|\int\right| D N_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right) d x-\int\left|D \hat{N}_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k-1}(x)\right)\right) d x \mid \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}} \\
& \left.\left.\left|\int\right| N_{k}\right|^{2} \frac{\phi^{\prime}\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right)}{d\left(\boldsymbol{\Phi}_{k}(x)\right)} d x-\int\left|\hat{N}_{k}\right|^{2^{\prime}} \frac{\phi^{\prime}\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k-1}(x)\right)\right)}{d\left(\boldsymbol{\Phi}_{k-1}(x)\right)} d x \right\rvert\, \leq C t_{k}^{3} \mathbf{E}_{k}^{1+\gamma_{2}} \tag{25.24}
\end{align*}
$$

For (25.24), notice that $N_{k}(p)=\sum_{i} \llbracket\left(F_{k}\right)_{i}(p)-p \rrbracket$. Hence, each component of $N_{k}$ satisfies

$$
\left|D\left(N_{k}\right)_{i}\left(\boldsymbol{\Phi}_{k}(x)\right)\right| \leq C\left|T_{\left(F_{k}\right)_{i}(x)} \mathbf{T}_{F_{k}}-T_{\boldsymbol{\Phi}_{k}(x)} \mathcal{M}_{k}\right|
$$

By the Lipschitz bound of $\boldsymbol{\varphi}_{k}$ (25.19) and of $F_{k}$, we thus have

$$
\begin{aligned}
& \int\left|D N_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right) \\
& \quad \leq C \int_{\mathbf{C}}\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right) d\|S\|(p)+O\left(t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}\right) \\
& \int\left|D \hat{N}_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k-1}(x)\right)\right) \\
& \quad \leq C \int_{\mathbf{C}}\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right|^{2} \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k-1}(p)\right)\right) d\|S\|(p)+O\left(t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}\right),
\end{aligned}
$$

where we denoted by $\mathbf{p}_{k}$ and $\mathbf{p}_{k-1}$ the nearest point projection on $\mathcal{M}_{k}$ and $\mathcal{M}_{k-1}$ respectively, while $\mathbf{C}$ is the vertical cylinder with base $B_{2 t_{k}}$. As we have from Theorem 18.16 that $\left\|\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right\|_{C^{2}} \leq C t_{k}^{-1} \mathbf{E}_{k}^{1 / 2}$, by the Lipschitz bound of $\phi$, we deduce for any $p \in \operatorname{spt}(S)$ and $q, q^{\prime} \in \mathcal{M}_{k}$,

$$
\begin{aligned}
\left|\phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right)-\phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k-1}(p)\right)\right)\right| & \leq C \mathbf{E}_{k}^{1 / 2}, \\
\left|T_{q} \mathcal{M}_{k}-T_{q^{\prime}} \mathcal{M}_{k}\right| & \leq C t_{k}^{-1} \mathbf{E}_{k}^{1 / 2}\left|q-q^{\prime}\right| .
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& \int_{\mathbf{C}}\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}\right|^{2} \mid \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right)-\phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k-1}(p)\right) \mid d\|S\|(p) \leq C t_{k}^{2} \mathbf{E}_{k}^{3 / 2},\right. \\
&\left|T_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}-T_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right| \leq C\left|D \boldsymbol{\varphi}_{k}\left(\mathbf{p}_{V_{0}}\left(\mathbf{p}_{k}(p)\right)\right)-D \boldsymbol{\varphi}_{k-1}\left(\mathbf{p}_{V_{0}}\left(\mathbf{p}_{k-1}(p)\right)\right)\right| \\
& \leq C \mathbf{E}_{k}+\left|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right|\left(\mathbf{p}_{V_{0}}(p)\right)
\end{aligned}
$$

where we used (18.18) in the last inequality. We therefore can conclude

$$
\begin{aligned}
& \left.\left|\int\right| D N_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k}(x)\right)\right) d x-\int_{B_{2 t_{k}}}\left|D \hat{N}_{k}\right|^{2} \phi\left(t_{k}^{-1} d\left(\boldsymbol{\Phi}_{k-1}(x)\right)\right) d x \mid \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}+C \int_{\mathbf{C}}\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}\right|^{2}\left|\phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right)-\phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k-1}(p)\right)\right)\right| d\|S\| \\
& \quad+C \int_{\mathbf{C}}| | \vec{S}(p)-\left.\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}\right|^{2}-\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right|^{2} \mid \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k-1}(p)\right)\right) d\|S\| \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}+C \int_{\mathbf{C}}\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}\right|\left|\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}-\vec{T}_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right| \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right) d\|S\| \\
& \quad+C \int_{\mathbf{C}}\left|\vec{S}(p)-\vec{T}_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right|\left|\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}-\vec{T}_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right| \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right) d\|S\| \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}+C t_{k} \mathbf{E}_{k}^{1 / 2}\left(\int_{\mathbf{C}}\left|\vec{T}_{\mathbf{p}_{k}(p)} \mathcal{M}_{k}-\vec{T}_{\mathbf{p}_{k-1}(p)} \mathcal{M}_{k-1}\right|^{2} \phi\left(t_{k}^{-1} d\left(\mathbf{p}_{k}(p)\right)\right) d\|S\|\right)^{\frac{1}{2}} \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}+C t_{k} \mathbf{E}_{k}^{1 / 2}\left(\int_{B_{2_{t}}}\left|D \boldsymbol{\varphi}_{k}-D \boldsymbol{\varphi}_{k-1}\right|^{2}\right)^{\frac{1}{2}} \\
& \leq C t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}},
\end{aligned}
$$

where we used (25.21) for the last inequality.
We finally turn to (25.25). For $x \in V_{0}$, denote by $z_{k}:=\left(x, \boldsymbol{\varphi}_{k}(x)\right)$ and $\hat{z}_{k}:=$ $\left(x, \boldsymbol{\varphi}_{k-1}(x)\right)$. Then we estimate

$$
\begin{aligned}
\left|\left|N_{k}\right|^{2}\left(z_{k}\right)-\left|\hat{N}_{k}\right|^{2}\left(\hat{z}_{k}\right)\right| \leq & \left|N_{k}\right|\left(z_{k}\right)\left|\left|N_{k}\right|\left(z_{k}\right)-\left|\hat{N}_{k}\right|\left(\hat{z}_{k}\right)\right| \\
& +\left|\hat{N}_{k}\right|(\hat{z})| | N_{k}\left|\left(z_{k}\right)-\left|\hat{N}_{k}\right|\left(\hat{z}_{k}\right)\right| .
\end{aligned}
$$

Moreover, using Cauchy-Schwarz and the fact that the $L^{2}$ norm of $N_{k}$ and $\hat{N}_{k}$ is bounded by $t_{k}^{2} \mathbf{E}_{k}^{1 / 2}$, we have

$$
\begin{align*}
& \left.\left.\left|\int\right| N_{k}\right|^{2} \frac{\phi^{\prime}\left(t_{k}^{-1} d\left(z_{k}\right)\right)}{d\left(z_{k}\right)} d x-\int\left|\hat{N}_{k}\right|^{2} \frac{\phi^{\prime}\left(d\left(\left(\hat{z}_{k}\right)\right)\right.}{d\left(\hat{z}_{k}\right)} d x \right\rvert\, \\
& \quad \leq C t_{k} \mathbf{E}_{k}^{1 / 2}\left(\int_{B_{2_{t_{k}}}}| | N_{k}\left|\left(z_{k}\right)-\left|\hat{N}_{k}\right|\left(\hat{z}_{k}\right)\right|^{2} d x\right)^{\frac{1}{2}} . \tag{25.26}
\end{align*}
$$

If we now define $p_{i}:=\left(F_{k}\right)_{i}(x)$ and $q_{i}:=\left(\hat{F}_{k}\right)_{i}(x):=\hat{z}_{k}+\left(\hat{N}_{k}\right)_{i}\left(\hat{Z}_{k}\right)$, we have (up to reordering the indices)

$$
\left|N_{k}\right|\left(z_{k}\right)=\left(\sum_{i}\left|p_{i}-z_{k}\right|^{2}\right)^{\frac{1}{2}}, \quad\left|\hat{N}_{k}\right|\left(\hat{z}_{k}\right)=\left(\sum_{i}\left|q_{i}-\hat{z}_{k}\right|^{2}\right)^{\frac{1}{2}} .
$$

Now we use the triangle inequality to see

$$
\begin{aligned}
\left|\left|N_{k}\right|\left(z_{k}\right)-\left|\hat{N}_{k}\right|\left(\hat{z}_{k}\right)\right|^{2} & =\left|\left(\sum_{i}\left|p_{i}-z_{k}\right|^{2}\right)^{\frac{1}{2}}-\left(\sum_{i}\left|q_{i}-\hat{z}_{k}\right|^{2}\right)^{\frac{1}{2}}\right|^{2} \\
& \leq C \sum_{i}\left|p_{i}-q_{\sigma(i)}\right|^{2}+C\left|z_{k}-\hat{z}_{k}\right|^{2} \\
& =C \mathcal{G}\left(\sum_{i} \llbracket p_{i} \rrbracket, \sum_{i} \llbracket q_{i} \rrbracket\right)^{2}+C\left|\boldsymbol{\varphi}_{k}(x)-\boldsymbol{\varphi}_{k-1}(x)\right|^{2},
\end{aligned}
$$

for $\sigma$ the permutation realizing the distance $\mathcal{G}\left(\sum_{i} \llbracket p_{i} \rrbracket, \sum_{i} \llbracket q_{i} \rrbracket\right)$.
Note that, since $\boldsymbol{\varphi}_{k}$ and $\boldsymbol{\varphi}_{k-1}$ agree on the boundary $\mathbf{p}_{V_{0}}(\Gamma)$, we can use (25.21) and the Poincaré inequality to conclude

$$
\begin{equation*}
\left\|\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right\|_{L^{2}\left(B_{2 t_{k}}\right)} \leq C t_{k}\left\|D \boldsymbol{\varphi}_{k}-D \boldsymbol{\varphi}_{k-1}\right\|_{L^{2}\left(B_{2 t_{k}}\right)} \leq C t_{k}^{3} \mathbf{E}_{k}^{1 / 2+\gamma_{2}} \tag{25.27}
\end{equation*}
$$



Figure 4: An illustration of how Lemma 25.3 is used.
To estimate further we split the distance $\mathcal{G}\left(\sum_{i} \llbracket p_{i} \rrbracket, \sum_{i} \llbracket q_{i} \rrbracket\right)$ into a horizontal and vertical part in the following sense. We define $V:=\hat{z}_{k}+T_{z_{k}} \mathcal{M}_{k-1}, \tilde{V}:=z_{k}+T_{z_{k}} \mathcal{M}_{k}$, $V^{\prime}:=\hat{z}_{k}+T_{z_{k}} \mathcal{M}_{k}$ and $\sum_{i} \llbracket q_{i}^{\prime} \rrbracket:=\left\langle S, \mathbf{p}_{V^{\prime}}, 0\right\rangle$. Observe that $V$ and $V^{\prime}$ differ by a rotation,
while $V^{\prime}$ and $\tilde{V}$ are parallel. We then apply the Lemma 25.3 to the shifted situation where $\hat{z}_{k}=0$ and deduce

$$
\begin{aligned}
\mathcal{G}\left(\sum_{i} \llbracket q_{i} \rrbracket, \sum_{i} \llbracket q_{i}^{\prime} \rrbracket\right) & \leq C \operatorname{Lip}\left(F_{k}\right)\left\|N_{k}\right\|_{C^{0}}\left(\left|V-V_{0}\right|+\left|V^{\prime}-V_{0}\right|\right) \\
& \leq C \operatorname{Lip}\left(F_{k}\right)\left\|N_{k}\right\|_{C^{0}}\left(\left|D \boldsymbol{\varphi}_{k}\right|+\left|D \boldsymbol{\varphi}_{k-1}\right|\right) \\
& \leq C t_{k} \mathbf{E}_{k}^{3 / 4+\gamma_{2}},
\end{aligned}
$$

where in the last inequality we used (25.16) and (25.19).
In order to estimate $\mathcal{G}\left(\sum_{i} \llbracket p_{i} \rrbracket, \sum_{i} \llbracket q_{i}^{\prime} \rrbracket\right)$, we call $f_{\tilde{v}}: T_{z_{k}} \mathcal{M}_{k} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ the function having the same graph as $F_{k}$ in $\mathrm{C}_{2 t_{k}}$. Observe that

$$
\left|T_{z_{k}} \mathcal{M}_{k}-V_{0}\right| \leq C t_{k}\left\|D^{2} \boldsymbol{\varphi}_{k}\right\|_{C^{0}} \leq C \mathbf{E}_{k}^{1 / 2}
$$

and by [14, Proposition 5.2]

$$
\operatorname{Lip}\left(f_{\tilde{V}}\right) \leq C \mathbf{E}_{k}^{\gamma_{2}} .
$$

Then we observe that $\sum_{i} \llbracket p_{i} \rrbracket=\sum_{i} \llbracket f_{\tilde{V} i}\left(z_{k}\right) \rrbracket$ and $\sum_{i} \llbracket q_{i}^{\prime} \rrbracket=\sum_{i} \llbracket f_{\tilde{V} i}\left(\mathbf{p}_{T_{p} \mathcal{M}_{k}}\left(\hat{z}_{k}\right)\right) \rrbracket$. Thus we have

$$
\begin{aligned}
\mathcal{G}\left(\sum_{i} \llbracket p_{i} \rrbracket, \sum_{i} \llbracket q_{i}^{\prime} \rrbracket\right) & \leq \operatorname{Lip}\left(f_{\tilde{V}}\right)\left|z_{k}-\mathbf{p}_{T_{z_{k}} \mathcal{M}_{k}}\left(\hat{z}_{k}\right)\right| \\
& \leq \operatorname{Lip}\left(f_{\tilde{V}}\right)\left(\left\|\boldsymbol{\varphi}_{k}\right\|_{C^{0}}+\left\|\boldsymbol{\varphi}_{k-1}\right\|_{C^{0}}\right) \leq C t_{k} \mathbf{E}_{k}^{1 / 2+\gamma_{2}} .
\end{aligned}
$$

Squaring and integrating (and using (25.27)), we deduce

$$
\int_{B_{2 t_{k}}}| | N_{k}\left|\left(z_{k}\right)-\left|\hat{N}_{k}\right|\left(\hat{z}_{k}\right)\right|^{2} \leq C t_{k}^{4} \mathbf{E}_{k}^{1+2 \gamma_{2}}
$$

Inserting in (25.26) we conclude

$$
\left.\left.\left|\int\right| N_{k}\right|^{2} \frac{\phi^{\prime}\left(t_{k}^{-1} d\left(z_{k}\right)\right.}{d\left(z_{k}\right)} d x-\int\left|\hat{N}_{k}\right|^{2} \frac{\phi^{\prime}\left(t_{k}^{-1} d((\hat{z}))\right.}{d(\hat{z})} d x \right\rvert\, \leq C t_{k}^{3} \mathbf{E}^{1+\gamma_{2}} .
$$

It remains to prove (25.15)-(25.21).
(25.19) and (25.20) follow from Theorem 18.16 using a simple rescaling and (25.14). Next, for $\boldsymbol{\varphi}_{k-1}$ the estimate on the second derivative derived from Theorem 18.16 and (25.14) is favourable, as it gives directly (25.17). However the estimate on the first derivative is not, as it would give

$$
\begin{equation*}
\left\|D \boldsymbol{\varphi}_{k-1}\right\|_{C^{0}\left(B_{5 t_{k}}\right)} \leq C m_{0}(k-1)^{1 / 2} \leq C\left(\frac{t_{k-1}}{t_{k}}\right)^{1-\delta_{1}} \mathbf{E}_{k}^{1 / 2} \tag{25.28}
\end{equation*}
$$

which is not good enough for our purposes.
Proof of (25.15), (25.16), and (25.18) In order to gain a more favorable estimate for the first derivative (and the $C^{0}$ norm of $\boldsymbol{\varphi}_{k-1}$ ) we first observe that by Lemma 18.4

$$
\mathbf{h}\left(T, \mathbf{C}_{10 t_{k}}\left(0, V_{0}\right)\right) \leq C \mathbf{E}_{k}^{1 / 2} t_{k}
$$

Arguing as in the proof of (25.3) it is not difficult to see that

$$
\begin{equation*}
\int_{\mathbf{C}_{5_{k}}\left(0, V_{0}\right) \cap \mathcal{M}_{k-1}}\left|N_{k-1}\right|^{2} \leq C \mathbf{E}_{k} t_{k}^{4} \tag{25.29}
\end{equation*}
$$

Since $\mathbf{T}_{F_{k-1}}$ coincides with $\operatorname{spt}(T)$ on a large set we can also infer

$$
\begin{equation*}
\int_{B_{5_{t_{k}}}}\left|\boldsymbol{\varphi}_{k-1}\right|^{2} \leq C \mathbf{E}_{k} t_{k}^{4} \tag{25.30}
\end{equation*}
$$

In order to see the latter estimate, consider first a point $p \in \mathcal{M}_{k-1}$ with the property that the support of $F_{k-1}(p)$ is a subset of the support of $T$. By the height bound we know that $\mathbf{h}\left(T, \mathbf{C}_{10 t_{k}}\left(0, V_{0}\right)\right) \leq C \mathbf{E}_{k}^{1 / 2} t_{k}$. In particular, if we let $\mathbf{p}_{0}^{\perp}$ be the projection on the orthogonal complement $V_{0}$, we conclude

$$
\left|\mathbf{p}_{0}^{\perp} \circ F_{k-1}\right|(p) \leq C \mathbf{E}_{k}^{1 / 2} t_{k} .
$$

Consider now that, if $x$ is such that $p=\left(x, \boldsymbol{\varphi}_{k-1}(x)\right)$, since $F_{k}(p)=\sum_{i} \llbracket F_{k}^{i}(p) \rrbracket=$ $\sum_{i} \llbracket N_{k}^{i}(p)+p \rrbracket$, we get

$$
\begin{align*}
\left|\boldsymbol{\varphi}_{k-1}(x)\right| & \leq\left|\mathbf{p}_{0}^{\perp} \circ F_{k-1}\right|\left(x, \boldsymbol{\varphi}_{k-1}(x)\right)+\left|\mathbf{p}_{0}^{\perp} \circ N_{k-1}\right|\left(x, \boldsymbol{\varphi}_{k-1}(x)\right) \\
& \leq C \mathbf{E}_{k}^{1 / 2} t_{k}+\left|N_{k-1}\right|\left(x, \boldsymbol{\varphi}_{k-1}(x)\right) . \tag{25.31}
\end{align*}
$$

Let now $\mathcal{K}$ be the set of such points $p$ (i.e. for which the support of $F_{k}(p)$ is contained in the support of $T$ ) and define $K:=\mathbf{p}_{0}(\mathcal{K}) \cap B_{5 t_{k}}$. Using the bounds (25.29) and (25.31) we easily obtain

$$
\begin{equation*}
\int_{K}\left|\boldsymbol{\varphi}_{k-1}(x)\right|^{2} \leq C \mathrm{E}_{k} t_{k}^{4} . \tag{25.32}
\end{equation*}
$$

In order to estimate the integral on the remaining portion (i.e. on $B_{5 t_{k}} \backslash K$ ), we apply (18.15) to $\overline{\mathcal{M}}_{k-1}$, sum over all the stopped squares in $B_{5 t_{k}} \backslash K$ (which by the stopping condition have side length comparable to $t_{k} / t_{k-1}$ ), scale it back to $\mathcal{M}_{k-1}$ and deduce

$$
\begin{align*}
\left|B_{5 t_{k}} \backslash K\right| & \leq \mathcal{H}^{2}\left(\mathbf{B}_{6 t_{k}} \cap \mathcal{M}_{k-1} \backslash \mathcal{K}\right) \leq C\left(\boldsymbol{m}_{0}(k-1)\right)^{1+\gamma_{2}}\left(\frac{t_{k}}{t_{k-1}}\right)^{4+\gamma_{2}} t_{k-1}^{2} \\
& \leq C\left(\frac{t_{k}}{t_{k-1}}\right)^{2+\gamma_{2}} \tag{25.33}
\end{align*}
$$

Then we observe that, by (25.32) and the classical Chebyshev inequality, there is at least one point $x \in B_{5 t_{k}}$ where $\left|\boldsymbol{\varphi}_{k-1}(x)\right| \leq C \mathbf{E}_{k}^{1 / 2} t_{k}$, and we use (25.28) to conclude that for all $y \in B_{5 t_{k}}$ we have

$$
\begin{equation*}
\left|\boldsymbol{\varphi}_{k-1}(y)\right| \leq C \mathbf{E}_{k}^{1 / 2} t_{k}+C \mathbf{E}_{k}^{1 / 2}\left(\frac{t_{k-1}}{t_{k}}\right)^{1-\delta_{1}}|x-y| \leq C \mathbf{E}_{k}^{1 / 2}\left(\frac{t_{k-1}}{t_{k}}\right)^{1-\delta_{1}} t_{k} \tag{25.34}
\end{equation*}
$$

Putting together (25.32), (25.33), and (25.34), we achieve

$$
\int_{B_{5_{t_{k}}}}\left|\boldsymbol{\varphi}_{k-1}\right|^{2} \leq C \mathbf{E}_{k} t_{k}^{4}+C \mathbf{E}_{k}\left(\frac{t_{k}}{t_{k-1}}\right)^{2+\gamma_{2}-2\left(1-\delta_{1}\right)} t_{k}^{4}
$$

Since $2+\gamma_{2} \geq 2-2 \delta_{1}$ and $t_{k} \leq t_{k-1}$, the latter clearly implies (25.30).
We next use Gagliardo-Nirenberg interpolation inequality and from (25.29) and (25.17) we get (25.15) and (25.16), namely

$$
\left\|\boldsymbol{\varphi}_{k-1}\right\|_{C^{0}\left(B_{2 t_{k}}\right)} \leq C t_{k} \mathbf{E}_{k}^{1 / 2}, \quad\left\|D \boldsymbol{\varphi}_{k-1}\right\|_{C^{0}\left(B_{2 t_{k}}\right)} \leq C \mathbf{E}_{k}^{1 / 2}
$$

We analogously conclude (25.18).
Proof of (25.21) We wish to show that

$$
\left\|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right\|_{L^{2}\left(B_{2 t_{k}}\right)}^{2} \leq C t_{k}^{2} \mathbf{E}_{k}^{1+2 \gamma_{2}} .
$$

We choose a suitable cut-off function $\psi$ which equals 1 on $B_{2 t_{k}}$ and is compactly supported in $B_{3 t_{k}}$ and write

$$
\int_{B_{2 t_{k}}}\left|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right|^{2} \leq \int_{B_{3_{t_{k}}}}\left|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right|^{2} \psi
$$

Integrating by parts, we can estimate

$$
\begin{aligned}
\int\left|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right|^{2} \psi= & \int\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right) \Delta\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right) \psi \\
& +\int\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right) \nabla\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right) \cdot \nabla \psi .
\end{aligned}
$$

We next use that $\|\nabla \psi\| \leq C t_{k}^{-1}$, (25.16), (25.17), (25.19), and (25.20) to estimate

$$
\begin{equation*}
\int_{B_{2 t_{k}}}\left|D\left(\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right)\right|^{2} \leq C \mathbf{E}_{k}^{1 / 2} t_{k}^{-1} \int_{B_{3 t_{k}}}\left|\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right| . \tag{25.35}
\end{equation*}
$$

We next consider the multivalued functions $f_{k}$ and $f_{k-1}$ on $B_{3 t_{k}}$ and taking values into $\mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ with the properties that

$$
\mathbf{G}_{f_{k}}=\mathbf{T}_{F_{k}}\left\llcorner\mathbf{C}_{0,3 t_{k}}, \quad \mathbf{G}_{f_{k-1}}=\mathbf{T}_{F_{k-1}}\left\llcorner\mathbf{C}_{0,3 t_{k}} .\right.\right.
$$

Note that the values of $f_{k}$ and $f_{k-1}$ coincide except for a set of measure at most $t_{k}^{2} \mathbf{E}_{k}^{1+\gamma_{2}}$ (again we use Theorem 18.21 and sum over the stopped squares). Moreover, because $\operatorname{Lip}\left(f_{k}\right), \operatorname{Lip}\left(f_{k-1}\right) \leq C \mathbf{E}_{k}^{\gamma_{2}}$, we immediately draw the conclusion

$$
\int_{B_{3_{k_{k}}}}\left|\eta \circ f_{k}-\eta \circ f_{k-1}\right| \leq \mathbf{E}_{k}^{1+2 \gamma_{2}} t_{k}^{3}
$$

On the other hand, appealing to Proposition 18.23 (and rescaling appropriately) we get

$$
\begin{aligned}
\int_{B_{3 t_{k}}}\left|\boldsymbol{\eta} \circ f_{k}-\boldsymbol{\varphi}_{k}\right| & \leq C \mathbf{E}_{k}^{3 / 4} t_{k}^{3}, \\
\int_{B_{3 t_{k}}}\left|\boldsymbol{\eta} \circ f_{k-1}-\boldsymbol{\varphi}_{k-1}\right| & \leq C\left(\left(\frac{t_{k-1}}{t_{k}}\right)^{2-2 \delta_{1}} \mathbf{E}_{k}\right)^{3 / 4}\left(\frac{t_{k}}{t_{k-1}}\right)^{4} t_{k-1}^{3} .
\end{aligned}
$$

While the first estimate is already suitable for our purposes, the second require some more care. We recall (25.10) to the effect that

$$
\left(\frac{t_{k-1}}{t_{k}}\right)^{2-2 \delta_{1}} \mathbf{E}_{k} \leq\left(\frac{t_{k-1}}{t_{k}}\right)^{2-2 \delta_{1}} m_{0}(k) \leq C
$$

for a geometric constant $C$. Since $\frac{1}{2-2 \delta_{1}} \geq \frac{3}{4}$, we can then estimate

$$
\int_{B_{3 t_{k}}}\left|\boldsymbol{\eta} \circ f_{k-1}-\boldsymbol{\varphi}_{k-1}\right| \leq C \mathbf{E}_{k}^{\frac{3}{4}} t_{k}^{3} .
$$

By possible choosing $\gamma_{2}$ sufficiently small we get

$$
\int_{B_{3_{k} t_{k}}}\left|\boldsymbol{\varphi}_{k}-\boldsymbol{\varphi}_{k-1}\right| \leq C \mathbf{E}_{k}^{1 / 2+2 \gamma_{2}} t_{k}^{3},
$$

which, by (25.35), gives (25.21).

### 25.1 LIPSCHITZ ESTIMATE USING 2D-ROTATIONS

Lemma 25.3. There is a constant $c>0$ such that the following holds. Let $F: V_{0} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$ be a Lipschitz map with $\operatorname{Lip}(F)<c$, let $V$ and $V^{\prime}$ be 2-dimensional subspaces with $\left|V-V_{0}\right|+$ $\left|V^{\prime}-V_{0}\right|<c$ and denote by $\mathbf{p}$ and $\mathbf{p}^{\prime}$ the orthogonal projection on $V$ and $V^{\prime}$ respectively. Then for $P:=\left\langle\mathbf{T}_{F}, \mathbf{p}, 0\right\rangle$ and $P^{\prime}:=\left\langle\mathbf{T}_{F}, \mathbf{p}^{\prime}, 0\right\rangle$ it holds

$$
\begin{equation*}
\mathcal{G}\left(P, P^{\prime}\right) \leq C \operatorname{Lip}(F)\|F\|_{C^{0}}\left(\left|V-V_{0}\right|+\left|V^{\prime}-V_{0}\right|\right) \tag{25.36}
\end{equation*}
$$

Proof. We use an argument already observed in more generality in [15, Lemma D.1]. However, we repeat here the parts needed for the previous lemma. First of all, we construct finitely many planes by using 2 d-rotations that will allow us to reduce (25.36)
to a one-dimensional situation. Recall the terminology: we say that $R \in \mathrm{SO}(n+2)$ is a 2d-rotation if there are two orthonormal vectors $e_{1}, e_{2}$ and an angle $\theta$ such that

$$
\left\{\begin{array}{l}
R\left(e_{1}\right)=\cos (\theta) e_{1}+\sin (\theta) e_{2} \\
R\left(e_{2}\right)=\cos (\theta) e_{1}-\sin (\theta) e_{2} \\
R(v)=v, \quad \text { for any } v \in\left\langle e_{1}, e_{2}\right\rangle^{\perp}
\end{array}\right.
$$

Now let us denote by $W_{1}=V \cap V^{\prime}$. If $\operatorname{dim}\left(W_{1}\right)=2$, then $V=V^{\prime}$ and there is nothing to prove. Otherwise $\operatorname{dim}\left(W_{1}\right)<2=\operatorname{dim}(V)=\operatorname{dim}\left(V^{\prime}\right)$ and we can write

$$
V=W_{1} \oplus \hat{V}, \quad V^{\prime}=W_{1} \oplus \hat{V}^{\prime},
$$

for some subspaces $\hat{V}$ and $\hat{V}^{\prime}$. Choose any unit vector $e_{1} \in \hat{V}=V \cap W_{1}^{\perp}$ and define

$$
e_{1}^{\prime}:=\frac{\mathbf{p}^{\prime}\left(e_{1}\right)}{\left|\mathbf{p}^{\prime}\left(e_{1}\right)\right|} \in V^{\prime} \cap W_{1}^{\perp} .
$$

Moreover, define $R_{1}$ to be the 2d-rotation mapping $e_{1}$ onto $e_{1}^{\prime}$ and

$$
\begin{aligned}
V_{2} & :=R_{1}(V), \\
W_{2} & :=V_{2} \cap V^{\prime} .
\end{aligned}
$$

Notice that $W_{1} \subset V_{1}$ is invariant under $R_{1}$, so clearly $W_{1}=\left(W_{1} \cap V^{\prime}\right) \subset\left(V_{2} \cap V^{\prime}\right)=W_{2}$. Moreover, $e_{1}^{\prime} \in V_{2} \cap V^{\prime}$, and hence

$$
W_{2} \supset\left\langle W_{1}, e_{1}^{\prime}\right\rangle .
$$

As $e_{1}^{\prime} \perp W_{1}$, we have $\operatorname{dim}\left(W_{2}\right) \geq \operatorname{dim}\left(W_{1}\right)+1$. Now, if $\operatorname{dim}\left(W_{2}\right)=2$, then $V_{2}=$ $R_{1}\left(V_{1}\right)=V^{\prime}$ and we define $R_{2}$ to be the identity. Otherwise $\operatorname{dim}\left(W_{2}\right)=1$ and we can again find a unit vector $e_{2} \in V_{2} \cap W_{2}^{\perp}$, define

$$
e_{2}^{\prime}:=\frac{\mathbf{p}^{\prime}\left(e_{2}\right)}{\left|\mathbf{p}^{\prime}\left(e_{2}\right)\right|} \in V^{\prime} \cap W_{1}^{\perp}
$$

and define $R_{2}$ to be the 2d-rotation mapping $e_{2}$ onto $e_{2}^{\prime}$. As before, we denote by $V_{3}:=R_{2}\left(V_{2}\right)$ and observe that $W_{3}:=V_{3} \cap V^{\prime}$ has at least one dimension more than $W_{2}$. Thus, in both cases we have

$$
V^{\prime}=R_{2} \circ R_{1}(V) .
$$

Next, denote by $V_{1}:=V$ and for $j \in\{1,2,3\}$ the orthogonal projection onto $V_{j}$ by $\mathbf{p}_{j}$ and $P_{j}:=\left\langle\mathbf{T}_{F}, \mathbf{p}_{j}, 0\right\rangle$. Notice that for $c>0$ small enough, $\operatorname{spt}\left(P_{j}\right)$ is a $Q$-valued point. We claim

$$
\mathcal{G}\left(P_{j}, P_{j+1}\right) \leq \operatorname{CLip}(F)\|F\|_{C^{0}}\left(\left|V_{j}-V_{0}\right|+\left|V_{j+1}-V_{0}\right|\right)
$$

concluding the lemma as $\left|V_{j}-V_{0}\right| \leq\left|V-V^{\prime}\right|+\left|V-V_{0}\right| \leq 2\left(\left|V-V_{0}\right|+\left|V^{\prime}-V_{0}\right|\right)$ for every $j$. Indeed, for each $j$, fix a unit vector $v_{j} \in V_{0}$ such that

$$
\left\langle e_{j}, e_{j}^{\prime}\right\rangle \cap V_{0}=\left\{t \cdot v_{j}: t \in \mathbb{R}\right\}
$$

Then we can apply the selection principle [12, Proposition 1.2] to the map $F^{j}(t):=F\left(t v_{j}\right)$ to get a selection

$$
F^{j}=\sum_{i} \llbracket F_{i}^{j} \rrbracket
$$

for some Lipschitz functions $F_{i}^{j}:[-1,1] \rightarrow \mathbb{R}^{n}$ satisfying

$$
\begin{equation*}
\left|D F_{i}^{j}\right| \leq|D F| \leq \operatorname{Lip}(F) \quad \text { a.e. } \tag{25.37}
\end{equation*}
$$

We therefore conclude the existence of points $s_{1}^{j}, \ldots, s^{j}{ }_{Q}, s_{1}^{j+1}, \ldots, s_{Q}^{j+1} \in[-1,1]$ such that

$$
\begin{aligned}
\mathcal{G}\left(P_{j}, P_{j+1}\right) & \leq \sum_{i}\left|F_{i}^{j}\left(s_{i}^{j}\right)-F_{i}^{j}\left(s_{i}^{j+1}\right)\right| \\
& \leq \operatorname{Lip}(F) \sum_{i}\left|s_{i}^{j}-s_{i}^{j+1}\right| \\
& \leq \operatorname{Lip}(F) \sum_{i}\left(\left|s_{i}^{j}\right|+\left|s_{i}^{j+1}\right|\right) \\
& \leq Q C \operatorname{Lip}(F)\|F\|_{C^{0}}\left(\left|V_{j}-V_{0}\right|+\left|V_{j+1}-V_{0}\right|\right)
\end{aligned}
$$

where we also have used (25.37).

In this section we complete the proof of Theorem 12.6, which in turn completes the proof of Theorem 1.7. We recall the $I_{0}$ from Corollary 23.6. The main point is the following conclusion.

Theorem 26.1. Let $T$ be as in Assumption 23.2 and assume that 0 is not a regular point. Then $I_{0}=1$ and for every $\mathrm{s}>0$

$$
\begin{equation*}
\lim _{r \downarrow 0} \frac{D(r)}{r^{2+\varsigma}}=\infty . \tag{26.1}
\end{equation*}
$$

The latter is in contradiction with the estimate (23.12) (i.e. $D(r) \leq C r^{2+\tau}$ ) for some positive constant $\tau$ which depends on the exponent $\alpha$ of Theorem 12.7.

### 26.1 BLOW-UP ANALYSIS

As already mentioned, Theorem 26.1 is reached through a suitable "blow-up" analysis. First of all, having fixed a sequence of $s_{j} \downarrow 0$ we define a suitable family of rescalings of the maps $N_{k}^{\prime} s$. First of all we choose any $k(j)$ with the property that

$$
\begin{equation*}
t_{k(j)+1}<s_{j} \leq t_{k(j)} \tag{26.2}
\end{equation*}
$$

Next we define the exponential map $\mathbf{e x}_{k}: T_{0} \mathcal{M}_{k} \rightarrow \mathcal{M}_{k}$ and we identify each tangent $T_{0} \mathcal{M}_{k}$ to $\mathbb{R}^{2}$ through a suitable rotation of the ambient Euclidean space which maps it onto $\mathbb{R}^{2} \times\{0\}$. We then consider the rescaled maps

$$
\begin{equation*}
\tilde{N}_{j}(x):=\frac{N_{k(j)}\left(\mathbf{e x}_{k(j)}\left(s_{j} x\right)\right)}{D\left(s_{j}\right)^{1 / 2}} . \tag{26.3}
\end{equation*}
$$

The main conclusion of our blow-up analysis is the following
Theorem 26.2. Let $T$ be as in Assumption 23.2 and assume that 0 is not a regular point. Let $s_{j} \downarrow 0$ be an arbitrary vanishing sequence of positive radii, let $k(j)$ be an arbitrary choice of integers satisfying (26.2) and let $\tilde{N}_{j}: B_{1}^{+} \rightarrow \mathcal{A}_{Q}\left(\mathbb{R}^{n}\right)$, where $B_{1}^{+}=B_{1} \cap\left\{\left(x_{1}, x_{2}\right): x_{2} \geq 0\right\}$. Then a subsequence, not relabeled, converges strongly in $W^{1,2}\left(B_{1}^{+}\right)$to a map $\tilde{N}_{\infty}$ satisfying the following conditions:
(i) $\tilde{N}_{\infty}\left(x_{1}, 0\right)=Q \llbracket 0 \rrbracket$ for all $x_{1}$;
(ii) $\tilde{N}_{\infty}$ is Dir-minimizing;
(iii) $\tilde{N}_{\infty}$ is $I_{0}$-homogeneous, where $I_{0}$ is the positive number in Corollary 23.6.
(iv) $\eta \circ \tilde{N}_{\infty} \equiv 0$;
(v) $\int_{B_{1}^{+}}\left|D \tilde{N}_{\infty}\right|^{2}=1$.

In particular $I_{0}=1$.
Then the arguments of Theorem 13.9 apply to $\tilde{N}_{\infty}$ and in particular give that $I_{0}=1$.
Proof of Theorem 26.2. Observe first that, following the computations of [21, Section 10.1] we conclude that

$$
e^{-C s_{j}} 8^{1+I_{0}} \leq \frac{H\left(4 s_{j}\right)}{H\left(s_{j} / 2\right)} \leq e^{C s_{j}} 8^{1+4 I_{0}}
$$

as long as $s_{j} \leq t_{k(j)}$. Since $I_{0}$ exists and is finite, there is a constant $C$ (depending only on $I_{0}$ ) such that

$$
D\left(4 s_{j}\right) \leq C D\left(s_{j} / 2\right)
$$

On the other hand, arguing as in the proof of Proposition 25.1, we easily see that

$$
D\left(t_{k(j)}\right) \geq C^{-1} t_{k(j)}^{2} \mathbf{E}\left(T, 24 t_{k(j)}\right)
$$

(we just need to choose the constant $M_{0}$ appropriately large to compensate for the larger radius in the right hand side) while $D\left(4 t_{k(j)}\right) \leq C t_{k(j)}^{2} \mathbf{E}\left(T, 24 t_{k(j)}\right)$. Now, since the geodesic ball $\mathcal{B}_{t_{k(j)}}$ in $\mathcal{M}_{k(j)}$ contains $\left\{d<t_{k(j)} / 2\right\}$ while the geodesic ball $\mathcal{B}_{2 t_{k(j)}} \subset\{d<$ $\left.4 t_{k(j)}\right\}$, using the fact that the rescaling of the manifolds converge smoothly to the flat plane $V_{0}$, we easily conclude that

$$
\int_{B_{2}^{+}}\left|D \tilde{N}_{j}\right|^{2} \leq C \int_{B_{1}^{+}}\left|D \tilde{N}_{j}\right|^{2} .
$$

We can then follow the argument of [21, Section 10.3] to conclude that, up to subsequences, $\tilde{N}_{j}$ converges strongly in the $W^{1,2}\left(B_{1}^{+}\right)$topology to a Dir-minimizing map $\tilde{N}_{\infty}$. Likewise we can follow the argument of [21, Section 10.2] to conclude that $\eta \circ \tilde{N}_{\infty}$ vanishes identically. Recall that the maps $N_{k(j)}$ vanish identically on $\Gamma$, while the rescalings of the latter converge smoothly to $T_{0} \Gamma=\left\{x_{2}=0\right\}$. The strong convergence then implies that $\tilde{N}_{\infty}=Q \llbracket 0 \rrbracket$ on $\left\{x_{2}=0\right\} \cap B_{1}$. We have thus proved (i), (ii), (iv), and (v). We can however also see that

$$
\frac{r \int \phi\left(r^{-1}|x|\right)\left|D \tilde{N}_{\infty}(x)\right|^{2} d x}{-\int \phi^{\prime}\left(r^{-1}|x|\right)|x|^{-1}\left|\tilde{N}_{\infty}(x)\right|^{2} d x}=\lim _{j \rightarrow \infty} \frac{r s_{j} D\left(r s_{j}\right)}{H\left(r s_{j}\right)}=I_{0}
$$

which means that the frequency function of $\tilde{N}_{\infty}$ is constant. This however happens if and only if $\tilde{N}_{\infty}$ is $I_{0}$-homogeneous.

As for the final statement, we invoke Theorem 13.3.

Now that we know that $I_{0}=1$, we can then conclude that by the strong convergence of $\left\{\tilde{N}_{j}\right\}_{j}$ in $W^{1,2}\left(B_{1}^{+}\right)$, we have

Corollary 26.3. If $T$ is as in Theorem 26.2, then

$$
\lim _{r \downarrow 0} \frac{D(2 r)}{D(r)}=4 .
$$

### 26.2 PROOF OF (26.1) AND CONCLUSION

Fix $\varsigma>0$ and consider the sequence of radii $r_{k}:=2^{-k}$. We know from Corollary 26.3 that, for $k$ sufficiently large

$$
D\left(r_{k}\right) \geq 2^{-2-\varsigma / 2} D\left(r_{k-1}\right) .
$$

In particular we conclude the existence of a $k_{0}$ such that for every $k \geq k_{0}$, we have

$$
D\left(2^{-k}\right) \geq 2^{-(2+\varsigma / 2)\left(k-k_{0}\right)} D\left(2^{-k_{0}}\right)
$$

In particular for every $r \leq 2^{-k_{0}}$ we can write

$$
D(r) \geq \frac{D\left(2^{-k_{0}}\right)}{2^{2+\varsigma / 2}} r^{2+\varsigma / 2}
$$

and since $D\left(2^{-k_{0}}\right)>0$, (26.1) readily follows.

Part III
APPENDIX

## APPENIX TO PART I

## A. 1 PROOF OF COROLLARY 2.4

Proof. By Lemma 2.3, we have for $0<r<2$

$$
\|T\|\left(\overline{\mathbf{B}}_{r}\right) \leq r^{n} \boldsymbol{\omega}_{n} \exp \left(C_{42}\left(\mathbf{A}_{\mathcal{M}}+\kappa_{T}\right)(2-r)\right) \frac{\|T\|\left(\overline{\mathbf{B}}_{2}\right)}{2^{n} \boldsymbol{\omega}_{n}} \leq 2^{-n} e^{4 C} \mathbb{M}(T) r^{n}
$$

and

$$
\|T\|\left(\overline{\mathbf{B}}_{r}\right) \geq r^{n} \boldsymbol{\omega}_{n} \lim _{s \downarrow 0}\left(\exp \left(C_{42}\left(\mathbf{A}_{\mathcal{M}}+\kappa_{T}\right)(s-r)\right) \frac{\|T\|\left(\overline{\mathbf{B}}_{s}\right)}{s^{n} \boldsymbol{\omega}_{n}}\right) \geq \boldsymbol{\omega}_{n} e^{-4 C_{42}} m r^{n}
$$

Hence, there is a constant $C_{43}>0$ such that

$$
\begin{equation*}
\frac{1}{C_{43}} r^{n} \leq\|T\|\left(\overline{\mathbf{B}}_{r}\right) \leq C_{43} r^{n} \tag{a.1}
\end{equation*}
$$

Recall that $C_{1}$ is such that $|\vec{H}| \leq C_{1} \mathbf{A}_{\mathcal{M}}$. Then we use Lemma 2.1 to estimate

$$
\begin{aligned}
& \left.\left.\left|\frac{\|T\|\left(\overline{\mathbf{B}}_{s}\right)}{s^{n}}-\frac{\|T\|\left(\overline{\mathbf{B}}_{r}\right)}{r^{n}}-\int_{\overline{\mathbf{B}}_{s} \backslash \overline{\mathbf{B}}_{r}}\right| X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\| \right\rvert\, \\
& \leq \int_{r}^{s} \rho^{-n-1}\left(C_{1} \rho \mathbf{A}_{\mathcal{M}}\|T\|\left(\overline{\mathbf{B}}_{r}\right)+\rho \boldsymbol{\omega}_{n-1} \alpha \kappa_{T} \rho^{n}\right) \mathrm{d} \rho \\
& \leq C_{3}\left(\mathbf{A}_{\mathcal{M}}+\kappa_{T}\right)(s-r) .
\end{aligned}
$$

## A. 2 PROOF OF LEMMA $3 \cdot 3$

The proof of Lemma 3.3 is based on the rather technical area comparison lemma: if we change slightly the $(n+1)$-component of a current, then its new mass stays close to its original mass.

In the following, we will denote points in $\mathbb{R}^{n+k}$ by $(x, y)$, where $x \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{k-1}$.

Lemma a.1. Let $0<\tau<1, \rho>0$ and $A \subset C_{1}$ be a Borel set which is a cylinder (i.e. $A=\boldsymbol{p}^{-1}(\boldsymbol{p}(A))$ ). Let $\mu: \mathbb{R}^{n} \rightarrow[0,1]$ be a $\mathcal{C}^{1}$-function satisfying $\sup _{p(A)}|D \mu| \leq \rho / \tau$ and consider the map

$$
\begin{aligned}
F: \mathbb{R}^{n+k} & \rightarrow \mathbb{R}^{n+k} \\
(x, y) & \mapsto\left(x_{1}, \ldots, x_{n}, \mu\left(x_{1}, \ldots, x_{n}\right) x_{n+1}, \boldsymbol{\Phi}\left(x_{1}, \ldots, x_{n}, \mu\left(x_{1}, \ldots, x_{n}\right) x_{n+1}\right)\right)
\end{aligned}
$$

Then there is a constant $C_{44}>0$ only depending on $n, k$ and $m$ such that for any current $T$ with $(T, \mathcal{M}) \in \mathcal{T}$ the following holds

$$
\mathbb{M}\left(F_{\#}(T\llcorner A))-\mathbb{M}\left(T\llcorner A) \leq C_{44}\left(\frac{1+\rho^{2}}{\tau^{2}} \int_{A} X_{n+1}^{2} \mathrm{~d}\|T\|+\frac{\kappa_{T}^{2}}{\tau^{2}}+\left(2+\frac{\rho^{2}}{\tau^{2}}\right) A\right)\right.\right.
$$

where $A_{\tau}:=\left\{x \in \mathbb{R}^{n+1}: \operatorname{dist}(x, A)<\tau\right\}$ is an enlargement of $A$ by $\tau$.
Proof. By [23, Section 4.1.30], we infer that for any $\omega \in \mathcal{D}^{n}\left(\mathbb{R}^{n+1}\right)$

$$
\left(F_{\#}(T\llcorner A))(\omega)=\int_{A}\left\langle F_{\#} \vec{T}(x), \omega(F(x))\right\rangle \mathrm{d}\|T\| .\right.
$$

We expand the tangent vector in the following basis for $T_{(x, \boldsymbol{\Phi}(x))} \mathcal{M}$

$$
\begin{equation*}
v_{j}(x):=\left(e_{j}, \partial_{j} \boldsymbol{\Phi}(x)\right) \quad \text { for } j \in\{1, \ldots, n+1\} \tag{a.2}
\end{equation*}
$$

where $e_{j}$ denotes the $j$-th standard basis vector in $\mathbb{R}^{n+1}$. Then there are real numbers $T_{j}$ such that

$$
\begin{equation*}
\vec{T}=\sum_{j=1}^{n+1} T_{j} v_{1} \wedge \cdots \wedge \widehat{v}_{j} \wedge \cdots \wedge v_{n+1} \tag{a.3}
\end{equation*}
$$

We compute

$$
\begin{aligned}
F_{\#} \vec{T}(x, y) & =T_{n+1} v_{1}(F(x)) \wedge \cdots \wedge v_{n}(F(x)) \\
& +\sum_{j=1}^{n}\left(T_{j} \mu-T_{n+1} x_{n+1} \partial_{j} \mu\right) v_{1}(F(x)) \wedge \cdots \wedge \widehat{v}_{j}(F(x)) \wedge \cdots \wedge v_{n+1}(F(x))
\end{aligned}
$$

and therefore, we have

$$
\begin{aligned}
\left|F_{\#} \vec{T}\right|^{2} & \leq\left(T_{n+1}^{2}+\sum_{j=1}^{n}\left(T_{j} \mu-T_{n+1} X_{n+1} \partial_{j} \mu\right)^{2}\right)\left(\sum_{j=1}^{n+1}\left|v_{1} \wedge \cdots \wedge \widehat{v}_{j} \wedge \cdots \wedge v_{n+1}\right|^{2}\right) \\
& \leq\left(T_{n+1}^{2}+\sum_{j=1}^{n}\left(T_{j} \mu-T_{n+1} X_{n+1} \partial_{j} \mu\right)^{2}\right)\left(1+C_{45}|D \boldsymbol{\Phi}|^{2}\right) \\
& \leq T_{n+1}^{2}+\sum_{j=1}^{n}\left(T_{j} \mu-T_{n+1} X_{n+1} \partial_{j} \mu\right)^{2}+C_{46}|D \boldsymbol{\Phi}|^{2}\left(1+\frac{\rho^{2}}{\tau^{2}}\right)
\end{aligned}
$$

We argue as in the original paper [27, Lemma 3.1.1] to deduce

$$
\begin{align*}
& \mathbb{M}\left(F_{\#}(T\llcorner A))-\mathbb{M}(T\llcorner A)\right. \\
& \leq 2 \frac{\rho^{2}}{\tau^{2}} \int_{A} X_{n+1}^{2} \mathrm{~d}\|T\|+\int_{A}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\|+C_{46} \mathbf{A}^{2}\left(1+\frac{\rho^{2}}{\tau^{2}}\right) \mathbb{M}(T) \tag{a.4}
\end{align*}
$$

In order to bound the second integral, we compute the first variation of $T$ with respect the following vectorfield

$$
\Xi: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}, \quad(x, y) \mapsto\left(x_{n+1}-\psi_{T}\left(x_{1}, \ldots, x_{n-1}\right)\right) \lambda^{2}(x) e_{n+1}
$$

where $e_{n+1}$ denotes the $(n+1)$-th basis vector of $\mathbb{R}^{n+k}$ and $\lambda: \mathbb{R}^{n+1} \rightarrow[0,1]$ is a $\mathcal{C}^{1}$ cut-off function with $\operatorname{spt}(\lambda) \subset A_{\tau},\left.\lambda\right|_{A}=1$ and $\sup |D \lambda| \leq C_{47} / \tau$. Notice that $\Xi$ vanishes on $\operatorname{spt}(\partial T)$ and therefore, by [21, Theorem 3.2]

$$
\begin{equation*}
\int \operatorname{div}_{\vec{T}} \Xi \mathrm{~d}\|T\|=-\int \Xi \cdot \vec{H}_{T}(x) \mathrm{d}\|T\|(x) \tag{a.5}
\end{equation*}
$$

where $\vec{H}_{T}$ is the mean curvature vector.
As $\operatorname{spt}(T) \subset \mathcal{M}$, we have $\operatorname{div}_{\vec{T}} \Xi=\operatorname{div}_{\mathcal{M}} \Xi-\operatorname{div}_{v} \Xi$ where $v \in T_{(x, \Phi(x))} \mathcal{M}$ is the outer normal vector to $\vec{T}$. We compute $v$ by expanding everything in the basis in (a.2):

$$
\begin{aligned}
v & =\sum_{j=1}^{n+1} v_{j} v_{j} \\
\vec{T} & =\tau_{1} \wedge \cdots \wedge \tau_{n} \quad \text { with } \quad \tau_{i}=\sum_{j=1}^{n+1} t_{i, j} v_{j}
\end{aligned}
$$

As $v$ is normal to $\vec{T}$, we can use the expansion (a.3) to find the following equalities for all $j \in\{1, \ldots, n+1\}$ and $t_{i}:=\left(t_{i, 1}, \ldots t_{i, n+1}\right)^{\top}$ with $i \in\{1, \ldots, n\}$ :

$$
\begin{align*}
T_{j} & =\operatorname{det}^{1, \ldots, \hat{j}, \ldots, n+1}\left(\begin{array}{ccc}
t_{1} & \cdots & t_{n}
\end{array}\right),  \tag{a.6}\\
0 & =\left\langle v, \tau_{i}\right\rangle=\left\langle\left(\begin{array}{c}
v_{1} \\
: \\
v_{n+1}
\end{array}\right), g \cdot t_{i}\right\rangle \tag{a.7}
\end{align*}
$$

where $g=\left(\left\langle v_{i}, v_{j}\right\rangle_{i, j}\right)=\operatorname{id}_{n+1}+\left(\left\langle\partial_{i} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle_{i, j}\right)=: \mathrm{id}_{n+1}+B$ is the metric.
From (a.7), we deduce that

$$
v_{j}=\star\left(\left(g \cdot t_{i}\right) \wedge \cdots \wedge\left(g \cdot t_{n}\right)\right)=(-1)^{j} \operatorname{det}^{1, \ldots, \hat{j}, \ldots, n+1}\left(\begin{array}{lll}
g \cdot t_{1} & \cdots & g \cdot t_{n}
\end{array}\right) .
$$

We compute

$$
\begin{align*}
\operatorname{div}_{v} \Xi & =\sum_{j=1}^{n+k}\left(D_{v} \Xi_{j}\right)_{j}=\left(D_{v} \Xi_{n+1}\right)_{n+1} \\
& =\left(\left\langle D\left(\left(x_{n+1}-\psi_{T}\left(x_{1}, \ldots, x_{n-1}\right)\right) \lambda^{2}(x)\right), \frac{v}{|v|}\right\rangle \frac{v}{|v|}\right)_{n+1}  \tag{a.8}\\
& =\frac{1}{|v|^{2}}\left(\lambda^{2} v_{n+1}^{2}-\lambda^{2} \sum_{j=1}^{n-1} v_{n+1} v_{j} \partial_{j} \psi_{T}+2 \lambda\left(X_{n+1}-\psi_{T}\right) \sum_{j=1}^{n+1} v_{n+1} v_{j} \partial_{j} \lambda\right) .
\end{align*}
$$

On the other hand, we need to compute the divergence with respect to $\mathcal{M}$. To do so, we compute the projection on $\mathcal{M}$ : Let $M$ be the matrix with column vectors $v_{1}, \ldots v_{n+1} \in \mathbb{R}^{n+k}$. Then we have

$$
\begin{aligned}
& \operatorname{div}_{\mathcal{M}} \Xi \\
& =\sum_{j=1}^{n+k}\left(D_{\mathcal{M}} \Xi_{j}\right)_{j}=\left(D_{\mathcal{M}} \Xi_{n+1}\right)_{n+1} \\
& =\left(M \cdot g^{-1} \cdot M^{T} \cdot D\left(\left(x_{n+1}-\psi_{T}\left(x_{1}, \ldots, x_{n-1}\right)\right) \lambda^{2}(x)\right)\right)_{n+1} \\
& =\left(\left(\begin{array}{ccc}
g^{1,1} & \cdots & g^{1, n+1} \\
: & & : \\
g^{1, n+1} & \cdots & g^{n+1, n+1} \\
\star & \star & \star
\end{array}\right)\left(\begin{array}{ccc}
1 & & 0 \\
& \ddots & \\
0 & & 1 \\
\partial_{1} \boldsymbol{\Phi} & \cdots & \partial_{n+1} \Phi
\end{array}\right)^{T}\left(\begin{array}{c}
: \\
2 \lambda\left(X_{n+1}-\psi_{T}\right) \partial_{i} \lambda-\lambda^{2} \partial_{i} \psi_{T} \\
: \\
2 \lambda\left(X_{n+1}-\psi_{T}\right) \partial_{n} \lambda \\
2 \lambda\left(X_{n+1}-\psi_{T}\right) \partial_{n+1} \lambda-\lambda^{2} \\
0
\end{array}\right)\right)_{n+1} \\
& =\left(\left(\begin{array}{ccc}
g^{1,1} & \cdots & g^{1, n+1} \\
: & & : \\
g^{1, n+1} & \cdots & g^{n+1, n+1} \\
\star & \star & \star
\end{array}\right)\left(\begin{array}{c}
: \\
2 \lambda\left(X_{n+1}-\psi_{T}\right) \partial_{i} \lambda-\lambda^{2} \partial_{i} \psi_{T} \\
: \\
2 \lambda\left(X_{n+1}-\psi_{T}\right) \partial_{n} \lambda \\
2 \lambda\left(X_{n+1}-\psi_{T}\right) \partial_{n+1} \lambda-\lambda^{2}
\end{array}\right)\right)_{n+1} \\
& =\lambda^{2} g^{n+1, n+1}-\lambda^{2} \sum_{j=1}^{n-1} g^{n+1, j} \partial_{j} \psi_{T}+2 \lambda\left(X_{n+1}-\psi_{T}\right) \sum_{j=1}^{n+1} g^{n+1, j} \partial_{j} \lambda .
\end{aligned}
$$

This together with (a.8) yields

$$
\begin{align*}
\operatorname{div}_{\vec{T}} \Xi= & \lambda^{2}\left(g^{n+1, n+1}-\frac{v_{n+1}^{2}}{|v|^{2}}\right)-\lambda^{2} \sum_{j=1}^{n-1}\left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \psi_{T} \\
& +2 \lambda\left(X_{n+1}-\psi_{T}\right) \sum_{j=1}^{n+1}\left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \lambda \tag{a.9}
\end{align*}
$$

Together with (a.5), we have

$$
\begin{align*}
-\int \Xi \cdot \vec{H}_{T} \mathrm{~d}\|T\|= & \int \lambda^{2}\left(\left(g^{n+1, n+1}-\frac{v_{n+1}^{2}}{|v|^{2}}\right)-\sum_{j=1}^{n-1}\left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \psi_{T}\right) \mathrm{d}\|T\| \\
& +2 \int \lambda\left(X_{n+1}-\psi_{T}\right) \sum_{j=1}^{n+1}\left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \lambda \mathrm{~d}\|T\| \tag{a.10}
\end{align*}
$$

In order to regain the term $1-T_{n+1}^{2}$, we first estimate $v_{n+1}$

$$
\begin{aligned}
& (-1)^{n+1} v_{n+1} \\
& =\operatorname{det}^{1, \ldots, n}\left(\begin{array}{lll}
g \cdot t_{1} & \cdots & g \cdot t_{n}
\end{array}\right) \\
& =\operatorname{det}^{1, \ldots, n}\left(\left(\operatorname{id}+\left(\left\langle\partial_{i} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle_{i, j}\right)\right) \cdot\left(\begin{array}{lll}
t_{1} & \cdots & t_{n}
\end{array}\right)\right) \\
& =\sum_{\sigma \in S_{n}} \operatorname{sgn}(\sigma)\left(t_{1, \sigma(1)}+\sum_{j=1}^{n+1} t_{1, j}\left\langle\partial_{\sigma(1)} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle\right) \cdots\left(t_{n, \sigma(n)}+\sum_{j=1}^{n+1} t_{n, j}\left\langle\partial_{\sigma(n)} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle\right) \\
& =\sum_{\sigma \in S_{n}} \operatorname{sgn}(\sigma) t_{1, \sigma(1)} \cdots t_{n, \sigma(n)}+O(|D \boldsymbol{\Phi}|) \\
& =T_{n+1}^{2}+O(|D \boldsymbol{\Phi}|) .
\end{aligned}
$$

Hence,

$$
\begin{equation*}
v_{n+1}^{2} \leq T_{n+1}^{2}+C_{48}|D \boldsymbol{\Phi}|^{2} . \tag{a.11}
\end{equation*}
$$

Now, we compute the norm of $v$. We use that the Hodge star is norm-preserving and therefore, we have for $\tilde{v}:=\left(v_{1}, \ldots, v_{n+1}\right)$

$$
\begin{aligned}
|\tilde{v}|^{2}=\operatorname{det}\left(\left\langle g \cdot t_{i}, g \cdot t_{j}\right\rangle_{i, j}\right) & =\operatorname{det}\left(\left\langle t_{i}, g^{2} t_{j}\right\rangle_{i, j}\right) \\
& =\operatorname{det}\left(\left(\left\langle t_{i}, t_{j}\right\rangle+2\left\langle t_{i}, B t_{j}\right\rangle+\left\langle t_{i}, B^{2} t_{j}\right\rangle\right)_{i, j}\right) .
\end{aligned}
$$

Notice that

$$
\begin{align*}
\left\langle t_{i}, t_{j}\right\rangle+2\left\langle t_{i}, B t_{j}\right\rangle+\left\langle t_{i}, B^{2} t_{j}\right\rangle & \geq\left\langle t_{i}, t_{j}\right\rangle-2\|B\|_{o p}\left|t_{i}\right|\left|t_{j}\right|-\|B\|_{o p}^{2}\left|t_{i}\right|\left|t_{j}\right| \\
& \geq\left\langle t_{i}, t_{j}\right\rangle-\left(2\|B\|+\|B\|^{2}\right)\left|t_{i}\right|\left|t_{j}\right| \\
& \geq\left\langle t_{i}, t_{j}\right\rangle-\left(2 \sqrt{n+1}|D \boldsymbol{\Phi}|^{2}+(n+1)|D \boldsymbol{\Phi}|^{4}\right)\left|t_{i} \| t_{j}\right|  \tag{a.12}\\
& \geq\left\langle t_{i}, t_{j}\right\rangle-2(n+1)|D \boldsymbol{\Phi}|^{2},
\end{align*}
$$

where we used in the last inequality the fact

$$
\left|t_{i}\right|^{2}=\left|\sum_{j=1}^{n+1} t_{i, j}\left(e_{j}, \partial_{j} \boldsymbol{\Phi}\right)\right|^{2}-\left|\sum_{j=1}^{n+1} t_{i, j} \partial_{j} \boldsymbol{\Phi}\right|^{2} \leq\left|\tau_{i}\right|^{2}=1 .
$$

Therefore, we estimate

$$
\begin{align*}
|\tilde{v}|^{2} & =\sum_{\sigma \in P_{n}} \prod_{i=1}^{n} \operatorname{sgn}(\sigma)\left\langle g \cdot t_{i}, g \cdot t_{\sigma(i)}\right\rangle \\
& \geq \sum_{\sigma \in P_{n}}\left(\prod_{i=1}^{n} \operatorname{sgn}(\sigma)\left\langle t_{i}, t_{\sigma(i)}\right\rangle-2^{n}(2(n+1))^{n}|D \boldsymbol{\Phi}|^{2}\right)  \tag{a.13}\\
& \geq \operatorname{det}\left(\left\langle t_{i}, t_{j}\right\rangle_{i, j}\right)-2^{2 n} n!(n+1)^{n}|D \boldsymbol{\Phi}|^{2} .
\end{align*}
$$

Now, we use that $\tau_{1}, \ldots \tau_{n}$ are orthonormal to deduce that

$$
\begin{aligned}
\delta_{i, j} & =\left\langle\tau_{i}, \tau_{j}\right\rangle=\left\langle\sum_{k=1}^{n+1} t_{i, k}\left(e_{k}, \partial_{k} \boldsymbol{\Phi}\right), \sum_{l=1}^{n+1} t_{i, l}\left(e_{l}, \partial_{l} \boldsymbol{\Phi}\right)\right\rangle \\
& =\left\langle\sum_{k=1}^{n+1} t_{i, k} e_{k}, \sum_{l=1}^{n+1} t_{i, l} e_{l}\right\rangle+\left\langle\sum_{k=1}^{n+1} t_{i, k} \partial_{k} \boldsymbol{\Phi}, \sum_{l=1}^{n+1} t_{i, l} \partial_{l} \boldsymbol{\Phi}\right\rangle \\
& =\left\langle t_{i}, t_{j}\right\rangle+\sum_{k, l=1}^{n+1} t_{i, k} t_{j, l}\left\langle\partial_{k} \boldsymbol{\Phi}, \partial_{l} \boldsymbol{\Phi}\right\rangle
\end{aligned}
$$

and hence,

$$
\left|\delta_{i, j}-\left\langle t_{i}, t_{j}\right\rangle\right| \leq 2(n+1)|D \boldsymbol{\Phi}|^{2}
$$

By a similar argument as in (a.13), it follows that

$$
\operatorname{det}\left(\left\langle t_{i}, t_{j}\right\rangle_{i, j}\right) \geq 1-2^{n} n!(n+1)^{n}|D \boldsymbol{\Phi}|^{2}
$$

Putting this into (a.13), we yield

$$
\begin{aligned}
|v|^{2} & =\left|\sum_{j=1}^{n+1} v_{j} v_{j}\right|^{2}=v_{1}^{2}+\cdots+v_{n+1}^{2}+\left|\sum_{j=1}^{n+1} v_{j} \partial_{j} \boldsymbol{\Phi}\right|^{2} \\
& \geq|\tilde{v}|^{2} \geq 1-2^{2 n+1} n!(n+1)^{n}|D \boldsymbol{\Phi}|^{2}
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{|v|^{2}} \leq \frac{1}{1-2^{2 n+1} n!(n+1)^{n}|D \boldsymbol{\Phi}|^{2}} \leq 1+C_{49}|D \boldsymbol{\Phi}|^{2} \tag{a.14}
\end{equation*}
$$

Now, we take care of $g^{-1}$. By the geometric series and the fact $g=\mathrm{id}+\left(\left\langle\partial_{i} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle_{i, j}\right)$, we have

$$
\begin{equation*}
g^{-1}=\mathrm{id}-\left(\left\langle\partial_{i} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle_{i, j}\right)+\sum_{l \geq 2}(-1)^{l}\left(\left\langle\partial_{i} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle_{i, j}\right)^{l} \tag{a.15}
\end{equation*}
$$

and hence,

$$
\begin{equation*}
\left|g^{i, j}\right| \leq \delta_{i, j}-\left\langle\partial_{i} \boldsymbol{\Phi}, \partial_{j} \boldsymbol{\Phi}\right\rangle+C_{12}|D \boldsymbol{\Phi}|^{4} . \tag{a.16}
\end{equation*}
$$

Now, we are ready to estimate piece by piece the right hand side of (a.10)

- We use (a.11), (a.14) and (a.16) to deduce

$$
\begin{aligned}
\int \lambda^{2} & \left(g^{n+\prime n+1}-\frac{v_{n+1}^{2}}{|v|^{2}}\right) \mathrm{d}\|T\| \\
& \geq \int \lambda^{2}\left(1-\left|\partial_{n+1} \boldsymbol{\Phi}\right|^{2}-C_{12}|D \boldsymbol{\Phi}|^{4}-T_{n+1}^{2}-C_{50}|D \boldsymbol{\Phi}|^{2}\right) \mathrm{d}\|T\| \\
& \geq \int \lambda^{2}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\|-C_{51} \mathbb{M}(T) \mathbf{A}^{2}
\end{aligned}
$$

- We use (a.11), (a.14) and (a.16) to deduce

$$
\begin{aligned}
\int \lambda^{2} \sum_{j=1}^{n-1} & \left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \psi_{T} \mathrm{~d}\|T\| \\
& \leq \int \lambda^{2}\left(C_{52}|D \boldsymbol{\Phi}|^{2}+\frac{\left|\left(v_{1}, \ldots, v_{n}\right)\right|}{|v|}\right) \kappa_{T} \mathrm{~d}\|T\| \\
& \leq \kappa_{T} \int \lambda^{2} \frac{\sqrt{|\tilde{v}|^{2}-v_{n+1}^{2}}}{|v|^{2}} \mathrm{~d}\|T\|+C_{53} \mathbf{M}(T)|D \boldsymbol{\Phi}|^{2} \\
& \leq \kappa_{T} \int \lambda^{2} \sqrt{1-T_{n+1}^{2}+C_{54}|D \boldsymbol{\Phi}|^{2}}\left(1+C_{49}|D \boldsymbol{\Phi}|^{2}\right) \mathrm{d}\|T\|+C_{53} \mathbf{M}(T) \mathbf{A}^{2} \\
& \leq \kappa_{T} \int \lambda^{2} \sqrt{1-T_{n+1}^{2}+C_{55}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\|+C_{53} \mathbf{M}(T) \mathbf{A}^{2} .
\end{aligned}
$$

- We use (a.11), (a.14), (a.16) and a similar argument as in (a.13) to deduce

$$
\begin{aligned}
& \int 2 \lambda\left(X_{n+1}-\psi_{T}\right) \sum_{j=1}^{n+1}\left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \lambda \mathrm{~d}\|T\| \\
& =\int 2 \lambda\left(\left|X_{n+1}\right|+\kappa_{T}\right)\left(\left(g^{n+1, n+1}-\frac{v_{n+1}^{2}}{|v|^{2}}\right) \partial_{n+1} \lambda+\sum_{j=1}^{n}\left(g^{n+1, j}-\frac{v_{n+1} v_{j}}{|v|^{2}}\right) \partial_{j} \lambda\right) \mathrm{d}\|T\| \\
& \leq 2 \int \lambda|D \lambda|\left(\left|X_{n+1}\right|+\kappa_{T}\right)\left(1-\left|\partial_{n+1} \boldsymbol{\Phi}\right|^{2}-T_{n+1}^{2}+C_{56}|D \boldsymbol{\Phi}|^{2}+\frac{\left|\left(v_{1}, \ldots, v_{n}\right)\right|}{|v|}\right) \mathrm{d}\|T\| \\
& \leq 2 \frac{C_{59}}{\tau}\left(\int \lambda\left(\left|X_{n+1}\right|+\kappa_{T}\right)\left(1-T_{n+1}^{2}+\frac{\sqrt{|\tilde{v}|^{2}-v_{n+1}^{2}}}{|v|^{2}}\right) \mathrm{d}\|T\|+C_{57} \mathbb{M}(T) \mathbf{A}^{2}\right) \\
& \leq 2 \frac{C_{59}}{\tau} \int \lambda\left(\left|X_{n+1}\right|+\kappa_{T}\right)\left(1-T_{n+1}^{2}+\sqrt{1-T_{n+1}^{2}+C_{58}|D \boldsymbol{\Phi}|^{2}}\left(1+C_{49}|D \boldsymbol{\Phi}|^{2}\right)\right) \mathrm{d}\|T\| \\
& \quad+C_{57} \frac{C_{59}}{\tau} \mathbb{M}(T) \mathbf{A}^{2} \\
& \leq 2 \frac{C_{59}}{\tau}\left(\int \lambda\left(\left|X_{n+1}\right|+\kappa_{T}\right) 2 \sqrt{1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\|+C_{61} \mathbb{M}(T) \mathbf{A}^{2}\right) .
\end{aligned}
$$

Putting all this into (a.5) yields

$$
\begin{align*}
& \int \lambda^{2}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\| \\
& \quad \leq \int \kappa_{T} \lambda^{2} \sqrt{1-T_{n+1}^{2}+C_{55}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\|+\frac{C_{59}}{\tau} \int \lambda\left|X_{n+1}\right| \sqrt{1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\| \\
& \quad+\frac{C_{59}}{\tau} \int \kappa_{T} \lambda \sqrt{1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\|+\int \Xi \cdot \vec{H} \mathrm{~d}\|T\|+C_{62} \mathbb{M}(T) \mathbf{A}^{2} . \tag{a.17}
\end{align*}
$$

Using three times the Cauchy inequality ( $2 a b \leq a^{2}+b^{2}$ ), we estimate

$$
\begin{aligned}
& \bullet \int \kappa_{T} \lambda^{2} \sqrt{1-T_{n+1}^{2}+C_{55}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\| \\
& \quad \leq \int_{A_{\tau}} \frac{\lambda^{2}}{4}\left(1-T_{n+1}^{2}+C_{55}|D \boldsymbol{\Phi}|^{2}\right) \mathrm{d}\|T\|+\int_{A_{\tau}} \kappa_{T}^{2} \lambda^{2} \mathrm{~d}\|T\|,
\end{aligned}
$$

- $\frac{C_{59}}{\tau} \int \lambda\left|X_{n+1}\right| \sqrt{1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\|$

$$
\leq \frac{1}{16} \int_{A_{\tau}} \lambda^{2}\left(1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}\right) \mathrm{d}\|T\|+\frac{C_{59}}{\tau^{2}} \int_{A_{\tau}} X_{n+1}^{2} \mathrm{~d}\|T\|,
$$

- $\frac{C_{59}}{\tau} \int \kappa_{T} \lambda \sqrt{1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}} \mathrm{~d}\|T\|$

$$
\leq \frac{1}{16} \int_{A_{\tau}} \lambda^{2}\left(1-T_{n+1}^{2}+C_{60}|D \boldsymbol{\Phi}|^{2}\right) \mathrm{d}\|T\|+\frac{C_{59}}{\tau^{2}} \int_{A_{\tau}} \kappa_{T}^{2} \mathrm{~d}\|T\| .
$$

Again putting this into (a.17) yields

$$
\begin{aligned}
\int_{A_{\tau}} \lambda^{2}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\| \leq & \frac{1}{2} \int_{A_{\tau}} \lambda^{2}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\|+\frac{C_{59}}{\tau^{2}} \int_{A_{\tau}} X_{n+1}^{2} \mathrm{~d}\|T\| \\
& +\int_{A_{\tau}} \Xi \cdot \vec{H} \mathrm{~d}\|T\|+C_{63} \mathrm{M}(T)\left(\mathbf{A}^{2}+\kappa_{T}^{2}+\frac{\kappa_{T}^{2}}{\tau^{2}}\right)
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\int_{A}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\| & \leq \int_{A_{\tau}} \lambda^{2}\left(1-T_{n+1}^{2}\right) \mathrm{d}\|T\| \\
& \leq 2 \frac{C_{59}}{\tau^{2}} \int_{A_{\tau}} X_{n+1}^{2} \mathrm{~d}\|T\|+C_{64} \mathbb{M}(T)\left(\kappa_{T}^{2}+\frac{\kappa_{T}^{2}}{\tau^{2}}+2 \mathbf{A}\right) .
\end{aligned}
$$

Using (a.4), we deduce the desired inequality

$$
\begin{aligned}
& \mathbb{M}\left(F_{\#}(T\llcorner A))-\mathbb{M}(T\llcorner A)\right. \\
& \quad \leq C_{65} \frac{1+\rho^{2}}{\tau^{2}} \int_{A} X_{n+1}^{2} \mathrm{~d}\|T\|+C_{64} \mathbb{M}(T) \frac{\kappa_{T}^{2}}{\tau^{2}}+C_{66} \mathbb{M}(T) \mathbf{A}\left(2+\frac{\rho^{2}}{\tau^{2}}\right) .
\end{aligned}
$$

Now we have all the tools to estimate the excess of $T$ with its height.
Proof of Lemma 3.3. The second inequality holds true with $C_{11} \geq 3^{n}\left(1+m \boldsymbol{\omega}_{n}\right) \geq \mathbb{M}(T)$. For the first inequality, we want to use Lemma a.1 for $A:=\mathbf{C}_{1+\tau} \backslash \mathbf{C}_{1}, \rho=3$ and
$\tau=\sigma / 2$. Consider $F$ as in the lemma for some $\mathcal{C}^{1}$-function $\mu: \mathbb{R}^{n} \rightarrow[0,1]$ satisfying $\sup |D \mu| \leq \rho / \tau$ and
$\mathbf{p}(A)$

$$
\begin{cases}\mu(z)=0 & \text { if }|z| \leq 1 \\ \mu(z)>0 & \text { if } 1<|z|<1+\tau \\ \mu(z)=1 & \text { if }|z| \geq 1+\tau .\end{cases}
$$

Moreover, we define for $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^{n+k}$ the homotopy

$$
H_{t}(x, y):=\left(\mathbf{p}(x),((1-t) \mu \circ \mathbf{p}(x)+t) x_{n+1}, \boldsymbol{\Phi}\left(\mathbf{p}(x),((1-t) \mu \circ \mathbf{p}(x)+t) x_{n+1}\right)\right) .
$$

Notice that $F$ is the identity on $\mathcal{M} \backslash \mathbf{C}_{1+\tau}$ and $F=(\mathbf{p}, 0, \boldsymbol{\Phi}(\mathbf{p}, 0))$ on $\mathbf{C}_{1}$.
Then for $R_{T}:=H_{\#}([0,1] \times \partial T)$ we have $\operatorname{spt}\left(R_{T}\right) \subset \mathcal{M}$ and

$$
\partial\left(T \left\llcorner\mathbf{C}_{1+\tau}-F_{\#}\left(T\left\llcorner\mathbf{C}_{1+\tau}\right)-R_{T}\right)=\partial\left(T-F_{\#} T-R_{T}\right)=0 .\right.\right.
$$

Hence, by the area minimality of $T$ in $\mathcal{M}$, we have

$$
\mathbb{M}\left(T\left\llcorner\mathbf{C}_{1+\tau}\right) \leq \mathbb{M}\left(F_{\#}\left(T\left\llcorner\mathbf{C}_{1+\tau}\right)\right)+\mathbb{M}\left(R_{T}\right)\right.\right.
$$

Moreover, by [33, Remark 26.21(2)], the following holds

$$
\mathbb{M}\left(R_{T}\right) \leq \sup _{\operatorname{spt}(\partial T)}\left|\partial_{t} H\right| \sup _{\operatorname{spt}(\partial T)}\left|\partial_{x} H\right| \mathbb{M}\left((\partial T)\left\llcorner\mathbf{C}_{2}\right) .\right.
$$

Therefore, we compute

$$
\begin{aligned}
&\left|\partial_{t} H\right| \leq\left(X_{n+1}-X_{n+1} \mu \circ \mathbf{p}\right)+|D \boldsymbol{\Phi}|\left(X_{n+1}-X_{n+1} \mu \circ \mathbf{p}\right) \\
& \leq\left(1+|D \boldsymbol{\Phi}|^{2}\right)\left|X_{n+1}\right|(1-\mu \circ \mathbf{p}) \\
& \leq \kappa_{T}\left(1+|D \boldsymbol{\Phi}|^{2}\right), \\
&\left|\partial_{x} H\right| \leq n+|D \mu| X_{n+1}+|D \boldsymbol{\Phi}|\left(n+|D \mu| X_{n+1}\right)+(|\mu|+1)+|D \boldsymbol{\Phi}|(|\mu|+1) \\
& \leq n+\left(\frac{6}{\sigma}\right) \kappa_{T}+|D \boldsymbol{\Phi}|\left(4+\left(n+\frac{6 \kappa_{T}}{\sigma}\right)\right)+4 \\
& \leq C_{67}\left(1+\frac{\kappa_{T}}{\sigma}\right), \\
& \mathbb{M}\left((\partial T)\left\llcorner\mathbf{C}_{2}\right) \leq \boldsymbol{\omega}_{n-1} 2^{n-1} \sqrt{n+\kappa_{T}^{2}+\mathbf{A}^{2}\left(1+\kappa_{T}\right)} \leq C_{68}\left(1+\kappa_{T}\right) .\right.
\end{aligned}
$$

Thus, we have

$$
\mathbb{M}\left(R_{T}\right) \leq C_{69} \frac{\kappa_{T}}{\sigma}(1+\mathbf{A})
$$

Now, we argue as originally in [27, Lemma 4.1] and use Lemma a. 1 to deduce

$$
\begin{aligned}
\mathbf{E}_{C}(T, 1) & \leq \mathbb{M}\left(F_{\#}(T\llcorner A))-\mathbb{M}\left(T\llcorner A)+C_{69} \frac{\kappa_{T}}{\sigma}(1+\mathbf{A})\right.\right. \\
& \leq \frac{C_{10}}{\sigma^{2}}\left(\kappa_{T}+\int_{\mathbf{C}_{1+\sigma}} X_{n+1}^{2} \mathrm{~d}\|T\|+\mathbf{A}\right) .
\end{aligned}
$$

## A. 3 Proof of lemma 3.4

Proof. We call a function $f T$-subharmonic if

$$
\int\left\langle D_{\vec{T}} f, D_{\vec{T}} \zeta\right\rangle \mathrm{d}\|T\| \leq 0 \quad \text { for all } \zeta \in \mathcal{C}^{1}\left(\mathbb{R}^{n+k} ; \mathbb{R}_{\geq 0}\right) \text { with } \operatorname{spt}(\zeta) \cap \operatorname{spt}(\partial T)=\varnothing \text {. }
$$

The functions

$$
h_{i}: \mathbb{R}^{n+k} \rightarrow \mathbb{R}, \quad(x, y) \mapsto(-1)^{i} x_{n+1}+x_{n+1}^{2}, \quad \text { for } i \in\{1,2\}
$$

are $T$-subharmonic as

$$
\begin{aligned}
\int\left\langle D_{\vec{T}} h_{i}, D_{\vec{T}} \zeta\right\rangle \mathrm{d}\|T\| & =\int\left\langle\pi \cdot D h_{i}, \pi \cdot D \zeta\right\rangle \mathrm{d}\|T\|=\int\left\langle D h_{i}, \pi \cdot D \zeta\right\rangle \mathrm{d}\|T\| \\
& =\int\left\langle(-1)^{i} \mathbf{e}_{n+1}+2 X_{n+1} \mathbf{e}_{n+1}, \pi \cdot D \zeta\right\rangle \mathrm{d}\|T\| \\
& =\int\left(\operatorname{div}_{\vec{T}}\left(\zeta\left((-1)^{i}+2 X_{n+1}\right) \mathbf{e}_{n+1}\right)-2 \zeta \pi_{n+1, n+1}\right) \mathrm{d}\|T\| \\
& =\int\left(-\zeta\left((-1)^{i}+2 X_{n+1}\right) \mathbf{e}_{n+1} \cdot \vec{H}-2 \zeta g^{n+1, n+1}\right) \mathrm{d}\|T\|, \\
& \leq \int \zeta\left(7 C_{1}\left|D^{2} \boldsymbol{\Phi}\right|-2\left(1-\left|\partial_{n+1} \boldsymbol{\Phi}\right|^{2}-C_{12}|D \boldsymbol{\Phi}|^{4}\right)\right) \mathrm{d}\|T\| \\
& \leq \int \zeta\left(7 C_{1}\left|D^{2} \boldsymbol{\Phi}\right|-2\left(1-\left(1+C_{12}\right)|D \boldsymbol{\Phi}|^{2}\right)\right) \mathrm{d}\|T\| \\
& \leq 0
\end{aligned}
$$

where $\pi(x)$ denotes the orthogonal projection to the tangent plane of $T$ at $x$ and we used (a.16), [21, Theorem 3.2] and the fact $\left(\operatorname{spt}\left(\zeta \mathbf{e}_{n+1}\right) \cap \operatorname{spt}(\partial T)\right) \subset(\operatorname{spt}(\zeta) \cap \operatorname{spt}(\partial T))=\varnothing$.

Consider the nonnegative, convex function

$$
f: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto \begin{cases}t-2 \kappa_{T}, & \text { if } t \geq 2 \kappa_{T} \\ -t-2 \kappa_{T}, & \text { if } t \leq-2 \kappa_{T} \\ 0, & \text { else }\end{cases}
$$

Notice that $f\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)$ vanishes on $\operatorname{spt}(\partial T)$. If $f$ were additionally smooth, than by [2, Lemma 7.5(3)] $f\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)$ would be $T$-subharmonic. Therefore,
we take a smooth nonnegative mollifier $\eta$ satisfying $\operatorname{spt}(\eta) \subset(-1,1)$ and $\int_{\mathbb{R}} \eta(x) d x=1$. Define $\eta_{\epsilon}(x):=\frac{1}{\epsilon} \eta(x / \epsilon)$ and $f_{\epsilon}:=f * \eta_{\epsilon} . f_{\epsilon}$ is smooth, convex and converges uniformly to $f$ when $\epsilon \downarrow 0$. Therefore $f_{\epsilon} \circ\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)$ is $T$-subharmonic and by [2, Theorem 7.5(6)]

$$
\begin{align*}
\sup _{\mathbf{C}_{1-\sigma} \cap \operatorname{spt}(T)} & f\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)^{2} \\
& =\sup _{a \in \mathbf{p}^{-1}(0)} \sup _{\tau_{a}\left(\overline{\mathbf{B}}_{1-\sigma}\right) \cap \operatorname{spt}(T)} f\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)^{2} \\
& =\sup _{a \in \mathbf{p}^{-1}(0)} \lim _{\epsilon \downarrow 0}\left(\sup _{\tau_{a}\left(\overline{\mathbf{B}}_{1-\sigma}\right) \cap \operatorname{spt}(T)} f_{\epsilon} \circ\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)\right)^{2}  \tag{a.18}\\
& \leq \sup _{a \in \mathbf{p}^{-1}(0)} \lim _{\epsilon \downarrow 0}\left(\frac{C_{70}}{\sigma^{n}} \int_{\tau_{a}\left(\overline{\mathbf{B}}_{1-\sigma / 2}\right)}\left(f_{\epsilon} \circ\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right)\right)^{2} \mathrm{~d}\|T\|\right) \\
& \leq \frac{C_{70}}{\sigma^{n}} \int_{\mathbf{C}_{1-\sigma / 2}} f^{2}\left((-1)^{i} X_{n+1}+X_{n+1}^{2}\right) \mathrm{d}\|T\| .
\end{align*}
$$

We deduce further that in $\overline{\mathbf{B}}_{2}$ the following holds

$$
\begin{align*}
X_{n+1}^{2}-40 \kappa_{T} & \leq\left(\left|X_{n+1}\right|+X_{n+1}^{2}\right)^{2}-40 \kappa_{T} \\
& \leq \begin{cases}\left(X_{n+1}+X_{n+1}^{2}\right)^{2}-20 \kappa_{T}, & \text { if }\left|X_{n+1}+X_{n+1}^{2}\right| \geq 2 \kappa_{T} \\
0, & \text { else }\end{cases}  \tag{a.19}\\
& + \begin{cases}\left(-X_{n+1}+X_{n+1}^{2}\right)^{2}-20 \kappa_{T}, & \text { if }\left|X_{n+1}-X_{n+1}^{2}\right| \geq 2 \kappa_{T} \\
0, & \text { else }\end{cases} \\
& \leq f^{2}\left(X_{n+1}+X_{n+1}^{2}\right)+f^{2}\left(-X_{n+1}+X_{n+1}^{2}\right)
\end{align*}
$$

and

$$
\begin{align*}
f^{2}\left(X_{n+1}+X_{n+1}^{2}\right)+f^{2}\left(-X_{n+1}+X_{n+1}^{2}\right) & \leq 2\left(\left(X_{n+1}+X_{n+1}^{2}\right)^{2}+\left(-X_{n+1}+X_{n+1}^{2}\right)^{2}+8 \kappa_{T}^{2}\right) \\
& \leq 4\left(\left|X_{n+1}\right|+X_{n+1}^{2}\right)^{2}+16 \kappa_{T}^{2} \\
& \leq 36\left(X_{n+1}^{2}+\kappa_{T}^{2}\right) . \tag{a.20}
\end{align*}
$$

Putting (a.18), (a.19) and (a.20), we conclude

$$
\begin{aligned}
\sup _{\mathbf{C}_{1-\sigma} \cap \operatorname{spt}(T)} X_{n+1}^{2} & \leq \frac{C_{70}}{\sigma^{n}} \int_{\mathbf{C}_{1-\sigma / 2}}\left(f^{2}\left(X_{n+1}+X_{n+1}^{2}\right)+f^{2}\left(-X_{n+1}+X_{n+1}^{2}\right)\right) \mathrm{d}\|T\|+40 \kappa_{T} \\
& \leq \frac{36 C_{70}}{\sigma^{n}} \int_{\mathbf{C}_{1-\sigma / 2}}\left(X_{n+1}^{2}+\kappa_{T}^{2}\right) \mathrm{d}\|T\|+40 \kappa_{T} \\
& \leq \frac{C_{13}}{\sigma^{n}}\left(\int_{\mathbf{C}_{1-\sigma / 2}} X_{n+1}^{2} \mathrm{~d}\|T\|+\kappa_{T}\right)
\end{aligned}
$$

For (ii.), we specify $C_{71}$ later and let

$$
\tilde{C}:=12 \cdot 3^{3 n+2}\left(7+2 m+2 C_{4}+C_{71}\right) C_{13}\left(1+m \omega_{n}\right) .
$$

Case 1: $\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A} \geq 3^{n+2}\left(1+m \omega_{n}\right) \frac{\sigma^{n+1}}{\tilde{C}}$.
In this case, as $\operatorname{spt}(T) \subset \overline{\mathbf{B}}_{3}$, we can bound

$$
\int X_{n+1}^{2} \mathrm{~d}\|T\| \leq 3^{n+2}\left(1+m \omega_{n}\right) \leq \frac{\tilde{C}}{\sigma^{n+1}}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right) .
$$

Case 2: $\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}<3^{n+2}\left(1+m \omega_{n}\right) \frac{\sigma^{n+1}}{\tilde{C}}$
Here, we aim to show that $\mathbf{C}_{1-\sigma / 2} \cap \operatorname{spt}(T) \subset \overline{\overline{\mathbf{B}}}_{1}$. If this were true, the following would conclude the lemma. Namely, recall the normal vector $v$ from the proof of Lemma a.1. Then, by Cauchy's inequality, we can deduce

$$
\begin{align*}
\int_{\overline{\mathbf{B}}_{1}} X_{n+1}^{2} \mathrm{~d}\|T\| & =\int_{\overline{\mathbf{B}}_{1}}\left(\left\langle X, \frac{v}{|v|}\right\rangle+\left\langle X, \mathbf{e}_{n+1}-\frac{v}{|v|}\right\rangle\right)^{2} \mathrm{~d}\|T\| \\
& \leq 2 \int_{\overline{\mathbf{B}}_{1}}\left(\left|X^{\perp}\right|^{2}+|X|^{2}\left|\mathbf{e}_{n+1}-\frac{v}{|v|}\right|^{2}\right) \mathrm{d}\|T\|  \tag{a.22}\\
& \leq 2 \int_{\overline{\mathbf{B}}_{1}}\left(\left|X^{\perp}\right|^{2}|X|^{-n-2}+\left\|\mathbf{e}_{n+1} \cdot \mathbf{e}_{n+1}^{\top}-\frac{1}{|v|^{2}} v \cdot v^{\top}\right\|^{2}\right) \mathrm{d}\|T\|
\end{align*}
$$

Now, we recall that the cylindrical excess can also be expressed by

$$
\frac{1}{r^{n}} \int_{\mathbf{C}_{r}}\|\pi-\mathbf{p}\|^{2} \mathrm{~d}\|T\|,
$$

where $\pi(x)$ still denotes the orthogonal projection to the tangent plane of $T$ at $x$ We compute for $(x, y) \in \overline{\mathbf{B}}_{1}$

$$
\begin{aligned}
(\pi-\mathbf{p})(x, y) & =\left(M \cdot g^{-1} \cdot M^{T}(x, y)^{T}-\left\langle(x, y), \frac{v}{|v|}\right\rangle \frac{v}{|v|}\right)-\sum_{j=1}^{n} x_{j} \mathbf{e}_{j} \\
& =B(x, y)+x_{n+1} \mathbf{e}_{n+1}-\left\langle(x, y), \frac{v}{|v|}\right\rangle \frac{v}{|v|},
\end{aligned}
$$

where

$$
B(x, y):=M \cdot g^{-1} \cdot M^{T}(x, y)^{T}-(x, 0)^{T} .
$$

Using (a.15) we estimate

$$
|B(x, y)| \leq C_{72}|D \Phi| .
$$

Hence, by Corollary 2.5 and the inequality (3.2), we can continue the estimate of (a.22) in the following way:

$$
\begin{align*}
\int_{\overline{\mathbf{B}}_{1}} X_{n+1}^{2} \mathrm{~d}\|T\| & \leq 2\left(\mathbf{E}_{S}(T, 1)+C_{4}\left(\mathbf{A}+\kappa_{T}\right)+\int_{\overline{\mathbf{B}}_{1}} 2\left(\|\pi-\mathbf{p}\|^{2}+\|B\|^{2}\right) \mathrm{d}\|T\|\right) \\
& \leq 2 \mathbf{E}_{S}(T, 1)+2 C_{4}\left(\mathbf{A}+\kappa_{T}\right)+4 \mathbf{E}_{C}(T, 1)+C_{72} \mathbf{A}^{2}  \tag{a.23}\\
& \leq\left(6+2 m+2 C_{4}\right)\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}\right)+\left(2 C_{4}+C_{71}\right) \mathbf{A} .
\end{align*}
$$

As $\left(6+2 m+2 C_{4}+C_{71}\right) \leq \tilde{C} \leq \tilde{C} \sigma^{-n-1}$, we are left with proving that

$$
\mathbf{C}_{1-\sigma / 2} \cap \operatorname{spt}(T) \subset \overline{\mathbf{B}}_{1} .
$$

First, we notice that due to a similar reasoning as we did for (i.) and using (a.23), we have

$$
\begin{align*}
\sup _{\overline{\mathbf{B}}_{1-\sigma / 6} \mathrm{spt}(T)} X_{n+1}^{2} & \leq \frac{6^{n}}{\sigma^{n}} C_{13}\left(\int_{\overline{\mathbf{B}}_{1}} X_{n+1}^{2} \mathrm{~d}\|T\|+\kappa_{T}\right) \\
& \leq \frac{6^{n} C_{13}}{\sigma^{n}}\left(\left(7+2 m+2 C_{4}\right)\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}\right)+\left(2 C_{4}+C_{71}\right) \mathbf{A}\right) \\
& \leq \frac{\sigma}{12} \tag{a.24}
\end{align*}
$$

As a next step, we show that $\operatorname{spt}\left((\partial T)\left\llcorner\mathbf{C}_{1-\sigma / 3}\right) \subset \mathbf{B}_{1-\sigma / 6}\right.$.
We argue by continuity: Assume by contradiction that this is not the case. Then we would find a $z \in \mathbb{R}^{n-1}$ such that $\left(z, \varphi_{T}(z), \psi_{T}(z), \boldsymbol{\Phi}\left(z, \varphi_{T}(z), \psi_{T}(z)\right)\right) \in \mathbf{C}_{1-\sigma / 3} \backslash \mathbf{B}_{1-\sigma / 6}$, hence, $\left|\left(z, \varphi_{T}(z)\right)\right|<1-\sigma / 3$ but $\left|\left(z, \varphi_{T}(z), \psi_{T}(z), \boldsymbol{\Phi}\left(z, \varphi_{T}(z), \psi_{T}(z)\right)\right)\right| \geq 1-\sigma / 6$. Then it must hold that

$$
\begin{equation*}
\psi_{T}(z)^{2}+\left|\boldsymbol{\Phi}\left(z, \varphi_{T}(z), \psi_{T}(z)\right)\right| \geq\left(1-\frac{\sigma}{6}\right)^{2}-\left(1-\frac{\sigma}{3}\right)^{2}=\frac{\sigma}{3}-\frac{\sigma^{2}}{12} . \tag{a.26}
\end{equation*}
$$

Consider now for $t \in[0,1]$ the curve $\gamma(t):=\left(t z, \varphi_{T}(t z), \psi_{T}(t z), \boldsymbol{\Phi}\left(t z, \varphi_{T}(t z), \psi_{T}(t z)\right)\right) \in$ $\mathbb{R}^{n+k}$. As $\gamma(0)=0$ and $\gamma(1) \notin \mathbf{B}_{1-\sigma / 6}$, there is by the mean value Theorem a $t \in[0,1]$ such that $|\gamma(t)|=1-\sigma / 6$. Let $\tilde{s}:=\min \{t \in[0,1]:|\gamma(t)|=1-\sigma / 6\}>0$. Then for all $0<s<\tilde{s}$, we have $\gamma(s) \in \mathbf{B}_{1-\sigma / 6}$ and by (a.24), $\psi_{T}(s z)^{2}<\sigma / 12$. But then we get by (a.26)

$$
\begin{aligned}
|\gamma(\tilde{s})-\gamma(s)| & \geq\left|\psi_{T}(\tilde{s} z)-\psi_{T}(s z)\right| \\
& \geq \sqrt{\frac{\sigma}{3}-\frac{\sigma^{2}}{12}-\left|\boldsymbol{\Phi}\left(\tilde{s} z, \varphi_{T}(\tilde{s} z), \psi_{T}(\tilde{s} z)\right)\right|^{2}}-\sqrt{\frac{\sigma}{12}} \\
& \geq \sqrt{\frac{\sigma}{4}-|D \boldsymbol{\Phi}|^{2}\left(1-\frac{\sigma}{3}\right)^{2}}-\sqrt{\frac{\sigma}{12}} \\
& \geq \frac{\sqrt{\sigma}}{24}
\end{aligned}
$$

where we used the assumption of the lemma in the last inequality. As $0<s<\tilde{s}$ was arbitrary, this contradicts the continuity of $\gamma$. Hence, (a.25) holds true.

And then $\operatorname{spt}(T)\left\llcorner\mathbf{C}_{1-\sigma / 2}\right.$ stays in the unit ball: We denote by $p$ to projection to $\mathbb{R}^{n+1}$. Then as $T$ is minimizing in $\mathcal{M}, p_{\#} T$ is minimizing a parametric integrand described Lemma 3.6. Then we can use [26, Corollary 4.2] to deduce that $\operatorname{spt}\left(p_{\#} T\right)$ is contained in the convex hull of $\operatorname{spt}\left(\partial\left(p_{\#} T\right)\right)$. Hence, $\operatorname{spt}\left(p_{\#} T\left\llcorner\mathbf{C}_{1-\sigma / 2}\right) \subset \overline{\mathbf{B}}_{1-\sigma / 6}\right.$. Using the fact that $T=(\mathrm{id}, \boldsymbol{\Phi})_{\#} p_{\#} T$ and $|D \boldsymbol{\Phi}| \leq \sigma / 6$, we conclude that $\operatorname{spt}(T)\left\llcorner\mathbf{C}_{1-\sigma / 2} \subset \overline{\mathbf{B}}_{1}\right.$.

## A. 4 PROOF OF REMARK 4.4

Proof. (i.) we choose $\sigma=1 / 4$ in Lemma 3.4 and get that

$$
\sup _{\mathrm{C}_{3 / 4} \cap \operatorname{spt}(T)} X_{n+1}^{2} \leq 4^{2 n+1} C_{13} C_{14}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right) \leq\left(\frac{1}{8}\right)^{2}
$$

(ii.) We first check, whether we created additional boundary while taking the intersection with $\mathbf{B}_{3}$. If this were the case, then for $|\omega| \leq \frac{1}{8}$, there is a point $(u, v)$ in

$$
\left\{x \in \gamma_{\omega}(\mathcal{M}):\left|\left(x_{1}, \ldots, x_{n-1}\right)\right| \leq \frac{1}{2},\left|x_{n}\right|<\frac{1}{2}\right\} \cap \gamma_{\omega}\left(X_{n+1}^{-1}\left(\left[-\frac{1}{8}, \frac{1}{8}\right]\right) \cap \partial \mathbf{B}_{3 / 4} \cap \mathcal{M}\right)
$$

with

- $u=\left(x_{1}, \ldots, x_{n-1}, x_{n} \cos (\omega)-x_{n+1} \sin (\omega), x_{n} \sin (\omega)+x_{n+1} \cos (\omega)\right)$
- $v=\boldsymbol{\Phi}\left(x_{1}, \ldots, x_{n-1}, x_{n} \cos (\omega)-x_{n+1} \sin (\omega), x_{n} \sin (\omega)+x_{n+1} \cos (\omega)\right)$
- $\left|x_{n+1}\right| \leq \frac{1}{8}$
- $x_{1}^{2}+\cdots+x_{n+1}^{2}+\left|\boldsymbol{\Phi}\left(x_{1}, \ldots x_{n+1}\right)\right|^{2}=\frac{9}{16}$
- $x_{1}^{2}+\cdots+x_{n-1}^{2} \leq \frac{1}{4}$
- $\left|x_{n} \cos (\omega)-x_{n+1} \sin (\omega)\right|<\frac{1}{2}$.

This implies that $x_{n}^{2} \geq \frac{19}{64}-\left|\boldsymbol{\Phi}\left(x_{1}, \ldots, x_{n+1}\right)\right|^{2} \geq \frac{9}{32}$ and hence,

$$
\begin{aligned}
\frac{1}{2} & >\left|x_{n} \cos (\omega)-x_{n+1} \sin (\omega)\right| \\
& \geq \sqrt{\frac{9}{32}} \cos (\omega)+\frac{1}{8}(\cos (\omega)-\sin (\omega)) \\
& \geq \frac{\sqrt{19}-1}{8} \cos \left(\frac{1}{8}\right)+\frac{1}{8}\left(\cos \left(\frac{1}{8}\right)-\sin \left(\frac{1}{8}\right)\right) \\
& >\frac{1}{2}
\end{aligned}
$$

Hence, there is no such $x$ and the intersection is trivial, thus we have

$$
\partial\left(\left(\boldsymbol{\mu}_{4 \#} \gamma_{\omega \#} T\right)\left\llcorner\mathbf{B}_{3}\right)=\left(\partial\left(\boldsymbol{\mu}_{4 \#} \gamma_{\omega \#} T\right)\right)\left\llcorner\mathbf{B}_{3} .\right.\right.
$$

The remaining conditions for $\left(\boldsymbol{\mu}_{4 \#} \gamma_{\omega \#} T\right)\left\llcorner\mathbf{B}_{3}\right.$ to belong to $\mathcal{T}$ follow like in the original paper [27, Remark 2.3].
(iii.) We write $\left(\boldsymbol{\mu}_{r \#} \gamma_{\omega \#} T\right)\left\llcorner\mathbf{B}_{3}=\left(\boldsymbol{\mu}_{r / 4 \#} \boldsymbol{\mu}_{4 \#} \gamma_{\omega \#} T\right)\left\llcorner\mathbf{B}_{3}\right.\right.$ in order to use Remark 4.4. As in the original paper [27, Remark 2.3], we deduce

$$
\sup \left\{x_{n+1}^{2}: x \in \operatorname{spt}\left(\left(\gamma_{\omega \#} T\right)\left\llcorner\mathbf{C}_{1 / 2}\right)\right\} \leq 4\left(\omega^{2}+\sup _{\mathbf{C}_{3 / 4} \cap \operatorname{spt}(T)}\left|X_{n+1}\right|\right) .\right.
$$

Hence, by using Lemma 3.3 (with $\sigma \uparrow 0$ and Lemma 3.4, we have

$$
\begin{align*}
\mathbf{E}_{C}\left(\left(\boldsymbol{\mu}_{4 \#} \gamma_{\omega \#} T\right)\right. & \left\llcorner\mathbf{B}_{3}, 1\right) \leq C_{10}\left(C_{11} \sup _{\mathrm{C}_{2} \cap \operatorname{spt}\left(\mu_{\mu \# \#} \gamma_{\omega \neq T)}\right.} X_{n+1}^{2}+\frac{\kappa_{T}+\mathbf{A}}{4}\right) \\
& \leq C_{10}\left(16 C_{11} \sup _{\mathrm{C}_{1 / 2} \cap \operatorname{spt}\left(\gamma_{\omega \# T} T\right)} X_{n+1}^{2}+\kappa_{T}+\mathbf{A}\right) \\
& \leq C_{10}\left(4^{3} C_{11} \omega^{2}+4^{3} C_{11} \sup _{\mathrm{C}_{3 / 4} \cap \operatorname{spt}(T)} X_{n+1}^{2}+\kappa_{T}+\mathbf{A}\right)  \tag{a.27}\\
& \leq \frac{C_{21}}{C_{20}}\left(\omega^{2}+\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right)  \tag{a.28}\\
& \leq \frac{1}{C_{20}} .
\end{align*}
$$

Thus, we can use Remark 4.4 and conclude.

## A. 5 proof of lemma 6.1

Proof. The plan to prove this lemma is as follows: First, we bound the excess with $\int X_{n+1}^{2} \mathrm{~d}\|T\|$ by Lemma 3.3. Then, we construct a vectorfield and compute the associated first variation. By minimality of $T$ this can be expressed by the mean curvature vector. Moreover, by the choice of the vectorfield, we can bound $\int X_{n+1}^{2} \mathrm{~d}\|T\|$ with $\int\left|X^{\perp}\right|^{2}|X|^{-2} \mathrm{~d}\|T\|$. By Corollary 2.5 this carries over to the spherical excess.
Let $T$ be as in the lemma and $C_{10}$ as in Lemma 3.3. Moreover, we define

$$
\begin{aligned}
& C_{32}=2^{2 n+2} C_{13} C_{14}, \\
& C_{33}=3^{2 n+8} C_{10}\left(1+m \omega_{n}\right) .
\end{aligned}
$$

We apply Lemma 3.4 with $\sigma=1 / 2$ to deduce

$$
\sup _{\mathrm{C}_{1 / 2} \operatorname{spt}(T)} X_{n+1}^{2} \leq 2^{2 n+1} C_{13} C_{14}\left(\mathbf{E}_{C}(T, 1)+\kappa_{T}+\mathbf{A}\right) \leq \frac{1}{2}
$$

Hence, for all $x=(\tilde{x}, \tilde{y}) \in \mathbf{C}_{1 / 2} \cap \operatorname{spt}(T)$ the following holds

$$
\begin{equation*}
|x|^{2} \leq\left(1+|D \boldsymbol{\Phi}(\tilde{x})|^{2}\right)\left(|\mathbf{p}(x)|^{2}+x_{n+1}^{2}\right) \leq \frac{4}{3}\left(\frac{1}{4}+\frac{1}{2}\right)=1 . \tag{a.29}
\end{equation*}
$$

For $x=(\tilde{x}, \tilde{y}) \in \mathbb{R}^{n+k}$ the projection to the tangent space of $\mathcal{M}$ at $(\tilde{x}, \boldsymbol{\Phi}(\tilde{x}))$ is given by

$$
P=P_{\tilde{x}}:=M g^{-1} M^{T}=\binom{\mathrm{id}}{D \boldsymbol{\Phi}} g^{-1}\left(\begin{array}{ll}
\mathrm{id} & D \boldsymbol{\Phi}
\end{array}\right)=\left(\begin{array}{cc}
g^{-1} & g^{-1} \cdot D \boldsymbol{\Phi} \\
\left(g^{-1} \cdot D \boldsymbol{\Phi}\right)^{T} & D \boldsymbol{\Phi}^{T} \cdot g^{-1} \cdot D \boldsymbol{\Phi}
\end{array}\right) .
$$

Therefore

$$
\begin{equation*}
\operatorname{tr}_{n+1}(P):=\sum_{i=1}^{n+1} P_{i i} \leq n+1+C_{73}|D \boldsymbol{\Phi}|^{2} \tag{a.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|(P-\mathrm{id})\binom{\tilde{x}}{0}\right|=\left|\binom{g^{-1} \tilde{x}-\tilde{x}}{D \boldsymbol{\Phi}\left(g^{-1} \tilde{x}\right)}\right| \leq C_{74}|D \boldsymbol{\Phi}(\tilde{x})|, \tag{a.31}
\end{equation*}
$$

where we used (a.15).
Denote by $v$ the outer unit normal vector being tangent to $\mathcal{M}$ and normal to the approximate tangent space of $T$. As $\boldsymbol{v}=\left(\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{n+k}\right) \in \operatorname{span}\left\{\left(\mathbf{e}_{i}, \partial_{i} \boldsymbol{\Phi}\right): i \leq n+1\right\}$, we have

$$
\boldsymbol{v}_{n+1+j}=\sum_{i=1}^{n+1} \boldsymbol{v}_{i} \partial_{i} \boldsymbol{\Phi}^{j} \quad \text { for all } j \leq k-1 .
$$

Denote by $\tilde{\boldsymbol{v}}=\left(\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{k+1}\right)$. Then the following holds

$$
\begin{equation*}
|\boldsymbol{v}| \leq(1+|D \boldsymbol{\Phi}|)|\tilde{\boldsymbol{v}}| . \tag{a.32}
\end{equation*}
$$

Moreover, define $A:=\mathbf{B}_{1} \backslash B_{1 / 4}$ where $B_{1 / 4}=\overline{\mathbf{B}}_{1 / 4}^{n+1} \times \mathbb{R}^{k-1}$. Denote $\kappa:=\kappa_{T}, \varepsilon:=$ $\sqrt{\mathbf{E}_{C}(T, 1 / 3)}, \beta:=4 C_{33}^{-1 / 2}$ and for all $x \in \mathbb{R}^{n+k}$ let

$$
\lambda(x):=\max \left\{0, \frac{x_{n+1}}{|\tilde{x}|}-\beta \varepsilon-\kappa\right\} .
$$

Then in $A$ we have

$$
\begin{align*}
\left|\left\langle(\tilde{X}, 0), D_{\vec{T}} \lambda\right\rangle\right| & \leq\left|\left\langle(\tilde{X}, 0), D_{\vec{T}}\left(\frac{X_{n+1}}{|\tilde{X}|}\right)\right\rangle\right| \\
& =\left|\left\langle(\tilde{X}, 0),\left((P-\boldsymbol{v} \otimes \boldsymbol{v})\left(\frac{\mathbf{e}_{n+1}}{|\tilde{X}|}-\frac{X_{n+1}}{|\tilde{X}|^{3}}(\tilde{X}, 0)\right)\right)\right\rangle\right| \\
& \leq\left|\frac{\boldsymbol{v}_{n+1}}{|\tilde{X}|}\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle-\frac{X_{n+1}}{|\tilde{X}|^{3}}\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle^{2}\right|+8 C_{74}|D \boldsymbol{\Phi}|  \tag{a.33}\\
& \leq 2\left|\frac{\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle}{|\tilde{X}|}\right|+8 C_{74}|D \boldsymbol{\Phi}|
\end{align*}
$$

Let $k \in \mathbb{N}$ with $k \geq 1$ and choose a $\mathcal{C}^{1}$ function $\mu_{k}: \mathbb{R} \rightarrow \mathbb{R}$ such that for $t \geq 1 / 4$ we have

$$
\mu_{k}(t)=\max \left\{0, t^{-n}-1\right\}^{1+1 / k}
$$

Moreover, let $h_{k}: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ be a $\mathcal{C}^{1}$ vectorfield satisfying $\left.h_{k}\right|_{B_{1 / 4} \cap \operatorname{spt}(T)} \equiv 0$ and

$$
h_{k}(x)=\lambda^{2}(x) \mu_{k}(|\tilde{x}|)(\tilde{x}, 0) \quad \text { for } x \notin B_{1 / 4}
$$

Notice that for $x \in\left(\operatorname{spt}(\partial T) \cap \mathbf{B}_{2}\right) \subset\left\{x \in \mathbb{R}^{n+k}: x_{n+1} \leq|\tilde{x}|(\beta \varepsilon+\kappa)\right\}$ we have $\lambda(x)=0$, and when $|\tilde{x}| \geq 1, \mu_{k}(|\tilde{x}|)=0$. Hence, $h_{k}$ vanishes on

$$
\begin{equation*}
\operatorname{spt}(\partial T) \cup\left(B_{1 / 4} \cap \operatorname{spt}(T)\right) \cup\left\{x \in \mathbb{R}^{n+k}: x_{n+1} \leq|\tilde{x}|(\beta \varepsilon+\kappa)\right\} \tag{a.34}
\end{equation*}
$$

and by [21, Thereom 3.2], $\int_{\mathbf{B}_{3}} \operatorname{div}_{\vec{T}} h_{k} \mathrm{~d}\|T\|=-\int h_{k} \cdot \vec{H}_{T} \mathrm{~d}\|T\|$.
We compute

$$
\begin{aligned}
\operatorname{div}_{\vec{T}} h_{k}= & \sum_{j=1}^{n+1}\left((P-\boldsymbol{v} \otimes \boldsymbol{v})\left(2 X_{j} \lambda \mu_{k} D \lambda+X_{j} \lambda^{2} \frac{\mu_{k}^{\prime}}{|\tilde{X}|}(\tilde{X}, 0)+\mathbf{e}_{j} \lambda^{2} \mu_{k}\right)\right)_{j} \\
= & 2 \lambda \mu_{k}\left\langle(\tilde{X}, 0), D_{\vec{T}} \lambda\right\rangle+\lambda^{2} \mu_{k}^{\prime}\left\langle(\tilde{X}, 0),(P-\boldsymbol{v} \otimes \boldsymbol{v}) \frac{(\tilde{X}, 0)}{|\tilde{X}|}\right\rangle \\
& +\operatorname{tr}_{n+1}(P-\boldsymbol{v} \otimes \boldsymbol{v}) \lambda^{2} \mu_{k}
\end{aligned}
$$

Using (a.34), (a.30), (a.31), (a.32) and (a.33) we find

$$
\begin{aligned}
\lim _{k \rightarrow \infty} & \int_{A} h_{k} \cdot \vec{H}_{T} \mathrm{~d}\|T\| \\
\quad \leq & \lim _{k \rightarrow \infty} \int_{A} 4 \lambda \mu_{k}\left|\frac{\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle}{|\tilde{X}|}\right|+\lambda^{2} \mu_{k}^{\prime}\left\langle\tilde{X},(\mathrm{id}-\tilde{\boldsymbol{v}} \otimes \tilde{\boldsymbol{v}}) \frac{\tilde{X}}{|\tilde{X}|}\right\rangle+n \lambda^{2} \mu_{k} \mathrm{~d}\|T\|+C_{75} \mathbf{A} \\
= & \int_{A} 4 \lambda\left(|\tilde{X}|^{-n}-1\right)\left|\frac{\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle}{|\tilde{X}|}\right|+\lambda^{2} n|\tilde{X}|^{-n}-\lambda^{2} n|\tilde{X}|^{-n-2}\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle^{2} \mathrm{~d}\|T\| \\
& +\int_{A} n \lambda^{2}\left(|\tilde{X}|^{-n}-1\right) \mathrm{d}\|T\|+C_{75} \mathbf{A} \\
= & \int_{A}\left(4 \lambda\left(|\tilde{X}|^{-n}-1\right)\left|\frac{\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle}{|\tilde{X}|}\right|-\lambda^{2} n|\tilde{X}|^{-n-2}\langle\tilde{\boldsymbol{v}}, \tilde{X}\rangle^{2}-n \lambda^{2}\right) \mathrm{d}\|T\|+C_{75} \mathbf{A}
\end{aligned}
$$

and hence,

$$
\begin{aligned}
n \int_{A} \lambda^{2} \mathrm{~d}\|T\| & \leq \int_{A}\left(4 \lambda\left(|\tilde{X}|^{-n}-1\right)\left|\frac{\langle\tilde{X}, \tilde{v}\rangle}{|\tilde{X}|}\right|-\lambda^{2} n|\tilde{X}|^{-n-2}\langle\tilde{v}, \tilde{X}\rangle^{2}\right) \mathrm{d}\|T\|+C_{76} \mathbf{A} \\
& \leq C_{77}\left(\int_{A} \lambda\left|\frac{\langle\tilde{X}, \tilde{v}\rangle}{|\tilde{X}|}\right| \mathrm{d}\|T\|+\mathbf{A}\right) \\
& \leq \frac{n}{2} \int_{A} \lambda^{2} \mathrm{~d}\|T\|+\frac{C_{78}}{2}\left(\int_{A}\left|\frac{\langle\tilde{X}, \tilde{\tilde{x}}\rangle}{|\tilde{X}|}\right|^{2} \mathrm{~d}\|T\|+\mathbf{A}\right)
\end{aligned}
$$

We conclude

$$
\int_{A} \lambda^{2} \mathrm{~d}\|T\| \leq C_{78}\left(\int_{A}\left|\frac{\langle\tilde{X}, \tilde{\boldsymbol{v}}\rangle}{|\tilde{X}|}\right|^{2} \mathrm{~d}\|T\|+\mathbf{A}\right)
$$

We argue in the same way to prove the same inequality for

$$
\tilde{\lambda}:=\min \left\{0, \frac{X_{n+1}}{|\tilde{X}|}+\beta \varepsilon+\kappa\right\} .
$$

As the $\operatorname{spt}(\lambda)=\left\{x \in \mathbb{R}^{n+k}: x_{n+1} \geq|\tilde{x}|(\beta \varepsilon+\kappa)\right\}$ and $\operatorname{spt}(\tilde{\lambda})=\left\{x \in \mathbb{R}^{n+k}: x_{n+1} \leq\right.$ $-|\tilde{x}|(\beta \varepsilon+\kappa)\}$, we see that $\operatorname{spt}\left(\lambda^{2}+\tilde{\lambda}^{2}\right)=\left\{x \in \mathbb{R}^{n+k}:\left|x_{n+1}\right| \geq|\tilde{x}|(\beta \varepsilon+\kappa)\right\}$ and hence

$$
\begin{aligned}
& \int_{A} X_{n+1}^{2} \mathrm{~d}\|T\| \\
& \leq \int_{A} \frac{X_{n+1}^{2}}{|\tilde{X}|^{2}} \mathrm{~d}\|T\| \\
& =\int_{A}\left(\frac{X_{n+1}}{|\tilde{X}|}-(\beta \varepsilon+\kappa)\right)\left(\frac{X_{n+1}}{|\tilde{X}|}+(\beta \varepsilon+\kappa)\right) \mathrm{d}\|T\|+(\beta \varepsilon+\kappa)^{2}\|T\|(A) \\
& \leq \int_{A}\left|\frac{X_{n+1}}{|\tilde{X}|}-(\beta \varepsilon+\kappa)\right|\left|\frac{X_{n+1}}{|\tilde{X}|}+(\beta \varepsilon+\kappa)\right| \mathbb{1}_{\text {spt }\left(\lambda^{2}+\tilde{\lambda}^{2}\right)} \mathrm{d}\|T\|+(\beta \varepsilon+\kappa)^{2}\|T\|(A) \\
& \leq \frac{1}{2} \int_{A}\left(\lambda^{2}+\tilde{\lambda}^{2}\right) \mathrm{d}\|T\|+2\left(\beta^{2} \varepsilon^{2}+\kappa^{2}\right)\|T\|(A) \\
& \leq C_{79}\left(\int_{A}\left|\frac{\langle\tilde{X}, v\rangle}{|\tilde{X}|}\right|^{2} \mathrm{~d}\|T\|+\mathbf{A}\right)+2\left(\beta^{2} \varepsilon^{2}+\kappa^{2}\right)\|T\|(A) \\
& \leq C_{78}\left(\int_{A}\left|X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\|+\mathbf{A}\right)+2\left(\beta^{2} \varepsilon^{2}+\kappa^{2}\right)\|T\|(A) .
\end{aligned}
$$

Notice that by the assumption of the lemma

$$
\int_{B_{1 / 4}} X_{n+1}^{2} \mathrm{~d}\|T\| \leq \frac{\mathbf{E}_{\mathcal{C}}(T, 1)}{C_{33}}\|T\|\left(B_{1 / 4}\right)=\frac{\mathbf{E}_{\mathcal{C}}(T, 1)}{16} \beta^{2}\|T\|\left(B_{1 / 4}\right) \leq \epsilon^{2} \beta^{2}\|T\|\left(B_{1 / 4}\right) .
$$

We use Lemma 3.3 (with $T, \sigma$ replaced by $\left(\mu_{3 \# T} T\right)\left\llcorner\mathbf{B}_{3}, 1 / 2\right.$ ), (a.29) and Corollary 2.5 (with $s=1$ ) to deduce

$$
\begin{aligned}
\varepsilon^{2} & =\mathbf{E}_{C}\left(\left(\boldsymbol{\mu}_{3 \#} T\right)\left\llcorner\mathbf{B}_{3}, 1\right)\right. \\
& \leq 4 C_{10}\left(\kappa_{\left(\mu_{3 \#} T\right)\left\llcorner\mathbf{B}_{3}\right.}+\int_{\mathbf{C}_{3 / 2}} X_{n+1}^{2} \mathrm{~d}\left\|\boldsymbol{\mu}_{3 \#} T\right\|+\mathbf{A}_{\mu_{3}(\mathcal{M})}\right) \\
& \leq 4 \cdot 3^{n} C_{10}\left(\kappa+\int_{\mathbf{C}_{1 / 2}} X_{n+1}^{2} \mathrm{~d}\|T\|+\mathbf{A}\right) \\
& \leq 3^{n+2} C_{10}\left(\kappa+\int_{\mathbf{B}_{1}} X_{n+1}^{2} \mathrm{~d}\|T\|+\mathbf{A}\right) \\
& \leq 3^{n+2} C_{10}\left(C_{78}\left(\int_{A}\left|X^{\perp}\right|^{2}|X|^{-n-2} \mathrm{~d}\|T\|+2 \mathbf{A}\right)+2 \mathbb{M}(T)\left(\beta^{2} \varepsilon^{2}+\kappa\right)\right) \\
& \leq 3^{n+2} C_{10}\left(C_{78}\left(\mathbf{E}_{S}(T, 1)+C_{4} \kappa+\left(2+C_{4}\right) \mathbf{A}\right)+2 \mathbb{M}(T)\left(\beta^{2} \varepsilon^{2}+\kappa\right)\right) \\
& \leq \frac{3^{2 n+3} C_{10}\left(1+m \boldsymbol{\omega}_{n}\right) 16}{3^{2 n+8}\left(1+m \omega_{n}\right) C_{10}} \varepsilon^{2}+\frac{C_{34}}{2}\left(\mathbf{E}_{S}(T, 1)+\kappa+\mathbf{A}\right) \\
& \leq \frac{\varepsilon^{2}}{2}+\frac{C_{34}}{2}\left(\mathbf{E}_{S}(T, 1)+\kappa+\mathbf{A}\right) .
\end{aligned}
$$

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[^0]:    1 Here and in the sequel we call a constant geometric if it depends only on $n, Q, N_{0}, M_{0}, C_{e}^{b}, C_{e}^{\natural}, C_{h}$ which we fixed.

